```
On[Assert];
$Path = Union[Append[$Path, NotebookDirectory[]]];
<< PotentialIntegration`
(* Show the double surface integrand *)
vol_rp@vol_r[A[r] B[rp] grad_r@grad_rp@g[r - rp]] //. vol2surface //. vol2surface2 //
  Simplify // disp
```

$$\mathrm{surf}_{r'}\left[\mathrm{surf}_r\left[A\, n_r \otimes \left(\frac{1}{2}\, g\, \left(n_{r'} \odot (r - r')\, \nabla B + 2\, B\, n_{r'}\right)\right) - \frac{1}{6}\, g\, \left(n_r \odot \mathrm{tr}_{1,2}[n_{r'} \otimes (r - r') \otimes (r - r')]\, \mathrm{rot}_{1,2}[\nabla B \otimes \nabla A] + 3\, B\, n_r \odot (r - r')\, \mathrm{rot}_{1,2}[n_{r'} \otimes \nabla A]\right)\right]\right]$$

```
(* Step 1: transform the double volume integral
  to a double surface integral with ∫dr' outside *)
ClearAll[extractIntegrand]
extractIntegrand[surf_rp[a_]] := a
doubleSurfaceIntegrand1 = extractIntegrand[
   vol_rp@vol_r[A[r] B[rp] grad_r@grad_rp@g[r - rp]] //. vol2surface //. vol2surface2];
(* Step 2: Rewrite the integrand as as a function of r-
  r' (only A(r) is affected ) *)
doubleSurfaceIntegrand2 = doubleSurfaceIntegrand1 /.
   A[r] → A[rp] + tr_{1,2}[∇A ** (r - rp)];
(* Step 3: Express the integral in terms of I0= ∫g,
I1=∫(r-r')g, I2=∫(r-r')^2 g *)
step3moveTermsOutside = Join[{
     (* n_r can be moved outside since its constant over a triangle *)
     n_r :→ n_rf,
     (* move trace outside of the surface integral *)
     c_ tr_{1,2}[a_ ** b_] /; rank[a] == 1 && rank[b] == 1 :→ tr_{1,2}[a ** b ** c],
     (* expand sums in the surface integral *)
     surf_r[k_ a_ ** (b_ + c_)] /; b + c =!= r - rp :→ surf_r[k a ** b] + surf_r[k a ** c]
    },
   moveOpOutside[tr],
   moveLinearOutside[surf],
   simplifications
  ];
step3replaceTerms = {
   surf_r[g[r - rp]] → I_0,
   surf_r[(r - rp) g[r - rp]] → I_1,
   surf_r[(r - rp) ** (r - rp) g[r - rp]] → I_2
  };
step3simplify = Join[{
     rot_{1,2}[a_ ** b_] /; rank[a] == 1 && rank[b] == 1 :→ b ** a,
     (* move tr inside *)
     tr_v__[c_ a_] /; rank[c] == 0 :→ c tr_v[a],
```

```
      tr_v__[a_ ** (b_ + c_)] :> tr_v[a ** b] + tr_v[a ** c],
      tr_{1,2}[a_ ** b_ ** c_] /; rank[a] == 1 && rank[b] == 1 :> tr_{1,2}[a ** b] c,
      tr_{1,2}[a_ ** b_ ** c_] /; rank[a] == 1 && rank[b] == 2 :> tr_{1,2}[a ** b] ** c
    }, simplifications ];
doubleSurfaceIntegrand3 =
  Expand[doubleSurfaceIntegrand2 //. step3moveTermsOutside /.
     step3replaceTerms] //. step3simplify;
(* Pretty print the result *)
rank[r' | n | n'] := 1
step3show = {
    rp → r',
    n_rf → n,
    n_rp → n',
    (op : A | B) [r'] :> op,
    tr_{1,2}[a_ ** tr_{1,2}[b_ ** I_2]] /; rank[a] == 1 && rank[b] == 1 :> I_2[a, b]
  };
doubleSurfaceIntegrand3 //. step3show // disp
```

$$\frac{1}{2}\,A\,n'\odot\mathcal{I}_1\,n\otimes\nabla B + B\,\nabla A\odot\mathcal{I}_1\,n\otimes n' - \frac{1}{2}\,B\,n\odot\mathcal{I}_1\,\nabla A\otimes n' +$$
$$A\,B\,n\otimes n'\,\mathcal{I}_0 - \frac{1}{6}\,\nabla A\otimes\nabla B\,\mathcal{I}_2[n,\ n'] + \frac{1}{2}\,n\otimes\nabla B\,\mathcal{I}_2[\nabla A,\ n']$$

```
doubleSurfaceIntegrand3
```

$$A[rp]\,B[rp]\,n_{rf} ** n_{rp}\,\mathcal{I}_0 + B[rp]\,n_{rf} ** n_{rp}\,tr_{1,2}[\nabla A ** \mathcal{I}_1] +$$
$$\frac{1}{2}\,n_{rf} ** \nabla B\,tr_{1,2}[\nabla A ** tr_{1,2}[n_{rp} ** \mathcal{I}_2]] - \frac{1}{2}\,B[rp]\,\nabla A ** n_{rp}\,tr_{1,2}[n_{rf} ** \mathcal{I}_1] -$$
$$\frac{1}{6}\,\nabla A ** \nabla B\,tr_{1,2}[n_{rf} ** tr_{1,2}[n_{rp} ** \mathcal{I}_2]] + \frac{1}{2}\,A[rp]\,n_{rf} ** \nabla B\,tr_{1,2}[n_{rp} ** \mathcal{I}_1]$$

```
(* Check that the transformations have not changed the result *)
checkTensor[vol_rp@vol_r[A[r] B[rp] grad_r@grad_rp@g[r - rp]],
 surf_rp[doubleSurfaceIntegrand1]]
checkTensor[doubleSurfaceIntegrand1 /. rp → r0,
 surf_r[doubleSurfaceIntegrand3 /. rp → r0 /. n_rf → n_r /. I_k_ :> surfaceIntegrand[I_k]]]
```

Checked (ϵ=6.1×10^{-8}, nIter=1) that

$$\mathrm{vol}_{r'}[\mathrm{vol}_r[A\,B\,\nabla_r[\nabla_{r'}[g]]]]==\mathrm{surf}_{r'}\Big[\mathrm{surf}_r\Big[A\,n_r\otimes\left(\frac{1}{2}\,g\,n_{r'}\odot(r-r')\,\nabla B+B\,g\,n_{r'}\right)-\frac{1}{6}\,g\,n_r\odot \mathrm{tr}_{1,2}[n_{r'}\otimes(r-r')\otimes(r-r')]\,\mathrm{rot}_{1,2}[\nabla B\otimes\nabla A]-\frac{1}{2}\,B\,g\,n_r\odot(r-r')\,\mathrm{rot}_{1,2}[n_{r'}\otimes\nabla A]\Big]\Big]$$

in 0.17 s

Checked (ϵ=3.5×10^{-12}, nIter=1) that

$$\mathrm{surf}_r\Big[A\,n_r\otimes\left(\frac{1}{2}\,n_{r0}\odot(r-r0)\,\nabla B\,g[r-r0]+B[r0]\,g[r-r0]\,n_{r0}\right)-\frac{1}{6}\,n_r\odot \mathrm{tr}_{1,2}[n_{r0}\otimes(r-r0)\otimes(r-r0)]\,g[r-r0]\,\mathrm{rot}_{1,2}[\nabla B\otimes\nabla A]-\frac{1}{2}\,B[r0]\,n_r\odot(r-r0)\,g[r-r0]\,\mathrm{rot}_{1,2}[n_{r0}\otimes\nabla A]\Big]==\mathrm{surf}_r\Big[-\frac{1}{6}\,n_r\odot \mathrm{tr}_{1,2}[n_{r0}\otimes((r-r0)\otimes(r-r0)\otimes 1\,g[r-r0])]\,\nabla A\otimes\nabla B-\frac{1}{2}\,B[r0]\,n_r\odot((r-r0)\otimes[r-r0])\,\nabla A\otimes n_{r0}+\frac{1}{2}\,\nabla A\odot \mathrm{tr}_{1,2}[n_{r0}\otimes((r-r0)\otimes(r-r0)\otimes 1\,g[r-r0])]\,n_r\otimes\nabla B+\frac{1}{2}\,A[r0]\,n_{r0}\odot((r-r0)\otimes 1\,g[r-r0])\,n_r\otimes\nabla B+B[r0]\,\nabla A\odot((r-r0)\otimes 1\,g[r-r0])\,n_r\otimes n_{r0}+A[r0]\,B[r0]\,n_r\otimes n_{r0}\,g[r-r0]\Big]$$

in 0.066 s

```
(* Step 4: Compute the integrals I0, I1, I2 analytically *)
```