$$-\frac{1}{2}\Delta\psi_i(\vec{r}) + V(r)\psi_i(\vec{r}) = E\psi_i(\vec{r})$$
$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]$$
$$\psi_i(\vec{r}) = \frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi)$$
$$\Delta\psi_i(\vec{r}) = \Delta\left[\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi) \right]$$
$$\Delta\psi_i(\vec{r}) = \left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]\right\}\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi)$$
$$\frac{\partial}{\partial r}\left[\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi) \right] = -\frac{P_{nl}(r)}{r^2}Y_{lm}(\theta,\phi) + \frac{1}{r}\frac{\partial P_{nl}(r)}{\partial r}Y_{lm}(\theta,\phi)$$
$$\frac{2}{r}\left[-\frac{P_{nl}(r)}{r^2}Y_{lm}(\theta,\phi) + \frac{1}{r}\frac{\partial P_{nl}(r)}{\partial r}Y_{lm}(\theta,\phi) \right] =$$
$$-2\frac{P_{nl}(r)}{r^3}Y_{lm}(\theta,\phi) + \frac{2}{r^2}\frac{\partial P_{nl}(r)}{\partial r}Y_{lm}(\theta,\phi)$$
$$\frac{\partial^2}{\partial r^2}\left[\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi) \right] =$$
$$2\frac{P_{nl}(r)}{r^3}Y_{lm}(\theta,\phi) - \frac{2}{r^2}\frac{\partial P_{nl}(r)}{\partial r}Y_{lm}(\theta,\phi) + \frac{1}{r}\frac{\partial^2_{nl}P(r)}{\partial r^2}Y_{lm}(\theta,\phi)$$
$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right]\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi) = 2\frac{P_{nl}(r)}{r^3}Y_{lm}(\theta,\phi) - \frac{2}{r^2}\frac{\partial P_{nl}(r)}{\partial r}Y_{lm}(\theta,\phi) +$$
$$\frac{1}{r}\frac{\partial^2_{nl}P(r)}{\partial r^2}Y_{lm}(\theta,\phi) - 2\frac{P_{nl}(r)}{r^3}Y_{lm}(\theta,\phi) + \frac{2}{r^2}\frac{\partial P_{nl}(r)}{\partial r}Y_{lm}(\theta,\phi)$$
$$= \frac{1}{r}\frac{\partial^2_{nl}P(r)}{\partial r^2}Y_{lm}(\theta,\phi)$$
$$\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi) =$$
$$\frac{P_{nl}(r)}{r^3}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]Y_{lm}(\theta,\phi)\right\}$$
$$-\frac{1}{2}\Delta\psi_i(\vec{r}) + V(r)\psi_i(\vec{r}) - E\psi_i(\vec{r}) = 0$$
$$\Delta\psi_i(\vec{r}) = 2[V(r) - E]\psi_i(\vec{r})$$
$$\frac{1}{r}\frac{\partial^2 P_{nl}(r)}{\partial r^2}Y_{lm}(\theta,\phi) + \frac{P_{nl}(r)}{r^3}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]Y_{lm}(\theta,\phi)\right\} =$$
$$2[V(r) - E]\frac{P_{nl}(r)}{r}Y_{lm}(\theta,\phi)$$
$$\frac{1}{P_{nl}(r)}\frac{\partial^2 P_{nl}(r)}{\partial r^2} + \frac{1}{r^2}\frac{1}{Y_{lm}(\theta,\phi)}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]Y_{lm}(\theta,\phi)\right\} =$$
$$2[V(r) - E]$$
$$\frac{1}{P_{nl}(r)}\frac{d^2 P_{nl}(r)}{dr^2} - \frac{l(l+1)}{r^2} = 2[V(r) - E]$$
$$\frac{1}{r^2}\frac{1}{Y_{lm}(\theta,\phi)}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]Y_{lm}(\theta,\phi)\right\} = -\frac{l(l+1)}{r^2}$$
$$\frac{d^2 P_{nl}(r)}{dr^2} - \frac{l(l+1)}{r^2}P_{nl}(r) = 2[V(r) - E]P_{nl}(r)$$
$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]=\hat{l}^2$$

$$\therefore\ \left\{\begin{array}{l}\dfrac{d^2P_{nl}(r)}{dr^2}-\dfrac{l(l+1)}{r^2}P_{nl}(r)=2[V(r)-E]P_{nl}(r)\\ \hat{l}^2Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\end{array}\right.$$