

$$\begin{aligned}
& \left[-\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} + \int d\vec{r}_2 |\phi(\vec{r}_2)|^2 \frac{1}{|\vec{r}_1 - \vec{r}_2|} \right] \phi(\vec{r}_1) = E' \phi(\vec{r}_1) \\
E' &= E - \int d\vec{r}_2 \phi^*(\vec{r}_2) \left(-\frac{1}{2}\nabla_2^2 - \frac{2}{r_2} \right) \phi(\vec{r}_2) \\
E' &= E - \int d\vec{r}_1 \phi^*(\vec{r}_1) \left(-\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} \right) \phi(\vec{r}_1) \\
E &= E' + \int d\vec{r}_1 \phi^*(\vec{r}_1) \left(-\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} \right) \phi(\vec{r}_1) \\
E &= \phi^*(\vec{r}_1) \left[-\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} + \int d\vec{r}_2 |\phi(\vec{r}_2)|^2 \frac{1}{|\vec{r}_1 - \vec{r}_2|} \right] \phi(\vec{r}_1) + \\
& \quad \int d\vec{r}_1 \phi^*(\vec{r}_1) \left(-\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} \right) \phi(\vec{r}_1) \\
E &= 2\phi^*(\vec{r}_1) \left[-\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} + \int d\vec{r}_2 |\phi(\vec{r}_2)|^2 \frac{1}{|\vec{r}_1 - \vec{r}_2|} \right] \phi(\vec{r}_1) - \int \int d\vec{r}_1 d\vec{r}_2 \frac{|\phi(\vec{r}_1)|^2 |\phi(\vec{r}_2)|^2}{|\vec{r}_1 - \vec{r}_2|} \\
E &= 2E' - \int \int d\vec{r}_1 d\vec{r}_2 \frac{|\phi(\vec{r}_1)|^2 |\phi(\vec{r}_2)|^2}{|\vec{r}_1 - \vec{r}_2|} \\
\therefore E &= 2E' - \int_0^{r_c} r U(r) |\phi(r)|^2 dr
\end{aligned}$$