$$-\frac{1}{2}\Delta\psi_i(\vec{r}) + V(r)\psi_i(\vec{r}) = E\psi_i(\vec{r})$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\psi_i(\vec{r}) = r^l L_{nl}(r) Y_{lm}(\theta,\phi)$$

$$\Delta\psi_i(\vec{r}) = \Delta[r^l L_{nl}(r) Y_{lm}(\theta,\phi)]$$

$$\Delta\psi_i(\vec{r}) = \left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right\} r^l L_{nl}(r) Y_{lm}(\theta,\phi)$$

$$\frac{\partial}{\partial r}[r^l L_{nl}(r) Y_{lm}(\theta,\phi)] = l r^{l-1} L_{nl}(r) Y_{lm}(\theta,\phi) + r^l \frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi)$$

$$\frac{2}{r}\left[l r^{l-1} L_{nl}(r) Y_{lm}(\theta,\phi) + r^l \frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi)\right] = 2l r^{l-2} L_{nl}(r) Y_{lm}(\theta,\phi) +$$

$$2r^{l-1}\frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi)$$

$$\frac{\partial^2}{\partial r^2}[r^l L_{nl}(r) Y_{lm}(\theta,\phi)] =$$

$$l(l-1) r^{l-2} L_{nl}(r) Y_{lm}(\theta,\phi) + 2l r^{l-1}\frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi) + r^l \frac{\partial^2 L_{nl}(r)}{\partial r^2} Y_{lm}(\theta,\phi)$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] r^l L_{nl}(r) Y_{lm}(\theta,\phi) = l(l-1) r^{l-2} L_{nl}(r) Y_{lm}(\theta,\phi) + 2l r^{l-1}\frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi) +$$

$$r^l \frac{\partial^2 L_{nl}(r)}{\partial r^2} Y_{lm}(\theta,\phi) + 2l r^{l-2} L_{nl}(r) Y_{lm}(\theta,\phi) + 2r^{l-1}\frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi)$$

$$= r^l \frac{\partial^2 L_{nl}(r)}{\partial r^2} Y_{lm}(\theta,\phi) + 2(l+1) r^{l-1}\frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi) + l(l+1) r^{l-2} L_{nl}(r) Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] r^l L_{nl}(r) Y_{lm}(\theta,\phi) =$$

$$r^{l-2} L_{nl}(r)\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi)\right\}$$

$$-\frac{1}{2}\Delta\psi_i(\vec{r}) + V(r)\psi_i(\vec{r}) - E\psi_i(\vec{r}) = 0$$

$$\Delta\psi_i(\vec{r}) = 2[V(r) - E]\psi_i(\vec{r})$$

$$r^l \frac{\partial^2 L_{nl}(r)}{\partial r^2} Y_{lm}(\theta,\phi) + 2(l+1) r^{l-1}\frac{\partial L_{nl}(r)}{\partial r} Y_{lm}(\theta,\phi) + l(l+1) r^{l-2} L_{nl}(r) Y_{lm}(\theta,\phi) +$$

$$r^{l-2} L_{nl}(r)\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi)\right\} =$$

$$2[V(r) - E] r^l L_{nl}(r) Y_{lm}(\theta,\phi)$$

$$\frac{1}{L_{nl}(r)}\left[\frac{\partial^2 L_{nl}(r)}{\partial r^2} + \frac{2(l+1)}{r}\frac{\partial L_{nl}(r)}{\partial r}\right] + \frac{l(l+1)}{r^2} +$$

$$\frac{1}{r^2}\frac{1}{Y_{lm}(\theta,\phi)}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi)\right\} = 2[V(r) - E]$$

$$\frac{1}{L_{nl}(r)}\left[\frac{\partial^2 L_{nl}(r)}{\partial r^2} + \frac{2(l+1)}{r}\frac{\partial L_{nl}(r)}{\partial r}\right] = 2[V(r) - E]$$

$$\frac{l(l+1)}{r^2} + \frac{1}{r^2}\frac{1}{Y_{lm}(\theta,\phi)}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi)\right\} = 0$$

$$\frac{\partial^2 L_{nl}(r)}{\partial r^2} + \frac{2(l+1)}{r}\frac{\partial L_{nl}(r)}{\partial r} = 2[V(r) - E] L_{nl}(r)$$

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi) = l(l+1) Y_{lm}(\theta,\phi)$$

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] = \hat{l}^2$$

$$\therefore \begin{cases} \dfrac{\partial^2 L_{nl}(r)}{\partial r^2} + \dfrac{2(l+1)}{r}\dfrac{\partial L_{nl}(r)}{\partial r} = 2[V(r) - E]L_{nl}(r) \\ \hat{l}^2 Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi) \end{cases}$$