変数変換

$r = el^e + yl^e$

$\dfrac{dy}{dr} = \dfrac{1}{l^e}$

$dr = l^e dy$

r	$nl^e \to (n+1)l^e$
y	$0 \to 1$

$$\psi^{(e)}(y) = \sum_{i=0}^{1} \psi_i^{(e)} \phi_i(y) \quad (0 \leq y \leq 1)$$

$$\begin{cases} \phi_0(y) = 1 - y \\ \phi_1(y) = y \end{cases}$$

ローカル行列$\mathbf{A}$の要素$A_{\alpha\beta}^{(e)}$の計算

・$A_{00}^{(e)}$の計算

$$\int_0^1 \frac{(el^e + yl^e)^2}{2} \frac{dy}{dr} \left[\frac{\phi_0(y)}{dy} \right] \frac{dy}{dr} \left[\frac{\phi_0(y)}{dy} \right] l^e dy$$

$$= \int_0^1 \frac{(el^e + yl^e)^2}{2} \frac{1}{l^e} dy$$

$$= \int_0^1 \frac{e^2(l^e)^2 + 2e(l^e)^2 y + y^2(l^e)^2}{2} \frac{1}{l^e} dy$$

$$= \frac{l^e}{2} \int_0^1 (e^2 + 2ey + y^2) dy$$

$$= \frac{l^e}{2} \left[e^2 y + ey^2 + \frac{1}{3} y^3 \right]_0^1$$

$$= \frac{l^e}{2} \left(e^2 + e + \frac{1}{3} \right)$$

$$-\int_0^1 (el^e + yl^e)(1-y)^2 l^e dy$$

$$= -(l^e)^2 \int_0^1 (e + y)(1-y)^2 dy$$

$$= -(l^e)^2 \int_0^1 \left[e(1-y)^2 + y(1-y)^2 \right] dy$$

$$= -(l^e)^2 \left\{ \int_0^1 \left[e(1-y)^2 \right] dy + \int_0^1 y(1-y)^2 dy \right\}$$

$$= -(l^e)^2 \left[-\frac{1}{3} e(1-y)^3 \Big|_0^1 + -\frac{1}{3} y(1-y)^3 \Big|_0^1 + \frac{1}{3} \int_0^1 (1-y)^3 dy \right]$$

$$= -(l^e)^2 \left[\frac{e}{3} - \frac{1}{12} (1-y)^4 \Big|_0^1 \right]$$

$$= -(l^e)^2 \left(\frac{e}{3} + \frac{1}{12} \right)$$

$$A_{00}^{(e)} = \int_0^1 \frac{(el^e + yl^e)^2}{2} \frac{dy}{dr} \left[\frac{\phi_0(y)}{dy} \right] \frac{dy}{dr} \left[\frac{\phi_1(y)}{dy} \right] (l^e) dy - \int_0^1 (el^e + yl^e)(1-y)^2 l^e dy$$

$$= \frac{l^e}{2} \left(e^2 + e + \frac{1}{3} \right) - (l^e)^2 \left(\frac{e}{3} + \frac{1}{12} \right)$$

・$A_{01}^{(e)} = A_{10}^{(e)}$の計算

$$\int_0^1 \frac{(el^e + yl^e)^2}{2} \frac{dy}{dr} \left[\frac{\phi_0(y)}{dy} \right] \frac{dy}{dr} \left[\frac{\phi_1(y)}{dy} \right] l^e dy$$

$$= -\frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right)$$
$$-\int_0^1 (el^e + yl^e)y(1-y)l^e dy$$
$$= -(l^e)^2 \int_0^1 (e+y)y(1-y)dy$$
$$= -(l^e)^2\left[\int_0^1 ey(1-y)dy + \int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{2}y^2(1-y)\Big|_0^1 + \int_0^1 \frac{e}{2}y^2 dy + \int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{6}y^3\Big|_0^1 + \int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{6} + \int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{6} + \frac{1}{3}y^3(1-y)\Big|_0^1 + \frac{1}{3}\int_0^1 y^3 dy\right]$$
$$= -(l^e)^2\left(\frac{e}{6} + \frac{1}{12}\right)$$
$$A_{01}^{e(e)} = A_{10}^{(e)} = \int_0^1 \frac{(el^e + yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy - \int_0^1 (el^e + yl^e)y(1-y)l^e dy$$
$$= -\frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right) - (l^e)^2\left(\frac{e}{6} + \frac{1}{12}\right)$$
・$A_{11}^{(e)}$の計算
$$\int_0^1 \frac{(el^e + yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy$$
$$= \frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right)$$
$$-\int_0^1 (el^e + yl^e)y^2 l^e dy$$
$$= -(l^e)^2 \int_0^1 (e+y)y^2 dy$$
$$= -(l^e)^2\left[\int_0^1 ey^2 dy + \int_0^1 y^3 dy\right]$$
$$= -(l^e)^2\left(\frac{e}{3} + \frac{1}{4}\right)$$
$$A_{11}^{(e)} = \int_0^1 \frac{(el^e + yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy - \int_0^1 (el^e + yl^e)y^2 l^e dy$$
$$= \frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right) - (l^e)^2\left(\frac{e}{3} + \frac{1}{4}\right)$$
ローカル行列$\mathbf{B}$の要素$B_{\alpha\beta}^{(e)}$の計算
・$B_{00}^{(e)}$の計算
$$B_{00}^{(e)} = \int_0^1 (el^e + yl^e)^2 \phi_0(y)\phi_0(y) l^e dy$$
$$= \int_0^1 (el^e + yl^e)^2 (1-y)^2 l^e dy$$
$$= \int_0^1 (el^e + yl^e)^2 (1-y)^2 l^e dy$$
$$= \int_0^1 \left[e^2(l^e)^2 + 2e(l^e)^2 y + y^2(l^e)^2\right](1-y)^2 l^e dy$$
$$= (l^e)^3 \int_0^1 \left[e(l^e)^2 + 2e(l^e)^2 y + y^2(l^e)^2\right](1-y)^2 dy$$

$$= (l^e)^3 \int_0^1 e^2(1-y)^2 dy + (l^e)^3 \int_0^1 2ey(1-y)^2 dy + (l^e)^3 \int_0^1 y^2(1-y)^2 dy$$
$$= \frac{(l^e)^3}{3} e^2 + \frac{(l^e)^3}{6} e + \frac{1}{30}(l^e)^3$$
$$= (l^e)^3 \left(\frac{e^2}{3} + \frac{e}{6} + \frac{1}{30} \right)$$
・$B_{01}^{(e)} = B_{10}^{(e)}$の計算
$$B_{01}^{(e)} = B_{10}^{(e)} = \int_0^1 (el^e + yl^e)^2 \phi_0(y)\phi_1(y) l^e dy$$
$$= (l^e)^3 \int_0^1 (e^2 + 2ey + y^2) y(1-y) dy$$
$$= \frac{(l^e)^3}{6} e^2 + \frac{(l^e)^3}{6} e + \frac{(l^e)^3}{20}$$
$$= (l^e)^3 \left(\frac{e^2}{6} + \frac{e}{6} + \frac{1}{20} \right)$$
・$B_{11}^{(e)}$の計算
$$B_{11}^{(e)} = \int_0^1 (el^e + yl^e)^2 \phi_1(y)\phi_1(y) l^e dy$$
$$= \int_0^1 (el^e + yl^e)^2 y^2 l^e dy$$
$$= (l^e)^3 \int_0^1 (e^2 + 2ey + y^2) y^2 dy$$
$$= (l^e)^3 \left(\frac{e^2}{3} + \frac{e}{2} + \frac{1}{5} \right)$$
結果
$$\left\langle \psi_\alpha^{(e)} \middle| A_{\alpha\beta}^{(e)} \middle| \psi_\beta^{(e)} \right\rangle = \left\langle \psi_\alpha^{(e)} \middle| B_{\alpha\beta}^{(e)} \middle| \psi_\beta^{(e)} \right\rangle$$
$$\begin{cases} A_{00}^{(e)} = \frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right) - (l^e)^2\left(\frac{e}{3} + \frac{1}{12}\right) \\ A_{01}^{(e)} = A_{10}^{(e)} = -\frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right) - (l^e)^2\left(\frac{e}{6} + \frac{1}{12}\right) \\ A_{11}^{(e)} = \frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right) - (l^e)^2\left(\frac{e}{3} + \frac{1}{4}\right) \end{cases}$$
$$\begin{cases} B_{00}^{(e)} = (l^e)^3\left(\frac{e^2}{3} + \frac{e}{6} + \frac{1}{30}\right) \\ B_{01}^{(e)} = B_{10}^{(e)} = (l^e)^3\left(\frac{e^2}{6} + \frac{e}{6} + \frac{1}{20}\right) \\ B_{11}^{(e)} = (l^e)^3\left(\frac{e^2}{3} + \frac{e}{2} + \frac{1}{5}\right) \end{cases}$$