変数変換

$r = el^e + yl^e$

$\dfrac{dy}{dr} = \dfrac{1}{l^e}$

$dr = l^e dy$

r	$nl^e \rightarrow (n+1)l^e$
y	$0 \rightarrow 1$

$$\psi^{(e)}(y) = \sum_{i=0}^{1} u_i^{(e)} \phi_i(y) \quad (0 \le y \le 1)$$

$$\begin{cases} \phi_0(y) = 1 - y \\ \phi_1(y) = y \end{cases}$$

ローカル行列$\mathbf{A}$の要素$A_{\alpha\beta}^{(e)}$の計算

・$A_{00}^{(e)}$の計算

$$\int_0^1 \frac{(el^e + yl^e)^2}{2} \frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right](l^e)dy$$

$$= \int_0^1 \frac{(el^e + yl^e)^2}{2}\frac{1}{l^e}dy$$

$$= \int_0^1 \frac{e^2(l^e)^2 + 2e(l^e)^2 y + y^2(l^e)^2}{2}\frac{1}{l^e}dy$$

$$= \frac{l^e}{2}\int_0^1 (e^2 + 2ey + y^2)dy$$

$$= \frac{l^e}{2}\left[e^2 y + ey^2 + \frac{1}{3}y^3\right]_0^1$$

$$= \frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right)$$

$$-\int_0^1 (el^e + yl^e)(1-y)^2 l^e dy$$

$$= -(l^e)^2\int_0^1 (e+y)(1-y)^2 dy$$

$$= -(l^e)^2\int_0^1 \left[e(1-y)^2 + y(1-y)^2\right]dy$$

$$= -(l^e)^2\left\{\int_0^1 \left[e(1-y)^2\right]dy + \int_0^1 y(1-y)^2 dy\right\}$$

$$= -(l^e)^2\left[-\frac{1}{3}e(1-y)^3\Big|_0^1 + -\frac{1}{3}y(1-y)^3\Big|_0^1 + \frac{1}{3}\int_0^1 (1-y)^3 dy\right]$$

$$= -(l^e)^2\left[\frac{e}{3} - \frac{1}{12}(1-y)^4\Big|_0^1\right]$$

$$= -(l^e)^2\left(\frac{e}{3} + \frac{1}{12}\right)$$

$$A_{00}^{(e)} = \int_0^1 \frac{(el^e + yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right](l^e)dy - \int_0^1 (el^e + yl^e)(1-y)^2 l^e dy$$

$$= \frac{l^e}{2}\left(e^2 + e + \frac{1}{3}\right) - (l^e)^2\left(\frac{e}{3} + \frac{1}{12}\right)$$

・$A_{01}^{(e)} = A_{10}^{(e)}$の計算

$$\int_0^1 \frac{(el^e+yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy$$
$$= -\frac{l^e}{2}\left(e^2+e+\frac{1}{3}\right)$$
$$-\int_0^1 (el^e+yl^e)y(1-y)l^e dy$$
$$= -(l^e)^2\int_0^1 (e+y)y(1-y)dy$$
$$= -(l^e)^2\left[\int_0^1 ey(1-y)dy+\int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{2}y^2(1-y)\Big|_0^1+\int_0^1 \frac{e}{2}y^2dy+\int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{6}y^3\Big|_0^1+\int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{6}+\int_0^1 y^2(1-y)dy\right]$$
$$= -(l^e)^2\left[\frac{e}{6}+\frac{1}{3}y^3(1-y)\Big|_0^1+\frac{1}{3}\int_0^1 y^3dy\right]$$
$$= -(l^e)^2\left(\frac{e}{6}+\frac{1}{12}\right)$$
$$Al_{01}^{e(e)} = A_{10}^{(e)} = \int_0^1 \frac{(el^e+yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy-\int_0^1 (el^e+yl^e)y(1-y)l^e$$
$$= -\frac{l^e}{2}\left(e^2+e+\frac{1}{3}\right)-(l^e)^2\left(\frac{e}{6}+\frac{1}{12}\right)$$

・$A_{11}^{(e)}$の計算

$$\int_0^1 \frac{(el^e+yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy$$
$$= \frac{l^e}{2}\left(e^2+e+\frac{1}{3}\right)$$
$$-\int_0^1 (el^e+yl^e)y^2l^e dy$$
$$= -(l^e)^2\int_0^1 (e+y)y^2dy$$
$$= -(l^e)^2\left[\int_0^1 ey^2dy+\int_0^1 y^3dy\right]$$
$$= -(l^e)^2\left(\frac{e}{3}+\frac{1}{4}\right)$$
$$A_{11}^{(e)} = \int_0^1 \frac{(el^e+yl^e)^2}{2}\frac{dy}{dr}\left[\frac{\phi_0(y)}{dy}\right]\frac{dy}{dr}\left[\frac{\phi_1(y)}{dy}\right]l^e dy-\int_0^1 (el^e+yl^e)y^2l^e dy$$
$$= \frac{l^e}{2}\left(e^2+e+\frac{1}{3}\right)-(l^e)^2\left(\frac{e}{3}+\frac{1}{4}\right)$$

ローカル行列**B**の要素$B_{\alpha\beta}^{(e)}$の計算

・$B_{00}^{(e)}$の計算

$$B_{00}^{(e)} = \int_0^1 (el^e+yl^e)^2\phi_0(y)\phi_0(y)l^e dy$$
$$= \int_0^1 (el^e+yl^e)^2(1-y)^2l^e dy$$

$$= \int_0^1 (el^e + yl^e)^2 (1-y)^2 l^e dy$$
$$= \int_0^1 \left[e^2 (l^e)^2 + 2e(l^e)^2 y + y^2 (l^e)^2 \right] (1-y)^2 l^e dy$$
$$= (l^e)^3 \int_0^1 \left[e(l^e)^2 + 2e(l^e)^2 y + y^2 (l^e)^2 \right] (1-y)^2 dy$$
$$= (l^e)^3 \int_0^1 e^2 (1-y)^2 dy + (l^e)^3 \int_0^1 2ey(1-y)^2 dy + (l^e)^3 \int_0^1 y^2 (1-y)^2 dy$$
$$= \frac{(l^e)^3}{3} e^2 + \frac{(l^e)^3}{6} e + \frac{1}{30} (l^e)^3$$
$$= (l^e)^3 \left(\frac{e^2}{3} + \frac{e}{6} + \frac{1}{30} \right)$$

・$B_{01}^{(e)} = B_{10}^{(e)}$の計算

$$B_{01}^{(e)} = B_{10}^{(e)} = \int_0^1 (el^e + yl^e)^2 \phi_0(y) \phi_1(y) l^e dy$$
$$= (l^e)^3 \int_0^1 (e^2 + 2ey + y^2) y (1-y) dy$$
$$= \frac{(l^e)^3}{6} e^2 + \frac{(l^e)^3}{6} e + \frac{(l^e)^3}{20}$$
$$= (l^e)^3 \left(\frac{e^2}{6} + \frac{e}{6} + \frac{1}{20} \right)$$

・$B_{11}^{(e)} = B_{11}^{(e)}$の計算

$$B_{11}^{(e)} = \int_0^1 (el^e + yl^e)^2 \phi_1(y) \phi_1(y) l^e dy$$
$$= \int_0^1 (el^e + yl^e)^2 y^2 l^e dy$$
$$= (l^e)^3 \int_0^1 (e^2 + 2ey + y^2) y^2 dy$$
$$= (l^e)^3 \left(\frac{e^2}{3} + \frac{e}{2} + \frac{1}{5} \right)$$

結果

$$\left\langle \psi_\alpha^{(e)} \middle| A_{\alpha\beta}^{(e)} \middle| \psi_\beta^{(e)} \right\rangle = \left\langle \psi_\alpha^{(e)} \middle| B_{\alpha\beta}^{(e)} \middle| \psi_\beta^{(e)} \right\rangle$$

$$\begin{cases} A_{00}^{(e)} = \dfrac{l^e}{2} \left(e^2 + e + \dfrac{1}{3} \right) - (l^e)^2 \left(\dfrac{e}{3} + \dfrac{1}{12} \right) \\ A_{01}^{(e)} = A_{10}^{(e)} = -\dfrac{l^e}{2} \left(e^2 + e + \dfrac{1}{3} \right) - (l^e)^2 \left(\dfrac{e}{6} + \dfrac{1}{12} \right) \\ A_{11}^{(e)} = \dfrac{l^e}{2} \left(e^2 + e + \dfrac{1}{3} \right) - (l^e)^2 \left(\dfrac{e}{3} + \dfrac{1}{4} \right) \end{cases}$$

$$\begin{cases} B_{00}^{(e)} = (l^e)^3 \left(\dfrac{e^2}{3} + \dfrac{e}{6} + \dfrac{1}{30} \right) \\ B_{01}^{(e)} = B_{10}^{(e)} = (l^e)^3 \left(\dfrac{e^2}{6} + \dfrac{e}{6} + \dfrac{1}{20} \right) \\ B_{11}^{(e)} = (l^e)^3 \left(\dfrac{e^2}{3} + \dfrac{e}{2} + \dfrac{1}{5} \right) \end{cases}$$