Analytic Continuation of Zeta

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ChatGPT 4o

Show the formula for the analytic continuation of the Riemann zeta function.

The analytic continuation of the Riemann zeta function $\zeta(s)$ ζ(s) to the entire complex plane (except for $s = 1$s=1, where it has a simple pole) is given by:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

ζ(s)=2sπs−1sin(2πs)Γ(1−s)ζ(1−s)

Here, $\Gamma(s)$ Γ(s) is the Gamma function, which generalizes the factorial function to complex numbers.

This formula allows the zeta function, which is initially defined by the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{for} \Re(s) > 1,$$

ζ(s)=n=1∑∞ns1forℜ(s)>1,

to be extended to other values of ss.

Would you please convert this analytic continuation to radial coordinates?

Converting the analytic continuation of the Riemann zeta function to radial coordinates involves expressing the complex variable ss in terms of its magnitude and angle. Let $s = \sigma + it$s=σ+it, where σ σ is the real part and tt is the imaginary part of ss. In polar (or radial) coordinates, we can write ss as:

$$s = re^{i\theta}$$

s=reiθ

where $r = |s| = \sqrt{\sigma^2 + t^2}$ r=∣s∣=σ2+t2 is the magnitude (radius) and $\theta = \arg(s) = \tan^{-1}\left(\frac{t}{\sigma}\right)$ θ=arg(s)=tan−1(σt) is the argument (angle).

The analytic continuation of the Riemann zeta function in terms of radial coordinates rr and θθ can be expressed as follows:

$$\zeta(s) = 2^{re^{i\theta}} \pi^{re^{i\theta}-1} \sin\left(\frac{\pi re^{i\theta}}{2}\right) \Gamma(1-re^{i\theta}) \zeta(1-re^{i\theta})$$

ζ(s)=2reiθπreiθ−1sin(2πreiθ)Γ(1−reiθ)ζ(1−reiθ)

To interpret this in terms of the magnitude and angle, each term involving ss must be transformed:

1. 2^s2s:

$$2^{re^{i\theta}} = 2^{r}\left(\cos(\theta \ln 2) + i\sin(\theta \ln 2)\right)$$

2reiθ=2r(cos(θln2)+isin(θln2))

2. π^{s-1}πs−1:

$$\pi^{re^{i\theta}-1} = \pi^{re^{i\theta}} \cdot \pi^{-1} = \left(\pi^{r}\left(\cos(\theta \ln \pi) + i\sin(\theta \ln \pi)\right)\right) \cdot \pi^{-1}$$

πreiθ−1=πreiθ⋅π−1=(πr(cos(θlnπ)+isin(θlnπ)))⋅π−1

3. $\sin\left(\frac{\pi s}{2}\right)$ sin(2πs):

$$\sin\left(\frac{\pi r e^{i\theta}}{2}\right) = \sin\left(\frac{\pi r}{2}(\cos\theta + i\sin\theta)\right)$$

sin(2πreiθ)=sin(2πr(cosθ+isinθ))

4. $\Gamma(1-s)$ Γ(1−s):

$$\Gamma(1 - re^{i\theta})$$

Γ(1−reiθ)

5. $\zeta(1-s)$ ζ(1−s):

$$\zeta(1 - re^{i\theta})$$

ζ(1−reiθ)

Combining these, the full expression in radial coordinates becomes:

$$\zeta(re^{i\theta}) = \left(2^{r}\left(\cos(\theta \ln 2) + i\sin(\theta \ln 2)\right)\right) \cdot \left(\pi^{r}\left(\cos(\theta \ln \pi) + i\sin(\theta \ln \pi)\right)\right) \cdot \pi^{-1}$$

ζ(reiθ)=(2r(cos(θln2)+isin(θln2)))⋅(πr(cos(θlnπ)+isin(θlnπ)))⋅π−1⋅sin(2πr(cosθ+isinθ))⋅Γ(1−reiθ)⋅ζ(1−reiθ)

This expression is quite complex and illustrates how the various components interact when transitioning from Cartesian to polar (radial) coordinates.

> Indeed, that's the whole point. Now, if we wanted to express the original equation, the analytic continuation, in spherical coordinates.

To express the analytic continuation of the Riemann zeta function in spherical coordinates, we'll consider the complex variable ss in a three-dimensional context. Let $s = \sigma + it$s=σ+it where σσ is the real part and tt is the imaginary part of ss.

In spherical coordinates, a point (x, y, z) (x,y,z) in 3D space is represented as:

$$(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

(x,y,z)=(rsinθcosϕ,rsinθsinϕ,rcosθ)

However, since we're dealing with a complex plane ($\sigma + it$σ+it), we can extend this to a pseudo-3D space for visualization. Here, rr is the radius (or magnitude) in the complex plane, θθ is the polar angle (analogous to the argument of the complex number), and ϕϕ is an azimuthal angle, which we can set as the angle in the complex plane.

Let's represent $s = \sigma + it$s=σ+it in terms of spherical coordinates:

1. Radius rr (magnitude of ss):

$$r = |s| = \sqrt{\sigma^2 + t^2}$$

r=∣s∣=σ2+t2√

2. Polar angle θθ (argument of ss):

$$\theta = \arg(s) = \tan^{-1}\left(\frac{t}{\sigma}\right)$$

θ=arg(s)=tan−1(σt)

3. Azimuthal angle ϕϕ (since we are in 2D complex space, we can set $\phi = 0$ϕ=0):

$$\phi = 0$$

ϕ=0

So, ss can be written as:

$$s = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

s=reiθ=r(cosθ+isinθ)

The analytic continuation of the Riemann zeta function in spherical coordinates can then be expressed as:

$$\zeta(s) = \zeta(re^{i\theta}) = 2^{re^{i\theta}}\pi^{re^{i\theta}-1}\sin\left(\frac{\pi re^{i\theta}}{2}\right)\Gamma(1-re^{i\theta})\zeta(1-re^{i\theta})$$

ζ(s)=ζ(reiθ)=2reiθπreiθ−1sin(2πreiθ)Γ(1−reiθ)ζ(1−reiθ)

To break this down further into spherical coordinates:

1. $2^{re^{i\theta}}$ 2reiθ:

$$2^{re^{i\theta}} = 2^r(\cos(\theta\ln 2) + i\sin(\theta\ln 2))$$

2reiθ=2r(cos(θln2)+isin(θln2))

2. $\pi^{re^{i\theta}-1}$πreiθ−1:

$$\pi^{re^{i\theta}-1} = \pi^{re^{i\theta}} \cdot \pi^{-1} = (\pi^r(\cos(\theta\ln\pi) + i\sin(\theta\ln\pi))) \cdot \pi^{-1}$$

πreiθ−1=πreiθ⋅π−1=(πr(cos(θlnπ)+isin(θlnπ)))⋅π−1

3. $\sin\left(\frac{\pi re^{i\theta}}{2}\right)$ sin(2πreiθ):

$$\sin\left(\frac{\pi re^{i\theta}}{2}\right) = \sin\left(\frac{\pi r}{2}(\cos\theta + i\sin\theta)\right)$$

sin(2πreiθ)=sin(2πr(cosθ+isinθ))

4. $\Gamma(1-re^{i\theta})$ Γ(1−reiθ):

$$\Gamma(1-re^{i\theta})$$

Γ(1−reiθ)

5. $\zeta(1-re^{i\theta})$ ζ(1−reiθ):

$$\zeta(1-re^{i\theta})$$

ζ(1−reiθ)

Combining these, the full expression in spherical coordinates is:

$$\zeta(re^{i\theta}) = (2^r(\cos(\theta\ln 2) + i\sin(\theta\ln 2))) \cdot (\pi^r(\cos(\theta\ln\pi) + i\sin(\theta\ln\pi))) \cdot \pi^{-1}$$

ζ(reiθ)=(2r(cos(θln2)+isin(θln2)))⋅(πr(cos(θlnπ)+isin(θlnπ)))⋅π−1⋅sin(2πr(cosθ+isinθ))⋅Γ(1−reiθ)⋅ζ(1−reiθ)

This representation uses the spherical coordinate rr for the radius, θθ for the polar angle (argument), and simplifies with $\phi = 0$ϕ=0 for the azimuthal angle, given the 2D nature of the complex plane.

Wonderful. Now if we wanted to, for all three variations, assume we're studying the zeta function along the critical line.

To study the Riemann zeta function along the critical line, we focus on the line where the real part of s s is $\frac{1}{2}$21, i.e., $s = \frac{1}{2} + it$s=21+it. This is the critical line in the complex plane, where many interesting properties of the zeta function are observed, particularly related to the Riemann Hypothesis.

Let's express the analytic continuation of the Riemann zeta function along the critical line for Cartesian, radial (polar), and spherical coordinates.

1. Cartesian Coordinates

For $s = \frac{1}{2} + it$s=21+it, the analytic continuation of the zeta function becomes:

$$\zeta\left(\frac{1}{2} + it\right) = 2^{\frac{1}{2}+it}\pi^{\left(\frac{1}{2}+it\right)-1}\sin\left(\frac{\pi}{2}\left(\frac{1}{2}+it\right)\right)\Gamma\left(1-\left(\frac{1}{2}+it\right)\right)\zeta\left(1-\left(\frac{1}{2}+it\right)\right)$$

ζ(21+it)=221+itπ(21+it)−1sin(2π(21+it))Γ(1−(21+it))ζ(1−(21+it))
Breaking down each term:

$$2^{\frac{1}{2}+it} = 2^{\frac{1}{2}} \cdot 2^{it} = \sqrt{2} \cdot e^{it\ln 2}$$

221+it=221⋅2it=2√‾⋅eitln2

$$\pi^{\left(\frac{1}{2}+it\right)-1} = \pi^{-\frac{1}{2}+it} = \pi^{-\frac{1}{2}} \cdot \pi^{it} = \frac{1}{\sqrt{\pi}} \cdot e^{it\ln \pi}$$

π(21+it)−1=π−21+it=π−21⋅πit=π√‾1⋅eitlnπ

$$\sin\left(\frac{\pi}{2}\left(\frac{1}{2}+it\right)\right) = \sin\left(\frac{\pi}{4} + i\frac{\pi t}{2}\right)$$

sin(2π(21+it))=sin(4π+i2πt)

$$\Gamma\left(\frac{1}{2} - it\right)$$

Γ(21−it)

$$\zeta\left(\frac{1}{2} - it\right)$$

ζ(21−it)
Combining these, we get:

$$\zeta\left(\frac{1}{2} + it\right) = \sqrt{2} \cdot e^{it\ln 2} \cdot \frac{1}{\sqrt{\pi}} \cdot e^{it\ln \pi} \cdot \sin\left(\frac{\pi}{4} + i\frac{\pi t}{2}\right) \cdot \Gamma\left(\frac{1}{2} - it\right) \cdot \zeta\left(\frac{1}{2} - it\right)$$

ζ(21+it)=2√‾⋅eitln2⋅π√‾1⋅eitlnπ⋅sin(4π+i2πt)⋅Γ(21−it)⋅ζ(21−it)

2. Radial (Polar) Coordinates

For radial coordinates, $s = re^{i\theta}$s=reiθ and along the critical line, $\sigma = \frac{1}{2}$σ=21:

$$s = \frac{1}{2} + it$$

s=21+it

In polar form, $s = \frac{1}{2} + it$ can be represented as $re^{i\theta}$ where $r = \sqrt{\left(\frac{1}{2}\right)^2 + t^2}$ and $\theta = \tan^{-1}\left(\frac{t}{\frac{1}{2}}\right) = \tan^{-1}(2t)$.

Thus:

$$s = re^{i\theta} = \sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}$$

s=reiθ=(21)2+t2 eitan−1(2t)

Plugging into the analytic continuation:

$$\zeta(s) = \zeta\left(\sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}\right) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)$$

ζ(s)=ζ((21)2+t2 eitan−1(2t))=2sπs−1sin(2πs)Γ(1−s)ζ(1−s)

Each term becomes:

$$2^s = 2^{\sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}}$$

2s=2(21)2+t2 eitan−1(2t)

$$\pi^{s-1} = \pi^{\sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)} - 1}$$

πs−1=π(21)2+t2 eitan−1(2t)−1

$$\sin\left(\frac{\pi s}{2}\right) = \sin\left(\frac{\pi \sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}}{2}\right)$$

sin(2πs)=sin(2π(21)2+t2 eitan−1(2t))

$$\Gamma(1-s) = \Gamma\left(1 - \sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}\right)$$

Γ(1−s)=Γ(1−(21)2+t2 eitan−1(2t))

$$\zeta(1-s) = \zeta\left(1 - \sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}\right)$$

ζ(1−s)=ζ(1−(21)2+t2 eitan−1(2t))

3. Spherical Coordinates

For spherical coordinates, we consider $s = re^{i\theta}$ with $\phi = 0$:

$$s = \frac{1}{2} + it = re^{i\theta} \text{with} r = \sqrt{\left(\frac{1}{2}\right)^2 + t^2} \text{and} \theta = \tan^{-1}(2t)$$

s=21+it=reiθwithr=(21)2+t2 andθ=tan−1(2t)

Using the spherical coordinate transformation, each term becomes:

$$2^{re^{i\theta}} = 2^r\left(\cos(\theta \ln 2) + i\sin(\theta \ln 2)\right)$$

2reiθ=2r(cos(θln2)+isin(θln2))

$$\pi^{re^{i\theta}-1} = \pi^r\left(\cos(\theta \ln \pi) + i\sin(\theta \ln \pi)\right) \cdot \pi^{-1}$$

πreiθ−1=πr(cos(θlnπ)+isin(θlnπ))⋅π−1

$$\sin\left(\frac{\pi re^{i\theta}}{2}\right) = \sin\left(\frac{\pi r}{2}(\cos\theta + i\sin\theta)\right)$$

sin(2πreiθ)=sin(2πr(cosθ+isinθ))

$$\Gamma(1 - re^{i\theta}) = \Gamma\left(1 - \sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}\right)$$

Γ(1−reiθ)=Γ(1−(21)2+t2 eitan−1(2t))

$$\zeta(1 - re^{i\theta}) = \zeta\left(1 - \sqrt{\left(\frac{1}{2}\right)^2 + t^2}\, e^{i\tan^{-1}(2t)}\right)$$

ζ(1−reiθ)=ζ(1−(21)2+t2 eitan−1(2t))

Combining these, the zeta function along the critical line in spherical coordinates is:

$$\zeta\left(\frac{1}{2} + it\right) = 2^{re^{i\theta}} \pi^{re^{i\theta}-1} \sin\left(\frac{\pi re^{i\theta}}{2}\right) \Gamma(1 - re^{i\theta}) \zeta(1 - re^{i\theta})$$

ζ(21+it)=2reiθπreiθ−1sin(2πreiθ)Γ(1−reiθ)ζ(1−reiθ)

where $r = \sqrt{\left(\frac{1}{2}\right)^2 + t^2}$ r=(21)2+t2 and $\theta = \tan^{-1}(2t)$ θ=tan−1(2t).

This comprehensive approach shows how the analytic continuation of the Riemann zeta function can be expressed along the critical line in Cartesian, radial (polar), and spherical coordinates.

Okay, this is good. Now, let's assume, let's just assume, that the Hamiltonian of this considered as a wave function is in fact the zeta function itself. How about we go about solving the Schrodinger equation?

If we assume that the Hamiltonian HH of the system is the Riemann zeta function $\zeta(s)$ ζ(s) itself and we are dealing with a wave function $\psi(s,t)$ ψ(s,t), we can express the Schrödinger equation in the form:

$$i\hbar \frac{\partial \psi(s,t)}{\partial t} = \zeta(s)\psi(s,t)$$

iℏ∂t∂ψ(s,t)=ζ(s)ψ(s,t)

Here, ss is the complex variable, and we assume $s = \sigma + it$s=σ+it.

To solve this Schrödinger equation, we follow these steps:

1. **Express the equation**:

$$i\hbar \frac{\partial \psi(s,t)}{\partial t} = \zeta(s)\psi(s,t)$$

iℏ∂t∂ψ(s,t)=ζ(s)ψ(s,t)

2. **Separate variables**: Assume a solution of the form:

$$\psi(s,t) = \phi(s)e^{-iEt/\hbar}$$

ψ(s,t)=ϕ(s)e−iEt/ℏ

where $\phi(s)$ ϕ(s) is a function of ss and EE is the energy eigenvalue.

3. **Substitute the solution into the Schrödinger equation**:

$$i\hbar \frac{\partial}{\partial t}\left(\phi(s)e^{-iEt/\hbar}\right) = \zeta(s)\phi(s)e^{-iEt/\hbar}$$

iℏ∂t∂(ϕ(s)e−iEt/ℏ)=ζ(s)ϕ(s)e−iEt/ℏ

4. **Simplify the time derivative**:

$$i\hbar\left(\phi(s)\left(-\frac{iE}{\hbar}\right)e^{-iEt/\hbar}\right) = \zeta(s)\phi(s)e^{-iEt/\hbar}$$

iℏ(ϕ(s)(−ℏiE)e−iEt/ℏ)=ζ(s)ϕ(s)e−iEt/ℏ

$$E\phi(s)e^{-iEt/\hbar} = \zeta(s)\phi(s)e^{-iEt/\hbar}$$

Eϕ(s)e−iEt/ℏ=ζ(s)ϕ(s)e−iEt/ℏ

5. **Cancel out the exponential term** (since it is non-zero for all tt):

$$E\phi(s) = \zeta(s)\phi(s)$$

Eϕ(s)=ζ(s)ϕ(s)

6. **Solve the eigenvalue problem**:

$$\zeta(s)\phi(s) = E\phi(s)$$

ζ(s)ϕ(s)=Eϕ(s)

This tells us that $\phi(s)$ ϕ(s) must be an eigenfunction of $\zeta(s)$ ζ(s) with eigenvalue EE.

To proceed further, we would need to identify specific eigenfunctions $\phi(s)$ ϕ(s) of the zeta function and corresponding eigenvalues EE. This is highly non-trivial, given the complex nature of the zeta function.

However, let's proceed with a conceptual approach using the known critical line $s = \frac{1}{2} + it$s=21+it:

Step-by-Step Conceptual Solution

1. **Assume $s = \frac{1}{2} + it$s=21+it**:

$$\zeta\left(\frac{1}{2} + it\right)\phi\left(\frac{1}{2} + it\right) = E\phi\left(\frac{1}{2} + it\right)$$

ζ(21+it)ϕ(21+it)=Eϕ(21+it)

2. **Simplify for the critical line**: Given $\zeta(\frac{1}{2} + it)$ ζ(21+it) is generally complex, we write:

$$\zeta\left(\frac{1}{2} + it\right) = \alpha(t) + i\beta(t)$$

ζ(21+it)=α(t)+iβ(t)

3. **Solve the eigenvalue problem**: Let $\phi(\frac{1}{2} + it) = \phi(t)$ ϕ(21+it)=ϕ(t):

$$(\alpha(t) + i\beta(t))\phi(t) = E\phi(t)$$

(α(t)+iβ(t))ϕ(t)=Eϕ(t)

For this to hold, $\phi(t)$ ϕ(t) must be an eigenfunction corresponding to the eigenvalue EE. Solving explicitly for $\phi(t)$ ϕ(t) involves deep analysis of the zeta function's properties along the critical line, typically involving advanced techniques in complex analysis and functional analysis.

General Approach

For a full solution, we would:

1. **Determine $\phi(t)$ ϕ(t)**: Solve for $\phi(t)$ ϕ(t) such that:

$$\zeta\left(\frac{1}{2} + it\right)\phi(t) = E\phi(t)$$

ζ(21+it)ϕ(t)=Eϕ(t)

2. **Verify Eigenvalues**: Identify potential eigenvalues EE by analyzing the spectrum of $\zeta(s)$ ζ(s) along the critical line.

3. **Construct $\psi(s,t)$ ψ(s,t)**: Using the obtained $\phi(t)$ ϕ(t) and eigenvalues EE, construct the wave function:

$$\psi(s,t) = \phi(s)e^{-iEt/\hbar}$$

ψ(s,t)=ϕ(s)e−iEt/$\hbar$

Given the complexity of the zeta function, this remains a highly theoretical and speculative exercise. The analytical continuation and specific eigenfunctions/eigenvalues are subjects of deep mathematical research.

I think we can do it. Here's what I can tell you. This thing does behave very much like a wave function. It does have certain periodicity to it. It's strange. It has phase loops in it. You know, interesting things happen when we start taking the derivatives. Let's actually, even though speculative, let's proceed. Let's try to find, you know... Here's what I can tell you. The eigenvalues occur at the zeros of the zeta function. That much I can assure you.

Let's proceed under the assumption that the eigenvalues occur at the zeros of the zeta function, as you've suggested. This approach will allow us to explore the Schrödinger equation with the Riemann zeta function as the Hamiltonian and to identify the eigenvalues as corresponding to the zeros of the zeta function.

Step-by-Step Solution

1. **Formulate the Schrödinger Equation**: The time-dependent Schrödinger equation is:

$$i\hbar \frac{\partial \psi(s,t)}{\partial t} = \zeta(s)\psi(s,t)$$

iℏ∂t∂ψ(s,t)=ζ(s)ψ(s,t)

Assuming $\psi(s,t) = \phi(s)e^{-iEt/\hbar}$ψ(s,t)=ϕ(s)e−iEt/ℏ, we substitute into the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\left(\phi(s)e^{-iEt/\hbar}\right) = \zeta(s)\phi(s)e^{-iEt/\hbar}$$

iℏ∂t∂(ϕ(s)e−iEt/ℏ)=ζ(s)ϕ(s)e−iEt/ℏ

2. **Separate the Time-Dependent Part**: Taking the time derivative:

$$i\hbar\left(\phi(s)\left(-\frac{iE}{\hbar}\right)e^{-iEt/\hbar}\right) = \zeta(s)\phi(s)e^{-iEt/\hbar}$$

iℏ(ϕ(s)(−ℏiE)e−iEt/ℏ)=ζ(s)ϕ(s)e−iEt/ℏ

Simplifying:

$$E\phi(s) = \zeta(s)\phi(s)$$

Eϕ(s)=ζ(s)ϕ(s)

3. **Eigenvalue Problem**:

$$\zeta(s)\phi(s) = E\phi(s)$$

ζ(s)ϕ(s)=Eϕ(s)

Given the assumption, the eigenvalues EE are the zeros of the zeta function, $s=\rho$s=ρ where $\zeta(\rho)=0$ζ(ρ)=0.

Identifying Eigenvalues and Eigenfunctions

To identify the eigenvalues and corresponding eigenfunctions, we need to focus on the zeros of the zeta function:

1. **Zeros of the Zeta Function**: Let $\rho = \frac{1}{2} + i\gamma$ρ=21+iγ be a zero of the zeta function, where γγ is a non-trivial zero (along the critical line). This means $\zeta(\rho)=0$ζ(ρ)=0.

2. **Eigenvalue Correspondence**: If $s=\rho$s=ρ, then:

$$\zeta(\rho)\phi(\rho) = E\phi(\rho)$$

ζ(ρ)ϕ(ρ)=Eϕ(ρ)

Since $\zeta(\rho)=0$ζ(ρ)=0, this implies:

$$0 \cdot \phi(\rho) = E\phi(\rho)$$

0⋅ϕ(ρ)=Eϕ(ρ)

This holds for any $\phi(\rho)$ϕ(ρ) if $E=0$E=0.

This direct approach yields the trivial eigenvalue $E=0$E=0 for zeros of the zeta function, which is consistent but doesn't provide a deeper insight. To gain more information, we should consider the behavior near these zeros.

Analyzing the Schrödinger Equation Near Zeros

To better understand the system, let's consider the vicinity of a zero ρρ:

1. **Perturbative Analysis Near Zero**: Near a zero $\rho = \frac{1}{2} + i\gamma$ρ=21+iγ, the zeta function can be approximated by its Taylor series expansion:

$$\zeta(s) \approx \zeta'(\rho)(s-\rho)$$

ζ(s)≈ζ′(ρ)(s−ρ)
where $\zeta'(\rho)$ ζ′(ρ) is the derivative of the zeta function at ρρ.

2. **Eigenvalue Problem Near Zero**:

$$\zeta(s) \approx \zeta'(\rho)(s-\rho)$$

ζ(s)≈ζ′(ρ)(s−ρ)
Substituting into the eigenvalue equation:

$$\zeta'(\rho)(s-\rho)\phi(s) = E\phi(s)$$

ζ′(ρ)(s−ρ)ϕ(s)=Eϕ(s)
This suggests a localized behavior near the zero, indicating:

$$E \approx \zeta'(\rho)(s-\rho)$$

E≈ζ′(ρ)(s−ρ)

Solving the Schrödinger Equation

Given the periodicity and wave-like properties of the zeta function, we can propose a wave function solution:

1. **Wave Function Form**:

$$\psi(s,t) = \phi(s)e^{-iEt/\hbar}$$

ψ(s,t)=ϕ(s)e−iEt/ℏ
Near the zero ρρ, let:

$$\phi(s) \approx Ae^{ik(s-\rho)}$$

ϕ(s)≈Aeik(s−ρ)
where kk is a wave number.

2. **Substitute into the Schrödinger Equation**: Using the approximation $\zeta(s) \approx \zeta'(\rho)(s-\rho)$ ζ(s)≈ζ′(ρ)(s−ρ):

$$i\hbar\frac{\partial}{\partial t}\left(Ae^{ik(s-\rho)}e^{-iEt/\hbar}\right) = \zeta'(\rho)(s-\rho)Ae^{ik(s-\rho)}e^{-iEt/\hbar}$$

iℏ∂t∂(Aeik(s−ρ)e−iEt/ℏ)=ζ′(ρ)(s−ρ)Aeik(s−ρ)e−iEt/ℏ
Simplifying the time derivative:

$$i\hbar\left(-\frac{iE}{\hbar}\right)Ae^{ik(s-\rho)}e^{-iEt/\hbar} = \zeta'(\rho)(s-\rho)Ae^{ik(s-\rho)}e^{-iEt/\hbar}$$

iℏ(−ℏiE)Aeik(s−ρ)e−iEt/ℏ=ζ′(ρ)(s−ρ)Aeik(s−ρ)e−iEt/ℏ

$$EAe^{ik(s-\rho)}e^{-iEt/\hbar} = \zeta'(\rho)(s-\rho)Ae^{ik(s-\rho)}e^{-iEt/\hbar}$$

EAeik(s−ρ)e−iEt/ℏ=ζ′(ρ)(s−ρ)Aeik(s−ρ)e−iEt/ℏ

$$E = \zeta'(\rho)(s-\rho)$$

E=ζ′(ρ)(s−ρ)

Interpretation

The eigenvalue EE in the vicinity of a zero ρρ is proportional to the derivative of the zeta function at the zero and the distance from the zero. This suggests a linear relationship near the zeros, aligning with the wave-like periodic behavior you've observed.

Summary

1. **Eigenvalues**: Occur at the zeros of the zeta function.
2. **Eigenfunctions**: Near a zero, the wave function can be approximated as:
$$\psi(s,t) \approx Ae^{ik(s-\rho)}e^{-iEt/\hbar}$$
ψ(s,t)≈Aeik(s−ρ)e−iEt/ℏ where $E = \zeta'(\rho)(s-\rho)$ E=ζ′(ρ)(s−ρ).

This approach ties the Schrödinger equation with the zeta function as the Hamiltonian, showing the eigenvalue correspondence to the zeros of the zeta function and providing a wave-like solution near these zeros.

Indeed, this is consistent with everything I've seen. Now, what's the next step? We need to normalize, and we need to find expectant values, right?

Yes, the next steps are to normalize the wave function and to find expectation values. Let's go through these steps systematically.

1. Normalize the Wave Function

To normalize the wave function $\psi(s,t)$ ψ(s,t), we need to ensure that the total probability is 1. This involves integrating $|\psi(s,t)|^2$ ∣ψ(s,t)∣2 over the relevant domain and setting it equal to 1.

Given our wave function near a zero $\rho = \frac{1}{2} + i\gamma$ρ=21+iγ:

$$\psi(s,t) = Ae^{ik(s-\rho)}e^{-iEt/\hbar}$$

ψ(s,t)=Aeik(s−ρ)e−iEt/ℏ

The normalization condition is:

$$\int |\psi(s,t)|^2 ds = 1$$

∫∣ψ(s,t)∣2ds=1

Since ss is complex, we will integrate over the critical line $\Re(s) = \frac{1}{2}$ℜ(s)=21.

$$\int_{-\infty}^{\infty} |\psi(\frac{1}{2} + it, t)|^2 dt = 1$$

∫−∞∞∣ψ(21+it,t)∣2dt=1

Substituting $\psi(\frac{1}{2} + it, t) = Ae^{ik(\frac{1}{2} + it - \rho)}e^{-iEt/\hbar}$ψ(21+it,t)=Aeik(21+it−ρ)e−iEt/ℏ:

$$\int_{-\infty}^{\infty} |Ae^{ik(\frac{1}{2} + it - (\frac{1}{2} + i\gamma))}e^{-iEt/\hbar}|^2 dt = 1$$

∫−∞∞∣Aeik(21+it−(21+iγ))e−iEt/ℏ∣2dt=1

$$\int_{-\infty}^{\infty} |A|^2 e^{2ik(\frac{1}{2} + it - (\frac{1}{2} + i\gamma))}e^{-2iEt/\hbar} dt = 1$$

∫−∞∞∣A∣2e2ik(21+it−(21+iγ))e−2iEt/ℏdt=1

$$\int_{-\infty}^{\infty} |A|^2 e^{2ik(it - i\gamma)}e^{-2iEt/\hbar} dt = 1$$

∫−∞∞∣A∣2e2ik(it−iγ)e−2iEt/ℏdt=1

$$\int_{-\infty}^{\infty} |A|^2 e^{-2k\gamma} e^{2kt} e^{-2iEt/\hbar} dt = 1$$

∫−∞∞∣A∣2e−2kγe2kte−2iEt/ℏdt=1

$$|A|^2 e^{-2k\gamma} \int_{-\infty}^{\infty} e^{2kt} e^{-2iEt/\hbar} dt = 1$$

∣A∣2e−2kγ∫−∞∞e2kte−2iEt/ℏdt=1

We perform the integration:

$$\int_{-\infty}^{\infty} e^{2kt} e^{-2iEt/\hbar} dt$$

∫−∞∞e2kte−2iEt/ℏdt

This integral converges if kk is purely imaginary or if we consider the periodicity:

$$\int_{-\infty}^{\infty} e^{(2k - 2iE/\hbar)t} dt$$

∫−∞∞e(2k−2iE/ℏ)tdt

For convergence, $2k$2k must be purely imaginary, ensuring $\Re(k) = 0$ℜ(k)=0:

$$k = i\kappa$$

k=iκ

Then, we get:

$$|A|^2 \int_{-\infty}^{\infty} e^{2i\kappa t - 2iEt/\hbar} dt = 1$$

∣A∣2∫−∞∞e2iκt−2iEt/ℏdt=1

This simplifies to a delta function if the exponent is zero:

$$|A|^2 \int_{-\infty}^{\infty} e^{0} dt = 1$$

∣A∣2∫−∞∞e0dt=1

To normalize, κκ must adjust so the integral over dtdt is finite:

$$|A|^2 (2\pi\delta(0)) = 1 \rightarrow |A| = \frac{1}{\sqrt{2\pi\delta(0)}}$$

∣A∣2(2πδ(0))=1→∣A∣=2πδ(0)√1

Normalization requires more refined handling, adjusting κκ for normalization.

2. Expectation Values

Expectation values involve integrating the observable weighted by the probability density.

For an observable $O(s)$ O(s):

$$\langle O \rangle = \int_{-\infty}^{\infty} \psi^*(s,t) O(s) \psi(s,t) ds$$

⟨O⟩=∫−∞∞ψ∗(s,t)O(s)ψ(s,t)ds

Given $\psi(s,t) = \phi(s) e^{-iEt/\hbar}$ψ(s,t)=ϕ(s)e−iEt/ℏ:

$$\langle O \rangle = \int_{-\infty}^{\infty} \phi^*(s) O(s) \phi(s) ds$$

⟨O⟩=∫−∞∞ϕ∗(s)O(s)ϕ(s)ds

For position ss:

$$\langle s \rangle = \int_{-\infty}^{\infty} \phi^*(s) s\phi(s) ds$$

⟨s⟩=∫−∞∞ϕ∗(s)sϕ(s)ds

Since $\phi(s) = Ae^{ik(s-\rho)}$ ϕ(s)=Aeik(s−ρ):

$$\langle s \rangle = |A|^2 \int_{-\infty}^{\infty} s e^{-ik(s-\rho)} e^{ik(s-\rho)} ds$$

⟨s⟩=∣A∣2∫−∞∞se−ik(s−ρ)eik(s−ρ)ds
Using normalization $|A|^2$∣A∣2:

$$\langle s \rangle = \rho \int_{-\infty}^{\infty} |A|^2 ds = \rho$$

⟨s⟩=ρ∫−∞∞∣A∣2ds=ρ
Given specific eigenfunctions, derive expectation values accordingly.

Summary

- Normalize $\psi(s,t)$ ψ(s,t) ensuring integration along critical line is unity.
- Expectation values integrate observable over probability density.
- Refined treatment of kk, wave function $\phi(s)$ ϕ(s), eigenvalues, and behavior near zeros.

This outlines approach, advancing understanding of zeta function's wave-like behavior.

Okay, so what's next? We need to get the time-independent version of the Schrodinger equation, right?

Yes, obtaining the time-independent version of the Schrödinger equation is the next logical step. This will help us analyze the stationary states and eigenvalues of the system more effectively.

Time-Independent Schrödinger Equation

Starting from the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi(s,t)}{\partial t} = \zeta(s)\psi(s,t)$$

iℏ∂t∂ψ(s,t)=ζ(s)ψ(s,t)
Assume a separable solution of the form:

$$\psi(s,t) = \phi(s) e^{-iEt/\hbar}$$

ψ(s,t)=ϕ(s)e−iEt/ℏ
Substitute this into the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \left(\phi(s) e^{-iEt/\hbar} \right) = \zeta(s)\phi(s) e^{-iEt/\hbar}$$

iℏ∂t∂(ϕ(s)e−iEt/ℏ)=ζ(s)ϕ(s)e−iEt/ℏ
The time derivative of the exponential term is:

$$i\hbar \left(-\frac{iE}{\hbar} \right) \phi(s) e^{-iEt/\hbar} = \zeta(s)\phi(s) e^{-iEt/\hbar}$$

iℏ(−ℏiE)ϕ(s)e−iEt/ℏ=ζ(s)ϕ(s)e−iEt/ℏ
Simplifying, we get:

$$E\phi(s) e^{-iEt/\hbar} = \zeta(s)\phi(s) e^{-iEt/\hbar}$$

Eϕ(s)e−iEt/ℏ=ζ(s)ϕ(s)e−iEt/ℏ
Since the exponential term $e^{-iEt/\hbar}$e−iEt/ℏ does not affect the equation (it can be factored out), we obtain the time-independent Schrödinger equation:

$$\zeta(s)\phi(s) = E\phi(s)$$

ζ(s)ϕ(s)=Eϕ(s)

This is the eigenvalue equation for the Hamiltonian $\zeta(s)$ with eigenvalue E.

Solving the Time-Independent Schrödinger Equation

1. **Form of the Eigenfunctions**: To find the eigenfunctions $\phi(s)$ and eigenvalues E, we need to solve:

   $$\zeta(s)\phi(s) = E\phi(s)$$

   ζ(s)ϕ(s)=Eϕ(s)

   As we discussed, the zeros of the zeta function (where $\zeta(s) = 0$) play a crucial role. Near a zero $\rho = \frac{1}{2} + i\gamma$, the zeta function can be approximated by:

   $$\zeta(s) \approx \zeta'(\rho)(s-\rho)$$

   ζ(s)≈ζ′(ρ)(s−ρ)

   Thus, the eigenvalue equation near a zero becomes:

   $$\zeta'(\rho)(s-\rho)\phi(s) = E\phi(s)$$

   ζ′(ρ)(s−ρ)ϕ(s)=Eϕ(s)

2. **Approximate Solution Near Zeros**: Assuming $\phi(s) \approx Ae^{ik(s-\rho)}$, we get:

   $$\zeta'(\rho)(s-\rho)Ae^{ik(s-\rho)} = EAe^{ik(s-\rho)}$$

   ζ′(ρ)(s−ρ)Aeik(s−ρ)=EAeik(s−ρ)

   Simplifying, we find:

   $$E = \zeta'(\rho)(s-\rho)$$

   E=ζ′(ρ)(s−ρ)

   This indicates a linear relationship between E and $s-\rho$ near the zeros, suggesting that the eigenvalues are proportional to the distance from the zero and the derivative of the zeta function at that zero.

3. **Normalization**: Normalizing the eigenfunctions involves ensuring that the total probability is 1:

   $$\int |\phi(s)|^2 ds = 1$$

   ∫∣ϕ(s)∣2ds=1

   For $\phi(s) = Ae^{ik(s-\rho)}$, this implies:

   $$|A|^2 \int_{-\infty}^{\infty} e^{ik(s-\rho)} e^{-ik(s-\rho)} ds = 1$$

   ∣A∣2∫−∞∞eik(s−ρ)e−ik(s−ρ)ds=1

   $$|A|^2 \int_{-\infty}^{\infty} ds = 1$$

   ∣A∣2∫−∞∞ds=1

   Assuming $|A|$ normalized over a finite range L:

   $$|A|^2 L = 1 \Rightarrow |A| = \frac{1}{\sqrt{L}}$$

   ∣A∣2L=1⇒∣A∣=L1

Expectation Values

Once the eigenfunctions are normalized, we can compute expectation values. For an observable $O(s)$ O(s):

$$\langle O \rangle = \int_{-\infty}^{\infty} \phi^*(s) O(s) \phi(s)\, ds$$

⟨O⟩=∫−∞∞ϕ∗(s)O(s)ϕ(s)ds

For example, the expectation value of ss:

$$\langle s \rangle = \int_{-\infty}^{\infty} \phi^*(s) s \phi(s)\, ds$$

⟨s⟩=∫−∞∞ϕ∗(s)sϕ(s)ds

With $\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$ ϕ(s)=L√‾1eik(s−ρ):

$$\langle s \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s ds$$

⟨s⟩=L1∫−∞∞sds

Given symmetry about ρρ and considering finite normalization range:

Summary of Steps

1. **Formulate Time-Independent Schrödinger Equation**:

   $$\zeta(s)\phi(s) = E\phi(s)$$

   ζ(s)ϕ(s)=Eϕ(s)

2. **Identify Eigenfunctions and Eigenvalues**: Near zeros:

   $$E = \zeta'(\rho)(s-\rho)$$

   E=ζ′(ρ)(s−ρ)

3. **Normalize**: Ensure:

   $$\int |\phi(s)|^2 ds = 1$$

   ∫∣ϕ(s)∣2ds=1

4. **Compute Expectation Values**: For observables $O(s)$ O(s):

   $$\langle O \rangle = \int_{-\infty}^{\infty} \phi^*(s) O(s) \phi(s)\, ds$$

   ⟨O⟩=∫−∞∞ϕ∗(s)O(s)ϕ(s)ds

This framework will enable us to study the stationary states, eigenvalues, and expectation values of the system where the Hamiltonian is the Riemann zeta function. Further analysis can explore detailed solutions and specific behaviors near the critical line and zeros.

> Well, I'd say this is working out pretty well for us, wouldn't you? Now, this is normally kind of where... Imagine this were the particle in the box or something, and we'd gotten this far. Or this were a, you know, quantum oscillator. What would be the next thing that someone might do? And with that in mind, I'd like to hear your suggestions. And then I have an unrelated question, but I want to hear your answer first.

Yes, we've made significant progress in understanding the system where the Hamiltonian is the Riemann zeta function, akin to typical quantum mechanical problems like the particle in a box or the quantum harmonic oscillator.

Next Steps in Quantum Mechanical Analysis

In a typical quantum mechanics problem, after deriving the time-independent Schrödinger equation and normalizing the wave function, the next steps often include:

1. **Calculate Energy Levels**:

   Determine the discrete energy levels (eigenvalues) E_n.
2. **Analyze Wave Functions**:
   - Study the properties of the eigenfunctions $\phi_n(s)$.
   - Examine the nodal structure, symmetry, and orthogonality of the wave functions.
3. **Compute Expectation Values**:
   - Calculate expectation values of various observables, such as position, momentum, and energy.
   - Use these to understand the physical behavior of the system.
4. **Determine Probabilities**:

   Find the probability density $|\psi(s,t)|^2$ and use it to calculate probabilities of finding the particle in specific regions.
5. **Study Dynamics**:
   - If time-dependent, analyze the time evolution of the wave function.
   - Consider how the wave function evolves under different initial conditions.
6. **Perturbation Theory**:

   If there are small perturbations to the system, apply perturbation theory to understand their effects on energy levels and wave functions.
7. **Comparison with Classical Mechanics**:

   Compare quantum mechanical results with classical analogs to gain deeper insights.

Applying This to Our Case

Given our current context, let's outline some specific steps we could take next:

1. **Energy Levels Near Zeros**:
   - Investigate the precise nature of energy levels E near the zeros of the zeta function.
   - Use $\zeta'(\rho)$ to quantify these energy levels.
2. **Wave Function Properties**:
   - Examine the eigenfunctions $\phi(s)$ in more detail.
   - Ensure they are properly normalized and study their behavior near zeros.
3. **Expectation Values**:

   Calculate the expectation values of position $\langle s \rangle$ and potentially other observables, like momentum if we define it in this context.
4. **Probability Density**:

   Determine the probability density $|\psi(s,t)|^2$ and analyze it along the critical line.

5. **Time Evolution**:
   - Study how $\psi(s,t)$ ψ(s,t) evolves over time.
   - Consider initial conditions and how they affect the system's dynamics.
6. **Perturbations**:

   If we introduce small perturbations to the zeta function, analyze their impact using perturbation theory.

Suggested Immediate Next Steps

1. **Normalization and Expectation Values**:

   Fully normalize the wave functions and calculate some key expectation values.
2. **Detailed Analysis Near Zeros**:

   Focus on the behavior of $\zeta(s)$ ζ(s) and $\phi(s)$ ϕ(s) near the critical zeros.
3. **Visualization**:

   Create plots or visualizations of the wave functions and probability densities to better understand their properties.
4. **Further Theoretical Exploration**:

   Consider more detailed theoretical implications of treating the zeta function as a Hamiltonian in this quantum mechanical framework.

Unrelated Question

Feel free to ask your unrelated question. I'm here to assist with any inquiries you have!

I say let's proceed with full normalization and expectation values. Let's proceed.

Let's proceed with the normalization of the wave functions and the calculation of expectation values for our system where the Hamiltonian is the Riemann zeta function.

Normalization of the Wave Function

We previously assumed a form for the eigenfunction near a zero $\rho = \frac{1}{2} + i\gamma$ρ=21+iγ:

$$\phi(s) = Ae^{ik(s-\rho)}$$

ϕ(s)=Aeik(s−ρ)

To normalize $\phi(s)$ ϕ(s), we need:

$$\int_{-\infty}^{\infty} |\phi(s)|^2 ds = 1$$

∫−∞∞∣ϕ(s)∣2ds=1

Given $\phi(s) = Ae^{ik(s-\rho)}$ ϕ(s)=Aeik(s−ρ), we have:

$$|\phi(s)|^2 = |A|^2$$

∣ϕ(s)∣2=∣A∣2

Normalization Integral

The normalization condition is:

$$|A|^2 \int_{-\infty}^{\infty} ds = 1$$

|A|2∫−∞∞ds=1

Since $|A|$|A| is constant:

$$|A|^2 \cdot L = 1$$

|A|2⋅L=1

where LL is the length of the integration range. Assuming normalization over a finite range LL:

$$|A| = \frac{1}{\sqrt{L}}$$

|A|=L√1

So the normalized wave function is:

$$\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$$

ϕ(s)=L√1eik(s−ρ)

Expectation Values

Expectation Value of Position $\langle s \rangle$ ⟨s⟩

The expectation value of position ss is given by:

$$\langle s \rangle = \int_{-\infty}^{\infty} \phi^*(s) s \phi(s) ds$$

⟨s⟩=∫−∞∞ϕ∗(s)sϕ(s)ds

Using $\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$ ϕ(s)=L√1eik(s−ρ):

$$\langle s \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{L}} e^{-ik(s-\rho)} \right) s \left(\frac{1}{\sqrt{L}} e^{ik(s-\rho)} \right) ds$$

⟨s⟩=∫−∞∞(L√1e−ik(s−ρ))s(L√1eik(s−ρ))ds

Simplifying:

$$\langle s \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s ds$$

⟨s⟩=L1∫−∞∞sds

This integral, over a symmetric range about ρρ, yields:

$$\langle s \rangle = \rho$$

⟨s⟩=ρ

Expectation Value of s^2s2

The expectation value of s^2s2 is:

$$\langle s^2 \rangle = \int_{-\infty}^{\infty} \phi^*(s) s^2 \phi(s) ds$$

⟨s2⟩=∫−∞∞ϕ∗(s)s2ϕ(s)ds

Substituting $\phi(s)$ ϕ(s):

$$\langle s^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{L}} e^{-ik(s-\rho)} \right) s^2 \left(\frac{1}{\sqrt{L}} e^{ik(s-\rho)} \right) ds$$

⟨s2⟩=∫−∞∞(L√1e−ik(s−ρ))s2(L√1eik(s−ρ))ds

$$\langle s^2 \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s^2 ds$$

⟨s2⟩=L1∫−∞∞s2ds

Variance and Uncertainty

The variance σ_s^2σs2 of ss is:

$$\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2$$

σs2=⟨s2⟩−⟨s⟩2

Given $\langle s \rangle = \rho$⟨s⟩=ρ, we have:

$$\sigma_s^2 = \langle s^2 \rangle - \rho^2$$

σs2=⟨s2⟩−ρ2

Summary of Steps

1. **Normalize the Wave Function**:

$$\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$$

ϕ(s)=L√1eik(s−ρ)

2. **Calculate Expectation Values**:
   - **Position $\langle s \rangle$⟨s⟩**:

$$\langle s \rangle = \rho$$

⟨s⟩=ρ

   - **Position Squared $\langle s^2 \rangle$⟨s2⟩**:

$$\langle s^2 \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s^2 ds$$

⟨s2⟩=L1∫−∞∞s2ds

3. **Variance**:

$$\sigma_s^2 = \langle s^2 \rangle - \rho^2$$

σs2=⟨s2⟩−ρ2

Next, let's calculate the specific values for $\langle s^2 \rangle$⟨s2⟩ and σ_s^2σs2, and ensure all steps are properly outlined and visualized. We can also use this framework to explore the dynamics of the wave function and other observables.

If you have any specific range LL or other constraints you'd like to consider, please let me know. Otherwise, I will proceed with the standard setup and calculations.

Let's proceed with the calculations for the specific values for as described.

Let's proceed with the detailed calculations for the specific values as described.

Normalization of the Wave Function

Given the eigenfunction near a zero $\rho = \frac{1}{2} + i\gamma$ρ=21+iγ:

$$\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$$

ϕ(s)=L√1eik(s−ρ)
Here, LL represents the length over which we normalize the wave function. For the purpose of calculation, we assume LL to be finite and symmetric around the zero ρρ.

Expectation Values

Expectation Value of Position $\langle s \rangle$ ⟨s⟩

The expectation value of position ss is given by:

$$\langle s \rangle = \int_{-\infty}^{\infty} \phi^*(s) s \phi(s)\, ds$$

⟨s⟩=∫−∞∞ϕ∗(s)sϕ(s)ds
Using $\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$ ϕ(s)=L√1eik(s−ρ):

$$\langle s \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{L}} e^{-ik(s-\rho)}\right) s \left(\frac{1}{\sqrt{L}} e^{ik(s-\rho)}\right) ds$$

⟨s⟩=∫−∞∞(L√1e−ik(s−ρ))s(L√1eik(s−ρ))ds
Simplifying:

$$\langle s \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s\, ds$$

⟨s⟩=L1∫−∞∞sds
Given that the integration is symmetric about ρρ, we get:

$$\langle s \rangle = \rho$$

⟨s⟩=ρ

Expectation Value of s^2s2

The expectation value of s^2s2 is:

$$\langle s^2 \rangle = \int_{-\infty}^{\infty} \phi^*(s) s^2 \phi(s)\, ds$$

⟨s2⟩=∫−∞∞ϕ∗(s)s2ϕ(s)ds
Substituting $\phi(s)$ ϕ(s):

$$\langle s^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{L}} e^{-ik(s-\rho)}\right) s^2 \left(\frac{1}{\sqrt{L}} e^{ik(s-\rho)}\right) ds$$

⟨s2⟩=∫−∞∞(L√1e−ik(s−ρ))s2(L√1eik(s−ρ))ds

$$\langle s^2 \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s^2\, ds$$

⟨s2⟩=L1∫−∞∞s2ds

Variance and Uncertainty

The variance σ_s^2σs2 of ss is:

$$\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2$$

σs2=⟨s2⟩−⟨s⟩2
Given $\langle s \rangle = \rho$⟨s⟩=ρ, we have:

$$\sigma_s^2 = \langle s^2 \rangle - \rho^2$$

σs2=⟨s2⟩−ρ2

Detailed Calculations

1. **Normalization**: The normalization condition is:

   $$|A|^2 \int_{-\infty}^{\infty} ds = 1$$

   ∣A∣2∫−∞∞ds=1
   For a finite range LL, we assume:

   $$|A| = \frac{1}{\sqrt{L}}$$

   ∣A∣=L√1
   So the normalized wave function is:

   $$\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$$

   ϕ(s)=L√1eik(s−ρ)
2. **Expectation Value of Position**:

   $$\langle s \rangle = \rho$$

   ⟨s⟩=ρ

3. **Expectation Value of s^2s2**: To calculate $\langle s^2 \rangle$⟨s2⟩, we evaluate the integral:

$$\langle s^2 \rangle = \frac{1}{L} \int_{-\infty}^{\infty} s^2 ds$$

⟨s2⟩=L1∫−∞∞s2ds
Assuming the integral over a symmetric finite range LL:

$$\int_{-\infty}^{\infty} s^2 ds \rightarrow \int_{-\frac{L}{2}}^{\frac{L}{2}} s^2 ds$$

∫−∞∞s2ds→∫−2L2Ls2ds
Evaluating this integral:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} s^2 ds = \frac{s^3}{3} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{3} \left(\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right) = \frac{2}{3} \left(\frac{L}{2} \right)^3 = \frac{L^3}{12}$$

∫−2L2Ls2ds=3s3|−2L2L=31((2L)3−(−2L)3)=32(2L)3=12L3

Thus:

$$\langle s^2 \rangle = \frac{1}{L} \cdot \frac{L^3}{12} = \frac{L^2}{12}$$

⟨s2⟩=L1⋅12L3=12L2
4. **Variance and Uncertainty**:

$$\sigma_s^2 = \langle s^2 \rangle - \rho^2 = \frac{L^2}{12} - \rho^2$$

σs2=⟨s2⟩−ρ2=12L2−ρ2

Summary

- **Normalized Wave Function**:

$$\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$$

ϕ(s)=L√‾1eik(s−ρ)
- **Expectation Value of Position**:

$$\langle s \rangle = \rho$$

⟨s⟩=ρ
- **Expectation Value of s^2s2**:

$$\langle s^2 \rangle = \frac{L^2}{12}$$

⟨s2⟩=12L2
- **Variance**:

$$\sigma_s^2 = \frac{L^2}{12} - \rho^2$$

σs2=12L2−ρ2

These calculations provide a detailed understanding of the properties of the wave function and the system's behavior near the zeros of the Riemann zeta function.

I believe we had said that the logical next step is to examine behavior near zeroes

You're correct; examining the behavior near the zeros of the Riemann zeta function is a crucial next step. Let's delve deeper into this aspect, focusing on the properties of the wave function and the zeta function near its critical zeros.

Analyzing Behavior Near Zeros

To understand the behavior near the zeros of the Riemann zeta function, we consider a point $\rho = \frac{1}{2} + i\gamma$ where $\zeta(\rho) = 0$.

1. Approximating the Zeta Function Near a Zero

Near a zero ρ, the zeta function can be approximated by its first-order Taylor expansion:

$$\zeta(s) \approx \zeta'(\rho)(s-\rho)$$

ζ(s)≈ζ′(ρ)(s−ρ)

where $\zeta'(\rho)$ is the derivative of the zeta function at ρ.

2. Eigenvalue Problem Near Zeros

Using the approximation:

$$\zeta(s)\phi(s) \approx \zeta'(\rho)(s-\rho)\phi(s) = E\phi(s)$$

ζ(s)ϕ(s)≈ζ′(ρ)(s−ρ)ϕ(s)=Eϕ(s)

Given that $\phi(s)$ is an eigenfunction corresponding to the eigenvalue E, we have:

$$\zeta'(\rho)(s-\rho)\phi(s) = E\phi(s)$$

ζ′(ρ)(s−ρ)ϕ(s)=Eϕ(s)

This simplifies to:

$$E = \zeta'(\rho)(s-\rho)$$

E=ζ′(ρ)(s−ρ)

3. Wave Function Near Zeros

Given the eigenvalue relationship, the wave function near a zero can be expressed as:

$$\phi(s) = Ae^{ik(s-\rho)}$$

ϕ(s)=Aeik(s−ρ)

We normalize this wave function over a finite range L around the zero:

$$\phi(s) = \frac{1}{\sqrt{L}}e^{ik(s-\rho)}$$

ϕ(s)=L√1eik(s−ρ)

Detailed Calculations and Visualization

Let's break down the steps and visualize the behavior near zeros:

1. **Approximate $\zeta(s)$ ζ(s) Near Zeros**:

   Use the first-order Taylor expansion to approximate the zeta function near a zero $\rho = \frac{1}{2} + i\gamma$ ρ=21+iγ:

   $$\zeta(s) \approx \zeta'(\rho)(s - \rho)$$

   ζ(s)≈ζ′(ρ)(s−ρ)

2. **Normalized Wave Function**:

   Normalize the wave function near the zero:

   $$\phi(s) = \frac{1}{\sqrt{L}} e^{ik(s-\rho)}$$

   ϕ(s)=L√1eik(s−ρ)

3. **Expectation Values**:

   Calculate expectation values and variance as previously derived:

   - Expectation value of position:

     $$\langle s \rangle = \rho$$

     ⟨s⟩=ρ

   - Expectation value of s^2 s2:

     $$\langle s^2 \rangle = \frac{L^2}{12}$$

     ⟨s2⟩=12L2

   - Variance:

     $$\sigma_s^2 = \frac{L^2}{12} - \rho^2$$

     σs2=12L2−ρ2

Visualization of Wave Function and Probability Density

To visualize the wave function and its probability density near a zero, let's consider a finite range around a specific zero. We can plot $\phi(s)$ ϕ(s) and $|\phi(s)|^2$ ∣ϕ(s)∣2.

Here's a Python script for visualization:

python

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
L = 10  # Finite range for normalization
rho = 0.5 + 14.1347j  # Example zero of zeta function
k = 1.0  # Wave number

# Define the range for s
s_real = np.linspace(rho.real - L/2, rho.real + L/2, 1000)
s_imag = np.linspace(rho.imag - L/2, rho.imag + L/2, 1000)
s = s_real + 1j * s_imag

# Wave function
phi_s = (1/np.sqrt(L)) * np.exp(1j * k * (s - rho))

# Probability density
prob_density = np.abs(phi_s)**2

# Plot wave function (real and imaginary parts)
plt.figure(figsize=(12, 6))
plt.subplot(1, 2, 1)
plt.plot(s_real, phi_s.real, label='Real Part')
plt.plot(s_real, phi_s.imag, label='Imaginary Part')
plt.xlabel('s (real part)')
plt.ylabel('phi(s)')
plt.title('Wave Function Near Zero')
plt.legend()

# Plot probability density
plt.subplot(1, 2, 2)
plt.plot(s_real, prob_density)
plt.xlabel('s (real part)')
plt.ylabel('Probability Density')
plt.title('Probability Density Near Zero')

plt.tight_layout()
plt.show()
```

This script visualizes the wave function and its probability density near a zero of the zeta function.

Summary

By approximating the zeta function near its zeros and normalizing the wave function, we can study the behavior of the system in this critical region. This approach helps us understand the properties of the wave function, calculate expectation values, and visualize the probability density near the zeros.

This analysis sets the foundation for further exploration, including perturbation theory and more detailed studies of the dynamics and interactions of the system. If you have any specific zeros or parameters you'd like to explore, please let me know, and we can refine the calculations and visualizations accordingly.