B

End-Actions for Restrained Members

A restrained member is one whose ends are restrained against displacements (translation and rotation), as in the case of a fixed-end beam. The end-actions for a restrained member are the reactive actions (forces and moments) developed at the ends when the member is subjected to loads, temperature changes, or other effects. Restrained members are encountered in the stiffness method of analysis and also in the determination of equivalent joint loads (see Chapters 3 and 4). In this appendix, formulas are given for end-actions in restrained members due to various causes. It is assumed in each case that the member is prismatic.

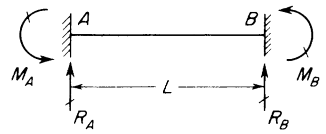
Table B-1 gives end-actions in fixed-end beams that are subjected to various conditions of loading. As shown in the figure at the top of the table, the length of the beam is L, the reactive moments at the left and right-hand ends are denoted M_A and M_B, respectively, and the reactive forces are denoted R_A and R_B, respectively. The moments are positive when counterclockwise, and the forces are positive when upward. Formulas for these quantities are given in Cases 1, 2, 5, 6, 7, and 8. However, Cases 3 and 4 differ slightly because of the special nature of the loads. In Case 3 the load is an axial force P, and therefore the only reactions are the two axial forces shown in the figure. In Case 4 the load is a twisting moment T, which produces reactions in the form of twisting moments only.

All of the formulas given in Table B-1 can be derived by standard methods of mechanics of materials. For instance, many of the formulas for beams can be obtained by integration of the differential equation for bending of a beam. The flexibility method, as described in Chapter 2, can also be used to obtain the formulas. Furthermore, the more complicated cases of loading frequently can be obtained from the simpler cases by using the principle of superposition.

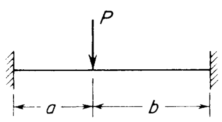
Fixed-end actions due to temperature changes are listed in Table B-2. Case 1 of this table is for a beam subjected to a uniform temperature increase of ΔT. The resulting end-actions consist of axial compressive forces that are independent of the length of the member. The second case is a beam subjected to a linear temperature gradient such that the top of the beam has a temperature change ΔT_2, while the bottom has a change ΔT_1. If the temperature at the centroidal axis remains unchanged, there is no tendency for the beam to change in length; and the end-actions consist of moments only. On the other hand, a nonzero change of temperature at the centroidal axis is covered by Case 1.

Table B-1

Fixed-End Actions Caused by Loads



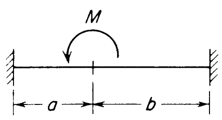
1



$$M_A = \frac{Pab^2}{L^2} \quad M_B = -\frac{Pa^2b}{L^2}$$

$$R_A = \frac{Pb^2}{L^3}(3a + b) \quad R_B = \frac{Pa^2}{L^3}(a + 3b)$$

2

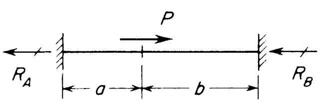


$$M_A = \frac{Mb}{L^2}(2a - b)$$

$$M_B = \frac{Ma}{L^2}(2b - a)$$

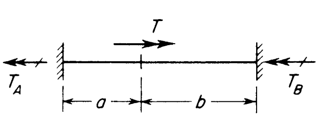
$$R_A = -R_B = \frac{6Mab}{L^3}$$

3



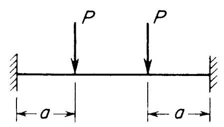
$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

4



$$T_A = \frac{Tb}{L} \qquad T_B = \frac{Ta}{L}$$

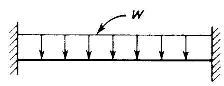
5



$$M_A = -M_B = \frac{Pa}{L}(L - a)$$

$$R_A = R_B = P$$

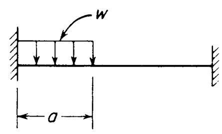
6



$$M_A = -M_B = \frac{wL^2}{12}$$

$$R_A = R_B = \frac{wL}{2}$$

7



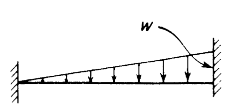
$$M_A = \frac{wa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = -\frac{wa^3}{12L^2}(4L - 3a)$$

$$R_A = \frac{wa}{2L^3}(2L^3 - 2a^2L + a^3)$$

$$R_B = \frac{wa^3}{2L^3}(2L - a)$$

8



$$M_A = \frac{wL^2}{30} \qquad M_B = -\frac{wL^2}{20}$$

$$R_A = \frac{3wL}{20} \qquad R_B = \frac{7wL}{20}$$

Table B-2

Fixed-End Actions Caused by Temperature Changes

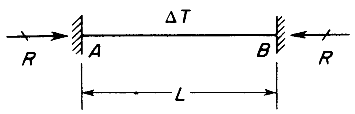
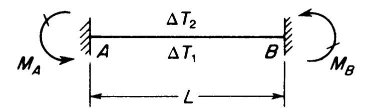
1 Uniform increase in temperature	2 Linear temperature gradient
	
$R = EA\alpha\,\Delta T$	$M_A = -M_B = \dfrac{\alpha EI(\Delta T_1 - \Delta T_2)}{d}$
E = modulus of elasticity A = cross-sectional area α = coefficient of thermal expansion ΔT = temperature increase	I = moment of inertia ΔT_1 = temperature change at bottom of beam ΔT_2 = temperature change at top of beam d = depth of beam

Table B-3 gives fixed-end actions due to prestrains in the members. A prestrain is an initial deformation of a member, causing end-actions to be developed when the ends of the member are held in the restrained positions. The simplest example of a prestrain is shown in Case 1, where member AB is assumed to have an initial length that is greater than the distance between supports by a small amount e. When the ends of the member are held in their final positions, the member will have been shortened by the distance e. The resulting fixed-end actions are the axial compressive forces shown in the table. Case 2 is a member with an initial bend in it, and the last case is a member having an initial circular curvature such that the deflection at the middle of the beam is equal to the small distance e.

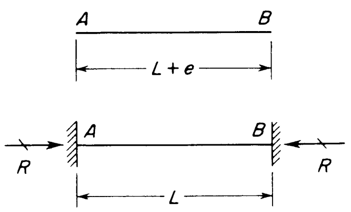
Table B-4 lists formulas for fixed-end actions caused by displacements of one end of the member. Cases 1 and 2 are for axial and lateral translations of the end B of the member through the small distance Δ, while Cases 3 and 4 are for rotations. The rotation through the angle θ shown in Case 3 produces bending of the member, while the rotation through the angle ϕ in Case 4 produces torsion. Formulas for the torsion constant J, which appears in the formulas of Case 4, are given in Appendix C for several cross-sectional shapes.

End-actions for truss members are listed in Table B-5 for three cases of loading: a uniform load, a concentrated load, and a moment. The members shown in the figures have pinned ends that are restrained against translation but not rotation, because only joint translations are of interest in a truss analysis. The members are shown inclined at an angle γ to the horizontal, in order to have a general orientation. However, the end-actions are independent of the angle of inclination, which may have any value (including 0 and 90 degrees). For both the uniform load and the concentrated load

Table B-3

Fixed-End Actions Caused by Prestrains

1 Bar with excess length



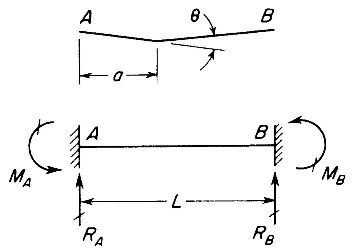
$$R = \frac{EAe}{L}$$

E = modulus of elasticity

A = cross-sectional area

e = excess length

2 Bar with a bend



$$M_A = \frac{2EI\theta}{L^2}(2L - 3a)$$

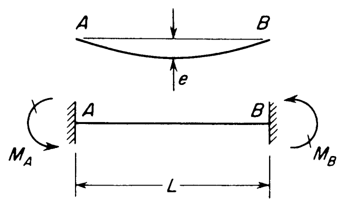
$$M_B = \frac{2EI\theta}{L^2}(L - 3a)$$

$$R_A = -R_B = \frac{6EI\theta}{L^3}(L - 2a)$$

I = moment of inertia

θ = angle of bend

3 Initial circular curvature



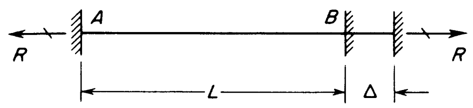
$$M_A = -M_B = \frac{8EIe}{L^2}$$

e = initial deflection at middle of bar

Table B-4

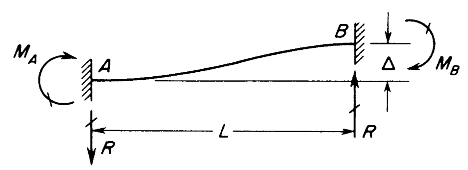
Fixed-End Actions Caused by End-Displacements

1



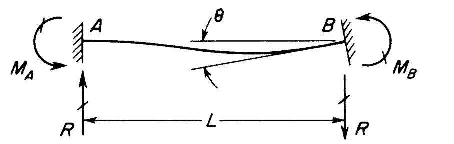
$$R = \frac{EA\Delta}{L}$$

2



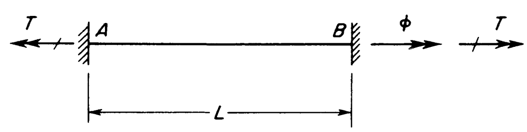
$$M_A = M_B = \frac{6EI\Delta}{L^2} \qquad R = \frac{12EI\Delta}{L^3}$$

3



$$M_A = \frac{2EI\theta}{L} \qquad M_B = \frac{4EI\theta}{L} \qquad R = \frac{6EI\theta}{L^2}$$

4



$$T = \frac{GJ\phi}{L}$$

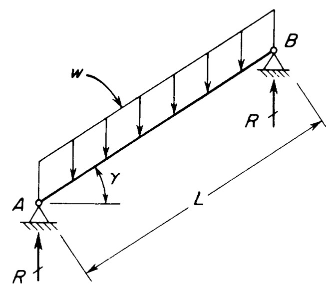
G = shear modulus of elasticity

J = torsion constant

Table B-5

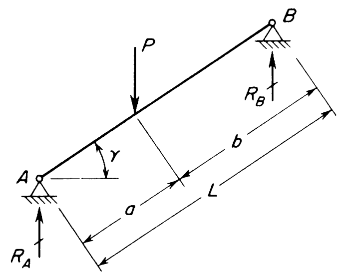
End-Actions for Truss Members

1



$$R = \frac{wL}{2}$$

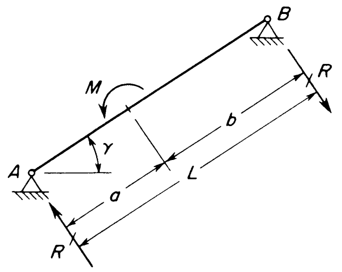
2



$$R_A = \frac{Pb}{L}$$

$$R_B = \frac{Pa}{L}$$

3



$$R = \frac{M}{L}$$

(Cases 1 and 2) the reactions are parallel to the lines of action of the loads, while in Case 3 the reactions are perpendicular to the axis of the member.

If a truss member is subjected to a uniform increase in temperature, Case 1 of Table B-2 can be used; if subjected to a prestrain consisting of an increase in length, Case 1 of Table B-3 can be used; and if subjected to a displacement in the axial direction, Case 1 of Table B-4 can be used.