B FLEXIBILITY METHOD

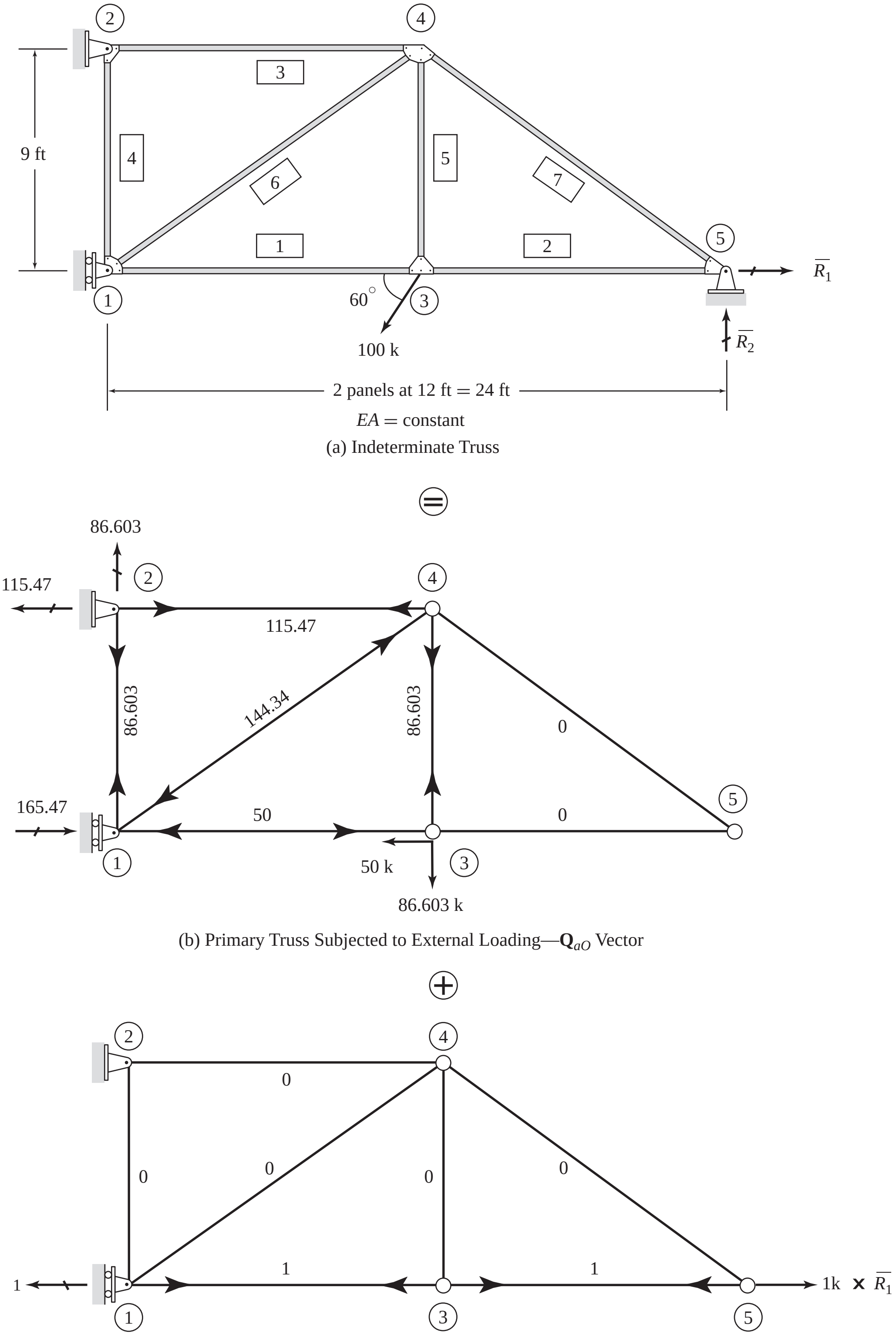
In this text, we have focused our attention on the matrix stiffness method of structural analysis, which is the most commonly used method in professional practice today, and which forms the basis for most of the currently available commercial software for structural analysis. However, as stated in Section 1.3, another type of matrix method, called the flexibility method, can also be used for structural analysis. While the stiffness method can be applied to both statically determinate and indeterminate structures, the flexibility method is applicable only to indeterminate structures. The flexibility method is essentially a generalization in matrix form of the classical method of consistent deformations, and is generally considered convenient for analyzing small structures with a few redundants.

In this appendix, we present the basic concept of the flexibility method, and illustrate its application to plane trusses. A more detailed treatment of this method can be found in [3] and [52].

Essentially, the flexibility method of analysis involves removing enough restraints from the indeterminate structure to render it statically determinate. This determinate structure, which must be statically stable, is called the *primary structure*; and the reactions or internal forces associated with the excess restraints removed from the given indeterminate structure to convert it into the determinate (primary) structure, are termed *redundants*. The redundants are then treated as unknown loads on the primary structure, and their values are determined by solving the compatibility equations based on the condition that the deformations of the primary structure due to the combined effect of the redundants and the given external loading must be the same as the deformations of the original indeterminate structure.

Consider, for example, a plane truss supported by five reaction components, as shown in Fig. B.1(a) on the next page. The truss is internally determinate, but externally indeterminate with two degrees of indeterminacy. This indicates that the truss has two more, or redundant, reactions than necessary for static stability. Thus, if we can determine two of the five reactions by using compatibility equations based on the geometry of the deformation of the truss, then the remaining three reactions and the member forces can be obtained from equilibrium considerations.

To analyze the truss by the flexibility method, we must select two of the unknown reactions and member forces to be the redundants. Suppose that we select the horizontal and vertical reactions, $\overline{R}_1$ and $\overline{R}_2$, at the hinged support at joint 5 to be the redundants. The hinged support at joint 5 is then removed from the given indeterminate truss to obtain the statically determinate and stable

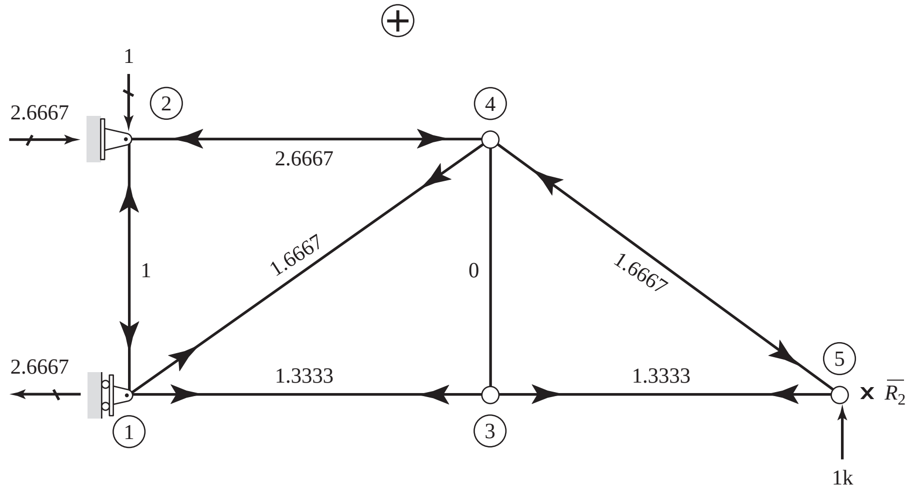


(a) Indeterminate Truss

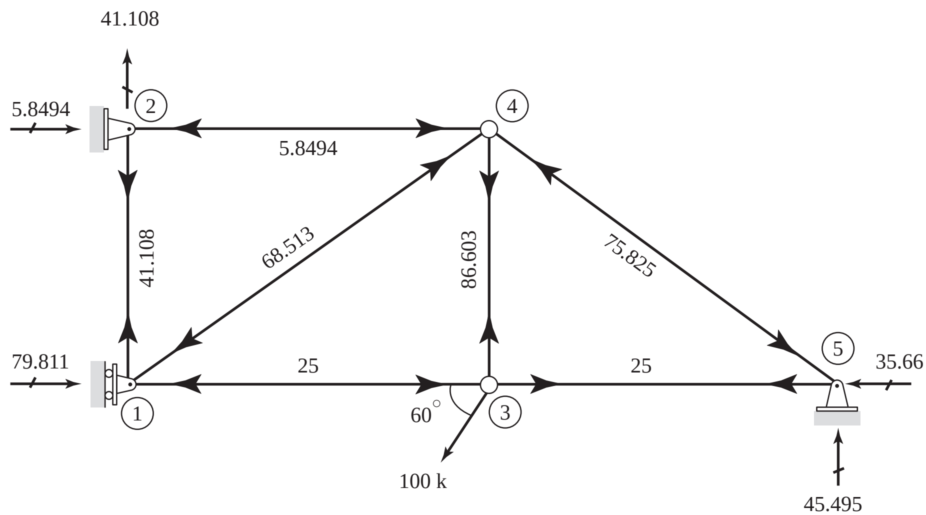
(b) Primary Truss Subjected to External Loading—$\mathbf{Q}_{aO}$ Vector

(c) Primary Truss Subjected to Unit Value of Redundant $\overline{R}_1$—First Column of **b** Matrix

Fig. B.1



(d) Primary Truss Subjected to Unit Value of Redundant $\overline{R_2}$—Second Column of **b** Matrix



(e) Support Reactions and Member Forces for Indeterminate Truss

Fig. B.1 (*continued*)

primary truss, as shown in Fig. B.1(b). The two redundants $\overline{R}_1$ and $\overline{R}_2$ are now treated as unknown loads on the primary truss, and their magnitudes can be determined from the compatibility conditions that the horizontal and vertical deflections at joint 5 of the primary truss due to the combined effect of the known 100 k load and the unknown redundants $\overline{R}_1$ and $\overline{R}_2$ must be equal to 0. This is because the deflections in the horizontal and vertical directions of the given indeterminate truss at the hinged support at joint 5 are 0.

The compatibility equations can be conveniently established by superimposing the deflections due to the external loading and the redundants, $\overline{R}_1$ and

$\overline{R}_2$, acting individually on the primary truss, as shown in Figs. B.1(b), (c), and (d), respectively. Thus,

$$\Delta_{O1} + f_{11}\overline{R}_1 + f_{12}\overline{R}_2 = 0 \tag{B.1a}$$

$$\Delta_{O2} + f_{21}\overline{R}_1 + f_{22}\overline{R}_2 = 0 \tag{B.1b}$$

in which Δ_{Oi} $(i = 1, 2)$ represents the deflection at joint 5 of the primary truss in the direction of the redundant $\overline{R}_i$, due to the external loading; and the *flexibility coefficient* f_{ij} $(i = 1, 2$ and $j = 1, 2)$ denotes the deflection of the primary truss at the location and in the direction of a redundant $\overline{R}_i$ due to a unit value of a redundant $\overline{R}_j$. Equations (B.1) can be expressed in matrix form as

$$\begin{bmatrix} \Delta_{O1} \\ \Delta_{O2} \end{bmatrix} + \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} \overline{R}_1 \\ \overline{R}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{B.2}$$

From the foregoing discussion for the example truss with two degrees of indeterminacy, we realize that the compatibility equations for a general indeterminate structure with n_i degrees of indeterminacy can be symbolically expressed as

$$\boxed{\boldsymbol{\Delta}_O + \mathbf{f}\overline{\mathbf{R}} = \mathbf{0}} \tag{B.3}$$

in which the $n_i \times 1$ vectors $\overline{\mathbf{R}}$ and $\boldsymbol{\Delta}_O$ denote, respectively, the unknown redundants, and the deflections of the primary structure at the locations and in the directions of the redundants due to external loads; and the $n_i \times n_i$ matrix $\mathbf{f}$ is called the *structure flexibility matrix*. The reader may recall from a previous course in mechanics of materials or *structural analysis* [18], that *Maxwell's law of reciprocal deflections* states that *for a linearly elastic structure, the deflection at a point i due to a unit load applied at a point j is equal to the deflection at j due to a unit load at i*. As the flexibility coefficient f_{ij} denotes the deflection of the primary structure at the location of the redundant $\overline{R}_i$ due to a unit value of the redundant $\overline{R}_j$, and the flexibility coefficient f_{ji} denotes the deflection corresponding to $\overline{R}_j$ due to a unit value of $\overline{R}_i$, according to *Maxwell's law* f_{ij} must be equal to f_{ji} (i.e., $f_{ij} = f_{ji}$). We can thus deduce that *for linearly elastic structures, the flexibility matrices are symmetric*.

From Eqs. (B.1) through (B.3), we can see that the elements of the vector $\boldsymbol{\Delta}_O$ and the flexibility matrix $\mathbf{f}$ represent deflections of the primary (statically determinate) structure. Once these deflections have been evaluated, the compatibility equations (Eqs. (B.3)) can be solved for the unknown redundants. With the redundants known, the other response characteristics of the structure can be evaluated, either by equilibrium or superposition.

The deflections (and the flexibility coefficients) of a primary structure can be conveniently expressed in terms of the forces and properties of its members, using the virtual work method. Recall from a previous course in mechanics of materials or structural analysis [18], that the expression of the virtual work method for truss deflections is given by

$$\Delta = \sum_{i=1}^{NM} \frac{Q_{ar}\,Q_{av}\,L}{EA} \tag{B.4}$$

in which NM denotes the number of members of the truss; Q_{ar} represents the axial forces in truss members due to the real loading that causes the deflection Δ, and Q_{av} represents the axial forces in the truss members due to a virtual unit load acting at the location and in the direction of the desired deflection Δ. Equation (B.4) can be expressed in matrix form as

$$\Delta = \mathbf{Q}_{av}^T \mathbf{f}_M \mathbf{Q}_{ar} \tag{B.5}$$

in which $\mathbf{Q}_{av}$ and $\mathbf{Q}_{ar}$ denote the $NM \times 1$ vectors containing member axial forces due to virtual (unit) and real (actual) loads, respectively; and $\mathbf{f}_M$ is a $NM \times NM$ diagonal matrix containing the member flexibilities (L/EA) on its main diagonal (i.e., $f_{Mij} = L_i/E_iA_i$ for $i = j$, and $f_{Mij} = 0$ for $i \neq j$). The diagonal matrix $\mathbf{f}_M$ is sometimes called the *unassembled flexibility matrix*. In order to develop the expressions for $\boldsymbol{\Delta}_O$ and $\mathbf{f}$ in terms of the member forces and properties, let us define a $NM \times 1$ vector $\mathbf{Q}_{aO}$ which contains the axial forces in the members of the primary truss due to the given external loading, and a $NM \times n_i$ matrix $\mathbf{b}$, the jth column of which contains member axial forces due to a unit value of the jth redundant (i.e., $\overline{R}_j = 1$). The matrix $\mathbf{b}$ is commonly referred to as an *equilibrium matrix*. In both $\mathbf{Q}_{aO}$ and $\mathbf{b}$ the member axial forces are stored in sequential order of member numbers; that is, the axial forces in the ith member are stored in the ith rows of $\mathbf{Q}_{aO}$ and $\mathbf{b}$, and so on. The member forces $\mathbf{Q}_{aO}$ for the example truss are shown in Fig. B.1(b). Note that since the primary truss is statically determinate, the forces in its members due to the given external loading can be conveniently evaluated by applying the method of joints. By using the member forces shown in Fig. B.1(b) and the member numbers given in Fig. B.1(a), we form the $\mathbf{Q}_{aO}$ vector for the truss as

$$\mathbf{Q}_{aO} = \begin{array}{c} \begin{bmatrix} -50 \\ 0 \\ 115.47 \\ 86.603 \\ 86.603 \\ -144.34 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \text{ k} \tag{B.6}$$

in which the tensile member axial forces are considered to be positive. The first column of $\mathbf{b}$ is obtained by subjecting the primary truss to a unit value of the redundant $\overline{R}_1$, as shown in Fig. B.1(c), and by computing the corresponding member forces by applying the method of joints. The second column of $\mathbf{b}$ is generated similarly by subjecting the primary truss to a unit value of the redundant $\overline{R}_2$, and by computing the corresponding member axial forces (see Fig. B.1(d)). The equilibrium matrix $\mathbf{b}$ thus obtained is

$$\mathbf{b} = \begin{bmatrix} 1 & 1.3333 \\ 1 & 1.3333 \\ 0 & -2.6667 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1.6667 \\ 0 & -1.6667 \end{bmatrix} \text{ k/k} \tag{B.7}$$

The first element, Δ_{O1}, of the $\mathbf{\Delta}_O$ vector represents deflection of the primary truss at the location and in the direction of $\overline{R}_1$ due to the given external loading. Therefore, for the purpose of calculating Δ_{O1} via the virtual work method, the real system consists of the given external loading as shown in Fig. B.1(b), and the virtual system consists of a unit load applied at the location and in the direction of the redundant $\overline{R}_1$, which is the same as the system shown in Fig. B.1(c) (without the multiplier $\overline{R}_1$). Thus, the virtual work expression for Δ_{O1} can be obtained by substituting $\mathbf{Q}_{aO}$ for $\mathbf{Q}_{ar}$ and the first column of $\mathbf{b}$ for $\mathbf{Q}_{av}$ in Eq. (B.5); that is,

$$\Delta_{O1} = \mathbf{b}_1^T \mathbf{f}_M \mathbf{Q}_{aO} \tag{B.8}$$

in which $\mathbf{b}_1$ denotes the first column of $\mathbf{b}$. The expression for the second element, Δ_{O2}, of $\mathbf{\Delta}_O$, in terms of member axial forces, can be obtained in a similar manner, and is given by

$$\Delta_{O2} = \mathbf{b}_2^T \mathbf{f}_M \mathbf{Q}_{aO} \tag{B.9}$$

with $\mathbf{b}_2$ denoting the second column of $\mathbf{b}$. By combining Eqs. (B.8) and (B.9), we obtain

$$\boxed{\mathbf{\Delta}_O = \mathbf{b}^T \mathbf{f}_M \mathbf{Q}_{aO}} \tag{B.10}$$

The expressions for the elements of the flexibility matrix $\mathbf{f}$, in terms of member forces and properties, can be obtained in a similar manner. For example, the virtual and real systems for the evaluation of f_{12} are shown in Figs. B.1(c) and (d), respectively, with the corresponding member forces stored in the first and second columns of $\mathbf{b}$. Therefore,

$$f_{12} = \mathbf{b}_1^T \mathbf{f}_M \mathbf{b}_2$$

Thus, the entire flexibility matrix $\mathbf{f}$ can be expressed in terms of the member forces and properties as

$$\boxed{\mathbf{f} = \mathbf{b}^T \mathbf{f}_M \mathbf{b}} \tag{B.11}$$

Finally, by substituting Eqs. (B.10) and (B.11) into Eq. (B.3), we obtain the structure's compatibility equations in terms of its member forces and properties, as

$$\boxed{\mathbf{b}^T \mathbf{f}_M \mathbf{Q}_{aO} + (\mathbf{b}^T \mathbf{f}_M \mathbf{b})\,\overline{\mathbf{R}} = \mathbf{0}} \tag{B.12}$$

To illustrate the application of the flexibility method to the analysis of plane trusses, let us reconsider the truss of Fig. B.1. The $\mathbf{Q}_{aO}$ vector and the $\mathbf{b}$ matrix for this truss were determined previously, and are given in Eqs. (B.6) and (B.7), respectively. The unassembled flexibility matrix for the structure is

$$\mathbf{f}_M = \frac{1}{EA}\begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15 \end{bmatrix} \tag{B.13}$$

Substituting Eqs. (B.6), (B.7), and (B.13) into Eq. (B.10), and performing the required matrix multiplications, we obtain

$$\mathbf{\Delta}_O = \mathbf{b}^T \mathbf{f}_M \mathbf{Q}_{aO} = \frac{1}{EA}\begin{bmatrix} -600 \\ -8{,}883 \end{bmatrix} \tag{B.14}$$

Substitution of Eqs. (B.7) and (B.13) into Eq. (B.11) yields the flexibility matrix:

$$\mathbf{f} = \mathbf{b}^T \mathbf{f}_M \mathbf{b} = \frac{1}{EA}\begin{bmatrix} 24 & 32 \\ 32 & 220.33 \end{bmatrix} \tag{B.15}$$

Next, we substitute Eqs. (B.14) and (B.15) into the compatibility equations (Eqs. (B.12)), and solve the resulting system of simultaneous equations for the unknown redundants. This yields

$$\overline{\mathbf{R}} = \begin{bmatrix} -35.66 \\ 45.495 \end{bmatrix} \text{k} \tag{B.16}$$

With the redundants known, the member axial forces in the actual indeterminate structure, $\mathbf{Q}_a$, can be conveniently evaluated by applying the superposition relationship (see Figs. B.1(a) through (d)):

$$\boxed{\mathbf{Q}_a = \mathbf{Q}_{aO} + \mathbf{b}\overline{\mathbf{R}}} \tag{B.17}$$

Substituting Eqs. (B.6), (B.7), and (B.16) into Eq. (B.17), we determine the axial forces in the members of the indeterminate truss to be

$$\mathbf{Q}_a = \begin{bmatrix} -25 \\ 25 \\ -5.8494 \\ 41.108 \\ 86.603 \\ -68.513 \\ -75.825 \end{bmatrix} \text{k}$$

These member axial forces are shown in Fig. B.1(e).