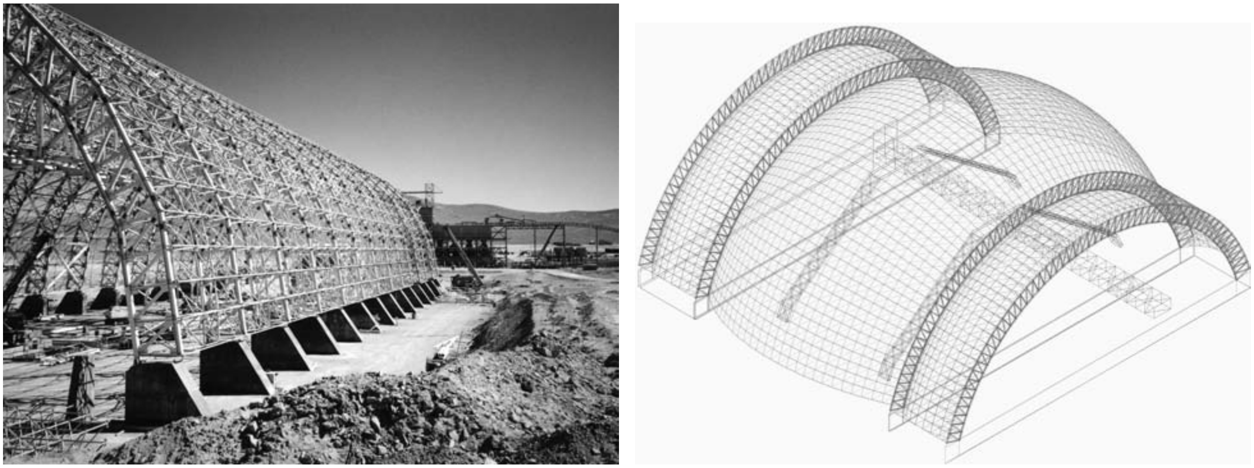
8

THREE-DIMENSIONAL FRAMED STRUCTURES

8.1 Space Trusses
8.2 Grids
8.3 Space Frames
Summary
Problems



Space Truss and its Analytical Model
(Courtesy of Triodetic)

Up to this point, we have focused our attention on the analysis of plane-framed structures. While many actual three-dimensional structures can be divided into planar parts for the purpose of analysis, there are others (e.g., lattice domes and transmission towers) that, because of the arrangement of their members or applied loading, cannot be divided into plane structures. Such structures are analyzed as space structures subjected to three-dimensional loadings. The matrix stiffness analysis of space structures is similar to that of plane structures—except, of course, that member stiffness and transformation matrices appropriate for the particular type of space structure under consideration are now used in the analysis.

In this chapter, we extend the matrix stiffness formulation, developed for plane structures, to the analysis of three-dimensional or space structures. Three types of space-framed structures are considered: space trusses, grids, and space frames, with methods for their analysis presented in Sections 8.1, 8.2, and 8.3, respectively.

The computer programs for the analysis of space-framed structures can be conveniently adapted from those for plane structures, via relatively straightforward modifications that should become apparent as the analysis of space structures is developed in this chapter. Therefore, the details of programming the analysis of space structures are not covered herein; they are, instead, left as exercises for the reader.

8.1 SPACE TRUSSES

A *space truss* is defined as *a three-dimensional assemblage of straight prismatic members connected at their ends by frictionless ball-and-socket joints, and subjected to loads and reactions that act only at the joints.* Like plane trusses, the members of space trusses develop only axial forces. The matrix stiffness analysis of space trusses is similar to that of plane trusses developed in Chapter 3 (and modified in Chapter 7).

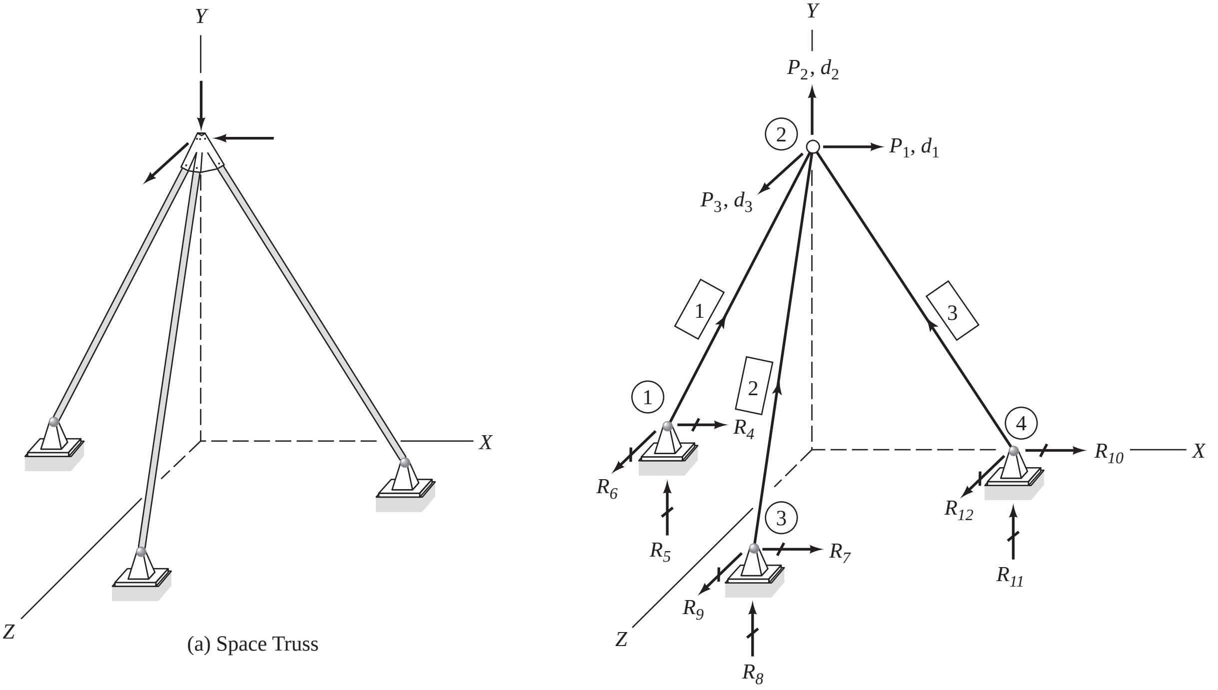
The process of developing the analytical models of space trusses (and numbering the degrees of freedom and restrained coordinates) is essentially the same as that for plane trusses (Chapter 3). The overall geometry of the space truss, and its joint loads and displacements, are described with reference to a global Cartesian or rectangular right-handed XYZ coordinate system, with three global (X, Y, and Z) coordinates now used to specify the location of each joint. Furthermore, since an unsupported joint of a space truss can translate in any direction in the three-dimensional space, three displacements—the translations in the X, Y, and Z directions—are needed to completely establish its deformed position. Thus, a free joint of a space truss has three degrees of freedom, and three structure coordinates (i.e., free and/or restrained coordinates) need to be defined at each joint, for the purpose of analysis. Thus,

$$\boxed{\left.\begin{array}{l} NCJT = 3 \\ NDOF = 3(NJ) - NR \end{array}\right\} \quad \text{for space trusses}} \tag{8.1}$$

The procedure for assigning numbers to the structure coordinates of a space truss is analogous to that for plane trusses. The degrees of freedom of the space truss are numbered first by beginning at the lowest-numbered joint with a degree of freedom, and proceeding sequentially to the highest-numbered joint. If a joint has more than one degree of freedom, then the translation in the X direction is numbered first, followed by the translation in the Y direction, and then the translation in the Z direction. After all the degrees of freedom have been numbered, the restrained coordinates of the space truss are numbered in the same manner as the degrees of freedom.

Consider, for example, the three-member space truss shown in Fig. 8.1(a). As the analytical model of the truss depicted in Fig. 8.1(b) indicates, the structure has three degrees of freedom ($NDOF = 3$), which are the translations d_1, d_2, and d_3 of joint 2 in the X, Y, and Z directions, respectively; and nine restrained coordinates ($NR = 9$), which are identified as R_4 through R_{12} at the support joints 1, 3, and 4.

As in the case of plane trusses, a local right-handed xyz coordinate system is established for each member of the space truss. The origin of the local coordinate system is located at one of the ends (which is referred to as the *beginning* of the member), with the x axis directed along the member's centroidal axis in its undeformed state. Since the space truss members can only develop axial forces, the positive directions of the y and z axes can be chosen



(a) Space Truss

(b) Analytical Model

Fig. 8.1

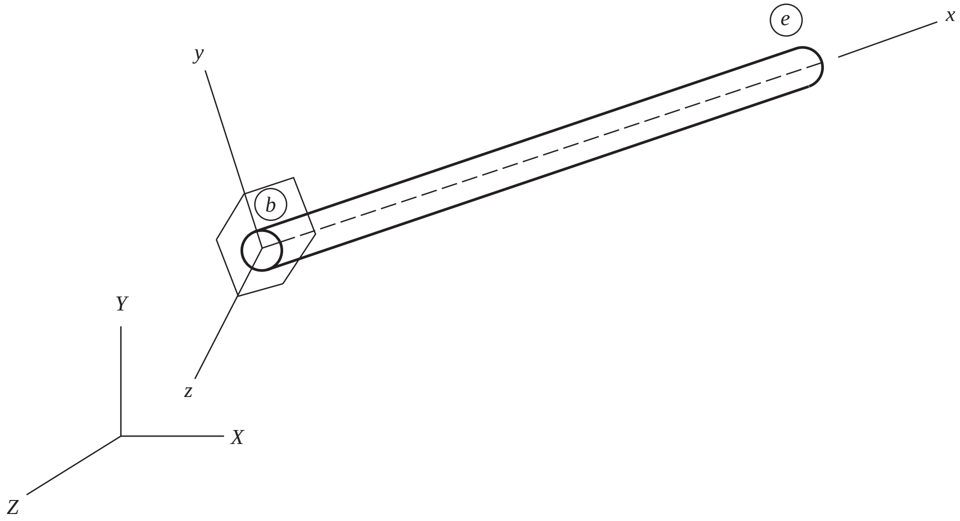


Fig. 8.2 *Local Coordinate System for Members of Space Trusses*

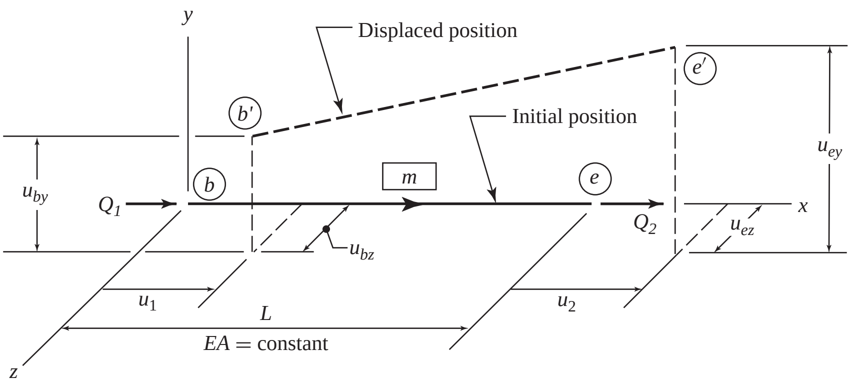
arbitrarily, provided that the x, y, and z axes are mutually perpendicular and form a right-handed coordinate system (Fig. 8.2).

Member Stiffness Relations in the Local Coordinate System

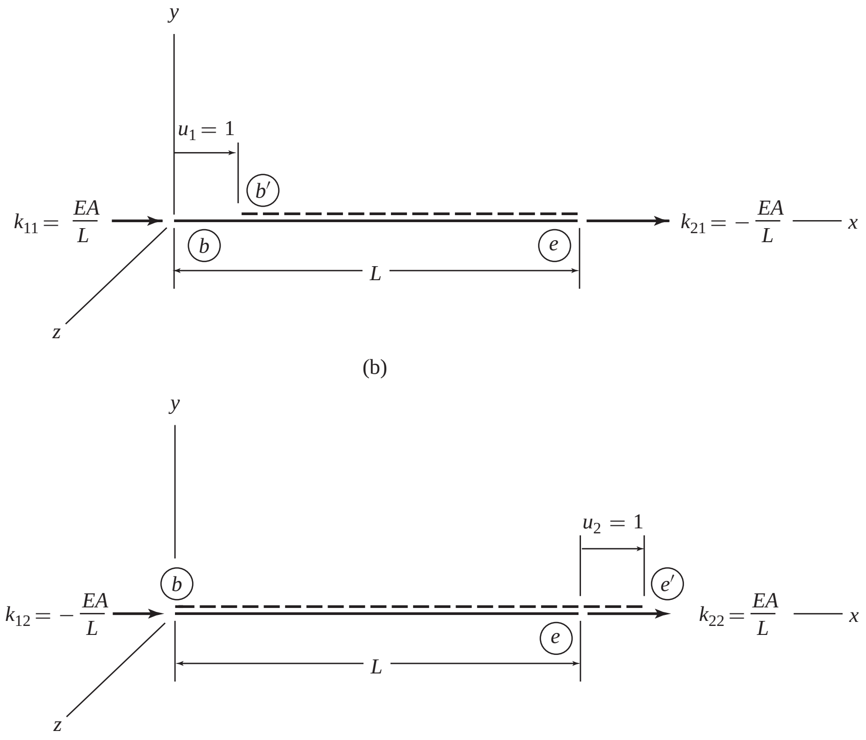
To establish the member local stiffness relations, let us focus our attention on an arbitrary prismatic member m of a space truss. When the truss is subjected to external loads, member m deforms and axial forces are induced at its ends. The initial and displaced positions of the member are shown in Fig. 8.3(a). As this figure indicates, three displacements—translations in the x, y, and z directions—are needed to completely specify the displaced position of each end of the member. Thus, the member has a total of six degrees of freedom or end displacements. However, as discussed in Section 3.3 (see Figs. 3.3(d) and (f)), small end displacements in the directions perpendicular to a truss member's centroidal axis do not cause any forces in the member. Thus, the end displacements u_{by}, u_{bz}, u_{ey}, and u_{ez} in the directions of the local y and z axes of the member, as shown in Fig. 8.3(a), are usually not evaluated in the analysis; and for analytical purposes, the member is considered to have only two degrees of freedom, u_1 and u_2, in its local coordinate system. Thus, the local end displacement vector $\mathbf{u}$ for a member of a space truss is expressed as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

in which u_1 and u_2 represent the displacements of the member ends b and e, respectively, in the direction of the member's local x axis, as shown in Fig. 8.3(a). As this figure also indicates, the member end forces corresponding to the end displacements u_1 and u_2 are denoted by Q_1 and Q_2, respectively.



(a) Member Forces and Displacements
in the Local Coordinate System



(b)

(c)

Fig. 8.3

The relationship between the local end forces **Q** and the end displacements **u**, for the members of space trusses, is written as

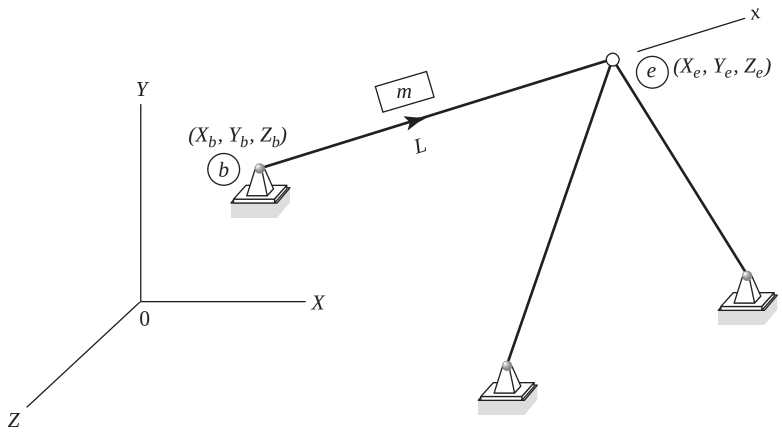
$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{u}} \qquad \textbf{(8.2)}$$

in which **k** represents the 2×2 member stiffness matrix in the local coordinate system. The explicit form of **k** can be obtained by subjecting the member to the unit end displacements, $u_1 = 1$ and $u_2 = 1$, as shown in Figs. 8.3(b) and (c), respectively, and evaluating the corresponding member end forces. Thus, the local stiffness matrix for the members of space trusses can be explicitly expressed as

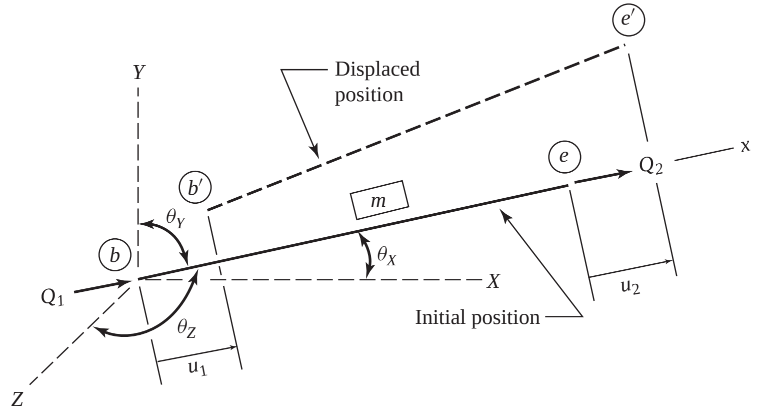
$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{8.3}$$

Coordinate Transformations

Consider an arbitrary member m of a space truss, as shown in Fig. 8.4(a), and let X_b, Y_b, Z_b, and X_e, Y_e, Z_e be the global coordinates of the joints to which the member ends b and e, respectively, are attached. The length and the direction

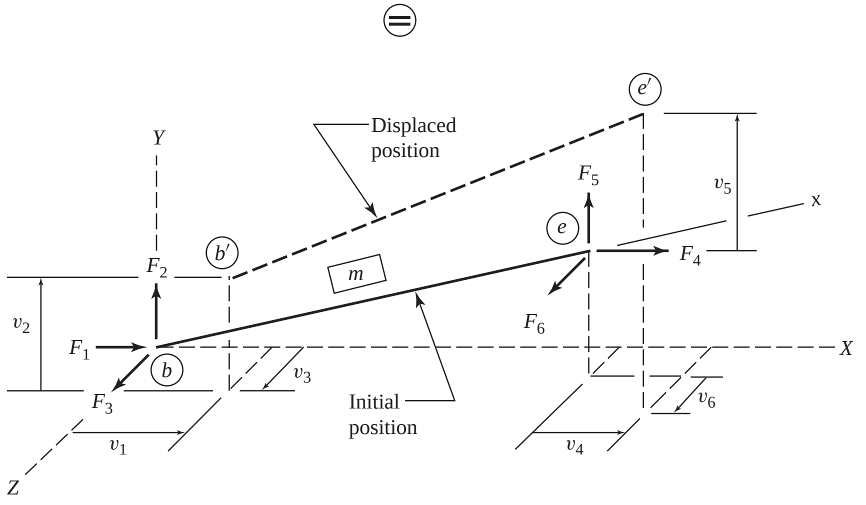


(a) Space Truss



(b) Member End Forces and End Displacements in the Local Coordinate System

Fig. 8.4



(c) Member End Forces and End Displacements
in the Global Coordinate System

Fig. 8.4 (*continued*)

cosines of the member can be expressed in terms of the global coordinates of its ends by the following relationships:

$$L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2 + (Z_e - Z_b)^2} \tag{8.4a}$$

$$\cos\theta_X = \frac{X_e - X_b}{L} \tag{8.4b}$$

$$\cos\theta_Y = \frac{Y_e - Y_b}{L} \tag{8.4c}$$

$$\cos\theta_Z = \frac{Z_e - Z_b}{L} \tag{8.4d}$$

in which θ_X, θ_Y, and θ_Z represent the angles between the positive directions of the global *X*, *Y*, and *Z* axes, respectively, and the positive direction of the member's local *x* axis, as shown in Fig. 8.4(b). Note that the origin of the global coordinate system is shown to coincide with that of the local coordinate system in this figure. With no loss in generality of the formulation, this convenient arrangement allows the angles between the local and global axes to be clearly visualized. It is important to realize that the member transformation matrix depends only on the angles between the local and global axes, regardless of whether or not the origins of the local and global coordinate systems coincide. Also shown in Fig. 8.4(b) are the member end displacements **u** and end forces **Q** in the local coordinate system; the equivalent systems of end displacements **v** and end forces **F**, in the global coordinate system, are depicted in Fig. 8.4(c). As indicated in Fig. 8.4(c), the global member end displacements **v** and end forces **F** are numbered by beginning at member end *b*, with the translation and force in the *X* direction numbered first, followed by the translation and force in the Y direction, and then the translation and force in the Z direction. The displacements and forces at the member's opposite end e are then numbered in the same sequential order.

Let us consider the transformation of member end forces and end displacements from a global to a local coordinate system. By comparing Figs. 8.4(b) and (c), we observe that at end b of the member, the local force Q_1 must be equal to the algebraic sum of the components of the global forces F_1, F_2, and F_3 in the direction of the local x axis; that is,

$$Q_1 = F_1 \cos\theta_X + F_2 \cos\theta_Y + F_3 \cos\theta_Z \tag{8.5a}$$

Similarly, at end e of the member, we can express Q_2 in terms of F_4, F_5, and F_6 as

$$Q_2 = F_4 \cos\theta_X + F_5 \cos\theta_Y + F_6 \cos\theta_Z \tag{8.5b}$$

Equations 8.5(a) and (b) can be written in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_X & \cos\theta_Y & \cos\theta_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_X & \cos\theta_Y & \cos\theta_Z \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} \tag{8.6}$$

Equation (8.6) can be symbolically expressed as $\mathbf{Q} = \mathbf{TF}$, with the 2×6 transformation matrix $\mathbf{T}$ given by

$$\mathbf{T} = \begin{bmatrix} \cos\theta_X & \cos\theta_Y & \cos\theta_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_X & \cos\theta_Y & \cos\theta_Z \end{bmatrix} \tag{8.7}$$

Since member end displacements, like end forces, are vectors, which are defined in the same directions as the corresponding forces, the foregoing transformation matrix $\mathbf{T}$ can also be used to transform member end displacements from the global to the local coordinate system; that is, $\mathbf{u} = \mathbf{Tv}$.

Next, we examine the transformation of member end forces from the local to the global coordinate system. A comparison of Figs. 8.4(b) and (c) indicates that at end b of the member, the global forces F_1, F_2, and F_3 must be the components of the local force Q_1 in the directions of the global X, Y, and Z axes, respectively; that is,

$$F_1 = Q_1 \cos\theta_X \qquad F_2 = Q_1 \cos\theta_Y \qquad F_3 = Q_1 \cos\theta_Z \tag{8.8a}$$

Similarly, at end e of the member, the global forces F_4, F_5, and F_6 can be expressed as the components of the local force Q_2, as

$$F_4 = Q_2 \cos\theta_X \qquad F_5 = Q_2 \cos\theta_Y \qquad F_6 = Q_2 \cos\theta_Z \tag{8.8b}$$

We can write Eqs. 8.8(a) and (b) in matrix form as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} \cos\theta_X & 0 \\ \cos\theta_Y & 0 \\ \cos\theta_Z & 0 \\ 0 & \cos\theta_X \\ 0 & \cos\theta_Y \\ 0 & \cos\theta_Z \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{8.9}$$

As the first matrix on the right side of Eq. (8.9) is the transpose of the transformation matrix **T** (Eq. (8.7)), the equation can be symbolically expressed as

$$\underset{6\times 1}{\mathbf{F}} = \underset{6\times 2}{\mathbf{T}^T}\ \underset{2\times 1}{\mathbf{Q}} \tag{8.10}$$

It may be of interest to note that the transformation relationship analogous to Eq. (8.10) for member end displacements (i.e., $\mathbf{v} = \mathbf{T}^T\mathbf{u}$) is not defined for space truss members, with two degrees of freedom, as used herein. This is because the local end displacement vectors **u** for such members do not contain the displacements of the member ends in the local *y* and *z* directions. As discussed previously, while the end forces in the local *y* and *z* directions of the members of space trusses are always 0, the displacements of the member ends in the local *y* and *z* directions are generally nonzero (see Fig. 8.3(a)). However, the foregoing limitation of the two-degree-of-freedom member model has no practical consequences, because the transformation relation $\mathbf{v} = \mathbf{T}^T\mathbf{u}$ is needed neither in the formulation of the matrix stiffness method of analysis, nor in its application.

Member Stiffness Relations in the Global Coordinate System

As in the case of plane trusses, the relationship between the global end forces **F** and the end displacements **v** for the members of space trusses is expressed as $\mathbf{F} = \mathbf{K}\mathbf{v}$, with the member global stiffness matrix **K** given by the equation

$$\underset{6\times 6}{\mathbf{K}} = \underset{6\times 2}{\mathbf{T}^T}\ \underset{2\times 2}{\mathbf{k}}\ \underset{2\times 6}{\mathbf{T}} \tag{8.11}$$

The explicit form of the 6 × 6 **K** matrix can be determined by substituting Eqs. (8.3) and (8.7) into Eq. (8.11) and performing the required matrix multiplications. The explicit form of the member global stiffness matrix **K**, thus obtained, is given in Eq. (8.12).

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta_X & \cos\theta_X\cos\theta_Y & \cos\theta_X\cos\theta_Z & -\cos^2\theta_X & -\cos\theta_X\cos\theta_Y & -\cos\theta_X\cos\theta_Z \\ \cos\theta_X\cos\theta_Y & \cos^2\theta_Y & \cos\theta_Y\cos\theta_Z & -\cos\theta_X\cos\theta_Y & -\cos^2\theta_Y & -\cos\theta_Y\cos\theta_Z \\ \cos\theta_X\cos\theta_Z & \cos\theta_Y\cos\theta_Z & \cos^2\theta_Z & -\cos\theta_X\cos\theta_Z & -\cos\theta_Y\cos\theta_Z & -\cos^2\theta_Z \\ -\cos^2\theta_X & -\cos\theta_X\cos\theta_Y & -\cos\theta_X\cos\theta_Z & \cos^2\theta_X & \cos\theta_X\cos\theta_Y & \cos\theta_X\cos\theta_Z \\ -\cos\theta_X\cos\theta_Y & -\cos^2\theta_Y & -\cos\theta_Y\cos\theta_Z & \cos\theta_X\cos\theta_Y & \cos^2\theta_Y & \cos\theta_Y\cos\theta_Z \\ -\cos\theta_X\cos\theta_Z & -\cos\theta_Y\cos\theta_Z & -\cos^2\theta_Z & \cos\theta_X\cos\theta_Z & \cos\theta_Y\cos\theta_Z & \cos^2\theta_Z \end{bmatrix} \tag{8.12}$$

Procedure for Analysis

The procedure for the analysis of plane trusses developed in Chapter 3 (see block diagram in Fig. 3.20), and modified in Chapter 7, can be used to analyze space trusses provided that: (a) three structure coordinates (i.e., degrees of freedom and/or restrained coordinates), in the global *X*, *Y*, and *Z* directions, are defined at each joint; and (b) the member stiffness and transformation matrices developed in this section (Eqs. (8.3), (8.7), and (8.12)) are used in the analysis. The procedure is illustrated by the following example.

EXAMPLE 8.1 Determine the joint displacements, member axial forces, and support reactions for the space truss shown in Fig. 8.5(a) by the matrix stiffness method.

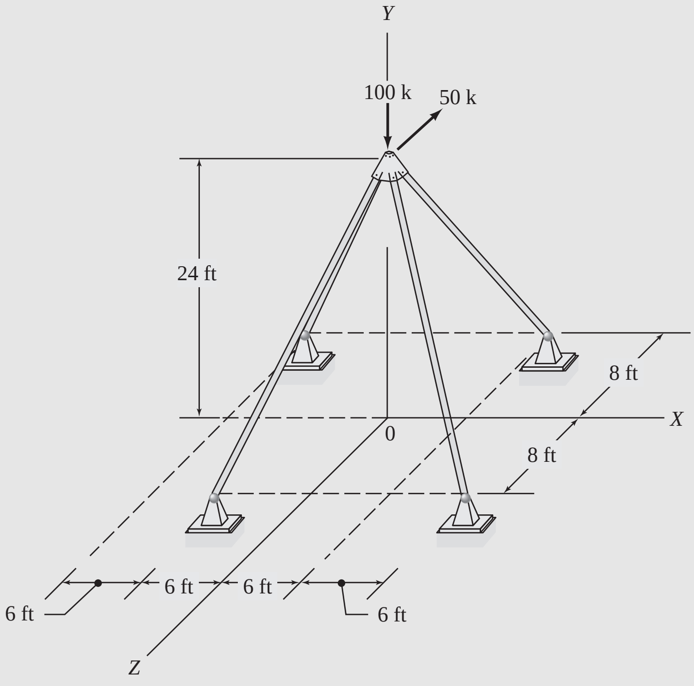
SOLUTION *Analytical Model:* See Fig. 8.5(b). The truss has three degrees of freedom, which are the translations of joint 5 in the *X*, *Y*, and *Z* directions. These are numbered 1, 2, and 3, respectively. The twelve restrained coordinates of the truss are identified by numbers 4 through 15 in the figure.

Structure Stiffness Matrix:

Member 1 From Fig. 8.5(b), we can see that joint 1 is the beginning joint, and joint 5 is the end joint, for this member. By applying Eqs. (8.4), we determine

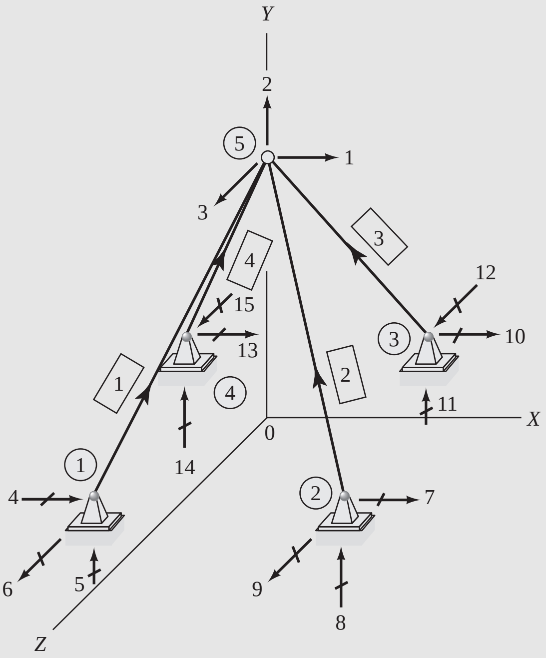
$$L = \sqrt{(X_5 - X_1)^2 + (Y_5 - Y_1)^2 + (Z_5 - Z_1)^2}$$

$$= \sqrt{(0+6)^2 + (24-0)^2 + (0-8)^2} = 26 \text{ ft} = 312 \text{ in.}$$



EA = constant
E = 10,000 ksi
$A = 8.4 \text{ in.}^2$
(a) Space Truss

Fig. 8.5



(b) Analytical Model

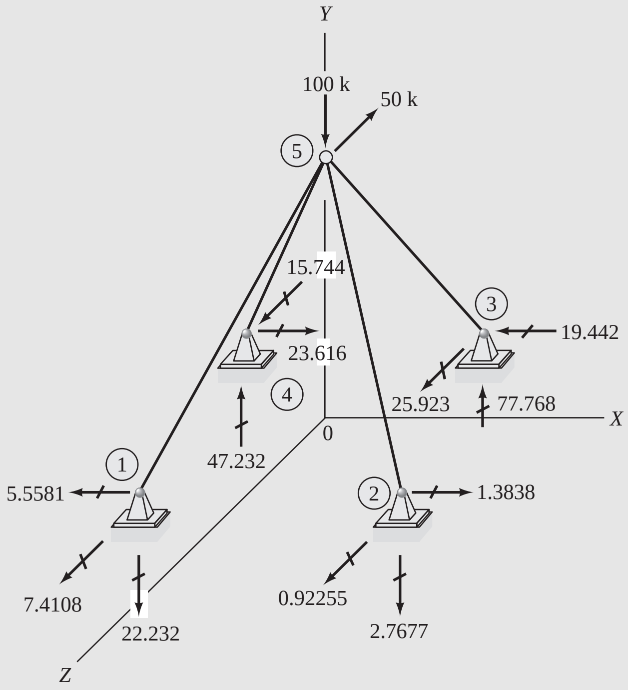
$$\mathbf{S} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left[\begin{array}{ccc} \begin{array}{c}(14.338 + 45.918 \\ +14.338 + 45.918)\end{array} & \begin{array}{c}(57.351 - 91.837 \\ -57.351 + 91.837)\end{array} & \begin{array}{c}(-19.117 + 30.612 \\ -19.117 + 30.612)\end{array} \\ \begin{array}{c}(57.351 - 91.837 \\ -57.351 + 91.837)\end{array} & \begin{array}{c}(229.4 + 183.67 \\ +229.4 + 183.67)\end{array} & \begin{array}{c}(-76.468 - 61.224 \\ +76.468 + 61.224)\end{array} \\ \begin{array}{c}(-19.117 + 30.612 \\ -19.117 + 30.612)\end{array} & \begin{array}{c}(-76.468 - 61.224 \\ +76.468 + 61.224)\end{array} & \begin{array}{c}(25.489 + 20.408 \\ +25.489 + 20.408)\end{array} \end{array}\right] & \begin{array}{c} 1 \\ \\ 2 \\ \\ 3 \end{array} \end{array} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left[\begin{array}{ccc} 120.51 & 0 & 22.99 \\ 0 & 826.14 & 0 \\ 22.99 & 0 & 91.794 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \text{ k/in.}$$

(c) Structure Stiffness Matrix

$$\mathbf{R} = \left[\begin{array}{c} -5.5581 \\ -22.232 \\ 7.4108 \\ 1.3838 \\ -2.7677 \\ 0.92255 \\ -19.442 \\ 77.768 \\ 25.923 \\ 23.616 \\ 47.232 \\ 15.744 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{array} \text{ k}$$

(d) Support Reaction Vector

Fig. 8.5 (*continued*)



(e) Support Reactions

Fig. 8.5 (*continued*)

$$\cos\theta_X = \frac{X_5 - X_1}{L} = \frac{0+6}{26} = 0.23077$$

$$\cos\theta_Y = \frac{Y_5 - Y_1}{L} = \frac{24-0}{26} = 0.92308$$

$$\cos\theta_Z = \frac{Z_5 - Z_1}{L} = \frac{0-8}{26} = -0.30769$$

By substituting $E = 10{,}000$ ksi, $A = 8.4$ in.^{2}, $L = 312$ in., and the foregoing direction cosines, into Eq. (8.12), we calculate the member's global stiffness matrix to be

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc:ccc} 14.338 & 57.351 & -19.117 & -14.338 & -57.351 & 19.117 \\ 57.351 & 229.4 & -76.468 & -57.351 & -229.4 & 76.468 \\ -19.117 & -76.468 & 25.489 & 19.117 & 76.468 & -25.489 \\ \hdashline -14.338 & -57.351 & 19.117 & 14.338 & 57.351 & -19.117 \\ -57.351 & -229.4 & 76.468 & 57.351 & 229.4 & -76.468 \\ 19.117 & 76.468 & -25.489 & -19.117 & -76.468 & 25.489 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \text{k/in.}$$

Next, by using the member code numbers 4, 5, 6, 1, 2, 3, we store the pertinent elements of $\mathbf{K}_1$ in the 3×3 structure stiffness matrix $\mathbf{S}$ in Fig. 8.5(c).

Member 2

$$L = \sqrt{(X_5 - X_2)^2 + (Y_5 - Y_2)^2 + (Z_5 - Z_2)^2}$$

$$= \sqrt{(0 - 12)^2 + (24 - 0)^2 + (0 - 8)^2} = 28 \text{ ft} = 336 \text{ in.}$$

$$\cos\theta_X = \frac{X_5 - X_2}{L} = \frac{0 - 12}{28} = -0.42857$$

$$\cos\theta_Y = \frac{Y_5 - Y_2}{L} = \frac{24 - 0}{28} = 0.85714$$

$$\cos\theta_Z = \frac{Z_5 - Z_2}{L} = \frac{0 - 8}{28} = -0.28571$$

$$\mathbf{K}_2 = \begin{array}{rrrrrrl} 7 & 8 & 9 & 1 & 2 & 3 & \\ 45.918 & -91.837 & 30.612 & -45.918 & 91.837 & -30.612 & 7 \\ -91.837 & 183.67 & -61.224 & 91.837 & -183.67 & 61.224 & 8 \\ 30.612 & -61.224 & 20.408 & -30.612 & 61.224 & -20.408 & 9 \\ -45.918 & 91.837 & -30.612 & 45.918 & -91.837 & 30.612 & 1 \\ 91.837 & -183.67 & 61.224 & -91.837 & 183.67 & -61.224 & 2 \\ -30.612 & 61.224 & -20.408 & 30.612 & -61.224 & 20.408 & 3 \end{array} \text{ k/in.}$$

Member 3

$$L = \sqrt{(X_5 - X_3)^2 + (Y_5 - Y_3)^2 + (Z_5 - Z_3)^2}$$

$$= \sqrt{(0 - 6)^2 + (24 - 0)^2 + (0 + 8)^2} = 26 \text{ ft} = 312 \text{ in.}$$

$$\cos\theta_X = \frac{X_5 - X_3}{L} = \frac{0 - 6}{26} = -0.23077$$

$$\cos\theta_Y = \frac{Y_5 - Y_3}{L} = \frac{24 - 0}{26} = 0.92308$$

$$\cos\theta_Z = \frac{Z_5 - Z_3}{L} = \frac{0 + 8}{26} = 0.30769$$

$$\mathbf{K}_3 = \begin{array}{rrrrrrl} 10 & 11 & 12 & 1 & 2 & 3 & \\ 14.338 & -57.351 & -19.117 & -14.338 & 57.351 & 19.117 & 10 \\ -57.351 & 229.4 & 76.468 & 57.351 & -229.4 & -76.468 & 11 \\ -19.117 & 76.468 & 25.489 & 19.117 & -76.468 & -25.489 & 12 \\ -14.338 & 57.351 & 19.117 & 14.338 & -57.351 & -19.117 & 1 \\ 57.351 & -229.4 & -76.468 & -57.351 & 229.4 & 76.468 & 2 \\ 19.117 & -76.468 & -25.489 & -19.117 & 76.468 & 25.489 & 3 \end{array} \text{ k/in.}$$

Member 4

$$L = \sqrt{(X_5 - X_4)^2 + (Y_5 - Y_4)^2 + (Z_5 - Z_4)^2}$$

$$= \sqrt{(0 + 12)^2 + (24 - 0)^2 + (0 + 8)^2} = 28 \text{ ft} = 336 \text{ in.}$$

$$\cos\theta_X = \frac{X_5 - X_4}{L} = \frac{0+12}{28} = 0.42857$$

$$\cos\theta_Y = \frac{Y_5 - Y_4}{L} = \frac{24-0}{28} = 0.85714$$

$$\cos\theta_Z = \frac{Z_5 - Z_4}{L} = \frac{0+8}{28} = 0.28571$$

$$\mathbf{K}_4 = \begin{array}{c} \begin{array}{cccccc} 13 & 14 & 15 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc:ccc} 45.918 & 91.837 & 30.612 & -45.918 & -91.837 & -30.612 \\ 91.837 & 183.67 & 61.224 & -91.837 & -183.67 & -61.224 \\ 30.612 & 61.224 & 20.408 & -30.612 & -61.224 & -20.408 \\ \hdashline -45.918 & -91.837 & -30.612 & 45.918 & 91.837 & 30.612 \\ -91.837 & -183.67 & -61.224 & 91.837 & 183.67 & 61.224 \\ -30.612 & -61.224 & -20.408 & 30.612 & 61.224 & 20.408 \end{array}\right] \end{array} \begin{array}{c} \\ 13 \\ 14 \\ 15 \\ 1 \\ 2 \\ 3 \end{array} \text{ k/in.}$$

The complete structure stiffness matrix **S**, obtained by assembling the pertinent stiffness coefficients of the four members of the truss, is given in Fig. 8.5(c).

Joint Load Vector: By comparing Fig. 8.5(a) and (b), we obtain

$$\mathbf{P} = \begin{bmatrix} 0 \\ -100 \\ -50 \end{bmatrix} \text{ k}$$

Joint Displacements: By substituting **P** and **S** into the structure stiffness relationship, **P** = **Sd**, we write

$$\begin{bmatrix} 0 \\ -100 \\ -50 \end{bmatrix} = \begin{bmatrix} 120.51 & 0 & 22.99 \\ 0 & 826.14 & 0 \\ 22.99 & 0 & 91.794 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

By solving the foregoing equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} 0.10913 \\ -0.12104 \\ -0.57202 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ in.}$$

Ans

Member End Displacements and End Forces:

Member 1 Using its code numbers, we determine the member's global end displacements to be

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.10913 \\ -0.12104 \\ -0.57202 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \text{ in.}$$

To determine the member's end displacements in the local coordinate system, we first evaluate its transformation matrix as defined in Eq. (8.7):

$$\mathbf{T}_1 = \begin{bmatrix} 0.23077 & 0.92308 & -0.30769 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.23077 & 0.92308 & -0.30769 \end{bmatrix}$$

The member local end displacements can now be calculated by using the relationship $\mathbf{u} = \mathbf{T}\mathbf{v}$, as

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0.089459 \end{bmatrix} \text{in.}$$

Before we can evaluate the member local end forces, we need to determine the local stiffness matrix $\mathbf{k}$, using Eq. (8.3):

$$\mathbf{k}_1 = \begin{bmatrix} 269.23 & -269.23 \\ -269.23 & 269.23 \end{bmatrix} \text{k/in.}$$

Now, we can compute the member local end forces by using the relationship $\mathbf{Q} = \mathbf{k}\mathbf{u}$, as

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 = \begin{bmatrix} -24.085 \\ 24.085 \end{bmatrix} \text{k}$$

in which the negative sign of the first element of $\mathbf{Q}_1$ indicates that the member axial force is tensile; that is,

$Q_{a1} = 24.085$ k (T) **Ans**

By applying the relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$, we determine the member end forces in the global coordinate system to be

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{array}{c} \left[\begin{array}{c} -5.5581 \\ -22.232 \\ 7.4108 \\ \hdashline 5.5581 \\ 22.232 \\ -7.4108 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \text{k}$$

Using the member code numbers 4, 5, 6, 1, 2, 3, the pertinent elements of $\mathbf{F}_1$ are stored in their proper positions in the support reaction vector $\mathbf{R}$, as shown in Fig. 8.5(d).

Member 2

$$\mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0.10913 \\ -0.12104 \\ -0.57202 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array} \text{in.}$$

$$\mathbf{T}_2 = \begin{bmatrix} -0.42857 & 0.85714 & -0.28571 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.42857 & 0.85714 & -0.28571 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{T}_2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0.012916 \end{bmatrix} \text{in.}$$

$$\mathbf{k}_2 = \begin{bmatrix} 250 & -250 \\ -250 & 250 \end{bmatrix} \text{k/in.}$$

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 = \begin{bmatrix} -3.2289 \\ 3.2289 \end{bmatrix} \text{k}$$

$Q_{a2} = 3.2289$ k (T) **Ans**

$$\mathbf{F}_2 = \mathbf{T}_2^T \mathbf{Q}_2 = \left[\begin{array}{r} 1.3838 \\ -2.7677 \\ 0.92255 \\ \hline -1.3838 \\ 2.7677 \\ -0.92255 \end{array}\right]\begin{array}{l} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array} \text{ k}$$

Member 3

$$\mathbf{v}_3 = \left[\begin{array}{r} 0 \\ 0 \\ 0 \\ 0.10913 \\ -0.12104 \\ -0.57202 \end{array}\right]\begin{array}{l} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} \text{ in.}$$

$$\mathbf{T}_3 = \begin{bmatrix} -0.23077 & 0.92308 & 0.30769 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.23077 & 0.92308 & 0.30769 \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{T}_3\mathbf{v}_3 = \begin{bmatrix} 0 \\ -0.31292 \end{bmatrix} \text{ in.}$$

$$\mathbf{k}_3 = \mathbf{k}_1$$

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 = \begin{bmatrix} 84.248 \\ -84.248 \end{bmatrix} \text{ k}$$

$Q_{a3} = 84.248$ k (C) **Ans**

$$\mathbf{F}_3 = \mathbf{T}_3^T \mathbf{Q}_3 = \left[\begin{array}{r} -19.442 \\ 77.768 \\ 25.923 \\ \hline 19.442 \\ -77.768 \\ -25.923 \end{array}\right]\begin{array}{l} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} \text{ k}$$

Member 4

$$\mathbf{v}_4 = \left[\begin{array}{r} 0 \\ 0 \\ 0 \\ 0.10913 \\ -0.12104 \\ -0.57202 \end{array}\right]\begin{array}{l} 13 \\ 14 \\ 15 \\ 1 \\ 2 \\ 3 \end{array} \text{ in.}$$

$$\mathbf{T}_4 = \begin{bmatrix} 0.42857 & 0.85714 & 0.28571 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.42857 & 0.85714 & 0.28571 \end{bmatrix}$$

$$\mathbf{u}_4 = \mathbf{T}_4\mathbf{v}_4 = \begin{bmatrix} 0 \\ -0.22042 \end{bmatrix} \text{ in.}$$

$$\mathbf{k}_4 = \mathbf{k}_2$$

$$\mathbf{Q}_4 = \mathbf{k}_4\mathbf{u}_4 = \begin{bmatrix} 55.104 \\ -55.104 \end{bmatrix} \text{k}$$

$$Q_{a4} = 55.104 \text{ k (C)}$$ **Ans**

$$\mathbf{F}_4 = \mathbf{T}_4^T\mathbf{Q}_4 = \left[\begin{array}{r} 23.616 \\ 47.232 \\ 15.744 \\ \hdashline -23.616 \\ -47.232 \\ -15.744 \end{array}\right]\begin{array}{c} 13 \\ 14 \\ 15 \\ 1 \\ 2 \\ 3 \end{array} \text{k}$$

Support Reactions: The completed reaction vector **R** is shown in Fig. 8.5(d), and the support reactions are depicted on a line diagram of the truss in Fig. 8.5(e). **Ans**

Equilibrium Check: Applying the equations of equilibrium to the free body of the entire space truss (Fig. 8.5(e)), we obtain

$$+ \rightarrow \sum F_X = 0 \qquad -5.5581 + 1.3838 - 19.442 + 23.616 \approx 0$$ **Checks**

$$+ \uparrow \sum F_Y = 0 \qquad -22.232 - 2.7677 + 77.768 + 47.232 - 100 \approx 0$$ **Checks**

$$+ \swarrow \sum F_Z = 0 \qquad 7.4108 + 0.92255 + 25.923 + 15.744 - 50 \approx 0$$ **Checks**

$$+\circlearrowleft \sum M_X = 0 \qquad 22.232(8) + 2.7677(8) + 77.768(8) + 47.232(8) - 50(24) \approx 0$$ **Checks**

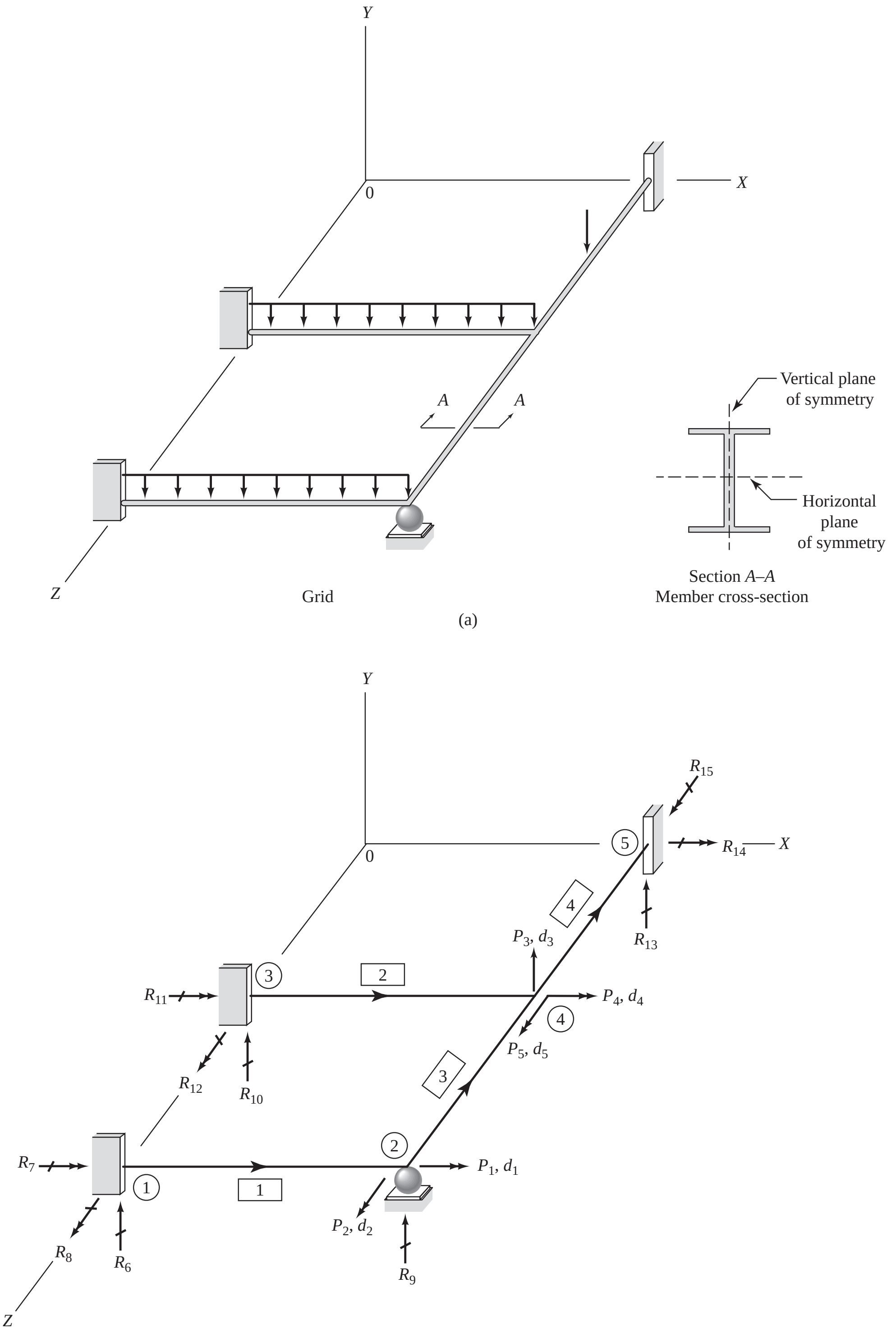
$$+\circlearrowleft \sum M_Y = 0 \qquad -5.5581(8) + 7.4108(6) + 1.3838(8) - 0.92255(12) + 19.442(8) - 25.923(6) - 23.616(8) + 15.744(12) \approx 0$$ **Checks**

$$+\circlearrowleft \sum M_Z = 0 \qquad 22.232(6) - 2.7677(12) + 77.768(6) - 47.232(12) \approx 0$$ **Checks**

8.2 GRIDS

A *grid* is defined as *a two-dimensional framework of straight members connected together by rigid and/or flexible connections, and subjected to loads and reactions perpendicular to the plane of the structure.* Because of their widespread use as supporting structures for long-span roofs and floors, the analysis of grids is usually formulated with the structural framework lying in a horizontal plane (unlike plane frames, which are oriented in a vertical plane), and subjected to external loads acting in the vertical direction, as shown in Fig. 8.6(a) on the next page.

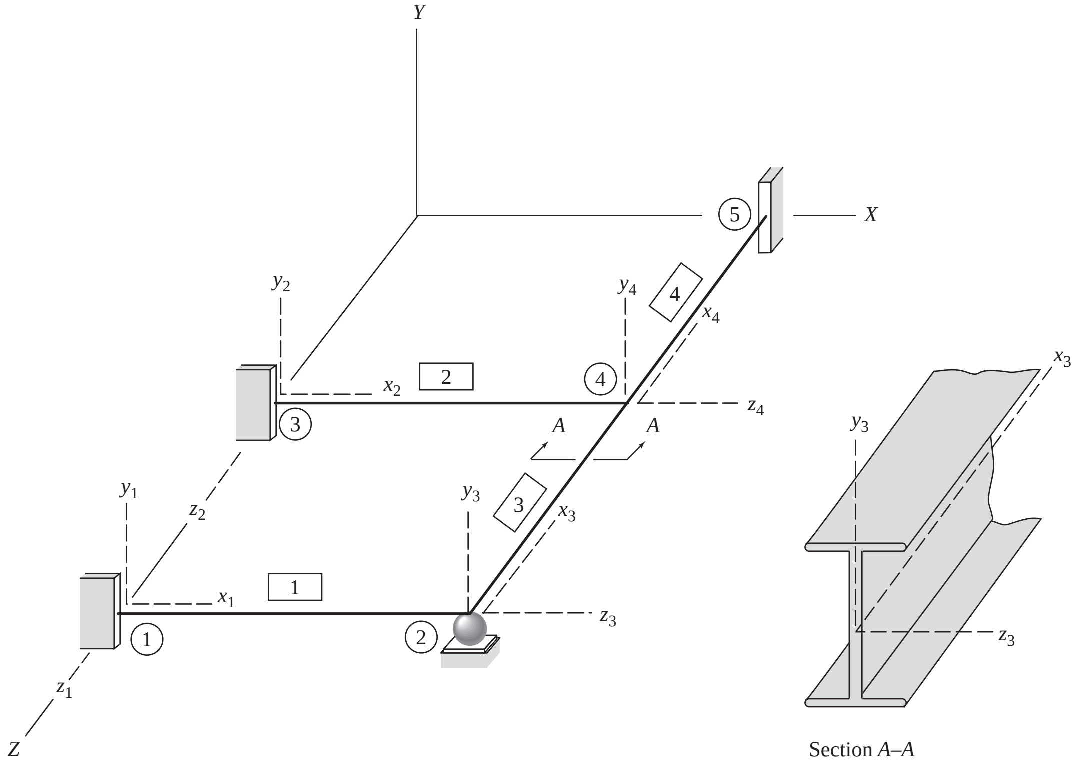
Grids are composed of members that have doubly symmetric cross-sections, with each member oriented so that one of the planes of symmetry of its cross-section is in the vertical direction; that is, perpendicular to the plane of the structure, and in (or parallel to) the direction of the external loads (Fig. 8.6(a)). Under the action of vertical external loads, the joints of a grid can translate in the vertical direction and can rotate about axes in the



(a)

(b) Analytical Model

Fig. 8.6



(c) Member Local Coordinate Systems

Fig. 8.6 (*continued*)

(horizontal) plane of the structure, while the grid members may be subjected to torsion, and uniaxial bending out of the plane of the structure.

Analytical Model and Degrees of Freedom

The process of dividing grids into members and joints, for the purpose of analysis, is the same as that for beams and plane frames—that is, a grid is divided into members and joints so that all of the members are straight and prismatic, and all the external reactions act only at the joints. Consider, for example, the grid of Fig. 8.6(a). The analytical model of the grid, as depicted in Fig. 8.6(b), shows that, for analysis, the grid is considered to be composed of four members and five joints. The overall geometry of the grid, and its joint loads and displacements, are described with reference to a global right-handed *XYZ* coordinate system, with the structure lying in the horizontal *XZ* plane, as shown in Fig. 8.6(b). Two global (*X* and *Z*) coordinates are needed to specify the location of each joint.

For each member of the grid, a local *xyz* coordinate system is established, with its origin at an end of the member and the *x* axis directed along the member's centroidal axis in the undeformed state. The local *y* and *z* axes are oriented, respectively, parallel to the vertical and horizontal axes of symmetry (or the principal axes of inertia) of the member cross-section. The positive direction of the local x axis is defined from the *beginning* toward the *end* of the member; the local y axis is considered positive upward (i.e., in the positive direction of the global Y axis); and the positive direction of the local z axis is defined so that the local xyz coordinate system is right-handed. The local coordinate systems selected for the four members of the example grid are depicted in Fig. 8.6(c).

As discussed previously, an unsupported joint of a grid can translate in the global Y direction and rotate about any axis in the XZ plane. Since small rotations can be treated as vector quantities, the foregoing joint rotation can be conveniently represented by its component rotations about the X and Z axes. Thus, a free joint of a grid has three degrees of freedom—the translation in the Y direction and the rotations about the X and Z axes. Therefore, three structure coordinates (i.e., free and/or restrained coordinates) need to be defined at each joint of the grid for the purpose of analysis; that is,

$$\boxed{\left.\begin{array}{l} NCJT = 3 \\ NDOF = 3(NJ) - NR \end{array}\right\} \quad \text{for grids}} \tag{8.13}$$

The procedure for numbering the structure coordinates of grids is analogous to that for other types of framed structures. The degrees of freedom are numbered before the restrained coordinates. In the case of a joint with multiple degrees of freedom, the translation in the Y direction is numbered first, followed by the rotation about the X axis, and then the rotation about the Z axis. After all the degrees of freedom have been numbered, the grid's restrained coordinates are numbered in the same manner as the degrees of freedom. In Fig. 8.6(b), the degrees of freedom and restrained coordinates of the example grid are numbered using this procedure. It should be noted from this figure that the rotations and moments are now represented by *double-headed arrows* (↠), instead of the curved arrows (↺) used previously for plane structures. The double-headed arrows provide a convenient and unambiguous means of representing rotations and moments in three-dimensional space. To represent a rotation (or a moment/couple), an arrow is drawn pointing in the positive direction of the axis about which the rotation occurs (or the moment/couple acts). The positive sense (i.e., clockwise or counterclockwise) of the rotation (or moment/couple) is indicated by the curved fingers of the right hand with the extended thumb pointing in the direction of the arrowheads, as shown in Fig. 8.7.

Member Stiffness Relations in the Local Coordinate System

When a member with a noncircular (e.g., rectangular or I-shaped) cross-section is subjected to torsion, its initially plane cross-sections become warped surfaces; restraint of this *warping*, or out-of-plane deformation, of cross-sections can induce bending stresses in the member. Thus, in the analysis of grids and space frames (to be developed in the next section), it is commonly

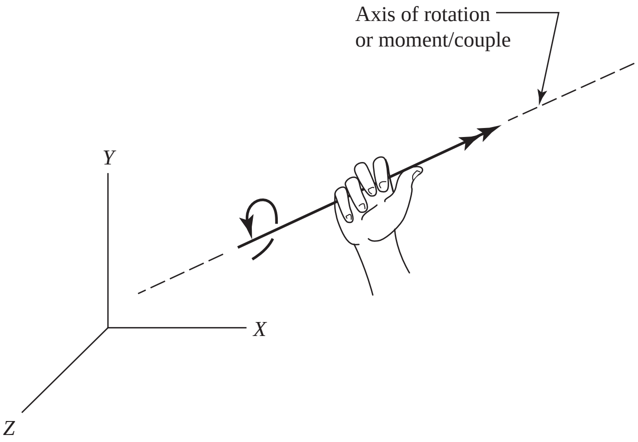


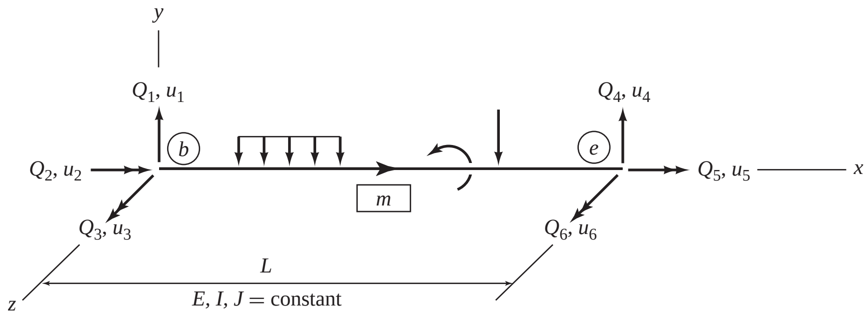
Fig. 8.7 *Representation of Rotation or Moment/Couple in Three-Dimensional Space*

assumed that *the cross sections of all the members are free to warp out of their planes under the action of torsional moments.* This assumption, together with the previously stated condition about the cross-sections of grid members being doubly symmetric with one of the planes of symmetry oriented parallel to the direction of applied loads, has the effect of uncoupling the member's torsional and bending stiffnesses so that a twisting (or torsional deformation) of the member induces only torsional moments but no bending moments, and vice versa. With the torsional and bending effects uncoupled, the local stiffness relations for the members of grids can be obtained by simply extending the stiffness relations for beams (Chapter 5) to include the familiar torsional stiffness relations found in textbooks on *mechanics of materials.*

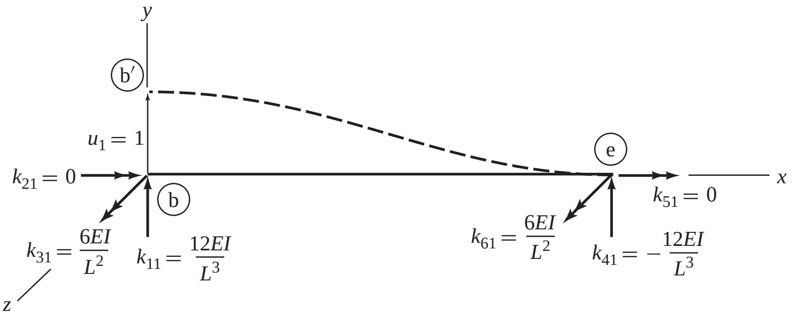
Consider an arbitrary member m of a grid, as shown in Fig. 8.8(a) on the next page. Like a joint of a grid, three displacements are needed to completely specify the displaced position of each end of the grid member. Thus, the member has a total of six degrees of freedom. In the local coordinate system of the member, the six member end displacements are denoted by u_1 through u_6, and the associated member end forces are denoted by Q_1 through Q_6, as shown in Fig. 8.8(a). As indicated in this figure, *a member's local end displacements and end forces are numbered by beginning at its end b, with the translation and the force in the y direction numbered first, followed by the rotation and moment about the x axis, and then the rotation and moment about the z axis. The displacements and forces at the member's opposite end e are then numbered in the same sequential order.*

The relationship between the end forces **Q** and the end displacements **u**, for the members of grids, can be expressed as

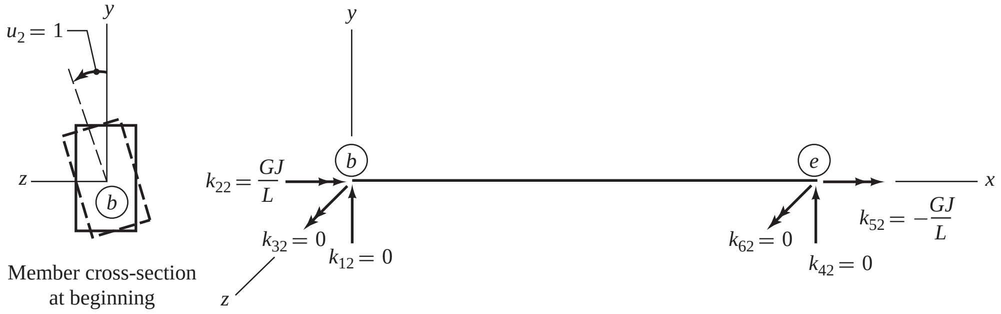
$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f} \tag{8.14}$$



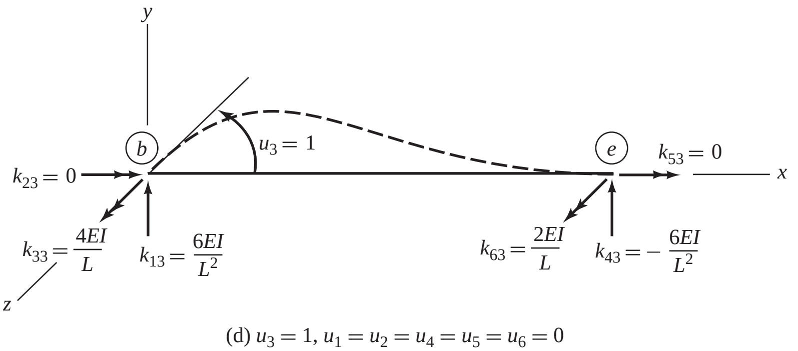
(a) Member Forces and Displacements in the Local Coordinate System



(b) $u_1 = 1, u_2 = u_3 = u_4 = u_5 = u_6 = 0$

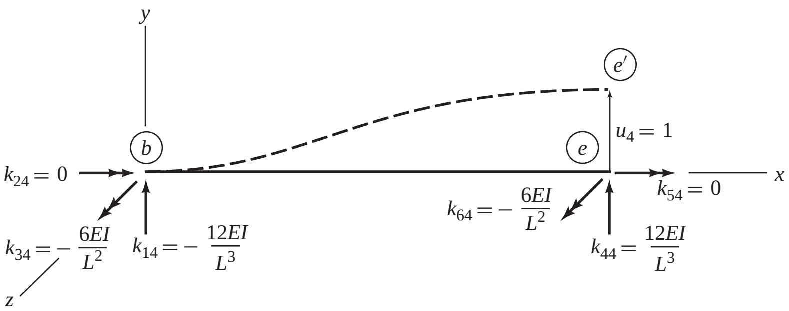


(c) $u_2 = 1, u_1 = u_3 = u_4 = u_5 = u_6 = 0$

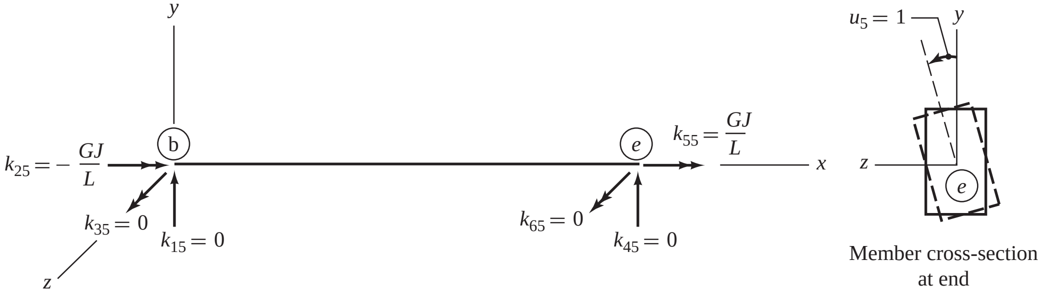


(d) $u_3 = 1, u_1 = u_2 = u_4 = u_5 = u_6 = 0$

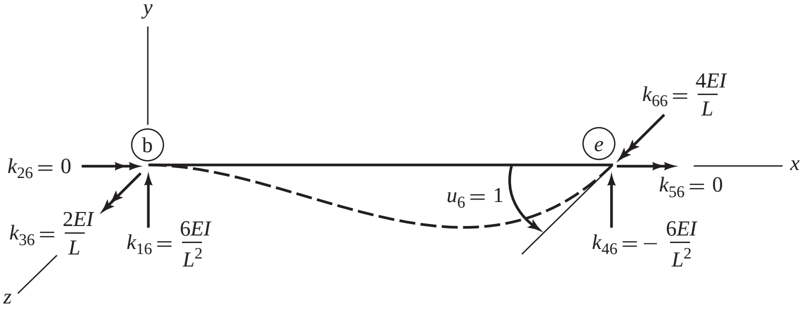
Fig. 8.8



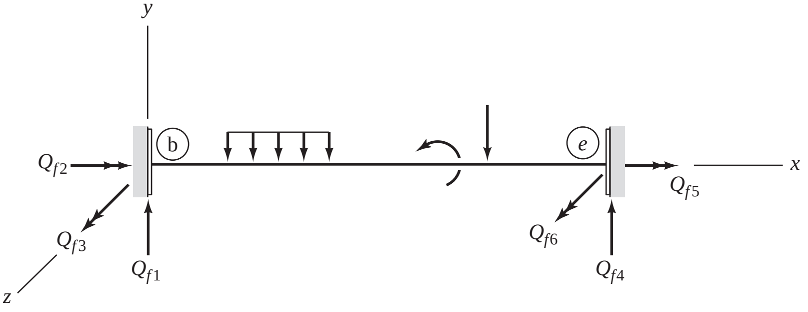
(e) $u_4 = 1, u_1 = u_2 = u_3 = u_5 = u_6 = 0$



(f) $u_5 = 1, u_1 = u_2 = u_3 = u_4 = u_6 = 0$



(g) $u_6 = 1, u_1 = u_2 = u_3 = u_4 = u_5 = 0$



(h) Member Fixed-End Forces in the Local
Coordinate System
$(u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 0)$

Fig. 8.8 (*continued*)

in which **k** represents the 6 × 6 member stiffness matrix in the local coordinate system, and $\mathbf{Q}_f$ denotes the 6 × 1 member local fixed-end force vector.

Like the other types of framed structures, the explicit form of **k** for grid members can be obtained by subjecting a member, separately, to unit values of each of the six end displacements, as shown in Figs. 8.8(b) through (g), and evaluating the corresponding member end forces.

The stiffness coefficients required to cause the unit values of the member end displacements u_1, u_3, u_4, and u_6, are shown in Figs. 8.8(b), (d), (e), and (g), respectively. The expressions for these stiffness coefficients were derived in Section 5.2. To derive the expressions for the torsional stiffness coefficients, recall from a previous course on mechanics of materials that the relationship between a torsional moment (or torque) M_T applied at the free end of a cantilever circular shaft, and the resulting angle of twist ϕ (see Fig. 8.9), can be expressed as

$$\boxed{\phi = \frac{M_T L}{G J}} \qquad \textbf{(8.15)}$$

in which G denotes the shear modulus of the material, and J denotes the polar moment of inertia of the shaft.

For members with noncircular cross sections, the relationship between the torsional moment M_T and the angle of twist ϕ can be quite complicated because of warping [40]. However, if warping is not restrained, then Eq. (8.15) can be used to approximate the torsional behavior of members with noncircular cross-sections—provided that J is now considered to be the *Saint-Venant's torsion constant*, or simply the *torsion constant*, of the member's cross-section, instead of its polar moment of inertia. Although the derivation of the expressions for torsion constant J for various cross-sectional shapes is beyond the scope of this text, such derivations can be found in textbooks on the *theory of elasticity* and *advanced mechanics of materials* [40]. The expressions for J for some common cross-sectional shapes are listed in Table 8.1. Furthermore, the torsion constant for any thin-walled, open cross-section can be approximated

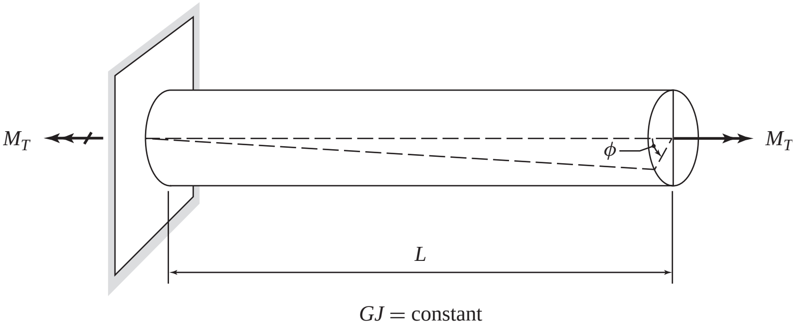
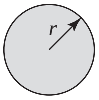
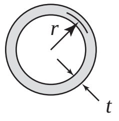
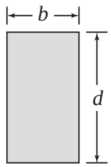
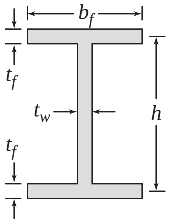
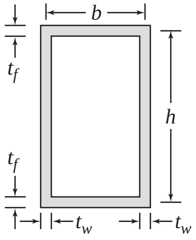


Fig. 8.9 *Circular Shaft Subjected to Torsional Moment*

Table 8.1 *Torsion Constants for Common Member Cross-Sections* [40, 52]

Cross-Section	Torsion Constant
	$J = \frac{1}{2}\pi r^4$
	$J = 2\pi r^3 t$
	$J = \beta b^3 d \quad \text{for} \quad b \leq d$ $\beta = \frac{1}{3} - 0.21\frac{b}{d}\left[1 - \frac{1}{12}\left(\frac{b}{d}\right)^4\right]$
	$J = \frac{1}{3}\left(2b_f t_f^3 + h t_w^3\right)$
	$J = \frac{2b^2h^2}{b/t_f + h/t_w}$

by the relationship

$$J = \frac{1}{3}\sum bt^3 \tag{8.16}$$

in which b and t denote, respectively, the width and thickness of each rectangular segment of the cross section.

Returning our attention to Fig. 8.8(c), we realize that the expression for the stiffness coefficient k_{22} can be obtained by substituting $\phi = u_2 = 1$ and

$M_T = k_{22}$ into Eq. (8.15), and solving the resulting equation for k_{22}. This yields

$$\boxed{k_{22} = \frac{GJ}{L}} \tag{8.17}$$

The other torsional stiffness coefficient k_{52} can now be determined by applying the following equilibrium equation.

$$\overset{+}{\twoheadrightarrow} \sum M_X = 0 \qquad \frac{GJ}{L} + k_{52} = 0$$

$$\boxed{k_{52} = -\frac{GJ}{L}} \tag{8.18}$$

The expressions for coefficients k_{25} and k_{55} (Fig. 8.8(f)) can be obtained in a similar manner. Substitution of $\phi = u_5 = 1$ and $M_T = k_{55}$ into Eq. (8.15) yields

$$\boxed{k_{55} = \frac{GJ}{L}} \tag{8.19}$$

and by considering the equilibrium of the free body of the member, we obtain k_{25} as

$$\overset{+}{\twoheadrightarrow} \sum M_x = 0 \qquad k_{25} + \frac{GJ}{L} = 0$$

$$\boxed{k_{25} = -\frac{GJ}{L}} \tag{8.20}$$

Thus, by arranging all the stiffness coefficients shown in Figs. 8.8(b) through (g) into a matrix, we obtain the following expression for the local stiffness matrix for the members of grids.

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 0 & 6L & -12 & 0 & 6L \\ 0 & \dfrac{GJL^2}{EI} & 0 & 0 & -\dfrac{GJL^2}{EI} & 0 \\ 6L & 0 & 4L^2 & -6L & 0 & 2L^2 \\ -12 & 0 & -6L & 12 & 0 & -6L \\ 0 & -\dfrac{GJL^2}{EI} & 0 & 0 & \dfrac{GJL^2}{EI} & 0 \\ 6L & 0 & 2L^2 & -6L & 0 & 4L^2 \end{bmatrix} \tag{8.21}$$

The local fixed-end force vector for the members of grids is expressed as (Fig. 8.8(h))

$$\mathbf{Q}_f = \begin{bmatrix} FS_b \\ FT_b \\ FM_b \\ FS_e \\ FT_e \\ FM_e \end{bmatrix} \tag{8.22}$$

in which the fixed-end shears (FS_b and FS_e) and bending moments (FM_b and FM_e) can be calculated by using the fixed-end force equations given for loading types 1 through 4 inside the front cover. (The procedure for deriving those fixed-end shear and bending moment equations was discussed in Section 5.4.)

The expressions for the fixed-end torsional moments (FT_b and FT_e), due to an external torque M_T applied to the member, are also given inside the front cover (see loading type 7). To derive these expressions, let us consider a fixed member of a grid, subjected to a torque M_T, as shown in Fig. 8.10(a). If the end e of the member were free to rotate, then its cross-section would twist clockwise as shown in Fig. 8.10(b). Let ϕ be the angle of twist at end e of the released member. As portion Ae of the released member (Fig. 8.10(b)) is not subjected to any torsional moments, the angle of twist, ϕ, at end e equals that at point A, and its magnitude can be obtained by substituting $L = l_1$ into Eq. (8.15); that is,

$$\phi = \frac{M_T l_1}{GJ} \tag{8.23}$$

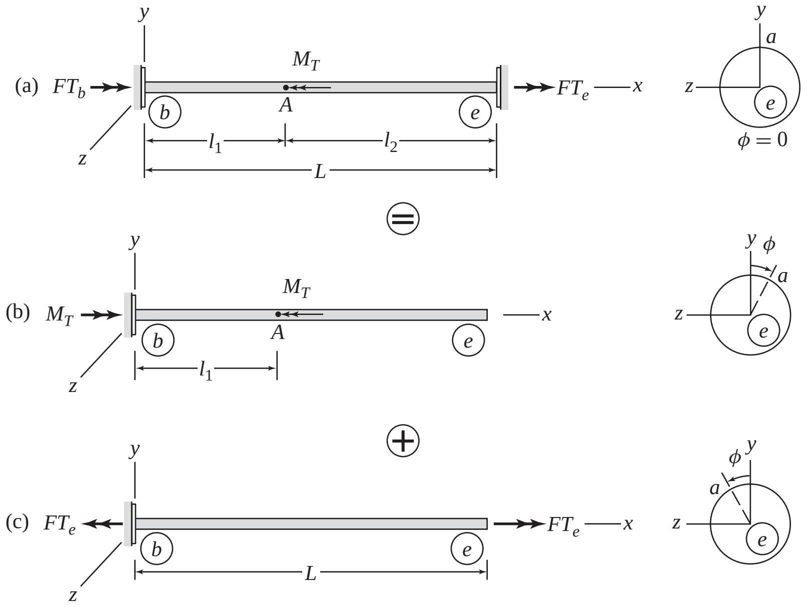


Fig. 8.10

Since the angle of twist at end e of the actual fixed member (Fig. 8.10(a)) is 0, the fixed-end torsional moment FT_e must be of such magnitude that, when applied to the released member as shown in Fig. 8.10(c), it should twist the cross-section at end e by an angle equal in magnitude to the angle ϕ due to the torque M_T, but in the opposite (i.e., counterclockwise) direction. The angle of twist due to FT_e can be obtained by substituting $M_T = FT_e$ into Eq. (8.15); that is,

$$\phi = \frac{FT_e L}{GJ} \tag{8.24}$$

and the relationship between FT_e and the external torque M_T can be established by equating Eqs. (8.23) and (8.24), as

$$\phi = \frac{FT_e L}{GJ} = \frac{M_T l_1}{GJ}$$

from which we obtain the expression for the fixed-end torsional moment FT_e:

$$FT_e = \frac{M_T l_1}{L} \tag{8.25}$$

The expression for the other fixed-end torsional moment, FT_b, can now be determined by applying the equilibrium condition that the algebraic sum of the three torsional moments acting on the fixed member (Fig. 8.10(a)) must be 0; that is,

$$\overset{+}{\longrightarrow\!\!\!\!\to} \sum M_x = 0 \qquad FT_b - M_T + FT_e = 0$$

By substituting Eq. (8.25) into the foregoing equation and rearranging terms, we obtain the expression for FT_b:

$$FT_b = M_T\left(\frac{L - l_1}{L}\right) = \frac{M_T l_2}{L} \tag{8.26}$$

Member Releases The expressions for the local stiffness matrix $\mathbf{k}$ and the fixed-end force vector $\mathbf{Q}_f$, as given in Eqs. (8.21) and (8.22), respectively, are valid only for members of type 0 (i.e., $MT = 0$), which are rigidly connected to joints at both ends. For grid members with moment releases, the foregoing expressions for $\mathbf{k}$ and $\mathbf{Q}_f$ need to be modified using the procedure described in Section 7.1. If the member releases are assumed to be in the form of spherical hinges (or ball-and-socket type of connections), so that both bending and torsional moments are 0 at the released member ends, then the modified local stiffness matrices $\mathbf{k}$ and fixed-end force vectors $\mathbf{Q}_f$ for the grid members with releases can be expressed as follows.

For a member with a hinge at the beginning ($MT = 1$):

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 3 & 0 & 0 & -3 & 0 & 3L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 3 & 0 & -3L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3L & 0 & 0 & -3L & 0 & 3L^2 \end{bmatrix} \tag{8.27}$$

$$\mathbf{Q}_f = \begin{bmatrix} FS_b - \dfrac{3}{2L} FM_b \\ 0 \\ 0 \\ FS_e + \dfrac{3}{2L} FM_b \\ FT_e + FT_b \\ FM_e - \dfrac{1}{2} FM_b \end{bmatrix} \tag{8.28}$$

For a member with a hinge at the end ($MT = 2$):

$$\mathbf{k} = \frac{EI}{L^3} \begin{bmatrix} 3 & 0 & 3L & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3L & 0 & 3L^2 & -3L & 0 & 0 \\ -3 & 0 & -3L & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.29}$$

$$\mathbf{Q}_f = \begin{bmatrix} FS_b - \dfrac{3}{2L} FM_e \\ FT_b + FT_e \\ FM_b - \dfrac{1}{2} FM_e \\ FS_e + \dfrac{3}{2L} FM_e \\ 0 \\ 0 \end{bmatrix} \tag{8.30}$$

For a member with hinges at both ends ($MT = 3$):

$$\mathbf{k} = \mathbf{0} \tag{8.31}$$

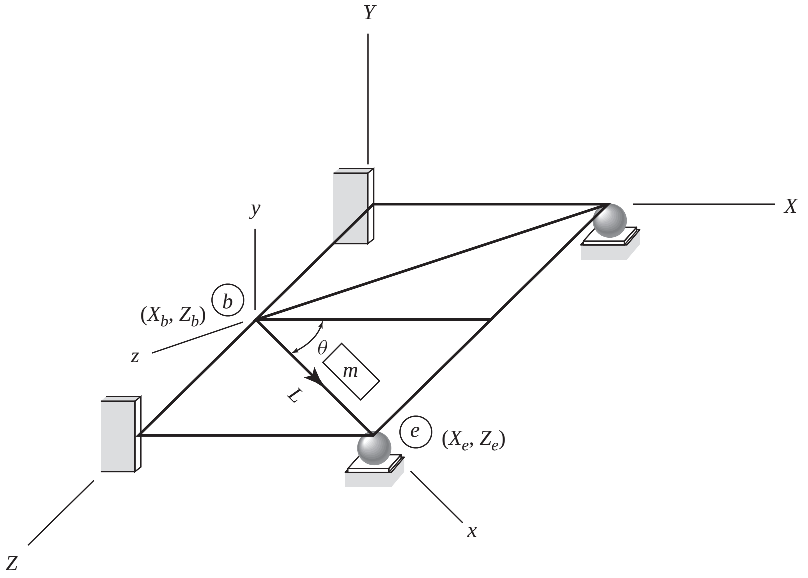
$$\mathbf{Q}_f = \begin{bmatrix} FS_b - \dfrac{1}{L} (FM_b + FM_e) \\ 0 \\ 0 \\ FS_e + \dfrac{1}{L} (FM_b + FM_e) \\ 0 \\ 0 \end{bmatrix} \tag{8.32}$$

Note that the members of type 3 offer no resistance against twisting and, therefore, cannot be subjected to any torques or torsional member loading.

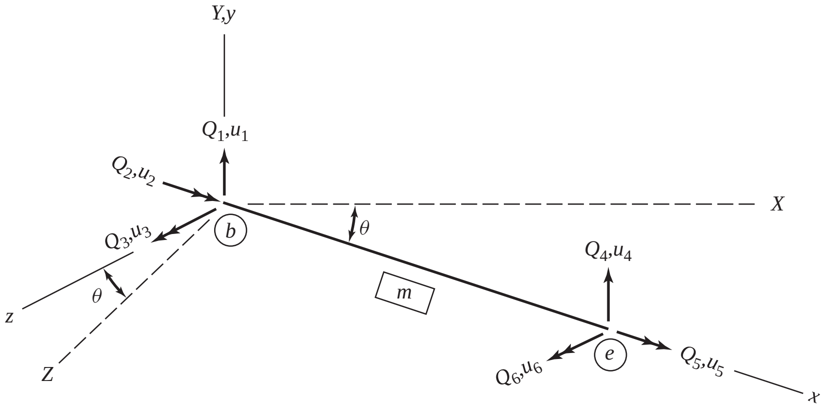
Coordinate Transformations

Consider an arbitrary member *m* of a grid, as shown in Fig. 8.11(a). The orientation of the member in the horizontal (XZ) plane is defined by an angle θ between the positive directions of the global X axis and the member's local x axis, as shown in the figure. The member's length, and its direction cosines, can be expressed in terms of the global coordinates of the member end joints, *b* and *e*, by the following relationships.

$$L = \sqrt{(X_e - X_b)^2 + (Z_e - Z_b)^2} \tag{8.33a}$$

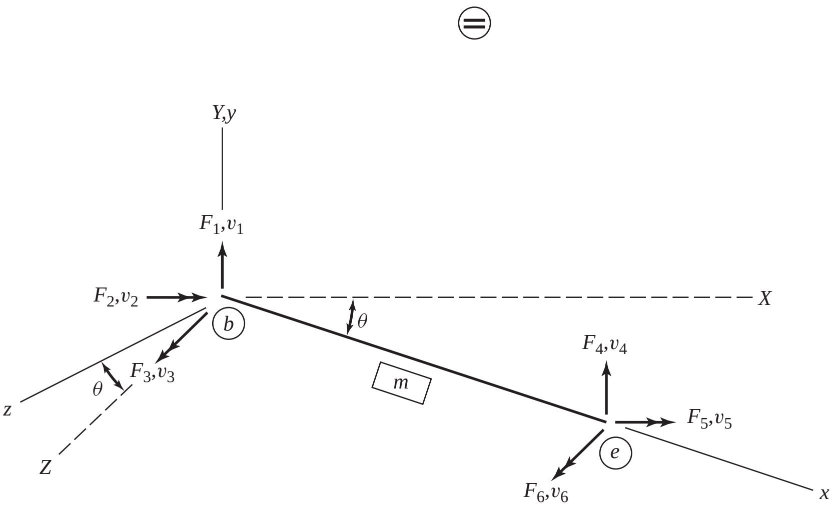


(a) Grid

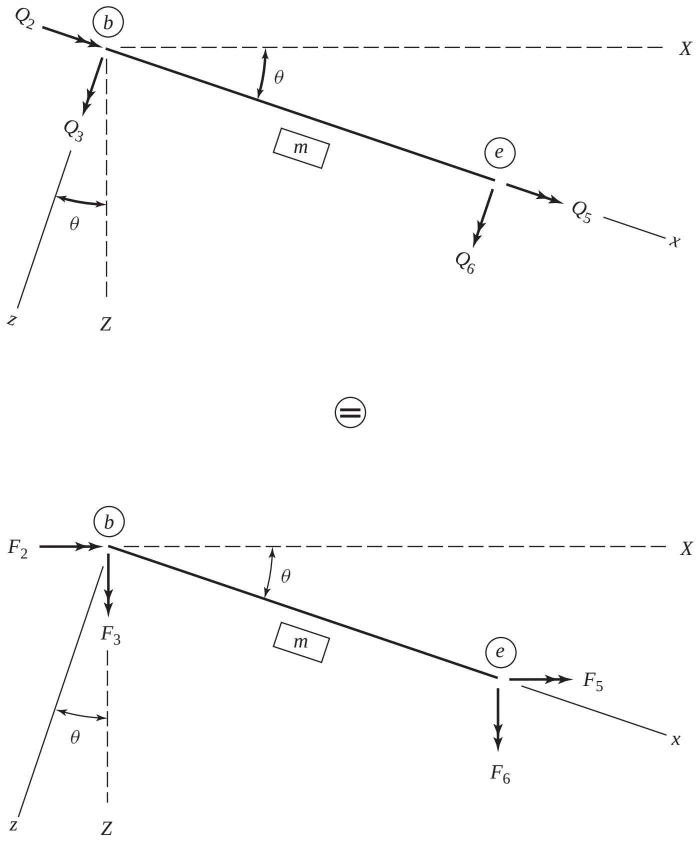


(b) Member End Forces and End Displacements in the Local Coordinate System

Fig. 8.11



(c) Member End Forces and End Displacements
in the Global Coordinate System



(d) Member Local and Global End Forces in the
Horizontal (XZ) or (xz) Plane

Fig. 8.11 (*continued*)

$$\cos\theta = \frac{X_e - X_b}{L} \tag{8.33b}$$

$$\sin\theta = \frac{Z_e - Z_b}{L} \tag{8.33c}$$

The member local end forces **Q** and end displacements **u** are shown in Fig. 8.11(b); Fig. 8.11(c) depicts the equivalent system of end forces **F** and end displacements **v**, in the global coordinate system. As indicated in Fig. 8.11(c), the global member end forces and end displacements are numbered by beginning at member end *b*, with the force and translation in the *Y* direction numbered first, followed by the moment and rotation about the *X* axis, and then the moment and rotation about the *Z* axis. The forces and displacements at the member's opposite end *e* are then numbered in the same sequential order.

By comparing Figs. 8.11(b) and (c), we realize that at member end *b*, the local forces Q_1, Q_2, and Q_3 must be equal to the algebraic sums of the components of the global forces F_1, F_2, and F_3 in the directions of the local *y*, *x*, and *z* axes, respectively; that is (also, see Fig. 8.11(d)),

$$Q_1 = F_1 \tag{8.34a}$$

$$Q_2 = F_2\cos\theta + F_3\sin\theta \tag{8.34b}$$

$$Q_3 = -F_2\sin\theta + F_3\cos\theta \tag{8.34c}$$

Similarly, the local forces at member end *e* can be expressed in terms of the global forces as

$$Q_4 = F_4 \tag{8.34d}$$

$$Q_5 = F_5\cos\theta + F_6\sin\theta \tag{8.34e}$$

$$Q_6 = -F_5\sin\theta + F_6\cos\theta \tag{8.34f}$$

Equations 8.34(a) through (f) can be expressed in matrix form as

$$\mathbf{Q} = \mathbf{TF} \tag{8.35}$$

with the transformation matrix **T** given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{8.36}$$

Because the translations and small rotations of the member ends can be treated as vector quantities, the foregoing transformation matrix also defines the transformation of member end displacements from the global to the local coordinate system; that is, $\mathbf{u} = \mathbf{Tv}$. Furthermore, the transformation matrix **T**, as given in Eq. (8.36), can be used to transform member end forces and displacements from the local to the global coordinate system via the relationships $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ and $\mathbf{v} = \mathbf{T}^T\mathbf{u}$, respectively.

Procedure for Analysis

The procedure for analysis of grids remains the same as that for plane frames developed in Chapter 6 (and modified in Chapter 7); provided, of course, that the member local stiffness and transformation matrices, and local fixed-end force vectors, developed in this section are used in the analysis. The procedure is illustrated by the following example.

EXAMPLE 8.2

Determine the joint displacements, member end forces, and support reactions for the three-member grid shown in Fig. 8.12(a), using the matrix stiffness method.

SOLUTION *Analytical Model:* The grid has three degrees of freedom and nine restrained coordinates, as shown in Fig. 8.12(b).

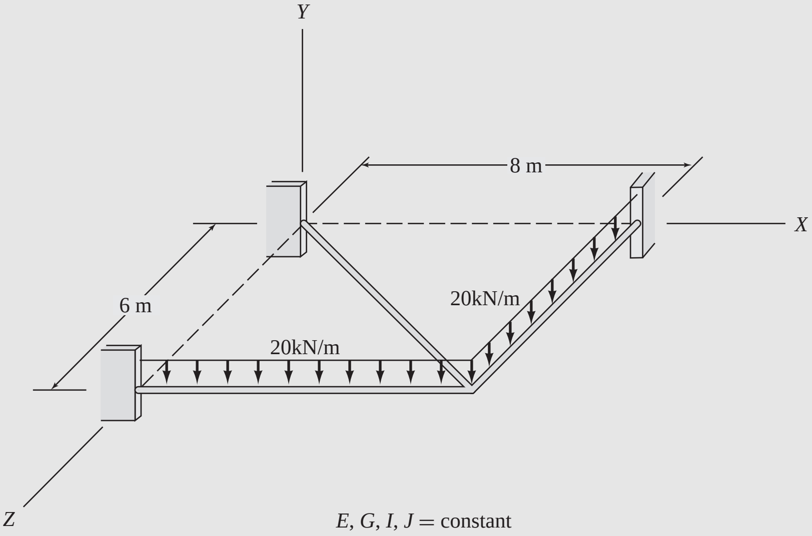
Structure Stiffness Matrix:

Member 1 From Fig. 8.12(b), we can see that joint 1 is the beginning joint, and joint 4 the end joint, for this member. By applying Eqs. (8.33), we determine the length, and the direction cosines, for the member to be

$$L = \sqrt{(X_4 - X_1)^2 + (Z_4 - Z_1)^2} = \sqrt{(8-0)^2 + (6-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{8-0}{10} = 0.8$$

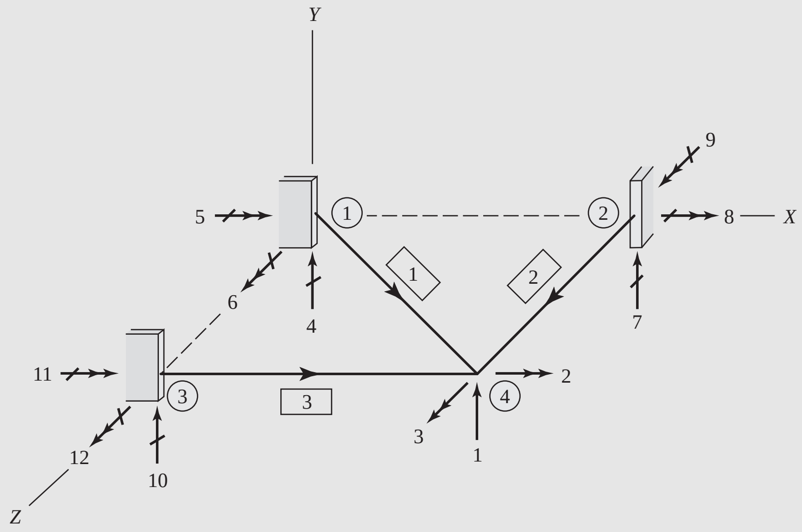
$$\sin\theta = \frac{Z_4 - Z_1}{L} = \frac{6-0}{10} = 0.6$$



E, G, I, J = constant
$E = 200$ GPa, $G = 76$ GPa
$I = 347(10^6)$ mm^4, $J = 115(10^6)$ mm^4

(a) Grid

Fig. 8.12

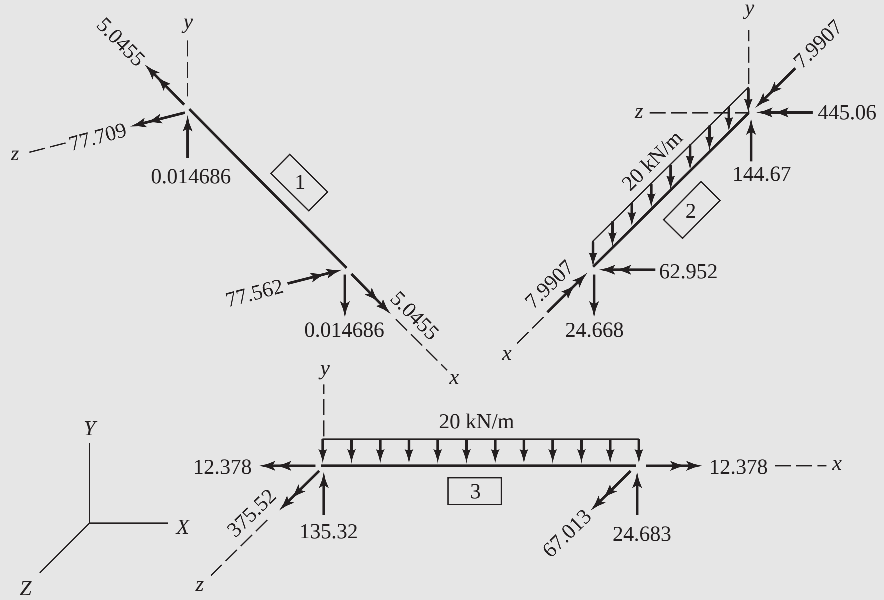


(b) Analytical Model

$$\mathbf{S} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 832.8+3{,}855.6+1{,}626.6 & 2{,}498.4+11{,}567 & -3{,}331.2-6{,}506.3 & 1 \\ 2{,}498.4+11{,}567 & 10{,}553+46{,}267+1{,}092.5 & -12{,}905 & 2 \\ -3{,}331.2-6{,}506.3 & -12{,}905 & 18{,}081+1{,}456.7+34{,}700 & 3 \end{array} = \begin{bmatrix} 6{,}315 & 14{,}065 & -9{,}837.5 \\ 14{,}065 & 57{,}912 & -12{,}905 \\ -9{,}837.5 & -12{,}905 & 54{,}238 \end{bmatrix}$$

$$\mathbf{P}_f = \begin{bmatrix} 60+80 \\ 60 \\ -106.67 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 140 \\ 60 \\ -106.67 \end{bmatrix}$$

(c) Structure Stiffness Matrix and Fixed-Joint Force Vector

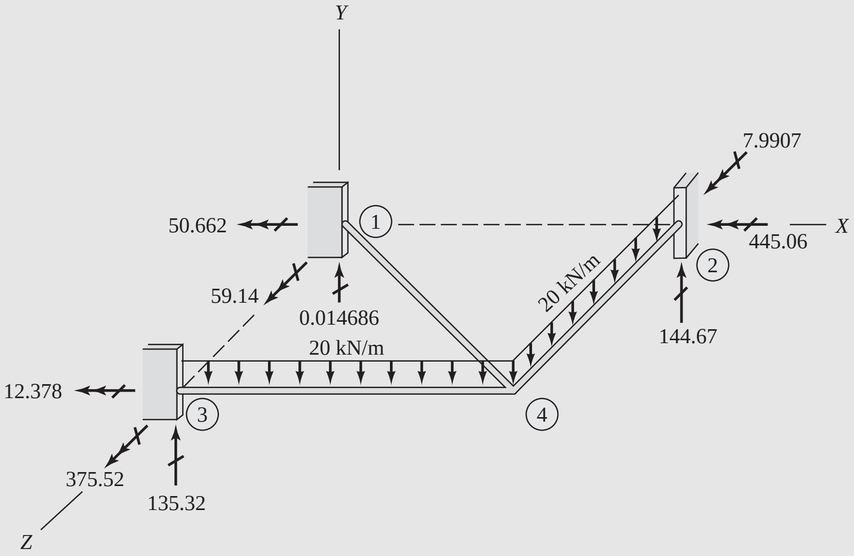


(d) Member Local End Forces

Fig. 8.12 (*continued*)

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} 0.014686 \text{ kN} \\ -50.662 \text{ kN}\cdot\text{m} \\ 59.14 \text{ kN}\cdot\text{m} \\ 144.67 \text{ kN} \\ -445.06 \text{ kN}\cdot\text{m} \\ 7.9907 \text{ kN}\cdot\text{m} \\ 135.32 \text{ kN} \\ -12.378 \text{ kN}\cdot\text{m} \\ 375.52 \text{ kN}\cdot\text{m} \end{array}\right] \end{array} \begin{array}{l} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 8.12 (*continued*)

Since $MT = 0$ for this member, we use Eq. (8.21) to determine its local stiffness matrix **k**. Thus, by substituting $E = 200(10^6)$ kN/m^2, $G = 76(10^6)$ kN/m^2, $L = 10$ m, $I = 347(10^{-6})$ m^4, and $J = 115(10^{-6})$ m^4 into Eq. (8.21), we obtain

$$\mathbf{k}_1 = \begin{bmatrix} 832.8 & 0 & 4{,}164 & -832.8 & 0 & 4{,}164 \\ 0 & 874 & 0 & 0 & -874 & 0 \\ 4{,}164 & 0 & 27{,}760 & -4{,}164 & 0 & 13{,}880 \\ -832.8 & 0 & -4{,}164 & 832.8 & 0 & -4{,}164 \\ 0 & -874 & 0 & 0 & 874 & 0 \\ 4{,}164 & 0 & 13{,}880 & -4{,}164 & 0 & 27{,}760 \end{bmatrix} \quad \textbf{(1)}$$

As the member is not subjected to any loads, its global and local fixed-end force vectors are 0; that is,

$$\mathbf{F}_{f1} = \mathbf{Q}_{f1} = \mathbf{0}$$

Before we can calculate the member global stiffness matrix **K**, we need to evaluate its transformation matrix **T**. Thus, by substituting $\cos\theta = 0.8$ and $\sin\theta = 0.6$ into

Eq. (8.36), we obtain

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0.6 & 0 & 0 & 0 \\ 0 & -0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & 0 & 0 & -0.6 & 0.8 \end{bmatrix} \tag{2}$$

Next, by substituting $\mathbf{k}_1$ (Eq. (1)) and $\mathbf{T}_1$ (Eq. (2)) into the relationship $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$, and performing the necessary matrix multiplications, we obtain the following global stiffness matrix for member 1:

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|ccc} 832.8 & -2{,}498.4 & 3{,}331.2 & -832.8 & -2{,}498.4 & 3{,}331.2 \\ -2{,}498.4 & 10{,}553 & -12{,}905 & 2{,}498.4 & 4{,}437.4 & -7{,}081.9 \\ 3{,}331.2 & -12{,}905 & 18{,}081 & -3{,}331.2 & -7{,}081.9 & 8{,}568.6 \\ \hline -832.8 & 2{,}498.4 & -3{,}331.2 & 832.8 & 2{,}498.4 & -3{,}331.2 \\ -2{,}498.4 & 4{,}437.4 & -7{,}081.9 & 2{,}498.4 & 10{,}553 & -12{,}905 \\ 3{,}331.2 & -7{,}081.9 & 8{,}568.6 & -3{,}331.2 & -12{,}905 & 18{,}081 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

From Fig. 8.12(b), we observe that the code numbers for member 1 are 4, 5, 6, 1, 2, 3. By using these code numbers, we store the pertinent elements of $\mathbf{K}_1$ in the 3×3 structure stiffness matrix $\mathbf{S}$, as shown in Fig. 8.12(c).

Member 2 $L = 6$ m, $\cos\theta = 0$, $\sin\theta = 1$

$$\mathbf{k}_2 = \begin{bmatrix} 3{,}855.6 & 0 & 11{,}567 & -3{,}855.6 & 0 & 11{,}567 \\ 0 & 1{,}456.7 & 0 & 0 & -1{,}456.7 & 0 \\ 11{,}567 & 0 & 46{,}267 & -11{,}567 & 0 & 23{,}133 \\ -3{,}855.6 & 0 & -11{,}567 & 3{,}855.6 & 0 & -11{,}567 \\ 0 & -1{,}456.7 & 0 & 0 & 1{,}456.7 & 0 \\ 11{,}567 & 0 & 23{,}133 & -11{,}567 & 0 & 46{,}267 \end{bmatrix} \tag{3}$$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{4}$$

$$\mathbf{K}_2 = \mathbf{T}_2^T\mathbf{k}_2\mathbf{T}_2 = \begin{array}{c} \begin{array}{cccccc} 7 & 8 & 9 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|ccc} 3{,}855.6 & -11{,}567 & 0 & -3{,}855.6 & -11{,}567 & 0 \\ -11{,}567 & 46{,}267 & 0 & 11{,}567 & 23{,}133 & 0 \\ 0 & 0 & 1{,}456.7 & 0 & 0 & -1{,}456.7 \\ \hline -3{,}855.6 & 11{,}567 & 0 & 3{,}855.6 & 11{,}567 & 0 \\ -11{,}567 & 23{,}133 & 0 & 11{,}567 & 46{,}267 & 0 \\ 0 & 0 & -1{,}456.7 & 0 & 0 & 1{,}456.7 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

To determine the local fixed-end force vector due to the 20 kN/m member load, we first evaluate the fixed-end shears and moments by using the expressions for loading type 3 given inside the front cover. This yields

$$FS_b = FS_e = 60 \text{ kN}$$
$$FM_b = -FM_e = 60 \text{ kN}\cdot\text{m}$$
$$FT_b = FT_e = 0$$

Since $MT = 0$ for this member, we use Eq. (8.22) to obtain its local fixed-end force vector:

$$\mathbf{Q}_{f2} = \begin{bmatrix} 60 \\ 0 \\ 60 \\ 60 \\ 0 \\ -60 \end{bmatrix} \tag{5}$$

Next, by substituting $\mathbf{T}_2$ (Eq. (4)) and $\mathbf{Q}_{f2}$ (Eq. (5)) into the transformation relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$, we obtain the global fixed-end force vector for member 2:

$$\mathbf{F}_{f2} = \left[\begin{array}{c} 60 \\ -60 \\ 0 \\ \hdashline 60 \\ 60 \\ 0 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array}$$

The relevant elements of $\mathbf{K}_2$ and $\mathbf{F}_{f2}$ are stored in $\mathbf{S}$ and the 3 × 1 structure fixed-joint force vector $\mathbf{P}_f$, respectively, as shown in Fig. 8.12(c).

Member 3 As the local x axis of this member is oriented in the positive direction of the global X axis, no coordinate transformations are needed; that is, $\mathbf{T_3} = \mathbf{I}.$ By using Eq. (8.21) with $L = 8$ m, we obtain

$$\mathbf{K}_3 = \mathbf{k}_3 = \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 12 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc:ccc} 1{,}626.6 & 0 & 6{,}506.3 & -1{,}626.6 & 0 & 6{,}506.3 \\ 0 & 1{,}092.5 & 0 & 0 & -1{,}092.5 & 0 \\ 6{,}506.3 & 0 & 34{,}700 & -6{,}506.3 & 0 & 17{,}350 \\ \hdashline -1{,}626.6 & 0 & -6{,}506.3 & 1{,}626.6 & 0 & -6{,}506.3 \\ 0 & -1{,}092.5 & 0 & 0 & 1{,}092.5 & 0 \\ 6{,}506.3 & 0 & 17{,}350 & -6{,}506.3 & 0 & 34{,}700 \end{array}\right] \end{array} \begin{array}{c} \\ 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} \tag{6}$$

$$FS_b = FS_e = 80 \text{ kN}$$
$$FM_b = -FM_e = 106.67 \text{ kN} \cdot \text{m}$$
$$FT_b = FT_e = 0$$

$$\mathbf{F}_{f3} = \mathbf{Q}_{f3} = \left[\begin{array}{c} 80 \\ 0 \\ 106.67 \\ \hdashline 80 \\ 0 \\ -106.67 \end{array}\right] \begin{array}{c} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} \tag{7}$$

The complete structure stiffness matrix **S** and the structure fixed-joint force vector $\mathbf{P}_f$ are given in Fig. 8.12(c).

Joint Load Vector: Because the grid is not subjected to any external loads at its joints, the joint load vector is 0; that is,

$$\mathbf{P} = \mathbf{0}$$

Joint Displacements: By substituting **P**, $\mathbf{P}_f$, and **S** into the structure stiffness relationship, $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$, we write

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 140 \\ 60 \\ -106.67 \end{bmatrix} = \begin{bmatrix} 6{,}315 & 14{,}065 & -9{,}837.5 \\ 14{,}065 & 57{,}912 & -12{,}905 \\ -9{,}837.5 & -12{,}905 & 54{,}238 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or

$$\begin{bmatrix} -140 \\ -60 \\ 106.67 \end{bmatrix} = \begin{bmatrix} 6{,}315 & 14{,}065 & -9{,}837.5 \\ 14{,}065 & 57{,}912 & -12{,}905 \\ -9{,}837.5 & -12{,}905 & 54{,}238 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

By solving the foregoing simultaneous equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} -55.951 \text{ m} \\ 11.33 \text{ rad} \\ -5.4856 \text{ rad} \end{bmatrix} \times 10^{-3}$$ **Ans**

Member End Displacements and End Forces:

Member 1

$$\mathbf{v}_1 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -55.951 \\ 11.33 \\ -5.4856 \end{bmatrix} \times 10^{-3}$$

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -55.951 \\ 5.7728 \\ -11.187 \end{bmatrix} \times 10^{-3}$$

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 = \begin{bmatrix} 0.014686 \text{ kN} \\ -5.0455 \text{ kN·m} \\ 77.709 \text{ kN·m} \\ -0.014686 \text{ kN} \\ 5.0455 \text{ kN·m} \\ -77.562 \text{ kN·m} \end{bmatrix}$$ **Ans**

The member local end forces are depicted in Fig. 8.12(d).

$$\mathbf{F}_1 = \mathbf{T}_1^T \mathbf{Q}_1 = \left[\begin{array}{c} 0.014686 \\ -50.662 \\ 59.14 \\ \hdashline -0.014686 \\ 50.574 \\ -59.022 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$

The pertinent elements of $\mathbf{F}_1$ are stored in the reaction vector **R**, as shown in Fig. 8.12(e).

Member 2

$$\mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -55.951 \\ 11.33 \\ -5.4856 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array} \times 10^{-3}, \qquad \mathbf{u}_2 = \mathbf{T}_2 \mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -55.951 \\ -5.4856 \\ -11.33 \end{array}\right] \times 10^{-3}$$

$$\mathbf{Q}_2 = \mathbf{k}_2 \mathbf{u}_2 + \mathbf{Q}_{f2} = \left[\begin{array}{c} 144.67 \text{ kN} \\ 7.9907 \text{ kN·m} \\ 445.06 \text{ kN·m} \\ -24.668 \text{ kN} \\ -7.9907 \text{ kN·m} \\ 62.952 \text{ kN·m} \end{array}\right]$$

Ans

$$\mathbf{F}_2 = \mathbf{T}_2^T \mathbf{Q}_2 = \left[\begin{array}{c} 144.67 \\ -445.06 \\ 7.9907 \\ \hdashline -24.668 \\ -62.952 \\ -7.9907 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array}$$

Member 3

$$\mathbf{u}_3 = \mathbf{v}_3 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -55.951 \\ 11.33 \\ -5.4856 \end{array}\right] \begin{array}{c} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} \times 10^{-3}$$

$$\mathbf{F}_3 = \mathbf{Q}_3 = \mathbf{k}_3 \mathbf{u}_3 + \mathbf{Q}_{f3} = \left[\begin{array}{c} 135.32 \text{ kN} \\ -12.378 \text{ kN·m} \\ 375.52 \text{ kN·m} \\ \hdashline 24.683 \text{ kN} \\ 12.378 \text{ kN·m} \\ 67.013 \text{ kN·m} \end{array}\right] \begin{array}{c} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array}$$

Ans

Support Reactions: The completed reaction vector **R** is shown in Fig. 8.12(e), and the support reactions are depicted on a line diagram of the grid in Fig. 8.12(f). **Ans**

Equilibrium Check: The three equilibrium equations ($\sum F_Y = 0$, $\sum M_X = 0$, and $\sum M_Z = 0$) are satisfied.

8.3 SPACE FRAMES

Space frames constitute the most general type of framed structures. The members of such frames may be oriented in any directions in three-dimensional space, and may be connected by rigid and/or flexible connections. Furthermore, external loads oriented in any arbitrary directions can be applied to the joints, as well as members, of space frames (Fig. 8.13(a)). Under the action of external loads, the members of a space frame are generally subjected to bending moments about both principal axes, shears in both principal directions, torsional moments, and axial forces.

As with grids, the analysis of space frames is commonly based on the assumption that the cross-sections of all the members are symmetric about at least two mutually perpendicular axes, and are free to warp out of their planes under the action of torsional moments. As discussed previously in the case of grids, the bending and torsional stiffnesses of a member are uncoupled if it satisfies the foregoing assumption.

The process of developing the analytical models, and numbering the degrees of freedom and restrained coordinates, of space frames is analogous to that for

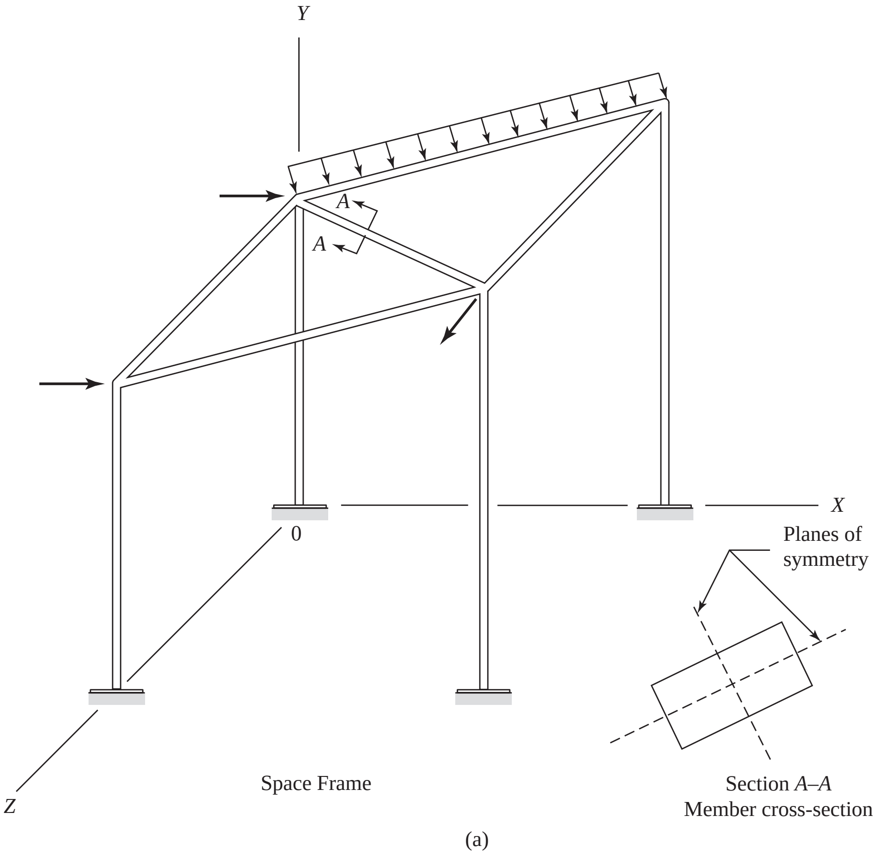
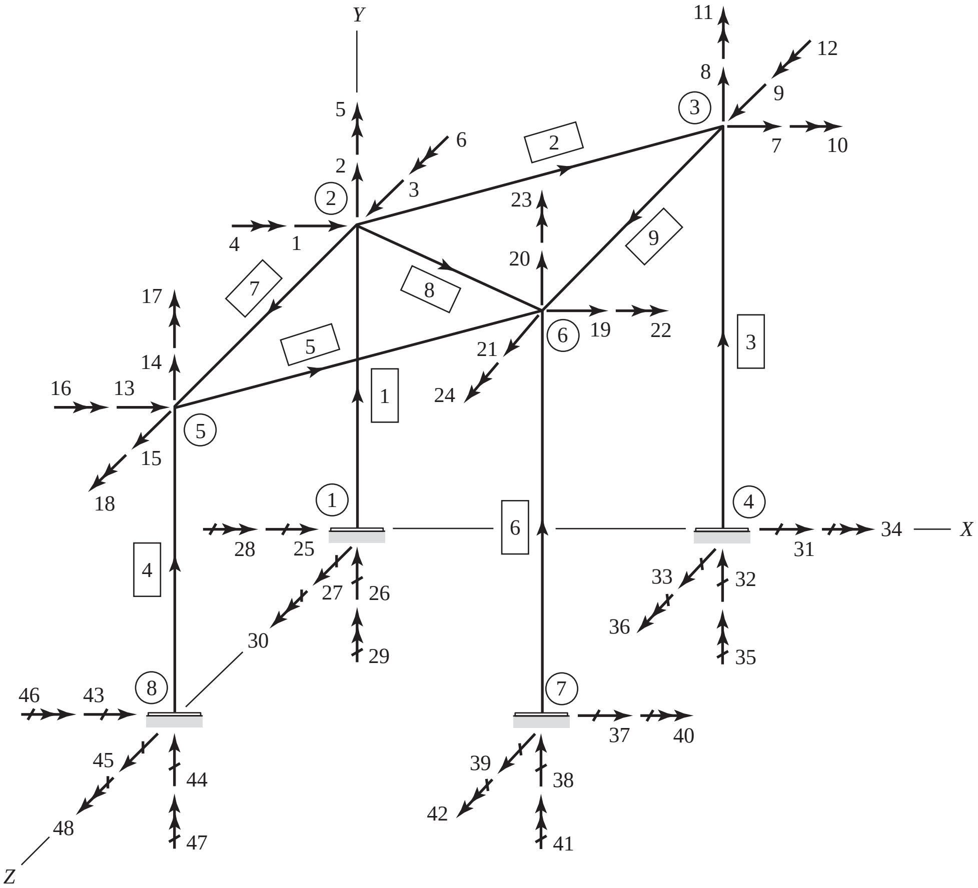


Fig. 8.13

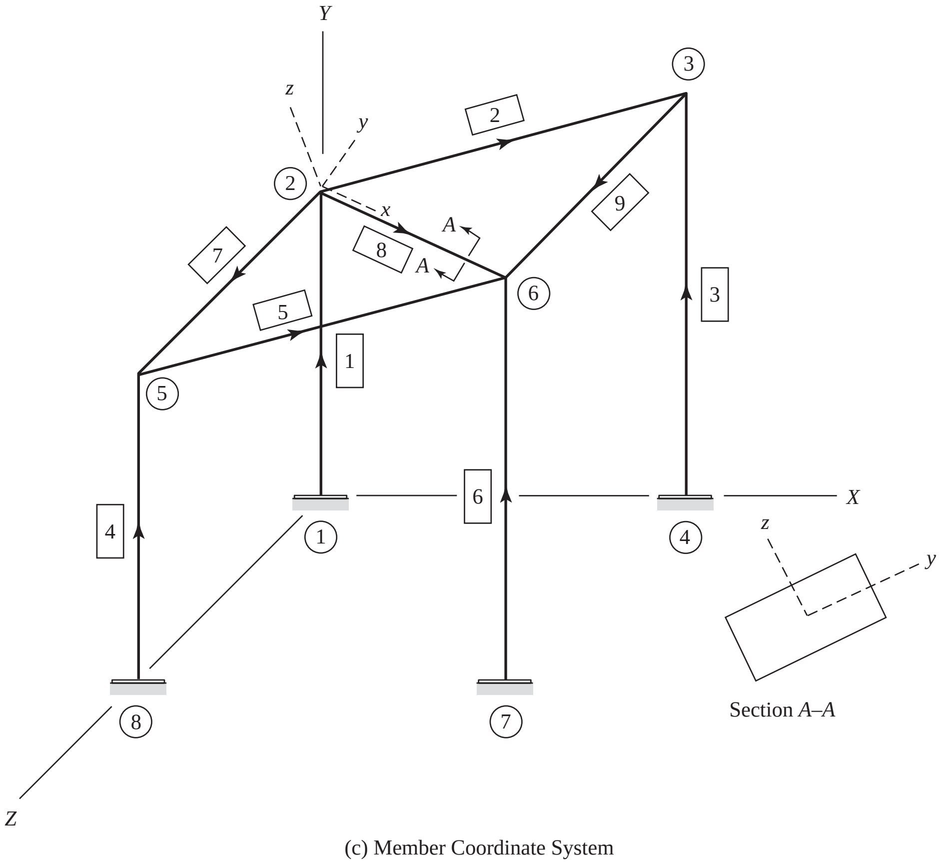


(b) Analytical Model (24 Degrees of Freedom and 24 Restrained Coordinates)

Fig. 8.13 (*continued*)

other types of framed structures. The overall geometry of the space frame, and its joint loads and displacements, are described with reference to a global right-handed *XYZ* coordinate system, with three global (*X*, *Y*, and *Z*) coordinates used to specify the location of each joint. An unsupported joint of a space frame can translate in any direction, and rotate about any axis, in three-dimensional space. Since small rotations can be treated as vector quantities, the rotation of a joint can be conveniently represented by its component rotations about the *X*, *Y*, and *Z* axes. Thus, a free joint of a space frame has six degrees of freedom—the translations in the *X*, *Y*, and *Z* directions and the rotations about the *X*, *Y*, and *Z* axes. Therefore, six structure coordinates (i.e., free and/or restrained coordinates) need to be defined at each joint of the space frame for the purpose of analysis; that is,

$$\boxed{\left.\begin{array}{l} NCJT = 6 \\ NDOF = 6(NJ) - NR \end{array}\right\} \quad \text{for space frames}} \tag{8.37}$$



(c) Member Coordinate System

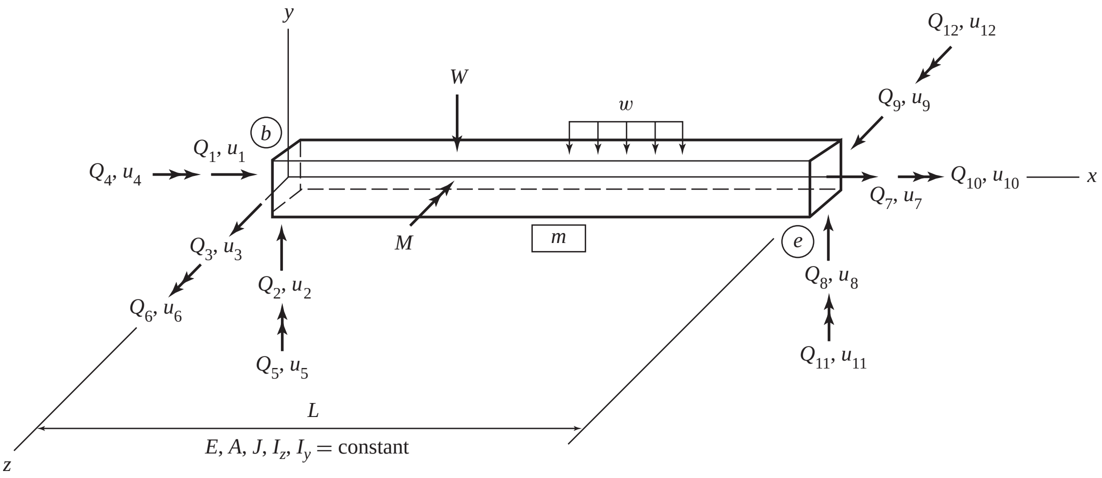
Fig. 8.13 (*continued*)

The procedure for assigning numbers to the structure coordinates of a space frame is similar to that for other types of framed structures, with the degrees of freedom numbered before the restrained coordinates. In the case of a joint with multiple degrees of freedom (or restrained coordinates), the translations (or forces) in the *X*, *Y*, and *Z* directions are numbered first in sequential order, followed by the rotations (or moments) about the *X*, *Y*, and *Z* axes, respectively, as shown in Fig. 8.13(b).

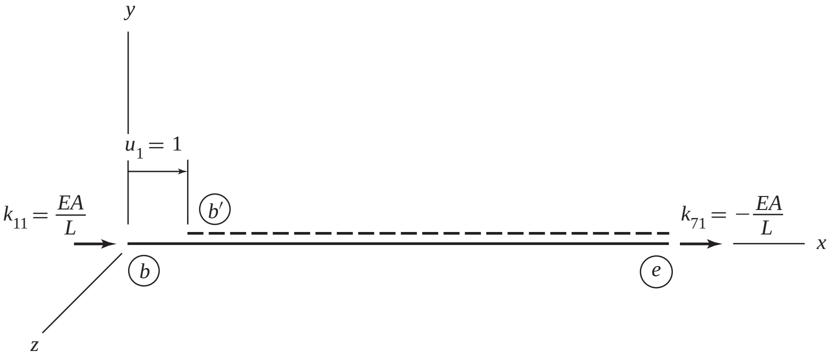
For each member of a space frame, a local *xyz* coordinate system is established, with its origin at an end of the member and the *x* axis directed along the member's centroidal axis in the undeformed state. The local *y* and *z* axes are oriented, respectively, parallel to the two axes of symmetry (or the principal axes of inertia) of the member cross-section, with their positive directions defined so that the local *xyz* coordinate system is right-handed (Fig. 8.13(c)).

Member Stiffness Relations in the Local Coordinate System

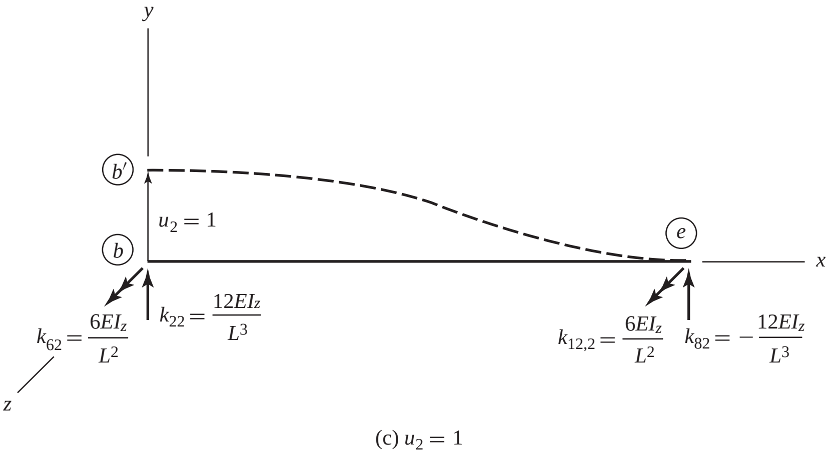
To establish the local stiffness relations, let us consider an arbitrary member *m* of a space frame, as shown in Fig. 8.14(a). Like a joint of a space frame, six



(a) Member Forces and Displacements in the Local Coordinate System

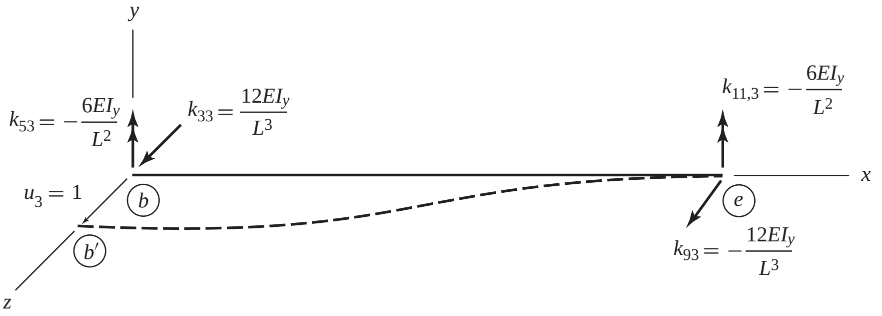


(b) $u_1 = 1$

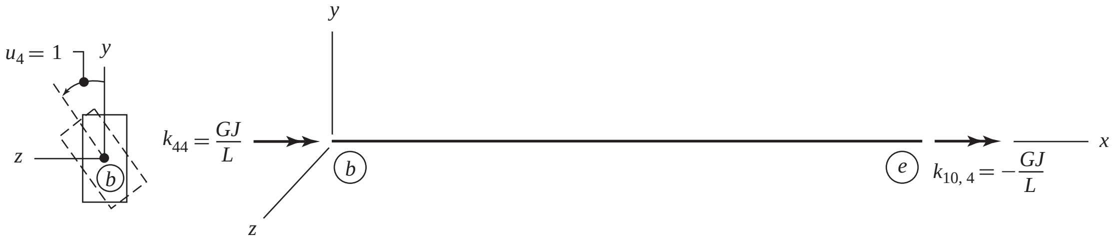


(c) $u_2 = 1$

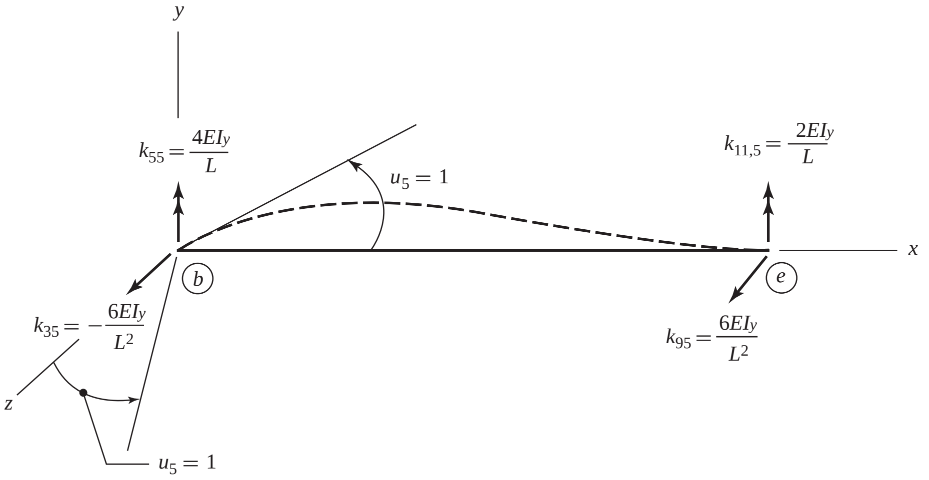
Fig. 8.14



(d) $u_3 = 1$



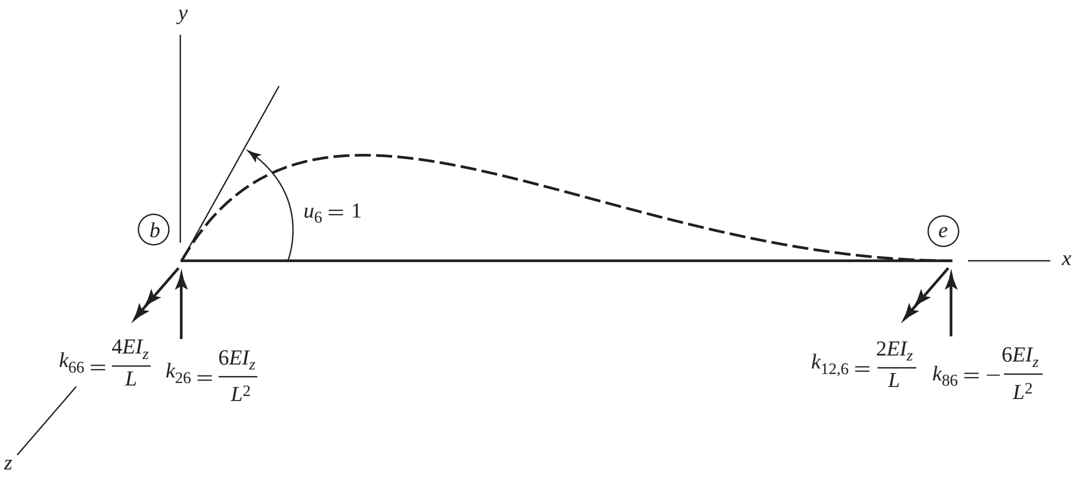
(e) $u_4 = 1$



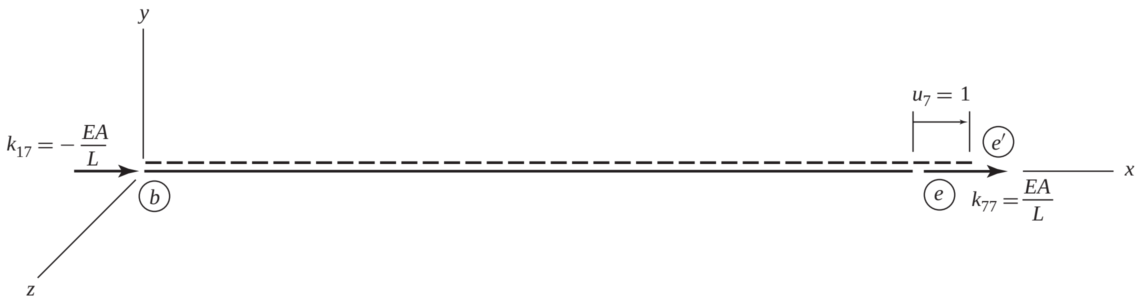
(f) $u_5 = 1$

Fig. 8.14 (*continued*)

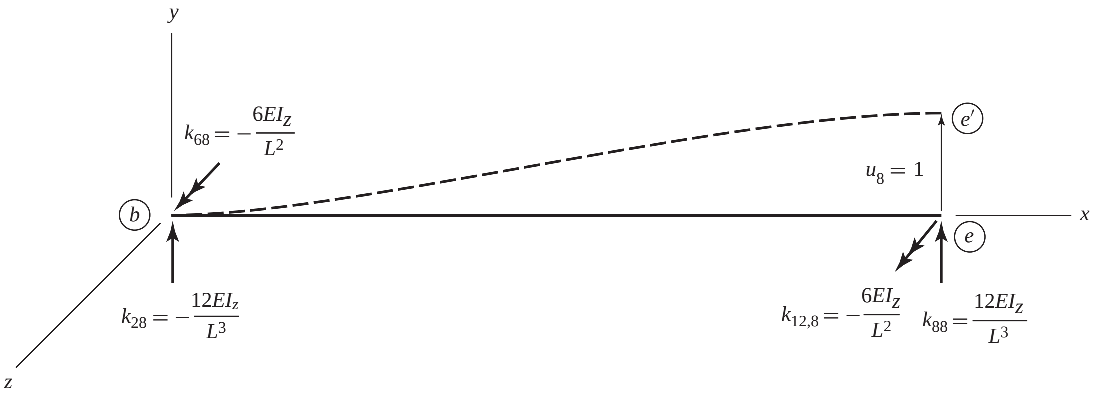
displacements are needed to completely specify the displaced position of each end of the space frame member. Thus, a member of a space frame has 12 degrees of freedom. In the member local coordinate system, the 12 end displacements are denoted by u_1 through u_{12}, and the corresponding member end forces are denoted by Q_1 through Q_{12}, as shown in Fig. 8.14(a). As indicated in this figure, *a member's local end displacements (or end forces) are numbered by beginning at its end b, with the translations (or forces) in the x, y, and z directions numbered first in sequential order, followed by the rotations (or moments) about the x, y, and z axes, respectively. The displacements (or forces) at the member's opposite end e are then numbered in the same sequential order.*



(g) $u_6 = 1$

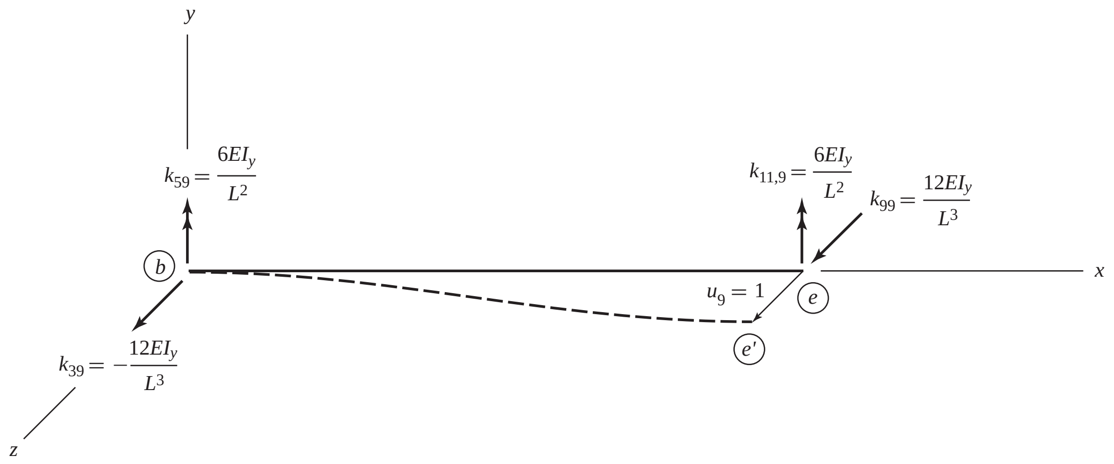


(h) $u_7 = 1$



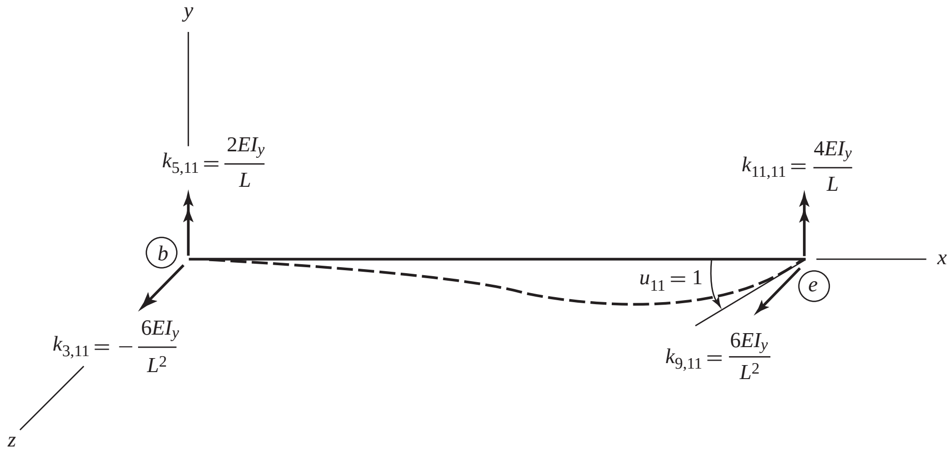
(i) $u_8 = 1$

Fig. 8.14 (*continued*)



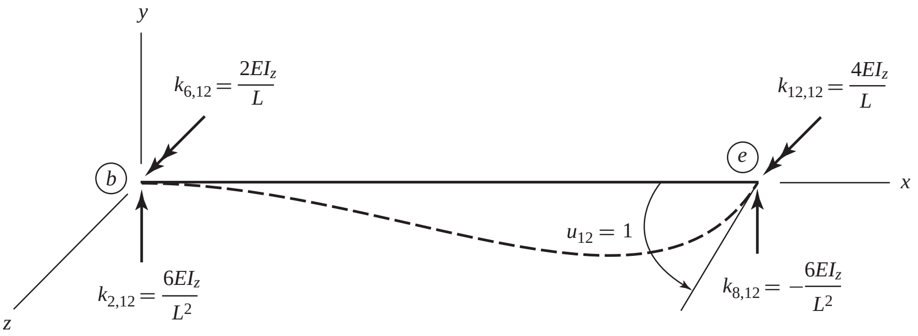
(j) $u_9 = 1$

(k) $u_{10} = 1$

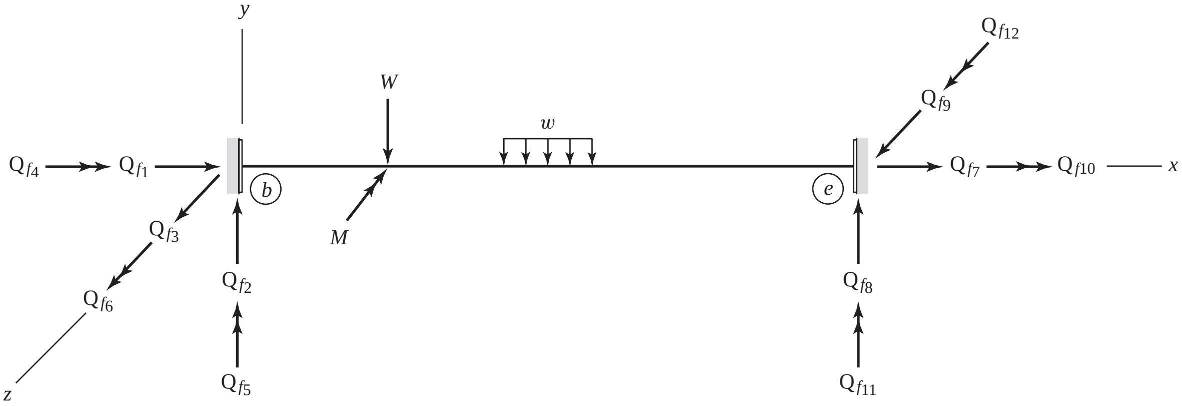


(l) $u_{11} = 1$

Fig. 8.14 (*continued*)



(m) $u_{12} = 1$



(n) Member Fixed-End Forces in the Local Coordinate System
($u_1 = u_2 = \cdots = u_{11} = u_{12} = 0$)

Fig. 8.14 (*continued*)

The relationship between the end forces **Q** and the end displacements **u**, for space frame members, can be expressed in the following, now familiar, form:

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f} \tag{8.38}$$

with **k** now representing the 12 × 12 member local stiffness matrix, and $\mathbf{Q}_f$ denoting the 12 × 1 member local fixed-end force vector.

The explicit form of **k** for members of space frames can be conveniently obtained using the expressions of the stiffness coefficients derived previously for prismatic members subjected to axial deformations (Section 3.3), bending deformations (Section 5.2), and torsional deformations (Section 8.2). The stiffness coefficients for a space frame member thus obtained, due to the unit values of the 12 end displacements (u_1 through u_{12}, respectively), are given in Figs. 8.14(b) through (m). Note that in Figs. 8.14(c), (g), (i), and (m), the moment of inertia of the member cross-section about its local z axis, I_z, is used in the expressions for the stiffness coefficients, because the end displacements u_2,

u_6, u_8, and u_{12} cause the member to bend about the z axis. However, in Figs. 8.14(d), (f), (j), and (l), because the end displacements u_3, u_5, u_9, and u_{11} cause the member to bend about its local y axis, the moment of inertia about the y axis, I_y, is used in the expressions for the corresponding stiffness coefficients. The explicit form of the local stiffness matrix **k** for members of space frames, obtained by arranging all the stiffness coefficients shown in Figs. 8.14(b) through (m) in a 12 × 12 matrix, is given in Eq. (8.39).

$$\mathbf{k} = \frac{E}{L^3}\begin{bmatrix}
AL^2 & 0 & 0 & 0 & 0 & 0 & -AL^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 12I_z & 0 & 0 & 0 & 6LI_z & 0 & -12I_z & 0 & 0 & 0 & 6LI_z \\
0 & 0 & 12I_y & 0 & -6LI_y & 0 & 0 & 0 & -12I_y & 0 & -6LI_y & 0 \\
0 & 0 & 0 & \frac{GJL^2}{E} & 0 & 0 & 0 & 0 & 0 & -\frac{GJL^2}{E} & 0 & 0 \\
0 & 0 & -6LI_y & 0 & 4L^2I_y & 0 & 0 & 0 & 6LI_y & 0 & 2L^2I_y & 0 \\
0 & 6LI_z & 0 & 0 & 0 & 4L^2I_z & 0 & -6LI_z & 0 & 0 & 0 & 2L^2I_z \\
-AL^2 & 0 & 0 & 0 & 0 & 0 & AL^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -12I_z & 0 & 0 & 0 & -6LI_z & 0 & 12I_z & 0 & 0 & 0 & -6LI_z \\
0 & 0 & -12I_y & 0 & 6LI_y & 0 & 0 & 0 & 12I_y & 0 & 6LI_y & 0 \\
0 & 0 & 0 & -\frac{GJL^2}{E} & 0 & 0 & 0 & 0 & 0 & \frac{GJL^2}{E} & 0 & 0 \\
0 & 0 & -6LI_y & 0 & 2L^2I_y & 0 & 0 & 0 & 6LI_y & 0 & 4L^2I_y & 0 \\
0 & 6LI_z & 0 & 0 & 0 & 2L^2I_z & 0 & -6LI_z & 0 & 0 & 0 & 4L^2I_z
\end{bmatrix} \tag{8.39}$$

The local fixed-end force vector for the members of space frames is expressed as follows (Fig. 8.14(n)).

$$\mathbf{Q}_f = \begin{bmatrix} FA_b \\ FS_{by} \\ FS_{bz} \\ FT_b \\ FM_{by} \\ FM_{bz} \\ FA_e \\ FS_{ey} \\ FS_{ez} \\ FT_e \\ FM_{ey} \\ FM_{ez} \end{bmatrix} \tag{8.40}$$

in which FS_{by} and FS_{bz} denote the fixed-end shears at member end b in the local y and z directions, respectively; and FM_{by} and FM_{bz} represent the fixed-end moments at the same member end about the y and z axes, respectively. The fixed-end shears and moments at the opposite end e of the member are defined in a similar manner. The fixed-end forces due to a prescribed member loading can be conveniently evaluated, using the fixed-end force expressions given inside the front cover. Any inclined member loads must be resolved into their components in the directions of the member's local x, y, and z axes before proceeding with the calculation of the fixed-end forces.

Member Releases The expressions for **k** and $\mathbf{Q}_f$, as given in Eqs. (8.39) and (8.40), respectively, are valid only for members rigidly connected to joints at both ends (i.e., members of type 0, or $MT = 0$). For members of space frames with moment releases, the foregoing expressions need to be modified, using the procedure described in Section 7.1. If the member releases are assumed to be in the form of spherical hinges (or ball-and-socket type of connections), so that all three moments (i.e., the bending moments about the y and z axes, and the torsional moment) are 0 at the released member ends, then the modified local stiffness matrices **k** and fixed-end force vectors $\mathbf{Q}_f$ for the members with releases can be expressed as follows.

For members with a hinge at the beginning ($MT = 1$), the modified **k** is given in Eq. (8.41):

$$\mathbf{k} = \frac{E}{L^3}\begin{bmatrix} AL^2 & 0 & 0 & 0 & 0 & 0 & -AL^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3I_z & 0 & 0 & 0 & 0 & 0 & -3I_z & 0 & 0 & 0 & 3LI_z \\ 0 & 0 & 3I_y & 0 & 0 & 0 & 0 & 0 & -3I_y & 0 & -3LI_y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -AL^2 & 0 & 0 & 0 & 0 & 0 & AL^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3I_z & 0 & 0 & 0 & 0 & 0 & 3I_z & 0 & 0 & 0 & -3LI_z \\ 0 & 0 & -3I_y & 0 & 0 & 0 & 0 & 0 & 3I_y & 0 & 3LI_y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3LI_y & 0 & 0 & 0 & 0 & 0 & 3LI_y & 0 & 3L^2I_y & 0 \\ 0 & 3LI_z & 0 & 0 & 0 & 0 & 0 & -3LI_z & 0 & 0 & 0 & 3L^2I_z \end{bmatrix} \tag{8.41}$$

and

$$\mathbf{Q}_f = \begin{bmatrix} FA_b \\ FS_{by} - \dfrac{3}{2L}FM_{bz} \\ FS_{bz} + \dfrac{3}{2L}FM_{by} \\ 0 \\ 0 \\ 0 \\ FA_e \\ FS_{ey} + \dfrac{3}{2L}FM_{bz} \\ FS_{ez} - \dfrac{3}{2L}FM_{by} \\ FT_b + FT_e \\ FM_{ey} - \dfrac{1}{2}FM_{by} \\ FM_{ez} - \dfrac{1}{2}FM_{bz} \end{bmatrix} \tag{8.42}$$

For members with a hinge at the end ($MT = 2$), the modified **k** is given in Eq. (8.43):

$$\mathbf{k} = \frac{E}{L^3}\begin{bmatrix} AL^2 & 0 & 0 & 0 & 0 & 0 & -AL^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3I_z & 0 & 0 & 0 & 3LI_z & 0 & -3I_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 3I_y & 0 & -3LI_y & 0 & 0 & 0 & -3I_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3LI_y & 0 & 3L^2I_y & 0 & 0 & 0 & 3LI_y & 0 & 0 & 0 \\ 0 & 3LI_z & 0 & 0 & 0 & 3L^2I_z & 0 & -3LI_z & 0 & 0 & 0 & 0 \\ -AL^2 & 0 & 0 & 0 & 0 & 0 & AL^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3I_z & 0 & 0 & 0 & -3LI_z & 0 & 3I_z & 0 & 0 & 0 & 0 \\ 0 & 0 & -3I_y & 0 & 3LI_y & 0 & 0 & 0 & 3I_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.43}$$

and

$$\mathbf{Q}_f = \begin{bmatrix} FA_b \\ FS_{by} - \dfrac{3}{2L}FM_{ez} \\ FS_{bz} + \dfrac{3}{2L}FM_{ey} \\ FT_b + FT_e \\ FM_{by} - \dfrac{1}{2}FM_{ey} \\ FM_{bz} - \dfrac{1}{2}FM_{ez} \\ FA_e \\ FS_{ey} + \dfrac{3}{2L}FM_{ez} \\ FS_{ez} - \dfrac{3}{2L}FM_{ey} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.44}$$

For members with hinges at both ends ($MT = 3$), the modified **k** is given in Eq. (8.45):

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.45}$$

and

$$
\mathbf{Q}_f = \begin{bmatrix} FA_b \\ FS_{by} - \dfrac{1}{L}(FM_{bz} + FM_{ez}) \\ FS_{bz} + \dfrac{1}{L}(FM_{by} + FM_{ey}) \\ 0 \\ 0 \\ 0 \\ FA_e \\ FS_{ey} + \dfrac{1}{L}(FM_{bz} + FM_{ez}) \\ FS_{ez} - \dfrac{1}{L}(FM_{by} + FM_{ey}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(8.46)}
$$

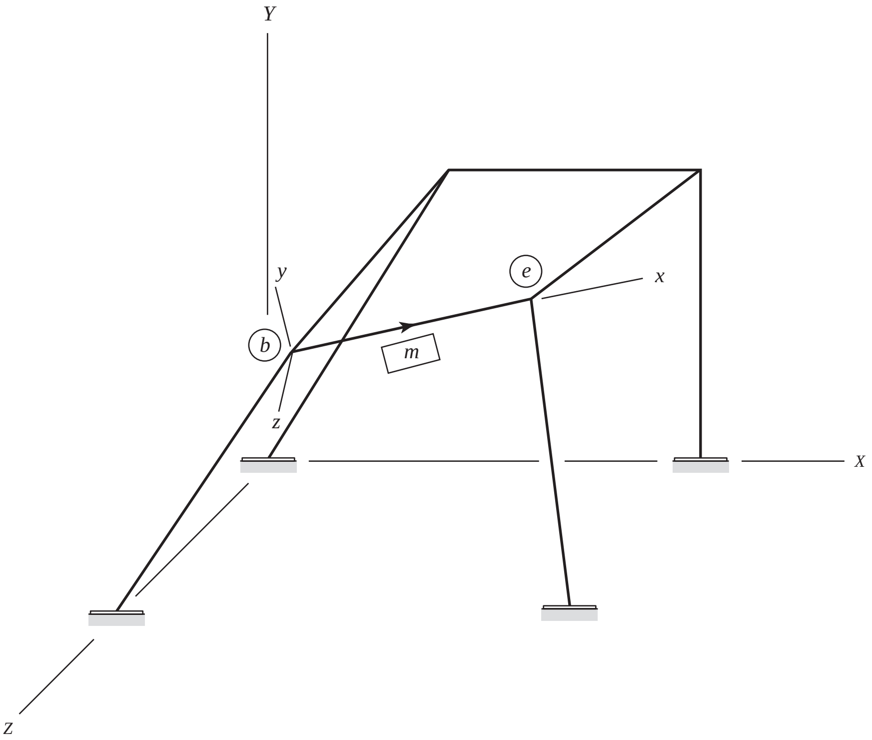
Coordinate Transformations

The expression of the transformation matrix **T** for members of space frames can be derived using a procedure essentially similar to that used previously for other types of framed structures. However, unlike the transformation matrices for trusses, plane frames, and grids, which contain direction cosines of only the member's longitudinal (or x) axis, the transformation matrix for members of space frames involves direction cosines of all three (x, y, and z) axes of the member local coordinate system with respect to the structure's global (XYZ) coordinate system.

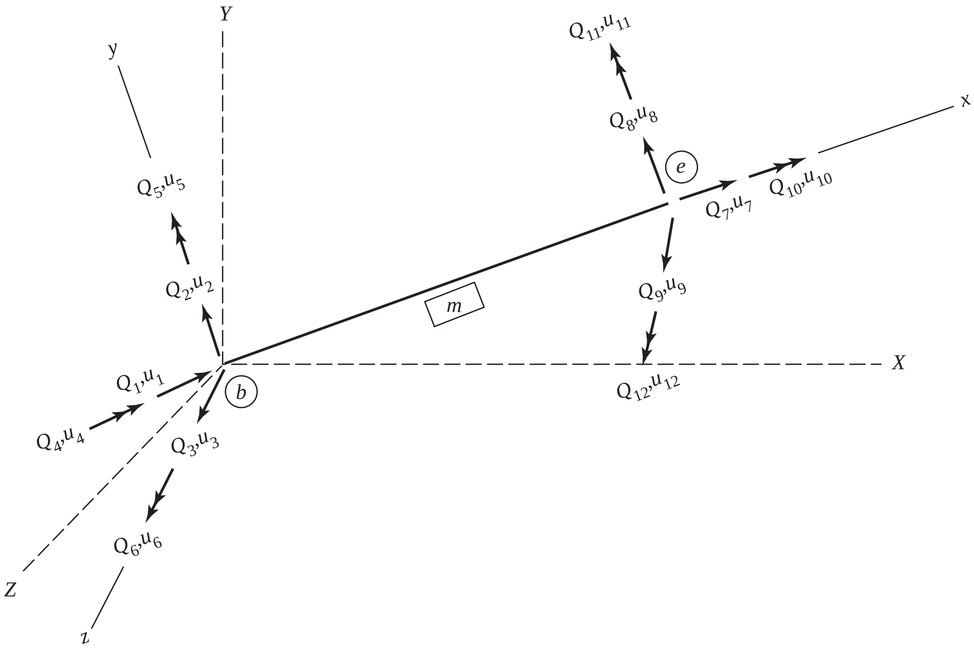
Consider an arbitrary member m of a space frame, as shown in Fig. 8.15(a) on the next page. The member end forces **Q** and end displacements **u**, in the local coordinate system, are shown in Fig. 8.15(b), and Fig. 8.15(c) depicts the equivalent system of member end forces **F** and end displacements **v**, in the global coordinate system. As indicated in Fig. 8.15(c), the global member end forces and displacements are numbered in a manner analogous to the local forces and displacements, except that they act in the directions of the global X, Y, and Z axes.

The orientation of a member of a space frame is defined by the angles between its local x, y, and z axes and the global X, Y, and Z axes. As shown in Fig. 8.16(a) on page 469, the angles between the local x axis and the global X, Y, and Z axes are denoted by θ_{xX}, θ_{xY}, and θ_{xZ}, respectively. Similarly, the angles between the local y axis and the global X, Y, and Z axes are denoted by θ_{yX}, θ_{yY}, and θ_{yZ}, respectively (Fig. 8.16(b)); and the angles between the local z axis and the global X, Y, and Z axes are denoted by θ_{zX}, θ_{zY}, and θ_{zZ}, respectively (Fig. 8.16(c)).

Now, let us consider the transformation of member end forces from the global to a local coordinate system. By comparing Figs. 8.15(b) and (c), we realize that, at member end b, the local forces Q_1, Q_2, and Q_3 must be equal to the

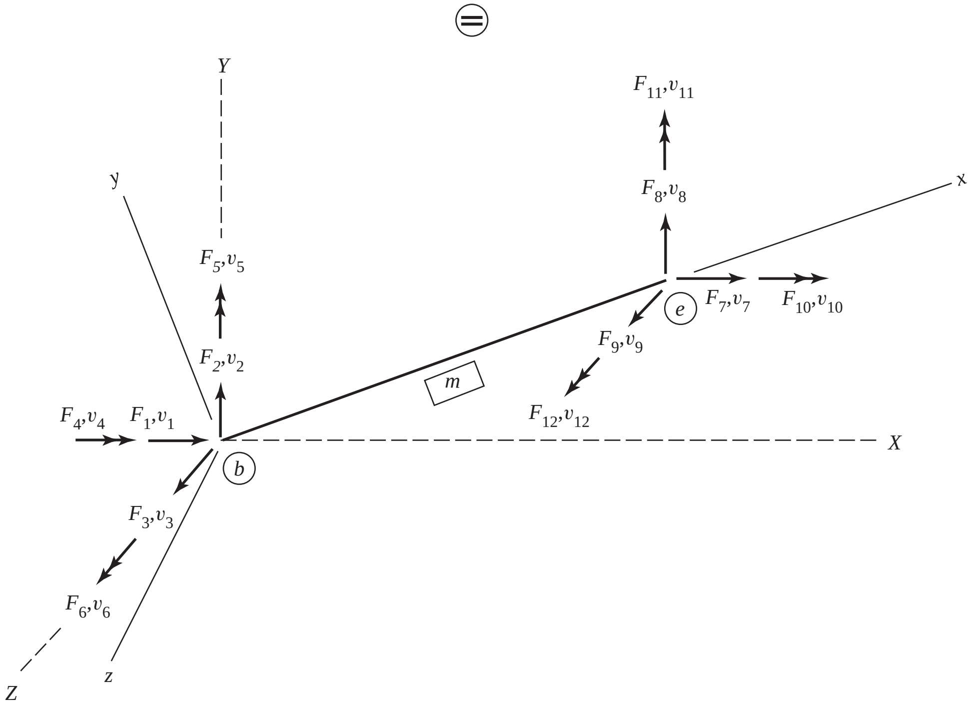


(a) Space Frame



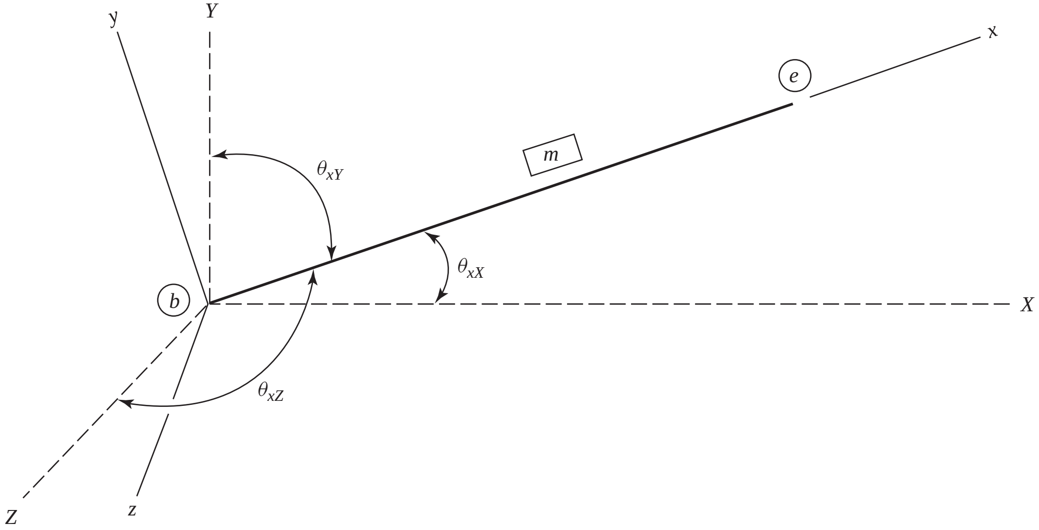
(b) Member End Forces and End Displacements in the Local Coordinate System

Fig. 8.15



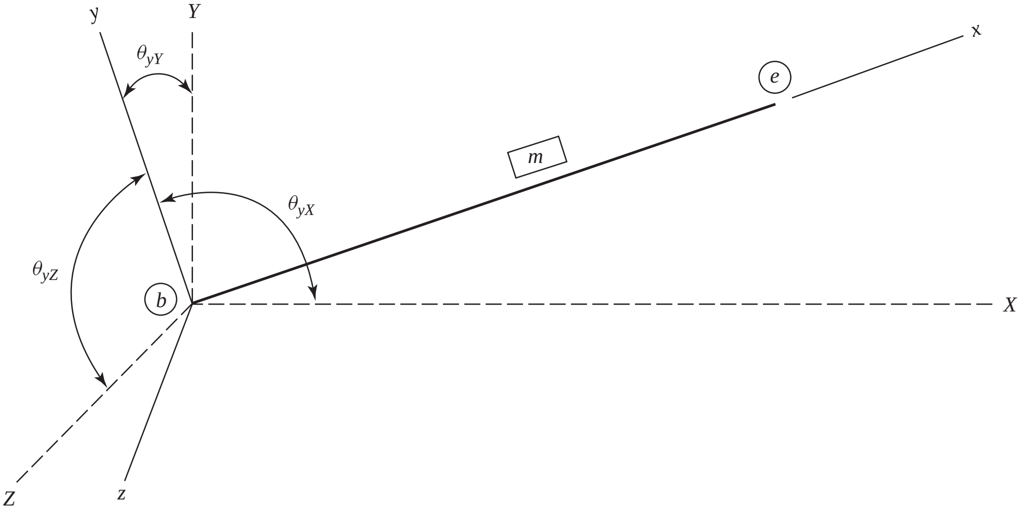
(c) Member End Forces and End Displacements in the Global Coordinate System

Fig. 8.15 (*continued*)

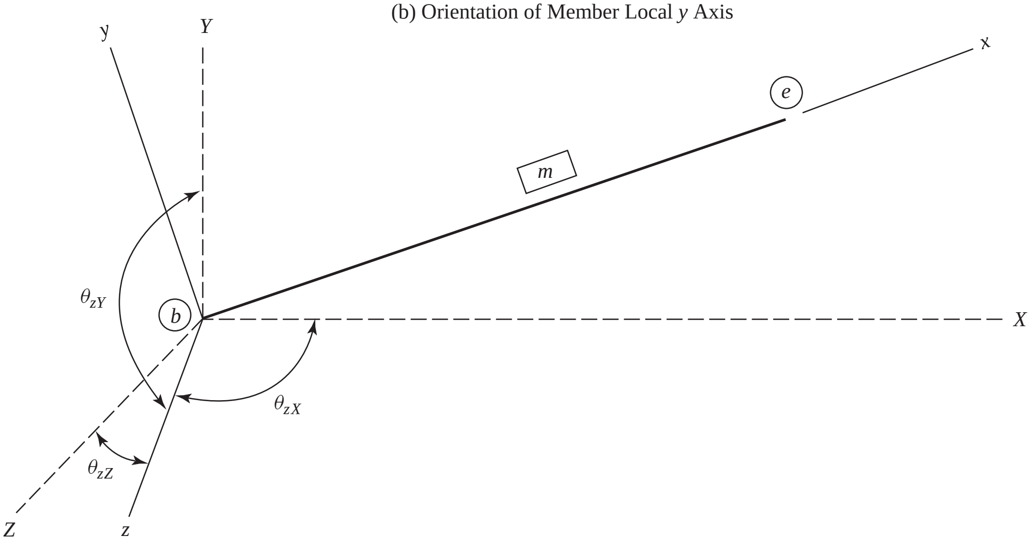


(a) Orientation of Member Local *x* Axis

Fig. 8.16



(b) Orientation of Member Local *y* Axis



(c) Orientation of Member Local *z* Axis

Fig. 8.16 (*continued*)

algebraic sums of the components of the global forces F_1, F_2, and F_3 in the directions of the local x, y, and z axes, respectively; that is (also, see Fig. 8.16),

$$Q_1 = F_1 \cos\theta_{xX} + F_2 \cos\theta_{xY} + F_3 \cos\theta_{xZ} \tag{8.47a}$$

$$Q_2 = F_1 \cos\theta_{yX} + F_2 \cos\theta_{yY} + F_3 \cos\theta_{yZ} \tag{8.47b}$$

$$Q_3 = F_1 \cos\theta_{zX} + F_2 \cos\theta_{zY} + F_3 \cos\theta_{zZ} \tag{8.47c}$$

Equations (8.47) can be written in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ r_{yX} & r_{yY} & r_{yZ} \\ r_{zX} & r_{zY} & r_{zZ} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{8.48}$$

in which

$$r_{iJ} = \cos\theta_{iJ} \qquad i = x, y, \text{ or } z \quad \text{and} \quad J = X, Y, \text{ or } Z \tag{8.49}$$

The local moments Q_4, Q_5, and Q_6, at member end *b*, can be similarly expressed in terms of their global counterparts F_4, F_5, and F_6, as

$$\begin{bmatrix} Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ r_{yX} & r_{yY} & r_{yZ} \\ r_{zX} & r_{zY} & r_{zZ} \end{bmatrix} \begin{bmatrix} F_4 \\ F_5 \\ F_6 \end{bmatrix} \tag{8.50}$$

Similarly, the local forces and moments at member end *e* can be expressed in terms of the global forces and moments by the following relationships.

$$\begin{bmatrix} Q_7 \\ Q_8 \\ Q_9 \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ r_{yX} & r_{yY} & r_{yZ} \\ r_{zX} & r_{zY} & r_{zZ} \end{bmatrix} \begin{bmatrix} F_7 \\ F_8 \\ F_9 \end{bmatrix} \tag{8.51}$$

and

$$\begin{bmatrix} Q_{10} \\ Q_{11} \\ Q_{12} \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ r_{yX} & r_{yY} & r_{yZ} \\ r_{zX} & r_{zY} & r_{zZ} \end{bmatrix} \begin{bmatrix} F_{10} \\ F_{11} \\ F_{12} \end{bmatrix} \tag{8.52}$$

By combining Eqs. (8.48) and Eqs. (8.50) through (8.52), we can now express the transformation relationship between the 12 × 1 member local end force vector **Q** and the 12 × 1 member global end force vector **F**, in the standard form of

$$\mathbf{Q} = \mathbf{TF} \tag{8.53}$$

in which **T** represents the 12 × 12 transformation matrix for the members of space frames. The explicit form of **T** is given in Eq. (8.54).

$$\mathbf{T} = \left[\begin{array}{ccc:ccc:ccc:ccc}
r_{xX} & r_{xY} & r_{xZ} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
r_{yX} & r_{yY} & r_{yZ} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
r_{zX} & r_{zY} & r_{zZ} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & r_{xX} & r_{xY} & r_{xZ} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_{yX} & r_{yY} & r_{yZ} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_{zX} & r_{zY} & r_{zZ} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & r_{xX} & r_{xY} & r_{xZ} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & r_{yX} & r_{yY} & r_{yZ} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & r_{zX} & r_{zY} & r_{zZ} & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{xX} & r_{xY} & r_{xZ} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{yX} & r_{yY} & r_{yZ} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{zX} & r_{zY} & r_{zZ}
\end{array}\right] \tag{8.54}$$

The transformation matrix **T** is usually expressed in a compact form in terms of its submatrices as

$$\mathbf{T} = \begin{bmatrix} \mathbf{r} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{r} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{r} \end{bmatrix} \tag{8.55}$$

in which **O** represents a 3 × 3 null matrix; and the 3 × 3 matrix **r**, which is commonly referred to as the *member rotation matrix*, is given by

$$\mathbf{r} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ r_{yX} & r_{yY} & r_{yZ} \\ r_{zX} & r_{zY} & r_{zZ} \end{bmatrix} \tag{8.56}$$

The rotation matrix **r** plays a key role in the analysis of space frames, and an alternate form of this matrix, which enables us to specify the member orientations more conveniently, is developed subsequently.

Since the member local and global end displacements, **u** and **v**, are also vector quantities, which are defined in the same directions as the corresponding forces, the foregoing transformation matrix **T** can also be used to transform member end displacements from the global to the local coordinate system; that is, $\mathbf{u} = \mathbf{T}\mathbf{v}$. Furthermore, by employing a procedure similar to that used in the preceding paragraphs, it can be shown that the inverse transformations of the member end forces and end displacements, from the local to the global coordinate system, are defined by the transpose of the transformation matrix given in Eq. (8.54) (or Eqs. (8.55) and (8.56)); that is, $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ and $\mathbf{v} = \mathbf{T}^T\mathbf{u}$. Once the transformation matrix **T** has been established for a member of a space frame, its global stiffness matrix and fixed-end force vector can be obtained via the standard relationships $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$ and $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$, respectively.

Member Rotation Matrix in Terms of the Angle of Roll From Eq. (8.56), we can see that the rotation matrix **r** consists of nine elements, with each element representing the direction cosine of a local axis with respect to a global axis, in accordance with Eq. (8.49). Of these nine direction cosines, the three in the first row of **r**, which represent the direction cosines of the local x axis, can be directly evaluated using the global coordinates of the two joints to which the member ends are attached. Thus, if X_b, Y_b, and Z_b and X_e, Y_e, and Z_e denote the global coordinates of the joints to which member ends b and e, respectively, are attached, then the direction cosines of the local x axis, with respect to the global X, Y, and Z axes, respectively, can be expressed as

$$r_{xX} = \cos\theta_{xX} = \frac{X_e - X_b}{L} \tag{8.57a}$$

$$r_{xY} = \cos\theta_{xY} = \frac{Y_e - Y_b}{L} \tag{8.57b}$$

$$r_{xZ} = \cos\theta_{xZ} = \frac{Z_e - Z_b}{L} \tag{8.57c}$$

in which the member length L is given by

$$L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2 + (Z_e - Z_b)^2} \tag{8.57d}$$

With the direction cosines of the member x axis now established, we focus our attention on the question of how to determine the direction cosines of the local y and z axes using the information about the member orientation that can be conveniently input by the user of the computer program. Since the x, y, and

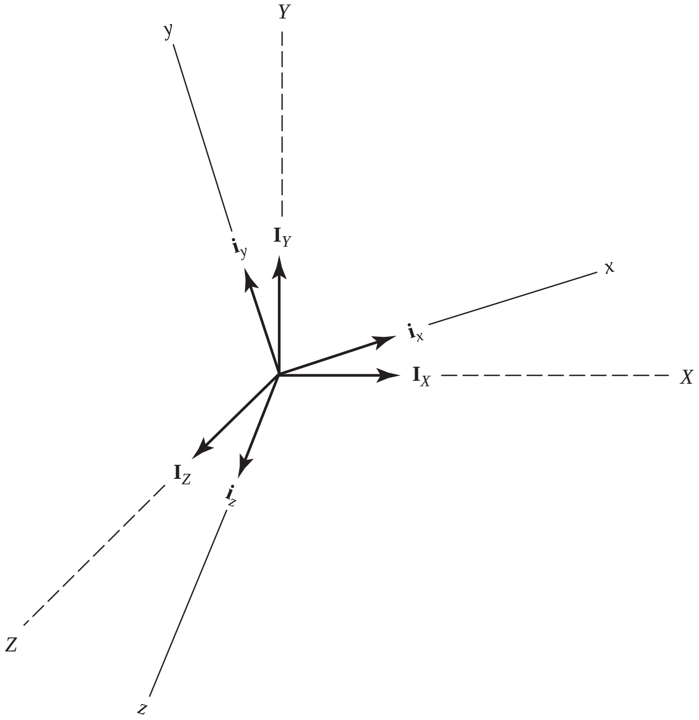


Fig. 8.17 *Unit Vectors in the Directions of the Local and Global Axes*

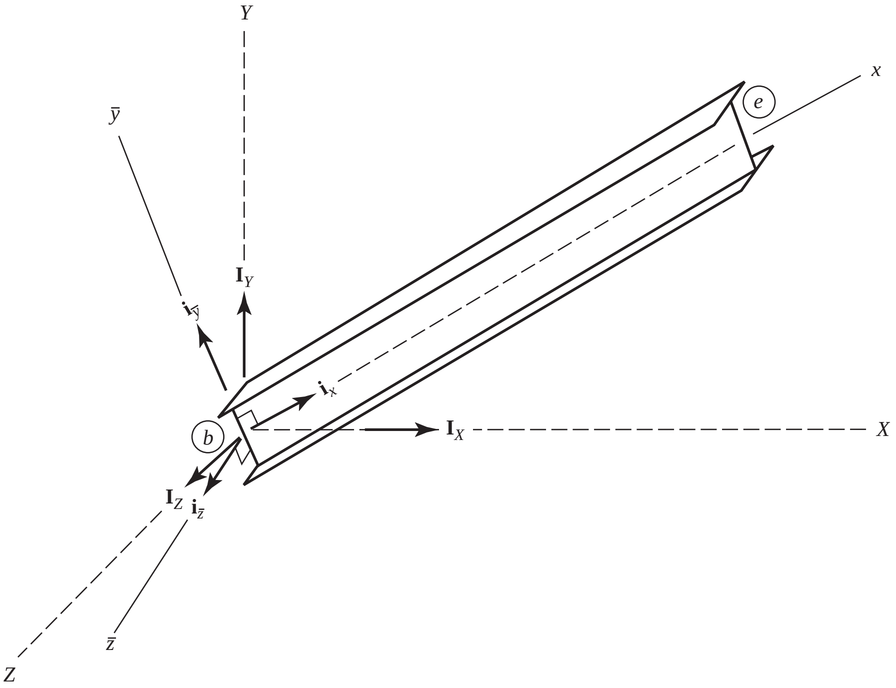
z axes form a mutually perpendicular right-handed coordinate system, it usually is convenient to define their directions by those of the unit vectors directed along these axes. Thus, if $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ denote, respectively, the unit vectors in the directions of the local x, y, and z axes, and $\mathbf{I}_X$, $\mathbf{I}_Y$, and $\mathbf{I}_Z$ denote, respectively, the unit vectors directed along the global X, Y, and Z axes (see Fig. 8.17), then the relationship between the local and global unit vectors is defined by the member rotation matrix $\mathbf{r}$, as

$$\begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ r_{yX} & r_{yY} & r_{yZ} \\ r_{zX} & r_{zY} & r_{zZ} \end{bmatrix} \begin{bmatrix} \mathbf{I}_X \\ \mathbf{I}_Y \\ \mathbf{I}_Z \end{bmatrix} \tag{8.58}$$

The reader may recall from a previous course in *statics* that if the direction cosines of two of the three unit vectors, directed along the axes of an orthogonal coordinate system, are known, then those of the third unit vector can be obtained by using a cross (or vector) product of the two known vectors. In the case under consideration, as discussed previously, the direction cosines of one of the unit vectors, $\mathbf{i}_x$, are defined by the global coordinates of the member ends (Eqs. (8.57)). Thus, if the user of the computer program can provide, as input, the direction cosines of either $\mathbf{i}_y$ or $\mathbf{i}_z$ (i.e., either the y or the z axis), then the direction cosines of the remaining third vector can be conveniently established via the cross product of the two known vectors. However, as the hand calculation of direction cosines of the *y* or *z* axis for each member of a structure can be a tedious and time-consuming chore, this approach is not considered user-friendly and is seldom used by practitioners.

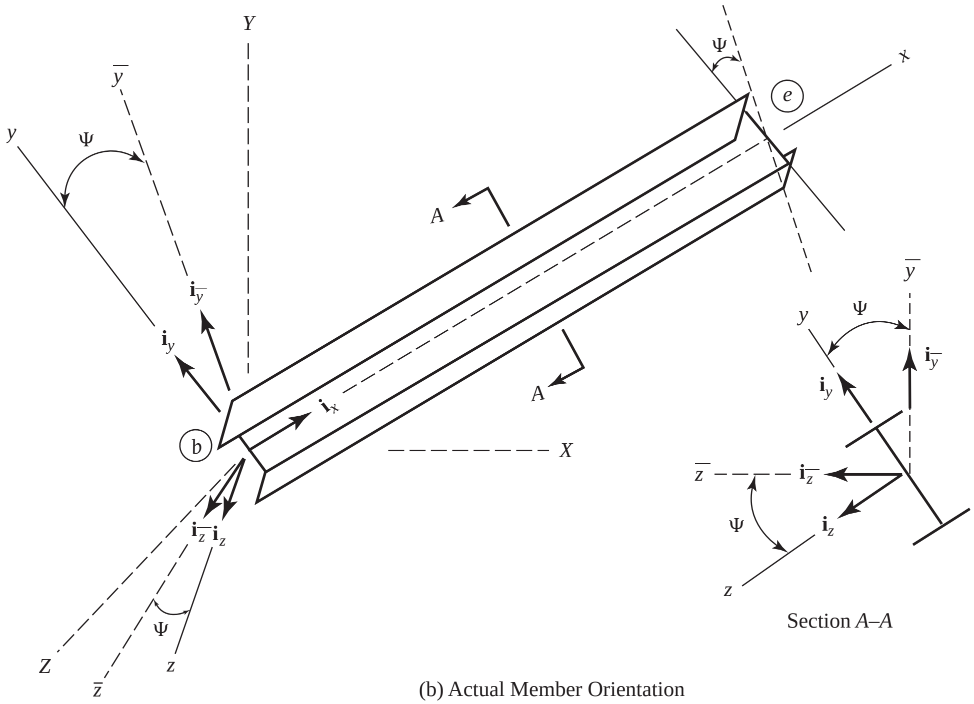
Instead, most computer programs allow the users to specify the orientation of the member *y* and *z* axes by means of the so-called *angle of roll* [3]. To define the angle of roll and to express the direction cosines of the member *y* and *z* axes in terms of this angle, we imagine that the member's desired (or actual design) orientation is reached in two steps, as shown in Figs. 8.18(a) and (b). In the first step, while the member's *x* axis is oriented in the desired direction, its *y* and *z* axes are oriented so that the *xy* plane is vertical and the *z* axis lies in a horizontal plane. The foregoing (imaginary) orientation of the member is depicted in Fig. 8.18(a), in which the member's principal axes are designated as $\bar{y}$ and $\bar{z}$ (instead of *y* and *z*, respectively), to indicate that they have not yet been positioned in their desired (or actual design) directions. As discussed previously, the direction of the local *x* axis is known from the global coordinates of the member ends. Since the $\bar{z}$ axis is perpendicular to the vertical $x\bar{y}$ plane, a vector $\bar{\mathbf{z}}$ directed along the $\bar{z}$ axis can be determined by the cross product of the vector $\mathbf{i}_x$ and a vertical unit vector $\mathbf{I}_Y$; that is,

$$\bar{\mathbf{z}} = \mathbf{i}_x \times \mathbf{I}_Y = \det \begin{vmatrix} \mathbf{I}_X & \mathbf{I}_Y & \mathbf{I}_Z \\ r_{xX} & r_{xY} & r_{xZ} \\ 0 & 1 & 0 \end{vmatrix} = -r_{xZ}\mathbf{I}_X + r_{xX}\mathbf{I}_Z \tag{8.59}$$



(a) Member Orientation with $x\bar{y}$ Plane Vertical

Fig. 8.18



(b) Actual Member Orientation

Fig. 8.18 (*continued*)

To obtain the unit vector $\mathbf{i}_{\bar{z}}$ along the local $\bar{z}$ axis, we divide the vector $\bar{\mathbf{z}}$ by its magnitude $\sqrt{r_{xX}^2 + r_{xZ}^2}$. This yields

$$\mathbf{i}_{\bar{z}} = -\frac{r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}\mathbf{I}_X + \frac{r_{xX}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}\mathbf{I}_Z \tag{8.60}$$

The unit vector $\mathbf{i}_{\bar{y}}$ can now be established by using the cross product $\mathbf{i}_{\bar{z}} \times \mathbf{i}_x$, as

$$\mathbf{i}_{\bar{y}} = \mathbf{i}_{\bar{z}} \times \mathbf{i}_x = \det\begin{vmatrix} \mathbf{I}_X & \mathbf{I}_Y & \mathbf{I}_Z \\ -\dfrac{r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & 0 & \dfrac{r_{xX}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \\ r_{xX} & r_{xY} & r_{xZ} \end{vmatrix}$$

from which we obtain

$$\mathbf{i}_{\bar{y}} = \left(-\frac{r_{xX}r_{xY}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}\right)\mathbf{I}_X + \left(\sqrt{r_{xX}^2 + r_{xZ}^2}\right)\mathbf{I}_Y - \left(\frac{r_{xY}r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}\right)\mathbf{I}_Z \tag{8.61}$$

From Eqs. (8.60) and (8.61), we can see that the transformation relationship between the global *XYZ* and the auxiliary local $x\bar{y}\bar{z}$ coordinate systems can be expressed as

$$\begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_{\bar{y}} \\ \mathbf{i}_{\bar{z}} \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ -\dfrac{r_{xX} r_{xY}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & \sqrt{r_{xX}^2 + r_{xZ}^2} & -\dfrac{r_{xY} r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \\ -\dfrac{r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & 0 & \dfrac{r_{xX}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \end{bmatrix} \begin{bmatrix} \mathbf{I}_X \\ \mathbf{I}_Y \\ \mathbf{I}_Z \end{bmatrix} \tag{8.62}$$

In the next step, we rotate the auxiliary $x\bar{y}\bar{z}$ coordinate system about its x axis, in a counterclockwise sense, by the angle of roll Ψ, until the member's principal axes are in their desired orientations. The final orientation of the member thus obtained is depicted in Fig. 8.18(b), in which the member's principal axes are now designated as y and z axes. From this figure, we can see that the unit vectors along the y and z axes can be expressed in terms of those directed along the $\bar{y}$ and $\bar{z}$ axes, as

$$\mathbf{i}_y = \cos\Psi \mathbf{i}_{\bar{y}} + \sin\Psi \mathbf{i}_{\bar{z}} \tag{8.63a}$$

$$\mathbf{i}_z = -\sin\Psi \mathbf{i}_{\bar{y}} + \cos\Psi \mathbf{i}_{\bar{z}} \tag{8.63b}$$

Thus, the transformation relationship between the auxiliary $x\bar{y}\bar{z}$ and the actual xyz coordinate systems is given by

$$\begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Psi & \sin\Psi \\ 0 & -\sin\Psi & \cos\Psi \end{bmatrix} \begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_{\bar{y}} \\ \mathbf{i}_{\bar{z}} \end{bmatrix} \tag{8.64}$$

Finally, to obtain the transformation relationship between the global XYZ and the actual local xyz coordinate systems, we substitute Eq. (8.62) into Eq. (8.64) and carry out the required matrix multiplication. This yields

$$\begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ \dfrac{-r_{xX} r_{xY} \cos\Psi - r_{xZ} \sin\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & \sqrt{r_{xX}^2 + r_{xZ}^2} \cos\Psi & \dfrac{-r_{xY} r_{xZ} \cos\Psi + r_{xX} \sin\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \\ \dfrac{r_{xX} r_{xY} \sin\Psi - r_{xZ} \cos\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & -\sqrt{r_{xX}^2 + r_{xZ}^2} \sin\Psi & \dfrac{r_{xY} r_{xZ} \sin\Psi + r_{xX} \cos\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \end{bmatrix} \begin{bmatrix} \mathbf{I}_X \\ \mathbf{I}_Y \\ \mathbf{I}_Z \end{bmatrix} \tag{8.65}$$

By comparing Eqs. (8.58) and (8.65), we can see that the member rotation matrix **r** can be expressed as

$$\mathbf{r} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ \dfrac{-r_{xX} r_{xY} \cos\Psi - r_{xZ} \sin\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & \sqrt{r_{xX}^2 + r_{xZ}^2} \cos\Psi & \dfrac{-r_{xY} r_{xZ} \cos\Psi + r_{xX} \sin\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \\ \dfrac{r_{xX} r_{xY} \sin\Psi - r_{xZ} \cos\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & -\sqrt{r_{xX}^2 + r_{xZ}^2} \sin\Psi & \dfrac{r_{xY} r_{xZ} \sin\Psi + r_{xX} \cos\Psi}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \end{bmatrix} \tag{8.66}$$

Note that the rotation matrix depends only on the global coordinates of the member ends and its angle of roll Ψ. Based on the foregoing derivation, the *angle of roll* Ψ is defined as *the angle, measured clockwise positive when looking in the negative x direction, through which the local xyz coordinate system must be rotated around its x axis, so that the xy plane becomes vertical with the y axis pointing upward (i.e., in the positive direction of the global Y axis).*

The expression of the rotation matrix **r**, as given by Eq. (8.66), can be used to determine the transformation matrices **T** for the members of space frames oriented in any arbitrary directions, except for vertical members. This is because for such members r_{xX} and r_{xZ} are zero, causing some elements of **r** in Eq. (8.66) to become undefined. This situation can be remedied by defining the angle of roll differently for vertical members, as follows. For the special case of vertical members (i.e., members with centroidal or local x axis parallel to the global Y axis), the angle of roll Ψ is defined as *the angle, measured clockwise positive when looking in the negative x direction, through which the local xyz coordinate system must be rotated around its x axis, so that the local z axis becomes parallel to, and points in the positive direction of, the global Z axis* (Fig. 8.19(b)).

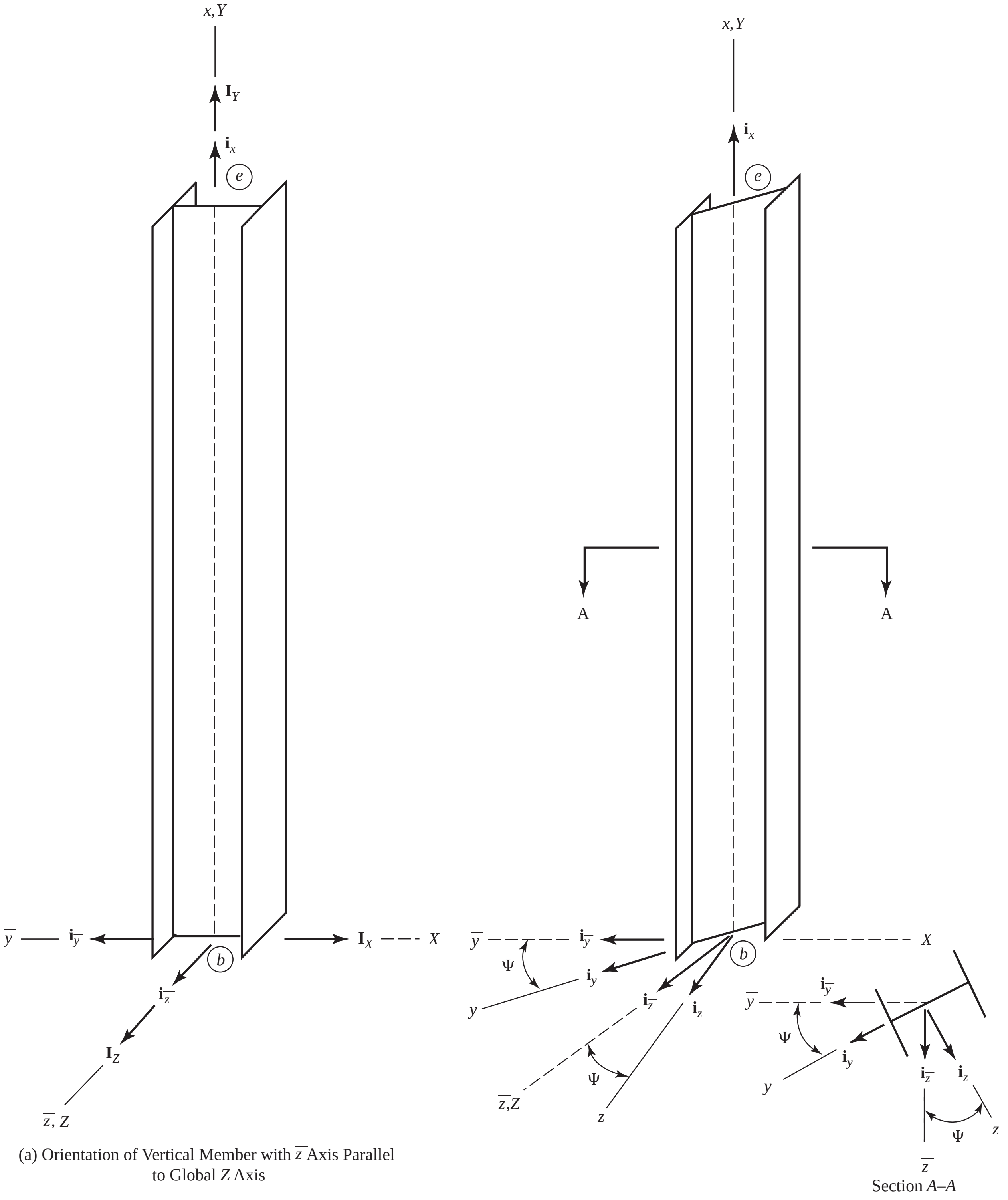
The expression of the rotation matrix **r** for vertical members can be derived using a procedure similar to that used previously for members with other orientations. We imagine that the vertical member's desired (or actual design) orientation is reached in two steps, as shown in Figs. 8.19(a) and (b) on the next page. In the first step, while the member's x axis is oriented in the desired (vertical) direction, its y and z axes are oriented so that the local z axis is parallel to the global Z axis, as shown in Fig. 8.19(a). As indicated there, the member's principal axes in this (imaginary) orientation are designated as $\bar{y}$ and $\bar{z}$ (instead of y and z, respectively). The direction of the local x axis (known from the global coordinates of the member ends) is represented by the unit vector $\mathbf{i}_x = r_{xY}\mathbf{I}_Y$, while the direction of the $\bar{z}$ axis is given by the unit vector $\mathbf{i}_{\bar{z}} = \mathbf{I}_Z$. The unit vector $\mathbf{i}_{\bar{y}}$, directed along the $\bar{y}$ axis, can therefore be conveniently established using the cross product $\mathbf{i}_{\bar{z}} \times \mathbf{i}_x$, as

$$\mathbf{i}_{\bar{y}} = \mathbf{i}_{\bar{z}} \times \mathbf{i}_x = \det \begin{vmatrix} \mathbf{I}_X & \mathbf{I}_Y & \mathbf{I}_Z \\ 0 & 0 & 1 \\ 0 & r_{xY} & 0 \end{vmatrix} = -r_{xY}\mathbf{I}_X \tag{8.67}$$

Thus, the transformation relationship between the global XYZ and the auxiliary local $x\bar{y}\bar{z}$ coordinate system is given by

$$\begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_{\bar{y}} \\ \mathbf{i}_{\bar{z}} \end{bmatrix} = \begin{bmatrix} 0 & r_{xY} & 0 \\ -r_{xY} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_X \\ \mathbf{I}_Y \\ \mathbf{I}_Z \end{bmatrix} \tag{8.68}$$

In the next step, we rotate the auxiliary $x\bar{y}\bar{z}$ coordinate system about its x axis, in a counterclockwise sense, by the angle of roll Ψ, until the member's principal axes are in their desired orientations. This final orientation of the member is depicted in Fig. 8.19(b), in which the member principal axes are



(a) Orientation of Vertical Member with $\bar{z}$ Axis Parallel to Global Z Axis

(b) Actual Orientation of Vertical Member

Fig. 8.19

now designated as the y and z axes. From this figure, we can see that the transformation relationship between the auxiliary $x\bar{y}\bar{z}$ and the actual xyz coordinate systems is the same as given previously in Eq. (8.64). Thus, the desired transformation from the global XYZ coordinate system to the local xyz coordinate system can be obtained by substituting Eq. (8.68) into Eq. (8.64) and performing the required matrix multiplication. This yields

$$\begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{bmatrix} = \begin{bmatrix} 0 & r_{xY} & 0 \\ -r_{xY}\cos\Psi & 0 & \sin\Psi \\ r_{xY}\sin\Psi & 0 & \cos\Psi \end{bmatrix} \begin{bmatrix} \mathbf{I}_X \\ \mathbf{I}_Y \\ \mathbf{I}_Z \end{bmatrix} \tag{8.69}$$

from which we obtain the rotation matrix $\mathbf{r}$ for vertical members:

$$\mathbf{r} = \begin{bmatrix} 0 & r_{xY} & 0 \\ -r_{xY}\cos\Psi & 0 & \sin\Psi \\ r_{xY}\sin\Psi & 0 & \cos\Psi \end{bmatrix} \tag{8.70}$$

Member Rotation Matrix in Terms of a Reference Point In most space frames, members are usually oriented so that their angles of roll can be found by inspection. There are structures, however, in which the orientations of some members may be such that their angles of roll cannot be conveniently determined. The orientation of such a member can alternatively be specified by means of the global coordinates of a reference point that lies in one of the principal (xy or xz) planes of the member, but not on its centroidal (x) axis.

To discuss the process of determining the member rotation matrix $\mathbf{r}$ using such a reference point, consider the space-frame member shown in Fig. 8.20 on the next page, and let X_P, Y_P, and Z_P denote the global coordinates of an arbitrarily chosen reference point P, which is located in the member's local xy plane, but not on its x axis. Since the global coordinates of the member end b are X_b, Y_b, and Z_b, the position vector $\mathbf{p}$, directed from member end b to reference point P, can be written as

$$\mathbf{p} = (X_P - X_b)\mathbf{I}_X + (Y_P - Y_b)\mathbf{I}_Y + (Z_P - Z_b)\mathbf{I}_z \tag{8.71}$$

Note that both points b and P are located in the local xy plane and, therefore, vector $\mathbf{p}$ also lies in that plane. Since the direction cosines of the local x axis are already known from the global coordinates of the member ends, the direction cosines of the local z axis can be conveniently established using the following relationship.

$$\mathbf{i}_z = \frac{\mathbf{i}_x \times \mathbf{p}}{|\mathbf{i}_x \times \mathbf{p}|} \tag{8.72}$$

in which $|\mathbf{i}_x \times \mathbf{p}|$ represents the magnitude of the vector that results from the cross product of the vectors $\mathbf{i}_x$ and $\mathbf{p}$. With both $\mathbf{i}_x$ and $\mathbf{i}_z$ now known, the direction cosines of the local y axis can be obtained via the cross product,

$$\mathbf{i}_y = \mathbf{i}_z \times \mathbf{i}_x \tag{8.73}$$

In the case that the reference point P is specified in the local xz plane of the member, the direction cosines of the local y axis need to be determined first

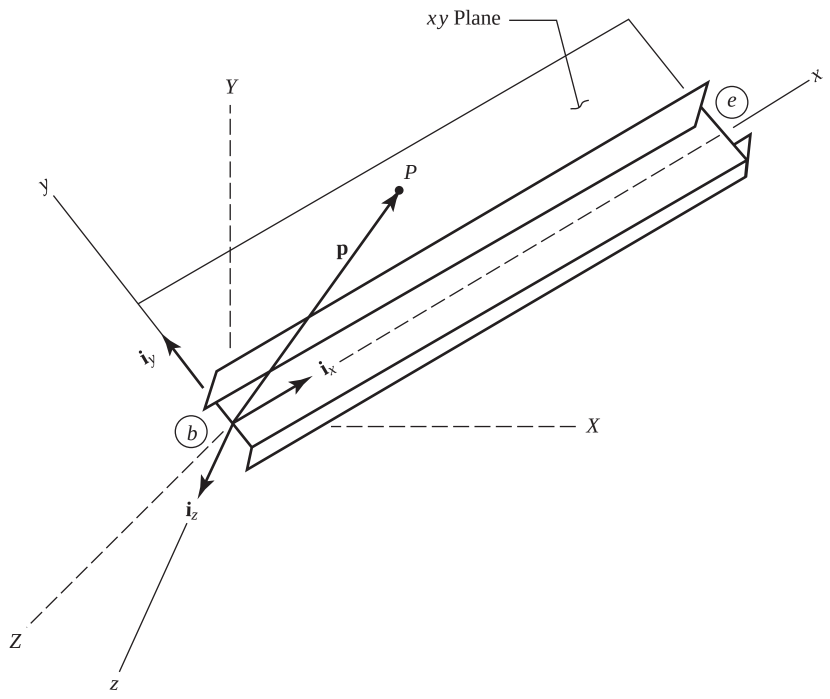


Fig. 8.20

using the relationship

$$\mathbf{i}_y = \frac{\mathbf{p} \times \mathbf{i}_x}{|\mathbf{p} \times \mathbf{i}_x|} \tag{8.74}$$

and then the direction cosines of the local z axis are obtained via the cross product

$$\mathbf{i}_z = \mathbf{i}_x \times \mathbf{i}_y \tag{8.75}$$

It should be realized that the procedure described by Eqs. (8.71) through (8.75) enables us to obtain the member rotation matrix **r** directly by means of a reference point, without involving the angle of roll of the member. However, if desired, the angle of roll can also be obtained from the global coordinates of a reference point. To establish the relationship between the angle of roll Ψ and a reference point P of a member, we first determine the components of the position vector **p** in the auxiliary $x\bar{y}\bar{z}$ coordinate system, by applying the transformation relationship given in Eq. (8.62), as

$$\begin{bmatrix} p_x \\ p_{\bar{y}} \\ p_{\bar{z}} \end{bmatrix} = \begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ -\dfrac{r_{xX}r_{xY}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & \sqrt{r_{xX}^2 + r_{xZ}^2} & -\dfrac{r_{xY}r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \\ -\dfrac{r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} & 0 & \dfrac{r_{xX}}{\sqrt{r_{xX}^2 + r_{xZ}^2}} \end{bmatrix} \begin{bmatrix} (X_P - X_b) \\ (Y_P - Y_b) \\ (Z_P - Z_b) \end{bmatrix}$$

from which we obtain

$$p_x = r_{xX}(X_P - X_b) + r_{xY}(Y_P - Y_b) + r_{xZ}(Z_P - Z_b) \tag{8.76a}$$

$$p_{\bar{y}} = -\frac{r_{xX}r_{xY}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}(X_P - X_b) + \sqrt{r_{xX}^2 + r_{xZ}^2}(Y_P - Y_b) - \frac{r_{xY}r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}(Z_P - Z_b) \tag{8.76b}$$

$$p_{\bar{z}} = -\frac{r_{xZ}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}(X_P - X_b) + \frac{r_{xX}}{\sqrt{r_{xX}^2 + r_{xZ}^2}}(Z_P - Z_b) \tag{8.76c}$$

in which p_x, $p_{\bar{y}}$, and $p_{\bar{z}}$ represent, respectively, the components of the position vector **p** in the directions of the local x axis and the $\bar{y}$ and $\bar{z}$ axes of the auxiliary $x\bar{y}\bar{z}$ coordinate system. Now, if the reference point P lies in the xy plane of the member as shown in Fig. 8.21, then we can see from this figure that the angle of roll Ψ and the components of **p** are related by the following equations:

$$\sin\Psi = \frac{p_{\bar{z}}}{\sqrt{p_{\bar{y}}^2 + p_{\bar{z}}^2}} \qquad \text{and} \qquad \cos\Psi = \frac{p_{\bar{y}}}{\sqrt{p_{\bar{y}}^2 + p_{\bar{z}}^2}} \tag{8.77}$$

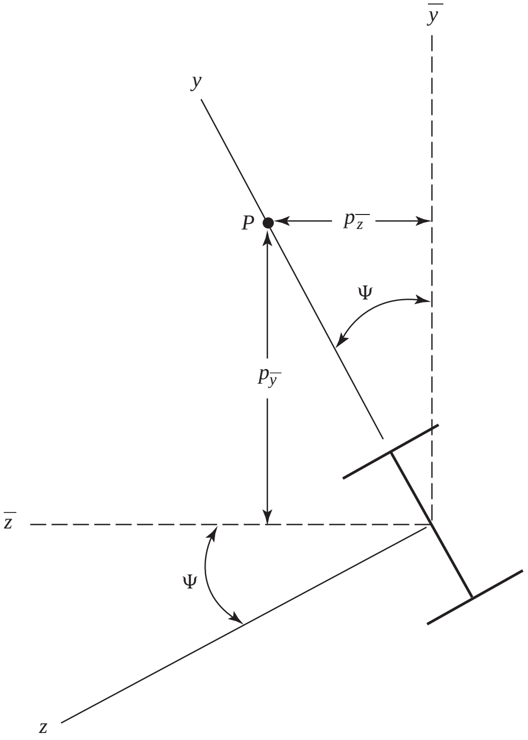


Fig. 8.21

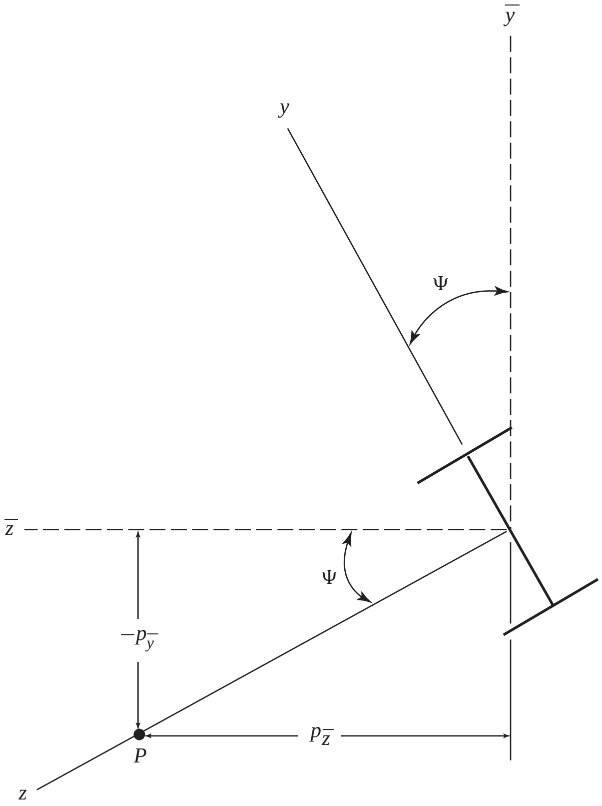


Fig. 8.22

In the case that the reference point P is specified in the local xz plane, then from Fig. 8.22 we can see that the relationships between the sine and cosine of Ψ and the components **p** can be expressed as

$$\sin\Psi = -\frac{p_{\bar{y}}}{\sqrt{p_{\bar{y}}^2 + p_{\bar{z}}^2}} \qquad \text{and} \qquad \cos\Psi = \frac{p_{\bar{z}}}{\sqrt{p_{\bar{y}}^2 + p_{\bar{z}}^2}} \tag{8.78}$$

Equations (8.77) and (8.78) are valid for space-frame members oriented in any arbitrary directions, including vertical members. However, since $r_{xX} = r_{xZ} = 0$ for vertical members, the expressions for $p_{\bar{y}}$ and $p_{\bar{z}}$, as given in Eqs. 8.76(b) and (c), cannot be used; appropriate expressions for the components of the position vector **p** in the auxiliary $x\bar{y}\bar{z}$ coordinate system must be derived by applying Eq. (8.68), as

$$\begin{bmatrix} p_x \\ p_{\bar{y}} \\ p_{\bar{z}} \end{bmatrix} = \begin{bmatrix} 0 & r_{xY} & 0 \\ -r_{xY} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (X_P - X_b) \\ (Y_P - Y_b) \\ (Z_P - Z_b) \end{bmatrix}$$

which yields

$$p_x = r_{xY}\,(Y_P - Y_b) \tag{8.79a}$$

$$p_{\bar{y}} = -r_{xY}(X_P - X_b) \quad \textbf{(8.79b)}$$
$$p_{\bar{z}} = Z_P - Z_b \quad \textbf{(8.79c)}$$

It is important to realize that, for vertical members, Eqs. 8.79(b) and (c) should be used to evaluate $p_{\bar{y}}$ and $p_{\bar{z}}$, whereas for members with other orientations, these components are obtained from Eqs. 8.76(b) and (c). After $p_{\bar{y}}$ and $p_{\bar{z}}$ have been evaluated, the sine and cosine of the member's angle of roll Ψ can be determined either by Eq. (8.77) if the reference point lies in the *xy* plane, or via Eq. (8.78) if the reference point is located in the *xz* plane. Once sin Ψ and cos Ψ are known, the member rotation matrix **r** can be determined by Eq. (8.66) if the member is not oriented in the vertical direction, or via Eq. (8.70) if the member is vertical.

EXAMPLE 8.3

The global coordinates of the joints to which the beginning and end of a space-frame member are attached are (4, 7, 6) ft and (20, 15, 17) ft, respectively. If the global coordinates of a reference point located in the local *xy* plane of the member are (10.75, 13.6, 13.85) ft, determine the rotation matrix of the member.

SOLUTION We determine the member rotation matrix **r** using the direct approach involving cross products of vectors, and then check our results using the angle-of-roll approach.

Using the given coordinates of the two ends of the member, we evaluate its length and the direction cosines of the local *x* axis, as (see Eqs. (8.57))

$$L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2 + (Z_b - Z_e)^2}$$
$$= \sqrt{(20-4)^2 + (15-7)^2 + (17-6)^2} = 21 \text{ ft}$$

$$r_{xX} = \frac{X_e - X_b}{L} = \frac{20-4}{21} = 0.7619 \quad \textbf{(1a)}$$

$$r_{xY} = \frac{Y_e - Y_b}{L} = \frac{15-7}{21} = 0.38095 \quad \textbf{(1b)}$$

$$r_{xZ} = \frac{Z_e - Z_b}{L} = \frac{17-6}{21} = 0.52381 \quad \textbf{(1c)}$$

Thus, the unit vector directed along the member local *x* (or centroidal) axis is

$$\mathbf{i}_x = 0.7619\ \mathbf{I}_X + 0.38095\ \mathbf{I}_Y + 0.52381\ \mathbf{I}_Z \quad \textbf{(2)}$$

Next, we form the position vector **p**, directed from member end *b* to reference point *P*, as (Eq. (8.71))

$$\mathbf{p} = (X_P - X_b)\mathbf{I}_X + (Y_P - Y_b)\mathbf{I}_Y + (Z_P - Z_b)\mathbf{I}_Z$$
$$= (10.75 - 4)\mathbf{I}_X + (13.6 - 7)\mathbf{I}_Y + (13.85 - 6)\mathbf{I}_Z$$

or

$$\mathbf{p} = 6.75\ \mathbf{I}_X + 6.6\ \mathbf{I}_Y + 7.85\ \mathbf{I}_Z$$

With $\mathbf{i}_x$ and **p** known, we can now apply Eq. (8.72) to determine the unit vector in the local *z* direction. For that purpose, we first obtain a vector **z** along the local *z* axis by evaluating the cross product of $\mathbf{i}_x$ and $\mathbf{p}$. Thus,

$$\mathbf{z} = \mathbf{i}_x \times \mathbf{p} = \det \begin{vmatrix} \mathbf{I}_X & \mathbf{I}_Y & \mathbf{I}_Z \\ 0.7619 & 0.38095 & 0.52381 \\ 6.75 & 6.6 & 7.85 \end{vmatrix}$$

$$= [(0.38095)(7.85) - (6.6)(0.52381)]\,\mathbf{I}_X$$
$$- [(0.7619)(7.85) - (6.75)(0.52381)]\,\mathbf{I}_Y$$
$$+ [(0.7619)(6.6) - (6.75)(0.38095)]\,\mathbf{I}_Z$$

or

$$\mathbf{z} = -0.46669\,\mathbf{I}_X - 2.4452\,\mathbf{I}_Y + 2.4571\,\mathbf{I}_Z$$

Note that $\mathbf{z}$ is not a unit vector. To obtain the unit vector $\mathbf{i}_z$, we need to divide $\mathbf{z}$ by its magnitude $|\mathbf{z}|$, which equals

$$|\mathbf{z}| = |\mathbf{i}_x \times \mathbf{p}| = \sqrt{(-0.46669)^2 + (-2.4452)^2 + (2.4571)^2} = 3.4977 \text{ ft}$$

Thus, the unit vector $\mathbf{i}_z$ is given by

$$\mathbf{i}_z = \frac{\mathbf{z}}{|\mathbf{z}|} = -0.13343\,\mathbf{I}_X - 0.69909\,\mathbf{I}_Y + 0.70249\,\mathbf{I}_Z \qquad \textbf{(3)}$$

The third unit vector, $\mathbf{i}_y$, can now be evaluated using the cross product of $\mathbf{i}_z$ (Eq. (3)) and $\mathbf{i}_x$ (Eq. (2)). Thus,

$$\mathbf{i}_y = \mathbf{i}_z \times \mathbf{i}_x = \det \begin{vmatrix} \mathbf{I}_X & \mathbf{I}_Y & \mathbf{I}_Z \\ -0.13343 & -0.69909 & 0.70249 \\ 0.7619 & 0.38095 & 0.52381 \end{vmatrix}$$

$$= [(-0.69909)(0.52381) - (0.38095)(0.70249)]\,\mathbf{I}_X$$
$$- [(-0.13343)(0.52381) - (0.7619)(0.70249)]\,\mathbf{I}_Y$$
$$+ [(-0.13343)(0.38095) - (0.7619)(-0.69909)]\,\mathbf{I}_Z$$

or

$$\mathbf{i}_y = -0.6338\,\mathbf{I}_X + 0.60512\,\mathbf{I}_Y + 0.48181\,\mathbf{I}_Z \qquad \textbf{(4)}$$

The member rotation matrix $\mathbf{r}$ can now be obtained by arranging the components of $\mathbf{i}_x$ (Eq. (2)), $\mathbf{i}_y$ (Eq. (4)) and $\mathbf{i}_z$ (Eq. (3)) in the first, second, and third rows, respectively, of a 3 × 3 matrix. The member rotation matrix thus obtained is

$$\mathbf{r} = \begin{bmatrix} 0.7619 & 0.38095 & 0.52381 \\ -0.6338 & 0.60512 & 0.48181 \\ -0.13343 & -0.69909 & 0.70249 \end{bmatrix} \qquad \textbf{Ans}$$

Alternative Method: The member rotation matrix $\mathbf{r}$ can alternatively be determined by applying Eq. (8.66), which contains the sine and cosine of the angle of roll Ψ. We first evaluate the components $p_{\bar{y}}$ and $p_{\bar{z}}$ of the position vector $\mathbf{p}$ using Eqs. 8.76(b) and (c), respectively. By substituting the numerical values of r_{xX}, r_{xY}, and r_{xZ} (from Eqs. (1)) and the given coordinates of member end b and reference point P into these equations, we obtain

$$p_{\bar{y}} = 2.2892 \text{ ft}$$
$$p_{\bar{z}} = 2.6446 \text{ ft}$$

By substituting these values of $p_{\bar{y}}$ and $p_{\bar{z}}$ into Eqs. (8.77), we obtain the sine and cosine of the angle of roll:

$$\sin\Psi = 0.75608 \qquad \cos\Psi = 0.65448 \tag{5}$$

Finally, by substituting the numerical values from Eqs. (1) and (5) into Eq. (8.66), we obtain the following rotation matrix for the member under consideration:

$$\mathbf{r} = \begin{bmatrix} 0.7619 & 0.38095 & 0.52381 \\ -0.6338 & 0.60512 & 0.48179 \\ -0.13343 & -0.69907 & 0.70249 \end{bmatrix} \qquad \textbf{Checks}$$

Procedure for Analysis

The general procedure for analysis of space frames remains the same as that for plane frames developed in Chapter 6 (and modified in Chapter 7)—provided that the member local stiffness and transformation matrices, and local fixed-end force vectors, developed in this section, are used in the analysis.

EXAMPLE 8.4 Determine the joint displacements, member end forces, and support reactions for the three-member space frame shown in Fig. 8.23(a) on the next page, using the matrix stiffness method.

SOLUTION *Analytical Model:* The space frame has six degrees of freedom and 18 restrained coordinates, as shown in Fig. 8.23(b).

Structure Stiffness Matrix:

Member 1 By substituting $L = 240$ in., and the material and cross-sectional properties given in Fig. 8.23(a), into Eq. (8.39), we obtain the local stiffness matrix $\mathbf{k}$ for member 1:

$$\mathbf{K}_1 = \mathbf{k}_1 = \begin{array}{cc} \begin{array}{cccccccccccc} 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \left[\begin{array}{cccccc|cccccc} 3{,}975.4 & 0 & 0 & 0 & 0 & 0 & -3{,}975.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 18.024 & 0 & 0 & 0 & 2{,}162.9 & 0 & -18.024 & 0 & 0 & 0 & 2{,}162.9 \\ 0 & 0 & 5.941 & 0 & -712.92 & 0 & 0 & 0 & -5.941 & 0 & -712.92 & 0 \\ 0 & 0 & 0 & 723.54 & 0 & 0 & 0 & 0 & 0 & -723.54 & 0 & 0 \\ 0 & 0 & -712.92 & 0 & 114{,}067 & 0 & 0 & 0 & 712.92 & 0 & 57{,}033 & 0 \\ 0 & 2{,}162.9 & 0 & 0 & 0 & 346{,}067 & 0 & -2{,}162.9 & 0 & 0 & 0 & 173{,}033 \\ \hline -3{,}975.4 & 0 & 0 & 0 & 0 & 0 & 3{,}975.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -18.024 & 0 & 0 & 0 & -2{,}162.9 & 0 & 18.024 & 0 & 0 & 0 & -2{,}162.9 \\ 0 & 0 & -5.941 & 0 & 712.92 & 0 & 0 & 0 & 5.941 & 0 & 712.92 & 0 \\ 0 & 0 & 0 & -723.54 & 0 & 0 & 0 & 0 & 0 & 723.54 & 0 & 0 \\ 0 & 0 & -712.92 & 0 & 57{,}033 & 0 & 0 & 0 & 712.92 & 0 & 114{,}067 & 0 \\ 0 & 2{,}162.9 & 0 & 0 & 0 & 173{,}033 & 0 & -2{,}162.9 & 0 & 0 & 0 & 346{,}067 \end{array}\right] & \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array} \tag{1}$$

Since the member's local x, y, and z axes are oriented in the directions of the global X, Y, and Z axes, respectively (see Fig. 8.23(a)), no coordinate transformations are necessary (i.e., $\mathbf{T}_1 = \mathbf{I}$); thus, $\mathbf{K}_1 = \mathbf{k}_1$.

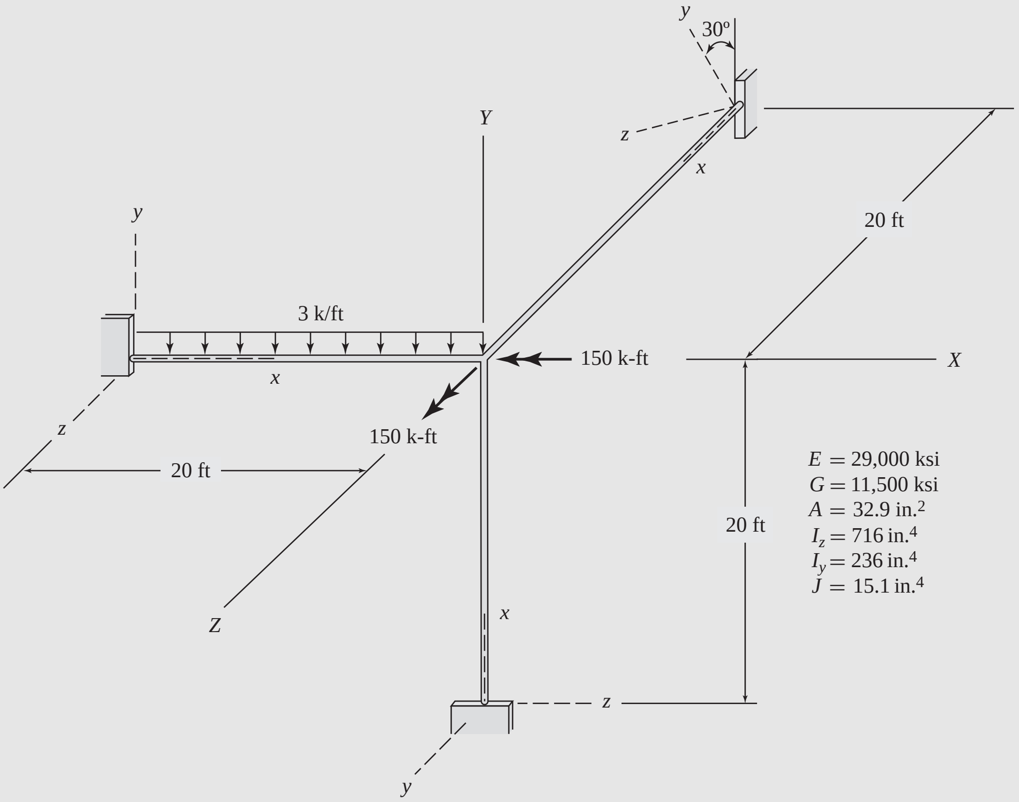
To determine the fixed-end force vector due to the 0.25 k/in. (= 3 k/ft) member load, we apply the fixed-end force expressions for loading type 3 given inside the front cover. This yields

$$FS_{by} = FS_{ey} = 30\text{ k}$$

$$FM_{bz} = -FM_{ez} = -1{,}200\text{ k-in.}$$

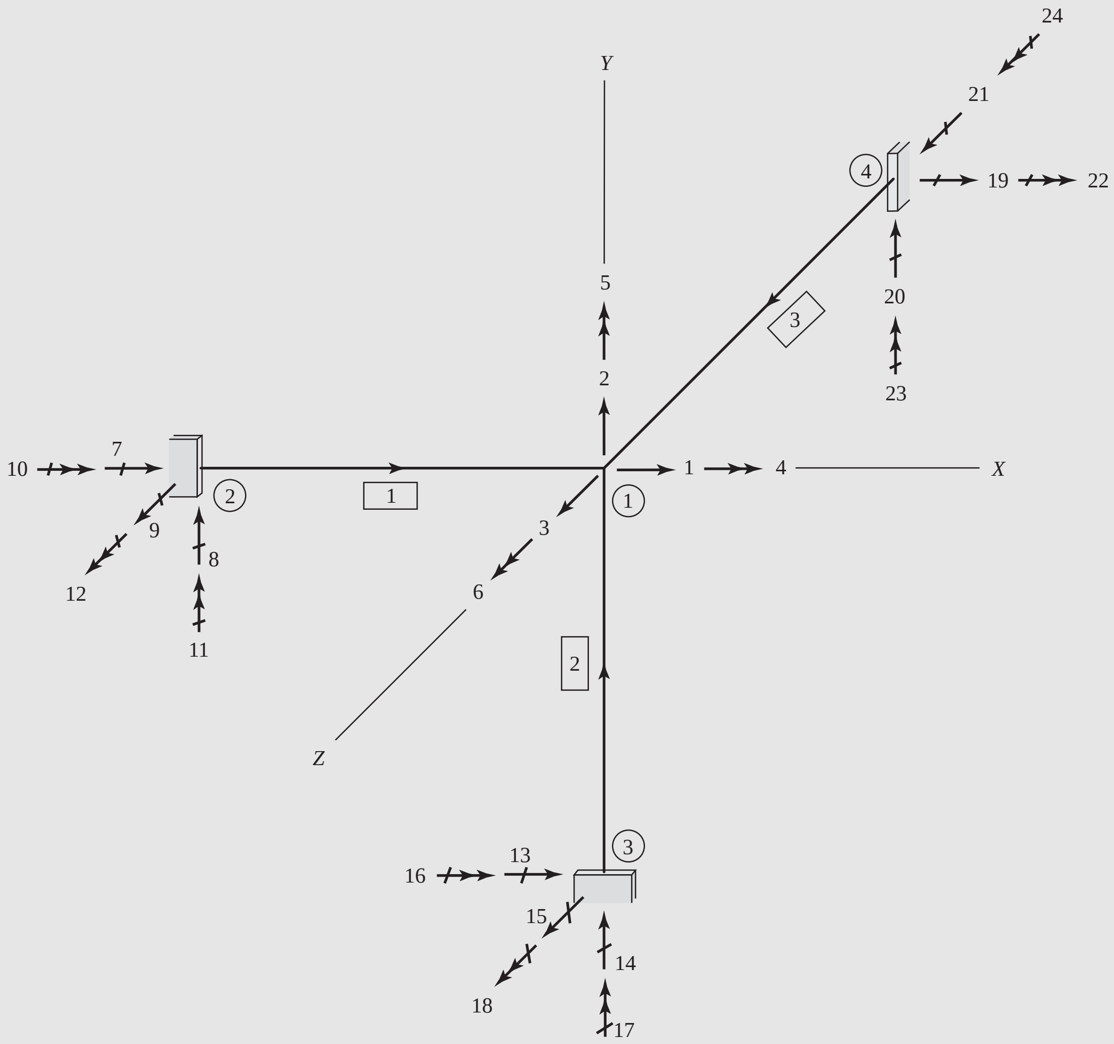
with the remaining fixed-end forces being 0. Thus, using Eq. (8.40), we obtain

$$\mathbf{F}_{f1} = \mathbf{Q}_{f1} = \left[\begin{array}{r} 0 \\ 30 \\ 0 \\ 0 \\ 0 \\ 1{,}200 \\ \hdashline 0 \\ 30 \\ 0 \\ 0 \\ 0 \\ -1{,}200 \end{array}\right]\begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \qquad \textbf{(2)}$$



(a) Space Frame

Fig. 8.23

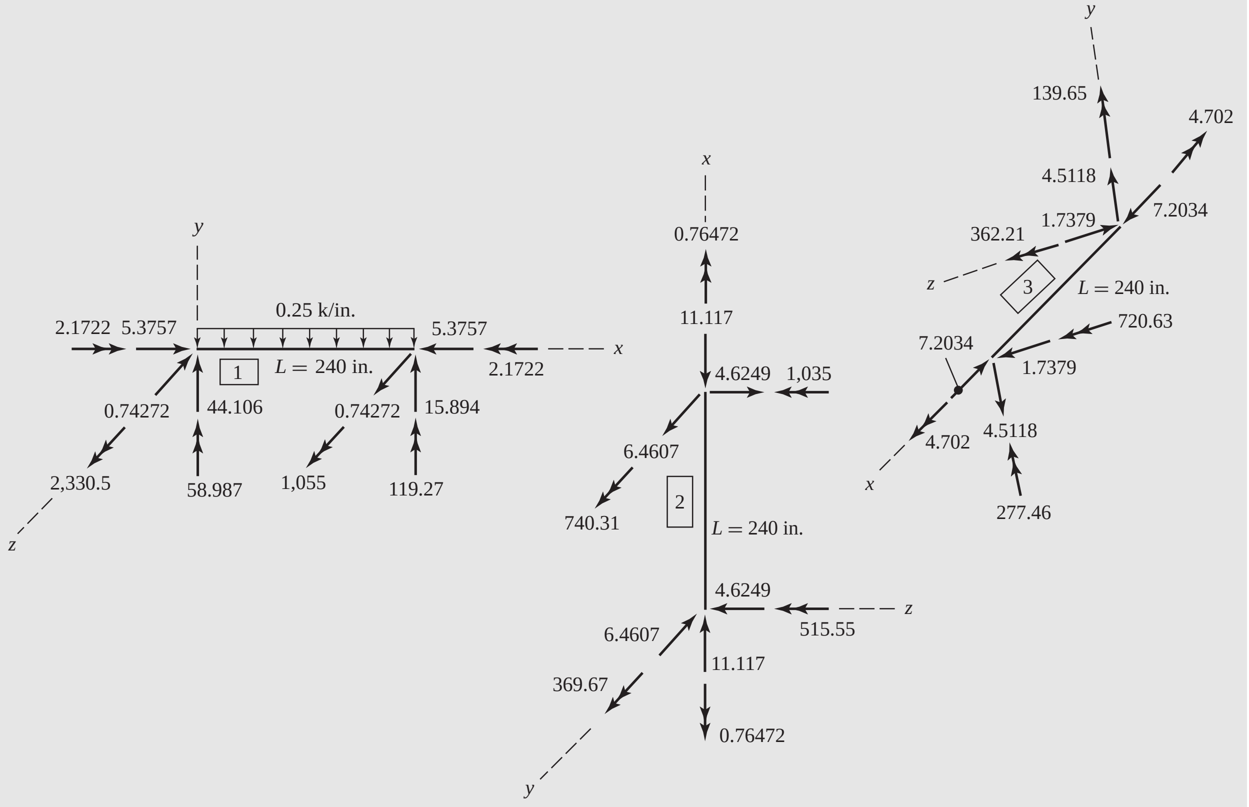


(b) Analytical Model

$$\mathbf{S} = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 3{,}990.3 & -5.2322 & 0 & -627.87 & -1{,}075.4 & 712.92 \\ -5.2322 & 4{,}008.4 & 0 & 1{,}800.4 & 627.87 & -2{,}162.9 \\ 0 & 0 & 3{,}999.4 & -2{,}162.9 & 712.92 & 0 \\ -627.87 & 1{,}800.4 & -2{,}162.9 & 634{,}857 & 100{,}459 & 0 \\ -1{,}075.4 & 627.87 & 712.92 & 100{,}459 & 286{,}857 & 0 \\ 712.92 & -2{,}162.9 & 0 & 0 & 0 & 460{,}857 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array} \qquad \mathbf{P}_f = \begin{array}{cc} \begin{bmatrix} 0 \\ 30 \\ 0 \\ 0 \\ 0 \\ -1{,}200 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

(c) Structure Stiffness Matrix and Fixed-Joint Force Vector

Fig. 8.23 (*continued*)

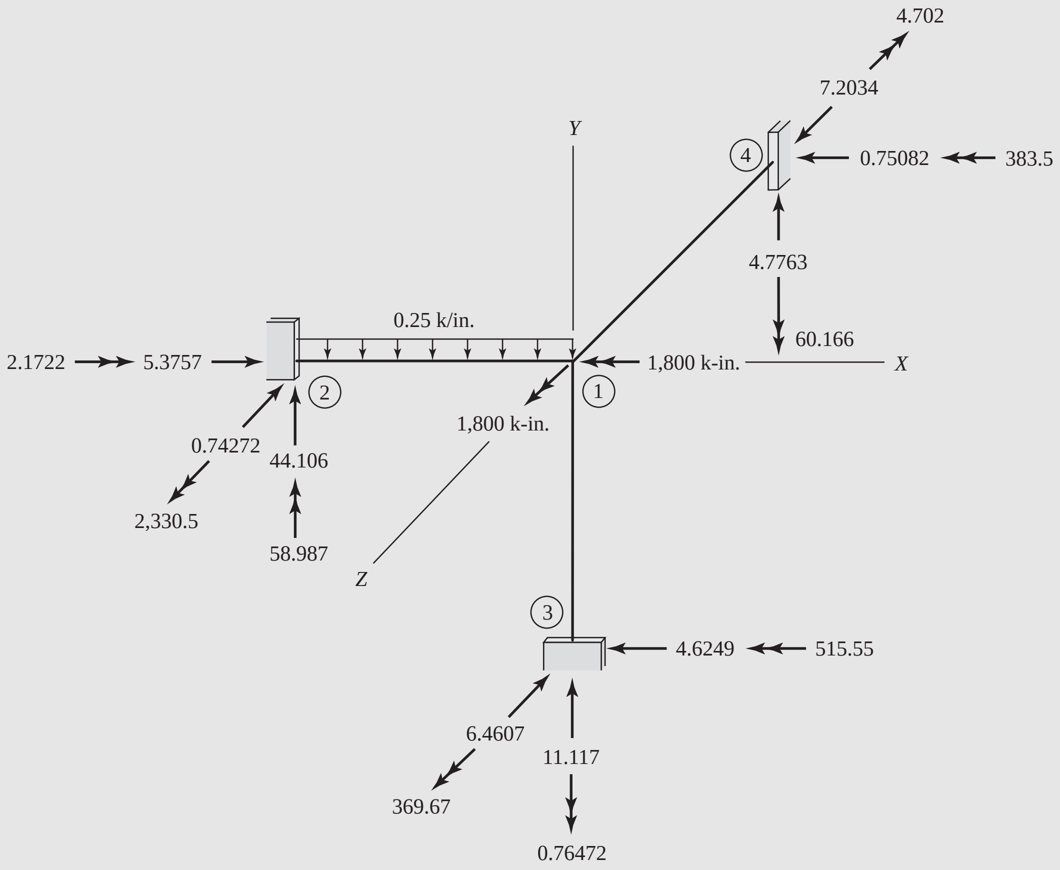


(d) Member Local End Forces

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} 5.3757 \text{ k} \\ 44.106 \text{ k} \\ -0.74272 \text{ k} \\ 2.1722 \text{ k-in.} \\ 58.987 \text{ k-in.} \\ 2{,}330.5 \text{ k-in.} \\ -4.6249 \text{ k} \\ 11.117 \text{ k} \\ -6.4607 \text{ k} \\ -515.55 \text{ k-in.} \\ -0.76472 \text{ k-in.} \\ 369.67 \text{ k-in.} \\ -0.75082 \text{ k} \\ 4.7763 \text{ k} \\ 7.2034 \text{ k} \\ -383.5 \text{ k-in.} \\ -60.166 \text{ k-in.} \\ -4.702 \text{ k-in.} \end{array}\right] \end{array} \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \end{array}$$

(e) Support Reaction Vector

Fig. 8.23 (*continued*)



(f) Support Reactions

Fig. 8.23 (*continued*)

From Fig. 8.23(b), we observe that the code numbers for member 1 are 7, 8, 9, 10, 11, 12, 1, 2, 3, 4, 5, 6. Using these code numbers, we store the pertinent elements of $\mathbf{K}_1$ (Eq. (1)) and $\mathbf{F}_{f1}$ (Eq. (2)) in the 6 × 6 structure stiffness matrix **S** and the 6 × 1 structure fixed-joint force vector $\mathbf{P}_f$, respectively (Fig. 8.23(c)).

Member 2 Because the length, as well as the material and cross-sectional properties, of member 2 are identical to those of member 1, $\mathbf{k}_2 = \mathbf{k}_1$ (Eq. (1)).

To obtain the transformation matrix **T** for member 2, we first determine the direction cosines of its local *x* axis using Eqs. (8.57), as

$$r_{xX} = \frac{X_e - X_b}{L} = 0$$

$$r_{xY} = \frac{Y_e - Y_b}{L} = \frac{0 - (-20)}{20} = 1$$

$$r_{xZ} = \frac{Z_e - Z_b}{L} = 0$$

From Fig. 8.23(a), we can see that the angle of roll Ψ for this vertical member is 90°. Thus,

$$\cos\Psi = 0 \qquad \text{and} \qquad \sin\Psi = 1$$

By substituting the foregoing numerical values of r_{xX}, r_{xY}, r_{xZ}, cos Ψ, and sin Ψ into Eq. (8.70), we determine the rotation matrix $\mathbf{r}$ for member 2 to be

$$\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

By substituting this rotation matrix into Eq. (8.55), we obtain the following 12 × 12 transformation matrix for member 2.

$$\mathbf{T}_2 = \left[\begin{array}{ccc|ccc|ccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right] \qquad \textbf{(3)}$$

The global stiffness matrix for member 2 can now be evaluated by substituting $\mathbf{k}_2$ (from Eq. (1)) and $\mathbf{T}_2$ (Eq. (3)) into the relationship $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$, and performing the necessary matrix multiplications. This yields

$$\mathbf{K}_2 = \begin{array}{cc}
\begin{array}{cccccccccccc}
13 & 14 & 15 & 16 & 17 & 18 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} & \\
\left[\begin{array}{cccccc|cccccc}
5.941 & 0 & 0 & 0 & 0 & -712.92 & -5.941 & 0 & 0 & 0 & 0 & -712.92 \\
0 & 3{,}975.4 & 0 & 0 & 0 & 0 & 0 & -3{,}975.4 & 0 & 0 & 0 & 0 \\
0 & 0 & 18.024 & 2{,}162.9 & 0 & 0 & 0 & 0 & -18.024 & 2{,}162.9 & 0 & 0 \\
0 & 0 & 2{,}162.9 & 346{,}067 & 0 & 0 & 0 & 0 & -2{,}162.9 & 173{,}033 & 0 & 0 \\
0 & 0 & 0 & 0 & 723.54 & 0 & 0 & 0 & 0 & 0 & -723.54 & 0 \\
-712.92 & 0 & 0 & 0 & 0 & 114{,}067 & 712.92 & 0 & 0 & 0 & 0 & 57{,}033 \\
\hline
-5.941 & 0 & 0 & 0 & 0 & 712.92 & 5.941 & 0 & 0 & 0 & 0 & 712.92 \\
0 & -3{,}975.4 & 0 & 0 & 0 & 0 & 0 & 3{,}975.4 & 0 & 0 & 0 & 0 \\
0 & 0 & -18.024 & -2{,}162.9 & 0 & 0 & 0 & 0 & 18.024 & -2{,}162.9 & 0 & 0 \\
0 & 0 & 2{,}162.9 & 173{,}033 & 0 & 0 & 0 & 0 & -2{,}162.9 & 346{,}067 & 0 & 0 \\
0 & 0 & 0 & 0 & -723.54 & 0 & 0 & 0 & 0 & 0 & 723.54 & 0 \\
-712.92 & 0 & 0 & 0 & 0 & 57{,}033 & 712.92 & 0 & 0 & 0 & 0 & 114{,}067
\end{array}\right] &
\begin{array}{c} 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}$$

The relevant elements of $\mathbf{K}_2$ are stored in $\mathbf{S}$ (Fig. 8.23(c)).

Member 3 $\mathbf{k}_3 = \mathbf{k}_1$ (given in Eq. (1)).

$$r_{xX} = \frac{X_e - X_b}{L} = 0$$

$$r_{xY} = \frac{Y_e - Y_b}{L} = 0$$

$$r_{xZ} = \frac{Z_e - Z_b}{L} = \frac{0 - (-20)}{20} = 1$$

From Fig. 8.23(a), we can see that $\Psi = 30^\circ$. Thus,

$$\cos\Psi = 0.86603 \qquad \text{and} \qquad \sin\Psi = 0.5$$

By applying Eq. (8.66), we determine the rotation matrix for member 3 to be

$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 1 \\ -0.5 & 0.86603 & 0 \\ -0.86603 & -0.5 & 0 \end{bmatrix}$$

Thus, the transformation matrix for this member is given by

$$\mathbf{T}_3 = \left[\begin{array}{ccc|ccc|ccc|ccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & 0.86603 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.86603 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5 & 0.86603 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.86603 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0.86603 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.86603 & -0.5 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0.86603 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.86603 & -0.5 & 0
\end{array}\right] \tag{4}$$

and the member global stiffness matrix $\mathbf{K}_3 = \mathbf{T}_3^T\mathbf{k}_3\mathbf{T}_3$ is

$$\mathbf{K}_3 = \begin{array}{cc}
\begin{array}{cccccccccccc} 19 & 20 & 21 & 22 & 23 & 24 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\left[\begin{array}{cccccc|cccccc}
8.9618 & -5.2322 & 0 & 627.87 & 1{,}075.4 & 0 & -8.9618 & 5.2322 & 0 & 627.87 & 1{,}075.4 & 0 \\
-5.2322 & 15.003 & 0 & -1{,}800.4 & -627.87 & 0 & 5.2322 & -15.003 & 0 & -1{,}800.4 & -627.87 & 0 \\
0 & 0 & 3{,}975.4 & 0 & 0 & 0 & 0 & 0 & -3{,}975.4 & 0 & 0 & 0 \\
627.87 & -1{,}800.4 & 0 & 288{,}067 & 100{,}459 & 0 & -627.87 & 1{,}800.4 & 0 & 144{,}033 & 50{,}229 & 0 \\
1{,}075.4 & -627.87 & 0 & 100{,}459 & 172{,}067 & 0 & -1{,}075.4 & 627.87 & 0 & 50{,}229 & 86{,}033 & 0 \\
0 & 0 & 0 & 0 & 0 & 723.54 & 0 & 0 & 0 & 0 & 0 & -723.54 \\ \hline
-8.9618 & 5.2322 & 0 & -627.87 & -1{,}075.4 & 0 & 8.9618 & -5.2322 & 0 & -627.87 & -1{,}075.4 & 0 \\
5.2322 & -15.003 & 0 & 1{,}800.4 & 627.87 & 0 & -5.2322 & 15.003 & 0 & 1{,}800.4 & 627.87 & 0 \\
0 & 0 & -3{,}975.4 & 0 & 0 & 0 & 0 & 0 & 3{,}975.4 & 0 & 0 & 0 \\
627.87 & -1{,}800.4 & 0 & 144{,}033 & 50{,}229 & 0 & -627.87 & 1{,}800.4 & 0 & 288{,}067 & 100{,}459 & 0 \\
1{,}075.4 & -627.87 & 0 & 50{,}229 & 86{,}033 & 0 & -1{,}075.4 & 627.87 & 0 & 100{,}459 & 172{,}067 & 0 \\
0 & 0 & 0 & 0 & 0 & -723.54 & 0 & 0 & 0 & 0 & 0 & 723.54
\end{array}\right] &
\begin{array}{c} 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}$$

The complete structure stiffness matrix $\mathbf{S}$ and the structure fixed-joint force vector $\mathbf{P}_f$ are given in Fig. 8.23(c).

Joint Load Vector: By comparing Figs. 8.23(a) and (b), we obtain

$$\mathbf{P} = \begin{array}{cc} \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -1{,}800 \text{ k-in.} \\ 0 \\ 1{,}800 \text{ k-in.} \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Joint Displacements: By substituting $\mathbf{P}$, $\mathbf{P}_f$, and $\mathbf{S}$ into the structure stiffness relationship, $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$, and solving the resulting simultaneous equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} -1.3522 \text{ in.} \\ -2.7965 \text{ in.} \\ -1.812 \text{ in.} \\ -3.0021 \text{ rad} \\ 1.0569 \text{ rad} \\ 6.4986 \text{ rad} \end{bmatrix} \times 10^{-3}$$

Ans

Member End Displacements and End Forces:

Member 1

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.3522 \\ -2.7965 \\ -1.812 \\ -3.0021 \\ 1.0569 \\ 6.4986 \end{bmatrix} \times 10^{-3} \tag{5}$$

$$\mathbf{F}_1 = \mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \left[\begin{array}{c} 5.3757 \text{ k} \\ 44.106 \text{ k} \\ -0.74272 \text{ k} \\ 2.1722 \text{ k-in.} \\ 58.987 \text{ k-in.} \\ 2{,}330.5 \text{ k-in.} \\ \hdashline -5.3757 \text{ k} \\ 15.894 \text{ k} \\ 0.74272 \text{ k} \\ -2.1722 \text{ k-in.} \\ 119.27 \text{ k-in.} \\ 1{,}055 \text{ k-in.} \end{array}\right] \begin{array}{l} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Ans

The member local end forces are depicted in Fig. 8.23(d), and the pertinent elements of $\mathbf{F}_1$ are stored in the support reaction vector $\mathbf{R}$ (Fig. 8.23(e)).

Member 2 $\mathbf{v}_2 = \mathbf{v}_1$ (see Eq. (5)).

$$\mathbf{u}_2 = \mathbf{T}_2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2.7965 \\ -1.812 \\ -1.3522 \\ 1.0569 \\ 6.4986 \\ -3.0021 \end{bmatrix} \times 10^{-3} \qquad \mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 = \begin{bmatrix} 11.117 \text{ k} \\ -6.4607 \text{ k} \\ -4.6249 \text{ k} \\ -0.76472 \text{ k-in.} \\ 369.67 \text{ k-in.} \\ -515.55 \text{ k-in.} \\ -11.117 \text{ k} \\ 6.4607 \text{ k} \\ 4.6249 \text{ k} \\ 0.76472 \text{ k-in.} \\ 740.31 \text{ k-in.} \\ -1{,}035 \text{ k-in.} \end{bmatrix}$$

Ans

$$\mathbf{F}_2 = \mathbf{T}_2^T\mathbf{Q}_2 = \left[\begin{array}{r} -4.6249 \\ 11.117 \\ -6.4607 \\ -515.55 \\ -0.76472 \\ 369.67 \\ \hdashline 4.6249 \\ -11.117 \\ 6.4607 \\ -1{,}035 \\ 0.76472 \\ 740.31 \end{array}\right]\begin{array}{l} 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Member 3 $\mathbf{v}_3 = \mathbf{v}_1$ (see Eq. (5)).

$$\mathbf{u}_3 = \mathbf{T}_3\mathbf{v}_3 = \left[\begin{array}{l} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.812 \\ -1.7457 \\ 2.5693 \\ 6.4986 \\ 2.4164 \\ 2.0714 \end{array}\right] \times 10^{-3} \qquad \mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 = \left[\begin{array}{c} 7.2034 \text{ k} \\ 4.5118 \text{ k} \\ -1.7379 \text{ k} \\ -4.702 \text{ k-in.} \\ 139.65 \text{ k-in.} \\ 362.21 \text{ k-in.} \\ -7.2034 \text{ k} \\ -4.5118 \text{ k} \\ 1.7379 \text{ k} \\ 4.702 \text{ k-in.} \\ 277.46 \text{ k-in.} \\ 720.63 \text{ k-in.} \end{array}\right]$$

Ans

$$\mathbf{F}_3 = \mathbf{T}_3^T\mathbf{Q}_3 = \left[\begin{array}{r} -0.75082 \\ 4.7763 \\ 7.2034 \\ -383.5 \\ -60.166 \\ -4.702 \\ \hdashline 0.75082 \\ -4.7763 \\ -7.2034 \\ -762.82 \\ -120.03 \\ 4.702 \end{array}\right]\begin{array}{l} 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Support Reactions: The completed reaction vector **R** is shown in Fig. 8.23(e), and the support reactions are depicted on a line diagram of the space frame in Fig. 8.23(f).

Ans

Equilibrium checks: The six equations of equilibrium ($\sum F_x = 0$, $\sum F_y = 0$, $\sum F_z = 0$, $\sum M_x = 0$, $\sum M_y = 0$, and $\sum M_z = 0$) are satisfied for each member of the space frame shown in Fig. 8.23(d). Furthermore, the six equilibrium equations in the directions of the global coordinate axes ($\sum F_X = 0$, $\sum F_Y = 0$, $\sum F_Z = 0$, $\sum M_X = 0$, $\sum M_Y = 0$, and $\sum M_Z = 0$) are satisfied for the entire structure shown in Fig. 8.23(f).

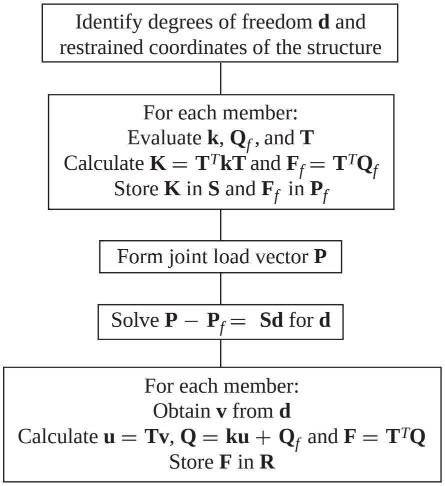


Fig. 8.24 *Stiffness Method of Analysis*

SUMMARY

In this chapter, we have extended the matrix stiffness method to the analysis of three-dimensional framed structures. The stiffness and transformation relationships for the members of space trusses, grids, and space frames are developed in Sections 8.1, 8.2, and 8.3, respectively. It should be noted that the overall format of the stiffness method of analysis remains the same for all types of (two- and three-dimensional) framed structures—provided that the member stiffness and transformation relations, appropriate for the particular type of structure being analyzed, are used in the analysis. A block diagram summarizing the overall format of the stiffness method is shown in Fig. 8.24.

PROBLEMS

Section 8.1

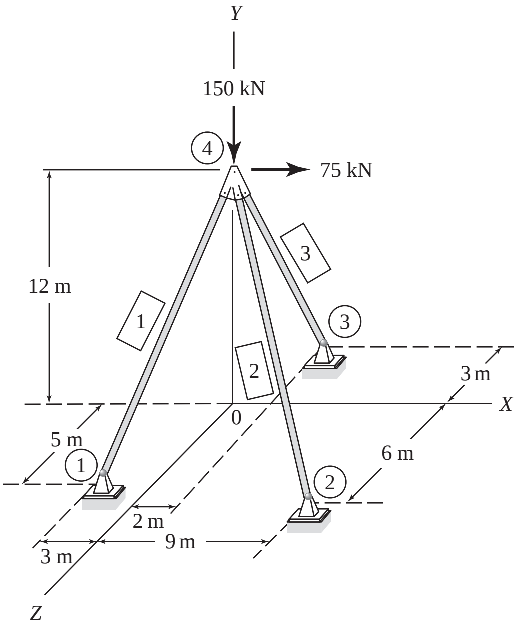
8.1 through 8.5 Determine the joint displacements, member axial forces, and support reactions for the space trusses shown in Figs. P8.1 through P8.5, using the matrix stiffness method. Check the hand-calculated results by using the computer program which can be downloaded from the publisher's website for this book, or by using any other general purpose structural analysis program available.

8.6 Develop a computer program for the analysis of space trusses by the matrix stiffness method. Use the program to analyze the trusses of Problems 8.1 through 8.5, and compare the computer-generated results to those obtained by hand calculations.

Section 8.2

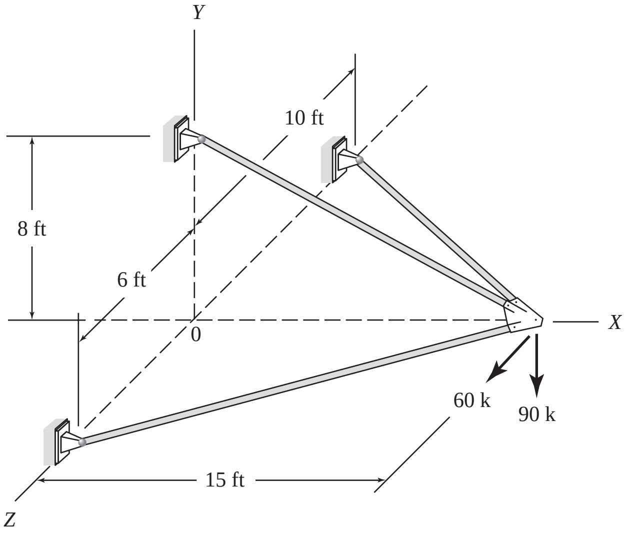
8.7 through 8.12 Determine the joint displacements, member local end forces, and support reactions for the grids shown in Figs. P8.7 through P8.12, using the matrix stiffness method. Check the hand-calculated results by using the computer program which can be downloaded from the publisher's website for this book, or by using any other general purpose structural analysis program available.

8.13 Develop a program for the analysis of grids by the matrix stiffness method. Use the program to analyze the grids of Problems 8.7 through 8.12, and compare the computer-generated results to those obtained by hand calculations.



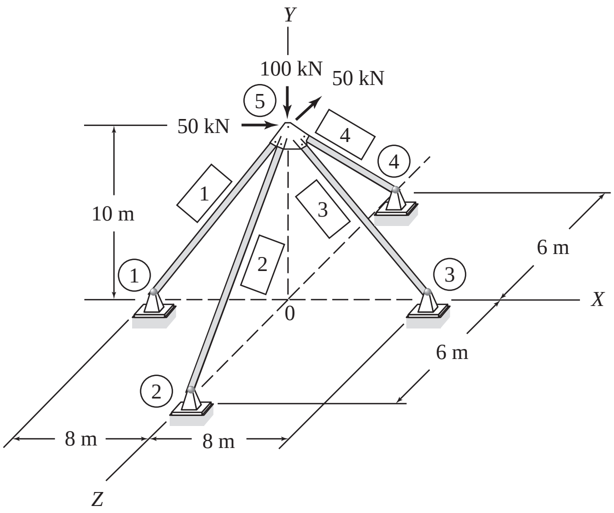
$EA =$ constant
$E \;= 70$ GPa
$A \;= 2{,}000$ mm^2

Fig. P8.1



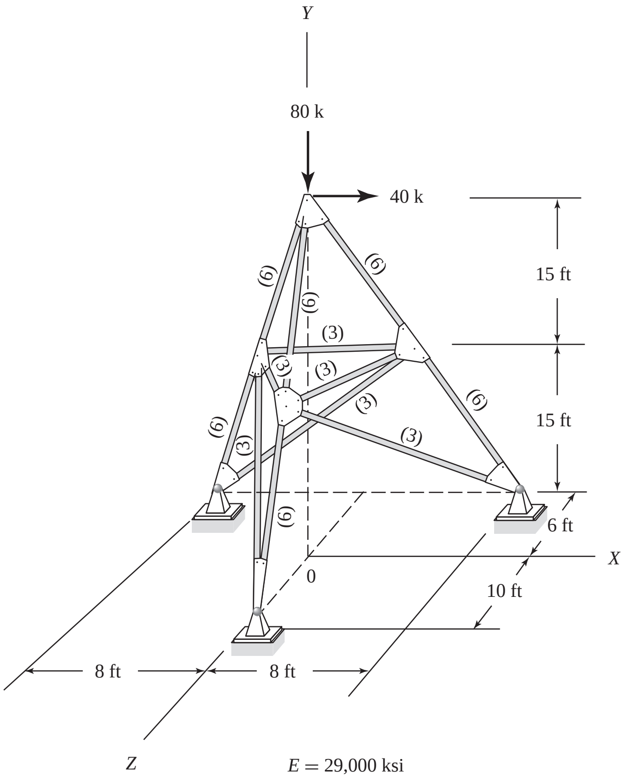
$EA =$ constant
$E \;= 29{,}000$ ksi
$A \;= 6$ in.^{2}

Fig. P8.2



$EA =$ constant
$E \;= 200$ GPa
$A \;= 3{,}800$ mm^2

Fig. P8.3



$E = 29{,}000$ ksi

Note: Numbers in parentheses represent cross-sectional areas in in.^{2}

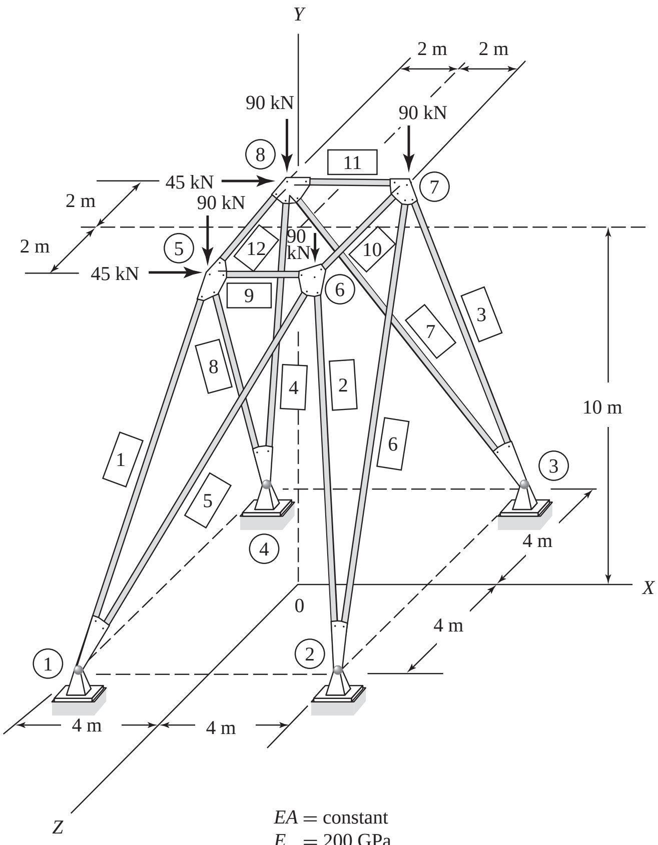
Fig. P8.4

Section 8.3

8.14 through 8.17 Determine the joint displacements, member local end forces, and support reactions for the space frames shown in Figs. 8.14 through 8.17, using the matrix stiffness method. Check the hand-calculated results by using the computer program which can be downloaded from the publisher's website for this book, or by using any other general purpose structural analysis program available.

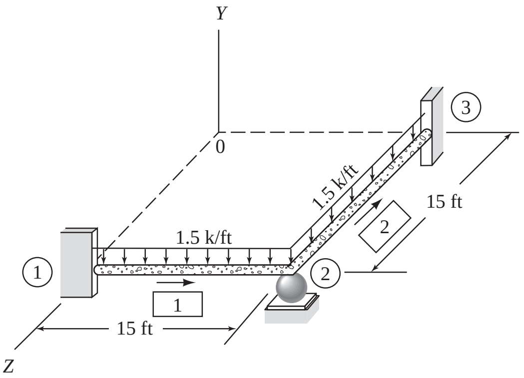
8.18 Develop a program for the analysis of space frames by the matrix stiffness method. Use the program to analyze the frames of Problems 8.14 through 8.17, and compare the computer-generated results to those obtained by hand calculations.

8.19 Develop a general computer program that can be used to analyze any type of framed structure.



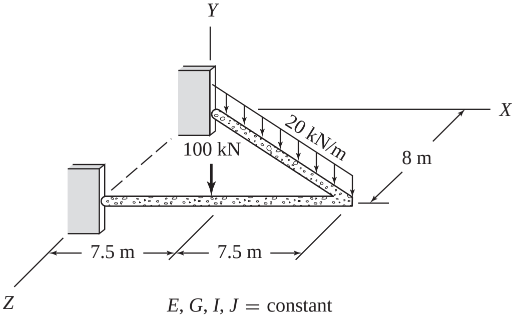
EA = constant
E = 200 GPa
A = 4,000 mm^2

Fig. P8.5



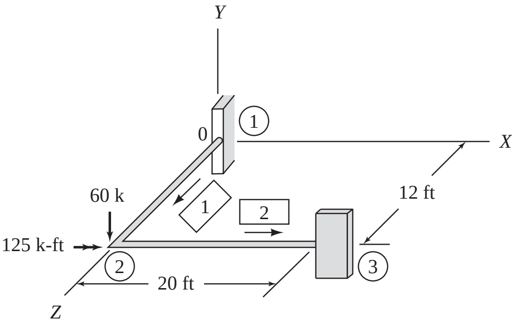
E, G, I, J = constant
E = 4,500 ksi
G = 1,800 ksi
I = 256 in.^{4}
J = 311 in.^{4}

Fig. P8.7



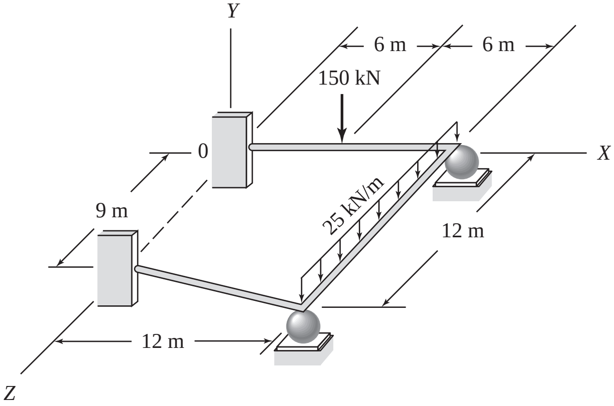
E, G, I, J = constant
E = 30 GPa
G = 12.5 GPa
I = 4.8(10^9) mm^4
J = 3.2(10^9) mm^4

Fig. P8.8



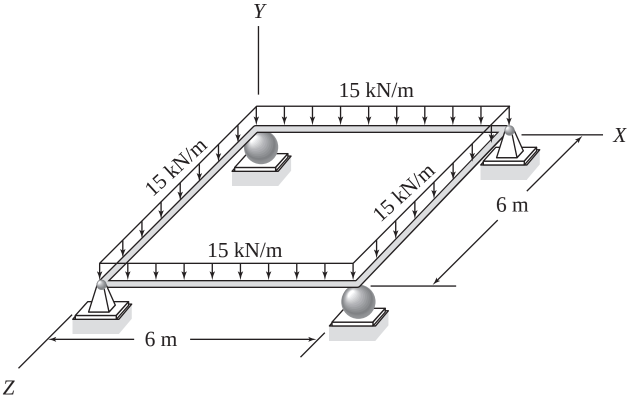
E, G, I, J = constant
E = 29,000 ksi
G = 11,500 ksi
I = 623 in.^{4}
J = 10.9 in.^{4}

Fig. P8.9



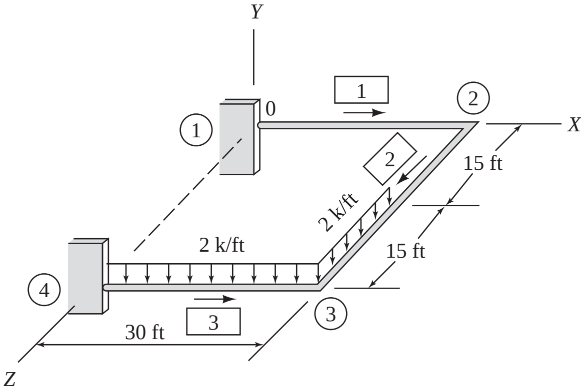
$E, G, I, J =$ constant
$E = 200$ GPa
$G = 76$ GPa
$I = 300(10^6)$ mm^4
$J = 7(10^6)$ mm^4

Fig. P8.10



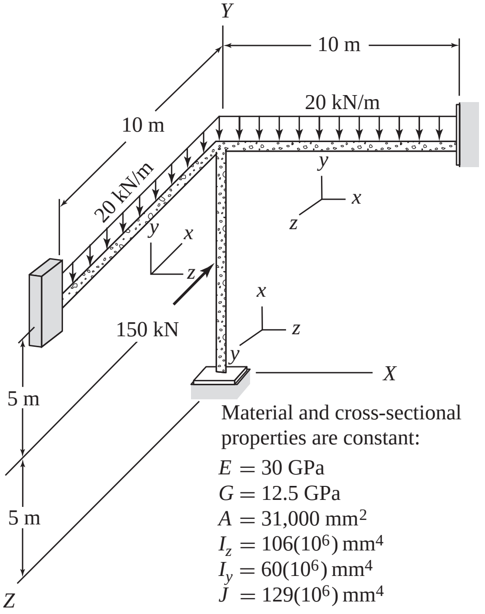
$E, G, I, J =$ constant
$E = 200$ GPa
$G = 76$ GPa
$I = 300(10^6)$ mm^4
$J = 7(10^6)$ mm^4

Fig. P8.12



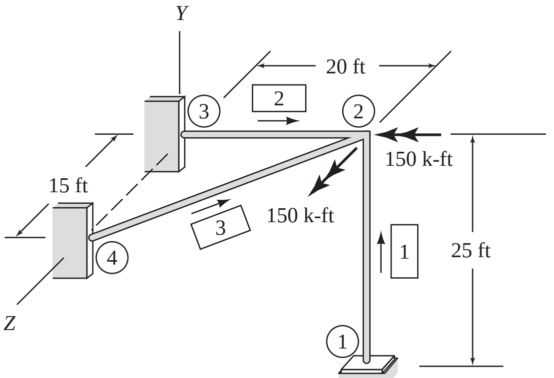
$E, G, I, J =$ constant
$E = 29{,}000$ ksi
$G = 11{,}500$ ksi
$I = 5{,}310$ in.^{4}
$J = 41.3$ in.^{4}

Fig. P8.11



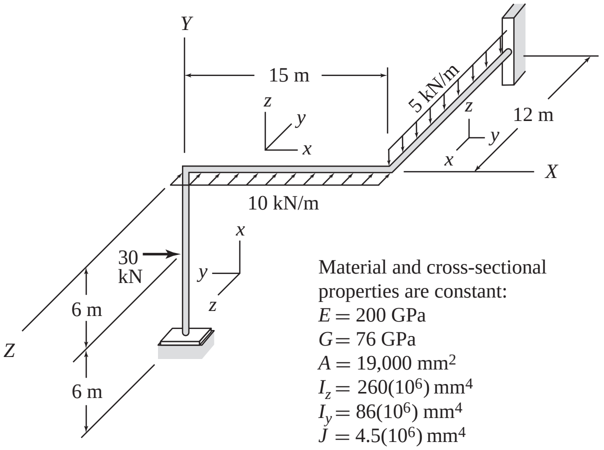
Material and cross-sectional properties are constant:
$E = 30$ GPa
$G = 12.5$ GPa
$A = 31{,}000$ mm^2
$I_z = 106(10^6)$ mm^4
$I_y = 60(10^6)$ mm^4
$J = 129(10^6)$ mm^4

Fig. P8.14



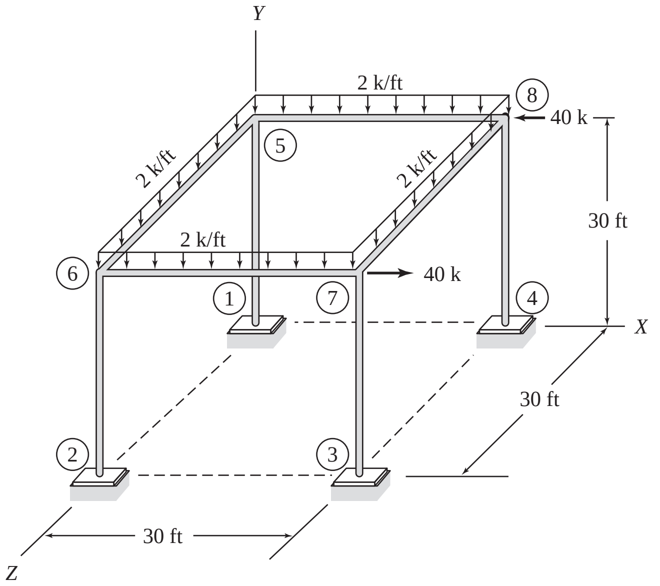
Material properties are constant:
$E = 10{,}000$ ksi
$G = 4{,}000$ ksi
All members have circular cross-sections with
$A = 4.52$ in.^{2}
$I_z = I_y = 18.7$ in.^{4}
$J = 37.4$ in.^{4}

Fig. P8.15

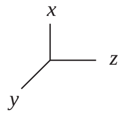


Material and cross-sectional properties are constant:
$E = 200$ GPa
$G = 76$ GPa
$A = 19{,}000$ mm^2
$I_z = 260(10^6)$ mm^4
$I_y = 86(10^6)$ mm^4
$J = 4.5(10^6)$ mm^4

Fig. P8.16



Member orientations:
Girders: local y axis is vertical
Columns:



Material and cross-sectional properties are constant:
$E = 29{,}000$ ksi
$G = 11{,}500$ ksi
$A = 47.7$ in.^{2}
$I_z = 5{,}170$ in.^{4}
$I_y = 443$ in.^{4}
$J = 18.5$ in.^{4}

Fig. P8.17