$$\mathbf{S}_{\mathrm{MS}} = \mathbf{R}_{\mathrm{T}}^{\mathrm{T}}\mathbf{S}_{\mathrm{M}}\mathbf{R}_{\mathrm{T}} \tag{4-51}$$

Evaluation of the matrix $\mathbf{S}_{\mathrm{MS}}$ from this equation is performed as follows:

$$\mathbf{S}_{\mathrm{MS}} = \left[\begin{array}{cc|cc} C_X & -C_Y & 0 & 0 \\ C_Y & C_X & 0 & 0 \\ \hline 0 & 0 & C_X & -C_Y \\ 0 & 0 & C_Y & C_X \end{array}\right]$$

$$\times \left[\begin{array}{cc|cc} \dfrac{EA_X}{L} & 0 & \dfrac{-EA_X}{L} & 0 \\ 0 & 0 & 0 & 0 \\ \hline \dfrac{-EA_X}{L} & 0 & \dfrac{EA_X}{L} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cc} C_X & C_Y & 0 & 0 \\ -C_Y & C_X & 0 & 0 \\ \hline 0 & 0 & C_X & C_Y \\ 0 & 0 & -C_Y & C_X \end{array}\right]$$

When this matrix multiplication is executed, the result is the matrix $\mathbf{S}_{\mathrm{MS}}$ (see Table 4-15), which was previously obtained by direct formulation.

In addition to the transformation of the member stiffness matrix from member axes to structural axes, the rotation of axes concept can also be used for other purposes in the stiffness method of analysis. One important application arises in the construction of the equivalent load vector $\mathbf{A}_{\mathrm{E}}$ from elements of the matrix $\mathbf{A}_{\mathrm{ML}}$. Contributions to the former array (the elements of which are in the directions of structural axes) from the latter array (which has elements in the directions of member axes) may be obtained by the following transformation:

$$\mathbf{A}_{\mathrm{MS}\,i} = \mathbf{R}_{\mathrm{T}\,i}^{\mathrm{T}}\mathbf{A}_{\mathrm{ML}\,i} \tag{4-52}$$

In this expression the vector $\mathbf{A}_{\mathrm{MS}\,i}$ represents fixed-end actions in the directions of structural axes, whereas the vector $\mathbf{A}_{\mathrm{ML}\,i}$ is the i-th column of the matrix $\mathbf{A}_{\mathrm{ML}}$. In expanded form, Eq. (4-52) becomes

$$\begin{bmatrix} (A_{MS})_{1,i} \\ (A_{MS})_{2,i} \\ (A_{MS})_{3,i} \\ (A_{MS})_{4,i} \end{bmatrix} = \begin{bmatrix} C_{Xi} & -C_{Yi} & 0 & 0 \\ C_{Yi} & C_{Xi} & 0 & 0 \\ 0 & 0 & C_{Xi} & -C_{Yi} \\ 0 & 0 & C_{Yi} & C_{Xi} \end{bmatrix} \begin{bmatrix} (A_{ML})_{1,i} \\ (A_{ML})_{2,i} \\ (A_{ML})_{3,i} \\ (A_{ML})_{4,i} \end{bmatrix}$$

$$= \begin{bmatrix} C_{Xi}(A_{ML})_{1,i} - C_{Yi}(A_{ML})_{2,i} \\ C_{Yi}(A_{ML})_{1,i} + C_{Xi}(A_{ML})_{2,i} \\ C_{Xi}(A_{ML})_{3,i} - C_{Yi}(A_{ML})_{4,i} \\ C_{Yi}(A_{ML})_{3,i} + C_{Xi}(A_{ML})_{4,i} \end{bmatrix}$$

The resulting terms in $\mathbf{A}_{\mathrm{MS}\,i}$, with the signs reversed, represent the incremental portions of $\mathbf{A}_{\mathrm{E}}$ given previously in Eqs. (4-33).

Another significant application of the rotation of axes concept appears in the calculation of final member end-actions. This computation consists of the superposition of the initial actions in member i and the effects of joint displacements. This superposition procedure is expressed by Eq. (4-5), which is repeated here.

$$\mathbf{A}_{\mathrm{M}i} = \mathbf{A}_{\mathrm{ML}i} + \mathbf{S}_{\mathrm{M}i}\mathbf{D}_{\mathrm{M}i} \tag{4-5}$$

The vector $\mathbf{D}_{\mathrm{M}i}$ in this equation must be ascertained from the vector of joint displacements $\mathbf{D}_{\mathrm{J}}$. The latter displacements are in the directions of structural axes, but the former displacements are in the directions of member axes. Therefore, the vector $\mathbf{D}_{\mathrm{M}i}$ may be obtained by the following transformation:

$$\mathbf{D}_{\mathrm{M}i} = \mathbf{R}_{\mathrm{T}i}\mathbf{D}_{\mathrm{J}i} \tag{4-53}$$

in which $\mathbf{D}_{\mathrm{J}i}$ is the vector of joint displacements for the ends of member i. Substitution of Eq. (4-53) into Eq. (4-5) produces the following expression:

$$\mathbf{A}_{\mathrm{M}i} = \mathbf{A}_{\mathrm{ML}i} + \mathbf{S}_{\mathrm{M}i}\mathbf{R}_{\mathrm{T}i}\mathbf{D}_{\mathrm{J}i} \tag{4-54}$$

Equation (4-54) may also be written in the expanded form

$$\begin{bmatrix} (A_M)_{1,i} \\ (A_M)_{2,i} \\ (A_M)_{3,i} \\ (A_M)_{4,i} \end{bmatrix} = \begin{bmatrix} (A_{ML})_{1,i} \\ (A_{ML})_{2,i} \\ (A_{ML})_{3,i} \\ (A_{ML})_{4,i} \end{bmatrix}$$
$$+ \frac{EA_{Xi}}{L_i}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} C_{Xi} & C_{Yi} & 0 & 0 \\ -C_{Yi} & C_{Xi} & 0 & 0 \\ 0 & 0 & C_{Xi} & C_{Yi} \\ 0 & 0 & -C_{Yi} & C_{Xi} \end{bmatrix}\begin{bmatrix} (D_J)_{j1} \\ (D_J)_{j2} \\ (D_J)_{k1} \\ (D_J)_{k2} \end{bmatrix}$$

In this expression the subscripts $j1$, $j2$, $k1$, and $k2$ carry the definitions given previously by Eqs. (4-27). When the matrix multiplications indicated above are performed, the resulting four equations are the same as Eqs. (4-34) obtained in Sec. 4.11 by direct formulation.

In summary, the concept of rotation of axes has several useful applications in the stiffness method of analysis. The member stiffness matrix for members axes $\mathbf{S}_{\mathrm{M}i}$ may be transformed into the member stiffness matrix for structural axes $\mathbf{S}_{\mathrm{MS}i}$ by means of Eq. (4-51). In addition, the contributions $\mathbf{A}_{\mathrm{MS}i}$ to the equivalent load vector $\mathbf{A}_{\mathrm{E}}$ from a given member may be evaluated conveniently by rotation of axes, as shown by Eq. (4-52). Also, the final member end-actions can be obtained by the rotation of axes formulation given by Eq. (4-54).

It will be seen later that the matrix equations given above for rotation of axes in plane trusses can be generalized and applied to more complicated types of structures.

4.15 Rotation of Axes in Three Dimensions. Consider the action A shown in three dimensions in Fig. 4-26. The two sets of orthogonal axes x_S, y_S, z_S and x_M, y_M, z_M are analogous to the two sets of axes in the two-