10

Free Vibration

PREVIEW

By *free vibration* we mean the motion of a structure without any dynamic excitation—external forces or support motion. Free vibration is initiated by disturbing the structure from its equilibrium position by some initial displacements and/or by imparting some initial velocities.

This chapter on free vibration of MDF systems is divided into three parts. In Part A we develop the notion of natural frequencies and natural modes of vibration of a structure; these concepts play a central role in the dynamic and earthquake analysis of linear systems (Chapters 12 and 13).

In Part B we describe the use of these vibration properties to determine the free vibration response of systems. Undamped systems are analyzed first. We then define systems with classical damping and systems with nonclassical damping. The analysis procedure is extended to systems with classical damping, recognizing that such systems possess the same natural modes as those of the undamped system.

Part C is concerned with numerical solution of the eigenvalue problem to determine the natural frequencies and modes of vibration. Vector iteration methods are effective in structural engineering applications, and we restrict this presentation to such methods. Only the basic ideas of vector iteration are included, without getting into subspace iteration or the Lanczos method. Although this limited treatment would suffice for many practical problems and research applications, the reader should recognize that a wealth of knowledge exists on the subject.

PART A: NATURAL VIBRATION FREQUENCIES AND MODES

10.1 SYSTEMS WITHOUT DAMPING

Free vibration of linear MDF systems is governed by Eq. (9.2.12) with $\mathbf{p}(t) = \mathbf{0}$, which for systems without damping is

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = 0 \tag{10.1.1}$$

Equation (10.1.1) represents N homogeneous differential equations that are coupled through the mass matrix, the stiffness matrix, or both matrices; N is the number of DOFs. It is desired to find the solution $\mathbf{u}(t)$ of Eq. (10.1.1) that satisfies the initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \tag{10.1.2}$$

at $t = 0$. A general procedure to obtain the desired solution for any MDF system is developed in Section 10.8. In this section the solution is presented in graphical form that enables us to understand free vibration of an MDF system in qualitative terms.

Figure 10.1.1 shows the free vibration of a two-story shear frame. The story stiffnesses and lumped masses at the floors are noted, and the free vibration is initiated by the deflections shown by curve a in Fig. 10.1.1b. The resulting motion u_j of the two masses is plotted in Fig. 10.1.1d as a function of time; T_1 will be defined later.

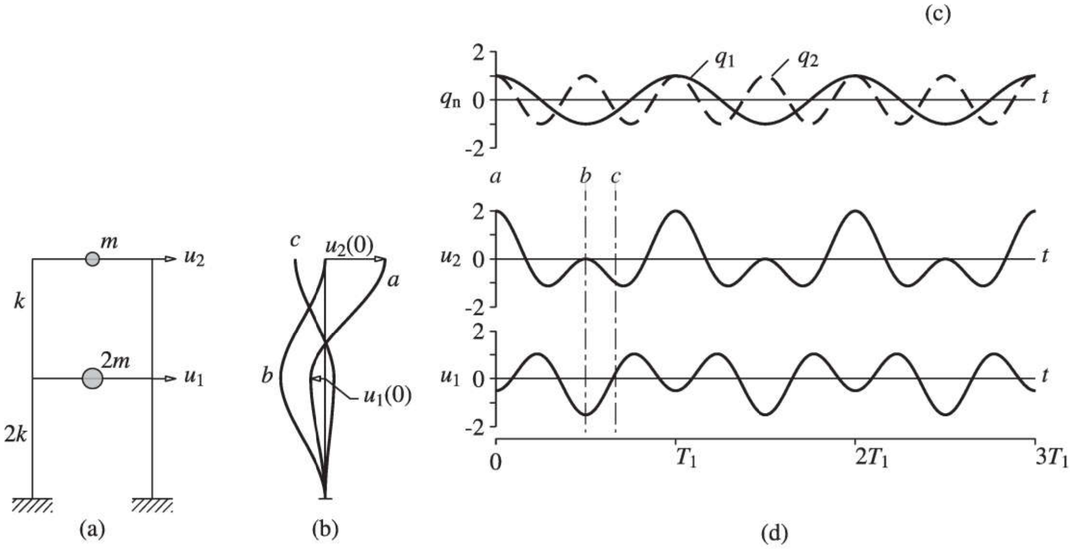


Figure 10.1.1 Free vibration of an undamped system due to arbitrary initial displacement: (a) two-story frame; (b) deflected shapes at time instants a, b, and c; (c) modal coordinates $q_n(t)$; (d) displacement history.

The deflected shapes of the structure at selected time instants a, b, and c are also shown; the $q_n(t)$ plotted in Fig.10.1.1c are discussed in Example 10.11. The displacement–time plot for the jth floor starts with the initial conditions $u_j(0)$ and $\dot{u}_j(0)$; the $u_j(0)$ are identified in Fig. 10.1.1b and $\dot{u}_j(0) = 0$ for both floors. Contrary to what we observed in Fig. 2.1.1 for SDF systems, the motion of each mass (or floor) is not a simple harmonic motion and the frequency of the motion cannot be defined. Furthermore, the deflected shape (i.e., the ratio u_1/u_2) varies with time, as is evident from the differing deflected shapes b and c, which are in turn different from the initial deflected shape a.

An undamped structure would undergo simple harmonic motion without change of deflected shape, however, if free vibration is initiated by appropriate distributions of displacements in the various DOFs. As shown in Figs. 10.1.2 and 10.1.3, two characteristic deflected shapes exist for this two-DOF system such that if it is displaced in one of these shapes and released, it will vibrate in simple harmonic motion, maintaining the initial deflected shape. Both floors vibrate in the same phase, i.e., they pass through their equilibrium, maximum, or minimum positions at the same instant of time. Each characteristic deflected shape is called a *natural mode of vibration* of an MDF system.

Observe that the displacements of both floors are in the same direction in the first mode but in opposite directions in the second mode. The point of zero displacement, called a *node,*† does not move at all (Fig. 10.1.3); as the mode number n increases, the number of nodes increases accordingly (see Fig. 12.8.2).

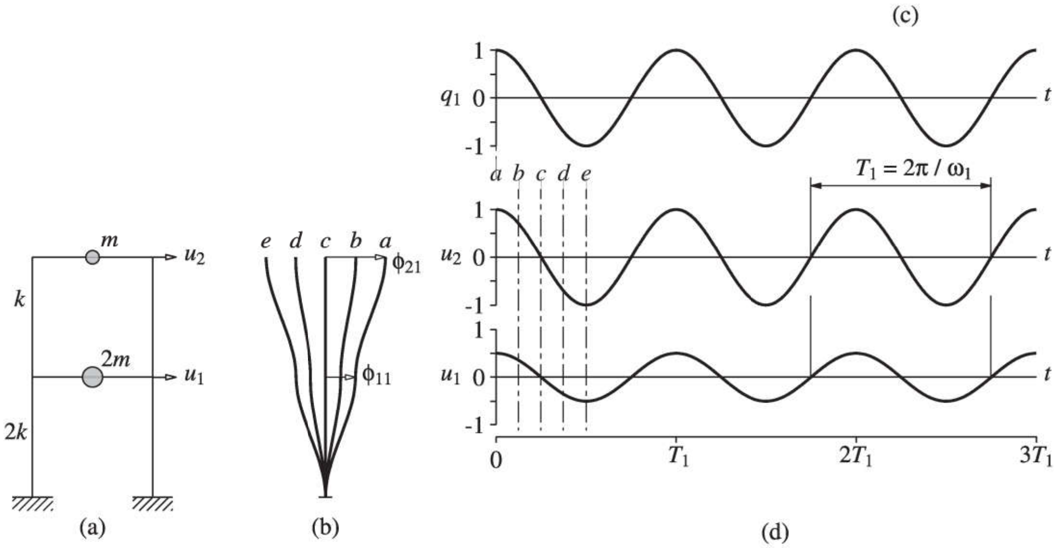


Figure 10.1.2 Free vibration of an undamped system in its first natural mode of vibration: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinate $q_1(t)$; (d) displacement history.

†Recall that we have already used the term *node* for nodal points in the structural idealization; the two different uses of *node* should be clear from the context.

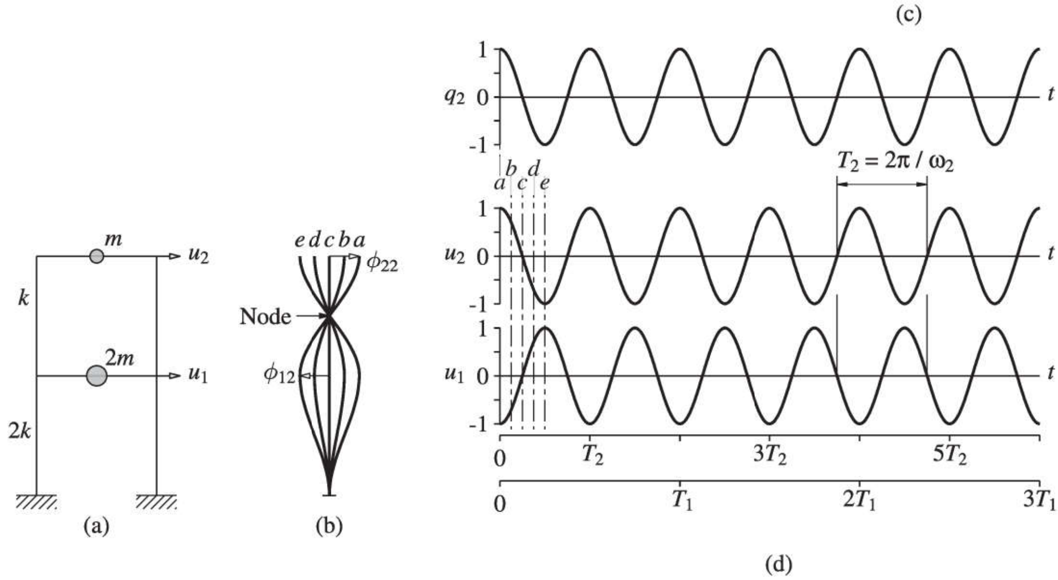


Figure 10.1.3 Free vibration of an undamped system in its second natural mode of vibration: (a) two-story frame; (b) deflected shapes at the time instants *a*, *b*, *c*, *d*, and *e*; (c) modal coordinate $q_2(t)$; (d) displacement history.

A *natural period of vibration* T_n of an MDF system is the time required for one cycle of the simple harmonic motion in one of these natural modes. The corresponding *natural circular frequency of vibration* is ω_n and the *natural cyclic frequency of vibration* is f_n, where

$$T_n = \frac{2\pi}{\omega_n} \qquad f_n = \frac{1}{T_n} \tag{10.1.3}$$

Figures 10.1.2 and 10.1.3 show the two natural periods T_n and natural frequencies ω_n $(n = 1, 2)$ of the two-story building vibrating in its natural modes $\boldsymbol{\phi}_n = \langle \phi_{1n} \quad \phi_{2n} \rangle^T$. The smaller of the two natural vibration frequencies is denoted by ω_1, and the larger by ω_2. Correspondingly, the longer of the two natural vibration periods is denoted by T_1 and the shorter one by T_2.

10.2 NATURAL VIBRATION FREQUENCIES AND MODES

In this section we introduce the eigenvalue problem whose solution gives the natural frequencies and modes of a system. The free vibration of an undamped system in one of its natural vibration modes, graphically displayed in Figs. 10.1.2 and 10.1.3 for a two-DOF system, can be described mathematically by

$$\mathbf{u}(t) = q_n(t)\boldsymbol{\phi}_n \tag{10.2.1}$$

where the deflected shape $\boldsymbol{\phi}_n$ does not vary with time. The time variation of the displacements is described by the simple harmonic function

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t \tag{10.2.2}$$

where A_n and B_n are constants that can be determined from the initial conditions that initiate the motion. Combining Eqs. (10.2.1) and (10.2.2) gives

$$\mathbf{u}(t) = \boldsymbol{\phi}_n \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \tag{10.2.3}$$

where ω_n and $\boldsymbol{\phi}_n$ are unknown.

Substituting this form of $\mathbf{u}(t)$ in Eq. (10.1.1) gives

$$\left[-\omega_n^2 \mathbf{m}\boldsymbol{\phi}_n + \mathbf{k}\boldsymbol{\phi}_n\right] q_n(t) = \mathbf{0}$$

This equation can be satisfied in one of two ways. Either $q_n(t) = 0$, which implies that $\mathbf{u}(t) = \mathbf{0}$ and there is no motion of the system (this is the so-called trivial solution), or the natural frequencies ω_n and modes $\boldsymbol{\phi}_n$ must satisfy the following algebraic equation:

$$\mathbf{k}\boldsymbol{\phi}_n = \omega_n^2 \mathbf{m}\boldsymbol{\phi}_n \tag{10.2.4}$$

which provides a useful condition. This algebraic equation is called the *matrix eigenvalue problem.* When necessary it is called the real eigenvalue problem to distinguish it from the complex eigenvalue problem mentioned in Chapter 14 for systems with damping. The stiffness and mass matrices $\mathbf{k}$ and $\mathbf{m}$ are known; the problem is to determine the scalar ω_n^2 and vector $\boldsymbol{\phi}_n$.

To indicate the formal solution to Eq. (10.2.4), it is rewritten as

$$\left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] \boldsymbol{\phi}_n = \mathbf{0} \tag{10.2.5}$$

which can be interpreted as a set of N homogeneous algebraic equations for the N elements ϕ_{jn} $(j = 1, 2, \ldots, N)$. This set always has the trivial solution $\boldsymbol{\phi}_n = \mathbf{0}$, which is not useful because it implies no motion. It has nontrivial solutions if

$$\det\left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] = 0 \tag{10.2.6}$$

When the determinant is expanded, a polynomial of order N in ω_n^2 is obtained. Equation (10.2.6) is known as the *characteristic equation* or *frequency equation.* This equation has N real and positive roots for ω_n^2 because $\mathbf{m}$ and $\mathbf{k}$, the structural mass and stiffness matrices, are symmetric and positive definite. The positive definite property of $\mathbf{k}$ is assured for all structures supported in a way that prevents rigid-body motion. Such is the case for civil engineering structures of interest to us, but not for unrestrained structures such as aircraft in flight—these are beyond the scope of this book. The positive definite property of $\mathbf{m}$ is also assured because the lumped masses are nonzero in all DOFs retained in the analysis after the DOFs with zero lumped mass have been eliminated by static condensation (Section 9.3).

The N roots, ω_n^2, of Eq. (10.2.6) determine the N natural frequencies ω_n $(n = 1, 2, \ldots, N)$ of vibration, conventionally arranged in sequence from smallest to largest $(\omega_1 < \omega_2 < \cdots < \omega_N)$. These roots of the characteristic equation are also known as *eigenvalues, characteristic values,* or *normal values.* When a natural frequency ω_n is known, Eq. (10.2.5) can be solved for the corresponding vector $\boldsymbol{\phi}_n$ to within a multiplicative constant. The eigenvalue problem does not fix the absolute amplitude of the vectors $\boldsymbol{\phi}_n$, only the shape of the vector given by the relative values of the N displacements ϕ_{jn} ($j = 1, 2, \ldots, N$). Corresponding to the N natural vibration frequencies ω_n of an N-DOF system, there are N independent vectors $\boldsymbol{\phi}_n$, which are known as *natural modes of vibration,* or *natural mode shapes of vibration.* These vectors are also known as *eigenvectors, characteristic vectors,* or *normal modes.* The term *natural* is used to qualify each of these vibration properties to emphasize the fact that these are natural properties of the structure in free vibration, and they depend only on its mass and stiffness properties. The subscript n denotes the mode number, and the first mode ($n = 1$) is also known as the fundamental mode.

As mentioned earlier, during free vibration in each natural mode, an undamped system oscillates at its natural frequency with all DOFs of the system vibrating in the same phase, passing through their equilibrium, maximum, or minimum positions at the same instant of time. Because this type of natural mode was the subject of Lagrange's classical (1811) treatise on mechanics, we will refer to such modes as *classical natural modes*. In damped systems, this property is generally violated and classical natural modes may not exist, as we shall see later.

10.3 MODAL AND SPECTRAL MATRICES

The N eigenvalues and N natural modes can be assembled compactly into matrices. Let the natural mode $\boldsymbol{\phi}_n$ corresponding to the natural frequency ω_n have elements ϕ_{jn}, where j indicates the DOFs. The N eigenvectors can then be displayed in a single square matrix, each column of which is a natural mode:

$$\boldsymbol{\Phi} = \left[\phi_{jn}\right] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix}$$

The matrix $\boldsymbol{\Phi}$ is called the *modal matrix* for the eigenvalue problem, Eq. (10.2.4). The N eigenvalues ω_n^2 can be assembled into a diagonal matrix $\boldsymbol{\Omega}^2$, which is known as the *spectral matrix* of the eigenvalue problem, Eq. (10.2.4):

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2 \end{bmatrix}$$

Each eigenvalue and eigenvector satisfies Eq. (10.2.4), which can be rewritten as the relation

$$\mathbf{k}\boldsymbol{\phi}_n = \mathbf{m}\boldsymbol{\phi}_n\omega_n^2 \tag{10.3.1}$$

By using the modal and spectral matrices, it is possible to assemble all of such relations ($n = 1,\ 2,\ \ldots,\ N$) into a single matrix equation:

$$\mathbf{k}\boldsymbol{\Phi} = \mathbf{m}\boldsymbol{\Phi}\boldsymbol{\Omega}^2 \tag{10.3.2}$$

Equation (10.3.2) provides a compact presentation of the equations relating all eigenvalues and eigenvectors.

10.4 ORTHOGONALITY OF MODES

The natural modes corresponding to different natural frequencies can be shown to satisfy the following orthogonality conditions. When $\omega_n \neq \omega_r$,

$$\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_r = 0 \qquad \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r = 0 \tag{10.4.1}$$

These important properties can be proven as follows: The nth natural frequency and mode satisfy Eq. (10.2.4); premultiplying it by $\boldsymbol{\phi}_r^T$, the transpose of $\boldsymbol{\phi}_r$, gives

$$\boldsymbol{\phi}_r^T \mathbf{k} \boldsymbol{\phi}_n = \omega_n^2 \boldsymbol{\phi}_r^T \mathbf{m} \boldsymbol{\phi}_n \tag{10.4.2}$$

Similarly, the rth natural frequency and mode satisfy Eq. (10.2.4); thus $\mathbf{k}\boldsymbol{\phi}_r = \omega_r^2 \mathbf{m} \boldsymbol{\phi}_r$. Premultiplying by $\boldsymbol{\phi}_n^T$ gives

$$\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_r = \omega_r^2 \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r \tag{10.4.3}$$

The transpose of the matrix on the left side of Eq. (10.4.2) will equal the transpose of the matrix on the right side of the equation; thus

$$\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_r = \omega_n^2 \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r \tag{10.4.4}$$

wherein we have utilized the symmetry property of the mass and stiffness matrices. Subtracting Eq. (10.4.3) from (10.4.4) gives

$$(\omega_n^2 - \omega_r^2) \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r = 0$$

Thus Eq. (10.4.1b) is true when $\omega_n^2 \neq \omega_r^2$, which for systems with positive natural frequencies implies that $\omega_n \neq \omega_r$. Substituting Eq. (10.4.1b) in (10.4.3) indicates that Eq. (10.4.1a) is true when $\omega_n \neq \omega_r$. This completes a proof for the orthogonality relations of Eq. (10.4.1).

We have established the orthogonality relations between modes with distinct frequencies (i.e., $\omega_n \neq \omega_r$). If the frequency equation (10.2.4) has a j-fold multiple root (i.e., the system has one frequency repeated j times), it is always possible to find j modes associated with this frequency that satisfy Eq. (10.4.1). If these j modes are included with the modes corresponding to the other frequencies, a set of N modes is obtained which satisfies Eq. (10.4.1) for $n \neq r$.

The orthogonality of natural modes implies that the following square matrices are diagonal:

$$\mathbf{K} \equiv \boldsymbol{\Phi}^T \mathbf{k} \boldsymbol{\Phi} \qquad \mathbf{M} \equiv \boldsymbol{\Phi}^T \mathbf{m} \boldsymbol{\Phi} \tag{10.4.5}$$

where the diagonal elements are

$$K_n = \boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_n \qquad M_n = \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n \tag{10.4.6}$$

Since $\mathbf{m}$ and $\mathbf{k}$ are positive definite, the diagonal elements of $\mathbf{K}$ and $\mathbf{M}$ are positive. They are related by

$$K_n = \omega_n^2 M_n \tag{10.4.7}$$

This can be demonstrated from the definitions of K_n and M_n as follows: Substituting Eq. (10.2.4) in (10.4.6a) gives

$$K_n = \phi_n^T(\omega_n^2 \mathbf{m}\phi_n) = \omega_n^2(\phi_n^T \mathbf{m}\phi_n) = \omega_n^2 M_n$$

10.5 INTERPRETATION OF MODAL ORTHOGONALITY

In this section we develop physically motivated interpretations of the orthogonality properties of natural modes. One implication of modal orthogonality is that the work done by the nth-mode inertia forces in going through the rth-mode displacements is zero. To demonstrate this result, consider a structure vibrating in the nth mode with displacements

$$\mathbf{u}_n(t) = q_n(t)\phi_n \tag{10.5.1}$$

The corresponding accelerations are $\ddot{\mathbf{u}}_n(t) = \ddot{q}_n(t)\phi_n$ and the associated inertia forces are

$$(\mathbf{f}_I)_n = -\mathbf{m}\ddot{\mathbf{u}}_n(t) = -\mathbf{m}\phi_n\ddot{q}_n(t) \tag{10.5.2}$$

Next, consider displacements of the structure in its rth natural mode:

$$\mathbf{u}_r(t) = q_r(t)\phi_r \tag{10.5.3}$$

The work done by the inertia forces of Eq. (10.5.2) in going through the displacements of Eq. (10.5.3) is

$$(\mathbf{f}_I)_n^T \mathbf{u}_r = -\left(\phi_n^T \mathbf{m}\phi_r\right)\ddot{q}_n(t)q_r(t) \tag{10.5.4}$$

which is zero because of the modal orthogonality relation of Eq. (10.4.1b). This completes the proof.

Another implication of the modal orthogonality properties is that the work done by the equivalent static forces associated with displacements in the nth mode in going through the rth-mode displacements is zero. These forces are

$$(\mathbf{f}_S)_n = \mathbf{k}\mathbf{u}_n(t) = \mathbf{k}\phi_n q_n(t)$$

and the work they do in going through the displacements of Eq. (10.5.3) is

$$(\mathbf{f}_S)_n^T \mathbf{u}_r = \left(\phi_n^T \mathbf{k}\phi_r\right) q_n(t)q_r(t)$$

which is zero because of the modal orthogonality relation of Eq. (10.4.1a). This completes the proof.

10.6 NORMALIZATION OF MODES

As mentioned earlier, the eigenvalue problem, Eq. (10.2.4), determines the natural modes to only within a multiplicative factor. If the vector ϕ_n is a natural mode, any vector proportional to ϕ_n is essentially the same natural mode because it also satisfies Eq. (10.2.4). Scale factors are sometimes applied to natural modes to standardize their elements associated with various DOFs. This process is called *normalization*. Sometimes it is convenient to normalize each mode so that its largest element is unity. Other times it may be advantageous to normalize each mode so that the element corresponding to a particular DOF, say the top floor of a multistory building, is unity. In theoretical discussions and computer programs it is common to normalize modes so that the M_n have unit values. In this case

$$M_n = \phi_n^T \mathbf{m} \phi_n = 1 \qquad \mathbf{\Phi}^T \mathbf{m} \mathbf{\Phi} = \mathbf{I} \tag{10.5.5}$$

where **I** is the identity matrix, a diagonal matrix with unit values along the main diagonal. Equation (10.5.5) states that the natural modes are not only orthogonal but are normalized with respect to **m**. They are then called a mass *orthonormal set.* When the modes are normalized in this manner, Eqs. (10.4.6a) and (10.4.5a) become

$$K_n = \phi_n^T \mathbf{k} \phi_n = \omega_n^2 M_n = \omega_n^2 \qquad \mathbf{K} = \mathbf{\Phi}^T \mathbf{k} \mathbf{\Phi} = \mathbf{\Omega}^2 \tag{10.5.6}$$

Example 10.1

(**a**) Determine the natural vibration frequencies and modes of the system of Fig. E10.1a using the first set of DOFs shown.

(**b**) Repeat part (a) using the second set of DOFs in Fig. E10.1b.

(**c**) Show that the natural frequencies and modes determined using the two sets of DOFs are the same.

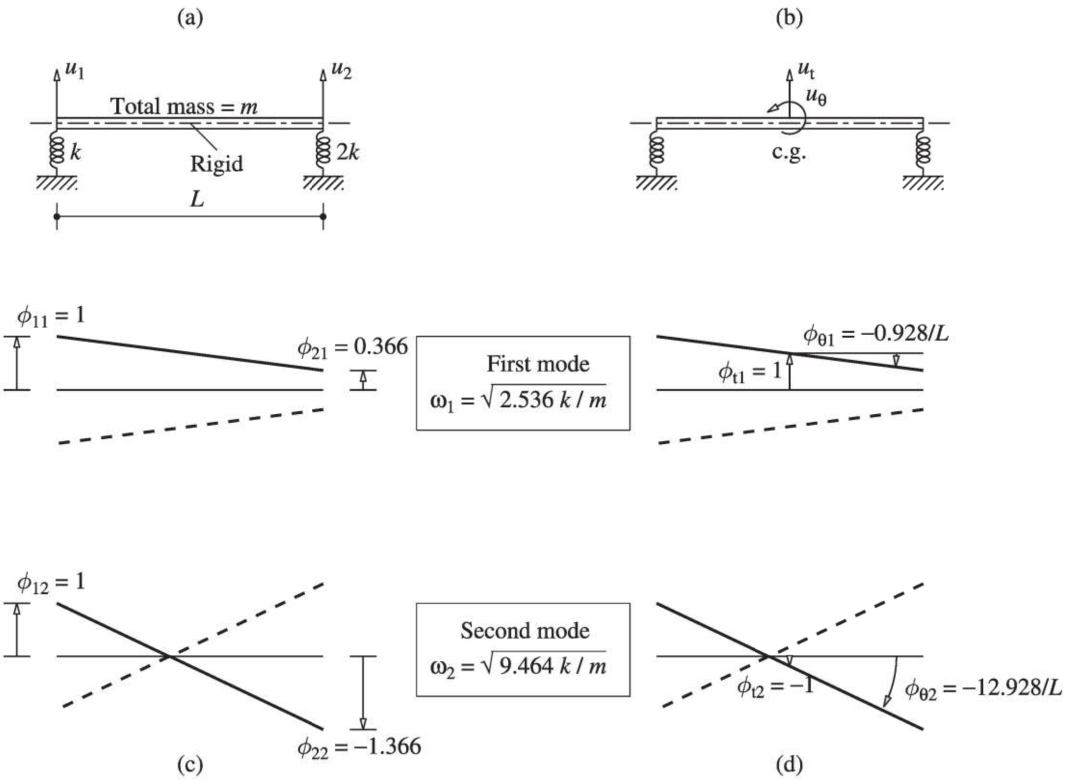


Figure E10.1

Solution (**a**) The mass and stiffness matrices for the system with the first set of DOFs were determined in Example 9.2:

$$\mathbf{m} = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then

$$\mathbf{k} - \omega_n^2 \mathbf{m} = \begin{bmatrix} k - m\omega_n^2/3 & -m\omega_n^2/6 \\ -m\omega_n^2/6 & 2k - m\omega_n^2/3 \end{bmatrix} \tag{a}$$

is substituted in Eq. (10.2.6) to obtain the frequency equation:

$$m^2\omega_n^4 - 12km\omega_n^2 + 24k^2 = 0$$

This is a quadratic equation in ω_n^2 that has the solutions

$$\omega_1^2 = \left(6 - 2\sqrt{3}\right)\frac{k}{m} = 2.536\frac{k}{m} \qquad \omega_2^2 = \left(6 + 2\sqrt{3}\right)\frac{k}{m} = 9.464\frac{k}{m} \tag{b}$$

Taking the square root of Eq. (b) gives the natural frequencies ω_1 and ω_2.

The natural modes are determined by substituting $\omega_n^2 = \omega_1^2$ in Eq. (a), and then Eq. (10.2.5) gives

$$k\begin{bmatrix} 0.155 & -0.423 \\ -0.423 & 1.155 \end{bmatrix}\begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{c}$$

Now select any value for one unknown, say $\phi_{11} = 1$. Then the first or second of the two equations gives $\phi_{21} = 0.366$. Substituting $\omega_n^2 = \omega_2^2$ in Eq. (10.2.5) gives

$$k\begin{bmatrix} -2.155 & -1.577 \\ -1.577 & -1.155 \end{bmatrix}\begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{d}$$

Selecting $\phi_{12} = 1$, either of these equations gives $\phi_{22} = -1.366$. In summary, the two modes plotted in Fig. E10.1c are

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ 0.366 \end{Bmatrix} \qquad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ -1.366 \end{Bmatrix} \tag{e}$$

(b) The mass and stiffness matrices of the system described by the second set of DOF were developed in Example 9.3:

$$\mathbf{m} = \begin{bmatrix} m & 0 \\ 0 & mL^2/12 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 3k & kL/2 \\ kL/2 & 3kL^2/4 \end{bmatrix} \tag{f}$$

Then

$$\mathbf{k} - \omega_n^2 \mathbf{m} = \begin{bmatrix} 3k - m\omega_n^2 & kL/2 \\ kL/2 & (9k - m\omega_n^2)L^2/12 \end{bmatrix} \tag{g}$$

is substituted in Eq. (10.2.6) to obtain

$$m^2\omega_n^4 - 12km\omega_n^2 + 24k^2 = 0$$

This frequency equation is the same as obtained in part (a); obviously, it gives the ω_1 and ω_2 of Eq. (b).

To determine the nth mode we go back to either of the two equations of Eq. (10.2.5) with $\left[\mathbf{k} - \omega^2\mathbf{m}\right]$ given by Eq. (g). The first equation gives

$$\left(3k - m\omega_n^2\right)\phi_{tn} + \frac{kL}{2}\phi_{\theta n} = 0 \quad \text{or} \quad \phi_{\theta n} = -\frac{3k - m\omega_n^2}{kL/2}\phi_{tn} \tag{h}$$

Substituting for $\omega_1^2 = 2.536k/m$ and $\omega_2^2 = 9.464k/m$ in Eq. (h) gives

$$\frac{L}{2}\phi_{\theta 1} = -0.464\phi_{t1} \qquad \frac{L}{2}\phi_{\theta 2} = 6.464\phi_{t2}$$

If $\phi_{t1} = 1$, then $\phi_{\theta 1} = -0.928/L$, and if $\phi_{t2} = -1$, then $\phi_{\theta 2} = -12.928/L$. In summary, the two modes plotted in Fig. E10.1d are

$$\phi_1 = \begin{Bmatrix} 1 \\ -0.928/L \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ -12.928/L \end{Bmatrix} \tag{i}$$

(c) The same natural frequencies were obtained using the two sets of DOFs. The mode shapes are given by Eqs. (e) and (i) for the two sets of DOFs. These two sets of results are plotted in Fig. E10.1c and d and can be shown to be equivalent on a graphical basis. Alternatively, the equivalence can be demonstrated by using the coordinate transformation from one set of DOFs to the other. The displacements $\mathbf{u} = \langle u_1 \quad u_2 \rangle^T$ are related to the second set of DOFs, $\bar{\mathbf{u}} = \langle u_t \quad u_\theta \rangle^T$ by

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -L/2 \\ 1 & L/2 \end{bmatrix} \begin{Bmatrix} u_t \\ u_\theta \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{a}\bar{\mathbf{u}} \tag{j}$$

The displacements $\bar{\mathbf{u}}$ in the first two modes are given by Eq. (i). Substituting the first mode in Eq. (j) leads to $\mathbf{u} = \langle 1.464 \quad 0.536 \rangle^T$. Normalizing the vector yields $\mathbf{u} = \langle 1 \quad 0.366 \rangle^T$, which is identical to ϕ_1 of Eq. (e). Similarly, substituting the second mode from Eq. (i) in Eq. (j) gives $\mathbf{u} = \langle 1 \quad -1.366 \rangle$, which is identical to ϕ_2 of Eq. (e).

Example 10.2

Determine the natural frequencies and modes of vibration of the system shown in Fig. E10.2a and defined in Example 9.5. Show that the modes satisfy the orthogonality properties.

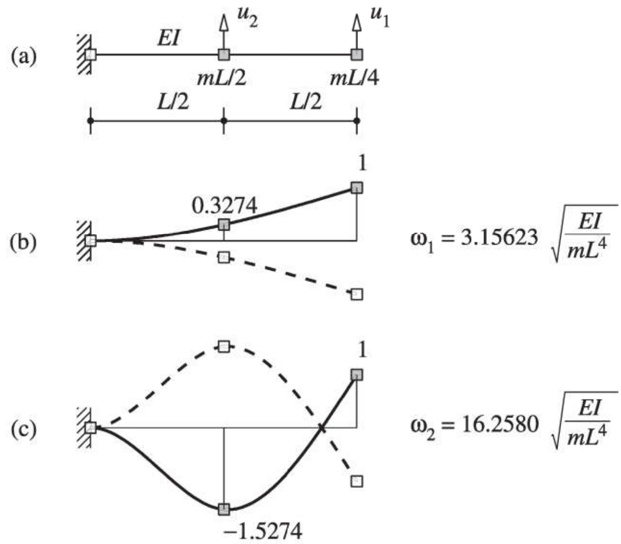


Figure E10.2

Solution The stiffness and mass matrices were determined in Example 9.5 with reference to the translational DOFs u_1 and u_2:

$$\mathbf{m} = \begin{bmatrix} mL/4 & \\ & mL/2 \end{bmatrix} \qquad \mathbf{k} = \frac{48EI}{7L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}$$

Then

$$\mathbf{k} - \omega^2\mathbf{m} = \frac{48EI}{7L^3}\begin{bmatrix} 2-\lambda & -5 \\ -5 & 16-2\lambda \end{bmatrix} \tag{a}$$

where

$$\lambda = \frac{7mL^4}{192EI}\omega^2 \tag{b}$$

Substituting Eq. (a) in (10.2.6) gives the frequency equation

$$2\lambda^2 - 20\lambda + 7 = 0$$

which has two solutions: $\lambda_1 = 0.36319$ and $\lambda_2 = 9.6368$. The natural frequencies corresponding to the two values of λ are obtained from Eq. (b)†:

$$\omega_1 = 3.15623\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 16.2580\sqrt{\frac{EI}{mL^4}} \tag{c}$$

The natural modes are determined from Eq. (10.2.5) following the procedure shown in Example 10.1 to obtain

$$\phi_1 = \begin{Bmatrix} 1 \\ 0.3274 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.5274 \end{Bmatrix} \tag{d}$$

These natural modes are plotted in Fig. E10.2b and c.

With the modes known we compute the left side of Eq. (10.4.1):

$$\phi_1^T\mathbf{m}\phi_2 = \frac{mL}{4}\langle 1 \quad 0.3274\rangle \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ -1.5274 \end{Bmatrix} = 0$$

$$\phi_1^T\mathbf{k}\phi_2 = \frac{48EI}{7L^3}\langle 1 \quad 0.3274\rangle \begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}\begin{Bmatrix} 1 \\ -1.5274 \end{Bmatrix} = 0$$

This verifies that the natural modes computed for the system are orthogonal.

Example 10.3

Determine the natural frequencies and modes of vibration of the system shown in Fig. E10.3a and defined in Example 9.6. Normalize the modes to have unit vertical deflection at the free end.

Solution The stiffness and mass matrices were determined in Example 9.6 with reference to DOFs u_1 and u_2:

$$\mathbf{m} = \begin{bmatrix} 3m & \\ & m \end{bmatrix} \qquad \mathbf{k} = \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$$

†Six significant digits are included so as to compare with the continuum model of a beam in Chapter 16.

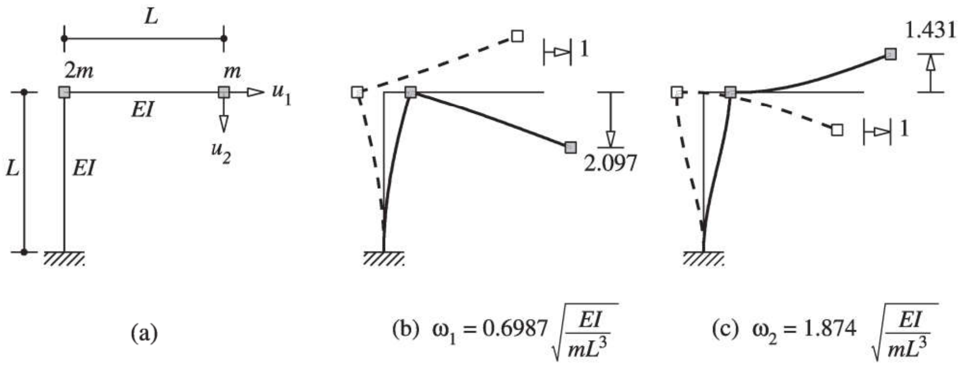


Figure E10.3

The frequency equation is Eq. (10.2.6), which, after substituting for **m** and **k**, evaluating the determinant, and defining

$$\lambda = \frac{7mL^3}{6EI}\omega^2 \tag{a}$$

can be written as

$$3\lambda^2 - 14\lambda + 7 = 0$$

The two roots are $\lambda_1 = 0.5695$ and $\lambda_2 = 4.0972$. The natural frequencies corresponding to the two values of λ are obtained from Eq. (a):

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}} \qquad \omega_2 = 1.874\sqrt{\frac{EI}{mL^3}} \tag{b}$$

The natural modes are determined from Eq. (10.2.5) following the procedure used in Example 10.1 to obtain

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} \tag{c}$$

These modes are plotted in Fig. E10.3b and c.

In computing the natural modes the mode shape value for the first DOF had been arbitrarily set as unity. The resulting mode is normalized to unit value in DOF u_2 by dividing ϕ_1 in Eq. (c) by 2.097. Similarly, the second mode is normalized by dividing ϕ_2 in Eq. (c) by -1.431. Thus the normalized modes are

$$\phi_1 = \begin{Bmatrix} 0.4769 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -0.6988 \\ 1 \end{Bmatrix} \tag{d}$$

Example 10.4

Determine the natural frequencies and modes of the system shown in Fig. E10.4a and defined in Example E9.1, a two-story frame idealized as a shear building. Normalize the modes so that $M_n = 1$.

Solution The mass and stiffness matrices of the system, determined in Example 9.1, are

$$\mathbf{m} = \begin{bmatrix} 2m & \\ & m \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \tag{a}$$

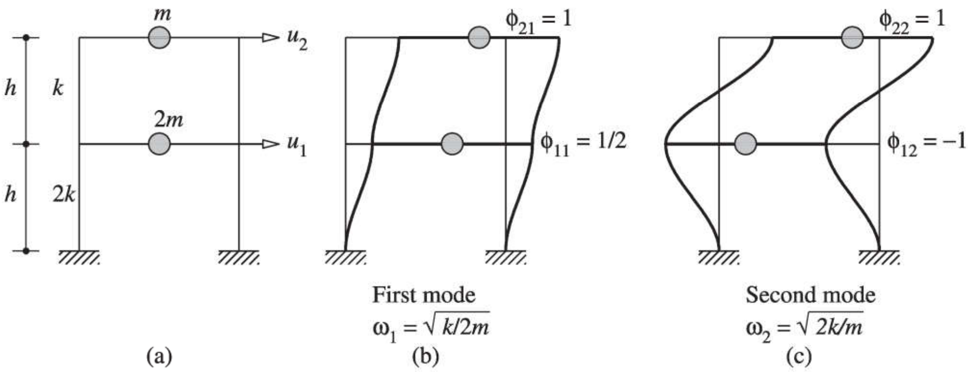


Figure E10.4

where $k = 24EI_c/h^3$. The frequency equation is Eq. (10.2.6), which, after substituting for **m** and **k** and evaluating the determinant, can be written as

$$(2m^2)\omega^4 + (-5km)\omega^2 + 2k^2 = 0 \tag{b}$$

The two roots are $\omega_1^2 = k/2m$ and $\omega_2^2 = 2k/m$, and the two natural frequencies are

$$\omega_1 = \sqrt{\frac{k}{2m}} \qquad \omega_2 = \sqrt{\frac{2k}{m}} \tag{c}$$

Substituting for k gives

$$\omega_1 = 3.464\sqrt{\frac{EI_c}{mh^3}} \qquad \omega_2 = 6.928\sqrt{\frac{EI_c}{mh^3}} \tag{d}$$

The natural modes are determined from Eq. (10.2.5) following the procedure used in Example 10.1 to obtain

$$\phi_1 = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{e}$$

These natural modes are shown in Fig. E10.4b and c.

To normalize the first mode, M_1 is calculated using Eq. (10.4.6), with ϕ_1 given by Eq. (e):

$$M_1 = \phi_1^T \mathbf{m} \phi_1 = m \left\langle \tfrac{1}{2} \quad 1 \right\rangle \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} = \frac{3}{2} m$$

To make $M_1 = 1$, divide ϕ_1 of Eq. (e) by $\sqrt{3m/2}$ to obtain the normalized mode,

$$\phi_1 = \frac{1}{\sqrt{6m}} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

For this ϕ_1 it can be verified that $M_1 = 1$. The second mode can be normalized similarly.

Example 10.5

Determine the natural frequencies and modes of the system shown in Fig. E10.5a and defined earlier in Example 9.9. The story height $h = 10$ ft.

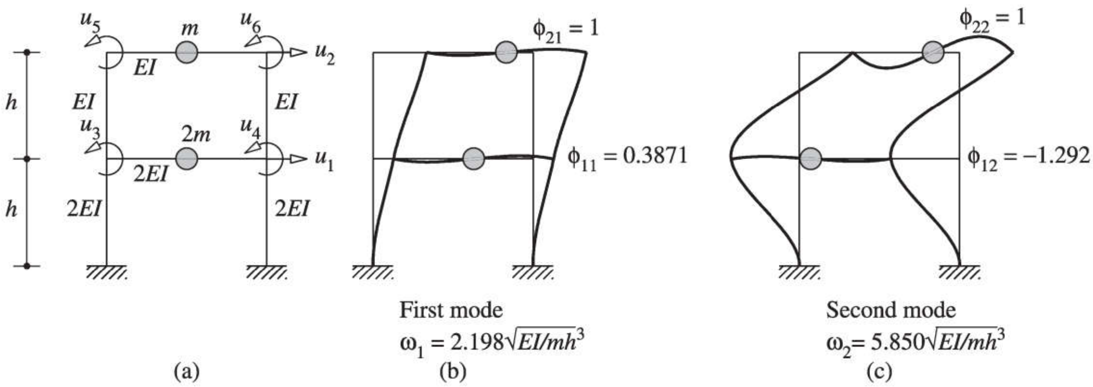


Figure E10.5

Solution With reference to the lateral displacements u_1 and u_2 of the two floors as the two DOFs, the mass matrix and the condensed stiffness matrix were determined in Example 9.9:

$$\mathbf{m}_{tt} = m\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \qquad \hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 54.88 & -17.51 \\ -17.51 & 11.61 \end{bmatrix} \tag{a}$$

The frequency equation is

$$\det(\hat{\mathbf{k}}_{tt} - \omega^2 \mathbf{m}_{tt}) = 0 \tag{b}$$

Substituting for $\mathbf{m}_{tt}$ and $\hat{\mathbf{k}}_{tt}$, evaluating the determinant, and obtaining the two roots just as in Example 10.4 leads to

$$\omega_1 = 2.198\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 5.850\sqrt{\frac{EI}{mh^3}} \tag{c}$$

It is of interest to compare these frequencies for a frame with flexible beams with those for the frame with flexurally rigid beams determined in Example 10.4. It is clear that beam flexibility has the effect of lowering the frequencies, consistent with intuition.

The natural modes are determined by solving

$$(\hat{\mathbf{k}}_{tt} - \omega_n^2 \mathbf{m}_{tt})\phi_n = \mathbf{0} \tag{d}$$

with ω_1 and ω_2 substituted successively from Eq. (c) to obtain

$$\phi_1 = \begin{Bmatrix} 0.3871 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1.292 \\ 1 \end{Bmatrix} \tag{e}$$

These vectors define the lateral displacements of each floor. They are shown in Fig. E10.5b and c together with the joint rotations. The joint rotations associated with the first mode are determined by substituting $\mathbf{u}_t = \phi_1$ from Eq. (e) in Eq. (d) of Example 9.9:

$$\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} -0.4426 & -0.2459 \\ -0.4426 & -0.2459 \\ 0.9836 & -0.7869 \\ 0.9836 & -0.7869 \end{bmatrix} \begin{Bmatrix} 0.3871 \\ 1.0000 \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} -0.4172 \\ -0.4172 \\ -0.4061 \\ -0.4061 \end{Bmatrix} \tag{f}$$

Similarly, the joint rotations associated with the second mode are obtained by substituting $\mathbf{u}_t = \phi_2$ from Eq. (e) in Eq. (d) of Example 9.9:

$$\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{h} \begin{Bmatrix} 0.3258 \\ 0.3258 \\ -2.0573 \\ -2.0573 \end{Bmatrix} \tag{g}$$

Example 10.6

Figure 9.5.1 shows the plan view of a one-story building. The structure consists of a roof, idealized as a rigid diaphragm, supported on three frames, A, B, and C, as shown. The roof weight is uniformly distributed and has a magnitude of 100 lb/ft.^{2} The lateral stiffnesses of the frames are $k_y = 75$ kips/ft for frame A, and $k_x = 40$ kips/ft for frames B and C. The plan dimensions are $b = 30$ ft and $d = 20$ ft, the eccentricity is $e = 1.5$ ft, and the height of the building is 12 ft. Determine the natural periods and modes of vibration of the structure.

Solution

Weight of roof slab: $w = 30 \times 20 \times 100 \text{ lb} = 60$ kips

Mass: $m = w/g = 1.863 \text{ kips-sec}^2/\text{ft}$

Moment of inertia: $I_O = \dfrac{m(b^2 + d^2)}{12} = 201.863 \text{ kips-ft-sec}^2$

Lateral motion of the roof diaphragm in the x-direction is governed by Eq. (9.5.18):

$$m\ddot{u}_x + 2k_x u_x = 0 \tag{a}$$

Thus the natural frequency of x-lateral vibration is

$$\omega_x = \sqrt{\frac{2k_x}{m}} = \sqrt{\frac{2(40)}{1.863}} = 6.553 \text{ rad/sec}$$

The corresponding natural mode is shown in Fig. E10.6c.

The coupled lateral (u_y)-torsional (u_θ) motion of the roof diaphragm is governed by Eq. (9.5.19). Substituting for m and I_O gives

$$\mathbf{m} = \begin{bmatrix} 1.863 & \\ & 201.863 \end{bmatrix}$$

From Eqs. (9.5.16) and (9.5.19) the stiffness matrix has four elements:

$$k_{yy} = k_y = 75 \text{ kips/ft}$$

$$k_{y\theta} = k_{\theta y} = ek_y = 1.5 \times 75 = 112.5 \text{ kips}$$

$$k_{\theta\theta} = e^2 k_y + \frac{d^2}{2} k_x = 8168.75 \text{ kips-ft}$$

Hence,

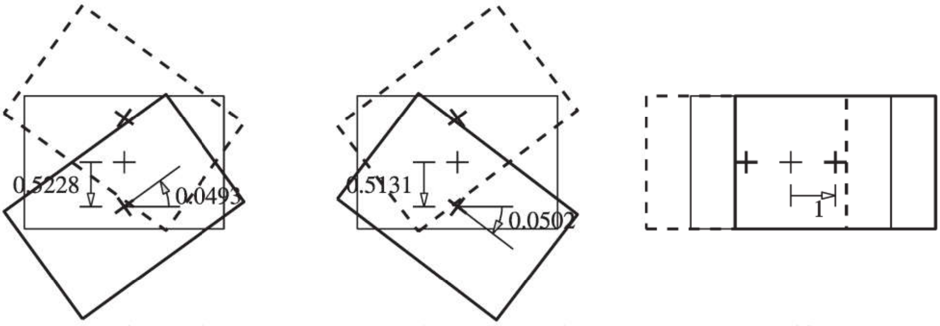
$$\mathbf{k} = \begin{bmatrix} 75.00 & 112.50 \\ 112.50 & 8168.75 \end{bmatrix}$$

With $\mathbf{k}$ and $\mathbf{m}$ known, the eigenvalue problem for this two-DOF system is solved by standard procedures to obtain:

Natural frequencies (rad/sec): $\omega_1 = 5.878$; $\omega_2 = 6.794$

Natural modes: $\phi_1 = \begin{Bmatrix} -0.5228 \\ 0.0493 \end{Bmatrix}$; $\phi_2 = \begin{Bmatrix} -0.5131 \\ -0.0502 \end{Bmatrix}$

These mode shapes are plotted in Fig. E10.6a and b. The motion of the structure in each mode consists of translation of the rigid diaphragm coupled with torsion about the vertical axis through the center of mass.



(a) First mode
$\omega_1 = 5.878$ rad/sec

(b) Second mode
$\omega_2 = 6.794$ rad/sec

(c) Third mode
$\omega_x = 6.553$ rad/sec

Figure E10.6

Example 10.7

Consider a special case of the system of Example 10.6 in which frame A is located at the center of mass (i.e., $e = 0$). Determine the natural frequencies and modes of this system.

Solution Equation (9.5.20) specialized for free vibration of this system gives three equations of motion:

$$m\ddot{u}_x + 2k_x u_x = 0 \qquad m\ddot{u}_y + k_y u_y = 0 \qquad I_O \ddot{u}_\theta + \frac{d^2}{2} k_x u_\theta = 0 \tag{a}$$

The first equation of motion indicates that translational motion in the x-direction would occur at the natural frequency

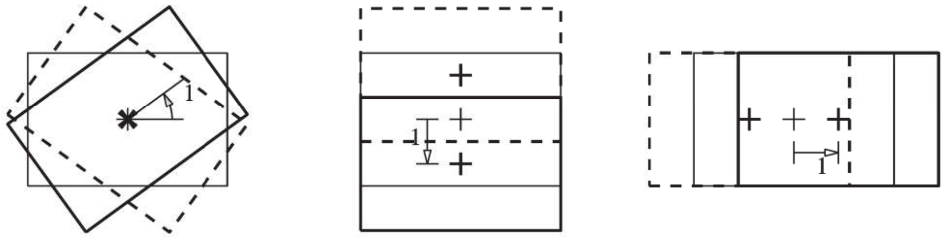
$$\omega_x = \sqrt{\frac{2k_x}{m}} = \sqrt{\frac{2(40)}{1.863}} = 6.553 \text{ rad/sec}$$

This motion is independent of lateral motion u_y or torsional motion u_θ (Fig. E10.7c). The second equation of motion indicates that translational motion in the y-direction would occur at the natural frequency

$$\omega_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{75}{1.863}} = 6.344 \text{ rad/sec}$$

This motion is independent of the lateral motion u_x or torsional motion u_θ (Fig. E10.7b). The third equation of motion indicates that torsional motion would occur at the natural frequency

$$\omega_\theta = \sqrt{\frac{d^2 k_x}{2I_O}} = \sqrt{\frac{(20)^2 40}{2(201.863)}} = 6.295 \text{ rad/sec}$$



(a) First mode
$\omega_\theta = 6.295$ rad/sec

(b) Second mode
$\omega_y = 6.344$ rad/sec

(c) Third mode
$\omega_x = 6.553$ rad/sec

Figure E10.7

The roof diaphragm would rotate about the vertical axis through its center of mass without any translation of this point in the x or y directions (Fig. E10.7a).

Observe that the natural frequencies ω_1 and ω_2 of the unsymmetric-plan system (Example 10.6) are different from and more separated than the natural frequencies ω_y and ω_θ of the symmetric-plan system (Example 10.7).

10.7 MODAL EXPANSION OF DISPLACEMENTS

Any set of N independent vectors can be used as a basis for representing any other vector of order N. In the following sections the natural modes are used as such a basis. Thus, a modal expansion of any displacement vector $\mathbf{u}$ has the form

$$\mathbf{u} = \sum_{r=1}^{N} \boldsymbol{\phi}_r q_r = \boldsymbol{\Phi}\mathbf{q} \tag{10.7.1}$$

where q_r are scalar multipliers called *modal coordinates* or *normal coordinates* and $\mathbf{q} = \langle q_1 \quad q_2 \quad \cdots \quad q_n \rangle^T$. When the $\boldsymbol{\phi}_r$ are known, for a given $\mathbf{u}$ it is possible to evaluate the q_r by multiplying both sides of Eq. (10.7.1) by $\boldsymbol{\phi}_n^T \mathbf{m}$:

$$\boldsymbol{\phi}_n^T \mathbf{m}\mathbf{u} = \sum_{r=1}^{N} (\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r) q_r$$

Because of the orthogonality relation of Eq. (10.4.1b), all terms in the summation above vanish except the $r = n$ term; thus

$$\boldsymbol{\phi}_n^T \mathbf{m}\mathbf{u} = (\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n) q_n$$

The matrix products on both sides of this equation are scalars. Therefore,

$$q_n = \frac{\boldsymbol{\phi}_n^T \mathbf{m}\mathbf{u}}{\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n} = \frac{\boldsymbol{\phi}_n^T \mathbf{m}\mathbf{u}}{M_n} \tag{10.7.2}$$

The modal expansion of the displacement vector **u**, Eq. (10.7.1), is employed in Section 10.8 to obtain solutions for the free vibration response of undamped systems. It also plays a central role in the analysis of forced vibration response and earthquake response of MDF systems (Chapters 12 and 13).

Example 10.8

For the two-story shear frame of Example 10.4, determine the modal expansion of the displacement vector $\mathbf{u} = \langle 1 \quad 1\rangle^T$.

Solution The displacement **u** is substituted in Eq. (10.7.2) together with $\phi_1 = \langle \frac{1}{2} \quad 1\rangle^T$ and $\phi_2 = \langle -1 \quad 1\rangle^T$, from Example 10.4, to obtain

$$q_1 = \frac{\langle \frac{1}{2} \quad 1\rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\langle \frac{1}{2} \quad 1\rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}} = \frac{2m}{3m/2} = \frac{4}{3}$$

$$q_2 = \frac{\langle -1 \quad 1\rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\langle -1 \quad 1\rangle \begin{bmatrix} 2m & \\ & m \end{bmatrix} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}} = \frac{-m}{3m} = -\frac{1}{3}$$

Substituting q_n in Eq. (10.7.1) gives the desired modal expansion, which is shown in Fig. E10.8.

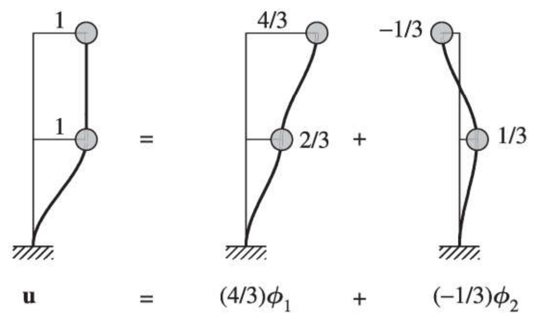


Figure E10.8

PART B: FREE VIBRATION RESPONSE

10.8 SOLUTION OF FREE VIBRATION EQUATIONS: UNDAMPED SYSTEMS

We now return to the problem posed by Eqs. (10.1.1) and (10.1.2) and find its solution. For the example structure of Fig. 10.1.1a, such a solution was shown in Fig. 10.1.1d. The differential equation (10.1.1) to be solved had led to the matrix eigenvalue problem of Eq. (10.2.4). Assuming that the eigenvalue problem has been solved for the natural frequencies and modes, the general solution of Eq. (10.1.1) is given by a superposition of the response in individual modes given by Eq. (10.2.3). Thus

$$\mathbf{u}(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_n (A_n \cos \omega_n t + B_n \sin \omega_n t) \tag{10.8.1}$$

where A_n and B_n are $2N$ constants of integration. To determine these constants, we will also need the equation for the velocity vector, which is

$$\dot{\mathbf{u}}(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_n \omega_n (-A_n \sin \omega_n t + B_n \cos \omega_n t) \tag{10.8.2}$$

Setting $t = 0$ in Eqs. (10.8.1) and (10.8.2) gives

$$\mathbf{u}(0) = \sum_{n=1}^{N} \boldsymbol{\phi}_n A_n \qquad \dot{\mathbf{u}}(0) = \sum_{n=1}^{N} \boldsymbol{\phi}_n \omega_n B_n \tag{10.8.3}$$

With the initial displacements $\mathbf{u}(0)$ and initial velocities $\dot{\mathbf{u}}(0)$ known, each of these two equation sets represents N algebraic equations in the unknowns A_n and B_n, respectively. Simultaneous solution of these equations is not necessary because they can be interpreted as a modal expansion of the vectors $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$. Following Eq. (10.7.1), we can write

$$\mathbf{u}(0) = \sum_{n=1}^{N} \boldsymbol{\phi}_n q_n(0) \qquad \dot{\mathbf{u}}(0) = \sum_{n=1}^{N} \boldsymbol{\phi}_n \dot{q}_n(0) \tag{10.8.4}$$

where, analogous to Eq. (10.7.2), $q_n(0)$ and $\dot{q}_n(0)$ are given by

$$q_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \mathbf{u}(0)}{M_n} \qquad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \dot{\mathbf{u}}(0)}{M_n} \tag{10.8.5}$$

Equations (10.8.3) and (10.8.4) are equivalent, implying that $A_n = q_n(0)$ and $B_n = \dot{q}_n(0)/\omega_n$. Substituting these in Eq. (10.8.1) gives

$$\mathbf{u}(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_n \left[q_n(0) \cos \omega_n t + \frac{\dot{q}_n(0)}{\omega_n} \sin \omega_n t \right] \tag{10.8.6}$$

or, alternatively,

$$\mathbf{u}(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_n q_n(t) \tag{10.8.7}$$

where

$$q_n(t) = q_n(0) \cos \omega_n t + \frac{\dot{q}_n(0)}{\omega_n} \sin \omega_n t \tag{10.8.8}$$

is the time variation of modal coordinates, which is analogous to the free vibration response of SDF systems [Eq. (2.1.3)]. Equation (10.8.6) provides the displacement $\mathbf{u}$ as a function of time due to initial displacement $\mathbf{u}(0)$ and velocity $\dot{\mathbf{u}}(0)$; the $\mathbf{u}(t)$ is independent of how the modes are normalized, although $q_n(t)$ are not. Assuming that the natural frequencies ω_n and modes $\boldsymbol{\phi}_n$ are available, the right side of Eq. (10.8.6) is known with $q_n(0)$ and $\dot{q}_n(0)$ defined by Eq. (10.8.5).

Example 10.9

Determine the free vibration response of the two-story shear frame of Example 10.4 due to initial displacement $\mathbf{u}(0) = \langle \frac{1}{2} \quad 1 \rangle^T$.

Solution The initial displacement and velocity vectors are

$$\mathbf{u}(0) = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \qquad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For the given $\mathbf{u}(0)$, $q_n(0)$ are calculated following the procedure of Example 10.8 and using ϕ_n from Eq. (e) of Example 10.4; the results are $q_1(0) = 1$ and $q_2(0) = 0$. Because the initial velocity $\dot{\mathbf{u}}(0)$ is zero, $\dot{q}_1(0) = \dot{q}_2(0) = 0$. Inserting $q_n(0)$ and $\dot{q}_n(0)$ in Eq. (10.8.8) gives the solution for modal coordinates

$$q_1(t) = 1 \cos \omega_1 t \qquad q_2(t) = 0$$

Substituting $q_n(t)$ and ϕ_n in Eq. (10.8.7) leads to

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \cos \omega_1 t$$

where $\omega_1 = \sqrt{k/2m}$ from Example 10.4. These solutions for $q_1(t)$, $u_1(t)$, and $u_2(t)$ had been plotted in Fig. 10.1.2c and d. Note that $q_2(t) = 0$ implies that the second mode has no contribution to the response, which is all due to the first mode. Such is the case because the initial displacement is proportional to the first mode and hence orthogonal to the second mode.

Example 10.10

Determine the free vibration response of the two-story shear frame of Example 10.4 due to initial displacement $\mathbf{u}(0) = \langle -1 \quad 1 \rangle^T$.

Solution The calculations proceed as in Example 10.9, leading to $q_1(0) = 0$, $q_2(0) = 1$, and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. Inserting these in Eq. (10.8.8) gives the solutions for modal coordinates:

$$q_1(t) = 0 \qquad q_2(t) = 1 \cos \omega_2 t$$

Substituting $q_n(t)$ and ϕ_n in Eq. (10.8.7) leads to

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cos \omega_2 t$$

where $\omega_2 = \sqrt{2k/m}$ from Example 10.4. These solutions for $q_2(t)$, $u_1(t)$, and $u_2(t)$ had been plotted in Fig. 10.1.3c and d. Note that $q_1(t) = 0$ implies that the first mode has no contribution to the response and the response is due entirely to the second mode. Such is the case because the initial displacement is proportional to the second mode and hence orthogonal to the first mode.

Example 10.11

Determine the free vibration response of the two-story shear frame of Example 10.4 due to initial displacements $\mathbf{u}(0) = \langle -\frac{1}{2} \quad 2 \rangle^T$.

Solution Following Example 10.8, $q_n(0)$ and $\dot{q}_n(0)$ are evaluated: $q_1(0) = 1$, $q_2(0) = 1$, and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. Substituting these in Eq. (10.8.8) gives the solution for modal coordinates

$$q_1(t) = 1 \cos \omega_1 t \qquad q_2(t) = 1 \cos \omega_2 t$$

Substituting $q_n(t)$ and ϕ_n in Eq. (10.8.7) leads to

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \cos\omega_1 t + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cos\omega_2 t$$

These solutions for $q_n(t)$ and $u_j(t)$ had been plotted in Fig. 10.1.1c and d. Observe that both natural modes contribute to the response due to these initial displacements.

10.9 SYSTEMS WITH DAMPING

When damping is included, the free vibration response of the system is governed by Eq. (9.2.12) with $\mathbf{p}(t) = \mathbf{0}$:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \tag{10.9.1}$$

It is desired to find the solution $\mathbf{u}(t)$ of Eq. (10.9.1) that satisfies the initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \tag{10.9.2}$$

at $t = 0$. Procedures to obtain the desired solution differ depending on the form of damping: classical or nonclassical; these terms are defined next.

If the damping matrix of a linear system satisfies the identity

$$\mathbf{c}\mathbf{m}^{-1}\mathbf{k} = \mathbf{k}\mathbf{m}^{-1}\mathbf{c} \tag{10.9.3}$$

all the natural modes of vibration are real-valued and identical to those of the associated undamped system; they were determined by solving the real eigenvalue problem, Eq. (10.2.4). Such systems are said to possess *classical damping* because they have classical natural modes, defined in Section 10.2.

To state an important property of classically damped systems, we express the displacement $\mathbf{u}$ in terms of the natural modes of the associated undamped system; thus we substitute Eq. (10.7.1) in Eq. (10.9.1):

$$\mathbf{m}\boldsymbol{\Phi}\ddot{\mathbf{q}} + \mathbf{c}\boldsymbol{\Phi}\dot{\mathbf{q}} + \mathbf{k}\boldsymbol{\Phi}\mathbf{q} = \mathbf{0}$$

Premultiplying by $\boldsymbol{\Phi}^T$ gives

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{10.9.4}$$

where the diagonal matrices $\mathbf{M}$ and $\mathbf{K}$ were defined in Eq. (10.4.5) and

$$\mathbf{C} = \boldsymbol{\Phi}^T\mathbf{c}\boldsymbol{\Phi} \tag{10.9.5}$$

For classically damped systems, the square matrix $\mathbf{C}$ is diagonal. Then, Eq. (10.9.4) represents N uncoupled differential equations in modal coordinates q_n, and classical modal analysis is applicable to such systems. Such a procedure to solve Eq. (10.9.1) is presented in Section 10.10.

A linear system is said to possess *nonclassical damping* if its damping matrix does not satisfy Eq. (10.9.3). For such systems, the natural modes of vibration are not real-valued, and the square matrix $\mathbf{C}$ of Eq. (10.9.5) is no longer diagonal, thus they are not amenable to classical modal analysis. Analytical solutions of Eq. (10.9.1) for nonclassically damped systems are presented in Chapter 14, and numerical solution methods in Chapter 16.

10.10 SOLUTION OF FREE VIBRATION EQUATIONS: CLASSICALLY DAMPED SYSTEMS

For classically damped systems, each of the N uncoupled differential equations in Eq. (10.9.4) is of the form

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = 0 \tag{10.10.1}$$

where M_n and K_n were defined in Eq. (10.4.6) and

$$C_n = \phi_n^T \mathbf{c} \phi_n \tag{10.10.2}$$

Equation (10.10.1) is of the same form as Eq. (2.2.1a) for an SDF system with damping. Thus the damping ratio can be defined for each mode in a manner analogous to Eq. (2.2.2) for an SDF system:

$$\zeta_n = \frac{C_n}{2M_n\omega_n} \tag{10.10.3}$$

Dividing Eq. (10.10.1) (t) by M_n gives

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = 0 \tag{10.10.4}$$

This equation is of the same form as Eq. (2.2.1b) governing the free vibration of an SDF system with damping for which the solution is Eq. (2.2.4). Adapting this result, the solution for Eq. (10.10.4) is

$$q_n(t) = e^{-\zeta_n\omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}_n(0) + \zeta_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right] \tag{10.10.5}$$

where the nth natural frequency of the system with damping

$$\omega_{nD} = \omega_n\sqrt{1-\zeta_n^2} \tag{10.10.6}$$

and ω_n is the nth natural frequency of the associated undamped system. The displacement response of the system is then obtained by substituting Eq. (10.10.5) for $q_n(t)$ in Eq. (10.8.7):

$$\mathbf{u}(t) = \sum_{n=1}^{N}\phi_n e^{-\zeta_n\omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}_n(0) + \zeta_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right] \tag{10.10.7}$$

This is the solution of the free vibration problem for classically damped MDF systems. It provides the displacement $\mathbf{u}$ as a function of time due to initial displacement $\mathbf{u}(0)$ and velocity $\dot{\mathbf{u}}(0)$. Assuming that the natural frequencies ω_n and modes ϕ_n of the system without damping are available together with the modal damping ratios ζ_n, the right side of Eq. (10.10.7) is known with $q_n(0)$ and $\dot{q}_n(0)$ defined by Eq. (10.8.5).

Damping influences the natural frequencies and periods of vibration of an MDF system according to Eq. (10.10.6), which is of the same form as Eq. (2.2.5) for an SDF system. Therefore, the effect of damping on the natural frequencies and periods of an MDF system is small for damping ratios ζ_n below 20% (Fig. 2.2.3), a range that includes most practical structures.

In a classically damped MDF system undergoing free vibration in its nth natural mode, the displacement amplitude at any DOF decreases with each vibration cycle. The rate of decay depends on the damping ratio ζ_n in that mode, in a manner similar to SDF systems. This similarity is apparent by comparing Eq. (10.10.5) with Eq. (2.2.4). Thus the ratio of two response peaks separated by j cycles of vibration is related to the damping ratio by Eq. (2.2.12) with appropriate change in notation.

Consequently, the damping ratio in a natural mode of an MDF system can be determined, in principle, from a free vibration test following the procedure presented in Section 2.2.4 for SDF systems. In such a test the structure would be deformed by pulling on it with a cable that is then suddenly released, thus causing the structure to undergo free vibration about its static equilibrium position. A difficulty in such tests is to apply the pull and release in such a way that the structure will vibrate in only one of its natural modes. For this reason this test procedure is not an effective means to determine damping except possibly for the fundamental mode. After the response contributions of the higher modes have damped out, the free vibration is essentially in the fundamental mode, and the damping ratio for this mode can be computed from the decay rate of vibration amplitudes.

Example 10.12

Determine the free vibration response of the two-story shear frame of Fig. E10.12.1a with $c = \sqrt{km/200}$ due to two sets of initial displacement (1) $\mathbf{u}(0) = \langle \frac{1}{2} \quad 1 \rangle^T$ and (2) $\mathbf{u}(0) = \langle -1 \quad 1 \rangle^T$.

Solution

Part 1 The $q_n(0)$ corresponding to this $\mathbf{u}(0)$ were determined in Example 10.9: $q_1(0) = 1$ and $q_2(0) = 0$; $\dot{q}_n(0) = 0$. The differential equations governing $q_n(t)$ are given by

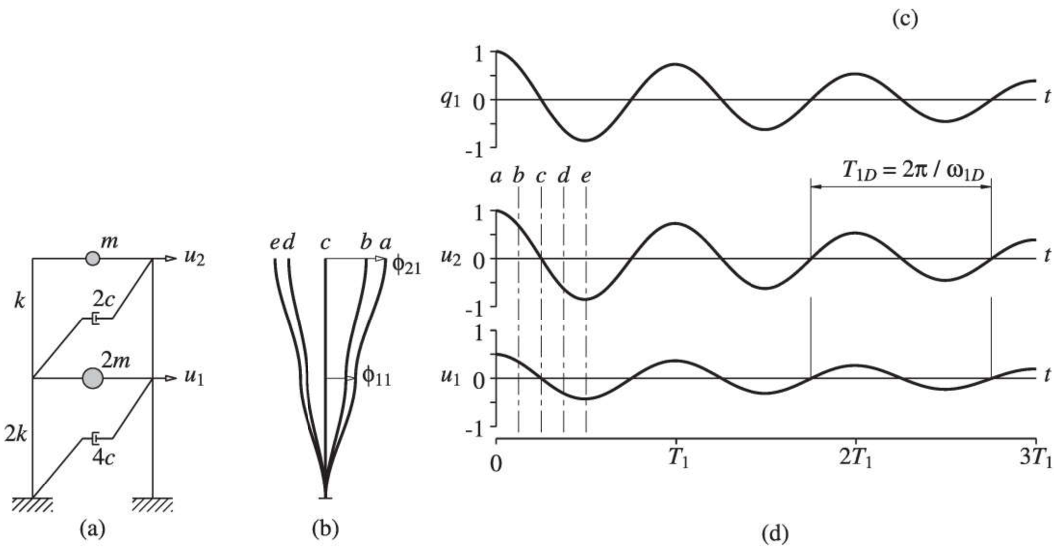


Figure E10.12.1 Free vibration of a classically damped system in the first natural mode of vibration: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinate $q_1(t)$; (d) displacement history.

Eq. (10.10.4). Because $q_2(0)$ and $\dot{q}_2(0)$ are both zero, $q_2(t) = 0$ for all times. The response is given by the $n = 1$ term in Eq. (10.10.7). Substituting the aforementioned values for $q_1(0)$, $\dot{q}_1(0)$, and $\phi_1 = \langle \frac{1}{2} \quad 1 \rangle^T$ gives

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} e^{-\zeta_1 \omega_1 t} \left(\cos \omega_{1D} t + \frac{\zeta_1}{\sqrt{1 - \zeta_1^2}} \sin \omega_{1D} t \right)$$

where $\omega_1 = \sqrt{k/2m}$ from Example 10.4 and $\zeta_1 = 0.05$ from Eq. (10.10.3).

Part 2 The $q_n(0)$ corresponding to this $\mathbf{u}(0)$ were determined in Example 10.10: $q_1(0) = 0$ and $q_2(0) = 1$; $\dot{q}_n(0) = 0$. The differential equations governing $q_n(t)$ are given by Eq. (10.10.4). Because $q_1(0)$ and $\dot{q}_1(0)$ are both zero, $q_1(t) = 0$ at all times. The response is given by the $n = 2$ term in Eq. (10.10.7). Substituting for $q_2(0)$, $\dot{q}_2(0)$, and $\phi_2 = \langle -1 \quad 1 \rangle^T$ gives

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} e^{-\zeta_2 \omega_2 t} \left(\cos \omega_{2D} t + \frac{\zeta_2}{\sqrt{1 - \zeta_2^2}} \sin \omega_{2D} t \right)$$

where $\omega_2 = \sqrt{2k/m}$ from Example 10.4 and $\zeta_2 = 0.10$ from Eq. (10.10.3).

Observations The results for free vibration due to initial displacements $\mathbf{u}(0) = \phi_1$ are presented in Fig. E10.12.1, and for $\mathbf{u}(0) = \phi_2$ in Fig. E10.12.2, respectively. The solutions for $q_n(t)$ are presented in part (c) of these figures; the floor displacements $u_j(t)$ in part (d); and the deflected shapes at selected time instants are plotted in part (b) of these figures.

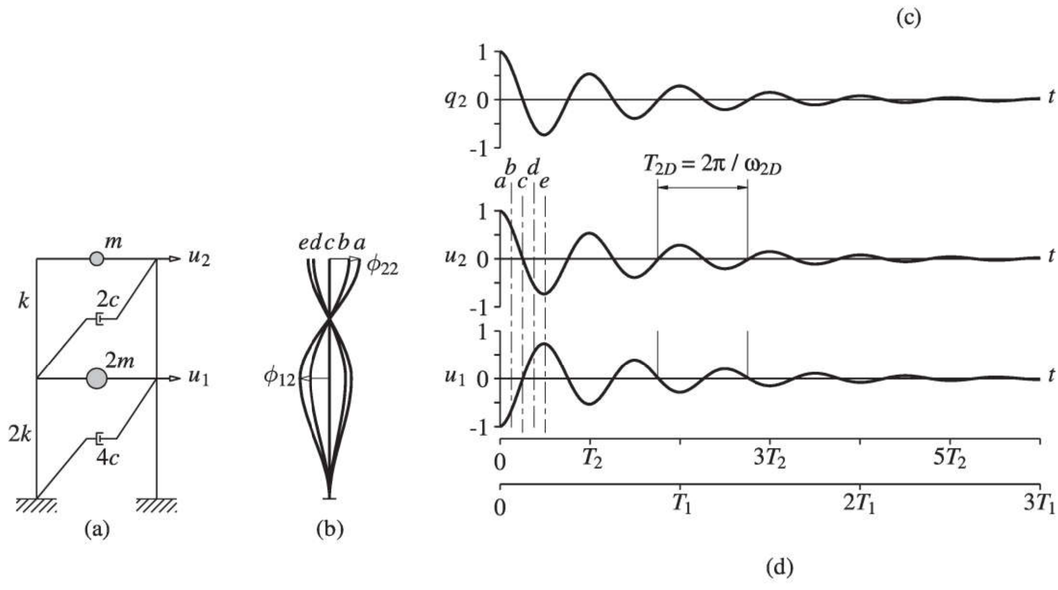


Figure E10.12.2 Free vibration of a classically damped system in the second natural mode of vibration: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinate $q_2(t)$; (d) displacement history.

These results permit the following observations: First, if the initial displacement is proportional to the nth mode, the response is due entirely to that mode; the other mode has no contribution. Second, the initial deflected shape is maintained during free vibration, just as in the case of undamped systems (Figs. 10.1.2 and 10.1.3). Third, the system oscillates at the frequency ω_{nD} with all floors (or DOFs) vibrating in the same phase, passing through their equilibrium, maximum, or minimum positions at the same instant of time. Thus, the system possesses classical natural modes of vibration, defined first in Section 10.2, as expected of classically damped systems. Although based on results for a system with two DOFs, these observations are valid for classically damped systems with any number of DOFs.

Example 10.13

Determine the free vibration response of the two-story shear frame of Example 10.12 due to initial displacements $\mathbf{u}(0) = \langle -\frac{1}{2} \;\; 2 \rangle^T$.

Solution The $q_n(0)$ corresponding to this $\mathbf{u}(0)$ were determined in Example 10.11: $q_n(0) = 1$ and $q_n(0) = 1$; $\dot{q}_n(0) = 0$. Substituting them in Eq. (10.10.5) gives the solution for modal coordinates:

$$q_1(t) = e^{-\zeta_1 \omega_1 t} \left[\cos \omega_{1D} t + \frac{\zeta_1}{\sqrt{1-\zeta_1^2}} \sin \omega_{1D} t \right] \tag{a}$$

$$q_2(t) = e^{-\zeta_2 \omega_2 t} \left[\cos \omega_{2D} t + \frac{\zeta_2}{\sqrt{1-\zeta_2^2}} \sin \omega_{2D} t \right] \tag{b}$$

where, as determined earlier, $\omega_1 = \sqrt{k/2m}$ and $\omega_2 = \sqrt{2k/m}$; ω_{nD} are given by Eq. (10.10.6), and $\zeta_1 = 0.05$ and $\zeta_2 = 0.10$ from Eq. (10.10.3).

Substituting $q_n(t)$ and ϕ_n in Eq. (10.8.7) leads to

$$\begin{aligned} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix} e^{-\zeta_1 \omega_1 t} \left[\cos \omega_{1D} t + \frac{\zeta_1}{\sqrt{1-\zeta_1^2}} \sin \omega_{1D} t \right] \\ + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} e^{-\zeta_2 \omega_2 t} \left[\cos \omega_{2D} t + \frac{\zeta_2}{\sqrt{1-\zeta_2^2}} \sin \omega_{2D} t \right] \end{aligned} \tag{c}$$

PART C: COMPUTATION OF VIBRATION PROPERTIES

10.11 SOLUTION METHODS FOR THE EIGENVALUE PROBLEM

Finding the vibration properties—natural frequencies and modes—of a structure requires solution of the matrix eigenvalue problem of Eq. (10.2.4), which is repeated for convenience:

$$\mathbf{k}\phi = \lambda \mathbf{m}\phi \tag{10.11.1}$$

As mentioned earlier, the eigenvalues $\lambda_n \equiv \omega_n^2$ are the roots of the characteristic equation (10.2.6):

$$p(\lambda) = \det[\mathbf{k} - \lambda\mathbf{m}] = 0 \tag{10.11.2}$$

where $p(\lambda)$ is a polynomial of order N, the number of DOFs of the system. This is not a practical method, especially for large systems (i.e., a large number of DOFs), because evaluation of the N coefficients of the polynomial requires much computational effort and the roots of $p(\lambda)$ are sensitive to numerical round-off errors in the coefficients.

Finding reliable and efficient methods to solve the eigenvalue problem has been the subject of much research, especially since development of the digital computer. Most of the methods available can be classified into three broad categories depending on which basic property is used as the basis of the solution algorithm: (1) Vector iteration methods work directly with the property of Eq. (10.11.1). (2) Transformation methods use the orthogonality property of modes, Eqs. (10.4.1). (3) Polynomial iteration techniques work on the fact that $p(\lambda_n) = 0$. A number of solution algorithms have been developed within each of the foregoing three categories. Combination of two or more methods that belong to the same or to different categories have been developed to deal with large systems. Two examples of such combined procedures are the determinant search method and the subspace iteration method.

All solution methods for eigenvalue problems must be iterative in nature because, basically, solving the eigenvalue problem is equivalent to finding the roots of the polynomial $p(\lambda)$. No explicit formulas are available for these roots when N is larger than 4, thus requiring an iterative solution. To find an eigenpair (λ_n, ϕ_n), only one of them is calculated by iteration; the other can be obtained without further iteration. For example, if λ_n is obtained by iteration, then ϕ_n can be evaluated by solving the algebraic equations $(\mathbf{k} - \lambda_n\mathbf{m})\phi_n = \mathbf{0}$. On the other hand, if ϕ_n is determined by iteration, λ_n can be obtained by evaluating Rayleigh's quotient (Section 10.12). Is it most economical to solve first for λ_n and then calculate ϕ_n (or vice versa), or to solve for both simultaneously? The answer to this question and hence the choice among the three procedure categories mentioned above depends on the properties of the mass and stiffness matrices—size N, bandwidth of $\mathbf{k}$, and whether $\mathbf{m}$ is diagonal or banded—and on the number of eigenpairs required.

In structural engineering we are usually analyzing systems with narrowly banded $\mathbf{k}$ and diagonal or narrowly banded $\mathbf{m}$ subjected to excitations that excite primarily the lower few (relative to N) natural modes of vibration. Inverse vector iteration methods are usually effective (i.e., reliable in obtaining accurate solutions and computationally efficient) for such situations, and this presentation is restricted to such methods. Only the basic ideas of vector iteration are included, without getting into subspace iteration or the Lanczos method. Similarly, transformation methods and polynomial iteration techniques are excluded. In short, this is a limited treatment of solution methods for the eigenvalue problem arising in structural dynamics. This is sufficient for our purposes, but more comprehensive treatments are available in other books.

10.12 RAYLEIGH'S QUOTIENT

In this section Rayleigh's quotient is presented because it is needed in vector iteration methods; its properties are also presented. If Eq. (10.11.1) is premultiplied by ϕ^T, the following scalar equation is obtained:

$$\phi^T \mathbf{k}\phi = \lambda \phi^T \mathbf{m}\phi$$

The positive definiteness of $\mathbf{m}$ guarantees that $\phi^T \mathbf{m}\phi$ is nonzero, so that it is permissible to solve for the scalar λ:

$$\lambda = \frac{\phi^T \mathbf{k}\phi}{\phi^T \mathbf{m}\phi} \tag{10.12.1}$$

which obviously depends on the vector ϕ. This quotient is called *Rayleigh's quotient.* It may also be derived by equating the maximum value of kinetic energy to the maximum value of potential energy under the assumption that the vibrating system is executing simple harmonic motion at frequency ω with the deflected shape given by ϕ (Section 8.5.3).

Rayleigh's quotient has the following properties, presented without proof:

1. When ϕ is an eigenvector ϕ_n of Eq. (10.11.1), Rayleigh's quotient is equal to the corresponding eigenvalue λ_n.
2. If ϕ is an approximation to ϕ_n with an error that is a first-order infinitesimal, Rayleigh's quotient is an approximation to λ_n with an error which is a second-order infinitesimal λ, i.e., Rayleigh's quotient is *stationary* in the neighborhoods of the true eigenvectors. The stationary value is actually a minimum in the neighborhood of the first eigenvector and a maximum in the vicinity of the *N*th eigenvector.
3. Rayleigh's quotient is bounded between $\lambda_1 \equiv \omega_1^2$ and $\lambda_N \equiv \omega_N^2$, the smallest and largest eigenvalues, i.e., it provides an upper bound for ω_1^2 and lower bound for ω_N^2.

A common engineering application of Rayleigh's quotient involves simply evaluating Eq. (10.12.1) for a trial vector ϕ that is selected on the basis of physical insight (Chapter 8). If the elements of an approximate eigenvector whose largest element is unity are correct to s decimal places, Rayleigh's quotient can be expected to be correct to about $2s$ decimal places. Several numerical procedures for solving eigenvalue problems make use of the stationary property of Rayleigh's quotient.

10.13 INVERSE VECTOR ITERATION METHOD

10.13.1 Basic Concept and Procedure

We restrict this presentation to systems with a stiffness matrix $\mathbf{k}$ that is positive definite, whereas the mass matrix $\mathbf{m}$ may be a banded mass matrix, or it may be a diagonal matrix with or without zero diagonal elements. The fact that vector iteration methods can handle zero diagonal elements in the mass matrix implies that these methods can be applied without requiring static condensation of the stiffness matrix (Section 9.3).

Our goal is to satisfy Eq. (10.11.1) by operating on it directly. We assume a trial vector for ϕ, say $\mathbf{x}_1$, and evaluate the right-hand side of Eq. (10.11.1). This we can do except for the eigenvalue λ, which is unknown. Thus we drop λ, which is equivalent to saying that we set $\lambda = 1$. Because eigenvectors can be determined only within a scale factor, the choice of λ will not affect the final result. With $\lambda = 1$ the right-hand side of Eq. (10.11.1) can be computed:

$$\mathbf{R}_1 = \mathbf{m}\mathbf{x}_1 \tag{10.13.1}$$

Since $\mathbf{x}_1$ was an arbitrary choice, in general $\mathbf{k}\mathbf{x}_1 \neq \mathbf{R}_1$. (If by coincidence we find that $\mathbf{k}\mathbf{x}_1 = \mathbf{R}_1$, the $\mathbf{x}_1$ chosen is an eigenvector.) We now set up an equilibrium equation

$$\mathbf{k}\mathbf{x}_2 = \mathbf{R}_1 \tag{10.13.2}$$

where $\mathbf{x}_2$ is the displacement vector corresponding to forces $\mathbf{R}_1$ and $\mathbf{x}_2 \neq \mathbf{x}_1$. Since we are using iteration to solve for an eigenvector, intuition may suggest that the solution of $\mathbf{x}_2$ of Eq. (10.13.2), obtained after one cycle of iteration, may be a better approximation to ϕ than was $\mathbf{x}_1$. This is indeed the case, as we shall demonstrate later, and by repeating the iteration cycle, we obtain an increasingly better approximation to the eigenvector. A corresponding value for the eigenvalue can be computed using Rayleigh's quotient, and the iteration can be terminated when two successive estimates of the eigenvalue are close enough. As the number of iterations increases, $\mathbf{x}_{i+1}$ approaches ϕ_1 and the eigenvalue approaches λ_1.

Thus the procedure starts with the assumption of a starting iteration vector $\mathbf{x}_1$ and consists of the following steps to be repeated for $j = 1, 2, 3, \ldots$ until convergence:

1. Determine $\bar{\mathbf{x}}_{j+1}$ by solving the algebraic equations:

$$\mathbf{k}\bar{\mathbf{x}}_{j+1} = \mathbf{m}\mathbf{x}_j \tag{10.13.3}$$

2. Obtain an estimate of the eigenvalue by evaluating Rayleigh's quotient:

$$\lambda^{(j+1)} = \frac{\bar{\mathbf{x}}_{j+1}^T \mathbf{k}\bar{\mathbf{x}}_{j+1}}{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1}} = \frac{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\mathbf{x}_j}{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1}} \tag{10.13.4}$$

3. Check convergence by comparing two successive values of λ:

$$\frac{|\,\lambda^{(j+1)} - \lambda^{(j)}\,|}{\lambda^{(j+1)}} \leq \text{tolerance} \tag{10.13.5}$$

4. If the convergence criterion is not satisfied, normalize $\bar{\mathbf{x}}_{j+1}$:

$$\mathbf{x}_{j+1} = \frac{\bar{\mathbf{x}}_{j+1}}{(\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1})^{1/2}} \tag{10.13.6}$$

   and go back to the first step and carry out another iteration using the next j.

5. Let l be the last iteration [i.e., the iteration that satisfies Eq. (10.13.5)]. Then

$$\lambda_1 \doteq \lambda^{(l+1)} \qquad \phi_1 \doteq \frac{\bar{\mathbf{x}}_{l+1}}{(\bar{\mathbf{x}}_{l+1}^T \mathbf{m}\bar{\mathbf{x}}_{l+1})^{1/2}} \tag{10.13.7}$$

The basic step in the iteration is the solution of Eq. (10.13.3)—a set of N algebraic equations—which gives a better approximation to ϕ_1. The calculation in Eq. (10.13.4) gives an approximation to the eigenvalue λ_1 according to Rayleigh's quotient. It is this approximation to λ_1 that we use to determine convergence in the iteration. Equation (10.13.6) simply assures that the new vector satisfies the mass orthonormality relation

$$\mathbf{x}_{j+1}^T \mathbf{m} \mathbf{x}_{j+1} = 1 \tag{10.13.8}$$

Although normalizing of the new vector does not affect convergence, it is numerically useful. If such normalizing is not included, the elements of the iteration vectors grow (or decrease) in each step, and this may cause numerical problems. Normalizing keeps the element values similar from one iteration to the next. The tolerance is selected depending on the accuracy desired. It should be 10^{-2s} or smaller when λ_1 is required to $2s$-digit accuracy. Then the eigenvector will be accurate to about s or more digits.

The inverse vector iteration algorithm can be organized a little differently for convenience in computer implementation, but such computational issues are not included in this presentation.

Example 10.14

The floor masses and story stiffnesses of the three-story frame, idealized as a shear frame, are shown in Fig. E10.14, where $m = 100$ kips/g and $k = 168$ kips/in. Determine the fundamental frequency ω_1 and mode shape ϕ_1 by inverse vector iteration.

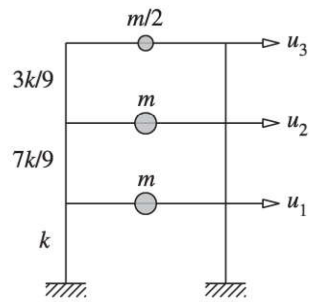


Figure E10.14

Solution The mass and stiffness matrices for the system are

$$\mathbf{m} = m \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix} \qquad \mathbf{k} = \frac{k}{9} \begin{bmatrix} 16 & -7 & 0 \\ -7 & 10 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

where $m = 0.259$ kip-sec^2/in. and $k = 168$ kips/in.

The inverse iteration algorithm of Eqs. (10.13.3) to (10.13.7) is implemented starting with an initial vector $\mathbf{x}_1 = \langle 1 \quad 1 \quad 1 \rangle^T$ leading to Table E10.14. The final result is $\omega_1 \doteq \sqrt{144.14} = 12.006$ and $\phi_1 \doteq \langle 0.6377 \quad 1.2752 \quad 1.9122 \rangle^T$.

10.13.2 Convergence of Iteration

In the preceding section we have merely presented the inverse iteration scheme and stated that it converges to the first eigenvector associated with the smallest eigenvalue. We now

TABLE E10.14 INVERSE VECTOR ITERATION FOR THE FIRST EIGENPAIR

Iteration	$\mathbf{x}_j$	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.0039 \\ 0.0068 \\ 0.0091 \end{bmatrix}$	147.73	$\begin{bmatrix} 0.7454 \\ 1.3203 \\ 1.7676 \end{bmatrix}$
2	$\begin{bmatrix} 0.7454 \\ 1.3203 \\ 1.7676 \end{bmatrix}$	$\begin{bmatrix} 0.0045 \\ 0.0089 \\ 0.0130 \end{bmatrix}$	144.29	$\begin{bmatrix} 0.6574 \\ 1.2890 \\ 1.8800 \end{bmatrix}$
3	$\begin{bmatrix} 0.6574 \\ 1.2890 \\ 1.8800 \end{bmatrix}$	$\begin{bmatrix} 0.0044 \\ 0.0089 \\ 0.0132 \end{bmatrix}$	144.15	$\begin{bmatrix} 0.6415 \\ 1.2785 \\ 1.9052 \end{bmatrix}$
4	$\begin{bmatrix} 0.6415 \\ 1.2785 \\ 1.9052 \end{bmatrix}$	$\begin{bmatrix} 0.0044 \\ 0.0089 \\ 0.0133 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6384 \\ 1.2758 \\ 1.9109 \end{bmatrix}$
5	$\begin{bmatrix} 0.6384 \\ 1.2758 \\ 1.9109 \end{bmatrix}$	$\begin{bmatrix} 0.0044 \\ 0.0088 \\ 0.0133 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6377 \\ 1.2752 \\ 1.9122 \end{bmatrix}$

demonstrate this convergence because the proof is instructive, especially in suggesting how to modify the procedure to achieve convergence to a higher eigenvector.

The modal expansion of vector $\mathbf{x}$ is [from Eqs. (10.7.1) and (10.7.2)]

$$\mathbf{x} = \sum_{n=1}^{N} \boldsymbol{\phi}_n q_n = \sum_{n=1}^{N} \boldsymbol{\phi}_n \frac{\boldsymbol{\phi}_n^T \mathbf{m}\mathbf{x}}{\boldsymbol{\phi}_n^T \mathbf{m}\boldsymbol{\phi}_n} \tag{10.13.9}$$

The nth term in this summation represents the nth modal component in $\mathbf{x}$.

The first iteration cycle involves solving the equilibrium equations (10.13.3) with $j = 1$: $\mathbf{k}\bar{\mathbf{x}}_2 = \mathbf{m}\mathbf{x}_1$, where $\mathbf{x}_1$ is a trial vector. This solution can be expressed as $\bar{\mathbf{x}}_2 = \mathbf{k}^{-1}\mathbf{m}\mathbf{x}_1$. Substituting the modal expansion of Eq. (10.13.9) for $\mathbf{x}_1$ gives

$$\bar{\mathbf{x}}_2 = \sum_{n=1}^{N} \mathbf{k}^{-1}\mathbf{m}\boldsymbol{\phi}_n q_n \tag{10.13.10}$$

By rewriting Eq. (10.11.1) for the nth eigenpair as $\mathbf{k}^{-1}\mathbf{m}\boldsymbol{\phi}_n = (1/\lambda_n)\boldsymbol{\phi}_n$ and substituting it in Eq. (10.13.10), we get

$$\bar{\mathbf{x}}_2 = \sum_{n=1}^{N} \frac{1}{\lambda_n}\boldsymbol{\phi}_n q_n = \frac{1}{\lambda_1}\sum_{n=1}^{N} \frac{\lambda_1}{\lambda_n}\boldsymbol{\phi}_n q_n \tag{10.13.11}$$

The second iteration cycle involves solving Eq. (10.13.3) with $j = 2$: $\bar{\mathbf{x}}_3 = \mathbf{k}^{-1}\mathbf{m}\bar{\mathbf{x}}_2$, wherein we have used the unnormalized vector $\bar{\mathbf{x}}_2$ instead of the normalized vector $\mathbf{x}_2$. This is acceptable for the present purpose because convergence is unaffected by normalization and eigenvectors are arbitrary within a multiplicative factor. Following the derivation of

Eqs. (10.13.10) and (10.13.11), it can be shown that

$$\bar{\mathbf{x}}_3 = \frac{1}{\lambda_1^2} \sum_{n=1}^{N} \left(\frac{\lambda_1}{\lambda_n}\right)^2 \phi_n q_n \tag{10.13.12}$$

Similarly, the vector $\bar{\mathbf{x}}_{j+1}$ after j iteration cycles can be expressed as

$$\bar{\mathbf{x}}_{j+1} = \frac{1}{\lambda_1^j} \sum_{n=1}^{N} \left(\frac{\lambda_1}{\lambda_n}\right)^j \phi_n q_n \tag{10.13.13}$$

Since $\lambda_1 < \lambda_n$ for $n > 1$, $(\lambda_1/\lambda_n)^j \to 0$ as $j \to \infty$, and only the $n = 1$ term in Eq. (10.13.13) remains significant, indicating that

$$\bar{\mathbf{x}}_{j+1} \to \frac{1}{\lambda_1^j} \phi_1 q_1 \quad \text{as} \quad j \to \infty \tag{10.13.14}$$

Thus $\bar{\mathbf{x}}_{j+1}$ converges to a vector proportional to ϕ_1. Furthermore, the normalized vector $\mathbf{x}_{j+1}$ of Eq. (10.13.6) converges to ϕ_1, which is mass orthonormal.

The rate of convergence depends on λ_1/λ_2, the ratio that appears in the second term in the summation of Eq. (10.13.13). The smaller this ratio is, the faster is the convergence; this implies that convergence is very slow when λ_2 is nearly equal to λ_1. For such situations the convergence rate can be improved by the procedures of Section 10.14.

If only the first natural mode ϕ_1 and the associated natural frequency ω_1 are required, there is no need to proceed further. This is an advantage of the iteration method. It is unnecessary to solve the complete eigenvalue problem to obtain one or two of the modes.

10.13.3 Evaluation of Higher Modes

To continue the solution after ϕ_1 and λ_1 have been determined, the starting vector is modified to make the iteration procedure converge to the second eigenvector. The necessary modification is suggested by the proof presented in Section 10.13.2 to show that the iteration process converges to the first eigenvector. Observe that after each iteration cycle the other modal components are reduced relative to the first modal component because its eigenvalue λ_1 is smaller than all other eigenvalues λ_n. The iteration process converges to the first mode for the same reason because $(\lambda_1/\lambda_n)^j \to 0$ as $j \to \infty$. In general, the iteration procedure will converge to the mode with the lowest eigenvalue contained in a trial vector $\mathbf{x}$.

To make the iteration procedure converge to the second mode, a trial vector $\mathbf{x}$ should therefore be chosen so that it does not contain any first-mode component [i.e., q_1 should be zero in Eq. (10.13.9)] and $\mathbf{x}$ is said to be orthogonal to ϕ_1. It is not possible to start *a priori* with such an $\mathbf{x}$, however. We therefore start with an arbitrary $\mathbf{x}$ and make it orthogonal to ϕ_1 by the Gram–Schmidt orthogonalization process. This process can also be used to orthogonalize a trial vector with respect to the first n eigenvectors that have already been determined so that iteration on the purified trial vectors will converge to the $(n+1)$th mode, the mode with the next eigenvalue in ascending sequence.

In principle, the Gram–Schmidt orthogonalization process, combined with the inverse iteration procedure, provides a tool for computing the second and higher eigenvalues and eigenvectors. This tool is not effective, however, as a general computer method for two reasons. First, if $\mathbf{x}_1$ were made orthogonal to ϕ_1 [i.e., $q_1 = 0$ in Eq. (10.13.9)], theoretically the iteration will not converge to ϕ_1 but to ϕ_2 (or some other eigenvector—the one with the next-higher eigenvalue which is contained in the modal expansion of $\mathbf{x}_1$). However, in practice this never occurs since the inevitable round-off errors in finite-precision arithmetic continuously reintroduce small components of ϕ_1 which the iteration process magnifies. Second, convergence of the iteration process becomes progressively slower for the higher modes. It is for these reasons that this method is not developed in this book.

10.14 VECTOR ITERATION WITH SHIFTS: PREFERRED PROCEDURE

The inverse vector iteration procedure of Section 10.13, combined with the concept of "shifting" the eigenvalue spectrum (or scale), provides an effective means to improve the convergence rate of the iteration process and to make it converge to an eigenpair other than (λ_1, ϕ_1). Thus this is the preferred method, as it provides a practical tool for computing as many pairs of natural vibration frequencies and modes of a structure as desired.

10.14.1 Basic Concept and Procedure

The solutions of Eq. (10.11.1) are the eigenvalues λ_n and eigenvectors ϕ_n; the number of such pairs equals N, the order of $\mathbf{m}$ and $\mathbf{k}$. Figure 10.14.1a shows the eigenvalue spectrum (i.e., a plot of $\lambda_1, \lambda_2, \ldots$ along the eigenvalue axis). Introducing a shift μ in the origin of the eigenvalue axis (Fig. 10.14.1b) and defining $\check{\lambda}$ as the shifted eigenvalue measured from the shifted origin gives $\lambda = \check{\lambda} + \mu$. Substituting this in Eq. (10.11.1) leads to

$$\check{\mathbf{k}}\phi = \check{\lambda}\mathbf{m}\phi \tag{10.14.1}$$

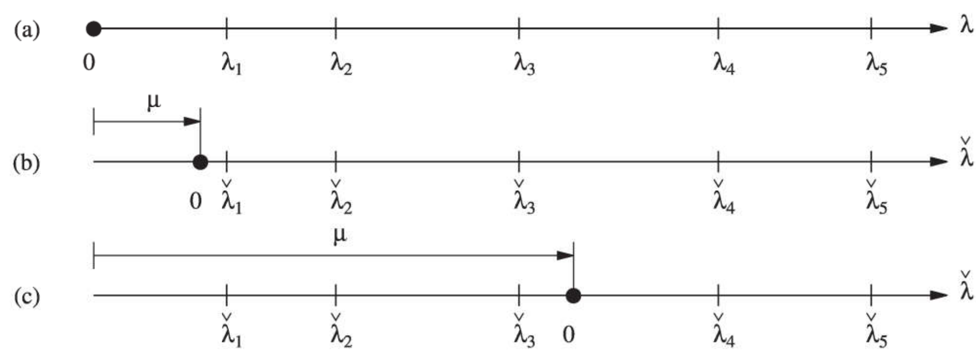


Figure 10.14.1 (a) Eigenvalue spectrum; (b) eigenvalue measured from a shifted origin; (c) location of shift point for convergence to λ_3.

where

$$\check{\mathbf{k}} = \mathbf{k} - \mu\mathbf{m} \qquad \check{\lambda} = \lambda - \mu \tag{10.14.2}$$

The eigenvectors of the two eigenvalue problems—original Eq. (10.11.1) and shifted Eq. (10.14.1)—are the same. This is obvious because if a ϕ satisfies one equation, it will also satisfy the other. However, the eigenvalues $\check{\lambda}$ of the shifted problem differ from the eigenvalues λ of the original problem by the shift μ [Eq. (10.14.2)]. The spectrum of the shifted eigenvalues $\check{\lambda}$ is also shown in Fig. 10.14.1b with the origin at μ. If the inverse vector iteration method of Section 10.13.1 were applied to the eigenvalue problem of Eq. (10.14.1), it obviously will converge to the eigenvector having the smallest magnitude of the shifted eigenvalue $|\check{\lambda}_n|$ (i.e., the eigenvector with original eigenvalue λ_n closest to the shift value μ).

If μ were chosen as in Fig. 10.14.1b, the iteration will converge to the first eigenvector. The rate of convergence depends on the ratio $\check{\lambda}_1/\check{\lambda}_2 = (\lambda_1 - \mu)/(\lambda_2 - \mu)$. The convergence rate has improved because this ratio is smaller than the ratio λ_1/λ_2 for the original eigenvalue problem. If μ were chosen between λ_n and λ_{n+1}, and μ is closer to λ_n than λ_{n+1}, the iteration will converge to λ_n. On the other hand, if μ is closer to λ_{n+1} than λ_n, the iteration will converge to λ_{n+1}. Thus the "shifting" concept enables computation of any pair (λ_n, ϕ_n). In particular, if μ were chosen as in Fig. 10.14.1c, the iteration will converge to the third eigenvector.

Example 10.15

Determine the natural frequencies and modes of vibration of the system of Example 10.14 by inverse vector iteration with shifting.

Solution Equation (10.14.1) with a selected shift μ is solved by inverse vector iteration. Selecting the shift $\mu_1 = 100$, $\check{\mathbf{k}}$ is calculated from Eq. (10.14.2) and the inverse vector iteration

TABLE E10.15a VECTOR ITERATION WITH SHIFT: FIRST EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	100	$\begin{bmatrix} 0.0114 \\ 0.0218 \\ 0.0313 \end{bmatrix}$	144.60	$\begin{bmatrix} 0.6759 \\ 1.2933 \\ 1.8610 \end{bmatrix}$
2	$\begin{bmatrix} 0.6759 \\ 1.2933 \\ 1.8610 \end{bmatrix}$	100	$\begin{bmatrix} 0.0145 \\ 0.0289 \\ 0.0432 \end{bmatrix}$	144.15	$\begin{bmatrix} 0.6401 \\ 1.2769 \\ 1.9083 \end{bmatrix}$
3	$\begin{bmatrix} 0.6401 \\ 1.2769 \\ 1.9083 \end{bmatrix}$	100	$\begin{bmatrix} 0.0144 \\ 0.0289 \\ 0.0433 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6377 \\ 1.2752 \\ 1.9122 \end{bmatrix}$
4	$\begin{bmatrix} 0.6377 \\ 1.2752 \\ 1.9122 \end{bmatrix}$	100	$\begin{bmatrix} 0.0144 \\ 0.0289 \\ 0.0433 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6375 \\ 1.2750 \\ 1.9125 \end{bmatrix}$

TABLE E10.15b VECTOR ITERATION WITH SHIFT: SECOND EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	600	$\begin{bmatrix} 0.0044 \\ 0.0028 \\ -0.0133 \end{bmatrix}$	605.11	$\begin{bmatrix} 0.8030 \\ 0.5189 \\ -2.4277 \end{bmatrix}$
2	$\begin{bmatrix} 0.8030 \\ 0.5189 \\ -2.4277 \end{bmatrix}$	600	$\begin{bmatrix} 0.0197 \\ 0.0201 \\ -0.0373 \end{bmatrix}$	648.10	$\begin{bmatrix} 1.0062 \\ 1.0221 \\ -1.8994 \end{bmatrix}$
3	$\begin{bmatrix} 1.0062 \\ 1.0221 \\ -1.8994 \end{bmatrix}$	600	$\begin{bmatrix} 0.0201 \\ 0.0201 \\ -0.0405 \end{bmatrix}$	648.64	$\begin{bmatrix} 0.9804 \\ 0.9778 \\ -1.9717 \end{bmatrix}$
4	$\begin{bmatrix} 0.9804 \\ 0.9778 \\ -1.9717 \end{bmatrix}$	600	$\begin{bmatrix} 0.0202 \\ 0.0202 \\ -0.0404 \end{bmatrix}$	648.65	$\begin{bmatrix} 0.9827 \\ 0.9829 \\ -1.9642 \end{bmatrix}$

algorithm of Eqs. (10.13.3) to (10.13.7) is implemented starting with an initial vector of $\mathbf{x}_1 = \langle 1 \quad 1 \quad 1\rangle^T$ leading to Table E10.15a. The final result is $\omega_1 = \sqrt{144.14} = 12.006$ and $\phi_1 = \langle 0.6375 \quad 1.2750 \quad 1.9125\rangle^T$. This is obtained in one iteration cycle less than in the iteration without shift in Example 10.14.

Starting with the shift $\mu_1 = 600$ and the same $\mathbf{x}_1$, the inverse iteration algorithm leads to Table E10.15b. The final result is $\omega_2 = \sqrt{648.65} = 25.468$ and $\phi_2 = \langle 0.9827 \quad 0.9829 \quad -1.9642\rangle^T$. Convergence is attained in four iteration cycles.

Starting with the shift $\mu_1 = 1500$ and the same $\mathbf{x}_1$, the inverse iteration algorithm leads to Table E10.15c. The final result is $\omega_3 = \sqrt{1513.5} = 38.904$ and $\phi_3 = \langle 1.5778 \quad -1.1270 \quad 0.4508\rangle^T$. Convergence is attained in three cycles.

TABLE E10.15c VECTOR ITERATION WITH SHIFT: THIRD EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	1500	$\begin{bmatrix} 0.0198 \\ -0.0156 \\ 0.0054 \end{bmatrix}$	1510.6	$\begin{bmatrix} 1.5264 \\ -1.2022 \\ 0.4148 \end{bmatrix}$
2	$\begin{bmatrix} 1.5264 \\ -1.2022 \\ 0.4148 \end{bmatrix}$	1500	$\begin{bmatrix} 0.1167 \\ -0.0832 \\ 0.0333 \end{bmatrix}$	1513.5	$\begin{bmatrix} 1.5784 \\ -1.1261 \\ 0.4509 \end{bmatrix}$
3	$\begin{bmatrix} 1.5784 \\ -1.1261 \\ 0.4509 \end{bmatrix}$	1500	$\begin{bmatrix} 0.1168 \\ -0.0834 \\ 0.0334 \end{bmatrix}$	1513.5	$\begin{bmatrix} 1.5778 \\ -1.1270 \\ 0.4508 \end{bmatrix}$

10.14.2 Rayleigh's Quotient Iteration

The inverse iteration method with shifts converges rapidly if a shift is chosen near enough to the eigenvalue of interest. However, selection of an appropriate shift is difficult without knowledge of the eigenvalue. Many techniques have been developed to overcome this difficulty; one of these is presented in this section.

The Rayleigh quotient calculated by Eq. (10.13.4) to estimate the eigenvalue provides an appropriate shift value, but it is not necessary to calculate and introduce a new shift at each iteration cycle. If this is done, however, the resulting procedure is called *Rayleigh's quotient iteration.*

This procedure starts with the assumption of a starting iteration vector $\mathbf{x}_1$ and starting shift $\lambda^{(1)}$ and consists of the following steps to be repeated for $j = 1, 2, 3, \ldots$ until convergence:

1. Determine $\bar{\mathbf{x}}_{j+1}$ by solving the algebraic equations:

$$[\mathbf{k} - \lambda^{(j)}\mathbf{m}]\bar{\mathbf{x}}_{j+1} = \mathbf{m}\mathbf{x}_j \tag{10.14.3}$$

2. Obtain an estimate of the eigenvalue and the shift for the next iteration from

$$\lambda^{(j+1)} = \frac{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\mathbf{x}_j}{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1}} + \lambda^{(j)} \tag{10.14.4}$$

3. Normalize $\bar{\mathbf{x}}_{j+1}$:

$$\mathbf{x}_{j+1} = \frac{\bar{\mathbf{x}}_{j+1}}{(\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1})^{1/2}} \tag{10.14.5}$$

This iteration converges to a particular eigenpair (λ_n, ϕ_n) depending on the starting vector $\mathbf{x}_1$ and the initial shift $\lambda^{(1)}$. If $\mathbf{x}_1$ includes a strong contribution of the eigenvector ϕ_n and $\lambda^{(1)}$ is close enough to λ_n, the iteration converges to the eigenpair (λ_n, ϕ_n). The rate of convergence is faster than the standard vector iteration with shift described in Section 10.14.1, but at the expense of additional computation because a new $[\mathbf{k}-\lambda^{(j)}\mathbf{m}]$ has to be factorized in each iteration.

Example 10.16

Determine all three natural frequencies and modes of the system of Example 10.14 by inverse vector iteration with the shift in each iteration cycle equal to Rayleigh's quotient from the previous cycle.

Solution The iteration procedure of Eqs. (10.14.3) to (10.14.5) is implemented with starting shifts of $\mu_1 = 100$, $\mu_2 = 600$, and $\mu_3 = 1500$, leading to Tables E10.16a, E10.16b, and E10.16c, respectively, where the final results are $\omega_1 = \sqrt{144.14} = 12.006$ and $\phi_1 = \langle 0.6375 \quad 1.2750 \quad 1.9125\rangle^T$, $\omega_2 = \sqrt{648.65} = 25.468$ and $\phi_2 = \langle 0.9825 \quad 0.9825 \quad -1.9649\rangle^T$, and $\omega_3 = \sqrt{1513.5} = 38.904$ and $\phi_3 = \langle 1.5778 \quad -1.1270 \quad 0.4508\rangle^T$.

Observe that convergence is faster when a new shift is used in each iteration cycle. Only two cycles are required instead of four (Example 10.15) for the first mode, and three instead of four for the second mode.

TABLE E10.16a RAYLEIGH'S QUOTIENT ITERATION: FIRST EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	100	$\begin{bmatrix} 0.0114 \\ 0.0218 \\ 0.0313 \end{bmatrix}$	144.60	$\begin{bmatrix} 0.6759 \\ 1.2933 \\ 1.8610 \end{bmatrix}$
2	$\begin{bmatrix} 0.6759 \\ 1.2933 \\ 1.8610 \end{bmatrix}$	144.60	$\begin{bmatrix} -1.3947 \\ -2.7895 \\ -4.1845 \end{bmatrix}$	144.14	$\begin{bmatrix} -0.6375 \\ -1.2750 \\ -1.9126 \end{bmatrix}$
3	$\begin{bmatrix} -0.6375 \\ -1.2750 \\ -1.9126 \end{bmatrix}$	144.14	$\begin{bmatrix} 1.9738 \times 10^6 \\ 3.9476 \times 10^6 \\ 5.9214 \times 10^6 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6375 \\ 1.2750 \\ 1.9125 \end{bmatrix}$

TABLE E10.16b RAYLEIGH'S QUOTIENT ITERATION: SECOND EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	λ^{j+1}	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	600	$\begin{bmatrix} 0.0044 \\ 0.0028 \\ -0.0133 \end{bmatrix}$	605.11	$\begin{bmatrix} 0.8030 \\ 0.5189 \\ -2.4277 \end{bmatrix}$
2	$\begin{bmatrix} 0.8030 \\ 0.5189 \\ -2.4277 \end{bmatrix}$	605.11	$\begin{bmatrix} 0.0220 \\ 0.0223 \\ -0.0418 \end{bmatrix}$	648.21	$\begin{bmatrix} 1.0036 \\ 1.0176 \\ -1.9070 \end{bmatrix}$
3	$\begin{bmatrix} 1.0036 \\ 1.0176 \\ -1.9070 \end{bmatrix}$	648.21	$\begin{bmatrix} 2.2624 \\ 2.2623 \\ -4.5249 \end{bmatrix}$	648.65	$\begin{bmatrix} 0.9825 \\ 0.9824 \\ -1.9650 \end{bmatrix}$
4	$\begin{bmatrix} 0.9825 \\ 0.9824 \\ -1.9650 \end{bmatrix}$	648.65	$\begin{bmatrix} 3.0372 \times 10^6 \\ 3.0372 \times 10^6 \\ -6.0745 \times 10^6 \end{bmatrix}$	648.65	$\begin{bmatrix} 0.9825 \\ 0.9825 \\ -1.9649 \end{bmatrix}$

TABLE E10.16c RAYLEIGH'S QUOTIENT ITERATION: THIRD EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	λ^{j+1}	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	1500	$\begin{bmatrix} 0.0198 \\ -0.0156 \\ 0.0054 \end{bmatrix}$	1510.6	$\begin{bmatrix} 1.5264 \\ -1.2022 \\ 0.4148 \end{bmatrix}$
2	$\begin{bmatrix} 1.5264 \\ -1.2022 \\ 0.4148 \end{bmatrix}$	1510.6	$\begin{bmatrix} 0.5431 \\ -0.3879 \\ 0.1552 \end{bmatrix}$	1513.5	$\begin{bmatrix} 1.5779 \\ -1.1268 \\ 0.4508 \end{bmatrix}$
3	$\begin{bmatrix} 1.5779 \\ -1.1268 \\ 0.4508 \end{bmatrix}$	1513.5	$\begin{bmatrix} 9.7061 \times 10^4 \\ -6.9329 \times 10^4 \\ 2.7732 \times 10^4 \end{bmatrix}$	1513.5	$\begin{bmatrix} 1.5778 \\ -1.1270 \\ 0.4508 \end{bmatrix}$

Application to structural dynamics. In modal analysis of the dynamic response of structures, we are interested in the lower J natural frequencies and modes (Chapters 12 and 13); typically, J is much smaller than N, the number of degrees of freedom. Although Rayleigh's quotient iteration may appear to be an effective tool for the necessary computation, it may not always work. For example, with the starting vector $\mathbf{x}_1$ and starting shift $\lambda^{(1)} = 0$, Eq. (10.14.4) may provide a value for Rayleigh's quotient (which, according to Section 10.12, is always higher than the first eigenvalue), which is also the next shift, closer to the second eigenvalue than the first, resulting in the iteration converging to the second mode. Thus it is necessary to supplement Rayleigh's quotient iteration by another technique to assure convergence to the lowest eigenpair $(\lambda_1,\ \boldsymbol{\phi}_1)$. One possibility is to use first the inverse iteration without shift, Eqs. (10.13.3) to (10.13.7), for a few cycles to obtain an iteration vector that is a good approximation (but has not converged) to $\boldsymbol{\phi}_1$, and then start with Rayleigh's quotient iteration.

Computer implementation of inverse vector iteration with shift should be reliable and efficient. By reliability we mean that it should give the desired eigenpair. Efficiency implies that with the fewest iterations and least computation, the method should provide results to the desired degree of accuracy. Both of these requirements are essential; otherwise, the computer program may skip a desired eigenpair, or the computations may be unnecessarily time consuming. The issues related to reliability and efficiency of computer methods for solving the eigenvalue problem are discussed further in other books.

10.15 TRANSFORMATION OF $\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi}$ TO THE STANDARD FORM

The standard eigenvalue problem $\mathbf{A}\mathbf{y} = \lambda\mathbf{y}$ arises in many situations in mathematics and in applications to problems in the physical sciences and engineering. It has therefore attracted much attention and many solution algorithms have been developed and are available in computer software libraries. These computer procedures could be used to solve the structural dynamics eigenvalue problem, $\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi}$, provided that it can be transformed to the standard form. Such a transformation is presented in this section.

We assume that $\mathbf{m}$ is positive definite; that is, it is either a diagonal matrix with nonzero masses or a banded matrix as in a consistent mass formulation (Chapter 17). If $\mathbf{m}$ is a diagonal matrix with zero mass in some degrees of freedom, these are first eliminated by static condensation (Section 9.3). Positive definiteness of $\mathbf{m}$ implies that $\mathbf{m}^{-1}$ can be calculated. Premultiplying the structural dynamics eigenvalue problem

$$\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi} \tag{10.15.1}$$

by $\mathbf{m}^{-1}$ gives the standard eigenvalue problem:

$$\mathbf{A}\boldsymbol{\phi} = \lambda\boldsymbol{\phi} \tag{10.15.2}$$

where

$$\mathbf{A} = \mathbf{m}^{-1}\mathbf{k} \qquad \lambda = \omega^2 \tag{10.15.3}$$

In general, $\mathbf{A}$ is not symmetric, although $\mathbf{m}$ and $\mathbf{k}$ are both symmetric matrices.

Because the computational effort could be greatly reduced if **A** were symmetric, we seek methods that yield a symmetric **A**. Consider that $\mathbf{m} = \text{diag}(m_j)$, a diagonal matrix with elements $m_{jj} = m_j$, and define $\mathbf{m}^{1/2} = \text{diag}(m_j^{1/2})$ and $\mathbf{m}^{-1/2} = \text{diag}(m_j^{-1/2})$. Then **m** and the identity matrix **I** can be expressed as

$$\mathbf{m} = \mathbf{m}^{1/2}\mathbf{m}^{1/2} \qquad \mathbf{I} = \mathbf{m}^{-1/2}\mathbf{m}^{1/2} \tag{10.15.4}$$

Using Eq. (10.15.4), Eq. (10.15.1) can be rewritten as

$$\mathbf{k}\mathbf{m}^{-1/2}\mathbf{m}^{1/2}\boldsymbol{\phi} = \omega^2\mathbf{m}^{1/2}\mathbf{m}^{1/2}\boldsymbol{\phi}$$

Premultiplying both sides by $\mathbf{m}^{-1/2}$ leads to

$$\mathbf{m}^{-1/2}\mathbf{k}\mathbf{m}^{-1/2}\mathbf{m}^{1/2}\boldsymbol{\phi} = \omega^2\mathbf{m}^{-1/2}\mathbf{m}^{1/2}\mathbf{m}^{1/2}\boldsymbol{\phi}$$

Utilizing Eq. (10.15.4b) to simplify the right-hand side of the equation above gives

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{y} \tag{10.15.5}$$

where

$$\mathbf{A} = \mathbf{m}^{-1/2}\mathbf{k}\mathbf{m}^{-1/2} \qquad \mathbf{y} = \mathbf{m}^{1/2}\boldsymbol{\phi} \qquad \lambda = \omega^2 \tag{10.15.6}$$

Equation (10.15.5) is the standard eigenvalue problem and **A** is now symmetric.

Thus if a computer program to solve $\mathbf{A}\mathbf{y} = \lambda\mathbf{y}$ were available, it could be utilized to determine the natural frequencies ω_n and modes ϕ_n of a system for which **m** and **k** were known as follows:

1. Compute **A** from Eq. (10.15.6a).
2. Determine the eigenvalues λ_n and eigenvectors $\mathbf{y}_n$ of **A** by solving Eq. (10.15.5).
3. Determine the natural frequencies and modes by

$$\omega_n = \sqrt{\lambda_n} \qquad \boldsymbol{\phi}_n = \mathbf{m}^{-1/2}\mathbf{y}_n \tag{10.15.7}$$

The transformation of Eq. (10.15.6) can be generalized to situations where the mass matrix is not diagonal but is banded like the stiffness matrix; such banding is typical of finite element formulations (Chapter 17). Then, **A** is a full matrix, although **k** and **m** are banded. This is a major computational disadvantage for large systems. For such situations the transformation of $\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi}$ to $\mathbf{A}\mathbf{y} = \lambda\mathbf{y}$ may not be an effective approach, and the inverse iteration method, which works directly with $\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi}$, may be more efficient.

FURTHER READING

Bathe, K. J., *Finite Element Procedures,* Prentice Hall, Englewood Cliffs, N.J., 1996, Chapters 10 and 11.

Crandall, S. H., and McCalley, R. B., Jr., "Matrix Methods of Analysis," Chapter 28 in *Shock and Vibration Handbook* (ed. C. M. Harris), McGraw-Hill, New York, 1988.

Humar, J. L., *Dynamics of Structures,* 2nd ed., A. A. Balkema Publishers, Lisse, The Netherlands, 2002, Chapter 11.

Parlett, B. N., *The Symmetric Eigenvalue Problem,* Prentice Hall, Englewood Cliffs, N.J., 1980.

PROBLEMS

Parts A and B

10.1 Determine the natural vibration frequencies and modes of the system of Fig. P9.1 with $k_1 = k$ and $k_2 = 2k$ in terms of the DOFs in the figure. Show that these results are equivalent to those presented in Fig. E10.1.

10.2 For the system defined in Problem 9.2:
(**a**) Determine the natural vibration frequencies and modes; express the frequencies in terms of m, EI, and L. Sketch the modes and identify the associated natural frequencies.
(**b**) Verify that the modes satisfy the orthogonality properties.
(**c**) Normalize each mode so that the modal mass M_n has unit value. Sketch these normalized modes. Compare these modes with those obtained in part (a) and comment on the differences.

10.3 Determine the free vibration response of the system of Problem 9.2 (and Problem 10.2) due to each of the three sets of initial displacements: (**a**) $u_1(0) = 1,\ u_2(0) = 0$; (**b**) $u_1(0) = 1,\ u_2(0) = 1$; (**c**) $u_1(0) = 1,\ u_2(0) = -1$. Comment on the relative contribution of the modes to the response in the three cases. Neglect damping in the system.

10.4 Repeat Problem 10.3(a) considering damping in the system. For each mode the damping ratio is $\zeta_n = 5\%$.

10.5 For the system defined in Problem 9.4:
(**a**) Determine the natural vibration frequencies and modes. Express the frequencies in terms of m, EI, and L, and sketch the modes.
(**b**) Determine the displacement response due to an initial velocity $\dot{u}_2(0)$ imparted to the top of the system.

10.6 For the two-story shear building shown in Problem 9.5:
(**a**) Determine the natural vibration frequencies and modes; express the frequencies in terms of m, EI, and h.
(**b**) Verify that the modes satisfy the orthogonality properties.
(**c**) Normalize each mode so that the roof displacement is unity. Sketch the modes and identify the associated natural frequencies.
(**d**) Normalize each mode so that the modal mass M_n has unit value. Compare these modes with those obtained in part (c) and comment on the differences.

10.7 The structure of Problem 9.5 is modified so that the columns are hinged at the base. Determine the natural vibration frequencies and modes of the modified system, and compare them with the vibration properties of the original structure determined in Problem 10.6. Comment on the effect of the column support condition on the vibration properties.

10.8 Determine the free vibration response of the structure of Problem 10.6 (and Problem 9.5) if it is displaced as shown in Fig. P10.8a and b and released. Comment on the relative contributions of the two vibration modes to the response that was produced by the two initial displacements. Neglect damping.

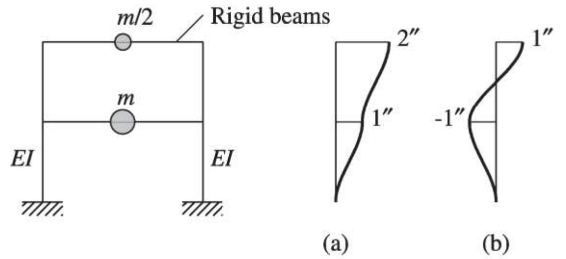


Figure P10.8

10.9 Repeat Problem 10.8 for the initial displacement of Fig. P10.8a, assuming that the damping ratio for each mode is 5%.

***10.10** Determine the natural vibration frequencies and modes of the system defined in Problem 9.6. Express the frequencies in terms of m, EI, and h and the joint rotations in terms of h. Normalize each mode to unit displacement at the roof and sketch it, identifying all DOFs.

10.11– 10.12 For the three-story shear buildings shown in Figs. P9.7 and P9.8:
(**a**) Determine the natural vibration frequencies and modes; express the frequencies in terms of m, EI, and h. Sketch the modes and identify the associated natural frequencies.
(**b**) Verify that the modes satisfy the orthogonality properties.
(**c**) Normalize each mode so that the modal mass M_n has unit value. Sketch these normalized modes. Compare these modes with those obtained in part (a) and comment on the differences.

10.13– 10.14 The structures of Figs. P9.7 and P9.8 are modified so that the columns are hinged at the base. Determine the natural vibration frequencies and modes of the modified system, and compare them with the vibration properties of the original structures determined in Problems 10.11 and 10.12. Comment on the effect of the column support condition on the vibration properties.

10.15– 10.16 Determine the free vibration response of the structures of Problems 10.11 and 10.12 (and Problems 9.7 and 9.8) if they are displaced as shown in Fig. P10.15–P10.16a, b, and c and released. Plot floor displacements versus t/T_1 and comment on the relative contributions of the three vibration modes to the response that was produced by each of the three initial displacements. Neglect damping.

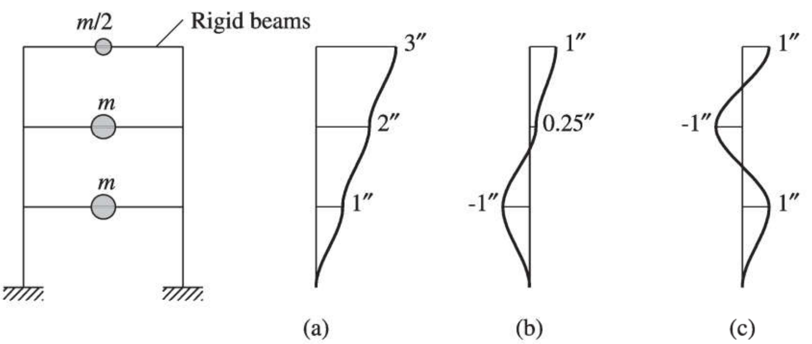


Figure P10.15–P10.16

*Denotes that a computer is necessary to solve this problem.

10.17– 10.18 Repeat Problems 10.15 and 10.16 for the initial displacement of Fig. P10.15a, assuming that the damping ratio for each mode is 5%.

***10.19– 10.22** Determine the natural vibration frequencies and modes of the systems defined in Problems 9.9 to 9.12. Express the frequencies in terms of m, EI, and h and the joint rotations in terms of h. Normalize each mode to unit displacement at the roof and sketch it, including all DOFs.

10.23 (**a**) For the system in Problem 9.13, determine the natural vibration frequencies and modes. Express the frequencies in terms of m, EI, and L, and sketch the modes.
(**b**) The structure is pulled through a lateral displacement $u_1(0)$ and released. Determine the free vibration response.

10.24 For the system defined in Problem 9.14, $m = 90$ kips/g, $k = 1.5$ kips/in., and $b = 25$ ft.
(**a**) Determine the natural vibration frequencies and modes.
(**b**) Normalize each mode so that the modal mass M_n has unit value. Sketch these modes.

10.25 Repeat Problem 10.24 using a different set of DOFs—those defined in Problem 9.15. Show that the natural vibration frequencies and modes determined using the two sets of DOFs are the same.

10.26 Repeat Problem 10.24 using a different set of DOFs—those defined in Problem 9.16. Show that the natural vibration frequencies and modes determined using the two sets of DOFs are the same.

10.27 Repeat Problem 10.24 using a different set of DOFs—those defined in Problem 9.17. Show that the natural vibration frequencies and modes determined using the two sets of DOFs are the same.

10.28 For the structure defined in Problem 9.18, determine the natural frequencies and modes. Normalize each mode such that $\phi_n^T \phi_n = 1$.

Part C

***10.29** The floor weights and story stiffnesses of the three-story shear frame are shown in Fig. P10.29, where $w = 100$ kips and $k = 326.32$ kips/in. Determine the fundamental natural vibration frequency ω_1 and mode ϕ_1 by inverse vector iteration.

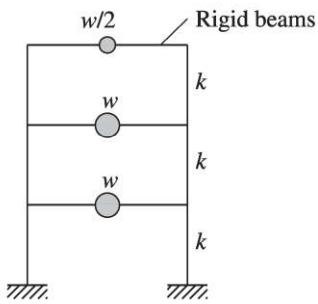


Figure P10.29

***10.30** For the system defined in Problem 10.29, there is concern for possible resonant vibrations due to rotating machinery mounted at the second-floor level. The operating speed of the

*Denotes that a computer is necessary to solve this problem.

motor is 430 rpm. Obtain the natural vibration frequency of the structure that is closest to the machine frequency.

*__10.31__ Determine the three natural vibration frequencies and modes of the system defined in Problem 10.29 by inverse vector iteration with shifting.

*__10.32__ Determine the three natural vibration frequencies and modes of the system defined in Problem 10.29 by inverse vector iteration with the shift in each iteration cycle equal to Rayleigh's quotient from the previous cycle.

*Denotes that a computer is necessary to solve this problem.

11

Damping in Structures

PREVIEW

Several issues that arise in defining the damping properties of structures are discussed in this chapter. It is impractical to determine the coefficients of the damping matrix directly from the structural dimensions, structural member sizes, and the damping properties of the structural materials used. Therefore, damping is generally specified by numerical values for the modal damping ratios, and these are sufficient for analysis of linear systems with classical damping. The experimental data that provide a basis for estimating these damping ratios are discussed in Part A of this chapter, which ends with recommended values for modal damping ratios. The damping matrix is needed, however, for analysis of linear systems with nonclassical damping and for analysis of nonlinear structures. Two procedures for constructing the damping matrix for a structure from the modal damping ratios are presented in Part B; classically damped systems as well as nonclassically damped systems are considered.

PART A: EXPERIMENTAL DATA AND RECOMMENDED MODAL DAMPING RATIOS

11.1 VIBRATION PROPERTIES OF MILLIKAN LIBRARY BUILDING

Chosen as an example to discuss damping, the Robert A. Millikan Library building is a nine-story reinforced-concrete building constructed in 1966–1967 on the campus of the California Institute of Technology in Pasadena, California. Figure 11.1.1 is a photograph of