ON THE MOTION OF WAVES

IN A VARIABLE CANAL OF SMALL DEPTH AND WIDTH*.

* From the *Transactions of the Cambridge Philosophical Society*, 1838. [Read *May* 15, 1837.]

ON THE MOTION OF WAVES IN A VARIABLE CANAL OF SMALL DEPTH AND WIDTH.

THE equations and conditions necessary for determining the motions of fluids in every case in which it is possible to subject them to Analysis, have been long known, and will be found in the First Edition of the *Mec. Anal.* of Lagrange. Yet the difficulty of integrating them is such, that many of the most important questions relative to this subject seem quite beyond the present powers of Analysis. There is, however, one particular case which admits of a very simple solution. The case in question is that of an indefinitely extended canal of small breadth and depth, both of which may vary very slowly, but in other respects quite arbitrarily. This has been treated of in the following paper, and as the results obtained possess considerable simplicity, perhaps they may not be altogether unworthy the Society's notice.

The general equations of motion of a non-elastic fluid acted on by gravity (g) in the direction of the axis z, are,

$$gz - \frac{p}{\rho} = \frac{d\phi}{dt} \quad \text{.........................} \quad (1),$$

$$0 = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} \quad \text{...................} \quad (2),$$

supposing the disturbance so small that the squares and higher powers of the velocities &c. may be neglected. In the above formulæ p = pressure, ρ = density, and ϕ is such a function of x, y, z and t, that the velocities of the fluid particles parallel to the three axes are

$$u = \left(\frac{d\phi}{dx}\right), \quad v = \left(\frac{d\phi}{dy}\right), \quad w = \left(\frac{d\phi}{dz}\right).$$

To the equations (1) and (2) it is requisite to add the conditions relative to the exterior surfaces of the fluid, and if $A = 0$ be the equation of one of these surfaces, the corresponding condition is [Lagrange, *Mec. Anal.* Tom. II. p. 303. (I.)],

$$0 = \frac{dA}{dt} + \frac{dA}{dx} u + \frac{dA}{dy} v + \frac{dA}{dz} w.$$

Hence

$$0 = \frac{dA}{dt} + \frac{dA}{dx} \cdot \frac{d\phi}{dx} + \frac{dA}{dy} \cdot \frac{d\phi}{dy} + \frac{dA}{dz} \cdot \frac{d\phi}{dz} \text{ (when } A = 0)\text{...(A).}$$

The equations (1) and (2) with the condition (A) applied to each of the exterior surfaces of the fluid will suffice to determine in every case the small oscillations of a non-elastic fluid, or at least in those where

$$udx + vdy + wdz$$

is an exact differential.

In what follows, however, we shall confine ourselves to the consideration of the motion of a non-elastic fluid, when two of the dimensions, viz. those parallel to y and z, are so small that ϕ may be expanded in a rapidly convergent series in powers of y and z, so that

$$\phi = \phi_0 + \phi' \frac{y}{1} + \phi_{,} \frac{z}{1} + \phi'' \frac{y^2}{1 \cdot 2} + \phi_{,}' yz + \phi_{,,} \frac{z^2}{1 \cdot 2} + \&c.$$

Then if we take the surface of the fluid in equilibrium as the plane of (x, y), and suppose the sides of the rectangular canal symmetrical with respect to the plane (x, z), ϕ will evidently contain none but even powers of y, and we shall have

$$\phi = \phi_0 + \phi_{,} z + \phi'' \frac{y^2}{1 \cdot 2} + \phi_{,,} \frac{z^2}{1 \cdot 2} + \&c. \text{ ....(3).}$$

Now if $$y = \pm \beta_x$$

represent the equation of the two sides of the canal, we need only satisfy one of them as

$$y - \beta_x = 0,$$

since the other will then be satisfied by the exclusion of the odd powers of y from ϕ.

The equation (A) gives, since here $A = y - \beta$,

$$0 = \frac{d\phi}{dy} - \frac{d\beta}{dx} \cdot \frac{d\phi}{dx} \text{ ...... (when } y = \beta) \text{ ....} (a).$$

Similarly, if $z - \gamma_x = 0$ is the equation of the bottom of the canal,

$$0 = \frac{d\phi}{dz} - \frac{d\gamma}{dx} \cdot \frac{d\phi}{dx} \text{ ...... (when } z = \gamma) \text{............} (b).$$

If moreover $z - \zeta_{x.t} = 0$ be the equation of the upper surface,

$$\left.\begin{array}{l} 0 = \dfrac{d\phi}{dz} - \dfrac{d\zeta}{dx}\dfrac{d\phi}{dx} - \dfrac{d\zeta}{dt} \\ \text{But here } p = 0\,;\ \therefore \text{ also by (2) } g\zeta = \dfrac{d\phi}{dt} \end{array}\right\} \text{... (when } z = \zeta) \text{...} (c).$$

Substituting from (3) in (b) we get

$$0 = \phi_{,} + \phi_{,,}\gamma + \&\text{c.} - \frac{d\gamma}{dx}\left\{\frac{d\phi_0}{dx} - \frac{d\phi_{,}}{dx}\frac{\gamma}{1} + \&\text{c.}\right\};$$

or neglecting quantities of the order γ^2,

$$0 = \phi_{,} + \phi_{,,}\gamma - \frac{d\gamma}{dx} \cdot \frac{d\phi_0}{dx} \text{ ....} (b').$$

Similarly (a) becomes

$$0 = \phi''\beta - \frac{d\beta}{dx} \cdot \frac{d\phi_0}{dx} \text{ ....} (a'),$$

and (c) becomes, since ζ is of the order of the disturbance,

$$\left.\begin{array}{l} 0 = \phi_{,} - \dfrac{d\zeta}{dt} \\ g\zeta = \dfrac{d\phi_0}{dt} \end{array}\right\} \left.\begin{array}{c} \text{when } z = \zeta, \\ \text{or neglecting (disturbance)}^2\ z = 0 \end{array}\right\} \text{...} (c')$$

provided as above we neglect (disturbance)^2.

Again, the condition (2) gives by equating separately the coefficients of powers and products of y and z,

$$\left.\begin{array}{l} 0 = \dfrac{d^2\phi_0}{dx^2} + \phi'' + \phi_{,,} \\ 0 = \&\text{c.} \end{array}\right\} \text{....................} (2').$$

If now by means of (a'), (b'), (c') we eliminate $\phi''\phi_{,,}$ from $(2')$, there results

$$0 = \frac{d^2\phi_0}{dx^2} + \left\{\frac{d\beta}{\beta dx} + \frac{d\gamma}{\gamma dx}\right\}\frac{d\phi_0}{dx} - \frac{1}{g\gamma}\left(\frac{d^2\phi_0}{dt^2}\right) \quad\text{............ (4).}$$

It now only remains to integrate this equation.

For this we shall suppose β and γ functions of x which vary very slowly, so that if written in their proper form we should have

$$\beta = \psi(\omega x),$$

where ω is a very small quantity. Then,

$$\frac{d\beta}{dx} = \omega\psi'(\omega x), \quad \frac{d^2\beta}{dx^2} = \omega^2\psi''(\omega x), \text{ \&c.}$$

Hence if we allow ourselves to omit quantities of the order ω^2, and assume, to satisfy (4),

$$\phi_0 = Af(t + X),$$

where A is a function of x of the same kind as β and γ, we have, omitting $\left(\frac{d^2A}{dx^2}\right)$,

$$\frac{d^2\phi_0}{dt^2} = Af'',$$

$$\frac{d\phi_0}{dx} = A\frac{dX}{dx}f' + \frac{dA}{dx}f,$$

$$\frac{d^2\phi_0}{dx^2} = A\left(\frac{dX}{dx}\right)^2 f'' + A\frac{d^2X}{dx^2}f' + 2\frac{dA}{dx}\cdot\frac{dX}{dx}f'.$$

Substituting these in (4), and still neglecting quantities of the order ω^2, we get

$$0 = \left\{A\left(\frac{dX}{dx}\right)^2 - \frac{A}{g\gamma}\right\}f''$$

$$+\left\{A\frac{d^2X}{dx^2} + 2\frac{dA}{dx}\frac{dX}{dx} + \left(\frac{d\beta}{\beta dx} + \frac{d\gamma}{\gamma dx}\right)A\frac{dX}{dx}\right\}f';$$

equating now separately the coefficients of f' and f'', we get

$$0=\left(\frac{dX}{dx}\right)^2-\frac{1}{g\gamma},$$

$$0=\frac{\frac{d^2X}{dx^2}}{\frac{dX}{dx}}+2\frac{dA}{Adx}+\frac{d\beta}{\beta dx}+\frac{d\gamma}{\gamma dx}.$$

The first, integrated, gives

$$X=\pm\int\frac{dx}{\sqrt{\gamma g}},$$

and the second

$$k=\frac{dX}{dx}\,A^2\beta\gamma=A^2\frac{\beta\gamma}{\sqrt{g\gamma}}=\frac{A^2\beta\gamma^{\frac{1}{2}}}{\sqrt{g}}.$$

Hence if we neglect the superfluous constant $k\sqrt{g}$, the general integral of (4) is, ($\because A=\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}$),

$$\phi_0=\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}\left\{f\left(t+\int\frac{dx}{\sqrt{g\gamma}}\right)+F\left(t-\int\frac{dx}{\sqrt{g\gamma}}\right)\right\};$$

therefore, by (c'),

$$\zeta=\frac{d\phi_0}{gdt}=\frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{g}\left\{f'\left(t+\int\frac{dx}{\sqrt{g\gamma}}\right)+F'\left(t-\int\frac{dx}{\sqrt{g\gamma}}\right)\right\},$$

and the actual velocity of the fluid particles in the direction of the axis of x, is

$$u=\frac{d\phi}{dx}=\frac{d\phi_0}{dx}=\frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{\sqrt{g\gamma}}\left\{f'\left(t+\int\frac{dx}{\sqrt{g\gamma}}\right)-F'\left(t-\int\frac{dx}{\sqrt{g\gamma}}\right)\right\},$$

neglecting quantities which are of the order (ω) compared with those retained.

If the initial values of ζ and u are given, we may then determine f' and F', and we thus see that a single wave, like a pulse of sound, divides into two, propagated in opposite directions. Considering, therefore, only that which proceeds in the direction of x positive, we have

$$\zeta=\frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{g}F'\left(t-\int\frac{dx}{\sqrt{g\gamma}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

$$u=\frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{3}{4}}}{g^{\frac{1}{2}}}F'\left(t-\int\frac{dx}{\sqrt{g\gamma}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Suppose now the value of $F'(x) = 0$, except from $x = a$ to $x = a + \alpha$, and δx to be the corresponding length of the wave, we have

$$t - \int \frac{dx}{\sqrt{g\gamma}} = a + \alpha,$$

$$\text{and} \quad t - \int \frac{dx}{\sqrt{g\gamma}} - \frac{\delta x}{\sqrt{g\gamma}} = a \text{ very nearly.}$$

Hence the variable length of the wave is

$$\delta x = \alpha \, . \sqrt{g\gamma} \quad \text{.......................... (7).}$$

Lastly, for any particular phase of the wave, we have

$$t - \int \frac{dx}{\sqrt{g\gamma}} = \text{const.}:$$

therefore

$$\frac{dx}{dt} = \sqrt{g\gamma} \quad \text{................................. (8)}$$

is the velocity with which the wave, or more strictly speaking the particular phase in question, progresses.

From (5), (6), (7), and (8) we see that if β represent the variable breadth of the canal and γ its depth,

ζ = height of the wave $\propto \beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}$,

u = actual velocity of the fluid particles $\propto \beta^{-\frac{1}{2}}\gamma^{-\frac{3}{4}}$.

dx = length of the wave $\propto \gamma^{\frac{1}{2}}$,

and $\dfrac{dx}{dt}$ = velocity of the wave's motion $= \sqrt{g\gamma}$.