In Eqs. (4-78) the subscripts $j1, j2, j3$, $k1$, $k2$, and $k3$ carry the meanings defined previously in Eqs. (4-66).

The next section contains an example analysis of a plane frame by the method described above, and a computer program for plane frames appears in Sec. 5.8.

4.18 Example. Figure 4-31a shows a plane frame having two members, three joints, six restraints, and three degrees of freedom. This frame is to be analyzed by the methods given in the previous section. For this purpose, assume that the cross-sectional area A_X and the moment of inertia I_Z are constant throughout the structure. Assume also that the parameters in the problem have the following numerical values:

$$E = 10{,}000 \text{ ksi} \quad L = 100 \text{ in.} \quad I_Z = 1000 \text{ in.}^4 \quad P = 10 \text{ kips} \quad A_X = 10 \text{ in.}^2$$

For this frame the material is aluminum, and units of kips, inches, and radians are used throughout the analysis.*

A numbering system for members, joints, and displacements is given in Fig. 4-31b, which shows the restrained structure. The sequence for numbering the joints is selected in such a manner that the matrices in the analysis do not require rearrangement.

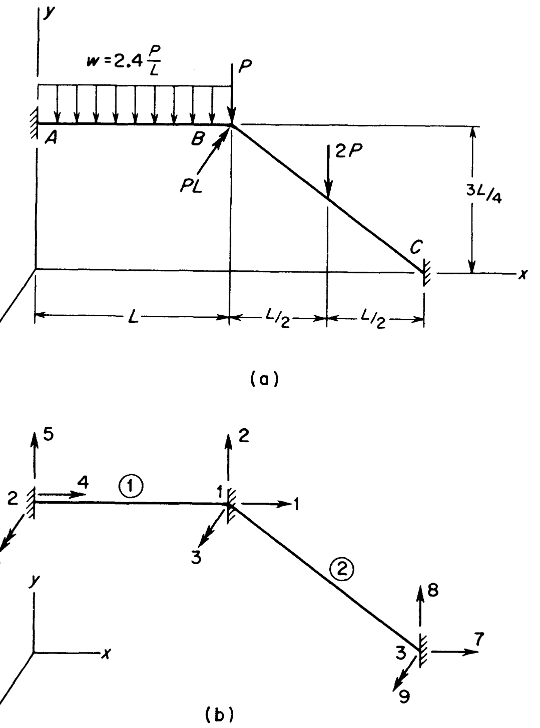

Fig. 4-31. Example (plane frame).

*Units for this example are in the United States (US) customary system. However, about half of the problems for Sec. 4.18 (given at the end of the chapter) are in the international (SI) system of units.

Table 4-28
Joint Information for Frame of Fig. 4-31

Joint	Coordinates (in.)		Restraint List		
	x	y	x	y	z
1	100	75	0	0	0
2	0	75	1	1	1
3	200	0	1	1	1

Joint information is summarized in Table 4-28, which contains the joint numbers, joint coordinates, and the conditions of restraint. The member information for the frame is presented in Table 4-29. Substitution of the direction cosines for the members into Eq. (4-62) results in the rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ shown in Table 4-30.

Table 4-29
Member Information for Frame of Fig. 4-31

Member	Joint j	Joint k	Area (in.^{2})	Moment of Inertia (in.^{4})	Length (in.)	Direction Cosines	
						C_X	C_Y
1	2	1	10	1000	100	1.0	0
2	1	3	10	1000	125	0.8	−0.6

Table 4-30
Rotation Matrices for Members of Frame of Fig. 4-31

$$
\mathbf{R}_1 = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}
$$

Table 4-31
Member Stiffness Matrices for Structural Axes

$$
\mathbf{S}_{MS1} = \begin{array}{cc} \left[\begin{array}{rrrrrr} 1000.0 & 0 & 0 & -1000.0 & 0 & 0 \\ 0 & 120.0 & 6000.0 & 0 & -120.0 & 6000.0 \\ 0 & 6000.0 & 400000.0 & 0 & -6000.0 & 200000.0 \\ -1000.0 & 0 & 0 & 1000.0 & 0 & 0 \\ 0 & -120.0 & -6000.0 & 0 & 120.0 & -6000.0 \\ 0 & 6000.0 & 200000.0 & 0 & -6000.0 & 400000.0 \end{array}\right] & \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{rrrrrr} 4 & \quad 5 & \quad 6 & \quad 1 & \quad 2 & \quad 3 \end{array} & \end{array}
$$

$$
\mathbf{S}_{MS2} = \begin{array}{cc} \left[\begin{array}{rrrrrr} 534.1 & -354.5 & 2304.0 & -534.1 & 354.5 & 2304.0 \\ -354.5 & 327.3 & 3072.0 & 354.5 & -327.3 & 3072.0 \\ 2304.0 & 3072.0 & 320000.0 & -2304.0 & -3072.0 & 160000.0 \\ -534.1 & 354.5 & -2304.0 & 534.1 & -354.5 & -2304.0 \\ 354.5 & -327.3 & -3072.0 & -354.5 & 327.3 & -3072.0 \\ 2304.0 & 3072.0 & 160000.0 & -2304.0 & -3072.0 & 320000.0 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \\ \begin{array}{rrrrrr} 1 & \quad 2 & \quad 3 & \quad 7 & \quad 8 & \quad 9 \end{array} & \end{array}
$$

Table 4-32

Joint Stiffness Matrix for Frame of Fig. 4-31

$$
\mathbf{S}_J = \left[\begin{array}{rrr|rrrrrr}
1534.1 & -354.5 & 2304.0 & -1000.0 & 0 & 0 & -534.1 & 354.5 & 2304.0 \\
-354.5 & 447.3 & -2928.0 & 0 & -120.0 & -6000.0 & 354.5 & -327.3 & 3072.0 \\
2304.0 & -2928.0 & 720000.0 & 0 & 6000.0 & 200000.0 & -2304.0 & -3072.0 & 160000.0 \\
\hline
-1000.0 & 0 & 0 & 1000.0 & 0 & 0 & 0 & 0 & 0 \\
0 & -120.0 & 6000.0 & 0 & 120.0 & 6000.0 & 0 & 0 & 0 \\
0 & -6000.0 & 200000.0 & 0 & 6000.0 & 400000.0 & 0 & 0 & 0 \\
-534.1 & 354.5 & -2304.0 & 0 & 0 & 0 & 534.1 & -354.5 & -2304.0 \\
354.5 & -327.3 & -3072.0 & 0 & 0 & 0 & -354.5 & 327.3 & -3072.0 \\
2304.0 & 3072.0 & 160000.0 & 0 & 0 & 0 & -2304.0 & -3072.0 & 320000.0
\end{array}\right]
$$

In preparation for generating the over-all joint stiffness matrix $\mathbf{S}_J$, the member stiffness matrices are calculated. This may be done for each member by first generating the member stiffness matrix $\mathbf{S}_{Mi}$ for member axes (see Table 4-26, Sec. 4.16) and then calculating the matrix $\mathbf{S}_{MSi}$ for structural axes by the rotation transformation of Eq. (4-64). For this purpose the matrix $\mathbf{R}_{Ti}$ for each member is composed as shown by Eq. (4-63). Alternatively, the matrix $\mathbf{S}_{MSi}$ may be calculated directly for each member using Table 4-27. The resulting matrices are given in Table 4-31. The indexes $j1$ through $k3$ (computed by Eqs. 4-66) are also indicated in Table 4-31 down the right-hand side and across the bottom of each matrix $\mathbf{S}_{MSi}$. These indexes may be used as a guide for the purpose of transferring elements to the matrix $\mathbf{S}_J$. After the transferring process is accomplished, the over-all joint stiffness matrix that results is shown in Table 4-32. This matrix is partitioned in the usual manner, thereby isolating the 3×3 stiffness matrix $\mathbf{S}_{FF}$. The inverse of this matrix is given in Table 4-33.

Table 4-33
Inverse of Stiffness Matrix

$$\mathbf{S}_{FF}^{-1} = \begin{bmatrix} 798.0 & 632.5 & 0.01877 \\ 632.5 & 2798.0 & 9.355 \\ 0.01877 & 9.355 & 1.426 \end{bmatrix} \times 10^{-6}$$

Next, the load information is processed, beginning with the joint loads shown in Table 4-34. The actions in this table are placed in the vector $\mathbf{A}_J$, as indicated by Eq. (4-73).

$$\mathbf{A}_J = \{0, -10, -1000, 0, 0, 0, 0, 0, 0\}$$

Table 4-34
Actions Applied at Joints

Joint	*Force in x Direction (kips)*	*Force in y Direction (kips)*	*Couple in z Sense (kip-in.)*
1	0	−10	−1000
2	0	0	0
3	0	0	0

In addition, the actions in Table 4-35 are placed in the matrix $\mathbf{A}_{ML}$ and then transferred to the vector of equivalent joint loads $\mathbf{A}_E$ in accordance with Eqs. (4-76). The following vector results:

$$\mathbf{A}_E = \{0, -22, -50, 0, -12, -200, 0, -10, 250\}$$

Adding the vectors $\mathbf{A}_J$ and $\mathbf{A}_E$ produces the combined load vector $\mathbf{A}_C$ as follows:

$$\mathbf{A}_C = \{0, -32, -1050, 0, -12, -200, 0, -10, 250\}$$

Table 4-35
Actions at Ends of Restrained Members Due to Loads

Member	$(A_{ML})_{1,i}$ *(kips)*	$(A_{ML})_{2,i}$ *(kips)*	$(A_{ML})_{3,i}$ *(kip-in.)*	$(A_{ML})_{4,i}$ *(kips)*	$(A_{ML})_{5,i}$ *(kips)*	$(A_{ML})_{6,i}$ *(kip-in.)*
1	0	12	200	0	12	−200
2	−6	8	250	−6	8	−250

The first three elements of this vector constitute the vector $\mathbf{A}_{FC}$

$$\mathbf{A}_{FC} = \{0, -32, -1050\}$$

and the last six elements are the vector $\mathbf{A}_{RC}$.

$$\mathbf{A}_{RC} = \{0, -12, -200, 0, -10, 250\}$$

Having all of the required matrices on hand, one may complete the solution by calculating the free joint displacements $\mathbf{D}_F$ by Eq. (4-3) with the following result:

$$\mathbf{D}_F = \mathbf{S}_{FF}^{-1}\mathbf{A}_{FC} = \{-0.02026, -0.09936, -0.001797\}$$

The first two elements in the vector $\mathbf{D}_F$ are the translations (inches) in the x and y directions at joint 1, and the last element is the rotation (radians) of the joint in the z sense.

The vector $\mathbf{D}_J$ for this structure consists of the vector $\mathbf{D}_F$ in the first part and zeros in the latter part.

$$\mathbf{D}_J = \{-0.02026, -0.09936, -0.001797, 0, 0, 0, 0, 0, 0\}$$

In the next step the support reactions are computed, using Eq. (4-4) with the matrix $\mathbf{S}_{RF}$ obtained from the lower left-hand portion of Table 4-32.

$$\mathbf{A}_R = -\mathbf{A}_{RC} + \mathbf{S}_{RF}\mathbf{D}_F = \{20.26, 13.14, 436.6, -20.26, 40.86, -889.5\}$$

Terms in the vector $\mathbf{A}_R$ consist of the forces (kips) in the x and y directions and the moments (kip-inches) in the z sense at points 2 and 3.

As the final step in the analysis, the member end-actions $\mathbf{A}_{Mi}$ are calculated, using either Eq. (4-77) or Eqs. (4-78). The results of these calculations are given in Table 4-36, which completes the analysis of the plane frame structure.

Table 4-36
Final Member End-Actions

Member	$(A_M)_{1,i}$ *(kips)*	$(A_M)_{2,i}$ *(kips)*	$(A_M)_{3,i}$ *(kip-in.)*	$(A_M)_{4,i}$ *(kips)*	$(A_M)_{5,i}$ *(kips)*	$(A_M)_{6,i}$ *(kip-in.)*
1	20.26	13.14	436.6	−20.26	10.86	−322.9
2	28.72	−4.53	−677.1	−40.73	20.53	−889.5

4.19 Grid Member Stiffnesses. A grid structure resembles a plane frame in several respects. All of the members and joints lie in the same plane, and the members are assumed to be rigidly connected at the joints