On Tidal Propagation in Shallow Rivers

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A reexamination of the momentum balance in shallow rivers, with scaling appropriate to the Saint Lawrence and the Fraser, shows that frictional forces exceed accelerations over most of the tidal cycle. Consequently, tidal propagation in shallow rivers is more properly envisaged as a diffusion than as a wave propagation phenomenon. The long time lags associated with low waters, unexplainable in terms of a simple wave propagation model, are easily accounted for by an equally simple diffusive model.

INTRODUCTION

The calculation of tides in unstratified rivers and estuaries is now routinely handled by numerical integration of the long gravity wave equations, following methods such as those presented by *Dronkers* [1964]. This technique has met with considerable success in predicting the amplitudes and phases of sea level displacements and, to a lesser degree, of tidal currents in many locations. There is little doubt that the basic formulation used includes the essential aspects of the physics of shallow water tides.

A numerical solution, however, is often of little help in understanding the qualitative aspects of a problem. A number of simple models, incorporating only those aspects of the physics which were thought to be the most important, have been put forward to explain the origin of many of the most striking distortions suffered by the tide as it propagates into shallow water. The earliest of these models is undoubtedly 'Green's Law,' which relates geometrical amplification to the conservation of energy flux in a nondissipative system [see *Lamb*, 1945, p. 273]. The gradual attenuation of the tide as it advances upstream is most simply studied by a linearization of the friction terms, an approximation first suggested by *Lorentz* [1926]. The asymmetry observed between flood currents and ebb flows in tidal rivers, whereby the flood current is stronger in speed but shorter in duration than the ebb flow, has been examined by *Kreiss* [1957] by means of a perturbation method and was found to arise from the combined effects of friction and nonlinearity. Finally, the occurrence of tidal bores has been shown to be a consequence of amplitude dispersion, related to the presence of the nonlinear advection terms, and was examined in detail by *Abbott* [1956]. To describe each of the above phenomena, a simplified physical model focuses on some qualitative feature of tidal propagation which is compatible with the conditions under which the restricted model is valid.

This paper concentrates on a flow regime which is relevant to tidal propagation in shallow rivers but which seems to have escaped attention in that connection. For the parameter range described below, the momentum balance of the tides is best described by a set of equations intermediate between the frictionless long gravity wave equations and the kinematic wave equations used in the study of flood waves [*Whitham*, 1974, p. 80]. It will be shown that these equations, in which frictional terms dominate the momentum balance, provide a simple qualitative explanation of the long phase lags associated with low water in, for example, the Fraser River. It also follows that the most appropriate conceptual model of the advancing tide in shallow rivers is a diffusive model (because the frictionally dominated equations are parabolic) and not a wave propagation model.

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THE SCALED EQUATIONS

Consider a narrow rectilinear channel of uniform depth and width. One-dimensional tidal propagation in this channel is governed by the momentum and mass continuity equations:

$$\hat{v}_{\hat{t}} + \hat{v}\hat{v}_{\hat{x}} + g\hat{\eta}_{\hat{x}} + \frac{K\hat{v}\,|\hat{v}\,|}{(H + \hat{\eta})} = 0 \tag{1}$$

$$[(H + \hat{\eta})\hat{v}]_{\hat{x}} + \hat{\eta}_{\hat{t}} = 0 \tag{2}$$

The coordinate $\hat{x}$ denotes position along the axis of the channel and increases upstream; $\hat{t}$ denotes time; $\hat{v}$ is the fluid velocity in the $\hat{x}$ direction, averaged over the depth; H is the depth of the channel below a horizontal datum; $\hat{\eta}$ is the height of the water surface above the same datum; g is the acceleration of gravity; and K is a friction coefficient. Derivatives are denoted by subscripts.

In a river the flow consists of a mean downstream discharge in addition to the tidal current; similarly, the height of the water surface above datum includes a mean slope as well as tidal fluctuations. Although both $\hat{v}$ and $\hat{\eta}$ should be split into mean and tidal parts, this step is not essential to the first part of the argument and will be delayed to a later stage of the analysis.

Introducing a length scale L, a time scale T, and a velocity scale U_0, we define the following nondimensional variables (without hats):

$$x = \frac{\hat{x}}{L} \qquad t = \frac{\hat{t}}{T} \qquad v = \frac{\hat{v}}{U_0} \qquad \eta = \frac{\hat{\eta}}{\epsilon H} \tag{3}$$

The scaling parameter ϵ is essentially the ratio of the tidal range to the mean depth. In terms of these scaled variables, (1) and (2) become

$$Sv_t + \frac{F^2}{\epsilon}vv_x + \eta_x + R\frac{v\,|v\,|}{1 + \epsilon\eta} = 0 \tag{4}$$

$$M[(1 + \epsilon\eta)v]_x + \eta_t = 0 \tag{5}$$

the nondimensional parameters S, F, R, and M being given by

$$\begin{aligned} F &= U_0/(gH)^{1/2} \\ R &= F^2KL/\epsilon H \\ S &= F^2L/\epsilon TU_0 \\ M &= U_0T/L\epsilon \end{aligned} \tag{6}$$

where F is the Froude number. Different flow regimes correspond to different ranges of relative values of the above parameters.

TABLE 1. Scales and Parameters for Shallow Tidal Reaches of the Fraser River and the Saint Lawrence River

	H, m	U_0, m s^{-1}	K	S/R	λ, km	L, km
Fraser	9.0	1.0	4.8×10^{-3}	0.045	408	84
Saint Lawrence	7.0	1.0	1.2×10^{-3}	0.11	357	120

THE FRICTIONAL REGIME

Let us consider the parameter range appropriate to tidal propagation in a shallow river; examples used will be the Fraser and the Saint Lawrence rivers. For tides of moderate amplitude, $\epsilon \leq 0(1)$, the tidal range is a fraction of the mean depth. One must then choose $M = 1$ to achieve a mass balance in (5); from (6) we then find that $S = F^2/\epsilon^2$. The ratio of acceleration (Sv_t) to frictional terms is given by

$$S/R = H/KTU_0 \quad (7)$$

For large values of this ratio the equations describe 'dynamic waves,' the usual undamped tides; for small values of S/R the frictional term dominates, and the system resembles the 'kinematic wave' equations describing flood waves in rivers.

Relevant parameters for the tidal reaches of the Fraser River and for the Saint Lawrence River above Quebec City are listed in Table 1. The values of the friction coefficients have been taken from the numerical models of *Ages and Woollard* [1976] for the Fraser River and of *Prandle and Crookshank* [1974] for the Saint Lawrence River. As the tide is predominantly semidiurnal in both cases, a 12-hour time scale has been used. In both cases, as is seen in Table 1, the ratio S/R is 0(0.1). In both rivers the ratio of tidal amplitude to mean depth is about 1/2, so that $\epsilon \simeq 1/2$, and the ratio of nonlinear to acceleration terms is also about 1/2. The nonlinear terms are thus even smaller than the local accelerations with respect to the frictional terms, and the dynamic balance in both cases is thus between friction and surface slope.

One of the ways in which the frictional model distinguishes itself from the wave model is in the value of the horizontal length scale. The appropriate length scale for a dynamic wave (corresponding to $R = 0$) is the wavelength $\lambda = (gH)^{1/2}T$. Wavelengths for both rivers examined are given in Table 1. The frictional model is characterized by $R = 1$ and hence, from (6) and $M = 1$, by the horizontal scale

$$L = [gH^2T/KU_0]^{1/2} \quad (8)$$

Calculated values of L for the Fraser and the Saint Lawrence are also shown in Table 1; they are smaller than λ by a factor of 3–5. The notable difference in length scales between the two physical models is, as we shall see below, of direct relevance to the understanding of the tidal phase lags in shallow rivers.

Equations (4) and (5) combine tidal and mean currents and slopes. To separate the two, we split the nondimensional velocity and surface displacement as follows:

$$\begin{aligned} v &= U(x) + u(x, t) \\ \eta &= Z(x) + \zeta(x, t) \end{aligned} \quad (9)$$

The tidal parts, u and ζ, are assumed to have zero mean over a suitable time interval: $\langle u \rangle = 0$, $\langle \zeta \rangle = 0$. Substituting (9) into (4) and (5) and averaging, we find that on the average,

$$\frac{F^2}{\epsilon}[UU_x + \langle uu_x \rangle] + Z_x + R\left\langle \frac{K(u+U)|u+U|}{1+\epsilon(\zeta+Z)} \right\rangle = 0 \quad (10)$$

$$[U(1 + \epsilon Z) + \epsilon\langle u\zeta \rangle]_x = 0 \quad (11)$$

The time-dependent momentum and mass balance equations are obtained by subtracting (10) and (11) from the complete equations to obtain

$$Su_t + \frac{F^2}{\epsilon}[uu_x - \langle uu_x \rangle + (uU)_x] + \zeta_x + R\left\{\frac{v|v|}{1+\epsilon\eta} - \left\langle \frac{v|v|}{1+\epsilon\eta} \right\rangle\right\} = 0 \quad (12)$$

$$M[u(1 + \epsilon Z) + \epsilon(u + U)\zeta - \epsilon\langle u\zeta \rangle]_x + \zeta_t = 0 \quad (13)$$

The friction terms have been left unexpanded for brevity. With $M = 1$ (and hence $F^2/\epsilon = S\epsilon$) the frictional regime tidal equations become

$$\zeta_x + R\left\{\frac{v|v|}{1+\epsilon\eta} - \left\langle \frac{v|v|}{1+\epsilon\eta} \right\rangle\right\} = 0 \quad (14)$$

$$u_x + \zeta_t + \epsilon[u(Z + \zeta) + U\zeta - \langle u\zeta \rangle]_x = 0 \quad (15)$$

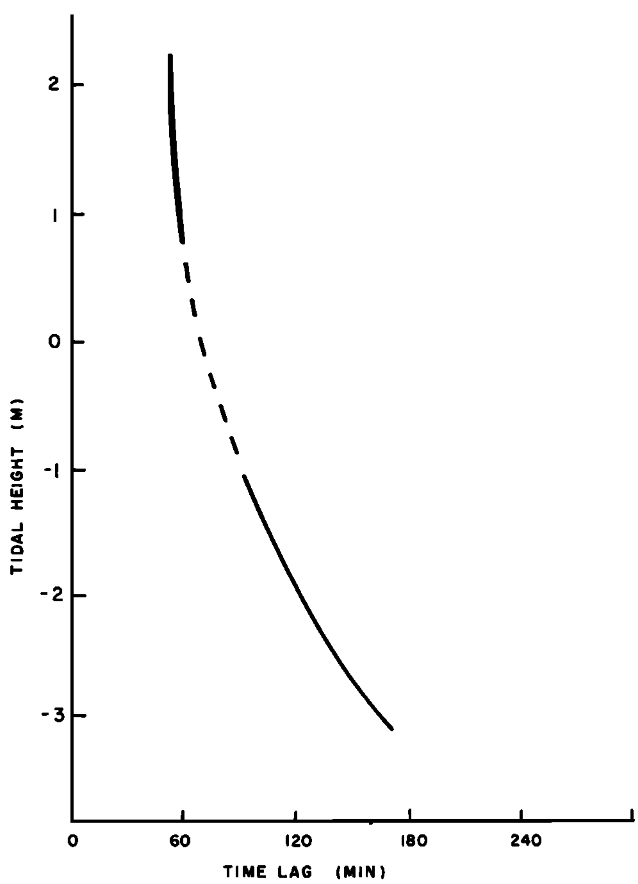

Fig. 1. Time lag (in minutes) for the propagation of the phase of sea level variations up the Fraser River, from Steveston (at the mouth) to New Westminster (30 km upstream). Tidal heights are referred to local geodetic datum [after *Ages and Woollard*, 1976].

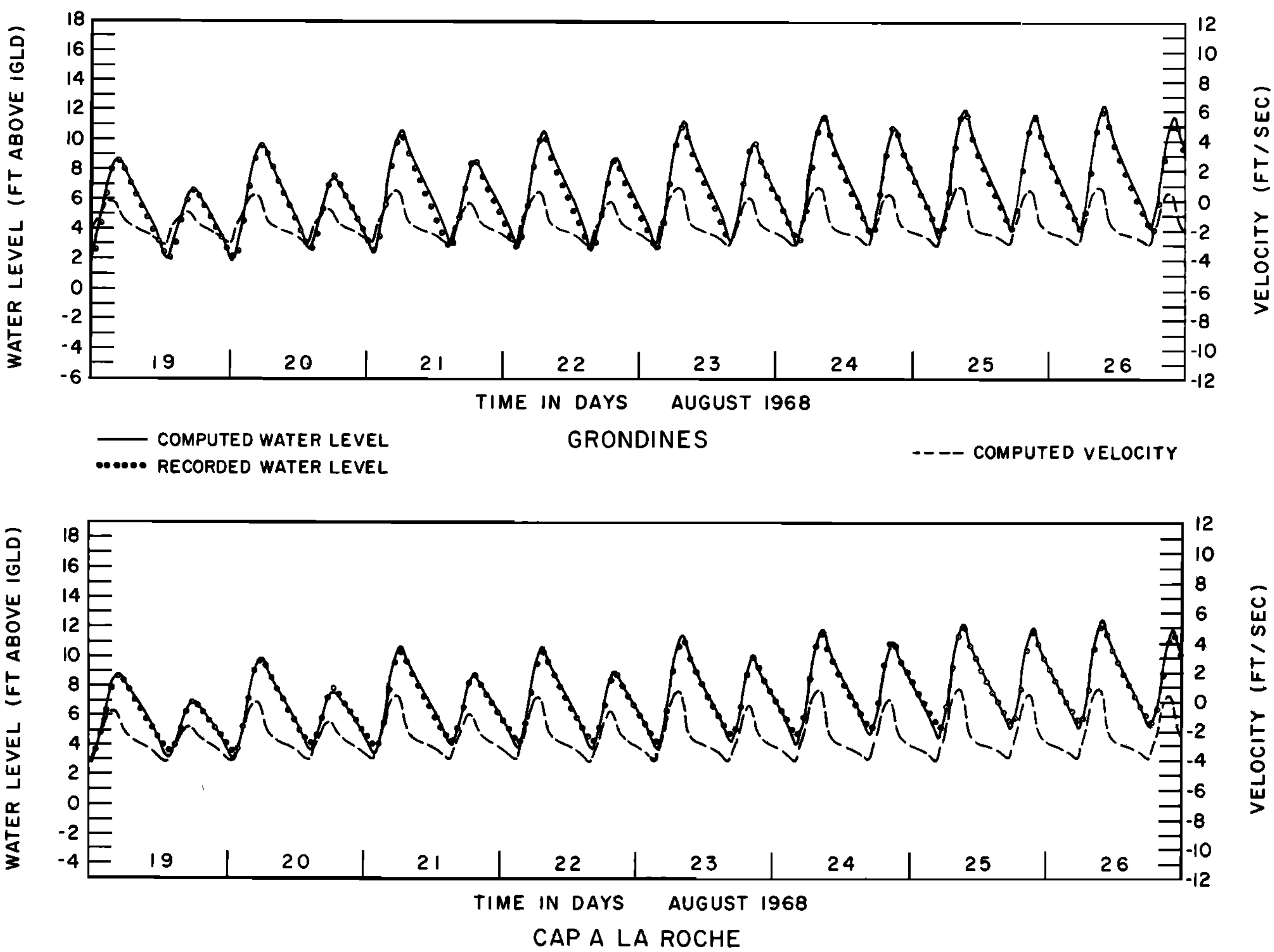

Fig. 2. Time series of water levels and of currents in the frictional reach of the Saint Lawrence River. Grondines and Cap à la Roche are 71 and 85 km, respectively, upstream of Quebec City. Note the near coincidence of slack tide with high water and of maximum ebb current with low water [from *Prandle and Crookshank,* 1972].

The flow regime described by (14) and (15) is intermediate between dynamic and kinematic waves. For dynamic waves ($S/R \gg 1$) the momentum and continuity equations are strongly coupled, and the equation system is hyperbolic with wave motion being possible in two directions. In kinematic waves the momentum equation reduces to a relationship between the flow speed and the depth, obtained by balancing friction with the mean slope of the free surface. This relation ($v(H)$, say) is then substituted in the continuity equation. The result is a one-dimensional wave equation, with propagation in one direction only. The system (14)-(15) resembles the kinematic wave equations in neglecting accelerations but differs from them in retaining the instantaneous surface slope ζ_x, thus keeping the momentum and continuity equations coupled. The frictional system (14)-(15) has only one characteristic [see *Whitham,* 1974, chapter 5] and is not hyperbolic but parabolic and partakes of the properties of the diffusion equation.

In neither the Fraser nor the Saint Lawrence is the parameter ϵ very small; the condition $M = 1$ (i.e., $\epsilon = U_0T/L$) yields $\epsilon \simeq 0.5$ in both cases. For a precise analysis of the frictional regime the nonlinear terms proportional to ϵ should thus be retained in (14) and (15). As a first approximation which will allow us to understand the simplest consequences of frictional dynamics, we neglect terms of the order of ϵ and adopt the scaling $R = 1$ to reduce (14) and (15) to

$$\zeta_x + (u + U)\,|u + U| - \langle (u + U)\,|u + U| \rangle = 0 \qquad (16)$$

$$u_x + \zeta_t = 0 \qquad (17)$$

Eliminating ζ, we obtain

$$u_t = \frac{1}{2\,|u + U|}\,u_{xx} \qquad (18)$$

a nonlinear diffusion equation governing tidal progression in a shallow river. We note that the frictional momentum balance, as expressed by (14), cannot hold at slack water, when $u + U = 0$; however, in the range of parameters considered it provides a useful approximation to tidal dynamics over most of the tidal cycle and is particularly relevant near full ebb, when $|u + U|$ is a maximum.

It is always advisable to have in mind a basically correct mental picture of a physical phenomenon in order to guide one's intuition. The form of the frictional regime equations clearly suggests that the best conceptual image of tidal motion in a shallow river is a diffusive process, the velocity field diffusing upriver from time-dependent boundary conditions at the mouth in the same way that heat diffuses in a rod at one end of which the temperature is varied. Although, for the parameter range examined, the diffusive model is applicable to most of the tidal cycle, it is definitely not valid at and near slack water, at which time the current (and the frictional force)

TABLE 2. Propagation Speeds and Lag Times for High- and Low-Water Conditions in the Fraser River and the Saint Lawrence River

	$H + \eta$, m	$\hat{u} + \hat{U}$, m s^{-1}	Observed V, m s^{-1}	Observed Lag, min	Wave Model $\hat{V}$, m s^{-1}	Wave Model Lag, min	Diffusive Model $\hat{V}_0$, m s^{-1}	Diffusive Model Lag, min	Diffusive Model $\hat{V}_1$, m s^{-1}	Diffusive Model Lag, min
					Fraser					
High water	11.5	0	10	50	10.6	47				
Low water	6.5	−2	3	160	6	83	3.4	145	2.5	197
					Saint Lawrence					
High water	8.5	0	9.5	90	9.1	91				
Low water	5.5	−1	5	160	6.3	132	7.0	119	5.5	151

Variables are defined in the text.

vanishes. An attempt to construct an approximate analytical model which would include both friction and advective nonlinearity has already been published by *Kreiss* [1957] under the conditions of relatively weak friction, which is not the case discussed here.

Near full ebb the total flow speed $|u + U|$ is stationary in time and is a maximum. Near that time one may thus approximate (18) by the linear diffusion equation

$$u_t = \kappa u_{xx} \qquad (19)$$

with a constant diffusion constant $\kappa = (2|u + U|)^{-1}$, i.e., inversely proportional to the maximum ebb current. Given a boundary condition $u = u_0 e^{-i\omega t}$ at $x = 0$, the solution of (19) which remains bounded at large values of x is

$$u(x, t) = u_0 \exp \{i[(\omega/2\kappa)^{1/2}x - \omega t] - (\omega/2\kappa)^{1/2}x\} \qquad (20)$$

The response is thus partly wavelike with speed of propagation $V_0 = (2\kappa\omega)^{1/2}$ and partly damped with an e folding length of $(2\kappa/\omega)^{1/2}$.

EXPLANATION OF THE PHASE LAGS

In the Fraser River the time lag in sea level displacements between the mouth of the river (Steveston) and New Westminster (30 km upstream) is only 50 min for very high waters but up to 160 min for very low tides (Figure 1). The corresponding propagation speeds are 10 m/s and 3 m/s. We also note that under the frictional regime, as illustrated in Figure 2, high water occurs very near slack water and low water occurs near maximum ebb.

A simple wave-dynamics model predicts a propagation speed (in dimensional variables) of

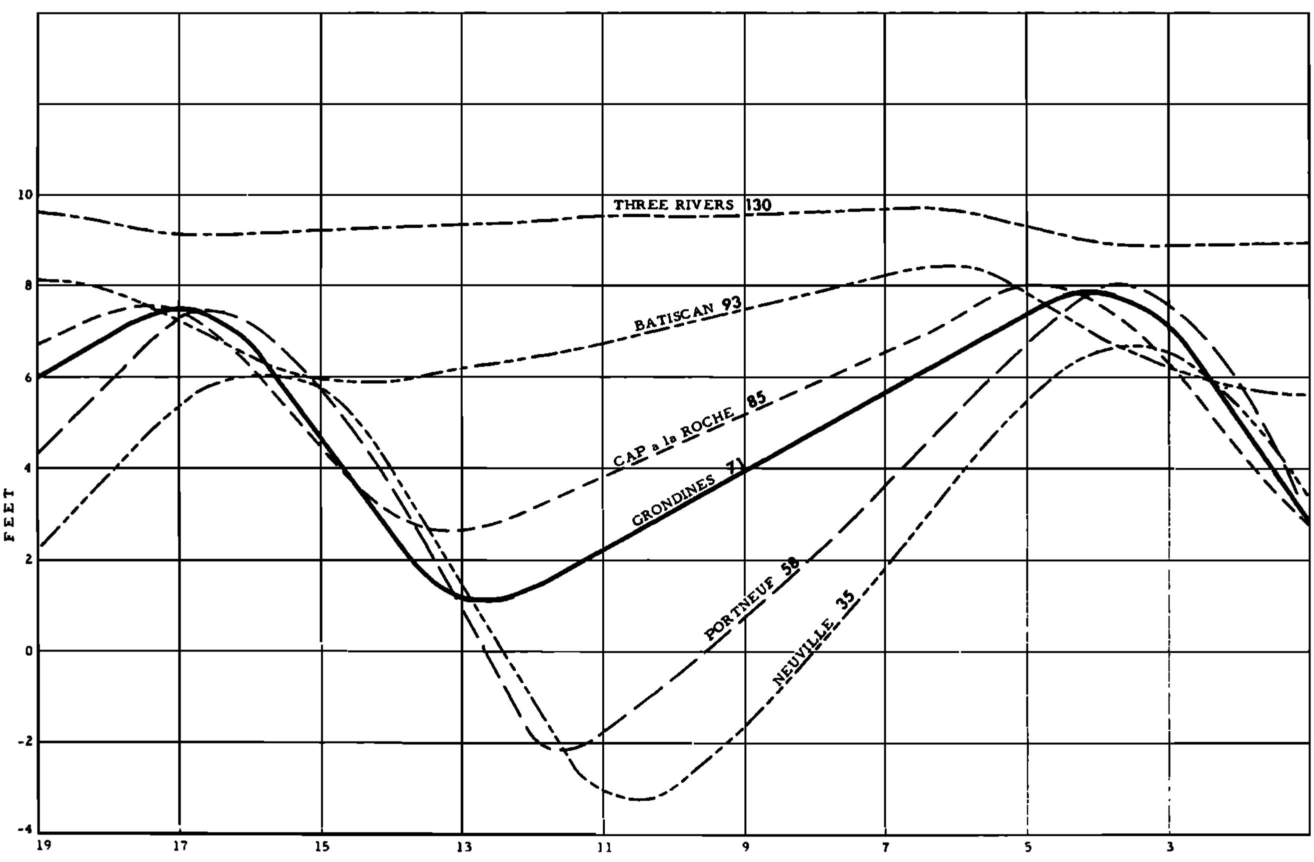

Fig. 3. Simultaneous records of sea level on the Saint Lawrence River. Each location is labeled with its distance upstream of Quebec City [from *Dohler*, 1964].

$$\hat{V} = \hat{u} + \hat{U} + [g(H + \hat{\eta})]^{1/2} \quad (21)$$

Speeds calculated for a tidal range of 5.0 m, i.e., with $\hat{\eta} = -2.5$ m at low water (full ebb, $\hat{u} = \hat{U} = -1$ m/s) and $\hat{\eta} = 2.5$ m at high water (slack tide, $\hat{u} + \hat{U} = 0$), are compared with the observed speeds in Table 2. The high-water propagation speed is well accounted for by the wave model (with calculated time lag to New Westminster of 47 min), but the low-water speed is greatly overestimated (and the time lag, of 83 min, grossly underestimated).

The frictional model based on the linearized solution (20) should be applicable to the calculation of the low-water time lags. The nondimensional diffusive speed $V_Q = (2\kappa\omega)^{1/2}$ may be converted to a dimensional speed by multiplying by the ratio of the length scale L (see Table 1) to the time scale (12 hours). The nondimensional frequency ω is of course equal to 2π, since the period of the tide has been chosen for the time scale. The same ebb velocity is used as in the wave model calculation. Two values of the diffusive speed are given in Table 2: $\hat{V}_0$, based on the length scale L of Table 1, and $\hat{V}_1$, calculated from a reduced length scale more appropriate to low-water conditions and obtained by replacing H by $H - \hat{\eta}$ in (8). It is clear from Table 2 that the diffusive model is more successful than the wave model in predicting low-water propagation speeds and corresponding time lags. The shorter length scale of the diffusive process leads to more rapid phase changes with distance from the river mouth than those in the wave model.

Similar calculations have been performed for the Saint Lawrence River, upstream of Quebec City, based on the information presented in Figure 3. Phase lags and speeds listed in Table 2 are for the Neuville to Cap à la Roche stretch of the river, 50 km long. The best results for the low-water propagation speed are obtained with the reduced velocity $\hat{V}_1$ based on the reduced length scale.

CONCLUSIONS

The observation that frictional forces greatly exceed the acceleration over most of the tidal cycle in shallow rivers implies that tidal motion in such circumstances is best represented in terms of a set of equations similar to the kinematic wave equations. One is thereby forced to abandon the simple conceptual model of the tide as a wave advancing up a river for an equally simple model in terms of a diffusive phenomenon, conditions upriver diffusing in from time-varying boundary conditions at the river mouth. This simplification is not appropriate near high-water slack (when the current, and hence the friction force, vanishes) but is most useful over most of the tidal cycle, especially near full ebb. The diffusive model is particularly successful in providing a simple explanation for the long phase lags observed in the upstream propagation of low water.

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