2 MATRIX ALGEBRA

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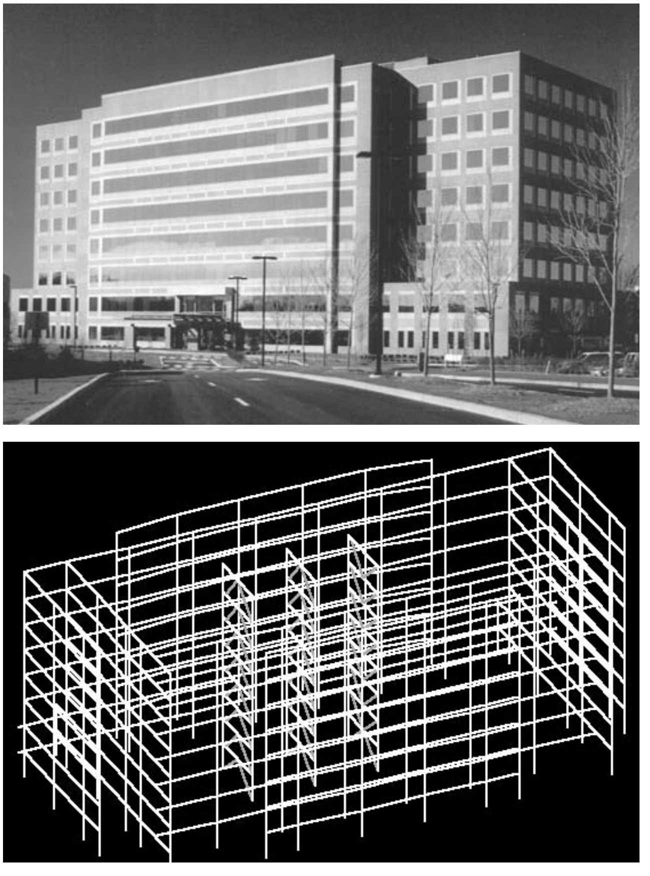

Somerset Corporate Center Office Building, New Jersey, and its Analytical Model
(Photo courtesy of Ram International. Structural Engineer: The Cantor Seinuk Group, P.C.)

In matrix methods of structural analysis, the fundamental relationships of equilibrium, compatibility, and member force–displacement relations are expressed in the form of matrix equations, and the analytical procedures are formulated by applying various matrix operations. Therefore, familiarity with the basic concepts of matrix algebra is a prerequisite to understanding matrix structural analysis. The objective of this chapter is to concisely present the basic concepts of matrix algebra necessary for formulating the methods of structural analysis covered in the text. A general procedure for solving simultaneous linear equations, the *Gauss–Jordan method*, is also discussed.

We begin with the basic definition of a matrix in Section 2.1, followed by brief descriptions of the various types of matrices in Section 2.2. The matrix operations of equality, addition and subtraction, multiplication, transposition, differentiation and integration, inversion, and partitioning are defined in Section 2.3; we conclude the chapter with a discussion of the Gauss–Jordan elimination method for solving simultaneous equations (Section 2.4).

2.1 DEFINITION OF A MATRIX

A *matrix* is defined as *a rectangular array of quantities arranged in rows and columns.* A matrix with m rows and n columns can be expressed as follows.

$$\mathbf{A} = [A] = \begin{array}{cl} \left[\begin{array}{ccccccc} A_{11} & A_{12} & A_{13} & \cdots & & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & & \cdots & A_{3n} \\ \cdots & \cdots & \cdots & \cdots & A_{ij} & \cdots & \\ A_{m1} & A_{m2} & A_{m3} & \cdots & & \cdots & A_{mn} \end{array}\right] & \begin{array}{l} \\ \\ \\ i\text{th row} \\ \\ \end{array} \\ \begin{array}{ccccccc} \qquad & \qquad & \qquad & & j\text{th column} & \qquad & m \times n \end{array} & \end{array} \tag{2.1}$$

As shown in Eq. (2.1), matrices are denoted either by boldface letters ($\mathbf{A}$) or by italic letters enclosed within brackets ($[A]$). The quantities forming a matrix are referred to as its *elements.* The elements of a matrix are usually numbers, but they can be symbols, equations, or even other matrices (called submatrices). Each element of a matrix is represented by a double-subscripted letter, with the first subscript identifying the row and the second subscript identifying the column in which the element is located. Thus, in Eq. (2.1), A_{23} represents the element located in the second row and third column of matrix $\mathbf{A}$. In general, A_{ij} refers to an element located in the ith row and jth column of matrix $\mathbf{A}$.

The size of a matrix is measured by the number of its rows and columns and is referred to as the *order* of the matrix. Thus, matrix $\mathbf{A}$ in Eq. (2.1), which has m rows and n columns, is considered to be of order $m \times n$ (m by n). As an example, consider a matrix **D** given by

$$\mathbf{D} = \begin{bmatrix} 3 & 5 & 37 \\ 8 & -6 & 0 \\ 12 & 23 & 2 \\ 7 & -9 & -1 \end{bmatrix}$$

The order of this matrix is 4×3, and its elements are symbolically denoted by D_{ij} with $i = 1$ to 4 and $j = 1$ to 3; for example, $D_{13} = 37$, $D_{31} = 12$, $D_{42} = -9$, etc.

2.2 TYPES OF MATRICES

We describe some of the common types of matrices in the following paragraphs.

Column Matrix (Vector)

If all the elements of a matrix are arranged in a single column (i.e., $n = 1$), it is called a *column matrix*. Column matrices are usually referred to as *vectors*, and are sometimes denoted by italic letters enclosed within braces. An example of a column matrix or vector is given by

$$\mathbf{B} = \{B\} = \begin{bmatrix} 35 \\ 9 \\ 12 \\ 3 \\ 26 \end{bmatrix}$$

Row Matrix

A matrix with all of its elements arranged in a single row (i.e., $m = 1$) is referred to as a *row matrix*. For example,

$$\mathbf{C} = [9 \quad 35 \quad -12 \quad 7 \quad 22]$$

Square Matrix

If a matrix has the same number of rows and columns (i.e., $m = n$), it is called a *square matrix*. An example of a 4×4 square matrix is given by

$$\mathbf{A} = \begin{bmatrix} 6 & 12 & 0 & 20 \\ 15 & -9 & -37 & 3 \\ -24 & 13 & 8 & 1 \\ 40 & 0 & 11 & -5 \end{bmatrix} \tag{2.2}$$

main diagonal

As shown in Eq. (2.2), the *main diagonal* of a square matrix extends from the upper left corner to the lower right corner, and it contains elements with matching subscripts—that is, $A_{11}, A_{22}, A_{33}, \ldots, A_{nn}$. The elements forming the main diagonal are referred to as the *diagonal elements;* the remaining elements of a square matrix are called the *off-diagonal elements*.

Symmetric Matrix

When the elements of a square matrix are symmetric about its main diagonal (i.e., $A_{ij} = A_{ji}$), it is termed a *symmetric matrix.* For example,

$$\mathbf{A} = \begin{bmatrix} 6 & 15 & -24 & 40 \\ 15 & -9 & 13 & 0 \\ -24 & 13 & 8 & 11 \\ 40 & 0 & 11 & -5 \end{bmatrix}$$

Lower Triangular Matrix

If all the elements of a square matrix above its main diagonal are zero, (i.e., $A_{ij} = 0$ for $j > i$), it is referred to as a *lower triangular matrix*. An example of a 4×4 lower triangular matrix is given by

$$\mathbf{A} = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 12 & -9 & 0 & 0 \\ 33 & 17 & 6 & 0 \\ -2 & 5 & 15 & 3 \end{bmatrix}$$

Upper Triangular Matrix

When all the elements of a square matrix below its main diagonal are zero (i.e., $A_{ij} = 0$ for $j < i$), it is called an *upper triangular matrix*. An example of a 3×3 upper triangular matrix is given by

$$\mathbf{A} = \begin{bmatrix} -7 & 6 & 17 \\ 0 & 12 & 11 \\ 0 & 0 & 20 \end{bmatrix}$$

Diagonal Matrix

A square matrix with all of its off-diagonal elements equal to zero (i.e., $A_{ij} = 0$ for $i \neq j$), is called a *diagonal matrix*. For example,

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 27 \end{bmatrix}$$

Unit or Identity Matrix

If all the diagonal elements of a diagonal matrix are equal to 1 (i.e., $I_{ij} = 1$ and $I_{ij} = 0$ for $i \neq j$), it is referred to as a *unit* (or *identity*) *matrix*. Unit matrices are commonly denoted by **I** or $[I]$. An example of a 3×3 unit matrix is given by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix

If all the elements of a matrix are zero (i.e., $O_{ij} = 0$), it is termed a *null matrix*. Null matrices are usually denoted by **O** or $[O]$. An example of a 3×4 null matrix is given by

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.3 MATRIX OPERATIONS

Equality

Matrices **A** and **B** are considered to be equal if they are of the same order and if their corresponding elements are identical (i.e., $A_{ij} = B_{ij}$). Consider, for example, matrices

$$\mathbf{A} = \begin{bmatrix} 6 & 2 \\ -7 & 8 \\ 3 & -9 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & 2 \\ -7 & 8 \\ 3 & -9 \end{bmatrix}$$

Since both **A** and **B** are of order 3×2, and since each element of **A** is equal to the corresponding element of **B**, the matrices **A** and **B** are equal to each other; that is, $\mathbf{A} = \mathbf{B}$.

Addition and Subtraction

Matrices can be added (or subtracted) only if they are of the same order. The addition (or subtraction) of two matrices **A** and **B** is carried out by adding (or subtracting) the corresponding elements of the two matrices. Thus, if $\mathbf{A} + \mathbf{B} = \mathbf{C}$, then $C_{ij} = A_{ij} + B_{ij}$; and if $\mathbf{A} - \mathbf{B} = \mathbf{D}$, then $D_{ij} = A_{ij} - B_{ij}$. The matrices **C** and **D** have the same order as matrices **A** and **B**.

EXAMPLE 2.1 Calculate the matrices $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 6 & 0 \\ -2 & 9 \\ 5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 7 & 5 \\ -12 & -1 \end{bmatrix}$$

SOLUTION

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} (6+2) & (0+3) \\ (-2+7) & (9+5) \\ (5-12) & (1-1) \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 5 & 14 \\ -7 & 0 \end{bmatrix} \qquad \textbf{Ans}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} (6-2) & (0-3) \\ (-2-7) & (9-5) \\ (5+12) & (1+1) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -9 & 4 \\ 17 & 2 \end{bmatrix} \qquad \textbf{Ans}$$

Multiplication by a Scalar

The product of a scalar c and a matrix **A** is obtained by multiplying each element of the matrix **A** by the scalar c. Thus, if $c\mathbf{A} = \mathbf{B}$, then $B_{ij} = cA_{ij}$.

EXAMPLE 2.2 Calculate the matrix $\mathbf{B} = c\mathbf{A}$ if $c = -6$ and

$$\mathbf{A} = \begin{bmatrix} 3 & 7 & -2 \\ 0 & 8 & 1 \\ 12 & -4 & 10 \end{bmatrix}$$

SOLUTION

$$\mathbf{B} = c\mathbf{A} = \begin{bmatrix} -6(3) & -6(7) & -6(-2) \\ -6(0) & -6(8) & -6(1) \\ -6(12) & -6(-4) & -6(10) \end{bmatrix} = \begin{bmatrix} -18 & -42 & 12 \\ 0 & -48 & -6 \\ -72 & 24 & -60 \end{bmatrix} \qquad \textbf{Ans}$$

Multiplication of Matrices

Two matrices can be multiplied only if the number of columns of the first matrix equals the number of rows of the second matrix. Such matrices are said to be *conformable* for multiplication. Consider, for example, the matrices

$$\underset{3 \times 2}{\mathbf{A} = \begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}} \quad \text{and} \quad \underset{2 \times 2}{\mathbf{B} = \begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}} \tag{2.3}$$

The product **AB** of these matrices is defined because the first matrix, **A**, of the sequence **AB** has two columns and the second matrix, **B**, has two rows. However, if the sequence of the matrices is reversed, then the product **BA** does not exist, because now the first matrix, **B**, has two columns and the second matrix, **A**, has three rows. The product **AB** is referred to either as **A** *postmultiplied* by **B**, or as **B** *premultiplied* by **A**. Conversely, the product **BA** is referred to either as **B** postmultiplied by **A**, or as **A** premultiplied by **B**.

When two conformable matrices are multiplied, the product matrix thus obtained has the number of rows of the first matrix and the number of columns of the second matrix. Thus, if a matrix **A** of order $l \times m$ is postmultiplied by a matrix **B** of order $m \times n$, then the product matrix $\mathbf{C} = \mathbf{AB}$ has the order $l \times n$; that is,

$$\underset{(l \times m)}{\mathbf{A}} \quad \underset{(m \times n)}{\mathbf{B}} = \underset{(l \times n)}{\mathbf{C}}$$

equal

$$\begin{array}{c} \\ i\text{th row} \\ \\ \end{array} \begin{bmatrix} & & & & \\ A_{i1} & A_{i2} & \cdots & \cdots & A_{im} \\ & & & & \end{bmatrix} \underset{j\text{th column}}{\begin{bmatrix} & B_{1j} & \\ & B_{2j} & \\ & \vdots & \\ & \vdots & \\ & B_{mj} & \end{bmatrix}} = \underset{j\text{th column}}{\begin{bmatrix} & & \\ & C_{ij} & \\ & & \end{bmatrix}} \begin{array}{c} \\ i\text{th row} \\ \\ \end{array} \tag{2.4}$$

Any element C_{ij} of the product matrix **C** can be determined by multiplying each element of the ith row of **A** by the corresponding element of the jth column of **B** (see Eq. 2.4), and by algebraically summing the products; that is,

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj} \tag{2.5}$$

Eq. (2.5) can be expressed as

$$\boxed{C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}} \tag{2.6}$$

in which m represents the number of columns of **A**, or the number of rows of **B**. Equation (2.6) can be used to determine all elements of the product matrix $\mathbf{C} = \mathbf{AB}$.

EXAMPLE 2.3 Calculate the product $\mathbf{C} = \mathbf{AB}$ of the matrices **A** and **B** given in Eq. (2.3).

SOLUTION

$$\mathbf{C} = \mathbf{AB} = \underset{(3 \times 2)}{\begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}} \underset{(2 \times 2)}{\begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}} = \underset{(3 \times 2)}{\begin{bmatrix} -2 & 9 \\ 26 & -32 \\ -33 & 41 \end{bmatrix}} \qquad \textbf{Ans}$$

The element C_{11} of the product matrix **C** is determined by multiplying each element of the first row of **A** by the corresponding element of the first column of **B** and summing the resulting products; that is,

$$C_{11} = 1(6) + 8(-1) = -2$$

Similarly, the element C_{12} is obtained by multiplying the elements of the first row of **A** by the corresponding elements of the second column of **B** and adding the resulting products; that is,

$$C_{12} = 1(-7) + 8(2) = 9$$

The remaining elements of **C** are computed in a similar manner:

$$C_{21} = 4(6) + (-2)(-1) = 26$$
$$C_{22} = 4(-7) - 2(2) = -32$$
$$C_{31} = -5(6) + 3(-1) = -33$$
$$C_{32} = -5(-7) + 3(2) = 41$$

A flowchart for programming the matrix multiplication procedure on a computer is given in Fig. 2.1. Any programming language (such as FORTRAN, BASIC, or C, among others) can be used for this purpose. The reader is encouraged to write this program in a general form (e.g., as a subroutine), so that it can be included in the structural analysis computer programs to be developed in later chapters.

An important application of matrix multiplication is to express simultaneous equations in compact matrix form. Consider the following system of linear simultaneous equations.

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 &= P_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{24}x_4 &= P_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + A_{34}x_4 &= P_3 \\ A_{41}x_1 + A_{42}x_2 + A_{43}x_3 + A_{44}x_4 &= P_4 \end{aligned} \tag{2.7}$$

in which xs are the unknowns and As and Ps represent the coefficients and constants, respectively. By using the definition of multiplication of matrices, this system of equations can be expressed in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \tag{2.8}$$

or, symbolically, as

$$\mathbf{Ax} = \mathbf{P} \tag{2.9}$$

Matrix multiplication is generally not commutative; that is,

$$\boxed{\mathbf{AB} \neq \mathbf{BA}} \tag{2.10}$$

Even when the orders of two matrices **A** and **B** are such that both products **AB** and **BA** are defined and are of the same order, the two products, in general, will

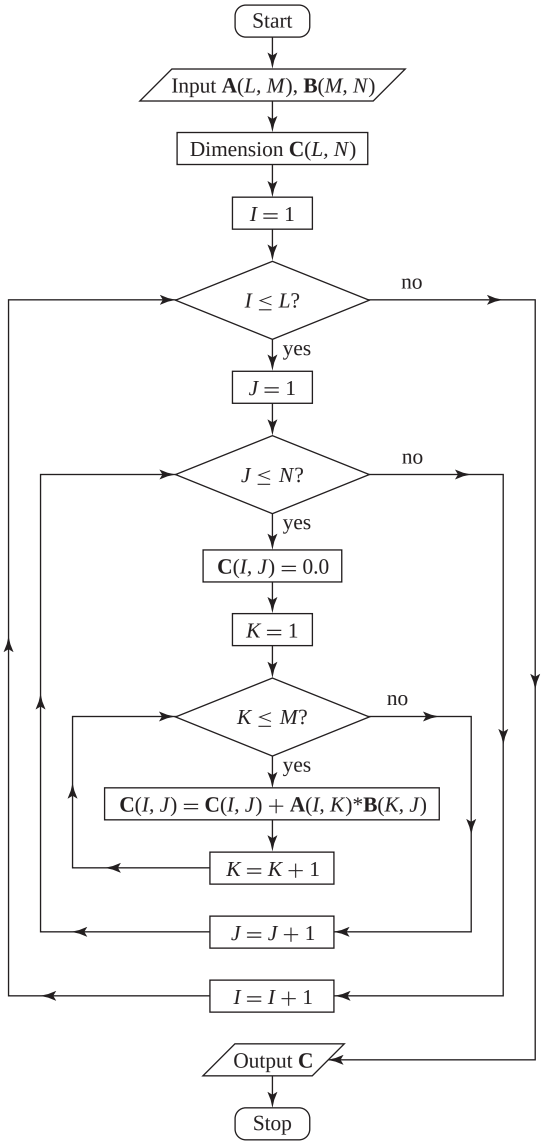

Fig. 2.1 *Flowchart for Matrix Multiplication*

not be equal. It is essential, therefore, to maintain the proper sequential order of matrices when evaluating matrix products.

EXAMPLE 2.4 Calculate the products **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 1 & -8 \\ -7 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -3 \\ 4 & -5 \end{bmatrix}$$

Are the products **AB** and **BA** equal?

SOLUTION

$$\mathbf{AB} = \begin{bmatrix} 1 & -8 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} -26 & 37 \\ -34 & 11 \end{bmatrix}$$ **Ans**

$$\mathbf{BA} = \begin{bmatrix} 6 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 27 & -54 \\ 39 & -42 \end{bmatrix}$$ **Ans**

Comparing products **AB** and **BA**, we can see that $\mathbf{AB} \neq \mathbf{BA}$. **Ans**

Matrix multiplication is associative and distributive, provided that the sequential order in which the matrices are to be multiplied is maintained. Thus,

$$\mathbf{ABC} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{2.11}$$

and

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{2.12}$$

The product of any matrix **A** and a conformable null matrix **O** equals a null matrix; that is,

$$\mathbf{AO} = \mathbf{O} \qquad \text{and} \qquad \mathbf{OA} = \mathbf{O} \tag{2.13}$$

For example,

$$\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The product of any matrix **A** and a conformable unit matrix **I** equals the original matrix **A**; thus,

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IA} = \mathbf{A} \tag{2.14}$$

For example,

$$\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$$

We can see from Eqs. (2.13) and (2.14) that the null and unit matrices serve purposes in matrix algebra that are similar to those of the numbers 0 and 1, respectively, in scalar algebra.

Transpose of a Matrix

The *transpose* of a matrix is obtained by interchanging its corresponding rows and columns. The transposed matrix is commonly identified by placing a superscript T on the symbol of the original matrix. Consider, for example, a 3×2 matrix

$$\mathbf{B} = \underset{3 \times 2}{\begin{bmatrix} 2 & -4 \\ -5 & 8 \\ 1 & 3 \end{bmatrix}}$$

The transpose of **B** is given by

$$\mathbf{B}^T = \underset{2 \times 3}{\begin{bmatrix} 2 & -5 & 1 \\ -4 & 8 & 3 \end{bmatrix}}$$

Note that the first row of **B** becomes the first column of $\mathbf{B}^T$. Similarly, the second and third rows of **B** become, respectively, the second and third columns of $\mathbf{B}^T$. The order of $\mathbf{B}^T$ thus obtained is 2×3.

As another example, consider the matrix

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 6 \\ -1 & 7 & -9 \\ 6 & -9 & 5 \end{bmatrix}$$

Because the elements of **C** are symmetric about its main diagonal (i.e., $C_{ij} = C_{ji}$ for $i \neq j$), interchanging the rows and columns of this matrix produces a matrix $\mathbf{C}^T$ that is identical to **C** itself; that is, $\mathbf{C}^T = \mathbf{C}$. Thus, *the transpose of a symmetric matrix equals the original matrix.*

Another useful property of matrix transposition is that *the transpose of a product of matrices equals the product of the transposed matrices in reverse order*. Thus,

$$\boxed{(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T} \tag{2.15}$$

Similarly,

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \tag{2.16}$$

EXAMPLE 2.5 Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ if

$$\mathbf{A} = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix}$$

SOLUTION

$$\mathbf{AB} = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 64 & -44 & 65 \\ 10 & 5 & 25 \\ -26 & 31 & -10 \end{bmatrix}$$

$$(\mathbf{AB})^T = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix} \tag{1}$$

$$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 6 & -2 \\ -1 & 7 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 9 & 2 & -3 \\ -5 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix} \tag{2}$$

By comparing Eqs. (1) and (2), we can see that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. **Ans**

Differentiation and Integration

A matrix can be differentiated (or integrated) by differentiating (or integrating) each of its elements.

EXAMPLE 2.6 Determine the derivative $d\mathbf{A}/dx$ if

$$\mathbf{A} = \begin{bmatrix} x^2 & 3\sin x & -x^4 \\ 3\sin x & -x & \cos^2 x \\ -x^4 & \cos^2 x & 7x^3 \end{bmatrix}$$

SOLUTION By differentiating the elements of $\mathbf{A}$, we obtain

$$\begin{aligned}
&A_{11} = x^2 && \frac{dA_{11}}{dx} = 2x \\
&A_{21} = A_{12} = 3\sin x && \frac{dA_{21}}{dx} = \frac{dA_{12}}{dx} = 3\cos x \\
&A_{31} = A_{13} = -x^4 && \frac{dA_{31}}{dx} = \frac{dA_{13}}{dx} = -4x^3 \\
&A_{22} = -x && \frac{dA_{22}}{dx} = -1 \\
&A_{32} = A_{23} = \cos^2 x && \frac{dA_{32}}{dx} = \frac{dA_{23}}{dx} = -2\cos x \sin x \\
&A_{33} = 7x^3 && \frac{dA_{33}}{dx} = 21x^2
\end{aligned}$$

Thus, the derivative $d\mathbf{A}/dx$ is given by

$$\frac{d\mathbf{A}}{dx} = \begin{bmatrix} 2x & 3\cos x & -4x^3 \\ 3\cos x & -1 & -2\cos x \sin x \\ -4x^3 & -2\cos x \sin x & 21x^2 \end{bmatrix}$$ **Ans**

EXAMPLE 2.7 Determine the partial derivative $\partial\mathbf{B}/\partial y$ if

$$\mathbf{B} = \begin{bmatrix} 2y^3 & -yz & -2xz \\ 3xy^2 & yz & -z^2 \\ 2x^2 & -2xz & 3xy^2 \end{bmatrix}$$

SOLUTION We determine the partial derivative, $\partial B_{ij}/\partial y$, of each element of $\mathbf{B}$ to obtain

$$\frac{\partial\mathbf{B}}{\partial y} = \begin{bmatrix} 6y^2 & -z & 0 \\ 6xy & z & 0 \\ 0 & 0 & 6xy \end{bmatrix}$$ **Ans**

EXAMPLE 2.8 Calculate the integral $\int_0^L \mathbf{A}\mathbf{A}^T\,dx$ if

$$\mathbf{A} = \begin{bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{bmatrix}$$

SOLUTION First, we calculate the matrix product $\mathbf{AA}^T$ as

$$\mathbf{B} = \mathbf{AA}^T = \begin{bmatrix} \left(1 - \dfrac{x}{L}\right) \\ \dfrac{x}{L} \end{bmatrix} \left[\left(1 - \dfrac{x}{L}\right) \; \dfrac{x}{L}\right] = \begin{bmatrix} \left(1 - \dfrac{x}{L}\right)^2 & \dfrac{x}{L}\left(1 - \dfrac{x}{L}\right) \\ \dfrac{x}{L}\left(1 - \dfrac{x}{L}\right) & \dfrac{x^2}{L^2} \end{bmatrix}$$

Next, we integrate the elements of **B** to obtain

$$\int_0^L B_{11} dx = \int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \int_0^L \left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right) dx$$

$$= \left(x - \frac{x^2}{L} + \frac{x^3}{3L^2}\right)_0^L = \frac{L}{3}$$

$$\int_0^L B_{21} dx = \int_0^L B_{12} dx = \int_0^L \frac{x}{L}\left(1 - \frac{x}{L}\right) dx = \int_0^L \left(\frac{x}{L} - \frac{x^2}{L^2}\right) dx$$

$$= \left(\frac{x^2}{2L} - \frac{x^3}{3L^2}\right)_0^L = \frac{L}{2} - \frac{L}{3} = \frac{L}{6}$$

$$\int_0^L B_{22} dx = \int_0^L \left(\frac{x^2}{L^2}\right) dx = \left(\frac{x^3}{3L^2}\right)_0^L = \frac{L}{3}$$

Thus,

$$\int_0^L \mathbf{AA}^T dx = \begin{bmatrix} \dfrac{L}{3} & \dfrac{L}{6} \\ \dfrac{L}{6} & \dfrac{L}{3} \end{bmatrix} = \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Ans

Inverse of a Square Matrix

The inverse of a square matrix **A** is defined as a matrix $\mathbf{A}^{-1}$ with elements of such magnitudes that the product of the original matrix **A** and its inverse $\mathbf{A}^{-1}$ equals a unit matrix **I**; that is,

$$\boxed{\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}} \tag{2.17}$$

The operation of inversion is defined only for square matrices, with the inverse of such a matrix also being a square matrix of the same order as the original matrix. A procedure for determining inverses of matrices will be presented in the next section.

EXAMPLE 2.9 Check whether or not matrix **B** is the inverse of matrix **A**, if

$$\mathbf{A} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0.5 & -1 \\ 1.5 & -2 \end{bmatrix}$$

SOLUTION

$$\mathbf{AB} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -1 \\ 1.5 & -2 \end{bmatrix} = \begin{bmatrix} (-2+3) & (4-4) \\ (-1.5+1.5) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also,

$$\mathbf{BA} = \begin{bmatrix} 0.5 & -1 \\ 1.5 & -2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} (-2+3) & (1-1) \\ (-6+6) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, $\mathbf{B}$ is the inverse of $\mathbf{A}$; that is,

$$\mathbf{B} = \mathbf{A}^{-1}$$ **Ans**

The operation of matrix inversion serves a purpose analogous to the operation of division in scalar algebra. Consider a system of simultaneous linear equations expressed in matrix form as

$$\mathbf{Ax} = \mathbf{P}$$

in which $\mathbf{A}$ is the square matrix of known coefficients; $\mathbf{x}$ is the vector of the unknowns; and $\mathbf{P}$ is the vector of the constants. As the operation of division is not defined in matrix algebra, the equation cannot be solved for $\mathbf{x}$ by dividing $\mathbf{P}$ by $\mathbf{A}$ (i.e., $\mathbf{x} = \mathbf{P}/\mathbf{A}$). However, we can determine $\mathbf{x}$ by premultiplying both sides of the equation by $\mathbf{A}^{-1}$, to obtain

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{P}$$

As $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{Ix} = \mathbf{x}$, we can write

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{P}$$

which shows that a system of simultaneous linear equations can be solved by premultiplying the vector of constants by the inverse of the coefficient matrix.

An important property of matrix inversion is that *the inverse of a symmetric matrix is also a symmetric matrix.*

Orthogonal Matrix

If the inverse of a matrix is equal to its transpose, the matrix is referred to as an *orthogonal matrix.* In other words, a matrix $\mathbf{A}$ is orthogonal if

$$\mathbf{A}^{-1} = \mathbf{A}^T$$

EXAMPLE 2.10 Determine whether matrix $\mathbf{A}$ given below is an orthogonal matrix.

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix}$$

SOLUTION

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} (0.64+0.36) & (-0.48+0.48) & 0 & 0 \\ (-0.48+0.48) & (0.36+0.64) & 0 & 0 \\ 0 & 0 & (0.64+0.36) & (-0.48+0.48) \\ 0 & 0 & (-0.48+0.48) & (0.36+0.64) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which shows that $\mathbf{A}\mathbf{A}^T = \mathbf{I}$. Thus,

$$\mathbf{A}^{-1} = \mathbf{A}^T$$

Therefore, matrix **A** is orthogonal. **Ans**

Partitioning of Matrices

In many applications, it becomes necessary to subdivide a matrix into a number of smaller matrices called *submatrices*. The process of subdividing a matrix into submatrices is referred to as *partitioning*. For example, a 4×3 matrix **B** is partitioned into four submatrices by drawing horizontal and vertical dashed partition lines:

$$\mathbf{B} = \left[\begin{array}{cc:c} 2 & -4 & -1 \\ -5 & 7 & 3 \\ 8 & -9 & 6 \\ \hdashline 1 & 3 & 8 \end{array}\right] = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \tag{2.18}$$

in which the submatrices are

$$\mathbf{B}_{11} = \begin{bmatrix} 2 & -4 \\ -5 & 7 \\ 8 & -9 \end{bmatrix} \qquad \mathbf{B}_{12} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$$

$$\mathbf{B}_{21} = [1 \quad 3] \qquad \mathbf{B}_{22} = [8]$$

Matrix operations (such as addition, subtraction, and multiplication) can be performed on partitioned matrices in the same manner as discussed previously by treating the submatrices as elements—provided that the matrices are partitioned in such a way that their submatrices are conformable for the particular operation. For example, suppose that the 4×3 matrix **B** of Eq. (2.18) is to be postmultiplied by a 3×2 matrix **C**, which is partitioned into two submatrices:

$$\mathbf{C} = \left[\begin{array}{cc} 9 & -6 \\ 4 & 2 \\ \hdashline -3 & 1 \end{array}\right] = \begin{bmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{bmatrix} \tag{2.19}$$

The product **BC** is expressed in terms of submatrices as

$$\mathbf{BC} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11}\mathbf{C}_{11} + \mathbf{B}_{12}\mathbf{C}_{21} \\ \mathbf{B}_{21}\mathbf{C}_{11} + \mathbf{B}_{22}\mathbf{C}_{21} \end{bmatrix} \tag{2.20}$$

It is important to realize that matrices **B** and **C** have been partitioned in such a way that their corresponding submatrices are conformable for multiplication; that is, the orders of the submatrices are such that the products $\mathbf{B}_{11}\mathbf{C}_{11}$, $\mathbf{B}_{12}\mathbf{C}_{21}$, $\mathbf{B}_{21}\mathbf{C}_{11}$, and $\mathbf{B}_{22}\mathbf{C}_{21}$ are defined. It can be seen from Eqs. (2.18) and (2.19) that this is achieved by partitioning the rows of the second matrix **C** of the product **BC** in the same way that the columns of the first matrix **B** are partitioned. The products of the submatrices are:

$$\mathbf{B}_{11}\mathbf{C}_{11} = \begin{bmatrix} 2 & -4 \\ -5 & 7 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 9 & -6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -17 & 44 \\ 36 & -66 \end{bmatrix}$$

$$\mathbf{B}_{12}\mathbf{C}_{21} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} [-3 \quad 1] = \begin{bmatrix} 3 & -1 \\ -9 & 3 \\ -18 & 6 \end{bmatrix}$$

$$\mathbf{B}_{21}\mathbf{C}_{11} = [1 \quad 3] \begin{bmatrix} 9 & -6 \\ 4 & 2 \end{bmatrix} = [21 \quad 0]$$

$$\mathbf{B}_{22}\mathbf{C}_{21} = [8][-3 \quad 1] = [-24 \quad 8]$$

By substituting the numerical values of the products of submatrices into Eq. (2.20), we obtain

$$\mathbf{BC} = \begin{bmatrix} \begin{bmatrix} 2 & -20 \\ -17 & 44 \\ 36 & -66 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -9 & 3 \\ -18 & 6 \end{bmatrix} \\ [21 \quad\quad 0] + [-24 \quad\quad 8] \end{bmatrix} = \begin{bmatrix} 5 & -21 \\ -26 & 47 \\ 18 & -60 \\ -3 & 8 \end{bmatrix}$$

2.4 GAUSS–JORDAN ELIMINATION METHOD

The *Gauss–Jordan elimination method* is one of the most commonly used procedures for solving simultaneous linear equations, and for determining inverses of matrices.

Solution of Simultaneous Equations

To illustrate the Gauss–Jordan method for solving simultaneous equations, consider the following system of three linear algebraic equations:

$$\begin{aligned} 5x_1 + 6x_2 - 3x_3 &= 66 \\ 9x_1 - x_2 + 2x_3 &= 8 \\ 8x_1 - 7x_2 + 4x_3 &= -39 \end{aligned} \tag{2.21a}$$

To determine the unknowns x_1, x_2, and x_3, we begin by dividing the first equation by the coefficient of its x_1 term to obtain

$$\begin{aligned} x_1 + 1.2x_2 - 0.6x_3 &= 13.2 \\ 9x_1 - x_2 + 2x_3 &= 8 \\ 8x_1 - 7x_2 + 4x_3 &= -39 \end{aligned} \tag{2.21b}$$

Next, we eliminate the unknown x_1 from the second and third equations by successively subtracting from each equation the product of the coefficient of its x_1 term and the first equation. Thus, to eliminate x_1 from the second equation, we multiply the first equation by 9 and subtract it from the second equation. Similarly, we eliminate x_1 from the third equation by multiplying the first equation by 8 and subtracting it from the third equation. This yields the system of equations

$$\begin{aligned} x_1 + 1.2x_2 - 0.6x_3 &= 13.2 \\ -11.8x_2 + 7.4x_3 &= -110.8 \\ -16.6x_2 + 8.8x_3 &= -144.6 \end{aligned} \tag{2.21c}$$

With x_1 eliminated from all but the first equation, we now divide the second equation by the coefficient of its x_2 term to obtain

$$\begin{aligned} x_1 + 1.2x_2 - 0.6x_3 &= 13.2 \\ x_2 - 0.6271x_3 &= 9.39 \\ -16.6x_2 + 8.8x_3 &= -144.6 \end{aligned} \tag{2.21d}$$

Next, the unknown x_2 is eliminated from the first and the third equations, successively, by multiplying the second equation by 1.2 and subtracting it from the first equation, and then by multiplying the second equation by -16.6 and subtracting it from the third equation. The system of equations thus obtained is

$$\begin{aligned} x_1 \qquad + 0.1525x_3 &= 1.932 \\ x_2 - 0.6271x_3 &= 9.39 \\ -1.61x_3 &= 11.27 \end{aligned} \tag{2.21e}$$

Focusing our attention now on the unknown x_3, we divide the third equation by the coefficient of its x_3 term (which is -1.61) to obtain

$$\begin{aligned} x_1 \qquad + 0.1525x_3 &= 1.932 \\ x_2 - 0.6271x_3 &= 9.39 \\ x_3 &= -7 \end{aligned} \tag{2.21f}$$

Finally, we eliminate x_3 from the first and the second equations, successively, by multiplying the third equation by 0.1525 and subtracting it from the first equation, and then by multiplying the third equation by -0.6271 and subtracting it from the second equation. This yields the solution of the given system of equations:

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 5 \\ x_3 &= -7 \end{aligned} \tag{2.21g}$$

or, equivalently,

$$x_1 = 3; \quad x_2 = 5; \quad x_3 = -7 \qquad \textbf{(2.21h)}$$

To check that this solution is correct, we substitute the numerical values of x_1, x_2, and x_3 back into the original equations (Eq. 2.21(a)):

$$\begin{aligned} 5(3) + 6(5) - 3(-7) &= 66 && \textbf{Checks} \\ 9(3) - 5 \quad + 2(-7) &= 8 && \textbf{Checks} \\ 8(3) - 7(5) + 4(-7) &= -39 && \textbf{Checks} \end{aligned}$$

As the foregoing example illustrates, the Gauss–Jordan method basically involves eliminating, in order, each unknown from all but one of the equations of the system by applying the following operations: dividing an equation by a scalar; and multiplying an equation by a scalar and subtracting the resulting equation from another equation. These operations (called the *elementary operations*) when applied to a system of equations yield another system of equations that has the same solution as the original system. In the Gauss–-Jordan method, the elementary operations are performed repeatedly until a system with each equation containing only one unknown is obtained.

The Gauss–Jordan elimination method can be performed more conveniently by using the matrix form of the simultaneous equations ($\mathbf{Ax} = \mathbf{P}$). In this approach, the coefficient matrix $\mathbf{A}$ and the vector of constants $\mathbf{P}$ are treated as submatrices of a partitioned *augmented matrix*,

$$\underset{n \times (n+1)}{\mathbf{G}} = [\underset{n \times n}{\mathbf{A}} \;\vdots\; \underset{n \times 1}{\mathbf{P}}] \qquad \textbf{(2.22)}$$

where n represents the number of equations. The elementary operations are then applied to the rows of the augmented matrix, until the coefficient matrix is reduced to a unit matrix. The elements of the vector, which initially contained the constant terms of the original equations, now represent the solution of the original system of equations; that is,

$$\mathbf{G} = \begin{cases} [\mathbf{A} \;\vdots\; \mathbf{P}] \\ \quad\big\downarrow \text{ elementary operations} \\ [\mathbf{I} \;\vdots\; \mathbf{x}] \end{cases} \qquad \textbf{(2.23)}$$

This procedure is illustrated by the following example.

EXAMPLE 2.11 Solve the system of simultaneous equations given in Eq. 2.21(a) by the Gauss–Jordan method.

SOLUTION The given system of equations can be written in matrix form as

$$\mathbf{Ax} = \mathbf{P}$$

$$\begin{bmatrix} 5 & 6 & -3 \\ 9 & -1 & 2 \\ 8 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 8 \\ -39 \end{bmatrix} \qquad \textbf{(1)}$$

from which we form the augmented matrix

$$\mathbf{G} = [\mathbf{A} \,\vdots\, \mathbf{P}] = \left[\begin{array}{rrr:r} 5 & 6 & -3 & 66 \\ 9 & -1 & 2 & 8 \\ 8 & -7 & 4 & -39 \end{array}\right] \tag{2}$$

We begin Gauss–Jordan elimination by dividing row 1 of the augmented matrix by $G_{11} = 5$ to obtain

$$\mathbf{G} = \left[\begin{array}{rrr:r} 1 & 1.2 & -0.6 & 13.2 \\ 9 & -1 & 2 & 8 \\ 8 & -7 & 4 & -39 \end{array}\right] \tag{3}$$

Next, we multiply row 1 by $G_{21} = 9$ and subtract it from row 2; then multiply row 1 by $G_{31} = 8$ and subtract it from row 3. This yields

$$\mathbf{G} = \left[\begin{array}{rrr:r} 1 & 1.2 & -0.6 & 13.2 \\ 0 & -11.8 & 7.4 & -110.8 \\ 0 & -16.6 & 8.8 & -144.6 \end{array}\right] \tag{4}$$

We now divide row 2 by $G_{22} = -11.8$ to obtain

$$\mathbf{G} = \left[\begin{array}{rrr:r} 1 & 1.2 & -0.6 & 13.2 \\ 0 & 1 & -0.6271 & 9.39 \\ 0 & -16.6 & 8.8 & -144.6 \end{array}\right] \tag{5}$$

Next, we multiply row 2 by $G_{12} = 1.2$ and subtract it from row 1, and then multiply row 2 by $G_{32} = -16.6$ and subtract it from row 3. Thus,

$$\mathbf{G} = \left[\begin{array}{rrr:r} 1 & 0 & 0.1525 & 1.932 \\ 0 & 1 & -0.6271 & 9.39 \\ 0 & 0 & -1.61 & 11.27 \end{array}\right] \tag{6}$$

By dividing row 3 by $G_{33} = -1.61$, we obtain

$$\mathbf{G} = \left[\begin{array}{rrr:r} 1 & 0 & 0.1525 & 1.932 \\ 0 & 1 & -0.6271 & 9.39 \\ 0 & 0 & 1 & -7 \end{array}\right] \tag{7}$$

Finally, we multiply row 3 by $G_{13} = 0.1525$ and subtract it from row 1; then multiply row 3 by $G_{23} = -0.6271$ and subtract it from row 2 to obtain

$$\mathbf{G} = \left[\begin{array}{rrr:r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array}\right] \tag{8}$$

Thus, the solution of the given system of equations is

$$\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \qquad \textbf{Ans}$$

To check our solution, we substitute the numerical value of **x** back into Eq. (1). This yields

$$\begin{bmatrix} 5 & 6 & -3 \\ 9 & -1 & 2 \\ 8 & -7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 15 + 30 + 21 = 66 \\ 27 - 5 - 14 = 8 \\ 24 - 35 - 28 = -39 \end{bmatrix} \qquad \textbf{Checks}$$

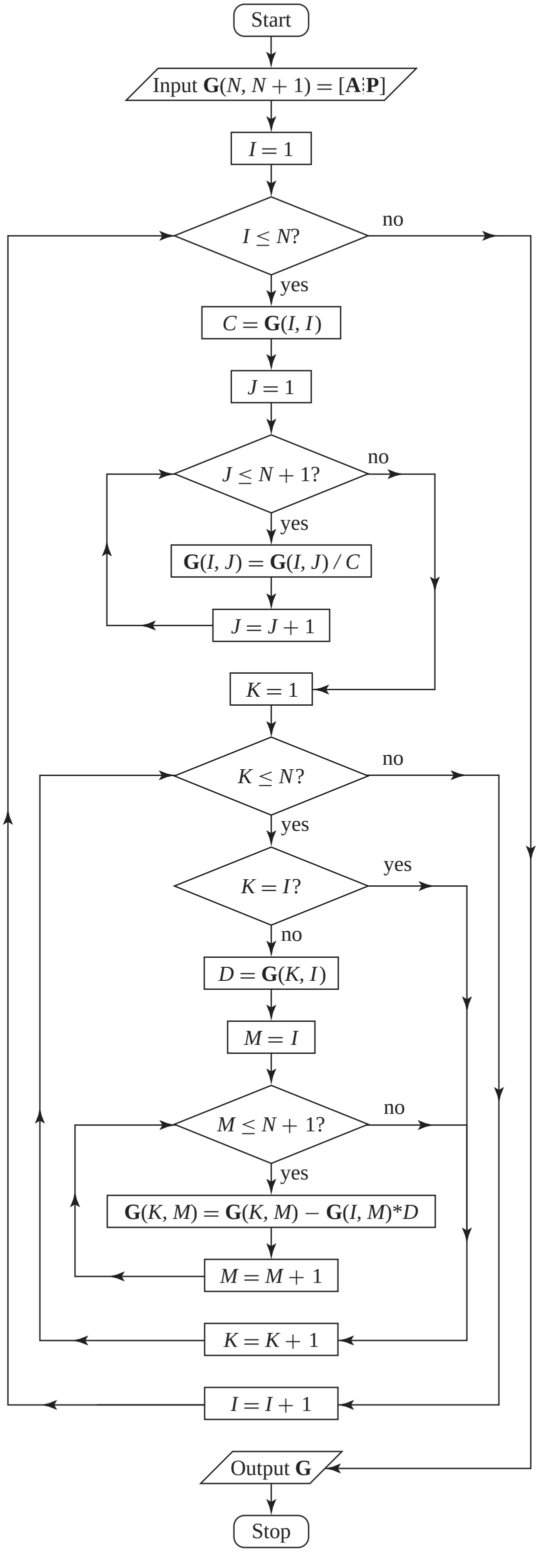

Fig. 2.2 *Flowchart for Solution of Simultaneous Equations by Gauss–Jordan Method*

The solution of large systems of simultaneous equations by the Gauss–Jordan method is usually carried out by computer, and a flowchart for programming this procedure is given in Fig. 2.2. The reader should write this program in a general form (e.g., as a subroutine), so that it can be conveniently included in the structural analysis computer programs to be developed in later chapters.

It should be noted that the Gauss–Jordan method as described in the preceding paragraphs breaks down if a diagonal element of the coefficient matrix **A** becomes zero during the elimination process. This situation can be remedied by interchanging the row of the augmented matrix containing the zero diagonal element with another row, to place a nonzero element on the diagonal; the elimination process is then continued. However, when solving the systems of equations encountered in structural analysis, the condition of a zero diagonal element should not arise; the occurrence of such a condition would indicate that the structure being analyzed is unstable [2]*.

Matrix Inversion

The procedure for determining inverses of matrices by the Gauss–Jordan method is similar to that described previously for solving simultaneous equations. The procedure involves forming an augmented matrix **G** composed of the matrix **A** that is to be inverted and a unit matrix **I** of the same order as **A**; that is,

$$\underset{n \times 2n}{\mathbf{G}} = [\underset{n \times n}{\mathbf{A}} \;\vdots\; \underset{n \times n}{\mathbf{I}}] \tag{2.24}$$

Elementary operations are then applied to the rows of the augmented matrix to reduce **A** to a unit matrix. Matrix **I**, which was initially the unit matrix, now represents the inverse matrix $\mathbf{A}^{-1}$; thus,

$$\mathbf{G} = \left\{ \begin{array}{l} [\mathbf{A} \;\vdots\; \mathbf{I}] \\ \quad\big\downarrow \text{ elementary operations} \\ [\mathbf{I} \;\vdots\; \mathbf{A}^{-1}] \end{array} \right. \tag{2.25}$$

EXAMPLE 2.12 Determine the inverse of the matrix shown using the Gauss–Jordan method.

$$\mathbf{A} = \begin{bmatrix} 13 & -6 & 6 \\ -6 & 12 & -1 \\ 6 & -1 & 9 \end{bmatrix}$$

SOLUTION The augmented matrix is given by

$$\mathbf{G} = [\mathbf{A} \;\vdots\; \mathbf{I}] = \left[\begin{array}{ccc:ccc} 13 & -6 & 6 & 1 & 0 & 0 \\ -6 & 12 & -1 & 0 & 1 & 0 \\ 6 & -1 & 9 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

*Numbers in brackets refer to items listed in the bibliography.

We begin the Gauss–Jordan elimination process by dividing row 1 of the augmented matrix by $G_{11} = 13$:

$$\mathbf{G} = \left[\begin{array}{ccc|ccc} 1 & -0.4615 & 0.4615 & 0.07692 & 0 & 0 \\ -6 & 12 & -1 & 0 & 1 & 0 \\ 6 & -1 & 9 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Next, we multiply row 1 by $G_{21} = -6$ and subtract it from row 2, and then multiply row 1 by $G_{31} = 6$ and subtract it from row 3. This yields

$$\mathbf{G} = \left[\begin{array}{ccc|ccc} 1 & -0.4615 & 0.4615 & 0.07692 & 0 & 0 \\ 0 & 9.231 & 1.769 & 0.4615 & 1 & 0 \\ 0 & 1.769 & 6.231 & -0.4615 & 0 & 1 \end{array}\right] \tag{3}$$

Dividing row 2 by $G_{22} = 9.231$, we obtain

$$\mathbf{G} = \left[\begin{array}{ccc|ccc} 1 & -0.4615 & 0.4615 & 0.07692 & 0 & 0 \\ 0 & 1 & 0.1916 & 0.04999 & 0.1083 & 0 \\ 0 & 1.769 & 6.231 & -0.4615 & 0 & 1 \end{array}\right] \tag{4}$$

Next, we multiply row 2 by $G_{12} = -0.4615$ and subtract it from row 1; then multiply row 2 by $G_{32} = 1.769$ and subtract it from row 3. This yields

$$\mathbf{G} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0.5499 & 0.09999 & 0.04998 & 0 \\ 0 & 1 & 0.1916 & 0.04999 & 0.1083 & 0 \\ 0 & 0 & 5.892 & -0.5499 & -0.1916 & 1 \end{array}\right] \tag{5}$$

Divide row 3 by $G_{33} = 5.892$:

$$\mathbf{G} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0.5499 & 0.09999 & 0.04998 & 0 \\ 0 & 1 & 0.1916 & 0.04999 & 0.1083 & 0 \\ 0 & 0 & 1 & -0.09333 & -0.03252 & 0.1697 \end{array}\right] \tag{6}$$

Multiply row 3 by $G_{13} = 0.5499$ and subtract it from row 1; then multiply row 3 by $G_{23} = 0.1916$ and subtract it from row 2 to obtain

$$\mathbf{G} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.1513 & 0.06787 & -0.09333 \\ 0 & 1 & 0 & 0.06787 & 0.1145 & -0.03252 \\ 0 & 0 & 1 & -0.09333 & -0.03252 & 0.1697 \end{array}\right] \tag{7}$$

Thus, the inverse of the given matrix $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.1513 & 0.06787 & -0.09333 \\ 0.06787 & 0.1145 & -0.03252 \\ -0.09333 & -0.03252 & 0.1697 \end{bmatrix}$$

Ans

Finally, we check our computations by using the relationship $\mathbf{AA}^{-1} = \mathbf{I}$:

$$\mathbf{AA}^{-1} = \begin{bmatrix} 13 & -6 & 6 \\ -6 & 12 & -1 \\ 6 & -1 & 9 \end{bmatrix} \begin{bmatrix} 0.1513 & 0.06787 & -0.09333 \\ 0.06787 & 0.1145 & -0.03252 \\ -0.09333 & -0.03252 & 0.1697 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9997 & 0.0002 & 0 \\ 0 & 0.9993 & 0 \\ 0 & 0 & 0.9998 \end{bmatrix} \approx \mathbf{I}$$

Checks

SUMMARY

In this chapter, we discussed the basic concepts of matrix algebra that are necessary for formulating the matrix methods of structural analysis:

1. A matrix is defined as a rectangular array of quantities (elements) arranged in rows and columns. The size of a matrix is measured by its number of rows and columns, and is referred to as its order.
2. Two matrices are considered to be equal if they are of the same order, and if their corresponding elements are identical.
3. Two matrices of the same order can be added (or subtracted) by adding (or subtracting) their corresponding elements.
4. The matrix multiplication $\mathbf{AB} = \mathbf{C}$ is defined only if the number of columns of the first matrix $\mathbf{A}$ equals the number of rows of the second matrix $\mathbf{B}$. Any element C_{ij} of the product matrix $\mathbf{C}$ can be evaluated by using the relationship

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} \tag{2.6}$$

   where m is the number of columns of $\mathbf{A}$, or the number of rows of $\mathbf{B}$. Matrix multiplication is generally not commutative; that is, $\mathbf{AB} \neq \mathbf{BA}$.
5. The transpose of a matrix is obtained by interchanging its corresponding rows and columns. If $\mathbf{C}$ is a symmetric matrix, then $\mathbf{C}^T = \mathbf{C}$. Another useful property of matrix transposition is that

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{2.15}$$

6. A matrix can be differentiated (or integrated) by differentiating (or integrating) each of its elements.
7. The inverse of a square matrix $\mathbf{A}$ is defined as a matrix $\mathbf{A}^{-1}$ which satisfies the relationship:

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{2.17}$$

8. If the inverse of a matrix equals its transpose, the matrix is called an orthogonal matrix.
9. The Gauss–Jordan method of solving simultaneous equations essentially involves successively eliminating each unknown from all but one of the equations of the system by performing the following operations: dividing an equation by a scalar; and multiplying an equation by a scalar and subtracting the resulting equation from another equation. These elementary operations are applied repeatedly until a system with each equation containing only one unknown is obtained.

PROBLEMS

Section 2.3

2.1 Determine the matrices $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 8 & -1 \\ 8 & -7 & -4 \\ -1 & -4 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 9 & -2 \\ -9 & 6 & 3 \\ 2 & -3 & -4 \end{bmatrix}$$

2.2 Determine the matrices $\mathbf{C} = 2\mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - 3\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 8 & -6 & -3 \\ 1 & -2 & 0 \\ -6 & 5 & -1 \\ -2 & 8 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 & -3 \\ -4 & 3 & 0 \\ 2 & -8 & 6 \\ -1 & 4 & -7 \end{bmatrix}$$

2.3 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = [4 \quad -6 \quad 2] \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$$

2.4 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 6 \\ -7 & -5 \\ 1 & -9 \\ -3 & 11 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 3 & -5 & 2 \\ -13 & -4 & 7 & 6 \end{bmatrix}$$

2.5 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & -6 & 1 \\ -6 & 5 & 7 \\ 1 & 7 & 8 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 5 & 0 \\ 5 & 7 & -2 \\ 0 & -2 & 9 \end{bmatrix}$$

2.6 Determine the products $\mathbf{C} = \mathbf{AB}$ if

$$\mathbf{A} = \begin{bmatrix} 12 & -11 & 10 \\ 0 & 2 & -4 \\ -7 & 9 & 8 \\ 6 & 15 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 13 & -1 & 5 \\ 16 & -9 & 0 \\ -3 & 20 & -7 \end{bmatrix}$$

2.7 Develop a computer program to determine the matrix product $\mathbf{C} = \mathbf{AB}$ of two conformable matrices $\mathbf{A}$ and $\mathbf{B}$ of any order. Check the program by solving Problems 2.4–2.6 and comparing the computer-generated results to those determined by hand calculations.

2.8 Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ by using the following matrices

$$\mathbf{A} = \begin{bmatrix} 21 & 10 & 16 \\ -15 & 11 & 0 \\ 13 & 20 & -9 \\ 7 & -17 & 14 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 7 & -4 \\ -1 & 9 \\ 3 & -6 \end{bmatrix}$$

2.9 Show that $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$ by using the following matrices

$$\mathbf{A} = \begin{bmatrix} -9 & 0 \\ 13 & 20 \\ 8 & -3 \\ -11 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 15 & -1 & -4 \\ 6 & 16 & 9 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -7 & 10 & 6 & 0 \\ -1 & 2 & -8 & -2 \\ 16 & 12 & 2 & 8 \end{bmatrix}$$

2.10 Determine the matrix triple product $\mathbf{C} = \mathbf{B}^T\mathbf{AB}$ if

$$\mathbf{A} = \begin{bmatrix} 40 & -10 & -25 \\ -10 & 15 & 12 \\ -25 & 12 & 30 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 7 & -3 \\ -7 & 8 & 4 \\ 3 & -4 & 9 \end{bmatrix}$$

2.11 Determine the matrix triple product $\mathbf{C} = \mathbf{B}^T\mathbf{AB}$ if

$$\mathbf{A} = \begin{bmatrix} 300 & -100 \\ -100 & 200 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.6 & 0.8 & -0.6 & -0.8 \\ -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix}$$

2.12 Develop a computer program to determine the matrix triple product $\mathbf{C} = \mathbf{B}^T\mathbf{AB}$, where $\mathbf{A}$ is a square matrix of any order. Check the program by solving Problems 2.10 and 2.11 and comparing the results to those determined by hand calculations.

2.13 Determine the derivative $d\mathbf{A}/dx$ if

$$\mathbf{A} = \begin{bmatrix} -2x^2 & 3\sin x & -7x \\ 3\sin x & \cos^2 x & -3x^3 \\ -7x & -3x^3 & 3\sin^2 x \end{bmatrix}$$

2.14 Determine the derivative $d(\mathbf{A} + \mathbf{B})/dx$ if

$$\mathbf{A} = \begin{bmatrix} -3x & 5 \\ 4x^2 & -x^3 \\ -7 & 5x \\ 2x^3 & -x^2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2x^2 & -x \\ -12x & 8 \\ 2x^3 & -3x^2 \\ -1 & 6x \end{bmatrix}$$

2.15 Determine the derivative $d(\mathbf{AB})/dx$ if

$$\mathbf{A} = \begin{bmatrix} 4x & 2 & -5x^2 \\ 2 & -3x^3 & -x \\ -5x^2 & -x & 7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -5x^3 & -x \\ 6 & -3x^2 \\ 2x^2 & 4x \end{bmatrix}$$

2.16 Determine the partial derivatives $\partial \mathbf{A}/\partial x$, $\partial \mathbf{A}/\partial y$, and $\partial \mathbf{A}/\partial z$, if

$$\mathbf{A} = \begin{bmatrix} x^2 & -y^2 & 2z^2 \\ -y^2 & 3xy & -yz \\ 2z^2 & -yz & 4xz \end{bmatrix}$$

2.17 Calculate the integral $\int_0^L \mathbf{A}\, dx$ if

$$\mathbf{A} = \begin{bmatrix} -5 & -3x^2 \\ 4x & -x^3 \\ 2x^4 & 6 \\ 5x^2 & -x \end{bmatrix}$$

2.18 Calculate the integral $\int_0^L \mathbf{A}\, dx$ if

$$\mathbf{A} = \begin{bmatrix} 2x & -\sin x & 2\cos^2 x \\ -\sin x & 5 & -4x^3 \\ 2\cos^2 x & -4x^3 & (1-x^2) \end{bmatrix}$$

2.19 Calculate the integral $\int_0^L \mathbf{AB}\, dx$ if

$$\mathbf{A} = \begin{bmatrix} -x^3 & 2x^2 & 3 \\ 2x & -x^2 & 2x^3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -2x & x^2 \\ 5 & -2x \\ 3x^3 & -3 \end{bmatrix}$$

2.20 Determine whether the matrices **A** and **B** given below are orthogonal matrices.

$$\mathbf{A} = \begin{bmatrix} -0.28 & -0.96 & 0 & 0 \\ 0.96 & -0.28 & 0 & 0 \\ 0 & 0 & -0.28 & -0.96 \\ 0 & 0 & 0.96 & -0.28 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.28 & 0.96 & 0 & 0 \\ 0.96 & -0.28 & 0 & 0 \\ 0 & 0 & -0.28 & 0.96 \\ 0 & 0 & 0.96 & -0.28 \end{bmatrix}$$

Section 2.4

2.21 through 2.25 Solve the following systems of simultaneous equations by the Gauss–Jordan method.

2.21
$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= -18 \\ -9x_1 + 5x_2 + 3x_3 &= 18 \\ 4x_1 + 7x_2 - 8x_3 &= 53 \end{aligned}$$

2.22
$$\begin{aligned} 20x_1 - 9x_2 + 15x_3 &= 354 \\ -9x_1 + 16x_2 - 5x_3 &= -275 \\ 15x_1 - 5x_2 + 18x_3 &= 307 \end{aligned}$$

2.23
$$\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 37.2 \\ 3x_1 + 5x_2 - x_3 &= -7.2 \\ x_1 - 4x_2 + 2x_3 &= 30.3 \end{aligned}$$

2.24
$$\begin{aligned} 6x_1 + 15x_2 - 24x_3 + 40x_4 &= 190.9 \\ 15x_1 + 9x_2 - 13x_3 &= 69.8 \\ -24x_1 - 13x_2 + 8x_3 - 11x_4 &= -96.3 \\ 40x_1 - 11x_3 + 5x_4 &= 119.35 \end{aligned}$$

2.25
$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 + 11x_4 &= 39 \\ 10x_1 + 7x_2 + 4x_3 - x_4 &= 127 \\ -3x_1 + 9x_2 + 5x_3 - 6x_4 &= 58 \\ x_1 - 4x_2 - 2x_3 + 9x_4 &= -14 \end{aligned}$$

2.26 Develop a computer program to solve a system of simultaneous equations of any size by the Gauss–Jordan method. Check the program by solving Problems 2.21 through 2.25 and comparing the computer-generated results to those determined by hand calculations.

2.27 through 2.30 Determine the inverse of the matrices shown by the Gauss–Jordan method.

2.27 $\mathbf{A} = \begin{bmatrix} 5 & 3 & -4 \\ 3 & 8 & -2 \\ -4 & -2 & 7 \end{bmatrix}$

2.28 $\mathbf{A} = \begin{bmatrix} 6 & -4 & 1 \\ -1 & 9 & 3 \\ 4 & 2 & 5 \end{bmatrix}$

2.29 $\mathbf{A} = \begin{bmatrix} 7 & -6 & 3 & -2 \\ -6 & 4 & -1 & 5 \\ 3 & -1 & 8 & 9 \\ -2 & 5 & 9 & 2 \end{bmatrix}$

2.30 $\mathbf{A} = \begin{bmatrix} 5 & -7 & -3 & 11 \\ 10 & -6 & -13 & 2 \\ -1 & 12 & 8 & -4 \\ -9 & 7 & -5 & 6 \end{bmatrix}$