7 MEMBER RELEASES AND SECONDARY EFFECTS

Earthquake-Damaged Bridge
(Courtesy of the USGS)

The matrix stiffness analysis of beams and plane frames, as developed in Chapters 5 and 6, is based on the assumption that each member of a structure is rigidly connected to joints at both ends, so that the member end rotations are equal to the rotations of the adjacent joints. Whereas these methods of analysis, as presented in preceding chapters, cannot be used to analyze beams and plane frames containing members connected by hinged connections, they can be modified relatively easily to include the effects of member hinges in the analysis. When the end of a member in a plane frame or beam is connected to the adjacent joint by a hinged connection, the moment at the hinged end must be zero. Because of this moment-releasing characteristic, member hinges are often referred to as *member releases*. In this chapter, we discuss modifications of the matrix stiffness methods that allow them to be used to analyze plane frames and beams containing members connected to joints by rigid (i.e., moment-resisting) and/or hinged (i.e., simple or shear) connections.

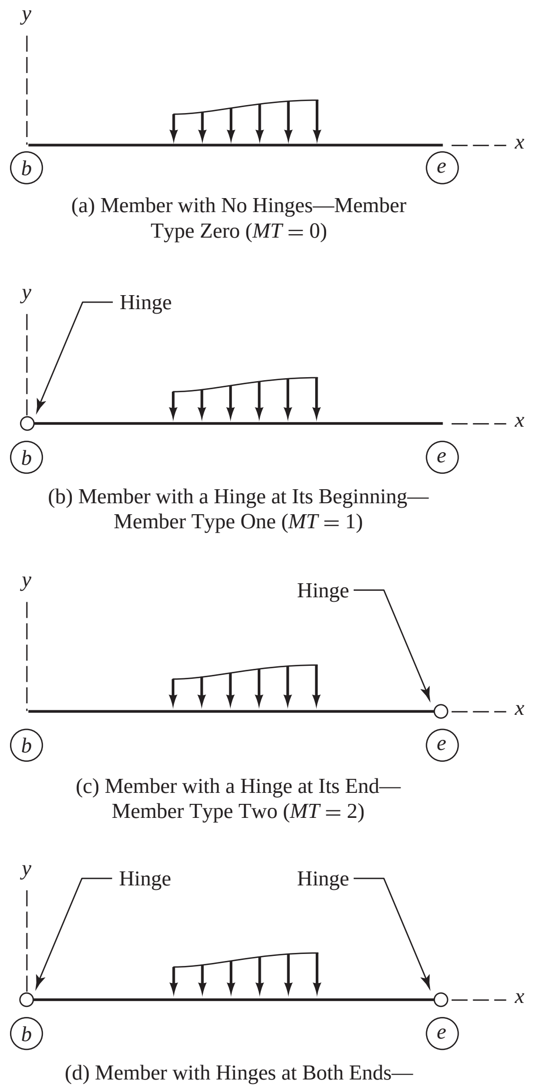
In this chapter, we also consider the procedures for including in matrix stiffness methods of analysis the effects of support displacements (due to weak foundations), temperature changes, and fabrication errors. Such secondary effects can induce significant stresses in statically indeterminate structures, and must be considered in their designs.

We begin the chapter by deriving the stiffness relationships for members of plane frames and beams with hinges. A procedure for the analysis of structures containing member releases is also developed in Section 7.1; the computer implementation of this procedure is presented in Section 7.2. We develop the analysis for the effects of support displacements in Section 7.3, and discuss the extension of the previously developed computer programs to include the effects of support displacements in Section 7.4. Finally, the procedure for including in the analysis the effects of temperature changes and fabrication errors is presented in Section 7.5.

7.1 MEMBER RELEASES IN PLANE FRAMES AND BEAMS

The effects of member releases can be conveniently incorporated in our stiffness methods by modifying the member local stiffness relationships to account for such releases. Only moment releases, in the form of hinges located at one or both ends of a member (see Fig. 7.1), are considered herein, because such releases are by far the most commonly encountered in civil engineering practice. However, the concepts presented can be readily used to introduce the effects of other types of member releases (e.g., shear and axial force releases) into the analysis.

Figure 7.1 depicts the types of member releases considered herein. From a computer programming viewpoint, it is usually convenient to classify each member of a beam or a plane frame into one of the four *member types* (MT) shown in the figure. Thus, as indicated in Fig. 7.1(a), a member that is rigidly connected to joints at both ends (i.e., has no hinges), is considered to be of type 0 (i.e., $MT = 0$). If end b of a member is connected to the adjacent joint by a hinged connection, while its opposite end e is rigidly connected to the



(a) Member with No Hinges—Member Type Zero ($MT = 0$)

(b) Member with a Hinge at Its Beginning—Member Type One ($MT = 1$)

(c) Member with a Hinge at Its End—Member Type Two ($MT = 2$)

(d) Member with Hinges at Both Ends—Member Type Three ($MT = 3$)

Fig. 7.1 *Member Releases*

adjacent joint (Fig. 7.1(b)), then the member is classified as type 1 (i.e., $MT = 1$). Conversely, if end b of a member is rigidly attached to the adjacent joint, but its end e is connected by a hinged connection to the adjacent joint (Fig. 7.1(c)), then the member is considered to be of type 2 (i.e., $MT = 2$). Finally, if a member is attached to joints at both ends by hinged connections (Fig. 7.1(d)), then it is classified as type 3 (i.e., $MT = 3$).

The expressions for the member local stiffness matrices $\mathbf{k}$ (Eqs. (5.53) and (6.6)) and the member local fixed-end force vectors $\mathbf{Q}_f$ (Eqs. (5.99) and (6.15)) derived for beams and plane frames can be used only for members of type 0

($MT = 0$), because they are based on the condition that the member is rigidly connected to joints at both ends, so that the member end rotations are equal to the rotations of the adjacent joints. When an end of a member is connected to the adjacent joint by a hinged connection, the moment at the hinged end must be zero. The previous expressions for $\mathbf{k}$ and $\mathbf{Q}_f$ can be easily modified to reflect the conditions of zero moments at the hinged member ends, as explained in the following paragraphs.

Local Stiffness Relations for Plane Frame Members with Hinges

We begin the development of the modified expressions by first writing the previously derived stiffness relations for a plane frame member with no hinges, in explicit form. By substituting the expressions for $\mathbf{k}$ and $\mathbf{Q}_f$ from Eqs. (6.6) and (6.15), respectively, into the member local stiffness relation $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$ (Eq. (6.4)), and carrying out the necessary matrix multiplication and addition, we obtain

$$Q_1 = \frac{EA}{L}(u_1 - u_4) + FA_b \tag{7.1a}$$

$$Q_2 = \frac{EI}{L^3}(12u_2 + 6Lu_3 - 12u_5 + 6Lu_6) + FS_b \tag{7.1b}$$

$$Q_3 = \frac{EI}{L^3}(6Lu_2 + 4L^2u_3 - 6Lu_5 + 2L^2u_6) + FM_b \tag{7.1c}$$

$$Q_4 = \frac{EA}{L}(-u_1 + u_4) + FA_e \tag{7.1d}$$

$$Q_5 = \frac{EI}{L^3}(-12u_2 - 6Lu_3 + 12u_5 - 6Lu_6) + FS_e \tag{7.1e}$$

$$Q_6 = \frac{EI}{L^3}(6Lu_2 + 2L^2u_3 - 6Lu_5 + 4L^2u_6) + FM_e \tag{7.1f}$$

***Members with a Hinge at the Beginning* ($MT = 1$)** When end b of a member is connected to the adjacent joint by a hinged connection, then from Fig. 6.3(b) we can see that its end moment Q_3 must be 0. By substituting $Q_3 = 0$ into Eq. (7.1c), and solving the resulting equation for the end rotation u_3, we obtain

$$u_3 = \frac{3}{2L}(-u_2 + u_5) - \frac{1}{2}u_6 - \frac{L}{4EI}FM_b \tag{7.2}$$

This equation indicates that the rotation u_3 (of the hinged end b of the member) is no longer an independent member coordinate (or degree of freedom), but is now a function of the end displacements u_2, u_5, and u_6. Thus, the number of independent member coordinates—that is, the independent end displacements required to define the displaced member configuration—is now reduced to five (i.e., u_1, u_2, u_4, u_5, and u_6). To eliminate the released coordinate u_3 from the member stiffness relations, we substitute Eq. (7.2) into Eqs. (7.1). This yields the following member stiffness equations:

$$Q_1 = \frac{EA}{L}(u_1 - u_4) + FA_b \tag{7.3a}$$

$$Q_2 = \frac{EI}{L^3}(3u_2 - 3u_5 + 3Lu_6) + \left(FS_b - \frac{3}{2L}FM_b\right) \tag{7.3b}$$

$$Q_3 = 0 \tag{7.3c}$$

$$Q_4 = \frac{EA}{L}(-u_1 + u_4) + FA_e \tag{7.3d}$$

$$Q_5 = \frac{EI}{L^3}(-3u_2 + 3u_5 - 3Lu_6) + \left(FS_e + \frac{3}{2L}FM_b\right) \tag{7.3e}$$

$$Q_6 = \frac{EI}{L^3}(3Lu_2 - 3Lu_5 + 3L^2u_6) + \left(FM_e - \frac{1}{2}FM_b\right) \tag{7.3f}$$

Equations (7.3), which represent the modified local stiffness relations for member type 1 ($MT = 1$), can be expressed in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 3 & 0 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & -3L \\ 0 & 3L & 0 & 0 & -3L & 3L^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} FA_b \\ FS_b - \frac{3}{2L}FM_b \\ 0 \\ FA_e \\ FS_e + \frac{3}{2L}FM_b \\ FM_e - \frac{1}{2}FM_b \end{bmatrix} \tag{7.4}$$

or, symbolically, as

$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

with

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 3 & 0 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & -3L \\ 0 & 3L & 0 & 0 & -3L & 3L^2 \end{bmatrix} \tag{7.5}$$

and

$$\mathbf{Q}_f = \begin{bmatrix} FA_b \\ FS_b - \dfrac{3}{2L}FM_b \\ 0 \\ FA_e \\ FS_e + \dfrac{3}{2L}FM_b \\ FM_e - \dfrac{1}{2}FM_b \end{bmatrix} \tag{7.6}$$

The **k** matrix in Eq. (7.5) and the $\mathbf{Q}_f$ vector in Eq. (7.6) now represent the modified local stiffness matrix and the modified local fixed-end force vector, respectively, for plane frame members of type 1 ($MT = 1$).

***Members with a Hinge at the End* (*MT* = 2)** When end e of a member is hinged, then its end moment Q_6 (Fig. 6.3(b)) must be 0. By substituting $Q_6 = 0$ into Eq. 7.1(f), and solving the resulting equation for the end rotation u_6, we obtain

$$u_6 = \frac{3}{2L}(-u_2 + u_5) - \frac{1}{2}u_3 - \frac{L}{4EI}FM_e \tag{7.7}$$

Next, we substitute Eq. (7.7) into Eqs. (7.1) to eliminate u_6 from the member stiffness relations. This yields

$$Q_1 = \frac{EA}{L}(u_1 - u_4) + FA_b \tag{7.8a}$$

$$Q_2 = \frac{EI}{L^3}(3u_2 + 3Lu_3 - 3u_5) + \left(FS_b - \frac{3}{2L}FM_e\right) \tag{7.8b}$$

$$Q_3 = \frac{EI}{L^3}(3Lu_2 + 3L^2u_3 - 3Lu_5) + \left(FM_b - \frac{1}{2}FM_e\right) \tag{7.8c}$$

$$Q_4 = \frac{EA}{L}(-u_1 + u_4) + FA_e \tag{7.8d}$$

$$Q_5 = \frac{EI}{L^3}(-3u_2 - 3Lu_3 + 3u_5) + \left(FS_e + \frac{3}{2L}FM_e\right) \tag{7.8e}$$

$$Q_6 = 0 \tag{7.8f}$$

The foregoing equations can be expressed in matrix form as

$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

with

$$
\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix}
\frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\
0 & 3 & 3L & 0 & -3 & 0 \\
0 & 3L & 3L^2 & 0 & -3L & 0 \\
-\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\
0 & -3 & -3L & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{7.9}
$$

and

$$
\mathbf{Q}_f = \begin{bmatrix}
FA_b \\
FS_b - \frac{3}{2L}FM_e \\
FM_b - \frac{1}{2}FM_e \\
FA_e \\
FS_e + \frac{3}{2L}FM_e \\
0
\end{bmatrix} \tag{7.10}
$$

The $\mathbf{k}$ matrix as given in Eq. (7.9) and the $\mathbf{Q}_f$ vector in Eq. (7.10) represent the modified local stiffness matrix and fixed-end force vector, respectively, for plane frame members of type 2 ($MT = 2$).

***Members with Hinges at Both Ends* (*MT* = 3)** If both ends of a member are hinged, then both of its end moments, Q_3 and Q_6, must be 0. Thus, by substituting $Q_3 = 0$ and $Q_6 = 0$ into Eqs. 7.1(c) and (f), respectively, and solving the resulting simultaneous equations for the end rotations u_3 and u_6, we obtain

$$
u_3 = \frac{1}{L}(-u_2 + u_5) - \frac{L}{6EI}(2FM_b - FM_e) \tag{7.11a}
$$

$$
u_6 = \frac{1}{L}(-u_2 + u_5) - \frac{L}{6EI}(2FM_e - FM_b) \tag{7.11b}
$$

Next, we substitute the foregoing equations into Eqs. (7.1) to obtain the local stiffness relations for the member type 3:

$$
Q_1 = \frac{EA}{L}(u_1 - u_4) + FA_b \tag{7.12a}
$$

$$
Q_2 = FS_b - \frac{1}{L}(FM_b + FM_e) \tag{7.12b}
$$

$$Q_3 = 0 \tag{7.12c}$$

$$Q_4 = \frac{EA}{L}(-u_1 + u_4) + FA_e \tag{7.12d}$$

$$Q_5 = FS_e + \frac{1}{L}(FM_b + FM_e) \tag{7.12e}$$

$$Q_6 = 0 \tag{7.12f}$$

Equations (7.12) can be expressed in matrix form as

$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

with

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.13}$$

and

$$\mathbf{Q}_f = \begin{bmatrix} FA_b \\ FS_b - \frac{1}{L}(FM_b + FM_e) \\ 0 \\ FA_e \\ FS_e + \frac{1}{L}(FM_b + FM_e) \\ 0 \end{bmatrix} \tag{7.14}$$

The foregoing $\mathbf{k}$ matrix (Eq. (7.13)) and $\mathbf{Q}_f$ vector (Eq. (7.14)) represent the modified local stiffness matrix and fixed-end force vector, respectively, for plane frame members of type 3 ($MT = 3$). Interestingly, from Eq. (7.13), we observe that the deletion of the third and sixth rows and columns (which correspond to the rotational coordinates of the plane frame members) from the $\mathbf{k}$ matrix for $MT = 3$ reduces it to the $\mathbf{k}$ matrix for members of plane trusses (Eq. (3.27)).

It should be realized that, although the number of independent member coordinates is reduced due to member releases, the orders of the modified stiffness matrices $\mathbf{k}$ (Eqs. (7.5), (7.9), and (7.13)) and the fixed-end force vector $\mathbf{Q}_f$ (Eqs. (7.6), (7.10), and (7.14)) are maintained as 6×6 and 6×1, respectively, with 0 elements in the rows and columns that correspond to the released coordinates. This form of $\mathbf{k}$ and $\mathbf{Q}_f$ eliminates the need to modify the expression for the member transformation matrix, $\mathbf{T}$, derived in Chapter 6 (Eq. (6.19)), and provides an efficient means of incorporating the effect of member releases in the computer program developed in Section 6.7.

Local Stiffness Relations for Beam Members with Hinges

As discussed in Chapter 5, in beams subjected to lateral loads, the axial displacements of members are 0. Thus, a member of a beam can have up to four degrees of freedom: namely, a translation perpendicular to the member's centroidal axis and a rotation, at each end. The modified stiffness relations for beam members with releases can be derived by applying the same procedure just used for members of plane frames. However, it is more convenient to obtain the modified member stiffness matrices **k** for beams by simply deleting the first and fourth rows and columns of the corresponding **k** matrices for plane-frame members. Similarly, the modified fixed-end force vectors $\mathbf{Q}_f$ for beam members can be obtained by deleting the first and fourth rows of the corresponding $\mathbf{Q}_f$ vectors for plane-frame members.

***Members with a Hinge at the Beginning* (*MT* = 1)** To obtain the modified stiffness matrix **k** for beam members of type 1, we delete rows 1 and 4 and columns 1 and 4 from the **k** matrix given in Eq. (7.5) for plane-frame members of the same type. This yields

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 3 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & -3L \\ 3L & 0 & -3L & 3L^2 \end{bmatrix} \tag{7.15}$$

Similarly, the modified fixed-end force vector $\mathbf{Q}_f$ for beam members of type 1 can be obtained by deleting rows 1 and 4 from the $\mathbf{Q}_f$ vector given in Eq. (7.6) for plane-frame members of type 1. Thus,

$$\mathbf{Q}_f = \begin{bmatrix} FS_b - \dfrac{3}{2L}FM_b \\ 0 \\ FS_e + \dfrac{3}{2L}FM_b \\ FM_e - \dfrac{1}{2}FM_b \end{bmatrix} \tag{7.16}$$

The rotation u_2 of the hinged end b of the member, if desired, can be evaluated by using the following relationship:

$$u_2 = \frac{3}{2L}(-u_1 + u_3) - \frac{1}{2}u_4 - \frac{L}{4EI}FM_b \tag{7.17}$$

Equation (7.17) is obtained simply by replacing u_2, u_3, u_5, and u_6 in Eq. (7.2) with u_1, u_2, u_3, and u_4, respectively.

***Members with a Hinge at the End* (*MT* = 2)** By deleting the first and fourth rows and columns from the **k** matrix given in Eq. (7.9), we obtain the modified stiffness matrix for the beam members of type 2 (i.e., $MT = 2$):

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 3 & 3L & -3 & 0 \\ 3L & 3L^2 & -3L & 0 \\ -3 & -3L & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.18}$$

and by deleting the first and fourth rows from the $\mathbf{Q}_f$ vector given in Eq. (7.10), we determine the modified fixed-end force vector for beam members of type 2 ($MT = 2$):

$$\mathbf{Q}_f = \begin{bmatrix} FS_b - \dfrac{3}{2L}FM_e \\ FM_b - \dfrac{1}{2}FM_e \\ FS_e + \dfrac{3}{2L}FM_e \\ 0 \end{bmatrix} \tag{7.19}$$

The expression for the rotation u_4, of the hinged end e of the member, can be obtained by substituting the subscripts 1, 2, 3, and 4 for the subscripts 2, 3, 5, and 6, respectively, in Eq. (7.7). This yields

$$u_4 = \frac{3}{2L}(-u_1 + u_3) - \frac{1}{2}u_2 - \frac{L}{4EI}FM_e \tag{7.20}$$

Members with Hinges at Both Ends (MT = 3) By deleting the first and fourth rows and columns from the $\mathbf{k}$ matrix given in Eq. (7.13), we realize that the modified stiffness matrix for beam members of type 3 (i.e., $MT = 3$) is a null matrix; that is,

$$\mathbf{k} = \mathbf{0} \tag{7.21}$$

which indicates that a beam member hinged at both ends offers no resistance against small end displacements in the direction perpendicular to its centroidal axis. (Recall from Section 3.3 that the members of trusses behave in a similar manner when subjected to lateral end displacements—see Figs. 3.3(d) and (f).) By deleting the first and fourth rows from the $\mathbf{Q}_f$ vector given in Eq. (7.14), we obtain the modified fixed-end force vector for beam members of type 3 (i.e., $MT = 3$):

$$\mathbf{Q}_f = \begin{bmatrix} FS_b - \dfrac{1}{L}(FM_b + FM_e) \\ 0 \\ FS_e + \dfrac{1}{L}(FM_b + FM_e) \\ 0 \end{bmatrix} \tag{7.22}$$

and from Eqs. (7.11) we obtain the following expressions for the rotations u_2 and u_4 of the hinged ends b and e, respectively, of the member.

$$u_2 = \frac{1}{L}(-u_1 + u_3) - \frac{L}{6EI}(2FM_b - FM_e) \tag{7.23a}$$

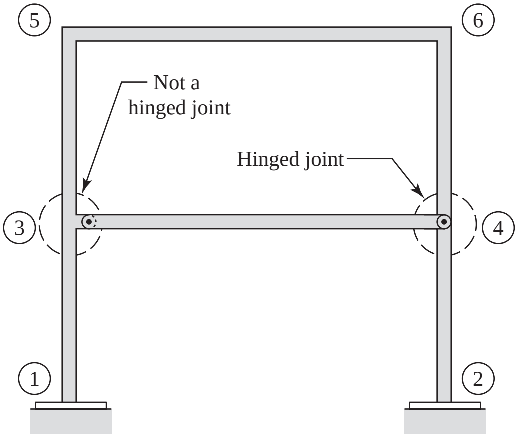
$$u_4 = \frac{1}{L}(-u_1 + u_3) - \frac{L}{6EI}(2FM_e - FM_b) \tag{7.23b}$$

Procedure for Analysis

The analysis procedures developed in Chapters 5 and 6 can be applied to beams and plane frames, respectively, containing member releases, provided that the modified expressions for the stiffness matrices $\mathbf{k}$ and fixed-end force vectors $\mathbf{Q}_f$, developed in this section are used for the members with releases (i.e., $MT = 1$, 2, or 3). Furthermore, in the analysis of plane frames, the global stiffness matrix $\mathbf{K}$ for members with releases is now evaluated using the matrix triple product $\mathbf{K} = \mathbf{T}^T\mathbf{kT}$ (Eq. (6.29)), instead of the explicit form of $\mathbf{K}$ given in Eq. (6.31), which is valid only for members with no releases (i.e., $MT = 0$). Similarly, the global fixed-end force vector $\mathbf{F}_f$ for plane frame members with releases is evaluated using the relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$ (Eq. (6.30)), instead of the explicit form given in Eq. (6.33). The rotations of the hinged member ends, if desired, can be evaluated using Eqs. (7.2), (7.7), and (7.11) when analyzing plane frames, and Eqs. (7.17), (7.20), and (7.23) in the case of beams.

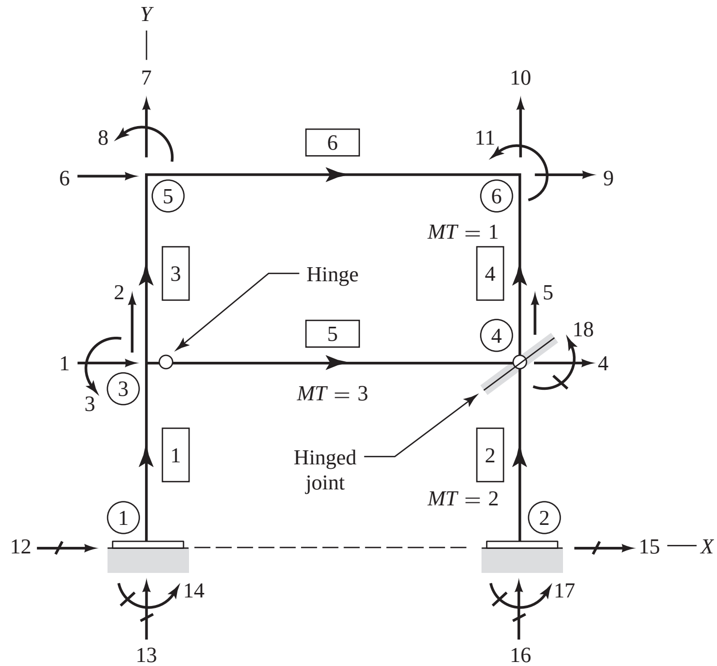
Hinged Joints in Beams and Plane Frames

If all the members meeting at a joint are connected to it by hinged connections, then the joint is considered to be a hinged joint. For example, joint 4 of the two-story plane frame shown in Fig. 7.2(a) is considered to be a hinged joint,

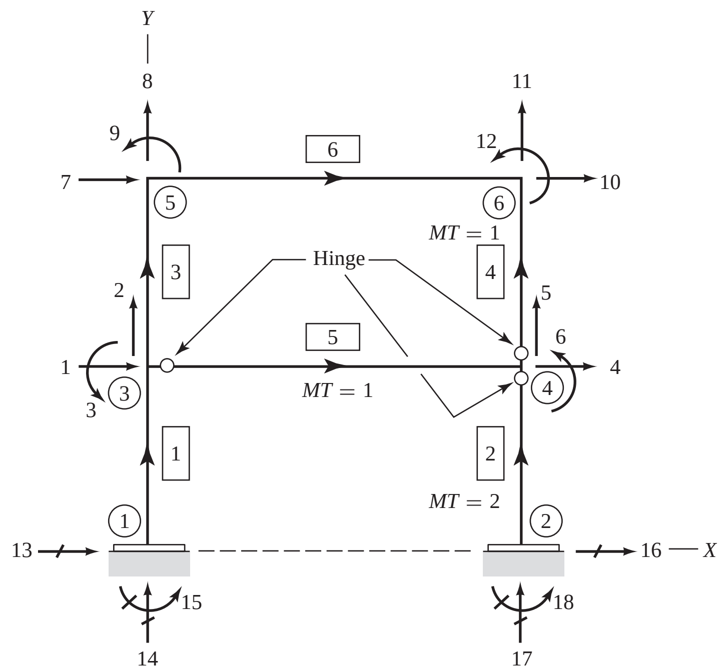


(a) Plane Frame

Fig. 7.2



(b) Analytical Model
(11 Degrees of Freedom)



(c) Alternative Analytical Model
(12 Degrees of Freedom)

Fig. 7.2 (*continued*)

because all three members meeting at this joint are attached to it by hinged connections. However, joint 3 of this frame is not considered to be hinged, because only one of the three members meeting at this joint is attached by a hinged connection; the remaining two members are rigidly connected to the joint.

As hinged joints cannot transmit any moments and are free to rotate, their rotational stiffnesses are 0. Thus, inclusion of the rotational degrees of freedom of such joints in the analysis causes the structure stiffness matrix **S** to become *singular,* with 0 elements in the rows and columns that correspond to the rotational degrees of freedom of the hinged joints. (Recall from your previous course in *college algebra* that the coefficient matrix of a system of linear equations is considered to be singular if its determinant is 0; and that such a system of equations does not yield a unique solution.) Perhaps the most straightforward and efficient way to remedy this difficulty is to eliminate the rotational degrees of freedom of hinged joints from the analysis by modeling such joints as restrained (or fixed) against rotations. This approach is based on the realization that because hinged joints are not subjected to any moments, their rotations are 0; even though the released ends of the members connected to such a joint can, and do, rotate. In Fig. 7.2(b), the hinged joint 4 of the example frame is modeled using this approach. As indicated in this figure, an imaginary clamp is applied to hinged joint 4 to restrain (or fix) it against rotation, while allowing it to freely translate in any direction. Joint 4, therefore, has two degrees of freedom—the translations in the X and Y directions—which are identified as d_4 and d_5, respectively; and one restrained coordinate, R_{18}, which represents the reaction moment that develops at the imaginary clamp. It should be realized that because hinged joints are not subjected to any external moments (or couples), the magnitudes of the imaginary reaction moments at such joints are always 0. However, the assignment of restrained coordinate numbers to these imaginary reactions, in accordance with the previously established scheme for numbering structure coordinates, enables us to include the effect of hinged joints in the computer programs developed in Chapters 5 and 6 without any reprogramming.

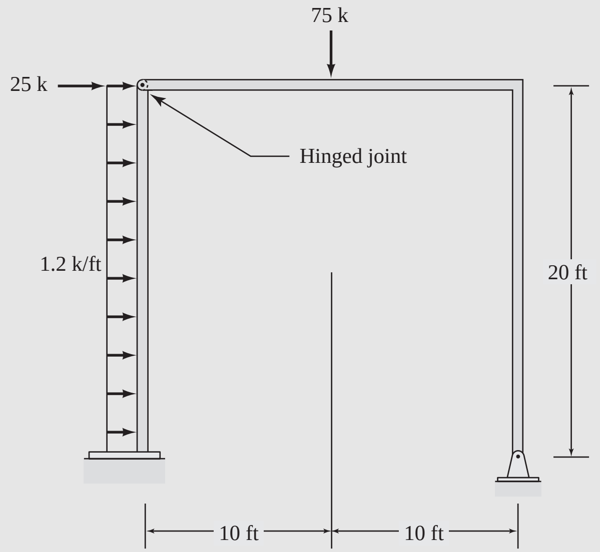
An alternative approach that can be used to overcome the problem of singularity (due to the lack of rotational stiffnesses of a hinged joint) is to model such a joint as rigidly connected to one (and only one) of the members meeting at the joint. This approach is based on the following concept: as no external moment is applied to the hinged joint, and because the moments at the ends of all but one of the members meeting at the joint are 0, the moment at the end of the one member that is rigidly connected to the joint must also be 0, to satisfy the moment equilibrium equation ($\sum M = 0$) for the joint. This alternative approach is used in Fig. 7.2(c) to model hinged joint 4 of the example frame. As shown in this figure, whereas members 2 and 4 are still attached by hinged connections to joint 4, the third member 5 is now rigidly connected to this joint. Note that because the end of member 5 is now rigidly connected, its member type, which was 3 (i.e., $MT = 3$) in the previous analytical model (Fig. 7.2(b)), is now 1 (i.e., $MT = 1$), as shown in Fig. 7.2(c). Joint 4 can now be treated as any other rigid joint of the plane frame, and is assigned three degrees of freedom, d_4, d_5, and d_6, as shown in the figure—with d_6 representing the rotation of joint 4, which in turn equals the rotation of the end of member 5.

Needless to state, the two approaches we have discussed for modeling hinged joints yield identical analysis results. However, the first approach generally provides a more efficient analytical model in terms of the number of degrees of freedom of the structure. From Figs. 7.2(b) and (c), we can see that the analytical models of the example frame, based on the first and the alternative approaches, involve 11 and 12 degrees of freedom, respectively.

EXAMPLE 7.1 Determine the joint displacements, member end forces, and support reactions for the plane frame shown in Fig. 7.3(a), using the matrix stiffness method.

SOLUTION *Analytical Model:* The analytical model of the frame is depicted in Fig. 7.3(b). Since both members 1 and 2, meeting at joint 2, are attached to it by hinged connections, joint 2 is modeled as a hinged joint with its rotation restrained by an imaginary clamp. Thus, joint 2 has two degrees of freedom—the translations in the X and Y directions—which are identified as d_1 and d_2, respectively. Also, for member 1, $MT = 2$, because the end of this member is hinged, whereas $MT = 1$ for member 2, which is hinged at its beginning.

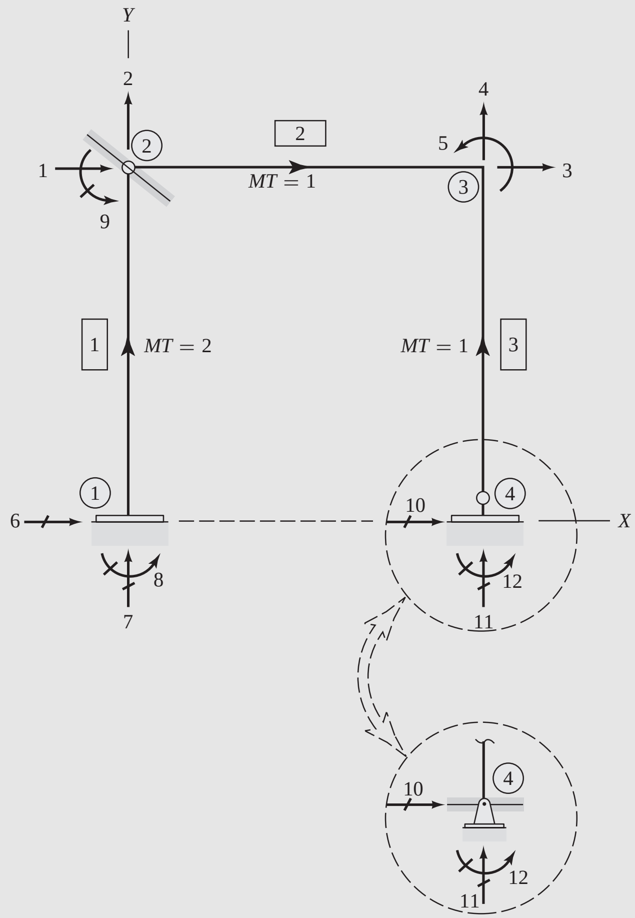
As far as the modeling of joint 4 is concerned, recall that in Chapters 5 and 6 (e.g., see Examples 5.7 and 6.5) we modeled such a joint as a rigid joint, free to rotate, with its rotation treated as a degree of freedom of the structure. However, in light of the discussion of member releases and hinged joints presented in this section, we can now eliminate the rotational degree of freedom of joint 4 from the analysis by modeling member 3 as hinged at its beginning (i.e., $MT = 1$), which allows us to model joint 4 as a hinged joint with its rotation restrained by an imaginary clamp. Note that the end



$E, A, I =$ constant
$E = 29{,}000$ ksi
$A = 14.7$ in.^{2}
$I = 800$ in.^{4}

(a) Frame

Fig. 7.3



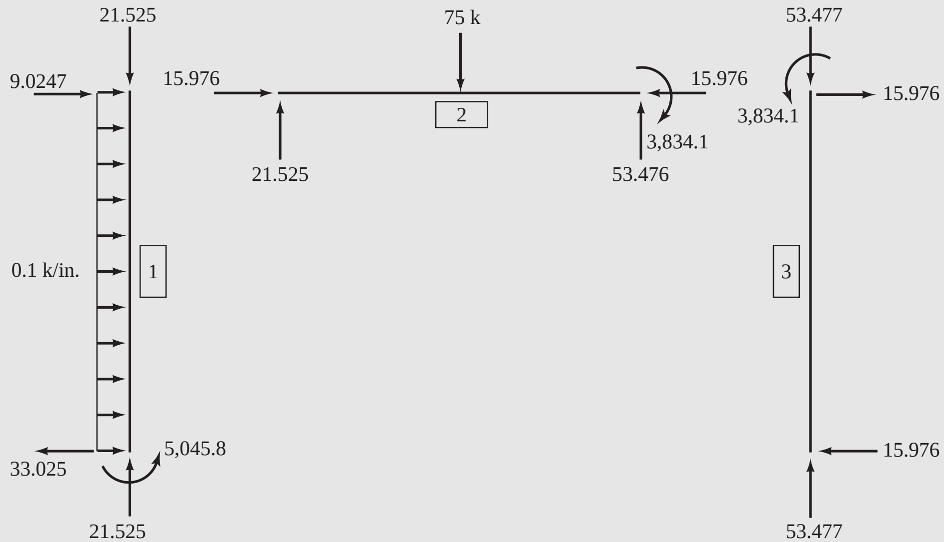
(b) Analytical Model

$$
\mathbf{S} = \begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
5.0347 + 1{,}776.3 & 0 & -1{,}776.3 & 0 & 0 \\
0 & 1{,}776.3 + 5.0347 & 0 & -5.0347 & 1{,}208.3 \\
-1{,}776.3 & 0 & 1{,}776.3 + 5.0347 & 0 & 1{,}208.3 \\
0 & -5.0347 & 0 & 5.0347 + 1{,}776.3 & -1{,}208.3 \\
0 & 1{,}208.3 & 1{,}208.3 & -1{,}208.3 & 290{,}000 + 290{,}000
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

$$
= \begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
1{,}781.3 & 0 & -1{,}776.3 & 0 & 0 \\
0 & 1{,}781.3 & 0 & -5.0347 & 1{,}208.3 \\
-1{,}776.3 & 0 & 1{,}781.3 & 0 & 1{,}208.3 \\
0 & -5.0347 & 0 & 1{,}781.3 & -1{,}208.3 \\
0 & 1{,}208.3 & 1{,}208.3 & -1{,}208.3 & 580{,}000
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\qquad
\mathbf{P}_f = \left[\begin{array}{c} -9 \\ 23.438 \\ 0 \\ 51.563 \\ -3{,}375 \end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

(c) Structure Stiffness Matrix and Fixed-Joint Force Vector

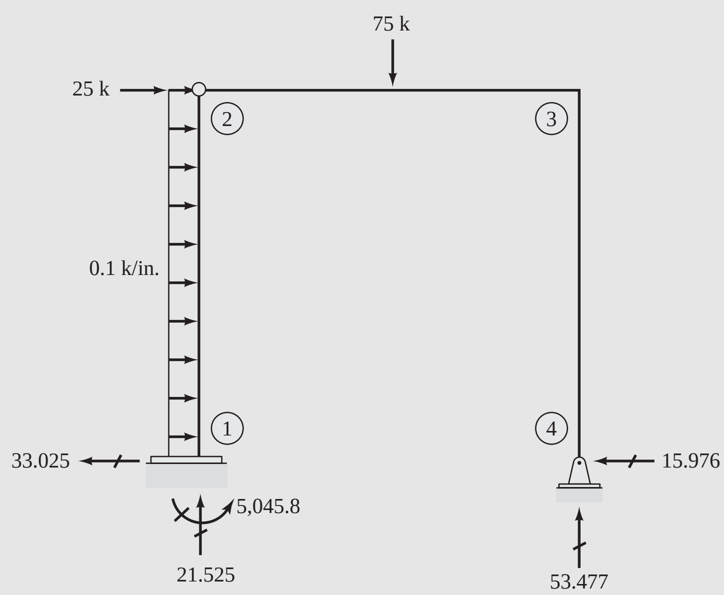
Fig. 7.3 (*continued*)



(d) Member Local End Forces

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} -33.025 \text{ k} \\ 21.525 \text{ k} \\ 5{,}045.8 \text{ k-in.} \\ 0 \\ -15.976 \text{ k} \\ 53.477 \text{ k} \\ 0 \end{array}\right] \end{array} \begin{array}{c} 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 7.3 (*continued*)

of member 3, which is connected to joint 4, can be considered to be hinged, because there is only one member connected to the joint that is not subjected to any external couple. Furthermore, the joint is supported by a hinged support which cannot exert any reaction moment at the joint. Thus, the moment at the end of member 3, which is connected to joint 4, must be 0; therefore, the member end can be treated as a hinged end. With its rotation restrained by the imaginary clamp, and its translations in the X and Y directions restrained by the actual hinged support, joint 4 is modeled as if it is attached to a fixed support, with no degrees of freedom, as depicted in Fig. 7.3(b).

Thus, the entire frame has five degrees of freedom and seven restrained coordinates, as shown in Fig. 7.3(b).

Structure Stiffness Matrix and Fixed-Joint Force Vector:

Member 1 ($MT = 2$) Because $MT = 2$ for this member, we use Eqs. (7.9) and (7.10) to determine its local stiffness matrix $\mathbf{k}$ and fixed-end force vector $\mathbf{Q}_f$, respectively. Thus, by substituting $E = 29{,}000$ ksi, $A = 14.7$ in.^{2}, $I = 800$ in.^{4}, and $L = 20$ ft $=$ 240 in. into Eq. (7.9), we obtain

$$\mathbf{k}_1 = \begin{bmatrix} 1{,}776.3 & 0 & 0 & -1{,}776.3 & 0 & 0 \\ 0 & 5.0347 & 1{,}208.3 & 0 & -5.0347 & 0 \\ 0 & 1{,}208.3 & 290{,}000 & 0 & -1{,}208.3 & 0 \\ -1{,}776.3 & 0 & 0 & 1{,}776.3 & 0 & 0 \\ 0 & -5.0347 & -1{,}208.3 & 0 & 5.0347 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

To determine the local fixed-end force vector due to the member load $w = 1.2$ k/ft $=$ 0.1 k/in., we first evaluate the fixed-end axial forces, shears, and moments in a corresponding rigidly connected member by using the expressions given inside the front cover:

$$FA_b = FA_e = 0$$

$$FS_b = FS_e = \frac{0.1(240)}{2} = 12 \text{ k}$$

$$FM_b = -FM_e = \frac{0.1(240)^2}{12} = 480 \text{ k-in.}$$

Next, we substitute the foregoing values into Eq. (7.10) to obtain the local fixed-end force vector for the released member under consideration:

$$\mathbf{Q}_{f1} = \begin{bmatrix} 0 \\ 12 - \dfrac{3(-480)}{2(240)} \\ 480 - \dfrac{1}{2}(-480) \\ 0 \\ 12 + \dfrac{3(-480)}{2(240)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \text{ k} \\ 720 \text{ k-in.} \\ 0 \\ 9 \text{ k} \\ 0 \end{bmatrix} \tag{2}$$

To obtain the member's stiffness matrix $\mathbf{K}$ and the fixed-end force vector $\mathbf{F}_f$ in the global coordinate system, we first substitute its direction cosines, $\cos\theta = 0$ and $\sin\theta = 1$, into Eq. (6.19), to obtain the transformation matrix.

$$\mathbf{T}_1 = \mathbf{T}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Next, by substituting $\mathbf{k}_1$ (Eq. (1)) and $\mathbf{T}_1$ (Eq. (3)) into the relationship $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$ (Eq. (6.29)), and performing the necessary matrix multiplications, we obtain

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccccc} 6 & 7 & 8 & 1 & 2 & 9 \end{array} \\ \left[\begin{array}{ccc|cc|c} 5.0347 & 0 & -1{,}208.3 & -5.0347 & 0 & 0 \\ 0 & 1{,}776.3 & 0 & 0 & -1{,}776.3 & 0 \\ -1{,}208.3 & 0 & 290{,}000 & 1{,}208.3 & 0 & 0 \\ -5.0347 & 0 & 1{,}208.3 & 5.0347 & 0 & 0 \\ 0 & -1{,}776.3 & 0 & 0 & 1{,}776.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array} \end{array}$$

Note that $\mathbf{K}_1$ is symmetric.

Similarly, by substituting $\mathbf{Q}_{f1}$ (Eq. (2)) and $\mathbf{T}_1$ (Eq. (3)) into the relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$ (Eq. (6.30)), we obtain

$$\mathbf{F}_{f1} = \begin{bmatrix} -15 \\ 0 \\ 720 \\ \hline -9 \\ 0 \\ \hline 0 \end{bmatrix} \begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array}$$

From Fig. 7.3(b), we observe that the code numbers for member 1 are 6, 7, 8, 1, 2, 9. Using these code numbers, we store the pertinent elements of $\mathbf{K}_1$ and $\mathbf{F}_{f1}$ in their proper positions in the 5×5 structure stiffness matrix $\mathbf{S}$ and the 5×1 structure fixed-joint force vector $\mathbf{P}_f$, respectively, as shown in Fig. 7.3(c).

Member 2 ($MT = 1$) No coordinate transformations are needed for this horizontal member; that is, $\mathbf{T}_2 = \mathbf{I}$, $\mathbf{K}_2 = \mathbf{k}_2$, and $\mathbf{F}_{f2} = \mathbf{Q}_{f2}$. As $MT = 1$, we use Eq. (7.5) to obtain

$$\mathbf{K}_2 = \mathbf{k}_2 = \mathbf{k}_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 9 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cc|c|ccc} 1{,}776.3 & 0 & 0 & -1{,}776.3 & 0 & 0 \\ 0 & 5.0347 & 0 & 0 & -5.0347 & 1{,}208.3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1{,}776.3 & 0 & 0 & 1{,}776.3 & 0 & 0 \\ 0 & -5.0347 & 0 & 0 & 5.0347 & -1{,}208.3 \\ 0 & 1{,}208.3 & 0 & 0 & -1{,}208.3 & 290{,}000 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array} \end{array} \tag{4}$$

Using the fixed-end force expressions given for loading type 1 for the 75 k member load, we obtain

$$FA_b = FA_e = 0$$

$$FS_b = FS_e = 37.5\,\text{k}$$

$$FM_b = -FM_e = 2{,}250\,\text{k-in.}$$

Substitution of the foregoing values into Eq. (7.6) yields

$$\mathbf{F}_{f2} = \mathbf{Q}_{f2} = \begin{bmatrix} 0 \\ 37.5 - \dfrac{3(2{,}250)}{2(240)} \\ 0 \\ 0 \\ 37.5 + \dfrac{3(2{,}250)}{2(240)} \\ -2{,}250 - \dfrac{1}{2}(2{,}250) \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 23.438\text{ k} \\ \hdashline 0 \\ \hdashline 0 \\ 51.563\text{ k} \\ -3{,}375\text{ k-in.} \end{array}\right] \end{array}\begin{array}{c} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array} \tag{5}$$

The relevant elements of $\mathbf{K}_2$ and $\mathbf{F}_{f2}$ are stored in $\mathbf{S}$ and $\mathbf{P}_f$, respectively, using the member code numbers 1, 2, 9, 3, 4, 5.

Member 3 ($MT = 1$) As E, A, I, L, and MT for member 3 are the same as for member 2, $\mathbf{k}_3 = \mathbf{k}_2$ as given in Eq. (4). Also, since the member is not subjected to any loads,

$$\mathbf{F}_{f3} = \mathbf{Q}_{f3} = \mathbf{0}$$

Furthermore, since the direction cosines of member 3 are identical to those of member 1, $\mathbf{T}_3 = \mathbf{T}_1$ as given in Eq. (3).

To determine the member global stiffness matrix, we substitute $\mathbf{k}_3$ from Eq. (4), and $\mathbf{T}_3$ from Eq. (3), into the relationship $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$ (Eq. (6.29)), and perform the necessary matrix multiplications. This yields

$$\mathbf{K}_3 = \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 12 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccc:ccc} 5.0347 & 0 & 0 & -5.0347 & 0 & -1{,}208.3 \\ 0 & 1{,}776.3 & 0 & 0 & -1{,}776.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -5.0347 & 0 & 0 & 5.0347 & 0 & 1{,}208.3 \\ 0 & -1{,}776.3 & 0 & 0 & 1{,}776.3 & 0 \\ -1{,}208.3 & 0 & 0 & 1{,}208.3 & 0 & 290{,}000 \end{array}\right] \end{array} \begin{array}{c} \\ 10 \\ 11 \\ 12 \\ 3 \\ 4 \\ 5 \end{array}$$

The pertinent elements of $\mathbf{K}_3$ are stored in $\mathbf{S}$ using the member code numbers 10, 11, 12, 3, 4, 5. The completed structure stiffness matrix $\mathbf{S}$ and the structure fixed-joint force vector $\mathbf{P}_f$ are given in Fig. 7.3(c).

Joint Load Vector: By comparing Figs. 7.3(a) and (b), we write

$$\mathbf{P} = \begin{bmatrix} 25 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Joint Displacements: By solving the system of simultaneous equations representing the structure stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$ (Eq. (6.42)), we obtain the following joint displacements:

$$\mathbf{d} = \begin{bmatrix} 3.5801\text{ in.} \\ -0.012118\text{ in.} \\ 3.5711\text{ in.} \\ -0.030106\text{ in.} \\ -0.0016582\text{ rad} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \qquad \textbf{Ans}$$

Member End Displacements and End Forces:

Member 1 ($MT = 2$) Using the member code numbers 6, 7, 8, 1, 2, 9, we write the global end displacement vector as

$$\mathbf{v}_1 = \begin{array}{c}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}\end{array}\begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.5801 \\ -0.012118 \\ 0 \end{bmatrix} \tag{6}$$

Next, we obtain the local end displacement vector $\mathbf{u}$ by substituting the foregoing $\mathbf{v}_1$ and $\mathbf{T}_1$ (Eq. (3)) into the relationship $\mathbf{u} = \mathbf{T}\mathbf{v}$ (Eq. (6.20)). This yields

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.012118 \\ -3.5801 \\ 0 \end{bmatrix} \tag{7}$$

We can now determine the member local end forces $\mathbf{Q}$ by substituting $\mathbf{u}_1$, $\mathbf{k}_1$ (Eq. (1)), and $\mathbf{Q}_{f1}$ (Eq. (2)) in the member stiffness relationship $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$ (Eq. (6.4)). Thus,

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 21.525 \text{ k} \\ 33.025 \text{ k} \\ 5{,}045.8 \text{ k-in.} \\ -21.525 \text{ k} \\ -9.0247 \text{ k} \\ 0 \end{bmatrix} \qquad \textbf{Ans}$$

These end forces for member 1 are depicted in Fig. 7.3(d).

To generate the support reaction vector $\mathbf{R}$ for the frame, we evaluate the global end forces $\mathbf{F}$ for the member by applying Eq. (6.23) as

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{bmatrix} -33.025 \\ 21.525 \\ 5{,}045.8 \\ \hdashline 9.0247 \\ -21.525 \\ \hdashline 0 \end{bmatrix}\begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array}$$

The pertinent elements of $\mathbf{F}_1$ are stored in $\mathbf{R}$, as shown in Fig. 7.3(e).

It should be realized that because the member end displacement vectors $\mathbf{v}$ and $\mathbf{u}$ are based on the compatibility of the joint and the member end displacements, such vectors (in the case of members with releases) contain 0 elements in the rows that correspond to the rotations of the released (or hinged) member ends. Thus, we can see from Eqs. (6) and (7) that the vectors $\mathbf{v}_1$ and $\mathbf{u}_1$ for member 1 (with $MT = 2$) contain 0 elements in their sixth rows. We can evaluate the rotation u_6 of the released end of this member by using Eq. (7.7), as

$$u_6 = \frac{3}{2(240)}(0 - 3.5801) - \frac{1}{2}(0) - \frac{240(-480)}{4(29{,}000)(800)}$$
$$= -0.021134 \text{ rad} = 0.021134 \text{ rad} \circlearrowright$$

Because the member end rotations are the same in the local and global coordinate systems,

$$v_6 = u_6 = 0.021134 \text{ rad}$$

Member 2 ($MT = 1$)

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{array}{c}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}\end{array}\begin{array}{l} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array} = \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} = \begin{bmatrix} 3.5801 \\ -0.012118 \\ 0 \\ 3.5711 \\ -0.030106 \\ -0.0016582 \end{bmatrix}$$

By using $\mathbf{k}_2$ from Eq. (4) and $\mathbf{Q}_{f2}$ from Eq. (5), we compute the member end forces to be

$$\mathbf{F}_2 = \mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} 15.976 \text{ k} \\ 21.525 \text{ k} \\ \hline 0 \\ \hline -15.976 \text{ k} \\ 53.476 \text{ k} \\ -3{,}834.1 \text{ k-in.} \end{bmatrix}\begin{array}{l} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array} \qquad \textbf{Ans}$$

The rotation, u_3, of the released end of this member, if desired, can be calculated by using Eq. (7.2).

Member 3 ($MT = 1$)

$$\mathbf{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}\begin{array}{l} 10 \\ 11 \\ 12 \\ 3 \\ 4 \\ 5 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.5711 \\ -0.030106 \\ -0.0016582 \end{bmatrix}$$

By using $\mathbf{T}_3$ from Eq. (3), we obtain

$$\mathbf{u}_3 = \mathbf{T}_3\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.030106 \\ -3.5711 \\ -0.0016582 \end{bmatrix}$$

Using $\mathbf{k}_3$ from Eq. (4) and $\mathbf{Q}_{f3} = \mathbf{0}$, we obtain the member local end forces as

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 = \begin{bmatrix} 53.477 \text{ k} \\ 15.976 \text{ k} \\ 0 \\ -53.477 \text{ k} \\ -15.976 \text{ k} \\ 3{,}834.1 \text{ k-in.} \end{bmatrix} \qquad \textbf{Ans}$$

The member local end forces are shown in Fig. 7.3(d).

$$\mathbf{F}_3 = \mathbf{T}_3^T \mathbf{Q}_3 = \begin{array}{c} \left[\begin{array}{r} -15.976 \\ 53.477 \\ 0 \\ \hdashline 15.976 \\ -53.477 \\ 3{,}834.1 \end{array}\right] \end{array} \begin{array}{r} 10 \\ 11 \\ 12 \\ 3 \\ 4 \\ 5 \end{array}$$

Support Reactions: See Figs. 7.3(e) and (f). **Ans**

7.2 COMPUTER IMPLEMENTATION OF ANALYSIS FOR MEMBER RELEASES

The computer programs developed in Chapters 5 and 6 for the analysis of rigidly connected beams and plane frames can be extended, with only minor modifications, to include the effects of member releases. In this section, we discuss the modifications in the program for the analysis of plane frames (Section 6.7) that are necessary to consider member releases. While the beam analysis program (Section 5.8) can be modified in a similar manner, the implementation of these modifications is left as an exercise for the reader.

The overall organization and format of the plane frame analysis program, as summarized in Table 6.1, remains the same when considering member releases. However, parts V, IX, and XII, and the subroutines ***MSTIFFL*** and ***MFEFLL***, must be revised as follows:

***Member Data* (*Part V*)** This part of the program (see flowchart in Fig. 4.3(e)) should be modified to include the reading and storing of the member type, *MT*, for each member of the frame. The number of columns of the *member data matrix* **MPRP** should be increased from four to five, with the value of *MT* (= 0, 1, 2, or 3) for a member *i* stored in the fifth column of the *i*th row of **MPRP**.

Generation of the Structure Stiffness Matrix and Equivalent Joint Load Vector* (*Part IX*), *and Calculation of Member Forces and Support Reactions* (*Part XII*)** In parts IX and XII of the computer program (see flowcharts in Figs. 6.24 and 6.31, respectively) a statement should be added to read, for each member, the value of *MT* from the fifth column of the **MPRP** matrix (i.e., *MT* = **MPRP** (*IM*, 5)), before the subroutines ***MSTIFFL and ***MFEFLL*** are called to form the member local stiffness matrix **BK**, and the local fixed-end force vector **QF**, respectively.

Subroutine MSTIFFL A flowchart for programming the modified version of this subroutine is given in Fig. 7.4 on the next page. As this flowchart indicates, the subroutine calculates the **BK** matrix using Eq. (6.6) if *MT* equals 0, Eq. (7.5) if *MT* equals 1, Eq. (7.9) if *MT* equals 2, or Eq. (7.13) if *MT* equals 3.

Subroutine MFEFLL A flowchart of the modified version of this subroutine is shown in Fig. 7.5 on page 363. The subroutine first calculates the fixed-end forces

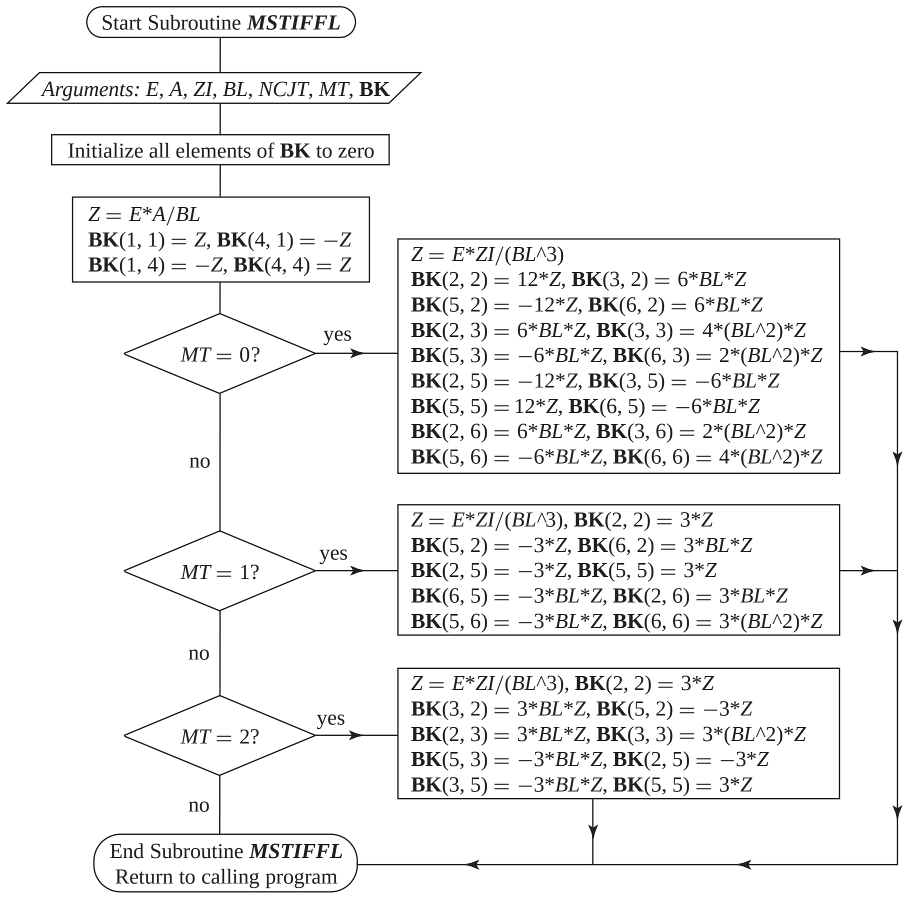


Fig. 7.4 *Flowchart of Subroutine **MSTIFFL** for Determining Member Local Stiffness Matrix for Plane Frames with Member Releases*

(*FAB*, *FSB*, *FMB*, *FAE*, *FSE*, and *FME*) in a corresponding rigidly connected member using the equations given inside the front cover. The **QF** vector is then formed in accordance with Eq. (6.15) if $MT = 0$, Eq. (7.6) if $MT = 1$, Eq. (7.10) if $MT = 2$, or Eq. (7.14) if $MT = 3$.

7.3 SUPPORT DISPLACEMENTS

The effect of small support displacements, due to weak foundations or other causes, can be conveniently included in the matrix stiffness method of analysis using the concept of equivalent joint loads [14]. This approach, which was discussed in Sections 5.6 and 6.5 for the case of member loads, essentially involves applying the prescribed external action (such as a system of member loads, support settlements, etc.) to the structure, with all of its joint displacements restrained by imaginary restraints. The structure fixed-joint forces that develop in the hypothetical fixed structure, as reactions at the imaginary

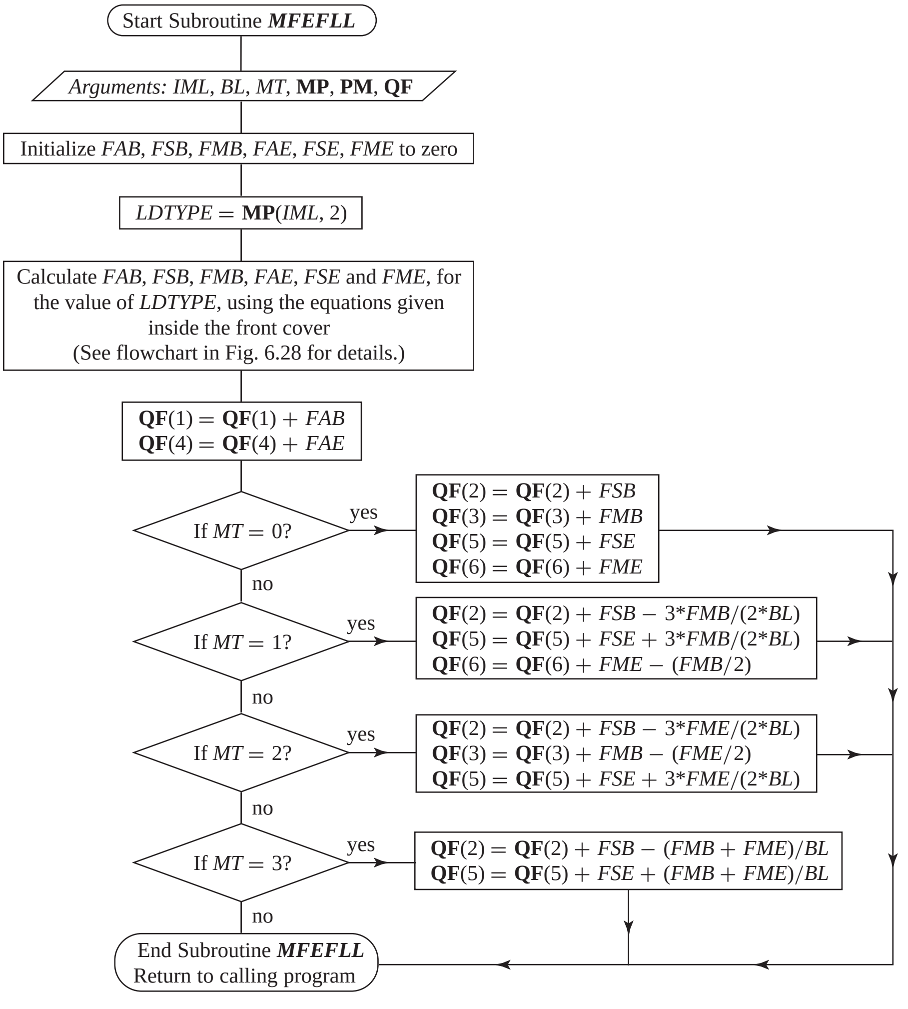
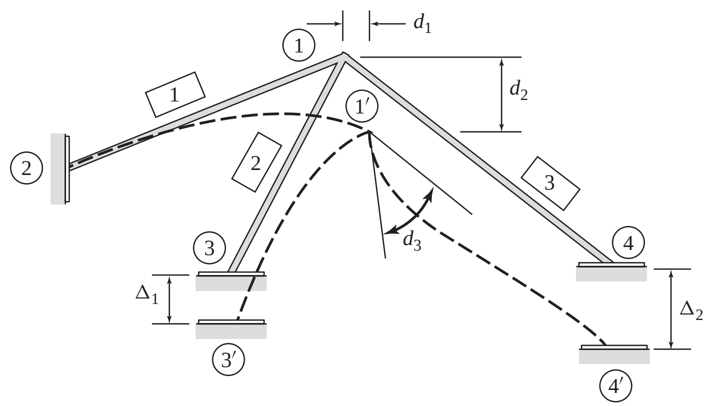


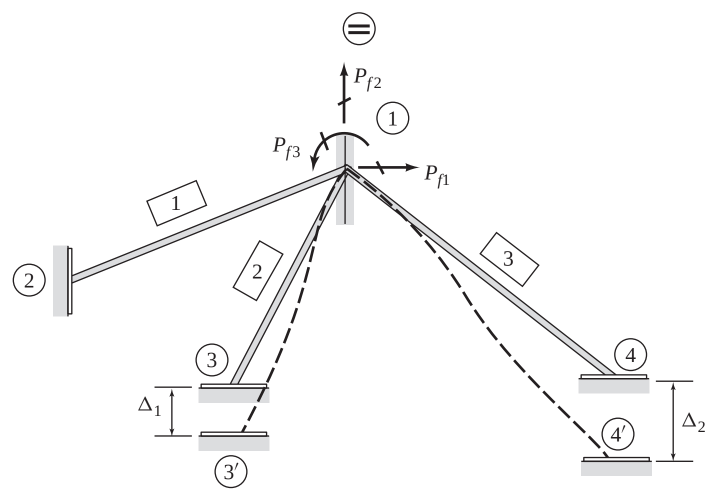
Fig. 7.5 *Flowchart of Subroutine **MFEFLL** for Determining Member Local Fixed-End Force Vector for Plane Frames with Member Releases*

restraints (i.e., at the location and in the direction of each degree of freedom of the actual structure), are then evaluated. The structure fixed-joint forces, with their directions reversed, now represent the equivalent joint loads, in the sense that when applied to the actual structure, they cause the same joint displacements as the original action (i.e., member loads, support settlements, etc.). Once the response of the structure to the equivalent joint loads has been determined, the actual structural response due to the original action is obtained by superposition of the responses of the fixed structure to the original action and the actual structure to the equivalent joint loads.

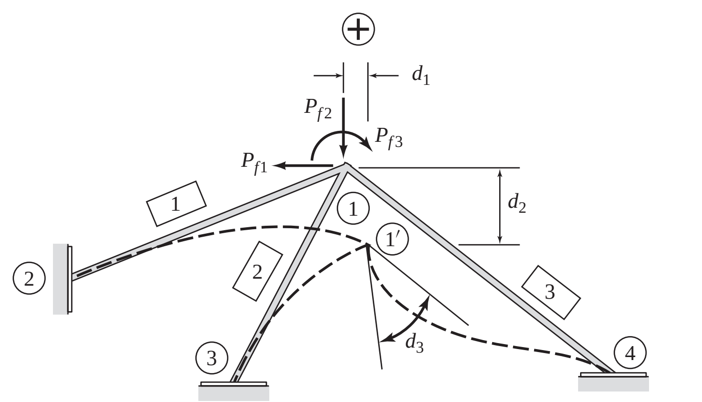
The foregoing approach is illustrated in Fig. 7.6 on the next page, for the case of support displacements, using an arbitrary three-degree-of-freedom frame as an example. Figure 7.6(a) shows the actual frame, whose supports 3 and 4 undergo small



(a) Actual Frame Subjected to Support Settlements



(b) Fixed Frame Subjected to Support Settlements



(c) Actual Frame Subjected to Equivalent Joint Loads

Fig. 7.6

settlements Δ_1 and Δ_2, respectively, causing the displacements d_1, d_2, and d_3 of free joint 1. To determine the response (i.e., joint displacements, member forces, and support reactions) of the frame to the support settlements, we first restrain all the joint displacements of the frame by applying an imaginary restraint at joint 1, and subject this completely fixed frame to the prescribed support settlements Δ_1 and Δ_2, as shown in Fig. 7.6(b). As joint 1 of the frame, initially free, is now restrained from translating and rotating by the imaginary restraint, the structure fixed-joint forces P_{f1}, P_{f2}, and P_{f3} develop at the imaginary restraint at this joint. (A procedure for evaluating structure fixed-joint forces due to support settlements is developed in a subsequent part of this section.) Next, as shown in Fig. 7.6(c), we apply the foregoing structure fixed-joint forces P_{f1}, P_{f2}, and P_{f3}, with their directions reversed, as external loads at joint 1 of the actual frame.

A comparison of Figs. 7.6(a), (b), and (c) indicates that the superposition of the support settlements and joint loads applied to the frame in Figs. 7.6(b) and (c) yields only the support settlements the frame is subjected to in Fig. 7.6(a), because each of the fixed-joint forces in Fig. 7.6(b) is canceled by its negative counterpart applied as a load in Fig. 7.6(c). Thus, according to the principle of superposition, the joint displacements d_1, d_2, and d_3 of the frame due to the support settlements Δ_1 and Δ_2 (Fig. 7.6(a)) must equal the algebraic sums of the corresponding joint displacements of the fixed frame subjected to the support settlements (Fig. 7.6(b)), and the actual frame, subjected to no settlements, but to the negatives of the fixed-joint forces (Fig. 7.6(c)). However, since the displacements of joint 1 of the fixed frame (Fig. 7.6(b)) are 0, the joint displacements of the frame subjected to the negatives of fixed-joint forces (Fig. 7.6(c)) must equal the actual joint displacements d_1, d_2, and d_3 of the frame due to the support settlements Δ_1 and Δ_2 (Fig. 7.6(a)). In other words, the negatives of the structure fixed-joint forces cause the same displacements at the locations and in the directions of the frame's degrees of freedom as the prescribed support settlements; and, in that sense, such forces can be considered as equivalent joint loads.

It should be realized that the foregoing equivalency is valid only for joint displacements. From Fig. 7.6(b), we can see that the end displacements of the members of the fixed frame are not 0. Therefore, the member end forces and support reactions of the actual frame due to settlements (Fig. 7.6(a)) must be obtained by superposition of the corresponding responses of the fixed frame (Fig. 7.6(b)) and the actual frame subjected to the equivalent joint loads (Fig. 7.6(c)).

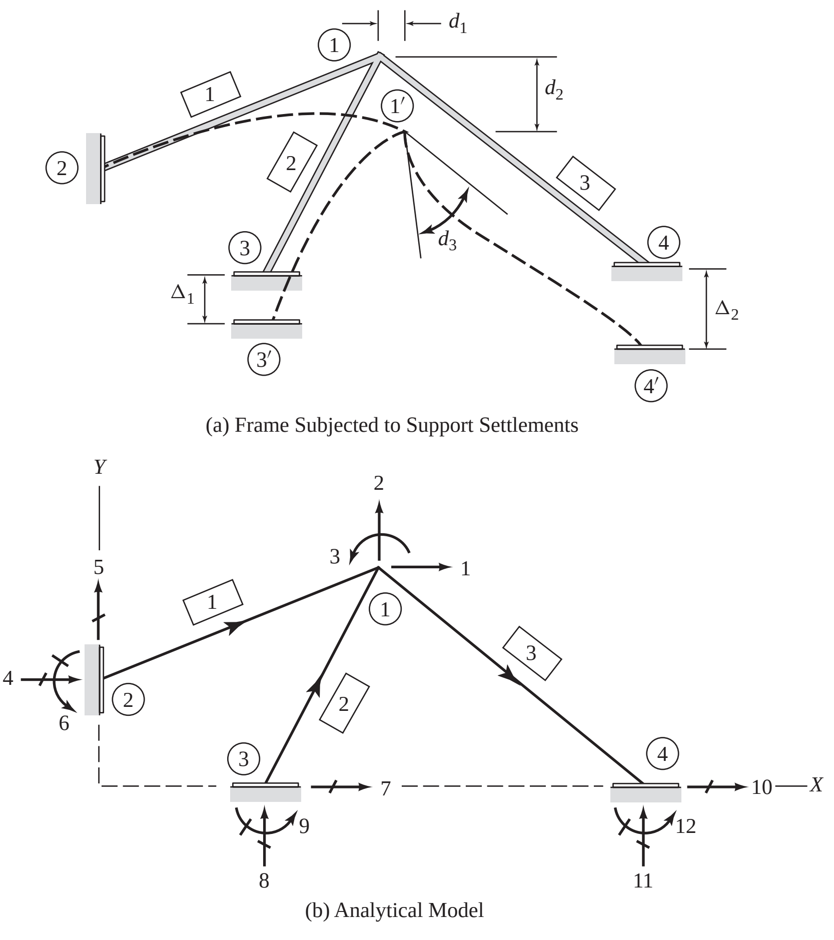
It may be recalled from Chapters 5 and 6 that, in the case of member loads, the response of the fixed structure was evaluated using the fixed-end force expressions for various types of member loads, as given inside the front cover; and that the fixed-joint force vector $\mathbf{P}_f$ was obtained by algebraically adding the fixed-end forces of members meeting at the joints (via the member code numbers). A procedure for evaluating the member fixed-end forces, and the structure fixed-joint forces, due to support settlements is presented in the following paragraphs. With the fixed-joint forces known, the response of the structure to the equivalent joint loads can be determined, using the standard matrix stiffness methods described in Chapters 3 through 6.

Evaluation of Structure Fixed-Joint Forces Due to Support Displacements

We begin by establishing a systematic way of identifying the support displacements of a structure. For that purpose, let us reconsider the three-degree-of-freedom frame of Fig. 7.6(a), subjected to the support settlements Δ_1 and Δ_2. The frame is redrawn in Fig. 7.7(a), with its analytical model depicted in Fig. 7.7(b). From Fig. 7.7(b), we observe that the frame has nine support reactions, which are identified by the restrained coordinate numbers 4 through 12. Thus, the frame can be subjected to a maximum of nine support displacements. The numbers assigned to the restrained coordinates are also used to identify the support displacements, with a support displacement at the location and in the direction of a support reaction R_i denoted by the symbol d_{si}. Thus, a comparison of Figs. 7.7(a) and (b) shows that for the frame under consideration,

$$d_{s8} = -\Delta_1 \qquad \text{and} \qquad d_{s11} = -\Delta_2 \tag{7.24}$$

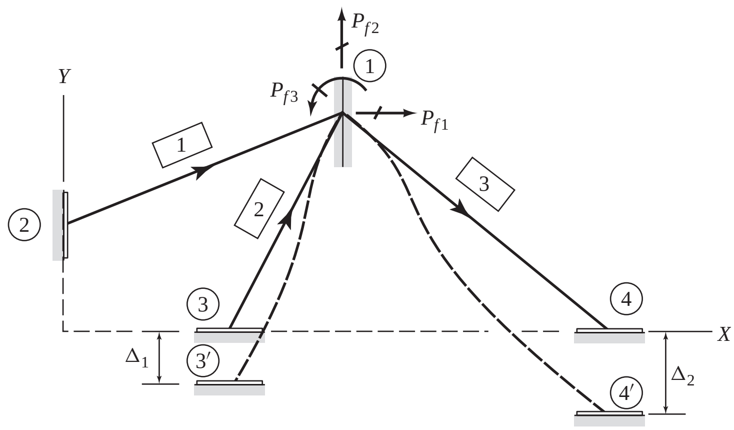
with the remaining seven support displacements being 0. The negative signs



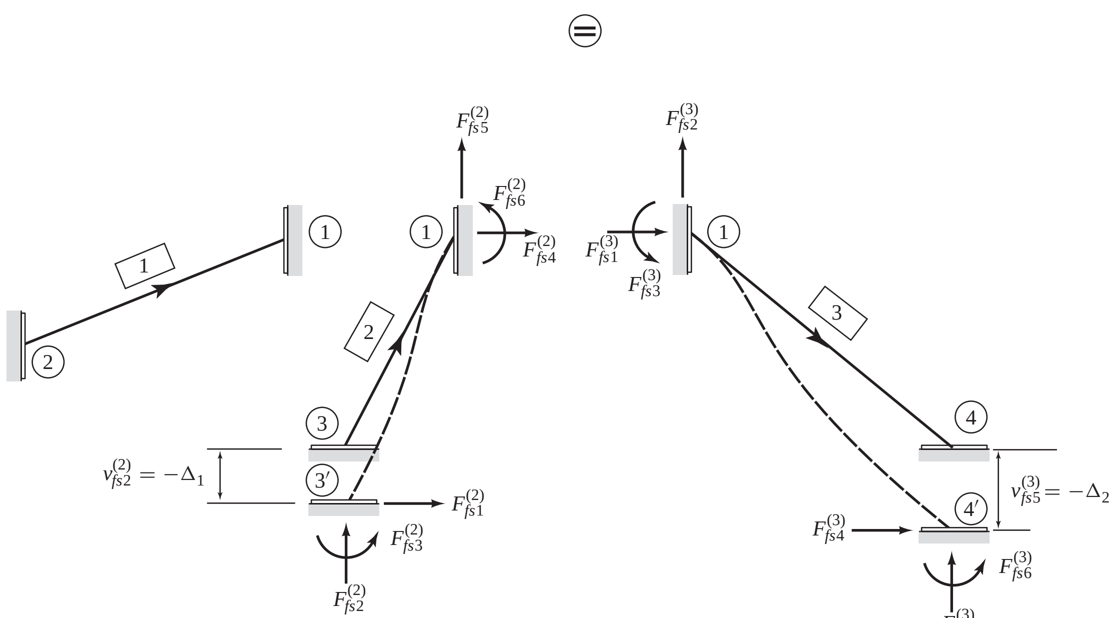
(a) Frame Subjected to Support Settlements

(b) Analytical Model

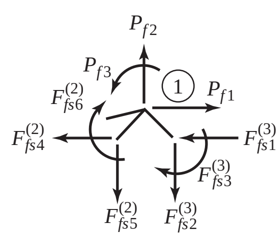
Fig. 7.7



(c) Fixed Frame Subjected to Support Settlements



(d) Member Global Fixed-End Displacements and Forces



(e) Free Body of Joint 1 —Fixed Frame

Fig. 7.7 (*continued*)

assigned to the magnitudes Δ_1 and Δ_2 of d_{s8} and d_{s11} indicate that these support displacements occur in the negative Y (i.e., downward) direction.

To illustrate the process of evaluating a structure's fixed-joint forces due to a prescribed set of support settlements, we restrain the joint displacements of the example frame by applying an imaginary restraint at joint 1, and subject this hypothetical completely fixed frame to the given support settlements Δ_1 and Δ_2, as shown in Fig. 7.7(c). The structure fixed-joint forces that develop at the imaginary restraint at joint 1 are denoted by P_{f1}, P_{f2}, and P_{f3} in the figure, with the fixed-joint force corresponding to an ith degree of freedom denoted by P_{fi}. To evaluate the fixed-joint forces, we first determine the displacements that the support settlements Δ_1 and Δ_2 cause at the ends of the members of the fixed frame. The free-body diagrams of the three members of the hypothetical fixed frame are depicted in Fig. 7.7(d). From Figs. 7.7(c) and (d), we observe that, while the settlements of supports 3 and 4 do not cause any displacement in member 1, they induce downward displacements of magnitudes Δ_1 and Δ_2, respectively, at the lower ends of members 2 and 3. Note that all other member end displacements are 0, because all the joint displacements of the fixed frame are 0, with the exception of the known support settlements. Thus, the end displacements of members 2 and 3, respectively, can be expressed in vector form, as

$$\mathbf{v}_{fs2} = \begin{array}{c} \begin{bmatrix} 0 \\ -\Delta_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array} \qquad \text{and} \qquad \mathbf{v}_{fs3} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\Delta_2 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 10 \\ 11 \\ 12 \end{array} \tag{7.25}$$

in which $\mathbf{v}_{fs}$ represents the *member global fixed-end displacement vector due to support displacements.* The foregoing member global fixed-end displacement vectors can be directly generated using the member code numbers, which define the member compatibility equations. For example, from Fig. 7.7(b), we can see that the code numbers for member 2 are 7, 8, 9, 1, 2, 3. By comparing these member code numbers with the support displacements of the frame, $d_{s8} = -\Delta_1$ and $d_{s11} = -\Delta_2$ (see Eq. (7.24)), we conclude that all the elements of $\mathbf{v}_{fs2}$ are zero, with the exception of the element in the second row which equals $-\Delta_1$ (i.e., $v_{fs2}^{(2)} = d_{s8} = -\Delta_1$). Similarly, by examining the code numbers 1, 2, 3, 10, 11, 12 of member 3, we realize that the only nonzero element of $\mathbf{v}_{fs3}$ is in the fifth row and it equals $-\Delta_2$ (i.e., $v_{fs5}^{(3)} = d_{s11} = -\Delta_2$).

Once the member fixed-end displacement vectors $\mathbf{v}_{fs}$ have been determined, they are used to calculate the corresponding *member global fixed-end force vectors due to support displacements*, $\mathbf{F}_{fs}$, through the member global stiffness relationship (Eq. (6.28)) derived in Chapter 6. By substituting $\mathbf{F} = \mathbf{F}_{fs}$, $\mathbf{v} = \mathbf{v}_{fs}$, and $\mathbf{F}_f = \mathbf{0}$ into Eq. (6.28), we obtain the following relationship between $\mathbf{F}_{fs}$ and $\mathbf{v}_{fs}$:

$$\boxed{\mathbf{F}_{fs} = \mathbf{K}\mathbf{v}_{fs}} \tag{7.26}$$

With the member global fixed-end forces known, the structure fixed-joint forces due to the support displacements can be evaluated using joint equilibrium equations. Thus, for the example frame, we apply the three equations of equilibrium, $\sum F_X = 0$, $\sum F_Y = 0$, and $\sum M = 0$, to the free body of joint 1 (see Fig. 7.7(e)) to obtain the following expressions for the fixed-joint forces in terms of the member fixed-end forces:

$$P_{f1} = F_{fs4}^{(2)} + F_{fs1}^{(3)} \tag{7.27a}$$

$$P_{f2} = F_{fs5}^{(2)} + F_{fs2}^{(3)} \tag{7.27b}$$

$$P_{f3} = F_{fs6}^{(2)} + F_{fs3}^{(3)} \tag{7.27c}$$

The structure fixed-joint force vector for the support settlements of the example frame can, therefore, be expressed as

$$\mathbf{P}_f = \begin{bmatrix} F_{fs4}^{(2)} + F_{fs1}^{(3)} \\ F_{fs5}^{(2)} + F_{fs2}^{(3)} \\ F_{fs6}^{(2)} + F_{fs3}^{(3)} \end{bmatrix} \tag{7.28}$$

As demonstrated in Chapters 5 and 6 for the case of member loads, the structure fixed-joint force vectors $\mathbf{P}_f$ can be conveniently generated by employing the member code number technique. The application of the technique remains the same in the case of support displacements, except that the elements of the member global fixed-end force vectors due to support displacements, $\mathbf{F}_{fs}$, must now be added into $\mathbf{P}_f$. When a structure is subjected to more than one type of action requiring evaluation of fixed-joint forces (e.g., member loads and support settlements), then the fixed-joint forces representing different types of actions can be conveniently combined into a single $\mathbf{P}_f$ vector. For example, in the case of a frame subjected to member loads and support settlements, the elements of the two types of member fixed-end force vectors—that is, due to member loads ($\mathbf{F}_f$) and support displacements ($\mathbf{F}_{fs}$)—can be stored in a single $\mathbf{P}_f$ vector using the member code number technique.

Once the structure fixed-joint forces due to support displacements have been evaluated, the structure stiffness relations $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$ (Eq. (6.42)) can be solved for the unknown joint displacements $\mathbf{d}$. With $\mathbf{d}$ known, the member global end displacement vector $\mathbf{v}$ for each member is determined by applying the compatibility equations defined by its code numbers. For members that are attached to the supports that undergo displacements, the displacements of the supported ends, due to the corresponding support displacements, must be included in the member global end displacement vectors $\mathbf{v}$. The inclusion of support displacements in the $\mathbf{v}$ vectors automatically adds the response of the fixed structure to support settlements (see, for example, Fig. 7.6(b)) into the analysis, thereby enabling us to evaluate the member local and global end forces, and support reactions, using the procedures developed in previous chapters.

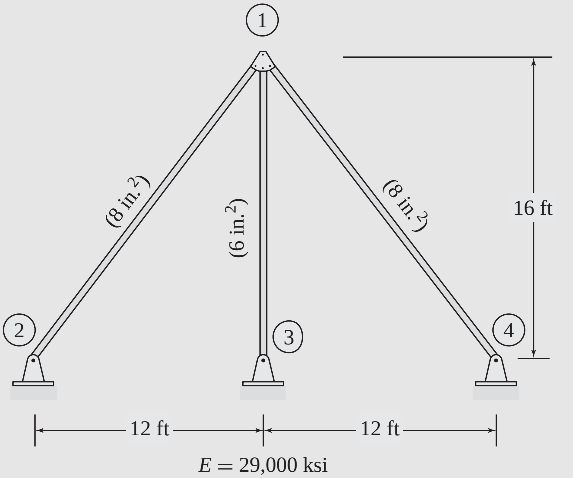
Procedure for Analysis

Based on the discussion presented in this section, we can develop the following step-by-step procedure for the matrix stiffness analysis of framed structures due to support displacements.

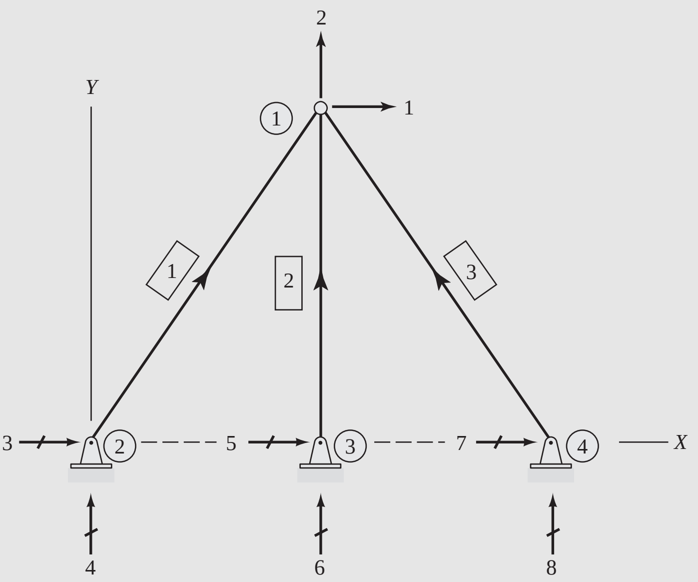
1. Prepare an analytical model of the structure, and determine its structure stiffness matrix $\mathbf{S}$. If the structure is subjected to member loads, then evaluate its fixed-joint force vector $\mathbf{P}_f$ due to the member loads. If the structure is subjected to joint loads, then form its joint load vector $\mathbf{P}$.
2. Calculate the structure fixed-joint force vector $\mathbf{P}_f$ ($NDOF \times 1$) due to the given support displacements. If a $\mathbf{P}_f$ vector was formed in step 1 for member loads, then store the member fixed-end forces due to support displacements in the previously formed $\mathbf{P}_f$ vector. For each member that is attached to the supports that undergo displacements, perform the following operations:
   a. Identify the member code numbers, and form the member global fixed-end displacement vector, $\mathbf{v}_{fs}$, from the specified support displacements, d_{si}. Note that the support translations are considered positive when in the positive directions of the global X and Y axes, and support rotations are considered positive when counterclockwise. For beams, form the member local fixed-end displacement vector due to support displacements, $\mathbf{u}_{fs}$, using the same process.
   b. Evaluate the member global fixed-end force vector due to support displacements, $\mathbf{F}_{fs}$, using the relationship $\mathbf{F}_{fs} = \mathbf{K}\mathbf{v}_{fs}$ (Eq. (7.26)). For beams, evaluate the member local fixed-end force vector due to support displacements, $\mathbf{Q}_{fs}$, using the relationship $\mathbf{Q}_{fs} = \mathbf{k}\mathbf{u}_{fs}$.
   c. Using member code numbers, store the pertinent elements of $\mathbf{F}_{fs}$, or $\mathbf{Q}_{fs}$ for beams, in their proper positions in the structure fixed-joint force vector $\mathbf{P}_f$.
3. Determine the unknown joint displacements $\mathbf{d}$ by solving the structure stiffness relationship, $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$.
4. Compute member end displacements and end forces, and support reactions. For each member of the structure, carry out the following steps.
   a. Obtain member end displacements in the global coordinate system, $\mathbf{v}$, from the joint displacements $\mathbf{d}$ and the specified support displacements d_{si}, by using the member code numbers. For beams, obtain the member local end displacements, $\mathbf{u}$, using the same process, and then go to step 4c.
   b. Determine the member end displacements in the local coordinate system, $\mathbf{u}$, by using the transformation relationship $\mathbf{u} = \mathbf{T}\mathbf{v}$.
   c. Calculate the member end forces in the local coordinate system, $\mathbf{Q}$, by using the stiffness relationship $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$. If the member is not subjected to any member loads, then $\mathbf{Q}_f = \mathbf{0}$. For beams, go to step 4e.
   d. Compute the member end forces in the global coordinate system, $\mathbf{F}$, using the transformation relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$.

e. If the member is attached to a support joint, then use the member code numbers to store the pertinent elements of **F**, or **Q** for beams, in their proper positions in the support reaction vector **R**.

EXAMPLE 7.2 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. 7.8(a) due to a settlement of $\frac{1}{2}$ in. of support 4. Use the matrix stiffness method.



(a) Truss

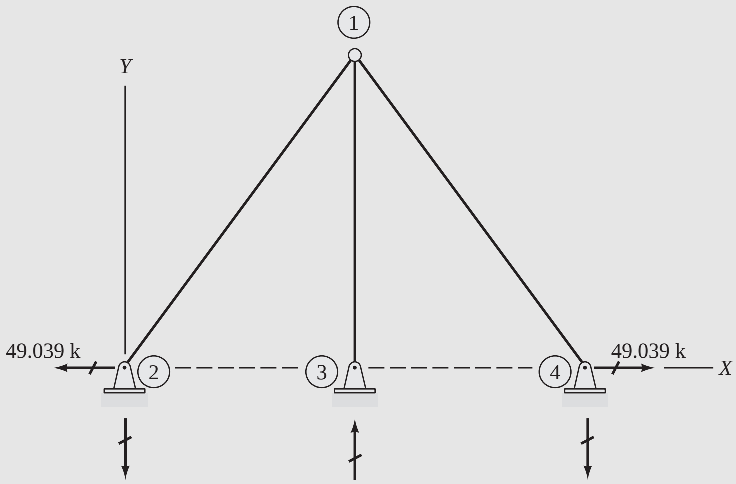


(b) Analytical Model

Fig. 7.8

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} -49.039 \\ -65.386 \\ 0 \\ 130.78 \\ 49.039 \\ -65.386 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \end{array} \text{ k}$$

(c) Support Reaction Vector



(d) Support Reactions

Fig. 7.8 (*continued*)

SOLUTION This truss was analyzed in Example 3.8 for joint loads. In this example, we use the same analytical model of the truss, so that the various member and structure matrices calculated in the previous example can be reused herein.

Analytical Model: See Fig. 7.8(b). The truss has two degrees of freedom and six restrained coordinates.

Structure Stiffness Matrix: From Example 3.8,

$$\mathbf{S} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \text{ k/in.} \tag{1}$$

Joint Load Vector: As the truss is not subjected to any loads,

$$\mathbf{P} = \mathbf{0} \tag{2}$$

Structure Fixed-Joint Force Vector Due to Support Displacements: From the analytical model of the truss in Fig. 7.8(b), we can see that the given .5 in. settlement (i.e., vertically downward displacement) of support joint 4 occurs at the location and in the direction of the reaction R_8. Thus, the given support displacement can be expressed as

$$d_{s8} = -0.5 \text{ in.}$$

From Fig. 7.8(b), we observe that member 3 is the only member attached to support 4 that undergoes displacement. Thus, using the member's code numbers 7, 8, 1, 2, we form its global fixed-end displacement vector due to support displacement as

$$\mathbf{v}_{fs3} = \begin{array}{c}\begin{bmatrix} v_{fs1} \\ v_{fs2} \\ v_{fs3} \\ v_{fs4} \end{bmatrix}\end{array}\begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ d_{s8} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0 \end{bmatrix} \text{ in.}$$

Next, we evaluate the global fixed-end force vector $\mathbf{F}_{fs3}$ due to the support settlement, for member 3, using the member global stiffness matrix $\mathbf{K}_3$ calculated in Example 3.8, and Eq. (7.26). Thus,

$$\mathbf{F}_{fs3} = \mathbf{K}_3\mathbf{v}_{fs3} = \begin{bmatrix} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{bmatrix}\begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 232 \\ -309.33 \\ \hdashline -232 \\ 309.33 \end{bmatrix}\begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k}$$

From the member code numbers, which are written on the right side of $\mathbf{F}_{fs3}$, we realize that the elements in the third and fourth rows of $\mathbf{F}_{fs3}$ should be stored in rows 1 and 2, respectively, of the $\mathbf{P}_f$ vector. Thus, the structure fixed-joint force vector, due to the support settlement, is given by

$$\mathbf{P}_f = \begin{bmatrix} -232 \\ 309.33 \end{bmatrix}\begin{array}{c} 1 \\ 2 \end{array} \text{ k} \tag{3}$$

Joint Displacements: By substituting $\mathbf{P}$ (Eq. (2)), $\mathbf{P}_f$ (Eq. (3)), and $\mathbf{S}$ (Eq. (1)) into the structure stiffness relationship, we write

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -232 \\ 309.33 \end{bmatrix} = \begin{bmatrix} 232 \\ -309.33 \end{bmatrix} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving these equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} 0.33333 \\ -0.14431 \end{bmatrix}\begin{array}{c} 1 \\ 2 \end{array} \text{ in.}$$ **Ans**

Member End Displacements and End Forces:

Member 1 Using the member code numbers 3, 4, 1, 2, we write the global end displacement vector as

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.33333 \\ -0.14431 \end{bmatrix} \text{ in.}$$

Next, we determine the member local end displacement vector $\mathbf{u}_1$, using the transformation matrix $\mathbf{T}_1$ from Example 3.8, and Eq. (3.63), as

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.33333 \\ -0.14431 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.08455 \\ -0.35325 \end{bmatrix} \text{ in.}$$

We can now calculate the member local end forces $\mathbf{Q}_1$ by applying the member stiffness relationship, $\mathbf{Q} = \mathbf{ku}$ (Eq. (3.7)). Thus, using $\mathbf{k}_1$ from Example 3.8, we obtain

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.08455 \\ -0.35325 \end{bmatrix} = \begin{bmatrix} -81.732 \\ 0 \\ 81.732 \\ 0 \end{bmatrix} \text{k}$$

Recall from Chapter 3 that the member axial force equals the first element of the $\mathbf{Q}_1$ vector; that is,

$$Q_{a1} = -81.732 \text{ k}$$

in which the negative sign indicates that the axial force is tensile, or

$$Q_{a1} = 81.732 \text{ k (T)}$$ **Ans**

By applying Eq. (3.66), we determine the member global end forces as

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -81.732 \\ 0 \\ 81.732 \\ 0 \end{bmatrix} = \left[\begin{array}{c} -49.039 \\ -65.386 \\ \hdashline 49.039 \\ 65.386 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{k}$$

Using the member code numbers 3, 4, 1, 2, we store the pertinent elements of $\mathbf{F}_1$ in the support reaction vector $\mathbf{R}$ (see Fig. 7.8(c)).

Member 2

$$\mathbf{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.33333 \\ -0.14431 \end{bmatrix} \text{in.}$$

Using $\mathbf{T}_2$ from Example 3.8, we calculate

$$\mathbf{u}_2 = \mathbf{T}_2\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.33333 \\ -0.14431 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.14431 \\ -0.33333 \end{bmatrix} \text{in.}$$

Next, using $\mathbf{k}_2$ from Example 3.8, we determine the member local end forces to be

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 = \begin{bmatrix} 906.25 & 0 & -906.25 & 0 \\ 0 & 0 & 0 & 0 \\ -906.25 & 0 & 906.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.14431 \\ -0.33333 \end{bmatrix} = \begin{bmatrix} 130.78 \\ 0 \\ -130.78 \\ 0 \end{bmatrix} \text{k}$$

$$Q_{a2} = 130.78 \text{ k (C)}$$ **Ans**

$$\mathbf{F}_2 = \mathbf{T}_2^T\mathbf{Q}_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 130.78 \\ 0 \\ -130.78 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 130.78 \\ \hdashline 0 \\ -130.78 \end{array}\right] \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \text{k}$$

Member 3

$$\mathbf{v}_3 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ d_{s8} \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.33333 \\ -0.14431 \end{bmatrix} \text{ in.}$$

Note that the support settlement $d_{s8} = -0.5$ in. is included in the foregoing global end displacement vector $\mathbf{v}_3$ for member 3. Next, using the member's direction cosines, $\cos\theta = -0.6$ and $\sin\theta = 0.8$, and Eq. (3.61), we evaluate its transformation matrix:

$$\mathbf{T}_3 = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix}$$

and determine the member local end displacements as

$$\mathbf{u}_3 = \mathbf{T}_3\mathbf{v}_3 = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ 0.33333 \\ -0.14431 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.3 \\ -0.31545 \\ -0.18008 \end{bmatrix} \text{ in.}$$

To obtain the member local stiffness matrix, we substitute $E = 29{,}000$ ksi, $A = 8$ in.^{2}, and $L = 240$ in. into Eq. (3.27):

$$\mathbf{k}_3 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.}$$

The member local end forces can now be computed as

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.4 \\ 0.3 \\ -0.31545 \\ -0.18008 \end{bmatrix} = \begin{bmatrix} -81.732 \\ 0 \\ 81.732 \\ 0 \end{bmatrix} \text{ k}$$

Thus,

$Q_{a3} = -81.732 \text{ k} = 81.732 \text{ k (T)}$ **Ans**

Finally, we calculate the member global end forces as

$$\mathbf{F}_3 = \mathbf{T}_3^T\mathbf{Q}_3 = \begin{bmatrix} -0.6 & -0.8 & 0 & 0 \\ 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & -0.8 \\ 0 & 0 & 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} -81.732 \\ 0 \\ 81.732 \\ 0 \end{bmatrix} = \begin{bmatrix} 49.039 \\ -65.386 \\ -49.039 \\ 65.386 \end{bmatrix} \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array}$$

and store the pertinent elements of $\mathbf{F}_3$ in the reaction vector $\mathbf{R}$, as shown in Fig. 7.8(c).

Support Reactions: The completed reaction vector $\mathbf{R}$ is shown in Fig. 7.8(c), and the support reactions are depicted on a line diagram of the truss in Fig. 7.8(d). **Ans**

Equilibrium Check: Applying the equilibrium equations to the free body of the entire structure (Fig. 7.8(d)), we write

$$+ \rightarrow \sum F_X = 0 \qquad -49.039 + 49.039 = 0 \qquad \textbf{Checks}$$

$$+ \uparrow \sum F_Y = 0 \qquad -65.386 + 130.78 - 65.386 = 0.008 \approx 0 \qquad \textbf{Checks}$$

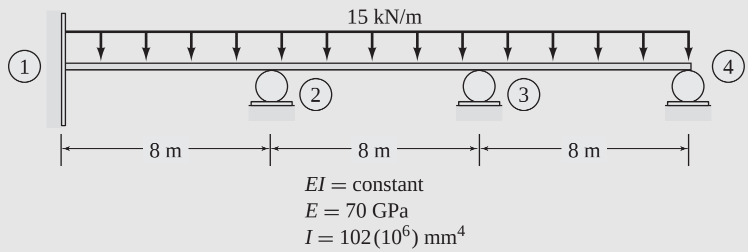
$$+ \circlearrowleft \sum M_{②} = 0 \qquad 130.78(12) - 65.386(24) = 0.096 \text{ k-ft} \approx 0 \qquad \textbf{Checks}$$

EXAMPLE 7.3

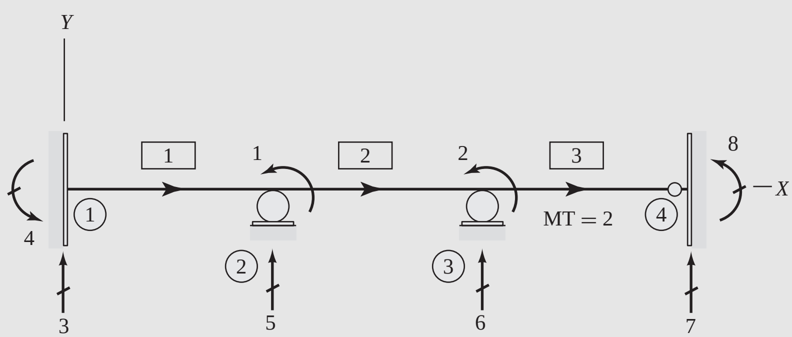
Determine the joint displacements, member end forces, and support reactions for the continuous beam shown in Fig. 7.9(a), due to the combined effect of the uniformly distributed load shown and the settlements of 45 mm and 15 mm, respectively, of supports 3 and 4. Use the matrix stiffness method.

SOLUTION

Analytical Model: See Fig. 7.9(b). The structure has two degrees of freedom and six restrained coordinates. Note that member 3 is modeled as being hinged at its right end



(a) Beam



(b) Analytical Model

$$\mathbf{S} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 3{,}570 + 3{,}570 & 1{,}785 \\ 1{,}785 & 3{,}570 + 2{,}677.5 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 7{,}140 & 1{,}785 \\ 1{,}785 & 6{,}247.5 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

(c) Structure Stiffness Matrix

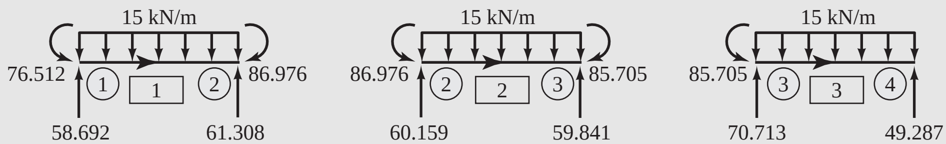
$$\mathbf{P}_f = \left[\begin{array}{c} -80 + 80 \\ -80 + 120 \end{array}\right] \begin{array}{c} 1 \\ 2 \end{array} = \left[\begin{array}{c} 0 \\ 40 \end{array}\right] \begin{array}{c} 1 \\ 2 \end{array}$$

(d) Structure Fixed-Joint Force Vector Due to Member Loads

Fig. 7.9

$$\mathbf{P}_f = \begin{bmatrix} 0 + 30.122 \\ 40 + 30.122 - 10.041 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} = \begin{bmatrix} 30.122 \\ 60.081 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

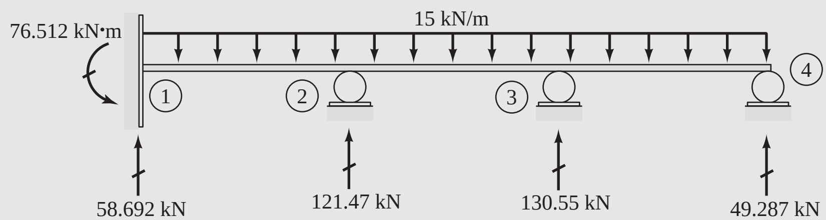
(e) Structure Fixed-Joint Force Vector Due to Member Loads and Support Displacements



(f) Member End Forces

$$\mathbf{R} = \begin{bmatrix} 58.692 \\ 76.512 \\ 61.308 + 60.159 \\ 59.841 + 70.713 \\ 49.287 \\ 0 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} = \begin{bmatrix} 58.692 \text{ kN} \\ 76.512 \text{ kN·m} \\ 121.47 \text{ kN} \\ 130.55 \text{ kN} \\ 49.287 \text{ kN} \\ 0 \end{bmatrix}$$

(g) Support Reaction Vector



(h) Support Reactions

Fig. 7.9 (*continued*)

(i.e., $MT = 2$), because the moment at that end of the member must be 0. This approach enables us to eliminate the rotational degrees of freedom of joint 4 from the analysis, by modeling it as a hinged joint with its rotation restrained by an imaginary clamp.

Structure Stiffness Matrix and Fixed-Joint Forces Due to Member Loads:

Members 1 and 2 ($MT = 0$) By substituting $E = 70(10^6)$ kN/m^2, $I = 102(10^{-6})$ m^4, and $L = 8$ m into Eq. (5.53), we evaluate the member stiffness matrices $\mathbf{k}$ as

$$\begin{array}{l} \text{Member 2} \longrightarrow \\ \text{Member 1} \longrightarrow \\ \mathbf{k}_1 = \mathbf{k}_2 = \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 2 \\ 3 & 4 & 5 & 1 \end{array} \\ \begin{bmatrix} 167.34 & 669.38 & -167.34 & 669.38 \\ 669.38 & 3{,}570 & -669.38 & 1{,}785 \\ -167.34 & -669.38 & 167.34 & -669.38 \\ 669.38 & 1{,}785 & -669.38 & 3{,}570 \end{bmatrix} \end{array} \begin{array}{cc} 3 & 5 \\ 4 & 1 \\ 5 & 6 \\ 1 & 2 \end{array} \qquad \textbf{(1)}$$

Using the equations given inside the front cover, we evaluate the fixed-end shears and moments due to the 15 kN/m uniformly distributed load as

$$FS_b = FS_e = 60 \text{ kN} \qquad FM_b = -FM_e = 80 \text{ kN} \cdot \text{m} \tag{2}$$

Thus, using Eq. (5.99), we obtain the member fixed-end force vectors:

$$\mathbf{Q}_{f1} = \mathbf{Q}_{f2} = \begin{array}{c} \left[\begin{array}{r} 60 \\ 80 \\ 60 \\ -80 \end{array}\right] \end{array} \begin{array}{c|c} 3 & 5 \\ 4 & 1 \\ 5 & 6 \\ 1 & 2 \\ \uparrow & \uparrow \\ \text{Member 1} & \text{Member 2} \end{array} \tag{3}$$

Next, using the code numbers for member 1 (3, 4, 5, 1) and member 2 (5, 1, 6, 2), we store the pertinent elements of $\mathbf{k}_1$ and $\mathbf{k}_2$ into the structure stiffness matrix $\mathbf{S}$, as shown in Fig. 7.9(c). Similarly, the pertinent elements of $\mathbf{Q}_{f1}$ and $\mathbf{Q}_{f2}$ are stored in the structure fixed-joint force vector $\mathbf{P}_f$, as shown in Fig. 7.9(d).

Member 3 (MT = 2) Because $MT = 2$ for this member, we use Eqs. (7.18) and (7.19) to determine its stiffness matrix $\mathbf{k}$ and fixed-end force vector $\mathbf{Q}_f$, respectively. Thus, by applying Eq. (7.18), we obtain

$$\mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 6 & 2 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc} 41.836 & 334.69 & -41.836 & 0 \\ 334.69 & 2{,}677.5 & -334.69 & 0 \\ -41.836 & -334.69 & 41.836 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 2 \\ 7 \\ 8 \end{array} \tag{4}$$

Next, by substituting the values of the fixed-end shears and moments from Eq. (2) into Eq. (7.19), we obtain the fixed-end force vector for the released member 3 as

$$\mathbf{Q}_{f3} = \left[\begin{array}{c} 60 - \dfrac{3(-80)}{2(8)} \\[2ex] 80 - \dfrac{1}{2}(-80) \\[2ex] 60 + \dfrac{3(-80)}{2(8)} \\[2ex] 0 \end{array}\right] = \left[\begin{array}{r} 75 \\[2ex] 120 \\[2ex] 45 \\[2ex] 0 \end{array}\right] \begin{array}{c} 6 \\[2ex] 2 \\[2ex] 7 \\[2ex] 8 \end{array} \tag{5}$$

The relevant elements of $\mathbf{k}_3$ and $\mathbf{Q}_{f3}$ are stored in $\mathbf{S}$ and $\mathbf{P}_f$, respectively, using the member code numbers 6, 2, 7, 8. The completed structure stiffness matrix $\mathbf{S}$, and the $\mathbf{P}_f$ vector containing the structure fixed-joint forces due to member loads, are shown in Figs. 7.9(c) and (d), respectively.

Structure Fixed-Joint Forces Due to Support Displacements: From the analytical model given in Fig. 7.9(b), we observe that the given support displacements can be expressed as

$$d_{s6} = -0.045 \text{ m} \qquad d_{s7} = -0.015 \text{ m}$$

As members 2 and 3 are attached to the supports that undergo displacements, we compute, for these members, the fixed-end forces due to support displacements, and add them to the previously formed $\mathbf{P}_f$ vector due to member loads.

Member 2 Using the member code numbers 5, 1, 6, 2, we form its fixed-end displacement vector due to support displacements, as

$$\mathbf{u}_{fs2} = \begin{array}{c} \begin{bmatrix} u_{fs1} \\ u_{fs2} \\ u_{fs3} \\ u_{fs4} \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_{s6} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.045 \\ 0 \end{bmatrix} \text{m}$$

Next, using the member stiffness matrix from Eq. (1) and the member stiffness relationship $\mathbf{Q}_{fs} = \mathbf{k}\mathbf{u}_{fs}$, we evaluate the fixed-end force vector due to support displacements, as

$$\mathbf{Q}_{fs2} = \mathbf{k}_2\mathbf{u}_{fs2} = \begin{bmatrix} 167.34 & 669.38 & -167.34 & 669.38 \\ 669.38 & 3{,}570 & -669.38 & 1{,}785 \\ -167.34 & -669.38 & 167.34 & -669.38 \\ 669.38 & 1{,}785 & -669.38 & 3{,}570 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.045 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7.53 \\ 30.122 \\ -7.53 \\ 30.122 \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array}$$

The relevant elements of $\mathbf{Q}_{fs2}$ are now added into the previously formed $\mathbf{P}_f$, using the member code numbers, as indicated in Fig. 7.9(e).

Member 3 Based on the member code numbers 6, 2, 7, 8, its fixed-end displacement vector, due to support displacements, is written as

$$\mathbf{u}_{fs3} = \begin{bmatrix} u_{fs1} \\ u_{fs2} \\ u_{fs3} \\ u_{fs4} \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array} = \begin{bmatrix} d_{s6} \\ 0 \\ d_{s7} \\ 0 \end{bmatrix} = \begin{bmatrix} -0.045 \\ 0 \\ -0.015 \\ 0 \end{bmatrix} \text{m}$$

Using $\mathbf{k}_3$ from Eq. (4), we calculate

$$\mathbf{Q}_{fs3} = \mathbf{k}_3\mathbf{u}_{fs3} = \begin{bmatrix} 41.836 & 334.69 & -41.836 & 0 \\ 334.69 & 2{,}677.5 & -334.69 & 0 \\ -41.836 & -334.69 & 41.836 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.045 \\ 0 \\ -0.015 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.2551 \\ -10.041 \\ 1.2551 \\ 0 \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array}$$

The pertinent elements of $\mathbf{Q}_{fs3}$ are stored in $\mathbf{P}_f$ using the member code numbers. The completed structure fixed-joint force vector $\mathbf{P}_f$, due to member loads and support displacements, is given in Fig. 7.9(e).

Joint Load Vector: Since no external loads are applied to the joints of the beam, its joint load vector is 0; that is

$$\mathbf{P} = \mathbf{0}$$

Joint Displacements: By substituting the numerical values of **P**, $\mathbf{P}_f$, and **S** into Eq. (5.109), we write the stiffness relations for the entire beam as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 30.122 \\ 60.081 \end{bmatrix} = \begin{bmatrix} -30.122 \\ -60.081 \end{bmatrix} = \begin{bmatrix} 7{,}140 & 1{,}785 \\ 1{,}785 & 6{,}247.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving these equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{array}{c} \begin{bmatrix} -1.9541 \\ -9.0585 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \times 10^{-3} \text{ rad}$$ **Ans**

Member End Displacements and End Forces:

Member 1 Using the member code numbers 3, 4, 5, 1, we write its end displacement vector as

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1.9541 \end{bmatrix} \times 10^{-3} \tag{6}$$

The member end forces can now be calculated using the member stiffness relationship $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$ (Eq. (5.4)). Thus, using $\mathbf{k}_1$ from Eq. (1), $\mathbf{Q}_{f1}$ from Eq. (3), and $\mathbf{u}_1$ from Eq. (6), we calculate

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 58.692 \text{ kN} \\ 76.512 \text{ kN}\cdot\text{m} \\ 61.308 \text{ kN} \\ \hdashline -86.976 \text{ kN}\cdot\text{m} \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array}$$ **Ans**

The end forces for member 1 are depicted in Fig. 7.9(f). To generate the support reaction vector **R**, we store the pertinent elements of $\mathbf{Q}_1$ in **R**, using the member code numbers, as shown in Fig. 7.9(g).

Member 2

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} = \begin{bmatrix} 0 \\ d_1 \\ d_{s6} \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.9541 \\ -45 \\ -9.0585 \end{bmatrix} \times 10^{-3} \tag{7}$$

Note that the support displacement d_{s6} is included in the foregoing end displacement vector for member 2. Using $\mathbf{k}_2$ from Eq. (1), $\mathbf{Q}_{f2}$ from Eq. (3), and $\mathbf{u}_2$ from Eq. (7), we determine

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} 60.159 \text{ kN} \\ \hdashline 86.976 \text{ kN}\cdot\text{m} \\ \hdashline 59.841 \text{ kN} \\ \hdashline -85.705 \text{ kN}\cdot\text{m} \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array}$$ **Ans**

Member 3

$$\mathbf{u}_3 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array} = \begin{bmatrix} d_{s6} \\ d_2 \\ d_{s7} \\ 0 \end{bmatrix} = \begin{bmatrix} -45 \\ -9.0585 \\ -15 \\ 0 \end{bmatrix} \times 10^{-3} \tag{8}$$

The rotation, u_4, of the released end of this member, if desired, can be evaluated using Eq. (7.20). Finally, using $\mathbf{k}_3$ from Eq. (4), $\mathbf{Q}_{f3}$ from Eq. (5), and $\mathbf{u}_3$ from Eq. (8), we calculate the member end forces as

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 + \mathbf{Q}_{f3} = \begin{array}{c}\left[\begin{array}{c} 70.713 \text{ kN} \\ \hdashline 85.705 \text{ kN} \cdot \text{m} \\ \hdashline 49.287 \text{ kN} \\ 0 \end{array}\right]\end{array}\begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array}$$ **Ans**

The member end forces are shown in Fig. 7.9(f).

Support Reactions: The completed reaction vector **R** is shown in Fig. 7.9(g), and the support reactions are depicted on a line diagram of the beam in Fig. 7.9(h). **Ans**

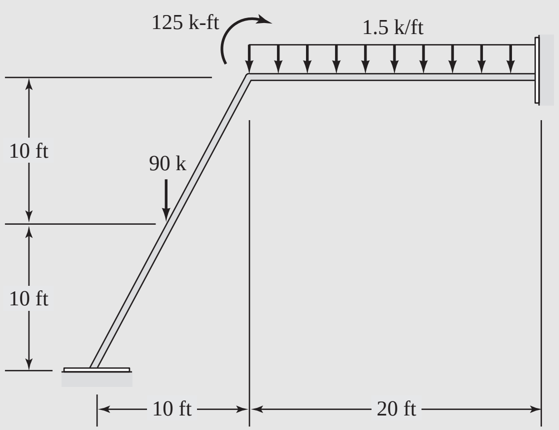
EXAMPLE 7.4

Determine the joint displacements, member local end forces, and support reactions for the plane frame of Fig. 7.10(a), due to the combined effect of the loading shown and a settlement of 1 in. of the left support. Use the matrix stiffness method.

SOLUTION

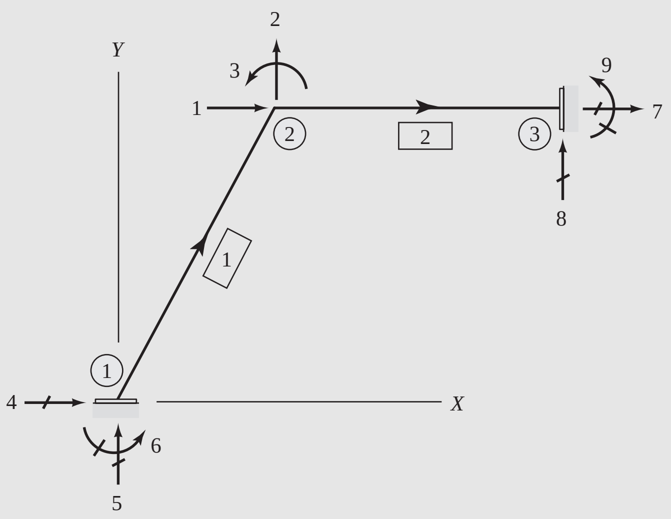
This frame was analyzed in Example 6.6 for external loading. In this example, we use the same analytical model of the frame, so that the various member and structure matrices calculated previously can be reused in the present example.

Analytical Model: See Fig. 7.10(b). The frame has three degrees of freedom and six restrained coordinates.



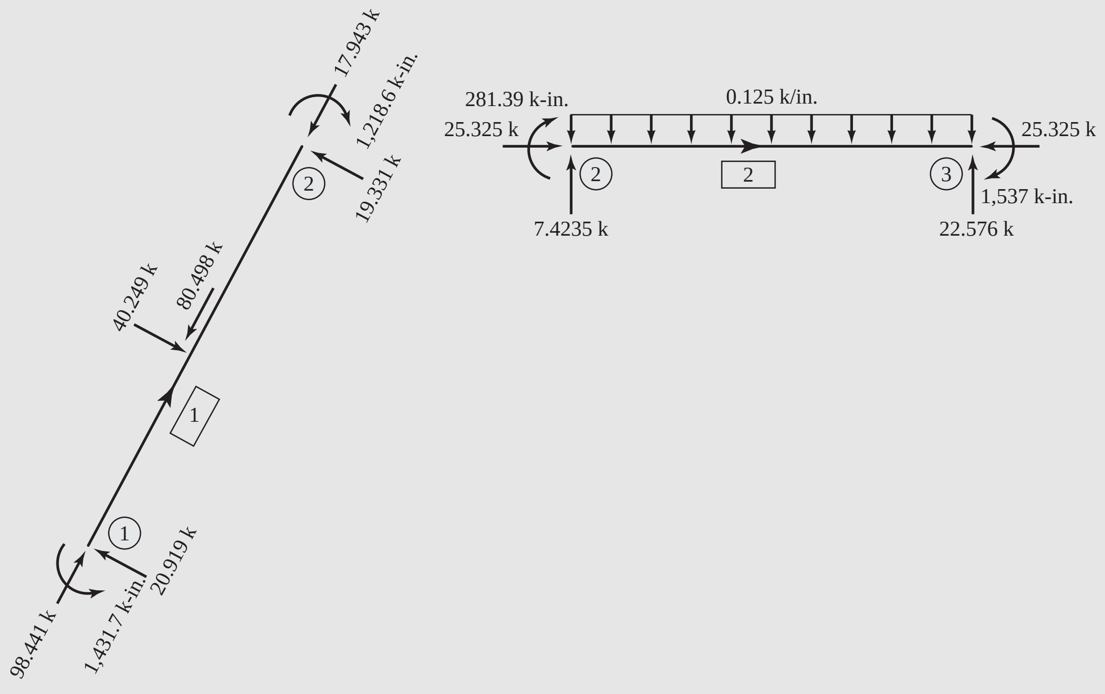
E, A, I = constant
E = 29,000 ksi
A = 11.8 in.^{2}
I = 310 in.^{4}

(a) Frame



(b) Analytical Model

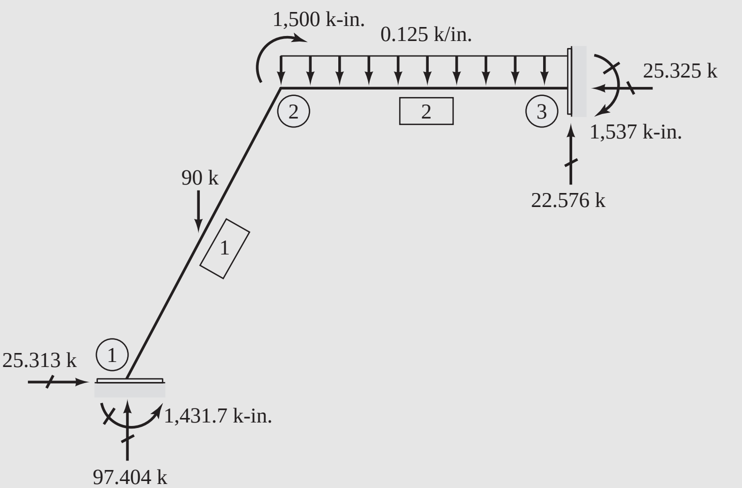
Fig. 7.10



(c) Member Local End Forces

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} 25.313 \text{ k} \\ 97.404 \text{ k} \\ 1{,}431.7 \text{ k-in.} \\ -25.325 \text{ k} \\ 22.576 \text{ k} \\ -1{,}537 \text{ k-in.} \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

(d) Support Reaction Vector



(e) Support Reactions

Fig. 7.10 (*continued*)

Structure Stiffness Matrix: As determined in Example 6.6, the structure stiffness matrix for the frame, in units of kips and inches, is given by

$$\mathbf{S} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 2{,}838.48 \end{bmatrix} \tag{1}$$

Structure Fixed-Joint Forces Due to Member Loads: From Example 6.6,

$$\mathbf{P}_f = \begin{array}{c} \begin{bmatrix} 0 \\ 60 \\ -750 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \tag{2}$$

Joint Load Vector: From Example 6.6,

$$\mathbf{P} = \begin{bmatrix} 0 \\ 0 \\ -1{,}500 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \tag{3}$$

Structure Fixed-Joint Forces Due to Support Displacement: From Fig. 7.10(b), we observe that the given 1 in. downward displacement of support 1 can be expressed as

$$d_{s5} = -1 \text{ in.}$$

As member 1 is the only member attached to support 1, we form its global fixed-end displacement vector due to support displacement, using the member code numbers 4, 5, 6, 1, 2, 3, as

$$\mathbf{v}_{fs1} = \begin{bmatrix} v_{fs1} \\ v_{fs2} \\ v_{fs3} \\ v_{fs4} \\ v_{fs5} \\ v_{fs6} \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 0 \\ d_{s5} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ in.}$$

Next, we substitute the member global stiffness matrix $\mathbf{K}_1$ (given in Example 6.6) and the foregoing $\mathbf{v}_{fs1}$ vector into Eq. (7.26), to evaluate the member global fixed-end force vector, $\mathbf{F}_{fs1}$, due to support settlement:

$$\mathbf{F}_{fs1} = \mathbf{K}_1 \mathbf{v}_{fs1} = \begin{bmatrix} -507.89 \\ -1{,}021.4 \\ -335.04 \\ \hdashline 507.89 \\ 1{,}021.4 \\ -335.04 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$

Based on the member code numbers, we add the elements in the fourth, fifth, and sixth rows of $\mathbf{F}_{fs1}$ into rows 1, 2, and 3, respectively, of the previously formed $\mathbf{P}_f$ vector (Eq. (2)), to obtain the structure fixed-joint force vector due to the combined effect of the member loads and support displacement, as

$$\mathbf{P}_f = \begin{bmatrix} 0 + 507.89 \\ 60 + 1{,}021.4 \\ -750 - 335.04 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 507.89 \\ 1{,}081.4 \\ -1{,}085.04 \end{bmatrix} \tag{4}$$

Joint Displacements: By substituting **P** (Eq. (3)), $\mathbf{P}_f$ (Eq. (4)), and **S** (Eq. (1)) into Eq. (6.42), we write the stiffness relations for the entire frame as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} 0 \\ 0 \\ -1{,}500 \end{bmatrix} - \begin{bmatrix} 507.89 \\ 1{,}081.4 \\ -1{,}085.04 \end{bmatrix} = \begin{bmatrix} -507.89 \\ -1{,}081.4 \\ -414.96 \end{bmatrix} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Solving these equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{array}{c} \begin{bmatrix} 0.017762 \text{ in.} \\ -1.0599 \text{ in.} \\ 0.00074192 \text{ rad} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$ **Ans**

Member End Displacements and End Forces:

Member 1

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 0 \\ d_{s5} \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.017762 \\ -1.0599 \\ 0.00074192 \end{bmatrix}$$

Using the member transformation matrix $\mathbf{T}_1$ from Example 6.6, and Eq. (6.20), we calculate

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} -0.89443 \\ -0.44721 \\ 0 \\ -0.94006 \\ -0.48988 \\ 0.00074192 \end{bmatrix}$$

Next, we use the member local stiffness matrix $\mathbf{k}_1$ and fixed-end force vector $\mathbf{Q}_{f1}$ from Example 6.6, and Eq. (6.4), to compute the local end forces as

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 98.441 \text{ k} \\ 20.919 \text{ k} \\ 1{,}431.7 \text{ k-in.} \\ -17.943 \text{ k} \\ 19.331 \text{ k} \\ -1{,}218.6 \text{ k-in.} \end{bmatrix}$$ **Ans**

The local member end forces are depicted in Fig. 7.10(c).

The member global end forces **F** can now be determined by applying Eq. (6.23), as

$$\mathbf{F}_1 = \mathbf{T}^T\mathbf{Q}_1 = \begin{bmatrix} 25.313 \\ 97.404 \\ 1{,}431.7 \\ -25.314 \\ -7.404 \\ -1{,}218.6 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$

Using the member code numbers, the pertinent elements of $\mathbf{F}_1$ are stored in the reaction vector $\mathbf{R}$ (see Fig. 7.10(d)).

Member 2 The global and local end displacements for this horizontal member are

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{array}{c}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}\end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.017762 \\ -1.0599 \\ 0.00074192 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using $\mathbf{k}_2$ and $\mathbf{Q}_{f2}$ from Example 6.6, we compute the member local and global end forces to be

$$\mathbf{F}_2 = \mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{array}{c}\begin{bmatrix} 25.325 \text{ k} \\ 7.4235 \text{ k} \\ -281.39 \text{ k-in.} \\ \hdashline -25.325 \text{ k} \\ 22.576 \text{ k} \\ -1{,}537 \text{ k-in.} \end{bmatrix}\end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \qquad \textbf{Ans}$$

The pertinent elements of $\mathbf{F}_2$ are stored in $\mathbf{R}$.

Support Reactions: See Figs. 7.10(d) and (e). **Ans**

7.4 COMPUTER IMPLEMENTATION OF SUPPORT DISPLACEMENT EFFECTS

The computer programs developed previously can be extended with relative ease, and without changing their overall organization, to include the effects of support displacements in the analysis. From the analysis procedure developed in Section 7.3, we realize that inclusion of support displacement effects essentially involves extension of the existing programs to perform three additional tasks: (a) reading and storing of the support displacement data, (b) evaluation of the structure fixed-joint forces due to support displacements, and (c) inclusion of support displacements in the member end displacement vectors, before calculation of the final member end forces and support reactions.

In this section, we consider the programming of these tasks, with particular reference to the program for the analysis of plane frames (Section 6.7). The modifications necessary in the plane truss and beam analysis programs are also described.

Input of Support Displacement Data The process of reading and storing the support displacements is similar to that for inputting the joint load data (e.g., see flowcharts in Figs. 4.3(f) and 5.20(b)). This process can be conveniently programmed using the flowchart given in Fig. 7.11 on the next page. The support displacement data consists of (a) the number of supports that undergo displacements (NSD), and (b) the joint number, and the magnitudes of the displacements, for each such support. As indicated in Fig. 7.11, the joint numbers of the

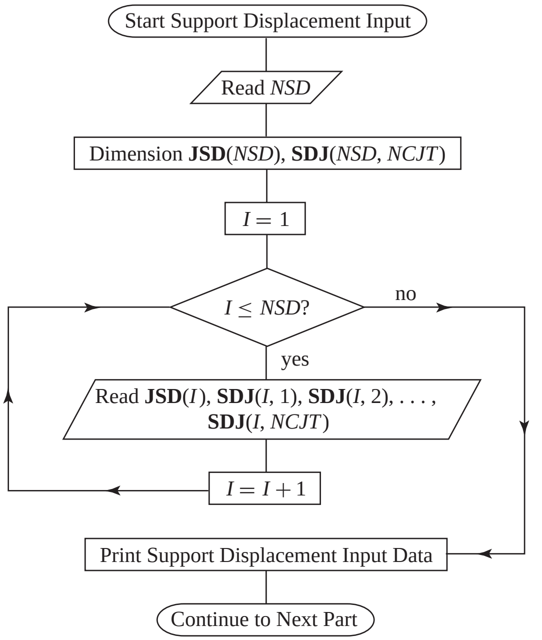


Fig. 7.11 *Flowchart for Reading and Storing Support Displacement Data*

supports that undergo displacements are stored in an integer vector **JSD** of order $NSD \times 1$, with their displacements stored in the corresponding rows of a real matrix **SDJ** of order $NSD \times NCJT$. For example, in the case of plane frames, the support displacement matrix **SDJ** would be of order $NSD \times 3$, with the support translations in the X and Y directions and the rotations being stored in the first, second, and third columns, respectively, of the matrix **SDJ**.

This subprogram for inputting support displacement data can be conveniently added as Part VIc in the computer programs for the analysis of plane frames (see Table 6.1) and beams (see Table 5.1); and it can be inserted between Parts VI and VII of the plane truss computer program (see Table 4.1).

Evaluation of Structure Equivalent Joint Loads Due to Support Displacements In this part of the program, the equivalent joint loads, or the negatives of the structure fixed-joint forces (i.e., $-\mathbf{P}_f$) due to support displacements, are added to the structure load vector **P**. A flowchart for constructing this part of the plane frame analysis program is presented in Fig. 7.12. As this flowchart indicates, the program essentially performs the following operations for each member of the structure.

1. First, the program determines whether the member under consideration, IM, is attached to a support that undergoes displacement, by comparing the member beginning and end joint numbers to those stored in the support displacement vector **JSD**. If the member is not attached to such a support, then no further action is taken for that member.

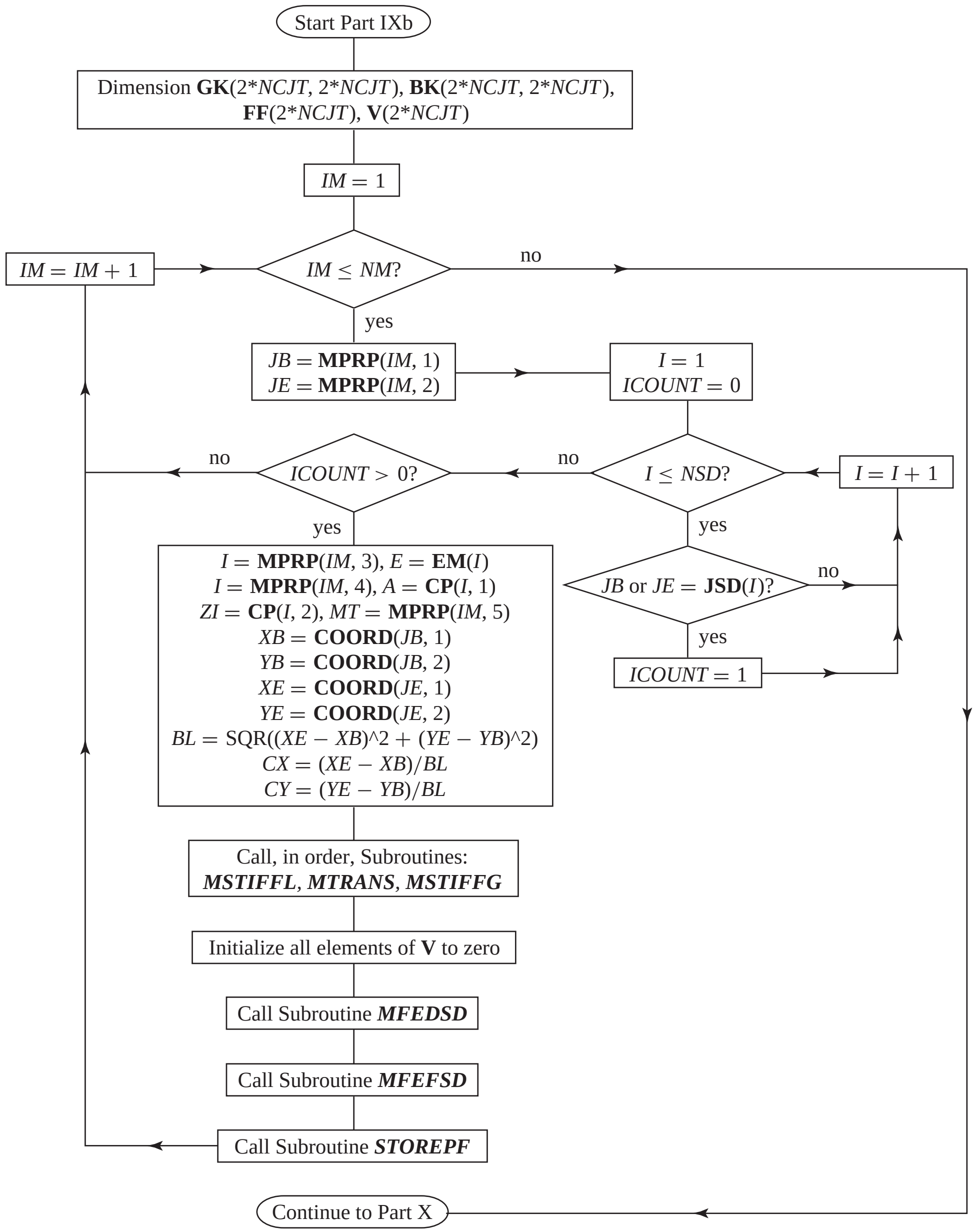


Fig. 7.12 *Flowchart for Generating Structure Equivalent Joint Load Vector Due to Support Displacements*

2. If the member is attached to a support that undergoes displacement(s), then its global stiffness matrix **GK** (= **K**) is obtained by calling, in order, the subroutines ***MSTIFFL*** (Fig. 7.4), ***MTRANS*** (Fig. 6.26), and ***MSTIFFG*** (Fig. 6.27).

3. Next, the program calls the subroutine ***MFEDSD*** to form the member global fixed-end displacement vector, **V** (= $\mathbf{v}_{fs}$), due to support displacements.

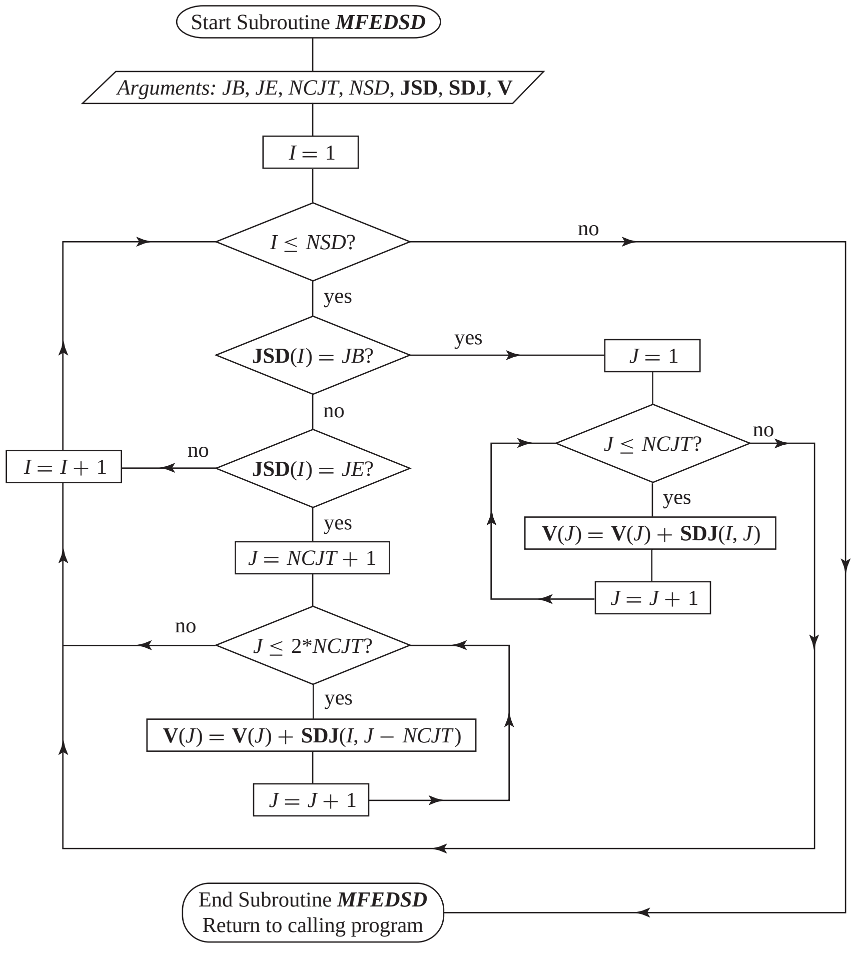


Fig. 7.13 *Flowchart of Subroutine* ***MFEDSD*** *for Determining Member Global Fixed-End Displacement Vector Due to Support Displacements*

As the flowchart in Fig. 7.13 indicates, this subroutine first checks the support displacement vector **JSD** to determine if the beginning joint of the member, *JB*, is a support joint subjected to displacements. If *JB* is such a joint, then the values of its displacements are read from the corresponding row of the support displacement matrix **SDJ**, and stored in the appropriate elements of the upper half of the member fixed-end displacement vector **V**. The process is then repeated for the end joint of the member, *JE*, with any corresponding support displacements being stored in the lower half of **V**.

4. Returning our attention to Fig. 7.12, we can see that the program then calls the subroutine ***MFEFSD*** (Fig. 7.14), which evaluates the member global fixed-end force vector due to support settlements **FF** ($= \mathbf{F}_{fs}$), using the relationship $\mathbf{F}_{fs} = \mathbf{K}\mathbf{v}_{fs}$ (Eq. (7.26)).

5. Finally, the negative values of the pertinent element of **FF** are added in their proper positions in the structural load vector **P**, using the subroutine

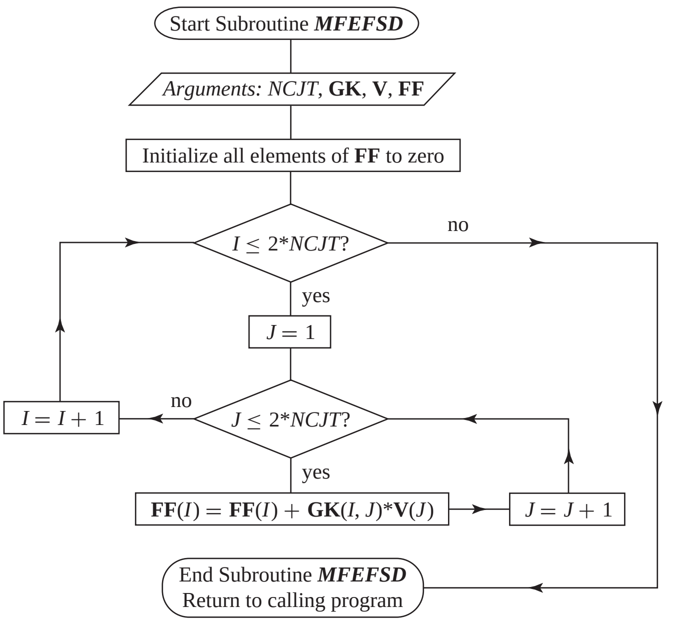


Fig. 7.14 *Flowchart of Subroutine **MFEFSD** for Determining Member Global Fixed-End Force Vector Due to Support Displacements*

STOREPF, which was developed in Chapter 6 (Fig. 6.30). When these operations have been completed for each member of the frame, the structure load vector **P** contains the equivalent joint loads (or the negatives of the structure fixed-joint forces) due to support displacements.

This subprogram, designated Part IXb in Fig. 7.12, can be conveniently inserted between Parts IX and X of the program for the analysis of plane frames (see Table 6.1). The flowcharts given in Figs. 7.12 through 7.14 can be used to develop the corresponding part of the beam analysis program (see Table 5.1), provided that: (a) the member global vectors **V** ($= \mathbf{v}_{fs}$) and **FF** ($= \mathbf{F}_{fs}$) are replaced by the local vectors **U** ($= \mathbf{u}_{fs}$) and **QF** ($= \mathbf{Q}_{fs}$), respectively; (b) the member local stiffness matrix **BK** ($= \mathbf{k}$) is used, instead of the global matrix **GK** ($= \mathbf{K}$), in subroutine ***MFEFSD***; and (c) the subroutine ***STOREPF*** developed in Chapter 5 (Fig. 5.28) is employed to store the negative elements of **QF** in the structure load vector **P**. The process of programming the corresponding part of the plane truss analysis program is essentially the same as discussed herein for the case of plane frames, except that the subroutine ***STOREPF*** (Fig. 6.30) should be copied from the plane frame analysis program and added to the plane truss program.

Calculation of Member Forces and Support Reactions (Part XII) Parts XII of the programs developed previously (see flowcharts in Figs. 6.31, 5.30, and 4.14) should be modified to include support displacements in the end displacement vectors of members attached to supports, before the end forces for such members are calculated. This can be achieved by simply calling the subroutine ***MFEDSD*** (Fig. 7.13) in these programs, to add the compatible support displacements to the member end displacement vectors. In the plane frame and truss analysis programs (Figs. 6.31 and 4.14, respectively), the subroutine ***MFEDSD*** should be called *after* the subroutine ***MDISPG*** has been used to form the member global end displacement vector $\mathbf{V}$ $(= \mathbf{v})$ due to the joint displacements $\mathbf{d}$, but *before* the subroutine ***MDISPL*** is called to evaluate the member local end displacement vector $\mathbf{U}$ $(= \mathbf{u})$. In the program for the analysis of beams (Fig. 5.30), however, the subroutine ***MFEDSD*** should be called *after* the subroutine ***MDISPL*** has been used to form the member end displacement vector $\mathbf{U}$ $(= \mathbf{u})$ from the joint displacements $\mathbf{d}$, but *before* the member end forces are calculated using the subroutine ***MFORCEL***. Furthermore, as discussed previously, before it can be used in the beam analysis program, the subroutine ***MFEDSD*** (as given in Fig. 7.13) must be modified to replace $\mathbf{V}$ with $\mathbf{U}$.

It may be of interest to note that the program for the analysis of plane frames, which was initially developed in Chapter 6 and has been extended in this chapter, is quite general, in the sense that it can also be used to analyze beams and plane trusses. When analyzing a truss using the frame analysis program, all of the truss members are modeled as hinged at both ends with $MT = 3$, and all the joints of the truss are modeled as hinged joints restrained against rotations by imaginary clamps.

7.5 TEMPERATURE CHANGES AND FABRICATION ERRORS

Like support displacements, changes in temperature and small fabrication errors can cause considerable stresses in statically indeterminate structures, which must be taken into account in their designs. However, unlike support displacements, which are generally specified with reference to the global coordinate systems of structures, temperature changes and fabrication errors, like member loads, are usually defined relative to the local coordinate systems of members. Therefore, the stiffness methods developed previously for the analysis of structures subjected to member loads, can be used without modifications to determine the structural responses to temperature changes and fabrication errors. The only difference is that the fixed-end forces, which develop in members due to temperature changes and fabrication errors, must now be included in the member local fixed-end force vectors $\mathbf{QF}$.

In this section, we derive the expressions for the fixed-end forces that develop in the members of framed structures due to temperature changes and two common types of fabrication errors. The application of these fixed-end force expressions in analysis is then illustrated by some examples.

Member Fixed-End Forces Due to Temperature Changes

We can develop the desired relationships by first determining the displacements caused by temperature changes at the ends of members that are free to

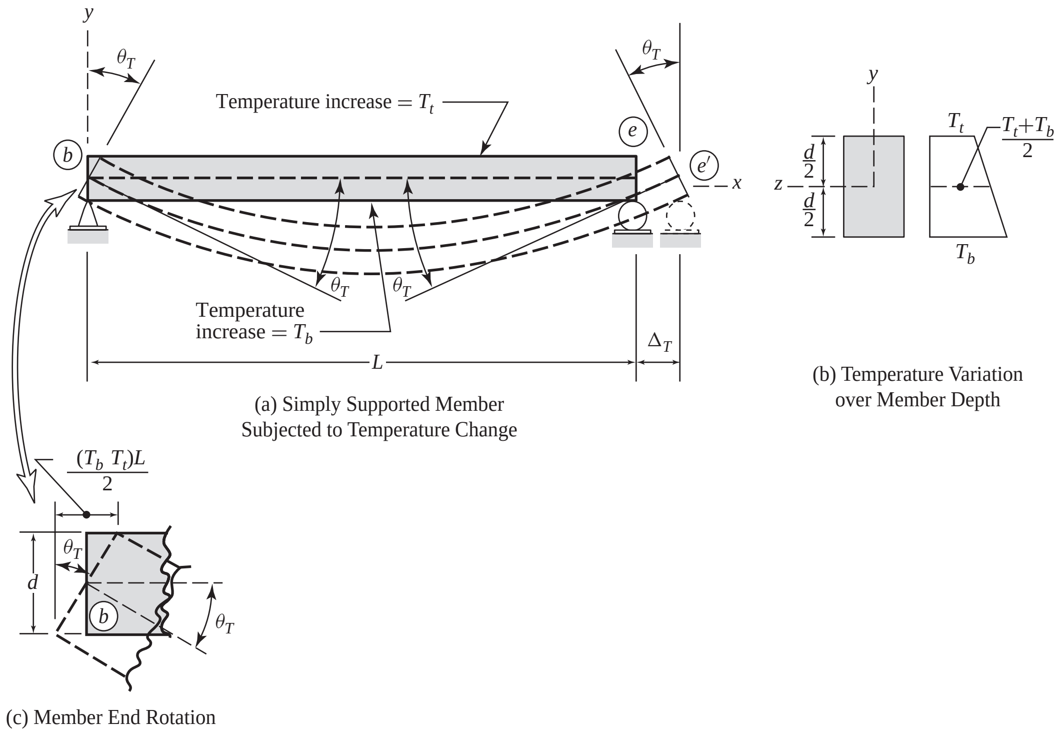


Fig. 7.15

deform. The fixed-end forces required to suppress these member end displacements can then be obtained, using the member stiffness matrices.

To examine member end displacements due to temperature changes, let us consider an arbitrary simply supported member of a plane frame, as shown in Fig. 7.15(a). Now, assume that the member is heated so that the temperature increase of its top surface is T_t and that of its bottom surface is T_b, with the temperature increase varying linearly between T_t and T_b over the depth d of the member cross-section, as shown in Fig. 7.15(b). Note that the temperature does not vary along the length of the member. Because the member is simply supported (so that it is statically determinate), it is free to expand in the longitudinal direction. If we assume that the member cross-section is symmetric about the *xz* plane (Fig. 7.15(b)) containing its centroidal axis, then the temperature increase at the level of the centroidal axis (i.e., at the distance $d/2$ from the top or bottom of the member) would be $(T_b + T_t)/2$. This temperature increase causes the member's centroidal axis to elongate by an amount Δ_T:

$$\Delta_T = \alpha \left(\frac{T_b + T_t}{2} \right) L \qquad \textbf{(7.29)}$$

in which α denotes the coefficient of thermal expansion.

In addition to the axial deformation Δ_T, the member also undergoes bending as its top and bottom surfaces elongate by different amounts (because they are subjected to different temperature increases). For example, as depicted in Fig. 7.15, if $T_b > T_t$, then the member bends concave upward, causing the cross-sections at its ends b and e to rotate inward, as shown. Since the temperature increase is uniform along the member's length, the rotations of its two end cross-sections must be equal in magnitude. From Fig. 7.15, we can see that these member end rotations can be related to the temperature change by dividing one-half of the difference between the elongations of the bottom and top fibers of the member, by its depth. Thus,

$$\theta_T = \frac{\alpha(T_b - T_t)L}{2d} \tag{7.30}$$

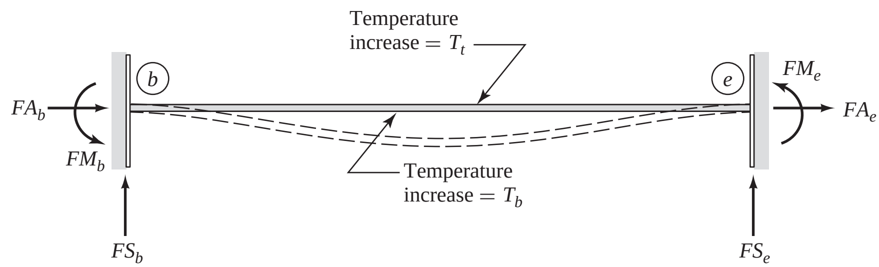
in which θ_T represents the magnitude of the rotations of the member end cross-sections which, in turn, equal the slopes of the elastic curve of the member at its ends, as shown in Figs. 7.15(a) and (c).

Using the sign convention for member local end displacements established in Chapter 6, we can express the local end displacement vector $\mathbf{u}_T$ for the simply supported member, due to the temperature change, as

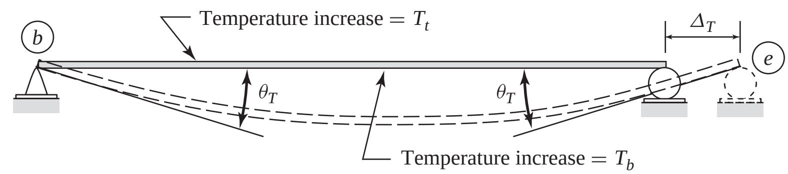
$$\mathbf{u}_T = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ -\theta_T \\ \Delta_T \\ 0 \\ \theta_T \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} = \frac{\alpha L}{2} \begin{bmatrix} 0 \\ 0 \\ -(T_b - T_t)/d \\ T_b + T_t \\ 0 \\ (T_b - T_t)/d \end{bmatrix} \tag{7.31}$$

in which the rotation of the beginning, b, of the member is negative, because it is clockwise, whereas the rotation of the member end, e, which has a counterclockwise sense, is positive.

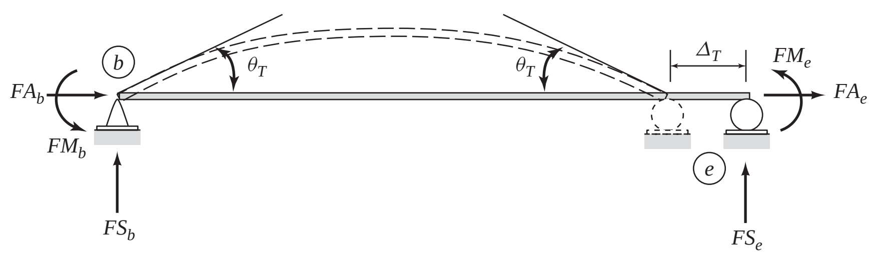
The member fixed-end forces necessary to suppress its end displacements $\mathbf{u}_T$ can now be established by applying the principle of superposition, as illustrated in Fig. 7.16. Figure 7.16(a) shows a fixed member of a plane frame subjected to a temperature increase, causing fixed-end forces to develop at its ends. In Fig. 7.16(b), the corresponding simply supported member is subjected to the same temperature change, causing the displacements $\mathbf{u}_T$ (Eq. (7.31)) at its ends, but no end forces; in Fig. 7.16(c), the simply supported member is subjected to the same fixed-end forces that develop in the fixed member of Fig. 7.16(a), but no temperature change. By comparing Figs. 7.16(a) through (c), we realize that the response of the fixed member of Fig. 7.16(a) must equal the superposition of the responses of the two simply supported members of Figs. 7.16(b) and (c). Therefore, since the end displacements of the fixed member due to the temperature change are 0 (Fig. 7.16(a)), its fixed-end forces, when applied to the simply supported beam (Fig. 7.16(c)), must cause the end displacements, $-\mathbf{u}_T$, that are equal in magnitude but opposite in direction to those due to the temperature change, $\mathbf{u}_T$ (Fig. 7.16(b)). The forces that can cause the end displacements $-\mathbf{u}_T$ in the simply supported member can be conveniently obtained by premultiplying the negative of the $\mathbf{u}_T$ vector given in



(a) Fixed Member Subjected to Temperature Change and Fixed-End Forces (No End Displacements)



(b) Simply Supported Member Subjected to Temperature Change Only (End Displacements = $\mathbf{u}_T$)



(c) Simply Supported Member Subjected to Fixed-End Forces Only (End Displacements = $-\mathbf{u}_T$)

Fig. 7.16

Eq. (7.31), by the member local stiffness matrix **k** (Eq. (6.6)). Thus,

$$\begin{bmatrix} FA_b \\ FS_b \\ FM_b \\ FA_e \\ FS_e \\ FM_e \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \left(-\frac{\alpha L}{2}\right) \begin{bmatrix} 0 \\ 0 \\ -(T_b - T_t)/d \\ T_b + T_t \\ 0 \\ (T_b - T_t)/d \end{bmatrix}$$

From which we obtain

$$\begin{bmatrix} FA_b \\ FS_b \\ FM_b \\ FA_e \\ FS_e \\ FM_e \end{bmatrix} = E\alpha \begin{bmatrix} A(T_b + T_t)/2 \\ 0 \\ I(T_b - T_t)/d \\ -A(T_b + T_t)/2 \\ 0 \\ -I(T_b - T_t)/d \end{bmatrix} \tag{7.32}$$

Thus, the fixed-end forces for the members of plane frames can be expressed as

$$\boxed{\begin{aligned} FA_b &= -FA_e = EA\alpha\left(\frac{T_b + T_t}{2}\right) \\ FM_b &= -FM_e = EI\alpha\left(\frac{T_b - T_t}{d}\right) \end{aligned}} \tag{7.33}$$

The expressions for the fixed-end moments, given in Eqs. (7.33), can also be used for the members of beams. However, as the beam members are free to expand axially, their fixed-end axial forces are 0 (i.e., $FA_b = FA_e = 0$). Similarly, the expressions for the fixed-end axial forces, given in Eqs. (7.33), can be used for the members of trusses; however, the fixed-end moments must now be set equal to 0 (i.e., $FM_b = FM_e = 0$) in Eqs. (7.33), because the ends of truss members are free to rotate.

The fixed-end force expressions given in Eqs. (7.33) are based on a linearly varying temperature change over the depth of the member cross-section. If the member is subjected to a uniform temperature increase, T_u, over its depth, then the corresponding expressions for fixed-end forces can be obtained by simply substituting $T_b = T_t = T_u$ into Eqs. (7.33). This yields

$$\boxed{FA_b = -FA_e = EA\alpha T_u} \tag{7.34}$$

As Eqs. (7.34) indicate, the member fixed-end moments would be 0 in the case of a uniform temperature change, because such a temperature change has no tendency to bend the member, but only to cause axial deformation. Equations (7.34) can be used to determine the fixed-end forces for the members of plane frames and trusses subjected to uniform temperature changes. As stated previously, the members of beams are free to expand in their axial directions; therefore, a uniform temperature change does not cause any fixed-end forces in such members.

Member Fixed-End Forces Due to Fabrication Errors

In structural analysis terminology, *fabrication error* is used to refer to a small initial deformation of a member in its unstressed state. The expressions for

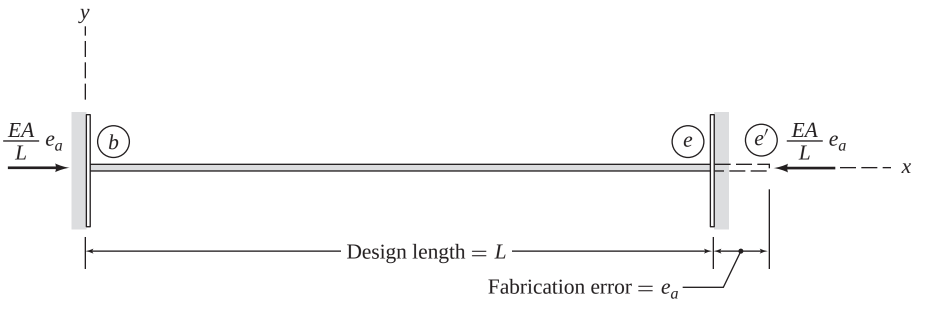


Fig. 7.17

member fixed-end forces due to fabrication errors can be derived in a manner similar to that for the case of temperature changes. In the following paragraphs, we develop the fixed-end force expressions for two common types of fabrication errors.

Errors in Initial Member Length Consider a member of a plane frame with a specified design length L. Now, suppose that the member is fabricated so that its initial unstressed length is longer than the specified length L by an amount e_a, as shown in Fig. 7.17. As the distance between the fixed supports is L, the supports must exert a compressive axial force of magnitude EAe_a/L on the member to reduce its length from $L + e_a$ to L, so that it can fit between the supports. Thus, the fixed-end forces that develop in the member due to its fabricated length being too long by an amount e_a are

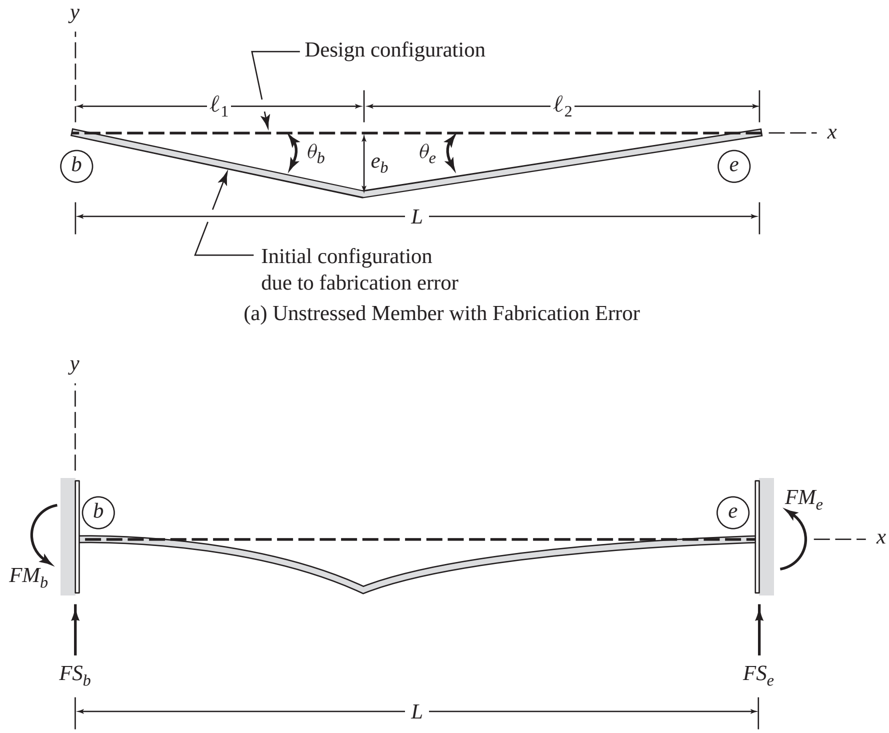
$$\boxed{FA_b = -FA_e = \frac{EA}{L}e_a} \tag{7.35}$$

Equations (7.35) can also be used to obtain fixed-end forces for the members of trusses due to fabrication errors in their lengths.

Errors in Initial Member Straightness Another type of fabrication error commonly encountered in structural design involves a lack of initial straightness of the members of beams and plane frames. Figure 7.18(a) on the next page shows such a member of a beam, which somehow has been fabricated with an initial bend, causing a small deflection e_b at a distance l_1 from the member's left end. To determine the fixed-end forces for this member, we first express the member end rotations θ_b and θ_e in terms of the fabrication error e_b, as (see Fig. 7.18(a))

$$\theta_b = \frac{e_b}{l_1} \qquad \text{and} \qquad \theta_e = \frac{e_b}{l_2} \tag{7.36}$$

Using the sign convention established for beam members in Chapter 5, we write the local end displacement vector $\mathbf{u}_e$ for the member, due to the fabrication



(a) Unstressed Member with Fabrication Error

(b) Fixed Member with Fabrication Error

Fig. 7.18

error, as

$$\mathbf{u}_e = \begin{array}{c}\left[\begin{array}{c} 0 \\ -\theta_b \\ 0 \\ \theta_e \end{array}\right]\end{array}\begin{array}{c}1\\2\\3\\4\end{array} = e_b \begin{bmatrix} 0 \\ -1/l_1 \\ 0 \\ 1/l_2 \end{bmatrix} \tag{7.37}$$

in which the rotation of the beginning, b, of the member is considered to be negative, because it has a clockwise sense. The member fixed-end forces (Fig. 7.18(b)) necessary to suppress the end displacements $\mathbf{u}_e$ can now be determined by premultiplying the negative of the $\mathbf{u}_e$ vector by the member local stiffness matrix $\mathbf{k}$ (Eq. (5.53)). Thus,

$$\begin{bmatrix} FS_b \\ FM_b \\ FS_e \\ FM_e \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} (-e_b) \begin{bmatrix} 0 \\ -1/l_1 \\ 0 \\ 1/l_2 \end{bmatrix}$$

$$= \frac{2EI\,e_b}{L^2 l_1 l_2} \begin{bmatrix} 3(l_2 - l_1) \\ L(2l_2 - l_1) \\ 3(l_1 - l_2) \\ L(l_2 - 2l_1) \end{bmatrix} \tag{7.38}$$

Therefore, the fixed-end forces for the members of beams are:

$$\boxed{\begin{aligned} FS_b &= -FS_e = \frac{6EI\,e_b}{L^2 l_1 l_2}(l_2 - l_1) \\ FM_b &= \frac{2EI\,e_b}{L l_1 l_2}(2l_2 - l_1) \\ FM_e &= \frac{2EI\,e_b}{L l_1 l_2}(l_2 - 2l_1) \end{aligned}} \tag{7.39}$$

As the fabrication error e_b is assumed to be small, it does not cause any axial deformation of the member; therefore, no axial force develops in the fixed member (i.e., $FA_b = FA_e = 0$). Thus, the expressions for the fixed-end forces given in Eq. (7.39) can also be used for the members of plane frames.

Procedure for Analysis

As stated at the beginning of this section, the procedures for the analysis of beams and plane frames, including the effects of temperature changes and fabrication errors, remain the same as developed in Chapters 5 and 6, respectively—provided that the member fixed-end forces caused by the temperature changes and fabrication errors are now included in the member local fixed-end force vectors $\mathbf{Q}_f$. In the case of plane trusses, however, the member and structure stiffness relationships must now be modified, to include the effects of temperature changes and fabrication errors, as follows: (a) the member local stiffness relationship given in Eq. (3.7) should be modified to $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$, (b) the member global stiffness relationship (Eq. (3.71)) now becomes $\mathbf{F} = \mathbf{Kv} + \mathbf{F}_f$, and (c) the structure stiffness relationship (Eq. (3.89)) should be updated to include the structure fixed-joint forces as $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$. The structure fixed-joint force vector $\mathbf{P}_f$ can be generated using the member code number technique as discussed in Chapter 6 for the case of plane frames.

EXAMPLE 7.5

Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. 7.19(a) on the next page, due to the combined effect of the following: (a) the joint loads shown in the figure, (b) a temperature drop of 30° F in member 1, and (c) the fabricated length of member 3 being $\frac{1}{8}$ in. too short. Use the matrix stiffness method.

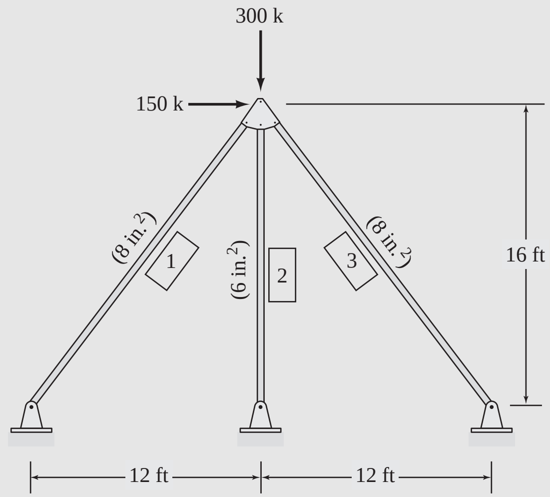
SOLUTION

This truss was analyzed in Example 3.8 for joint loads only, and in Example 7.2 for a support displacement.

Analytical Model: See Fig. 7.19(b). The analytical model used herein is the same as used in Examples 3.8 and 7.2.

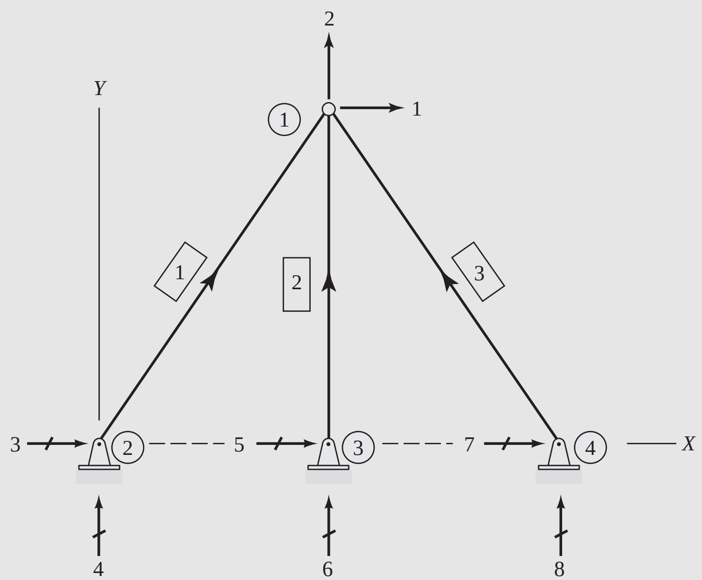
Structure Stiffness Matrix: From Example 3.8,

$$\mathbf{S} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \text{ k/in.} \tag{1}$$



$E = 29{,}000$ ksi
$\alpha = 6.5(10^{-6})/°\text{F}$

(a) Truss



(b) Analytical Model

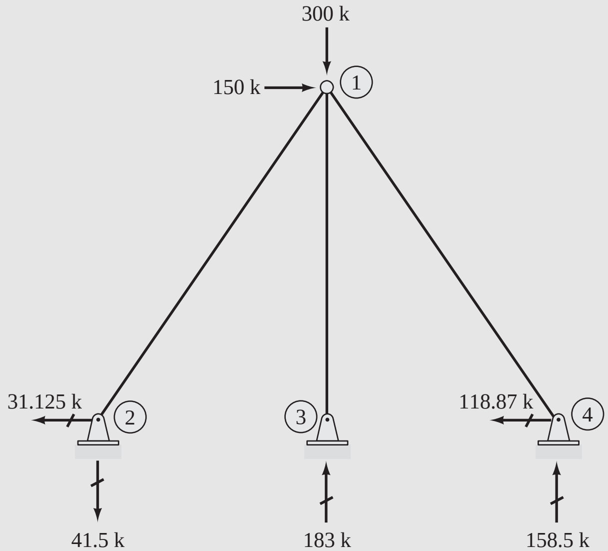
$$\mathbf{P}_f = \begin{array}{c} \begin{bmatrix} 27.144 - 72.5 \\ 36.192 + 96.667 \end{bmatrix} \end{array}\begin{array}{l} 1 \\ 2 \end{array} = \begin{bmatrix} -45.356 \\ 132.86 \end{bmatrix}\begin{array}{l} 1 \\ 2 \end{array} \text{ k}$$

(c) Structure Fixed-Joint Force Vector Due to Temperature Changes and Fabrication Errors

Fig. 7.19

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} -31.125 \\ -41.5 \\ 0 \\ 183 \\ -118.87 \\ 158.5 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \text{ k}$$

(d) Support Reaction Vector



(e) Support Reactions

Fig. 7.19 (*continued*)

Joint Load Vector: From Example 3.8,

$$\mathbf{P} = \begin{bmatrix} 150 \\ -300 \end{bmatrix} \text{ k} \tag{2}$$

Structure Fixed-Joint Force Vector Due to Temperature Changes and Fabrication Errors:

Member 1 By substituting $E = 29{,}000$ ksi, $A = 8$ in.^{2}, $\alpha = 6.5(10^{-6})/^\circ$ F, and $T_u = -30^\circ$ F into Eqs. (7.34), we evaluate the member fixed-end forces, due to the specified temperature change, as

$$FA_b = -FA_e = -45.24 \text{ k}$$

Thus, the local fixed-end force vector for member 1 can be expressed as

$$\mathbf{Q}_{f1} = \begin{bmatrix} FA_b \\ FS_b \\ FA_e \\ FS_e \end{bmatrix} = \begin{bmatrix} -45.24 \\ 0 \\ 45.24 \\ 0 \end{bmatrix} \text{ k} \tag{3}$$

Next, we obtain the global fixed-end force vector for this member by applying the transformation relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$, while using the transformation matrix $\mathbf{T}_1$ from

Example 3.8. Thus,

$$\mathbf{F}_{f1} = \mathbf{T}_1^T \mathbf{Q}_{f1} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -45.24 \\ 0 \\ 45.24 \\ 0 \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c} -27.144 \\ -36.192 \\ \hdashline 27.144 \\ 36.192 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{ k}$$

From the member code numbers 3, 4, 1, 2, which are written on the right side of $\mathbf{F}_{f1}$, we realize that the elements in the third and fourth rows of $\mathbf{F}_{f1}$ should be stored in rows 1 and 2, respectively, of the 2 × 1 structure fixed-joint force vector $\mathbf{P}_f$, as shown in Fig. 7.19(c).

Member 3 By substituting $e_a = -\frac{1}{8}$ in. into Eq. (7.35), we obtain

$$FA_b = -FA_e = \frac{29{,}000(8)}{20(12)}\left(-\frac{1}{8}\right) = -120.83 \text{ k}$$

Thus,

$$\mathbf{Q}_{f3} = \begin{bmatrix} FA_b \\ FS_b \\ FA_e \\ FS_e \end{bmatrix} = \begin{bmatrix} -120.83 \\ 0 \\ 120.83 \\ 0 \end{bmatrix} \text{ k} \qquad \textbf{(4)}$$

Using $\mathbf{T}_3$ from Example 7.2, we calculate

$$\mathbf{F}_{f3} = \mathbf{T}_3^T \mathbf{Q}_{f3} = \begin{bmatrix} -0.6 & -0.8 & 0 & 0 \\ 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & -0.8 \\ 0 & 0 & 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} -120.83 \\ 0 \\ 120.83 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 72.5 \\ -96.667 \\ \hdashline -72.5 \\ 96.667 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k}$$

The relevant elements of $\mathbf{F}_{f3}$ are stored in $\mathbf{P}_f$ using the member code numbers. The completed structure fixed-joint force vector $\mathbf{P}_f$, due to temperature change and fabrication error, is given in Fig. 7.19(c).

Joint Displacements: By substituting $\mathbf{P}$ (Eq. (2)), $\mathbf{P}_f$ (Fig. 7.19(c)), and $\mathbf{S}$ (Eq. (1)) into the structure stiffness relationship, we write

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} 150 \\ -300 \end{bmatrix} - \begin{bmatrix} -45.356 \\ 132.86 \end{bmatrix} = \begin{bmatrix} 195.36 \\ -432.86 \end{bmatrix} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Solving the foregoing equations,

$$\mathbf{d} = \begin{bmatrix} 0.28068 \\ -0.20193 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} \text{ in.} \qquad \textbf{Ans}$$

Member End Displacements and End Forces:

Member 1

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0.28068 \\ -0.20193 \end{bmatrix} \text{ in.}$$

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.006864 \\ -0.3457 \end{bmatrix} \text{in.}$$

Next, we calculate the member local end forces by applying the member stiffness relationship $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$. Thus, using $\mathbf{k}_1$ from Example 3.8, and $\mathbf{Q}_{f1}$ from Eq. (3), we obtain

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.006864 \\ -0.3457 \end{bmatrix} + \begin{bmatrix} -45.24 \\ 0 \\ 45.24 \\ 0 \end{bmatrix}$$

from which,

$$\mathbf{Q}_1 = \begin{bmatrix} -51.875 \\ 0 \\ 51.875 \\ 0 \end{bmatrix} \text{k}$$

Thus,

$$Q_{a1} = -51.875 \text{ k} = 51.875 \text{ k (T)}$$ **Ans**

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{array}{c} \begin{bmatrix} -31.125 \\ -41.5 \\ \hdashline 31.125 \\ 41.5 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{k}$$

The pertinent elements of $\mathbf{F}_1$ are stored in the support reaction vector $\mathbf{R}$ in Fig. 7.19(d).

Member 2

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0.28068 \\ -0.20193 \end{bmatrix} \text{in.}$$

Using $\mathbf{T}_2$ from Example 3.8, we obtain

$$\mathbf{u}_2 = \mathbf{T}_2\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.28068 \\ -0.20193 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.20193 \\ -0.28068 \end{bmatrix} \text{in.}$$

With $\mathbf{Q}_{f2} = \mathbf{0}$ and $\mathbf{k}_2$ obtained from Example 3.8, we determine the member local end forces to be

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 = \begin{bmatrix} 906.25 & 0 & -906.25 & 0 \\ 0 & 0 & 0 & 0 \\ -906.25 & 0 & 906.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.20193 \\ -0.28068 \end{bmatrix} = \begin{bmatrix} 183 \\ 0 \\ -183 \\ 0 \end{bmatrix} \text{k}$$

$$Q_{a2} = 183 \text{ k (C)}$$ **Ans**

$$\mathbf{F}_2 = \mathbf{T}_2^T \mathbf{Q}_2 = \begin{array}{c} \left[\begin{array}{r} 0 \\ 183 \\ \hdashline 0 \\ -183 \end{array}\right] \end{array} \begin{array}{l} 5 \\ 6 \\ 1 \\ 2 \end{array}$$

Member 3

$$\mathbf{v}_3 = \left[\begin{array}{c} 0 \\ 0 \\ d_1 \\ d_2 \end{array}\right] \begin{array}{l} 7 \\ 8 \\ 1 \\ 2 \end{array} = \left[\begin{array}{r} 0 \\ 0 \\ 0.28068 \\ -0.20193 \end{array}\right] \text{in.}$$

$$\mathbf{u}_3 = \mathbf{T}_3 \mathbf{v}_3 = \left[\begin{array}{r} 0 \\ 0 \\ -0.32995 \\ -0.10339 \end{array}\right] \text{in.}$$

Using $\mathbf{k}_3$ from Example 7.2 and $\mathbf{Q}_{f3}$ from Eq. (4), we calculate

$$\mathbf{Q}_3 = \mathbf{k}_3 \mathbf{u}_3 + \mathbf{Q}_{f3} = \left[\begin{array}{rrrr} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{r} 0 \\ 0 \\ -0.32995 \\ -0.10339 \end{array}\right] + \left[\begin{array}{r} -120.83 \\ 0 \\ 120.83 \\ 0 \end{array}\right] = \left[\begin{array}{r} 198.12 \\ 0 \\ -198.12 \\ 0 \end{array}\right] \text{k}$$

$Q_{a3} = 198.12$ k (C) **Ans**

$$\mathbf{F}_3 = \mathbf{T}_3^T \mathbf{Q}_3 = \left[\begin{array}{r} -118.87 \\ 158.5 \\ \hdashline 118.87 \\ -158.5 \end{array}\right] \begin{array}{l} 7 \\ 8 \\ 1 \\ 2 \end{array}$$

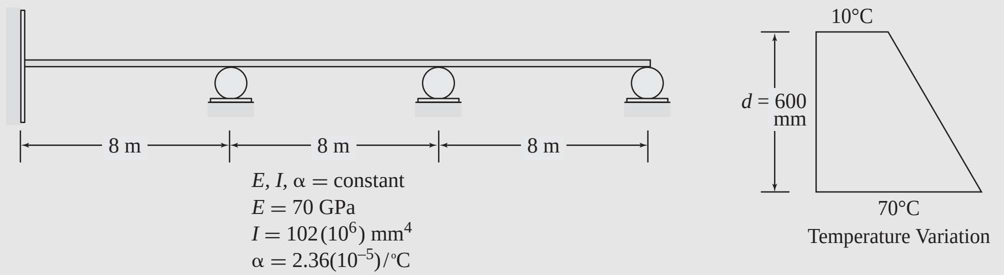
Support Reactions: See Figs. 7.19(d) and (e). **Ans**

EXAMPLE 7.6

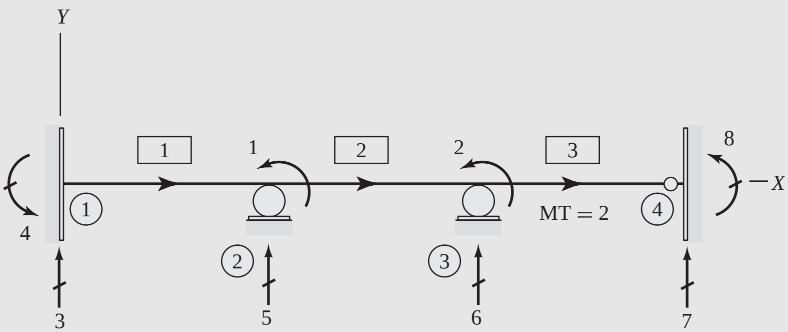
Determine the joint displacements, member end forces, and support reactions for the three-span continuous beam shown in Fig. 7.20(a), due to a temperature increase of 10° C at the top surface and 70° C at the bottom surface, of all spans. The temperature increase varies linearly over the depth $d = 600$ mm of the beam cross-section. Use the matrix stiffness method.

SOLUTION This beam was analyzed in Example 7.3 for member loads and support settlements.

Analytical Model: See Fig. 7.20(b). The analytical model used herein is the same as used in Example 7.3.



(a) Beam



(b) Analytical Model

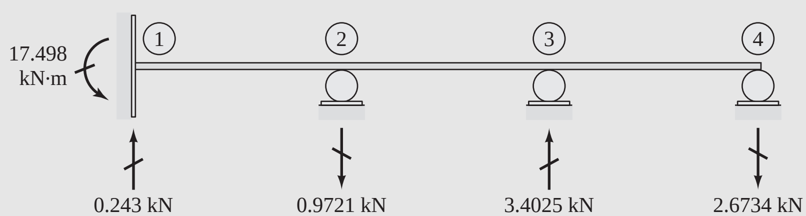
$$\mathbf{P}_f = \begin{bmatrix} -16.85 + 16.85 \\ -16.85 + 25.276 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} = \begin{bmatrix} 0 \\ 8.4252 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(c) Structure Fixed-Joint Force Vector Due to Temperature Change

(d) Member End Forces

$$\mathbf{R} = \begin{bmatrix} 0.243 \\ 17.498 \\ -0.243 - 0.7291 \\ 0.7291 + 2.6734 \\ -2.6734 \\ 0 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} = \begin{bmatrix} 0.243 \text{ kN} \\ 17.498 \text{ kN.m} \\ -0.9721 \text{ kN} \\ 3.4025 \text{ kN} \\ -2.6734 \text{ kN} \\ 0 \end{bmatrix}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 7.20

Structure Stiffness Matrix: As determined in Example 7.3, the structure stiffness matrix for the beam, in units of kN and meters, is

$$\mathbf{S} = \begin{bmatrix} 7{,}140 & 1{,}785 \\ 1{,}785 & 6{,}247.5 \end{bmatrix} \tag{1}$$

Joint Load Vector:

$$\mathbf{P} = \mathbf{0} \tag{2}$$

Structure Fixed-Joint Force Vector Due to Temperature Change:

Members 1 and 2 ($MT = 0$) By substituting the numerical values of E, I, L, α, $d = 0.6$ m, $T_t = 10°$C and $T_b = 70°$C into Eq. (7.33), we evaluate the member fixed-end moments due to the given temperature change as

$$FM_b = -FM_e = 70(10^6)(102)(10^{-6})(2.36)(10^{-5})\left(\frac{70-10}{0.6}\right) = 16.85\ \text{kN}\cdot\text{m} \tag{3a}$$

$$FS_b = FS_e = 0 \tag{3b}$$

Thus,

$$\mathbf{Q}_{f1} = \mathbf{Q}_{f2} = \begin{array}{c} \begin{bmatrix} 0 \\ 16.85 \\ 0 \\ -16.85 \end{bmatrix} \end{array} \begin{array}{c|c} 3 & 5 \\ 4 & 1 \\ 5 & 6 \\ 1 & 2 \end{array} \tag{4}$$

Member 1 Member 2

Next, using the member code numbers, we store the pertinent elements of $\mathbf{Q}_{f1}$ and $\mathbf{Q}_{f2}$ in their proper positions in the structure fixed-joint force vector $\mathbf{P}_f$, as shown in Fig. 7.20(c).

Member 3 ($MT = 2$) Because $MT = 2$ for this member, we substitute the values of fixed-end moments and shears from Eqs. (3) into Eq. (7.19) to obtain the fixed-end force vector for the released member 3 as

$$\mathbf{Q}_{f3} = \begin{bmatrix} 0 - \dfrac{3(-16.85)}{2(8)} \\ 16.85 - \dfrac{1}{2}(-16.85) \\ 0 + \dfrac{3(-16.85)}{2(8)} \\ 0 \end{bmatrix} = \begin{bmatrix} 3.1595 \\ 25.276 \\ -3.1595 \\ 0 \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array} \tag{5}$$

The relevant elements of $\mathbf{Q}_{f3}$ are stored in $\mathbf{P}_f$ using the member code numbers. The completed structure fixed-joint force vector $\mathbf{P}_f$ due to the temperature change, is shown in Fig. 7.20(c).

Joint Displacements: The structure stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$ for the entire beam can be written as

$$\begin{bmatrix} 0 \\ -8.4252 \end{bmatrix} = \begin{bmatrix} 7{,}140 & 1{,}785 \\ 1{,}785 & 6{,}247.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving these equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{array}{c} \begin{bmatrix} 3.6308 \\ -14.523 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \times 10^{-4} \text{ rad}$$ **Ans**

Member End Displacements and End Forces:

Member 1

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.6308 \end{bmatrix} \times 10^{-4}$$

Using $\mathbf{k}_1$ from Example 7.3 and $\mathbf{Q}_{f1}$ from Eq. (4),

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 0.243 \text{ kN} \\ 17.498 \text{ kN·m} \\ -0.243 \text{ kN} \\ -15.554 \text{ kN·m} \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array}$$ **Ans**

The end forces for member 1 are depicted in Fig. 7.20(d). To generate the support reaction vector **R**, the pertinent elements of $\mathbf{Q}_1$ are stored in **R**, as shown in Fig. 7.20(e).

Member 2

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ d_1 \\ 0 \\ d_2 \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 3.6308 \\ 0 \\ -14.523 \end{bmatrix} \times 10^{-4}$$

Using $\mathbf{k}_2$ from Example 7.3 and $\mathbf{Q}_{f2}$ from Eq. (4),

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} -0.7291 \text{ kN} \\ 15.554 \text{ kN·m} \\ 0.7291 \text{ kN} \\ -21.387 \text{ kN·m} \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array}$$ **Ans**

Member 3

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array} = \begin{bmatrix} 0 \\ -14.523 \\ 0 \\ 0 \end{bmatrix} \times 10^{-4}$$

Using $\mathbf{k}_3$ from Example 7.3 and $\mathbf{Q}_{f3}$ from Eq. (5),

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 + \mathbf{Q}_{f3} = \begin{bmatrix} 2.6734 \text{ kN} \\ 21.387 \text{ kN·m} \\ -2.6734 \text{ kN} \\ 0 \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array}$$ **Ans**

The member end forces are shown in Fig. 7.20(d).

Support Reactions: See Figs. 7.20(e) and (f). **Ans**

EXAMPLE 7.7 Determine the joint displacements, member end forces, and support reactions for the plane frame shown in Fig. 7.21(a) due to the combined effect of the following: (a) a temperature increase of 75° F in the girder, and (b) the fabricated length of the left column being $\frac{1}{4}$ in. too short. Use the matrix stiffness method.

SOLUTION This frame, subjected to joint and member loads, was analyzed in Example 7.1.

Analytical Model: See Fig. 7.21(b). The analytical model used herein is the same as used in Example 7.1.

Structure Stiffness Matrix: As determined in Example 7.1, the structure stiffness matrix for the frame, in units of kips and inches, is

$$\mathbf{S} = \begin{bmatrix} 1{,}781.3 & 0 & -1{,}776.3 & 0 & 0 \\ 0 & 1{,}781.3 & 0 & -5.0347 & 1{,}208.3 \\ -1{,}776.3 & 0 & 1{,}781.3 & 0 & 1{,}208.3 \\ 0 & -5.0347 & 0 & 1{,}781.3 & -1{,}208.3 \\ 0 & 1{,}208.3 & 1{,}208.3 & -1{,}208.3 & 580{,}000 \end{bmatrix} \quad (1)$$

Joint Load Vector:

$$\mathbf{P} = \mathbf{0} \quad (2)$$

Structure Fixed-Joint Force Vector Due to Temperature Changes and Fabrication Errors:

Member 1 ($MT = 2$) By substituting the numerical values of $E, A, L,$ and $e_a = -\frac{1}{4}$ in. into Eq. (7.35), we obtain the member fixed-end forces, due to the specified fabrication error, as

$$FA_b = -FA_e = -444.06 \text{ k}$$
$$FS_b = FS_e = FM_b = FM_e = 0$$

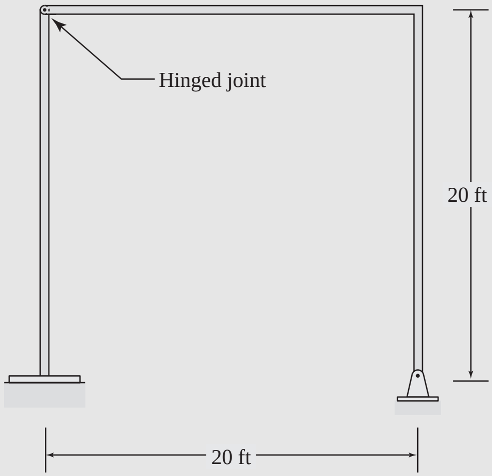
As $MT = 2$, we use Eq. (7.10) to form the member local fixed-end force vector. Thus,

$$\mathbf{Q}_{f1} = \begin{bmatrix} -444.06 \\ 0 \\ 0 \\ 444.06 \\ 0 \\ 0 \end{bmatrix} \quad (3)$$

Using the member transformation matrix, $\mathbf{T}_1$, from Example 7.1, we evaluate the global fixed-end force vector as

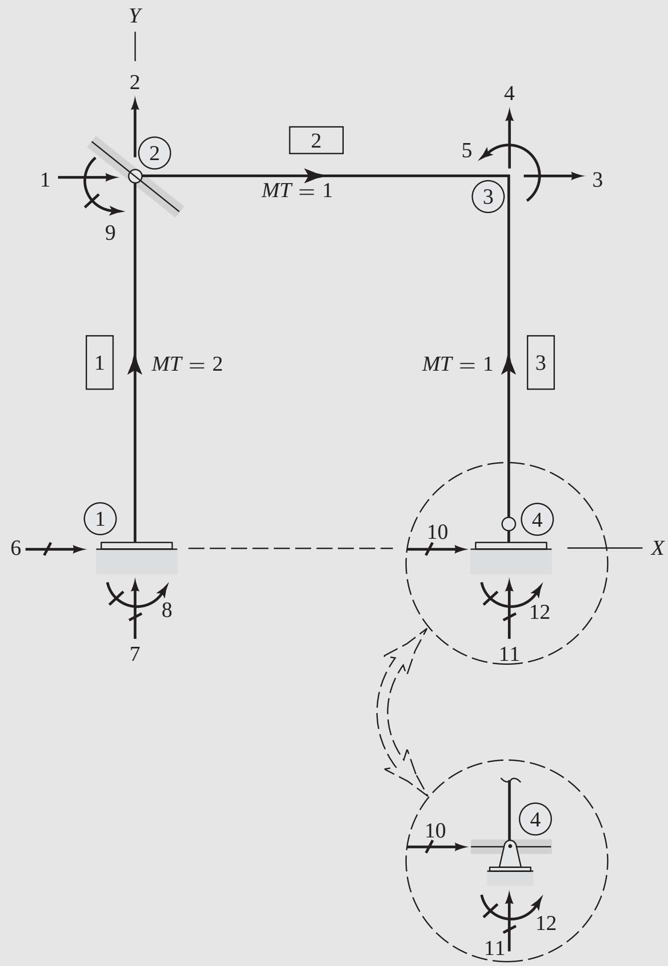
$$\mathbf{F}_{f1} = \mathbf{T}_1^T \mathbf{Q}_{f1} = \begin{array}{c} \begin{bmatrix} 0 \\ -444.06 \\ 0 \\ \hdashline 0 \\ 444.06 \\ \hdashline 0 \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array}$$

Next, using the member code numbers, we store the pertinent elements of $\mathbf{F}_{f1}$ in their proper positions in the 5×1 structure fixed-joint force vector $\mathbf{P}_f$, as shown in Fig. 7.21(c).



E, A, I, α = constant
$E = 29{,}000$ ksi
$A = 14.7$ in.^{2}
$I = 800$ in.^{4}
$\alpha = 6.5(10^{-6})/°$F

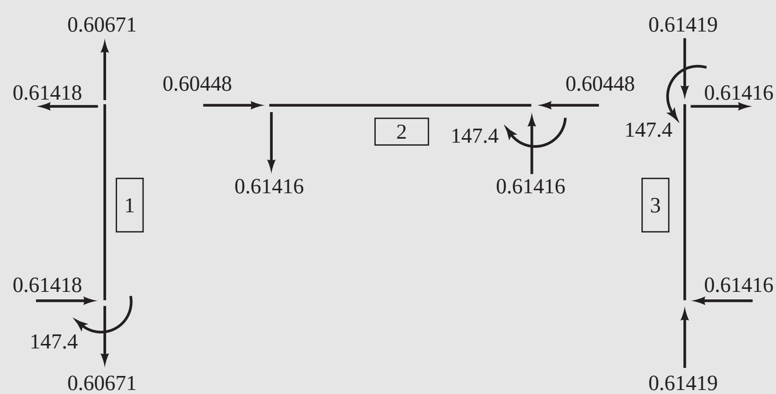
(a) Frame



(b) Analytical Model

$$\mathbf{P}_f = \begin{array}{c} \begin{bmatrix} 207.82 \\ 444.06 \\ -207.82 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

(c) Structure Fixed-Joint Force Vector Due to Temperature Changes and Fabrication Errors

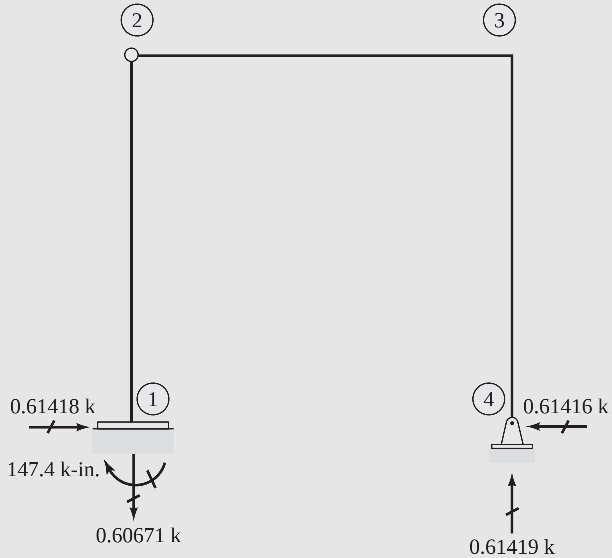


(d) Member Local End Forces

Fig. 7.21

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} 0.61418 \text{ k} \\ -0.60671 \text{ k} \\ -147.4 \text{ k-in.} \\ 0 \\ -0.61416 \text{ k} \\ 0.61419 \text{ k} \\ 0 \end{array}\right] \end{array} \begin{array}{l} 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 7.21 (*continued*)

Member 2 ($MT = 1$) By substituting the numerical values of E, A, α, and $T_u = 75°$ F into Eq. (7.34), we evaluate the member fixed-end forces, due to the given temperature change:

$$FA_b = -FA_e = 207.82 \text{ k}$$
$$FS_b = FS_e = FM_b = FM_e = 0$$

As $MT = 1$, we use Eq. (7.6) to form this horizontal member's local and global fixed-end force vectors:

$$\mathbf{F}_{f2} = \mathbf{Q}_{f2} = \left[\begin{array}{c} 207.82 \\ 0 \\ \hdashline 0 \\ \hdashline -207.82 \\ 0 \\ 0 \end{array}\right] \begin{array}{l} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array} \qquad \textbf{(4)}$$

The relevant elements of $\mathbf{F}_{f2}$ are stored in $\mathbf{P}_f$ using the member code numbers. The completed structure fixed-joint force vector $\mathbf{P}_f$, due to the temperature change and fabrication error, is given in Fig. 7.21(c).

Joint Displacements: Solving the structure stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$, we obtain the following joint displacements.

$$\mathbf{d} = \left[\begin{array}{c} -0.12199 \text{ in.} \\ -0.24965 \text{ in.} \\ -0.0053343 \text{ in.} \\ -0.00034577 \text{ in.} \\ 0.00053051 \text{ rad} \end{array}\right] \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \qquad \textbf{Ans}$$

Member End Displacements and End Forces:

Member 1 ($MT = 2$)

$$\mathbf{v}_1 = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.12199 \\ -0.24965 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.24965 \\ 0.12199 \\ 0 \end{bmatrix}$$

Using $\mathbf{k}_1$ from Example 7.1 and $\mathbf{Q}_{f1}$ from Eq. (3),

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} -0.60671 \text{ k} \\ -0.61418 \text{ k} \\ -147.4 \text{ k-in.} \\ 0.60671 \text{ k} \\ 0.61418 \text{ k} \\ 0 \end{bmatrix} \qquad \textbf{Ans}$$

These end forces are depicted in Fig. 7.21(d).

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{bmatrix} 0.61418 \\ -0.60671 \\ -147.4 \\ \hdashline -0.61418 \\ 0.60671 \\ \hdashline 0 \end{bmatrix} \begin{array}{l} 6 \\ 7 \\ 8 \\ 1 \\ 2 \\ 9 \end{array}$$

The pertinent elements of $\mathbf{F}_1$ are stored in $\mathbf{R}$, as shown in Fig. 7.21(e).

Member 2 ($MT = 1$)

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array} = \begin{bmatrix} -0.12199 \\ -0.24965 \\ 0 \\ -0.0053343 \\ -0.00034577 \\ 0.00053051 \end{bmatrix}$$

Using $\mathbf{k}_2$ from Example 7.1 and $\mathbf{Q}_{f2}$ from Eq. (4),

$$\mathbf{F}_2 = \mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{array}{c} \left[\begin{array}{c} 0.60448 \text{ k} \\ -0.61416 \text{ k} \\ \hdashline 0 \\ \hdashline -0.60448 \text{ k} \\ 0.61416 \text{ k} \\ -147.4 \text{ k-in.} \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 9 \\ 3 \\ 4 \\ 5 \end{array}$$ **Ans**

Member 3 ($MT = 1$)

$$\mathbf{v}_3 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ d_3 \\ d_4 \\ d_5 \end{array}\right] \begin{array}{c} 10 \\ 11 \\ 12 \\ 3 \\ 4 \\ 5 \end{array} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -0.0053343 \\ -0.00034577 \\ 0.00053051 \end{array}\right]$$

Using $\mathbf{T}_3$ from Example 7.1,

$$\mathbf{u}_3 = \mathbf{k}_3\mathbf{v}_3 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -0.00034577 \\ 0.0053343 \\ 0.00053051 \end{array}\right]$$

Using $\mathbf{k}_3$ from Example 7.1 and $\mathbf{Q}_{f3} = \mathbf{0}$, we calculate,

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 = \left[\begin{array}{c} 0.61419 \text{ k} \\ 0.61416 \text{ k} \\ 0 \\ -0.61419 \text{ k} \\ -0.61416 \text{ k} \\ 147.4 \text{ k-in.} \end{array}\right]$$ **Ans**

The member local end forces are shown in Fig. 7.21(d).

$$\mathbf{F}_3 = \mathbf{T}_3^T\mathbf{Q}_3 = \left[\begin{array}{c} -0.61416 \\ 0.61419 \\ 0 \\ \hdashline 0.61416 \\ -0.61419 \\ 147.4 \end{array}\right] \begin{array}{c} 10 \\ 11 \\ 12 \\ 3 \\ 4 \\ 5 \end{array}$$

Support Reactions: See Figs. 7.21(e) and (f). **Ans**

SUMMARY

In this chapter, we have extended the matrix stiffness formulation so that it can be used to analyze plane-framed structures containing member releases. Furthermore, the formulation has been extended to include in the analysis, the secondary effects of support displacements, temperature changes, and fabrication errors.

In the presence of member releases, the overall analysis procedure remains the same as before, except that the modified expressions for the member local stiffness matrices **k** and the fixed-end force vectors $\mathbf{Q}_f$, developed in Section 7.1, must be used for members with releases. If all the members meeting at a joint are connected to it by hinged connections, then such a joint can be modeled as a hinged joint with its rotation restrained by an imaginary clamp.

The effects of support displacements are included in the analysis using the concept of equivalent joint loads. The structure fixed-joint forces, due to the support displacements, are added to the $\mathbf{P}_f$ vector by performing the following operations for each member that is attached to a support that undergoes displacements: (a) forming the fixed-end displacement vector $\mathbf{v}_{fs}$ from the support displacements, (b) evaluating the fixed-end force vector $\mathbf{F}_{fs} = \mathbf{K}\mathbf{v}_{fs}$, and (c) storing the relevant elements of $\mathbf{F}_{fs}$ in $\mathbf{P}_f$ using the member code numbers. Once the structure's joint displacements have been determined by solving its stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$, the member end displacement vectors **v** are formed using both the joint displacements **d** and the specified support displacements. The rest of the procedure for evaluating member forces and support reactions remains the same as for the case of external loads.

The effects of temperature changes and fabrication errors can be included in the analysis methods developed previously, simply by including the member fixed-end forces due to these actions in the local fixed-end force vectors $\mathbf{Q}_f$. The expressions for member fixed-end forces, due to temperature changes and fabrication errors, are given in Section 7.5.

PROBLEMS

Section 7.1

7.1 and 7.2 Determine the joint displacements, member end forces, and support reactions for the beams shown in Figs. P7.1 and P7.2, using the matrix stiffness method.

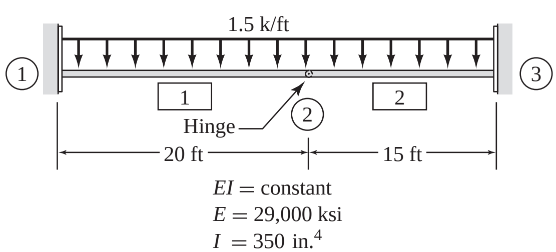


Fig. P7.1

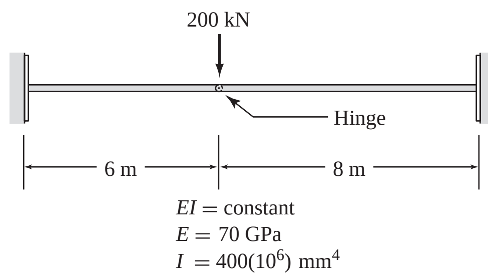


Fig. P7.2

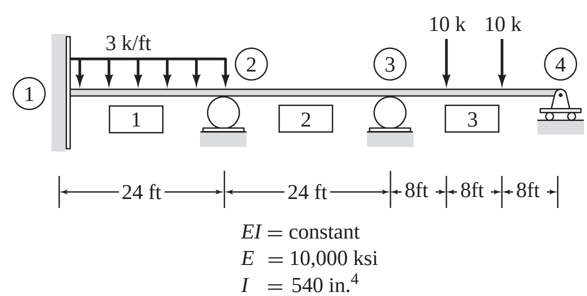


Fig. P7.3

7.3 Determine the joint displacements, member end forces, and support reactions for the beam shown in Fig. P7.3, by modeling member 3 as being hinged at its right end.

7.4 Determine the joint displacements, member end forces, and support reactions for the beam shown in Fig. P7.4, by modeling member 1 as being hinged at its left end and member 3 as being hinged at its right end.

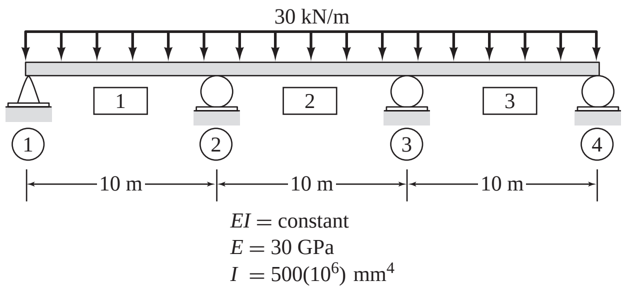


Fig. P7.4, P7.20

7.5 Determine the joint displacements, member end forces, and support reactions for the beam shown in Fig. P7.5 by modeling member 1 as being hinged at its left end.

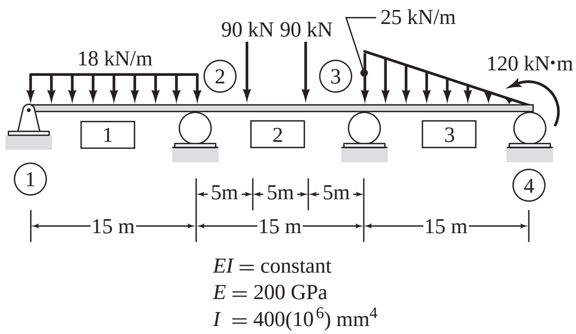


Fig. P7.5

7.6 Determine the joint displacements, member local end forces, and support reactions for the plane frame shown in Fig. P7.6, using the matrix stiffness method.

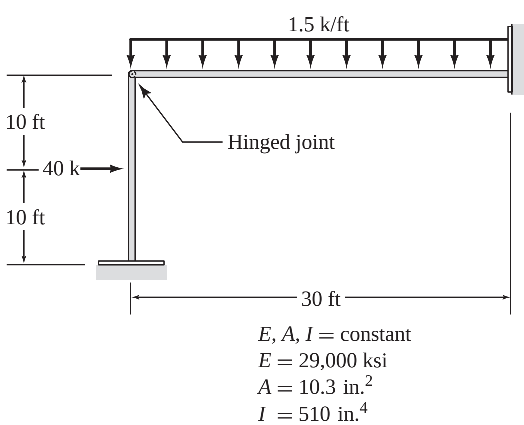


Fig. P7.6

7.7 and 7.8 Determine the joint displacements, member local end forces, and support reactions for the plane frames shown in Figs. P7.7 and P7.8, by modeling the horizontal member as being hinged at its right end.

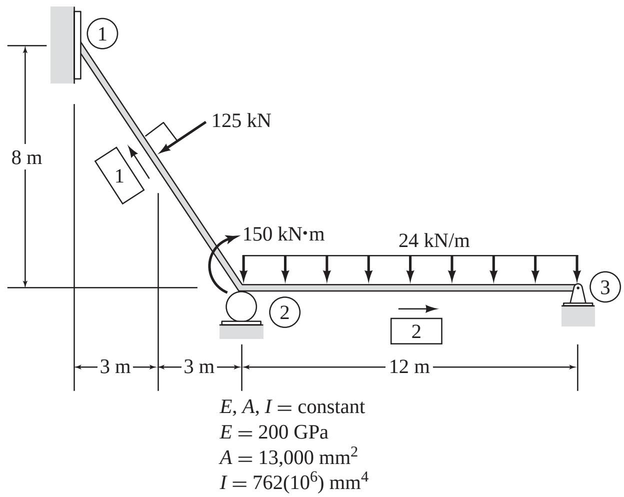


Fig. P7.7

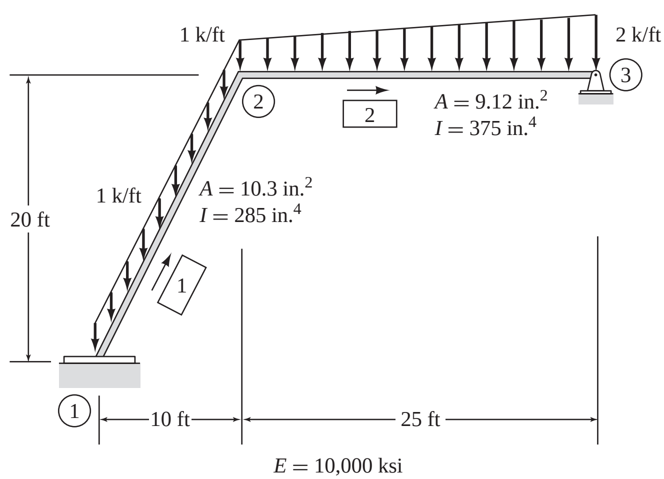


Fig. P7.8

7.9 Determine the joint displacements, member local end forces, and support reactions for the plane frame shown in Fig. P7.9, by modeling the horizontal member as being hinged at its left end and the inclined member as being hinged at its lower end.

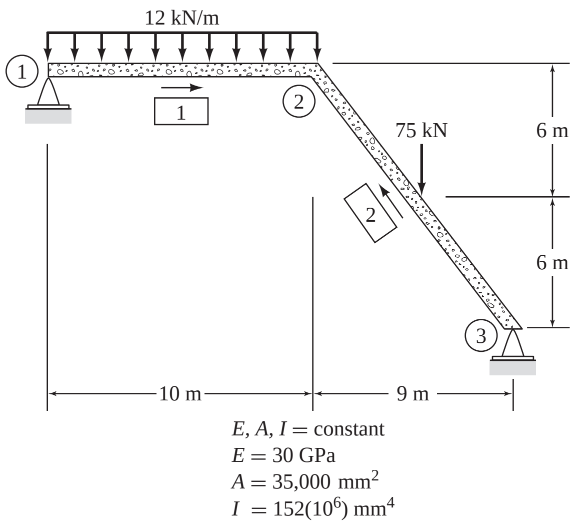


Fig. P7.9, P7.23, P7.33

7.10 Determine the joint displacements, member local end forces, and support reactions for the plane frame shown in Fig. P7.10, using the matrix stiffness method.

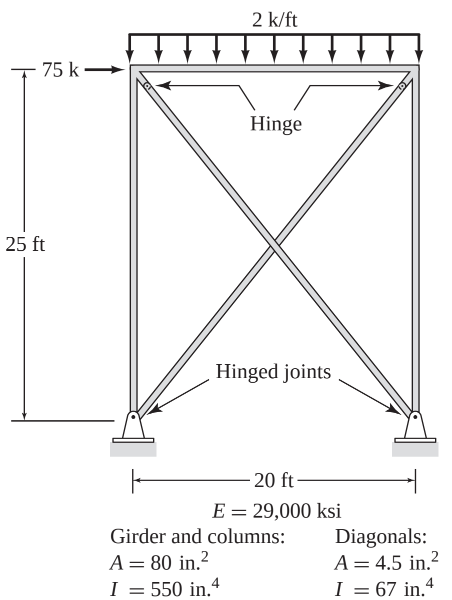


Fig. P7.10

7.11 Using a structural analysis computer program, determine the joint displacements, member local end forces, and reactions for the frame shown in Fig. P7.11 for the value of the load parameter $P = 300$ kN. What is the largest value of P that can be applied to the frame without exceeding the drift (maximum horizontal deflection) limitation of one percent of the frame height? Assume that the braces (inclined members) are connected by hinged connections at both ends.

7.12 Solve Problem 7.11 by assuming that the braces are connected by rigid (moment-resisting) connections at both ends.

7.13 Solve Problem 7.11 by assuming that the frame is unbraced. Note that, instead of developing a new analytical model for the unbraced frame, the previously developed models of the corresponding braced frames can be modified to eliminate the effect of bracing by simply using a very small value for the modulus of elasticity, E, of the bracing members (e.g., $E = 0.000001$ kN/m^2).

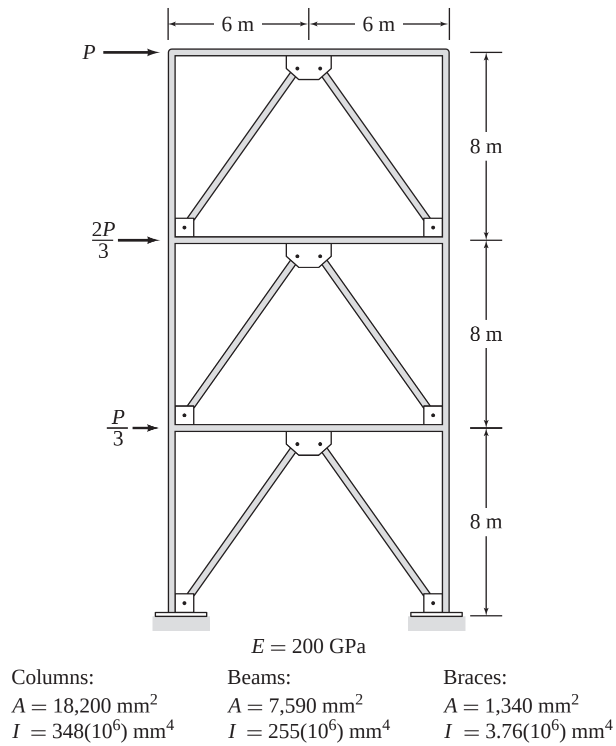


Fig. P7.11, P7.12, P7.13

Section 7.2

7.14 Modify the computer program developed in Chapter 5 for the analysis of rigidly connected beams, to include the effect of member releases. Use the modified program to analyze the beams of Problems 7.1 through 7.5, and compare the computer-generated results to those obtained by hand calculations.

7.15 Modify the program developed in Chapter 6 for the analysis of rigidly connected plane frames, to include the effect of member releases. Use the modified program to analyze the frames of Problems 7.6 through 7.10, and compare the computer-generated results to those obtained by hand calculations.

Section 7.3

7.16 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. P7.16, due to the combined effect of the loading shown and a settlement of $\frac{1}{2}$ in. of support 2. Use the matrix stiffness method.

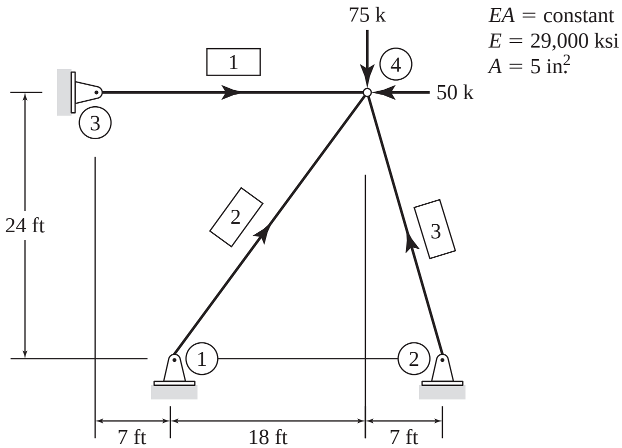


Fig. P7.16, P7.28

7.17 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. P7.17, due to the combined effect of the loading shown and a settlement of $\frac{1}{4}$ in. of support 3. Use the matrix stiffness method.

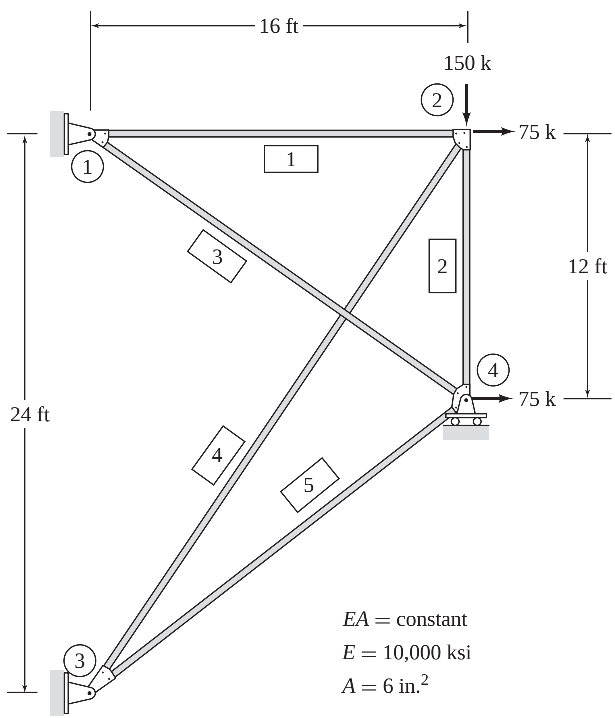


Fig. P7.17, P7.29, P7.30

7.18 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. P7.18, due to the combined effect of the loading shown and settlements of $\frac{1}{2}$ and $\frac{1}{4}$ in., respectively, of supports 3 and 4. Use the matrix stiffness method.

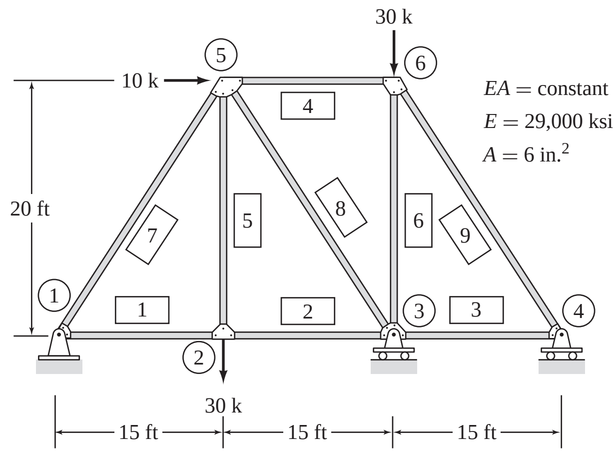


Fig. P7.18

7.19 Determine the joint displacements, member end forces, and support reactions for the three-span continuous beam shown in Fig. P7.19, due to settlements of 8 and 30 mm, respectively, of supports 2 and 3. Use the matrix stiffness method.

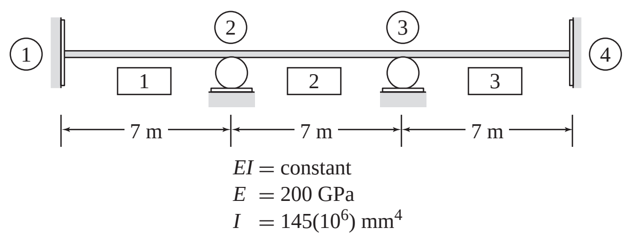


Fig. P7.19, P7.34

7.20 Solve Problem 7.4 for the loading shown in Fig. P7.4 and settlements of 12,75,60 and 25 mm, respectively, of supports 1, 2, 3, and 4.

7.21 Determine the joint displacements, member end forces, and support reactions for the beam shown in Fig. P7.21, due to the combined effect of the loading shown and a settlement of $1\frac{1}{4}$ in. of the middle support. Use the matrix stiffness method.

7.22 Determine the joint displacements, member local end forces, and support reactions for the plane frame shown in

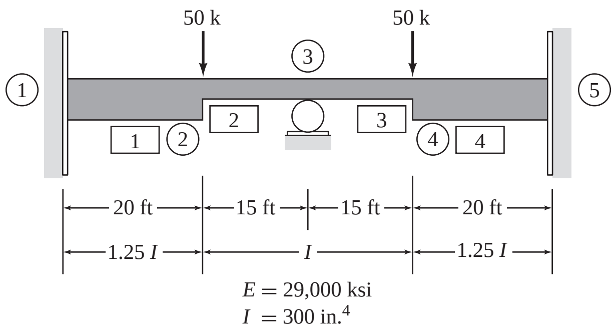


Fig. P7.21

Fig. P7.22, due to a settlement of 25 mm of the right support. Use the matrix stiffness method.

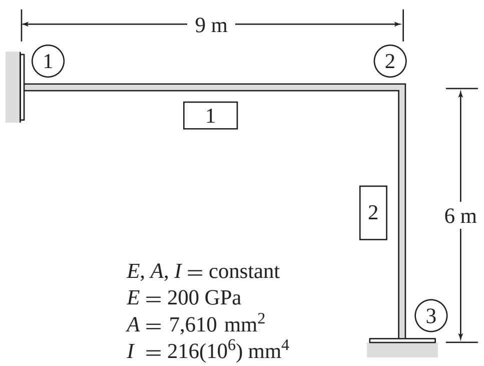


Fig. P7.22, P7.31, P7.32

7.23 Solve Problem 7.9 for the loading shown in Fig. P7.9 and a settlement of 50 mm of the right support.

7.24 Determine the joint displacements, member local end forces, and support reactions for the plane frame shown in Fig. P7.24, due to the combined effect of the following: (a) the loading shown in the figure, (b) a clockwise rotation of 0.017 radians of the left support, and (c) a settlement of $\frac{3}{4}$ in. of the right support. Use the matrix stiffness method.

Section 7.4

7.25 Extend the program developed in Chapter 4 for the analysis of plane trusses subjected to joint loads, to include the effect of support displacements. Use the modified program to analyze the trusses of Problems 7.16 through 7.18, and compare the computer-generated results to those obtained by hand calculations.

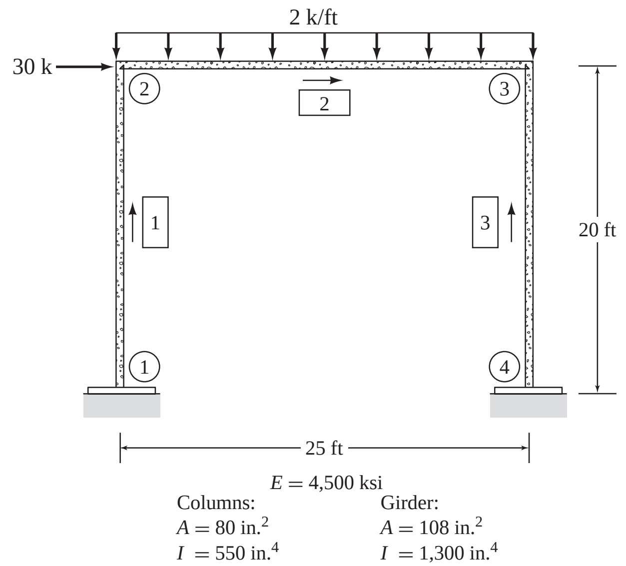


Fig. P7.24

7.26 Extend the program developed in Problem 7.14 for the analysis of beams subjected to external loads, to include the effect of support displacements. Use the modified program to analyze the beams of Problems 7.19 through 7.21, and compare the computer-generated results to those obtained by hand calculations.

7.27 Extend the program developed in Problem 7.15 for the analysis of plane frames subjected to external loads, to include the effect of support displacements. Use the modified program to analyze the frames of Problems 7.22 through 7.24, and compare the computer-generated results to those obtained by hand calculations.

Section 7.5

7.28 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. P7.28, due to a temperature drop of 100° F in member 2. Neglect the joint loads shown in the figure. Use the matrix stiffness method; $\alpha = 6.5(10^{-6})/°$ F.

7.29 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. P7.29, due to the combined effect of the following: (a) the joint loads shown in the figure, (b) a temperature increase of 70° F in member 2, (c) a temperature drop of 30° F in member 5, and (d) the fabricated length of member 4 being $\frac{1}{4}$ in. too long. Use the matrix stiffness method; $\alpha = 1.3(10^{-5})/°$ F.

7.30 Determine the joint displacements, member axial forces, and support reactions for the plane truss shown in Fig. P7.30, due to the fabricated lengths of members 3 and 4 being $\frac{1}{4}$ in. too short. Neglect the joint loads shown in the figure, and use the matrix stiffness method.

7.31 Determine the joint displacements, member local end forces, and support reactions for the plane frame of Fig. P7.31, due to a temperature increase of 50° C in the two members. Use the matrix stiffness method; $\alpha = 1.2(10^{-5})/°$ C.

7.32 Determine the joint displacements, member local end forces, and support reactions for the plane frame of Fig. P7.32, due to the fabricated lengths of the two members being 15 mm too short. Use the matrix stiffness method.

7.33 Determine the joint displacements, member local end forces, and support reactions for the plane frame of Fig. P7.33, due to the combined effect of the following: (a) the external loads shown in the figure, and (b) a temperature drop of 60° C in the two members. Use the matrix stiffness method; $\alpha = 10^{-5}/°$ C.

7.34 Determine the joint displacements, member end forces, and support reactions for the beam of Fig. P7.34, due to a linearly varying temperature increase of 55° C at the top surface and 5° C at the bottom surface, of all the members. Use the matrix stiffness method; $\alpha = 1.2(10^{-5})/°$ C and $d = 300$ mm.