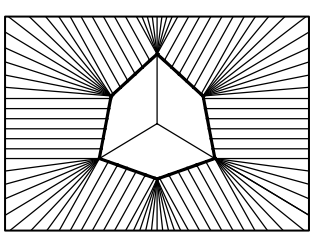
PLAXIS Version 8
Scientific Manual



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1 INTRODUCTION

In this part of the manual some scientific background is given of the theories and numerical methods on which the PLAXIS program is based. The manual contains chapters on deformation theory, groundwater flow theory and consolidation theory, as well as the corresponding finite element formulations and integration rules for the various types of elements used in PLAXIS. In the Appendix a global calculation scheme is provided for a plastic deformation analysis.

In addition to the specific information given in this part of the manual, more information on backgrounds of theory and numerical methods can be found in the literature, as amongst others referred to in Chapter 6. For detailed information on stresses, strains, constitutive modelling and the types of soil models used in the PLAXIS program, the reader is referred to the Material Models Manual.

2 DEFORMATION THEORY

In this chapter the basic equations for the static deformation of a soil body are formulated within the framework of continuum mechanics. A restriction is made in the sense that deformations are considered to be small. This enables a formulation with reference to the original undeformed geometry. The continuum description is discretised according to the finite element method.

2.1 BASIC EQUATIONS OF CONTINUUM DEFORMATION

The static equilibrium of a continuum can be formulated as:

$$\underline{\underline{L}}^T \underline{\sigma} + \underline{p} = \underline{0} \tag{2.1}$$

This equation relates the spatial derivatives of the six stress components, assembled in vector $\underline{\sigma}$, to the three components of the body forces, assembled in vector $\underline{p}$. $\underline{\underline{L}}^T$ is the transpose of a differential operator, defined as:

$$\underline{\underline{L}}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{2.2}$$

In addition to the equilibrium equation, the kinematic relation can be formulated as:

$$\underline{\varepsilon} = \underline{\underline{L}}\, \underline{u} \tag{2.3}$$

This equation expresses the six strain components, assembled in vector $\underline{\varepsilon}$, as the spatial derivatives of the three displacement components, assembled in vector $\underline{u}$, using the previously defined differential operator $\underline{\underline{L}}$. The link between Eq. (2.1) and (2.3) is formed by a constitutive relation representing the material behaviour. Constitutive relations, i.e. relations between rates of stress and strain, are extensively discussed in the Material Models Manual. The general relation is repeated here for completeness:

$$\dot{\underline{\sigma}} = \underline{\underline{M}}\, \dot{\underline{\varepsilon}} \tag{2.4}$$

The combination of Eqs. (2.1), (2.3) and (2.4) would lead to a second-order partial differential equation in the displacements $\underline{u}$.

However, instead of a direct combination, the equilibrium equation is reformulated in a weak form according to Galerkin's variation principle (see among others Zienkiewicz, 1967):

$$\int \delta \underline{u}^T \left(\underline{\underline{L}}^T \underline{\sigma} + \underline{p} \right) dV = 0 \tag{2.5}$$

In this formulation $\delta \underline{u}$ represents a kinematically admissible variation of displacements. Applying Green's theorem for partial integration to the first term in Eq. (2.5) leads to:

$$\int \delta \underline{\varepsilon}^T \underline{\sigma} \, dV = \int \delta \underline{u}^T \underline{p} \, dV + \int \delta \underline{u}^T \underline{t} \, dS \tag{2.6}$$

This introduces a boundary integral in which the boundary traction appears. The three components of the boundary traction are assembled in the vector $\underline{t}$. Eq. (2.6) is referred to as the virtual work equation.

The development of the stress state $\underline{\sigma}$ can be regarded as an incremental process:

$$\underline{\sigma}^i = \underline{\sigma}^{i-1} + \Delta\underline{\sigma} \qquad\qquad \Delta\underline{\sigma} = \int \dot{\underline{\sigma}} \, d\, t \tag{2.7}$$

In this relation $\underline{\sigma}^i$ represents the actual state of stress which is unknown and $\underline{\sigma}^{i-1}$ represents the previous state of stress which is known. The stress increment $\Delta\underline{\sigma}$ is the stress rate integrated over a small time increment.

If Eq. (2.6) is considered for the actual state *i*, the unknown stresses $\underline{\sigma}^i$ can be eliminated using Eq. (2.7):

$$\int \delta \underline{\varepsilon}^T \, \Delta\underline{\sigma} \, dV = \int \delta \underline{u}^T \underline{p}^i \, dV + \int \delta \underline{u}^T \underline{t}^{\,i} \, dS - \int \delta \underline{\varepsilon}^T \underline{\sigma}^{i-1} \, dV \tag{2.8}$$

It should be noted that all quantities appearing in Eqs. (2.1) to (2.8) are functions of the position in the three-dimensional space.

2.2 FINITE ELEMENT DISCRETISATION

According to the finite element method a continuum is divided into a number of (volume) elements. Each element consists of a number of nodes. Each node has a number of degrees of freedom that correspond to discrete values of the unknowns in the boundary value problem to be solved. In the present case of deformation theory the degrees of freedom correspond to the displacement components. Within an element the displacement field $\underline{u}$ is obtained from the discrete nodal values in a vector $\underline{v}$ using interpolation functions assembled in matrix $\underline{\underline{N}}$:

$$\underline{u} = \underline{\underline{N}}\,\underline{v} \tag{2.9}$$

The interpolation functions in matrix $\underline{\underline{N}}$ are often denoted as shape functions. Substitution of Eq. (2.9) in the kinematic relation (2.3) gives:

$$\underline{\varepsilon} = \underline{\underline{L}}\,\underline{\underline{N}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \tag{2.10}$$

In this relation $\underline{\underline{B}}$ is the strain interpolation matrix, which contains the spatial derivatives of the interpolation functions. Eqs. (2.9) and (2.10) can be used in variational, incremental and rate form as well.

Eq. (2.8) can now be reformulated in discretised form as:

$$\int \left(\underline{\underline{B}}\,\delta\underline{v}\right)^T \Delta\underline{\sigma}\, dV = \int \left(\underline{\underline{N}}\,\delta\underline{v}\right)^T \underline{p}^i\, dV + \int \left(\underline{\underline{N}}\,\delta\underline{v}\right)^T \underline{t}^i\, dS - \int \left(\underline{\underline{B}}\,\delta\underline{v}\right)^T \underline{\sigma}^{i-1}\, dV \tag{2.11}$$

The discrete displacements can be placed outside the integral:

$$\delta\underline{v}^T \int \underline{\underline{B}}^T \Delta\underline{\sigma} dV = \delta\underline{v}^T \int \underline{\underline{N}}^T \underline{p}^i\, dV + \delta\underline{v}^T \int \underline{\underline{N}}^T \underline{t}^i\, dS - \delta\underline{v}^T \int \underline{\underline{B}}^T \underline{\sigma}^{i-1} dV \tag{2.12}$$

Provided that Eq. (2.12) holds for any kinematically admissible displacement variation $\delta\underline{v}^T$, the equation can be written as:

$$\int \underline{\underline{B}}^T \Delta\underline{\sigma}\; dV = \int \underline{\underline{N}}^T\, \underline{p}^i\, dV + \int \underline{\underline{N}}^T\, \underline{t}^i\, dS - \int \underline{\underline{B}}^T\, \underline{\sigma}^{i-1}\, dV \tag{2.13}$$

The above equation is the elaborated equilibrium condition in discretised form. The first term on the right-hand side together with the second term represent the current external force vector and the last term represents the internal reaction vector from the previous step. A difference between the external force vector and the internal reaction vector should be balanced by a stress increment $\Delta\underline{\sigma}$.

The relation between stress increments and strain increments is usually non-linear. As a result, strain increments can generally not be calculated directly, and global iterative procedures are required to satisfy the equilibrium condition (2.13) for all material points. Global iterative procedures are described later in Section 2.4, but the attention is first focussed on the (local) integration of stresses.

2.3 IMPLICIT INTEGRATION OF DIFFERENTIAL PLASTICITY MODELS

The stress increments $\Delta\underline{\sigma}$ are obtained by integration of the stress rates according to Eq. (2.7). For differential plasticity models the stress increments can generally be written as:

$$\Delta\underline{\sigma} = \underline{\underline{D}}^e \left(\Delta\underline{\varepsilon} - \Delta\underline{\varepsilon}^p\right) \tag{2.14}$$

In this relation $\underline{\underline{D}}^e$ represents the elastic material matrix for the current stress increment. The strain increments $\Delta\underline{\varepsilon}$ are obtained from the displacement increments $\Delta\underline{v}$ using the strain interpolation matrix $\underline{\underline{B}}$, similar to Eq. (2.10).

For elastic material behaviour, the plastic strain increment $\Delta\underline{\varepsilon}^p$ is zero. For plastic material behaviour, the plastic strain increment can be written, according to Vermeer (1979), as:

$$\Delta\underline{\varepsilon}^p = \Delta\lambda \left[(1-\omega)\left(\frac{\partial g}{\partial \underline{\sigma}}\right)^{i\text{-}1} + \omega\left(\frac{\partial g}{\partial \underline{\sigma}}\right)^{i} \right] \tag{2.15}$$

In this equation $\Delta\lambda$ is the increment of the plastic multiplier and ω is a parameter indicating the type of time integration. For $\omega = 0$ the integration is called explicit and for $\omega = 1$ the integration is called implicit.

Vermeer (1979) has shown that the use of implicit integration ($\omega = 1$) has some major advantages, as it overcomes the requirement to update the stress to the yield surface in the case of a transition from elastic to elastoplastic behaviour. Moreover, it can be proven that implicit integration, under certain conditions, leads to a symmetric and positive differential matrix $\partial\underline{\varepsilon}$ / $\partial\underline{\sigma}$, which has a positive influence on iterative procedures. Because of these major advantages, restriction is made here to implicit integration and no attention is given to other types of time integration.

Hence, for $\omega = 1$ Eq. (2.15) reduces to:

$$\Delta\underline{\varepsilon}^p = \Delta\lambda \left(\frac{\partial g}{\partial \underline{\sigma}}\right)^{i} \tag{2.16}$$

Substitution of Eq. (2.16) into Eq. (2.14) and successively into Eq. (2.7) gives:

$$\underline{\sigma}^i = \underline{\sigma}^{tr} - \Delta\lambda\, \underline{\underline{D}}^e \left(\frac{\partial g}{\partial \underline{\sigma}}\right)^{i} \qquad \text{with:} \qquad \underline{\sigma}^{tr} = \underline{\sigma}^{i\text{-}1} + \underline{\underline{D}}^e\, \Delta\underline{\varepsilon} \tag{2.17}$$

In this relation $\underline{\sigma}^{tr}$ is an auxiliary stress vector, referred to as the *elastic stresses* or *trial stresses*, which is the new stress state when considering purely linear elastic material behaviour.

The increment of the plastic multiplier $\Delta\lambda$, as used in Eq. (2.17), can be solved from the condition that the new stress state has to satisfy the yield condition:

$$f\left(\underline{\sigma}^i\right) = 0 \tag{2.18}$$

For perfectly-plastic and linear hardening models the increment of the plastic multiplier can be written as:

$$\Delta\lambda = \frac{f\left(\underline{\sigma}^{tr}\right)}{d+h} \tag{2.19}$$

where:

$$d = \left(\frac{\partial f}{\partial \underline{\sigma}}\right)^{\underline{\sigma}^{tr}} \underline{\underline{D}}^{e} \left(\frac{\partial g}{\partial \underline{\sigma}}\right)^{i} \tag{2.20}$$

The symbol h denotes the hardening parameter, which is zero for perfectly-plastic models and constant for linear hardening models. In the latter case the new stress state can be formulated as:

$$\underline{\sigma}^{i} = \underline{\sigma}^{tr} - \frac{\left\langle f\left(\underline{\sigma}^{tr}\right)\right\rangle}{d+h} \underline{\underline{D}}^{e} \left(\frac{\partial g}{\partial \underline{\sigma}}\right)^{i} \tag{2.21}$$

The $\langle\ \rangle$-brackets are referred to as McCauley brackets, which have the following convention:

$$\langle x\rangle = 0 \quad \text{for:} \quad x \le 0 \qquad \text{and:} \qquad \langle x\rangle = x \quad \text{for:} \quad x > 0$$

2.4 GLOBAL ITERATIVE PROCEDURE

Substitution of the relationship between increments of stress and increments of strain, $\Delta\underline{\sigma} = \underline{\underline{M}}\,\Delta\underline{\varepsilon}$, into the equilibrium equation (2.13) leads to:

$$\underline{\underline{K}}^{i}\,\Delta\underline{v}^{i} = \underline{f}_{ex}^{\,i} - \underline{f}_{in}^{\,i\text{-}1} \tag{2.22}$$

In this equation $\underline{\underline{K}}$ is a stiffness matrix, $\Delta\underline{v}$ is the incremental displacement vector, $\underline{f}_{ex}$ is the external force vector and $\underline{f}_{in}$ is the internal reaction vector. The superscript i refers to the step number. However, because the relation between stress increments and strain increments is generally non-linear, the stiffness matrix cannot be formulated exactly beforehand. Hence, a global iterative procedure is required to satisfy both the equilibrium condition and the constitutive relation. The global iteration process can be written as:

$$\underline{\underline{K}}^{j}\,\delta\underline{v}^{j} = \underline{f}_{ex}^{\,i} - \underline{f}_{in}^{\,j\text{-}1} \tag{2.23}$$

The superscript j refers to the iteration number. $\delta\underline{v}$ is a vector containing sub-incremental displacements, which contribute to the displacement increments of step i:

$$\Delta \underline{v}^i = \sum_{j=1}^{n} \delta \underline{v}^j \tag{2.24}$$

where n is the number of iterations within step i. The stiffness matrix $\underline{\underline{K}}$, as used in Eq. (2.23), represents the material behaviour in an approximated manner. The more accurate the stiffness matrix, the fewer iterations are required to obtain equilibrium within a certain tolerance.

In its simplest form $\underline{\underline{K}}$ represents a linear-elastic response. In this case the stiffness matrix can be formulated as:

$$\underline{\underline{K}} = \int \underline{\underline{B}}^T \underline{\underline{D}}^e \underline{\underline{B}} \, dV \qquad \text{(elastic stiffness matrix)} \tag{2.25}$$

where $\underline{\underline{D}}^e$ is the elastic material matrix according to Hooke's law and $\underline{\underline{B}}$ is the strain interpolation matrix. The use of an elastic stiffness matrix gives a robust iterative procedure as long as the material stiffness does not increase, even when using non-associated plasticity models. Special techniques such as arc-length control (Riks, 1979), over-relaxation and extrapolation (Vermeer & Van Langen, 1989) can be used to improve the iteration process. Moreover, the automatic step size procedure, as introduced by Van Langen & Vermeer (1990), can be used to improve the practical applicability. For material models with linear behaviour in the elastic domain, such as the standard Mohr-Coulomb model, the use of an elastic stiffness matrix is particularly favourable, as the stiffness matrix needs only be formed and decomposed before the first calculation step. This calculation procedure is summarised in Appendix A.

3 GROUNDWATER FLOW THEORY

In this chapter we will review the theory of groundwater flow as used in PLAXIS. In addition to a general description of groundwater flow, attention is focused on the finite element formulation.

3.1 BASIC EQUATIONS OF STEADY FLOW

Flow in a porous medium can be described by Darcy's law. Considering flow in a vertical *x*-*y*-plane the following equations apply:

$$q_x = - k_x \frac{\partial \phi}{\partial x} \qquad q_y = - k_y \frac{\partial \phi}{\partial y} \tag{3.1}$$

The equations express that the specific discharge, q, follows from the permeability, k, and the gradient of the groundwater head. The head , ϕ, is defined as follows:

$$\phi = y - \frac{p}{\gamma_w} \tag{3.2}$$

where y is the vertical position, p is the stress in the pore fluid (negative for pressure) and γ_w is the unit weight of the pore fluid. For steady flow the continuity condition applies:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \tag{3.3}$$

Eq. (3.3) expresses that there is no net inflow or outflow in an elementary area, as illustrated in Figure 3.1.

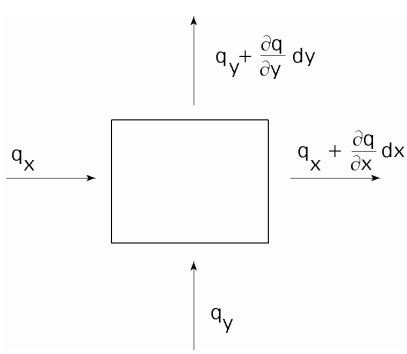


Figure 3.1 Illustration of continuity condition

3.2 FINITE ELEMENT DISCRETISATION

The groundwater head in any position within an element can be expressed in the values at the nodes of that element:

$$\phi(\xi,\eta) = \underline{N}\,\underline{\phi}^{e} \tag{3.4}$$

where $\underline{N}$ is the vector with interpolation functions and ξ and η are the local coordinates within the element. According to Eq. (3.1) the specific discharge is based on the gradient of the groundwater head. This gradient can be determined by means of the $\underline{\underline{B}}$-matrix, which contains the spatial derivatives of the interpolation functions. In order to describe flow for saturated soil (underneath the phreatic line) as well as non-saturated soil (above the phreatic line), a reduction function K^r is introduced in Darcy's law (Desai, 1976; Li & Desai, 1983; Bakker, 1989):

$$q_x = - K^r k_x \frac{\partial \phi}{\partial x} \qquad\qquad q_y = - K^r k_y \frac{\partial \phi}{\partial y} \tag{3.5}$$

The reduction function has a value of 1 below the phreatic line (compressive pore pressures) and has lower values above the phreatic line (tensile pore pressures). In the transition zone above the phreatic line, the function value decreases to the minimum of 10^{-4}.

In the transition zone the function is described using a log-linear relation:

$$K^r = 10^{-4h/h_k} \qquad\qquad 10^{-4} \le K^r \le 1$$

or

$$^{10}\log(K^r) = -\frac{4h}{h_k} \tag{3.6}$$

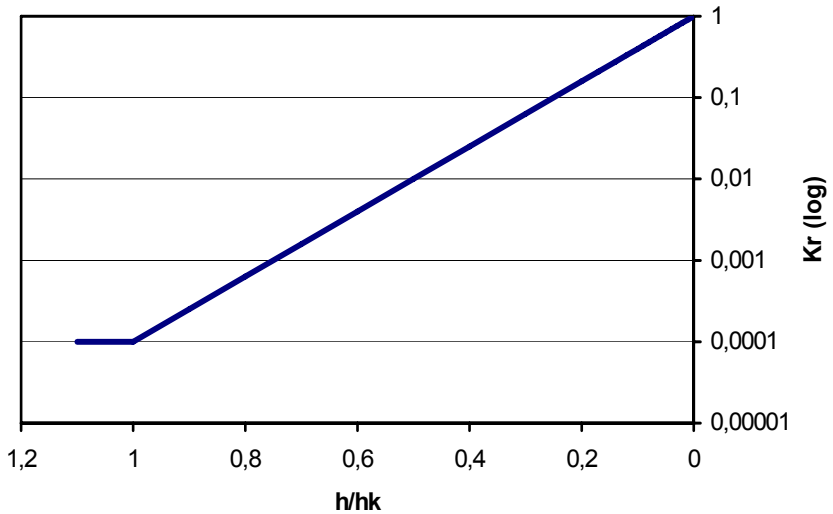
where h is the pressure head and h_k is the pressure head where the reduction function has reached the minimum of 10^{-4}. In PLAXIS h_k has a default value of 0.7 m (independent of the chosen length unit).

In the numerical formulation, the specific discharge, $\underline{q}$, is written as:

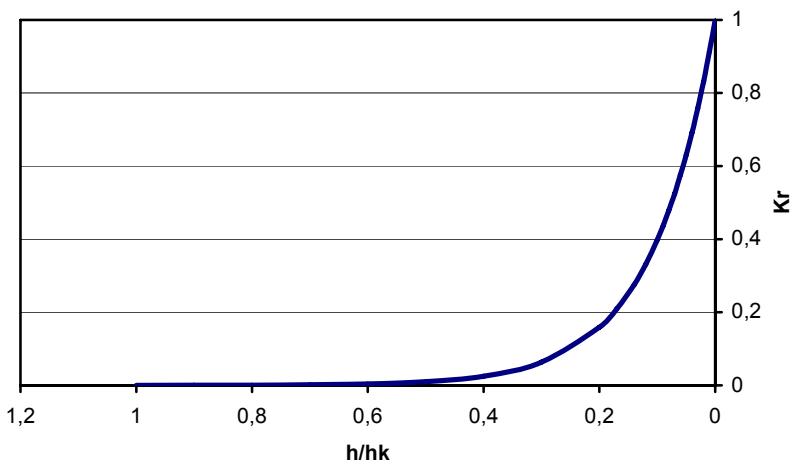
$$\underline{q} = -K^r \underline{\underline{R}}\,\underline{\underline{B}}\,\underline{\phi}^{e} \tag{3.7}$$

where:

$$\underline{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix} \qquad \text{and:} \qquad \underline{\underline{R}} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{3.8}$$



(a)



(b)

Figure 3.2 Adjustment of the permeability between saturated (a) and unsaturated (b) zones (K^r = ratio of permeability over saturated permeability)

From the specific discharges in the integration points, $\underline{q}$, the nodal discharges $\underline{Q}^e$ can be integrated according to:

$$\underline{Q}^e = - \int \underline{\underline{B}}^T \, \underline{q} \, dV \tag{3.9}$$

in which $\underline{\underline{B}}^T$ is the transpose of the *B*-matrix. On the element level the following equations apply:

$$\underline{Q}^e = \underline{\underline{K}}^e \underline{\phi}^e \qquad \text{with:} \qquad \underline{\underline{K}}^e = \int K^r \, \underline{\underline{B}}^T \, \underline{\underline{R}} \, \underline{\underline{B}} \, dV \quad (3.10)$$

On a global level, contributions of all elements are added and boundary conditions (either on the groundwater head or on the discharge) are imposed. This results in a set of *n* equations with *n* unknowns:

$$\underline{Q} = \underline{\underline{K}} \underline{\phi} \quad (3.11)$$

in which $\underline{\underline{K}}$ is the global flow matrix and $\underline{Q}$ contains the prescribed discharges that are given by the boundary conditions.

In the case that the phreatic line is unknown (unconfined problems), a Picard scheme is used to solve the system of equations iteratively. The linear set is solved in incremental form and the iteration process can be formulated as:

$$\underline{\underline{K}}^{j-1} \delta \underline{\phi}^j = \underline{Q} - \underline{\underline{K}}^{j-1} \underline{\phi}^{j-1} \qquad \underline{\phi}^j = \underline{\phi}^{j-1} + \delta \underline{\phi}^j \quad (3.12)$$

in which *j* is the iteration number and $\underline{r}$ is the unbalance vector. In each iteration increments of the groundwater head are calculated from the unbalance in the nodal discharges and added to the active head.

From the new distribution of the groundwater head the new specific discharges are calculated according to Eq. (3.7), which can again be integrated into nodal discharges. This process is continued until the norm of the unbalance vector, i.e. the error in the nodal discharges, is smaller than the tolerated error.

3.3 FLOW IN INTERFACE ELEMENTS

Interface elements are treated specially in groundwater calculations. The elements can be on or off. When the elements are switched off, there is a full coupling of the pore pressure degrees of freedom. When the interface elements are switched on, there is no flow from one side of the interface element to the other (impermeable screen).

4 CONSOLIDATION THEORY

In this chapter we will review the theory of consolidation as used in PLAXIS. In addition to a general description of Biot's theory for coupled consolidation, attention is focused on the finite element formulation. Moreover, a separate section is devoted to the use of advanced soil models in a consolidation analysis (elastoplastic consolidation).

4.1 BASIC EQUATIONS OF CONSOLIDATION

The governing equations of consolidation as used in PLAXIS follow Biot's theory (Biot, 1956). Darcy's law for fluid flow and elastic behaviour of the soil skeleton are also assumed. The formulation is based on small strain theory. According to Terzaghi's principle, stresses are divided into effective stresses and pore pressures:

$$\underline{\sigma} = \underline{\sigma'} + \underline{m}\left(p_{steady} + p_{excess}\right) \tag{4.1}$$

where:

$$\underline{\sigma} = \left(\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{xy}\ \sigma_{yz}\ \sigma_{zx} \right)^T \quad \text{and:} \quad \underline{m} = \left(1\ 1\ 1\ 0\ 0\ 0 \right)^T \tag{4.2}$$

$\underline{\sigma}$ is the vector with total stresses, $\underline{\sigma'}$ contains the effective stresses, p_{excess} is the excess pore pressure and m is a vector containing unity terms for normal stress components and zero terms for the shear stress components. The steady state solution at the end of the consolidation process is denoted as p_{steady}. Within PLAXIS p_{steady} is defined as:

$$p_{steady} = \sum Mweight \cdot p_{input} \tag{4.3}$$

where p_{input} is the pore pressure generated in the input program based phreatic lines or on a groundwater flow calculation. Note that within PLAXIS compressive stresses are considered to be negative; this applies to effective stresses as well as to pore pressures. In fact it would be more appropriate to refer to p_{excess} and p_{steady} as pore stresses, rather than pressures. However, the term pore pressure is retained, although it is positive for tension.

The constitutive equation is written in incremental form. Denoting an effective stress increment as $\dot{\underline{\sigma}}'$ and a strain increment as $\dot{\underline{\varepsilon}}$, the constitutive equation is:

$$\dot{\underline{\sigma}}' = \underline{\underline{M}}\,\dot{\underline{\varepsilon}} \tag{4.4}$$

where:

$$\dot{\underline{\varepsilon}} = \left(\dot{\varepsilon}_{xx}\ \dot{\varepsilon}_{yy}\ \dot{\varepsilon}_{zz}\ \dot{\gamma}_{xy}\ \dot{\gamma}_{yz}\ \dot{\gamma}_{zx} \right)^T \tag{4.5}$$

and $\underline{\underline{M}}$ represents the material stiffness matrix.

4.2 FINITE ELEMENT DISCRETISATION

To apply a finite element approximation we use the standard notation:

$$\underline{u} = \underline{\underline{N}}\,\underline{v} \qquad \underline{p} = \underline{\underline{N}}\,\underline{p}_n \qquad \underline{\varepsilon} = \underline{\underline{B}}\,\underline{v} \tag{4.6}$$

where $\underline{v}$ is the nodal displacement vector, $\underline{p}_n$ is the excess pore pressure vector, $\underline{u}$ is the continuous displacement vector within an element and $\underline{p}$ is the (excess) pore pressure. The matrix $\underline{\underline{N}}$ contains the interpolation functions and $\underline{\underline{B}}$ is the strain interpolation matrix.

In general the interpolation functions for the displacements may be different from the interpolation functions for the pore pressure. In PLAXIS, however, the same functions are used for displacements and pore pressures.

Starting from the incremental equilibrium equation and applying the above finite element approximation we obtain:

$$\int \underline{\underline{B}}^T \, d\underline{\sigma} \, dV = \int \underline{\underline{N}}^T \, d\underline{f} \, dV + \int \underline{\underline{N}}^T \, d\underline{t} \, dS + \underline{r}_0 \tag{4.7}$$

with:

$$\underline{r}_0 = \int \underline{\underline{N}}^T \, \underline{f}_0 \, dV + \int \underline{\underline{N}}^T \, \underline{t}_0 \, dS - \int \underline{\underline{B}}^T \, \underline{\sigma}_0 \, dV \tag{4.8}$$

where $\underline{f}$ is a body force due to self-weight and $\underline{t}$ represents the surface tractions. In general the residual force vector $\underline{r}_0$ will be equal to zero, but solutions of previous load steps may have been inaccurate. By adding the residual force vector the computational procedure becomes self-correcting. The term dV indicates integration over the volume of the body considered and dS indicates a surface integral.

Dividing the total stresses into pore pressure and effective stresses and introducing the constitutive relationship gives the nodal equilibrium equation:

$$\underline{\underline{K}} \, d\underline{v} + \underline{\underline{L}} \, d\underline{p}_n = d\underline{f}_n \tag{4.9}$$

where $\underline{\underline{K}}$ is the stiffness matrix, $\underline{\underline{L}}$ is the coupling matrix and $d\underline{f}_n$ is the incremental load vector:

$$\underline{\underline{K}} = \int \underline{\underline{B}}^T \, \underline{\underline{M}} \, \underline{\underline{B}} \, dV \tag{4.10a}$$

$$\underline{\underline{L}} = \int \underline{\underline{B}}^T \, \underline{m} \, \underline{\underline{N}} \, dV \tag{4.10b}$$

$$d\underline{f}_n = \int \underline{\underline{N}}^T \, d\underline{f} \, dV + \int \underline{\underline{N}}^T \, d\underline{t} \, dS \tag{4.10c}$$

To formulate the flow problem, the continuity equation is adopted in the following form:

$$\nabla^T \; \underline{\underline{R}} \; \nabla \; (\gamma_w \;\; y \; - \; p_{steady} \; - \; p \;) \; / \; \gamma_w - \underline{m}^T \;\; \frac{\partial \underline{\varepsilon}}{\partial t} + \frac{n}{K_w} \; \frac{\partial p}{\partial t} = 0 \qquad (4.11)$$

where $\underline{\underline{R}}$ is the permeability matrix:

$$\underline{\underline{R}} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \qquad (4.12)$$

n is the porosity, K_w is the bulk modulus of the pore fluid and γ_w is the unit weight of the pore fluid. This continuity equation includes the sign convention that p_{steady} and p are considered positive for tension.

As the steady state solution is defined by the equation:

$$\nabla^T \; \underline{\underline{R}} \; \nabla \; (\gamma_w \;\; y \; - \; p_{steady} \;) \; / \; \gamma_w = 0 \qquad (4.13)$$

the continuity equation takes the following form:

$$\nabla^T \; \underline{\underline{R}} \; \nabla \, p \; / \; \gamma_w + \underline{m}^T \;\; \frac{\partial \underline{\varepsilon}}{\partial t} \; - \; \frac{n}{K_w} \; \frac{\partial p}{\partial t} = 0 \qquad (4.14)$$

Applying finite element discretisation using a Galerkin procedure and incorporating prescribed boundary conditions we obtain:

$$- \; \underline{\underline{H}} \;\; \underline{p}_n + \underline{\underline{L}}^T \;\; \frac{d \, \underline{v}}{d \, t} - \underline{\underline{S}} \;\; \frac{d \, \underline{p}_n}{d \, t} = \underline{q} \qquad (4.15)$$

where:

$$\underline{\underline{H}} \; = \int (\nabla \, \underline{\underline{N}})^T \;\; \underline{\underline{R}} \; \nabla \, \underline{\underline{N}} \; / \; \gamma_w \;\; dV \, , \;\; \underline{\underline{S}} \; = \int \frac{n}{K_w} \;\; \underline{\underline{N}}^T \;\; \underline{\underline{N}} \;\; dV \qquad (4.16)$$

and $\underline{q}$ is a vector due to prescribed outflow at the boundary. However within PLAXIS Version 8 it is not possible to have boundaries with non-zero prescribed outflow. The boundary is either closed or open with zero excess pore pressure. Hence $\underline{q} = \underline{0}$. In reality the bulk modulus of water is very high and so the compressibility of water can be neglected in comparison to the compressibility of the soil skeleton.

In PLAXIS the bulk modulus of the pore fluid is taken automatically according to (also see Reference Manual):

$$\frac{K_w}{n} \; = \; \frac{3 \, (\, \nu_u - \nu \,)}{(\, 1 - 2 \, \nu_u \,) (\, 1 + \nu \,)} \, K_{skeleton} \qquad (4.17)$$

Where ν_u has a default value of 0.495. The value can be modified in the input program on the basis of Skempton's B-parameter. For drained material and material just switched on, the bulk modulus of the pore fluid is neglected.

The equilibrium and continuity equations may be compressed into a block matrix equation:

$$\begin{bmatrix} \underline{\underline{K}} & \underline{\underline{L}} \\ \underline{\underline{L}}^T & -\underline{\underline{S}} \end{bmatrix} \begin{bmatrix} \frac{d\,\underline{v}}{d\,t} \\ \\ \frac{d\,\underline{p}_n}{d\,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \underline{\underline{H}} \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{p}_n \end{bmatrix} + \begin{bmatrix} \frac{d\,\underline{f}_n}{d\,t} \\ \\ \underline{q}_n \end{bmatrix} \tag{4.18}$$

A simple step-by-step integration procedure is used to solve this equation. Using the symbol Δ to denote finite increments, the integration gives:

$$\begin{bmatrix} \underline{\underline{K}} & \underline{\underline{L}} \\ \underline{\underline{L}}^T & -\underline{\underline{S}}^* \end{bmatrix} \begin{bmatrix} \Delta\underline{v} \\ \Delta\underline{p}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \Delta t\ \underline{\underline{H}} \end{bmatrix} \begin{bmatrix} \underline{v}_0 \\ \underline{p}_{n0} \end{bmatrix} + \begin{bmatrix} \Delta\,\underline{f}_n \\ \\ \Delta t\ \underline{q}_n^* \end{bmatrix} \tag{4.19}$$

where:

$$\underline{\underline{S}}^* = \alpha\ \Delta t\ \underline{\underline{H}} + \underline{\underline{S}} \qquad\qquad \underline{q}_n^* = \underline{q}_{n0} + \alpha\ \Delta\,\underline{q}_n \tag{4.20}$$

and $\underline{v}_0$ and $\underline{p}_{n0}$ denote values at the beginning of a time step. The parameter α is the time integration coefficient. In general the integration coefficient α can take values from 0 to 1. In PLAXIS the fully implicit scheme of integration is used with $\alpha = 1$.

4.3 ELASTOPLASTIC CONSOLIDATION

In general, when a non-linear material model is used, iterations are needed to arrive at the correct solution. Due to plasticity or stress-dependent stiffness behaviour the equilibrium equations are not necessarily satisfied using the technique described above. Therefore the equilibrium equation is inspected here. Instead of Eq. (4.9) the equilibrium equation is written in sub-incremental form:

$$\underline{\underline{K}}\ \delta\,\underline{v} + \underline{\underline{L}}\ \delta\,\underline{p}_n = \underline{r}_n \tag{4.21}$$

where $\underline{r}_n$ is the global residual force vector. The total displacement increment $\Delta\underline{v}$ is the summation of sub-increments $\delta\underline{v}$ from all iterations in the current step:

$$\underline{r}_n = \int \underline{\underline{N}}^T\ \underline{f}\ dV + \int \underline{\underline{N}}^T\ \underline{t}\ dS - \int \underline{\underline{B}}^T\ \underline{\sigma}\ dV \tag{4.22}$$

with:

$$\underline{f} = \underline{f}_0 + \Delta \underline{f} \qquad \text{and:} \qquad \underline{t} = \underline{t}_0 + \Delta \underline{t} \tag{4.23}$$

In the first iteration we consider $\underline{\sigma} = \underline{\sigma}_0$, i.e. the stress at the beginning of the step. Successive iterations are used on the current stresses that are computed from the appropriate constitutive model.

5 ELEMENT FORMULATIONS

In this chapter the interpolation functions of the finite elements used in PLAXIS are described. Each element consists of a number of nodes. Each node has a number of degrees-of-freedom that correspond to discrete values of the unknowns in the boundary value problem to be solved. In the case of deformation theory the degrees-of-freedom correspond to the displacement components, whereas in the case of groundwater flow the degrees-of-freedom are the groundwater heads. For consolidation problems degrees-of-freedom are both displacement components and (excess) pore pressures. In addition to the interpolation functions it is described which type of numerical integration over elements is used in PLAXIS.

5.1 INTERPOLATION FUNCTIONS FOR LINE ELEMENTS

Within an element the displacement field $\underline{u} = (u_x\ u_y)^T$ is obtained from the discrete nodal values in a vector $\underline{v} = (v_1\ v_2\ ...\ v_n)^T$ using interpolation functions assembled in matrix $\underline{\underline{N}}$:

$$\underline{u} = \underline{\underline{N}}\,\underline{v} \tag{5.1}$$

Hence, interpolation functions $\underline{\underline{N}}$ are used to interpolate values inside an element based on known values in the nodes. Interpolation functions are also denoted as shape functions.

Let us first consider a line element. Line elements are the basis for geotextile elements, plate elements and distributed loads. When the local position, ξ, of a point (usually a stress point or an integration point) is known, one can write for a displacement component u:

$$u(\xi) = \sum_{i=1}^{n} N_i(\xi) v_i \tag{5.2}$$

where:

v_i the nodal values,

$N_i(\xi)$ the value of the shape function of node i at position ξ,

$u(\xi)$ the resulting value at position ξ and

n the number of nodes per element.

In the graph, an example of a 3-node line element is given, which is compatible with the 6-node triangle elements in PLAXIS, since 6-node triangles have three nodes at a side. The shape functions N_i have the property that the function is equal to one at node i and zero at the other nodes. For 3-node line elements, where the nodes 1, 2 and 3 are located at ξ = -1, 0 and 1 respectively, the shape functions are given by:

$$N_1 = -\tfrac{1}{2}(1-\xi)\,\xi \qquad (5.3)$$

$$N_2 = (1+\xi)(1-\xi)$$

$$N_3 = \tfrac{1}{2}(1+\xi)\,\xi$$

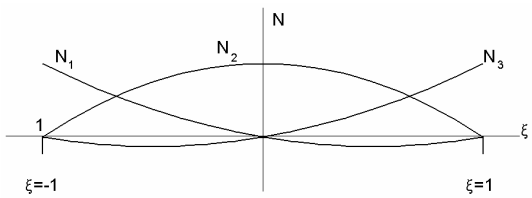


Figure 5.1 Shape functions for 3-node line element

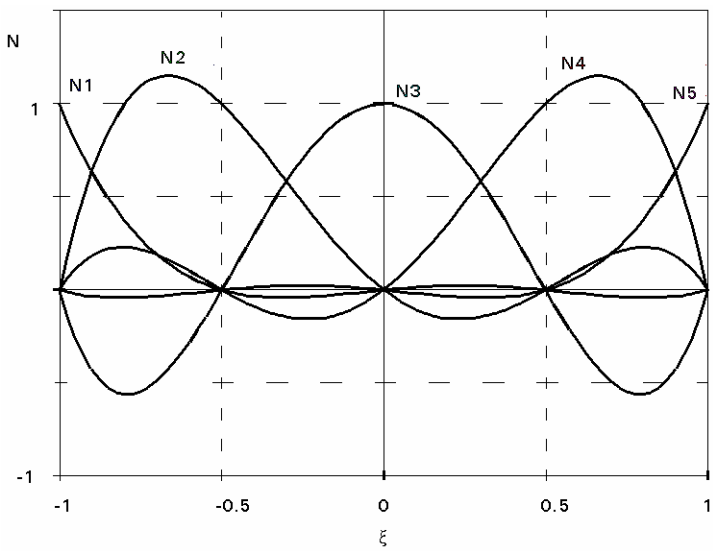


Figure 5.2 Shape functions for 5-node line element

When using 15-noded triangles, there are five nodes at a side. For 5-node line elements, where nodes 1 to 5 are at ξ = -1, -½, 0, ½ and 1 respectively, we have:

$$N_1 = -(1-\xi)(1-2\xi)\,\xi(-1-2\xi)/6 \qquad (5.4)$$

$$N_2 = 4(1-\xi)(1-2\xi)\,\xi(-1-\xi)/3$$

$$N_3 = (1-\xi)(1-2\xi)(-1-2\xi)(-1-\xi)$$

$$N_4 = 4(1-\xi)\,\xi(-1-2\xi)(-1-\xi)/3$$

$$N_5 = (1-2\xi)\,\xi(-1-2\xi)(-1-\xi)/6$$

5.2 INTERPOLATION FUNCTIONS FOR TRIANGULAR ELEMENTS

For triangular elements there are two local coordinates (ξ and η). In addition we use an auxiliary coordinate $\zeta = 1-\xi-\eta$. For 15-node triangles the shape functions can be written as (see the local node numbering as shown in Figure 5.3) :

$$
\begin{aligned}
N_1 &= \zeta(4\zeta-1)(4\zeta-2)(4\zeta-3)/6 \\
N_2 &= \xi(4\xi-1)(4\xi-2)(4\xi-3)/6 \\
N_3 &= \eta(4\eta-1)(4\eta-2)(4\eta-3)/6 \\
N_4 &= 4\,\zeta\xi(4\zeta-1)(4\xi-1) \\
N_5 &= 4\,\xi\eta(4\xi-1)(4\eta-1) \\
N_6 &= 4\,\eta\zeta(4\eta-1)(4\zeta-1) \\
N_7 &= \xi\zeta(4\zeta-1)(4\zeta-2)*8/3 \\
N_8 &= \zeta\xi(4\xi-1)(4\xi-2)*8/3 \\
N_9 &= \eta\xi(4\xi-1)(4\xi-2)*8/3 \\
N_{10} &= \xi\eta(4\eta-1)(4\eta-2)*8/3 \\
N_{11} &= \zeta\eta(4\eta-1)(4\eta-2)*8/3 \\
N_{12} &= \eta\zeta(4\zeta-1)(4\zeta-2)*8/3 \\
N_{13} &= 32\,\eta\xi\zeta(4\zeta-1) \\
N_{14} &= 32\,\eta\xi\zeta(4\xi-1) \\
N_{15} &= 32\,\eta\xi\zeta(4\eta-1)
\end{aligned}
\qquad (5.5)
$$

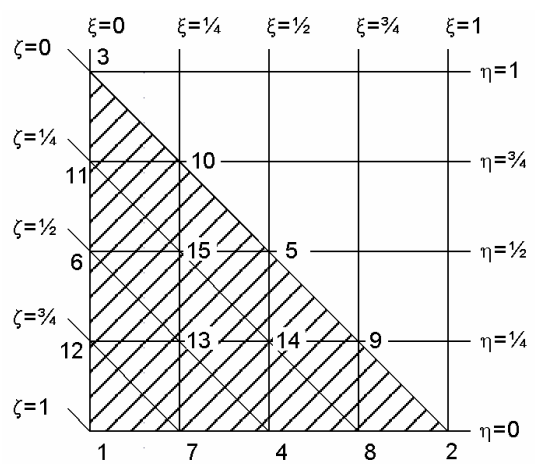


Figure 5.3 Local numbering and positioning of nodes

Similarly, for 6-node elements the shape functions are:

$$\begin{aligned} N_1 &= \zeta(2\zeta\text{-}1) \\ N_2 &= \xi(2\xi\text{-}1) \\ N_3 &= \eta(2\eta\text{-}1) \\ N_4 &= 4\,\zeta\,\xi \\ N_5 &= 4\,\xi\,\eta \\ N_6 &= 4\,\eta\,\zeta \end{aligned} \tag{5.6}$$

5.3 NUMERICAL INTEGRATION OF LINE ELEMENTS

In order to obtain the integral over a certain line or area, the integral is numerically estimated as:

$$\int_{\xi=-1}^{1} F(\xi)d\xi \approx \sum_{i=1}^{k} F(\xi_i)\; w_i \tag{5.7}$$

where $F(\xi_i)$ is the value of the function F at position ξ_i and w_i the weight factor for point i. A total of k sampling points is used.

Two methods are frequently used in PLAXIS, firstly Newton-Cotes integration, where the points ξ_i are chosen at the position of the nodes, and secondly Gauss integration where fewer points at special locations can be used to obtain high accuracy. The position and weight factors of the two types of integration are given in Table 5.1 and 5.2 respectively. Note that the sum of the weight factors is equal to 2.

Using Newton-Cotes integration one can exactly integrate polynominal functions one order below the number of points used. For Gauss-integration a polynomial function of degree $2k$-1 can be integrated exactly by using k points. For interface elements and geotextile elements PLAXIS uses Newton-Cotes integration, whereas for beam elements and the integration of boundary loads Gaussian integration is used.

Table 5.1 Newton-Cotes integration

	ξ_i	w_i
2 nodes	± 1	1
3 nodes	± 1, 0	1/3, 4/3
4 nodes	± 1, ± 1/3	1/4, ¾
5 nodes	± 1, ± 1/2, 0	7/45, 32/45, 12/45

Table 5.2 Gauss integration

	ξ_i	w_i
1 point	0.000000...	2
2 points	±0.577350...(±1/√3)	1
3 points	±0.774596... (±√0.6)	0.55555... (5/9)
	0.000000...	0.88888... (8/9)
4 points	±0.861136...	0.347854...
	±0.339981...	0.652145...
5 points	±0.906179...	0.236926...
	±0.538469...	0.478628...
	0.000000...	0.568888...

5.4 NUMERICAL INTEGRATION OF TRIANGULAR ELEMENTS

As for line elements, one can formulate the numerical integration over triangular elements as:

$$\int\int F(\xi,\eta)\ d\xi\ d\eta \approx \sum_{i=1}^{k} F(\xi_i,\eta_i)\ w_i \tag{5.8}$$

PLAXIS uses Gaussian integration within the triangular elements. For 6-node elements the integration is based on 3 sample points, whereas for 15-node elements 12 sample points are used. The position and weight factors of the integration points are given in Table 5.3 and 5.4. Note that, in contrast to the line elements, the sum of the weight factors is equal to 1.

Table 5.3 3-point integration for 6-node elements

Point	ξ_i	η_i	ζ_i	w_i
1, 2 & 3	1/6	1/6	2/3	1/3

Table 5.4 12-point integration for 15-node elements

Point	ξ_i	η_i	ζ_i	w_i
1,2 & 3	0.063089...	0.063089...	0.873821...	0.050845...
4 .. 6	0.249286...	0.249286...	0.501426...	0.116786...
7..12	0.310352...	0.053145...	0.636502...	0.082851...

5.5 DERIVATIVES OF SHAPE FUNCTIONS

In order to calculate Cartesian strain components from displacements, such as formulated in Eq. (2.10), derivatives need to be taken with respect to the global system of axes (x,y,z).

$$\underline{\varepsilon} = \underline{\underline{B}}_i \, \underline{v}_i \tag{5.9}$$

where

$$\underline{\underline{B}}_i = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial N_i}{\partial z} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial z} & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} & 0 & \dfrac{\partial N_i}{\partial x} \end{bmatrix} \tag{5.10}$$

Within the elements, derivatives are calculated with respect to the local coordinate system (ξ,η,ζ). The relationship between local and global derivatives involves the Jacobian J:

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \varsigma} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \varsigma} & \dfrac{\partial y}{\partial \varsigma} & \dfrac{\partial z}{\partial \varsigma} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{bmatrix} \tag{5.11}$$

Or inversely:

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \varsigma} \end{bmatrix} \tag{5.12}$$

The local derivatives $\partial N_i/\partial \xi$, etc., can easily be derived from the element shape functions, since the shape functions are formulated in local coordinates. The components of the Jacobian are obtained from the differences in nodal coordinates. The inverse Jacobian J^{-1} is obtained by numerically inverting J.

The Cartesian strain components can now be calculated by summation of all nodal contributions:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \sum_i \underline{\underline{B}}_i \begin{bmatrix} v_{x,i} \\ v_{y,i} \\ v_{z,i} \end{bmatrix} \tag{5.13}$$

where v_i are the displacement components in node i.

For a plane strain analysis, strain components in z-direction are zero by definition, i.e. $\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$. For an axisymmetric analysis, the following conditions apply:

$\varepsilon_{zz} = u_x / r$ and $\gamma_{yz} = \gamma_{zx} = 0$ (r = radius)

5.6 CALCULATION OF ELEMENT STIFFNESS MATRIX

The element stiffness matrix, $\underline{\underline{K}}^e$, is calculated by the integral (see also Eq. 2.25):

$$\underline{\underline{K}}^e = \int \underline{\underline{B}}^T \underline{\underline{D}}^e \underline{\underline{B}} \, dV \tag{5.14}$$

The integral is estimated by numerical integration as described in section 5.3. In fact, the element stiffness matrix is composed of submatrices $\underline{\underline{K}}^e_{ij}$ where i and j are the local nodes. The process of calculating the element stiffness matrix can be formulated as:

$$\underline{\underline{K}}^e_{ij} = \sum_k \underline{\underline{B}}^T_i \underline{\underline{D}}^e \underline{\underline{B}}_j w_k \tag{5.15}$$

6 THEORY OF SENSITIVITY ANALYSIS & PARAMETER PARIATION

This chapter presents some of the theoretical backgrounds of the sensitivity analysis and parameter variation module. The chapter does not give a full theoretical description of the methods of interval analysis. For a more detailed description you are referred to the literature e.g. [17] to [23].

6.1 SENSITIVITY ANALYSIS

A method for quantifying sensitivity in the sense discussed in this Section and in the Section *Sensitivity analysis and Parameter variation* of the Reference Manual is the sensitivity ratio η_{SR} [15]. The ratio is defined as the percentage change in output divided by the percentage change in input for a specific input variable, as shown in Eq. 6.1:

$$\eta_{SR} = \frac{\left[\frac{f(x_{L,R}) - f(x)}{f(x)}\right] \cdot 100\%}{\left[\frac{x_{L,R} - x}{x}\right] \cdot 100\%} \tag{6.1}$$

$f(x)$ is the reference value of the output variable using reference values of the input variables and $f(x_{L,R})$ is the value of the output variable after changing the value of one input variable, whereas x and $x_{L,R}$ in the denominator are the respective input variables. For the sensitivity ratio, an input variable, $x_{L,R}$, is varied individually across the entire range requiring $2N+1$ calculations, N being the number of varied parameters considered.

An extension to the sensitivity ratio and a more robust method of evaluating important sources of uncertainty is the sensitivity score, η_{SS}, which is the sensitivity ratio η_{SR} weighted by a normalized measure of the variability in an input variable, as given by Eq. 6.2:

$$\eta_{SS} = \eta_{SR} \cdot \frac{(\max x_R - \min x_R)}{x} \tag{6.2}$$

By normalising the measure of variability (i.e. the range divided by the reference value), this method effectively weights the ratios in a manner that is independent of the units of the input variable.

Performing a sensitivity analysis as described above, the sensitivity score of each variable, $\eta_{SS,i}$, on respective results $A,B,\ldots,Z$, (e.g. displacements, forces, factor of safety, etc.) at each construction step (calculation phase) can be quantified as shown in Table 6.1 (*Sensitivity matrix*). The total sensitivity score of each variable, $\Sigma\eta_{SS,i}$, results from summation of all sensitivity scores for each respective result at each construction step.

Table 6.1 Sensitivity matrix

	Respective results					
	A	B	…	Z	Σ	α
Input variables						%
x_1	$\eta_{SS,A1}$	$\eta_{SS,B1}$	…	$\eta_{SS,Z1}$	$\Sigma\eta_{SS,1}$	$\alpha(x_1)$
x_2	$\eta_{SS,A2}$	$\eta_{SS,B2}$	…	$\eta_{SS,Z2}$	$\Sigma\eta_{SS,2}$	$\alpha(x_2)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_N	$\eta_{SS,AN}$	$\eta_{SS,BN}$	…	$\eta_{SS,ZN}$	$\Sigma\eta_{SS,N}$	$\alpha(x_N)$

It should be noted, that the results of the sensitivity analysis appeared to be strongly dependent on the respective results used and thus results relevant for the problem investigated have to be chosen based on sound engineering judgement. Finally, the total relative sensitivity $\alpha(x_i)$ for each input variable is then given by [16] as

$$\alpha(x_i) = \frac{\Sigma\eta_{SS,i}}{\sum_{i=1}^{N}\Sigma\eta_{SS,i}} \tag{6.3}$$

Figure 6.1 shows the total relative sensitivity of each parameter $\alpha(x_i)$ in diagram form in order to illustrate the 'major' variables.

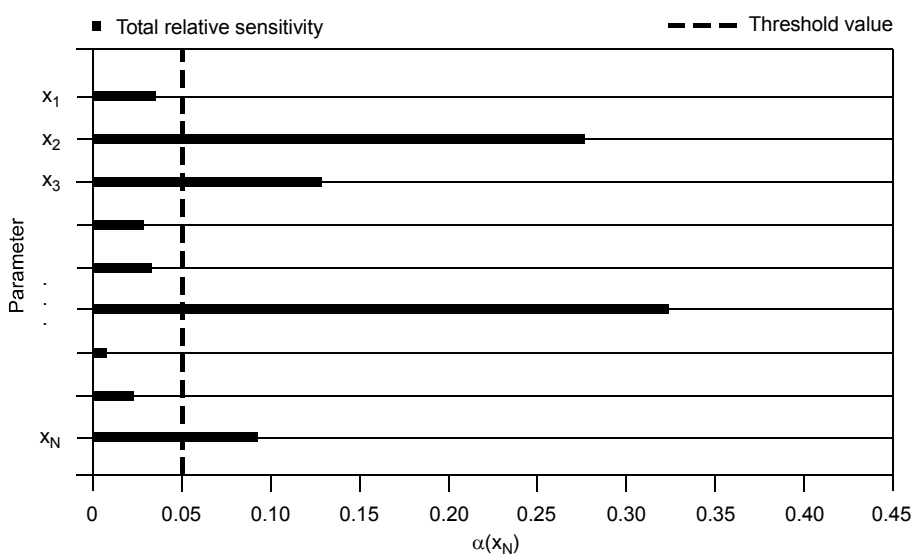


Figure 6.1 Total relative sensitivity in diagram form

6.1.1 DEFINITION OF THRESHOLD VALUE

The benefit of such an analysis is twofold: Firstly, the results are the basis for a decision-making in order to reduce the computational effort involved when utilizing a parameter variation, i.e. at this end a decision has to be made (definition of a threshold value), which variables (parameters) should be used in further calculations and which one can be treated as deterministic values as their influence on the result is not significant (Figure 6.1). Secondly, sensitivity analysis can be applied for example to design further investigation programs to receive additional information about parameters with high sensitivity in order to reduce the uncertainty of the system response, i.e. the result may act as a basis for the design of an investigation program (laboratory and/or in situ tests).

6.2 THEORY OF PARAMETER VARIATION

The parameter variation used in the PLAXIS parameter variation module refer to classical set theory where uncertainty is represented in terms of closed intervals (bounds) assuming that the true value of the relevant unknown quantity is captured ($X \in [x_{min},x_{max}]$). In general, an interval is defined as a pair of elements of some (at least partially) ordered sets [24]. An interval is identified with the set of elements lying between the interval endpoints (including the endpoints) and using the set of real numbers as the underlying ordered set (real intervals). Hence, all intervals are closed sets. Thus, a (proper) real interval **x** is a subset of the set of real numbers R such that:

$$\mathbf{x} = \left[x_{\min}, x_{\max}\right] = \left\{x' \in \mathrm{R} \mid x_{\min} \le x' \le x_{\max}\right\} \tag{6.4}$$

where $x_{min} \le x_{max}$ and $x_{min} = \sup(\mathbf{x})$, $x_{max} = \inf(\mathbf{x})$ are endpoints of the interval **x**. In general, x' denotes any element of the interval **x**. If the true values of the parameters of interest are bounded by intervals, this will always ensure a reliable estimate (worst/best-case analysis) based on the information available.

For the parameter variation, the input parameters x_i are treated as interval numbers ($x_{i,min}/x_{i,max}$) whose ranges contain the uncertainties in those parameters. The resulting computations, carried out entirely in interval form, would then literally carry the uncertainties associated with the data through the analysis. Likewise, the final outcome in interval form would contain all possible solutions due to the variations in input.

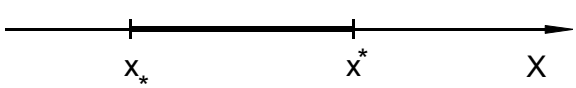


Figure 6.2 Total relative sensitivity in diagram form

6.2.1 BOUNDS ON THE SYSTEM RESPONSE

Let X be a non-empty set containing all the possible values of a parameter x and $y = f(x)$, $f : X \rightarrow Y$ be a real-valued function of x. The interval of the system response through f, can be calculated by means of a function used in set theory. In fact, if x belongs to set A, then the range of y is

$$f(A) = \{f(x) : x \in A\}. \tag{6.5}$$

Here, the set A is called the focal element. The basic step is the calculation of the system response through function f which represents here a numerical model. In general, this involves two global optimisation problems which can be solved by applying twice the techniques of global optimisation (e.g. [25],[26]) to find the lower and upper bound, $y_{\min}$ and $y_{\max}$, respectively, of the system response:

$$f(A) = [y_{\min}, y_{\max}] \tag{6.6}$$

where

$$y_{\min} = \min_{x \in A} f(x) \tag{6.7}$$

$$y_{\max} = \max_{x \in A} f(x) \tag{6.8}$$

In the absence of any further information a so-called *calculation matrix*, can be constructed by assuming independence between parameters x [11]. A is the Cartesian product of N finite intervals $x_1 \times \ldots \times x_N$ (calculation matrix), therefore it is a N-dimensional box (interval vector) whose 2^N vertices are indicated as v_k, $k = 1,\ldots,2^N$, N being the number of parameters considered. The lower and upper bounds $y_{\min}$ and $y_{\max}$ of the system response can be obtained as follows:

$$y_{\min} = \min_k \{f(v_k) : k = 1,...2^N\} \tag{6.9}$$

$$y_{\max} = \max_k \{f(v_k) : k = 1,...2^N\} \tag{6.10}$$

If $f(A)$ has no extreme value in the interior of A, except at the vertices, Equations (6.9) and (6.10) are correct in which case the methods of interval analysis are applicable, e.g. the Vertex method [27]. If, on the other hand, $f(A)$ has one or more extreme values in the interior of A, then Equations (6.9) and (6.10) can be taken as approximations to the true global minimum and maximum value.

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APPENDIX A - CALCULATION PROCESS
Finite element calculation process based on the elastic stiffness matrix

Read input data

Form stiffness matrix $\underline{\underline{K}} = \int \underline{\underline{B}}^T \; \underline{\underline{D}}^e \; \underline{\underline{B}} \; d\,V$

New step $i \rightarrow i + 1$

Form new load vector $\underline{f}_{ex}^{\,i} = \underline{f}_{ex}^{\,i-1} + \Delta \underline{f}_{ex}$

Form reaction vector $\underline{f}_{in} = \int \underline{\underline{B}}^T \; \underline{\sigma}_c^{i-1} \, d\,V$

Calculate unbalance $\Delta \underline{f} = \underline{f}_{ex}^{\,i} - \underline{f}_{in}$

Reset displacement increment $\Delta \underline{v} = 0$

New iteration $j \rightarrow j + 1$

Solve displacements $\delta \underline{v} = \underline{\underline{K}}^{-1} \Delta \underline{f}$

Update displacement increments $\Delta \underline{v}^{\,j} = \Delta \underline{v}^{\,j-1} + \delta \underline{v}$

Calculate strain increments $\Delta \underline{\varepsilon} = \underline{\underline{B}} \, \Delta \underline{v}$; $\delta \underline{\varepsilon} = \underline{\underline{B}} \, \delta \underline{v}$

Calculate stresses: Elastic $\underline{\sigma}^{tr} = \underline{\sigma}_c^{i-1} + \underline{\underline{D}}^e \, \Delta \underline{\varepsilon}$

Equilibrium $\underline{\sigma}^{eq} = \underline{\sigma}_c^{i,j-1} + \underline{\underline{D}}^e \, \delta \underline{\varepsilon}$

Constitutive $\underline{\sigma}_c^{i,j} = \underline{\sigma}^{tr} - \dfrac{\left\langle f\left(\underline{\sigma}^{tr}\right)\right\rangle}{d} \underline{\underline{D}}^e \dfrac{\partial g}{\partial \underline{\sigma}}$

Form reaction vector $\underline{f}_{in} = \int \underline{\underline{B}}^T \; \underline{\sigma}_c^{i,j} \, d\,V$

Calculate unbalance $\Delta \underline{f} = \underline{f}_{ex}^{\,i} - \underline{f}_{in}$

Calculate error $e = \dfrac{|\Delta \underline{f}|}{|\underline{f}_{ex}^{\,i}|}$

Accuracy check if $e > e_{tolerated} \rightarrow$ new iteration

Update displacements $\underline{v}^{\,i} = \underline{v}^{\,i-1} + \Delta \underline{v}$

Write output data (results)

If not finished $\rightarrow$ new step

Finish

APPENDIX B - SYMBOLS

$\underline{\underline{B}}$	:	Strain interpolation matrix
$\underline{\underline{D}}^e$	:	Elastic material stiffness matrix representing Hooke's law
f	:	Yield function
$\underline{f}$	:	Load vector
g	:	Plastic potential function
k	:	Permeability
K^r	:	Permeability reduction function
$\underline{\underline{K}}$	:	Stiffness matrix
$\underline{\underline{K}}$	:	Flow matrix
$\underline{\underline{L}}$	:	Differential operator
$\underline{L}$	:	Coupling matrix
$\underline{\underline{M}}$	:	Material stiffness matrix
$\underline{\underline{N}}$	:	Matrix with shape functions
p	:	Pore pressure (negative for pressure)
$\underline{p}$	:	Body forces vector
q	:	Specific discharge
$\underline{Q}$	:	Vector with nodal discharges
$\underline{r}$	:	Unbalance vector
$\underline{\underline{R}}$	:	Permeability matrix
t	:	Time
$\underline{t}$	:	Boundary tractions
$\underline{u}$	:	Vector with displacement components
$\underline{v}$	:	Vector with nodal displacements
V	:	Volume
w	:	Weight factor
γ	:	Volumetric weight
$\underline{\varepsilon}$	:	Vector with strain components
λ	:	Plastic multiplier
$\xi\ \eta\ \zeta$	:	Local coordinates
$\underline{\sigma}$	:	Vector with stress components
ϕ	:	Groundwater head
ω	:	Integration constant (explicit: ω=0; implicit: ω=1)