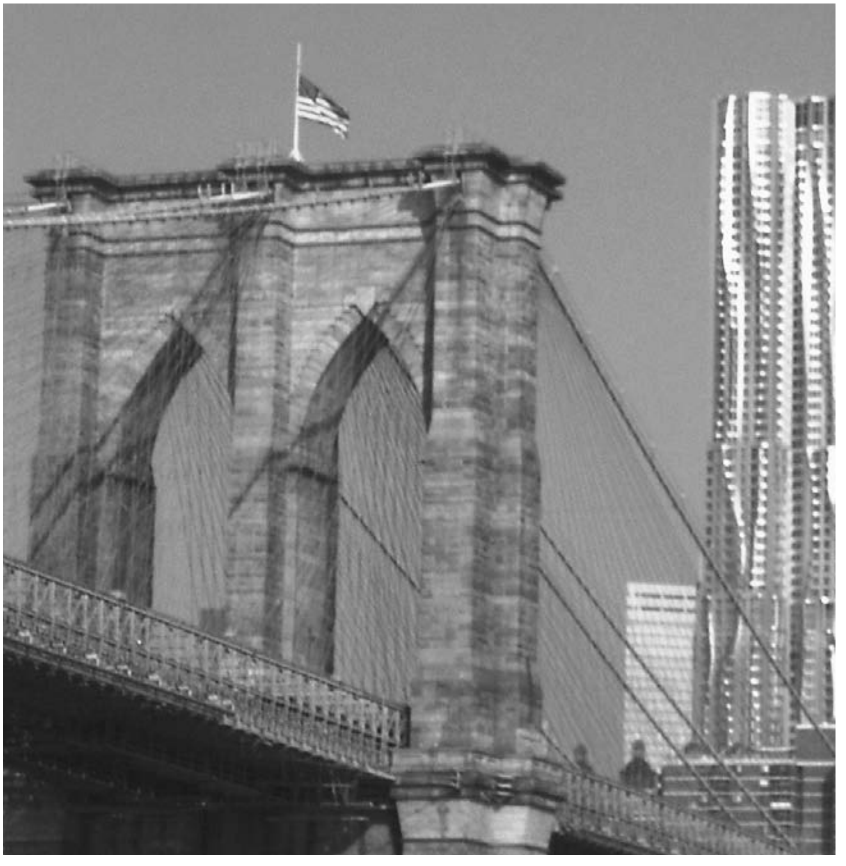
10 INTRODUCTION TO NONLINEAR STRUCTURAL ANALYSIS



Beekman Tower and Brooklyn Bridge, New York
(Courtesy of Ed Fitzgerald)

Thus far in this text, we have focused our attention on the linear analysis of structures, which it may be recalled, is based on two fundamental assumptions, namely, (a) *geometric linearity* implying that the structure's deformations are so small that the member strains can be expressed as linear functions of joint displacements and the equilibrium equations can be based on the undeformed geometry of the structure, and (b) *material linearity*, represented by the linearly elastic stress-strain relationship for the structural material. The linear analysis (sometimes also referred to as the *first-order analysis*) generally proves adequate for predicting the performance of most common types of engineering structures under service (working) loading conditions. However, as the loads increase beyond service levels into the failure range, the accuracy of the linear analysis gradually deteriorates because the response of the structure usually becomes increasingly nonlinear as its deformations increase and/or its material is strained beyond the yield point. In some structures, such as cable suspension systems, the load carrying capacity relies on geometric nonlinearity even under normal service conditions. Because of its inherent limitations, linear analysis cannot be used to predict instability phenomena and ultimate load capacities of structures.

With the recent introduction of design specifications based on the ultimate strengths of structures, the use of nonlinear analysis in structural design is increasing. In a nonlinear analysis, the restrictions of linear analysis are removed by formulating the equations of equilibrium on the deformed geometry of the structure that is not known in advance, and/or taking into account the effects of inelasticity of the structural material. The load-deformation (stiffness) relationships thus obtained for the structure are nonlinear, and are usually solved using iterative techniques.

The objective of this chapter is to introduce the reader to the exciting and still-evolving field of nonlinear structural analysis. Because of space limitations, only the basic concepts of geometrically nonlinear analysis of plane trusses are covered herein. However, it should be realized that a realistic prediction of structural response in the failure range generally requires consideration of the effects of both geometric and material nonlinearities in the analysis. For a more detailed study, the reader should refer to one of the books devoted entirely to the subject of nonlinear structural analysis, such as [8, 9].

We begin this chapter with an intuitive discussion of the basic concept of geometrically nonlinear analysis, and how it differs from the conventional linear analysis in Section 10.1. A matrix stiffness formulation for geometrically nonlinear analysis of plane trusses is then developed in Section 10.2. While a block diagram summarizing the various steps of nonlinear analysis is provided, the programming details are not covered herein; they are, instead, left as an exercise for the reader. The computer program for geometrically nonlinear analysis of plane trusses can be conveniently adapted from that for the linear analysis of such structures, via relatively straightforward modifications that should become apparent as the nonlinear analysis is developed in this chapter.

10.1 BASIC CONCEPT OF GEOMETRICALLY NONLINEAR ANALYSIS

As stated before, in the linear analysis, the structure's deformations are assumed to be so small that the member strains are expressed as linear functions of joint displacements and the equilibrium equations are based on the undeformed geometry of the structure. In geometrically nonlinear analysis, the restrictions of small deformations are removed by formulating the strain-displacement relations and the equilibrium equations on the deformed geometry of the structure.

To illustrate the basic concept of geometrically nonlinear analysis, consider the two-member plane truss composed of a linearly elastic material, shown in Fig. 10.1(a). Note that the truss is symmetric and is loaded symmetrically with a vertical load P. Thus, it is considered to have only one degree of freedom, which is the vertical displacement δ of the free joint 2.

As shown in Fig. 10.1(b), in the linear analysis, the joint displacement δ is assumed to be so small that the member axial deformations u equal the components of δ in the undeformed directions of the members, that is

$$u \cong \delta \sin\theta \tag{10.1}$$

in which, θ denotes the angle of inclination of members in the undeformed configuration. The member axial strain ε can now be expressed as a linear function of joint displacement δ as

$$\varepsilon = \frac{u}{L} \cong \left(\frac{\sin\theta}{L}\right)\delta \tag{10.2}$$

with $L =$ undeformed length of members. Recall that in linear analysis, the equilibrium equations are based on the undeformed geometry of the structure. Figure 10.1(b) shows the free body diagram of joint 2 of the truss in the undeformed configuration, with the member axial forces Q inclined in the undeformed member directions (i.e., at angles θ with the horizontal). By considering the equilibrium of the joint in the vertical direction, we write

$$Q \cong \frac{P}{2\sin\theta} \tag{10.3}$$

from which we obtain the expression for member axial stress σ,

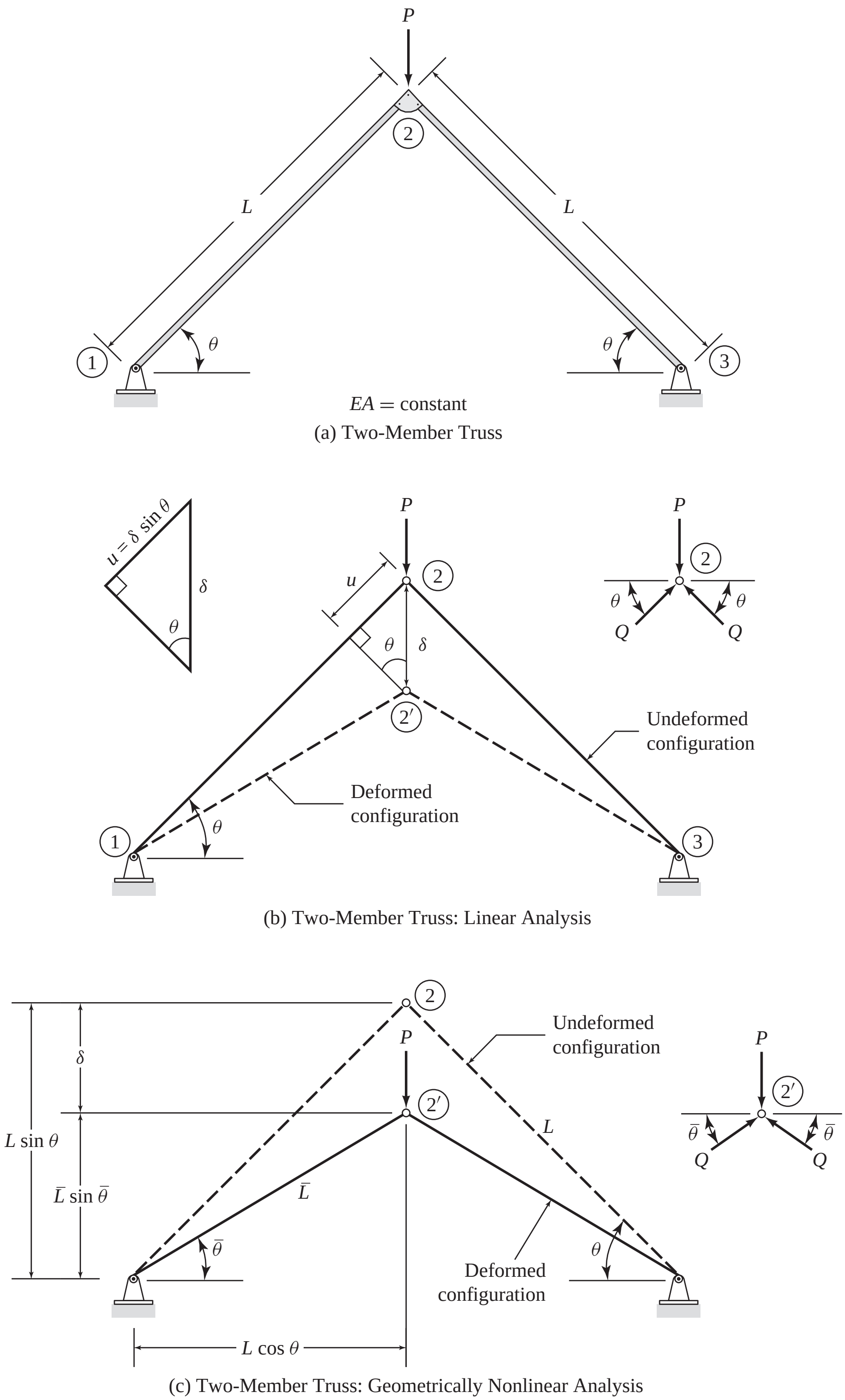
$$\sigma = \frac{Q}{A} \cong \frac{P}{2A\sin\theta} \tag{10.4}$$

For linearly elastic material,

$$\sigma = E\varepsilon \tag{10.5}$$

By substituting Eqs. (10.2) and (10.4) into Eq. (10.5), we obtain the desired (stiffness) relationship between the load P and the displacement δ of the truss based on the linear analysis:

$$\boxed{P \cong \left(\frac{2EA\sin^2\theta}{L}\right)\delta} \tag{10.6}$$

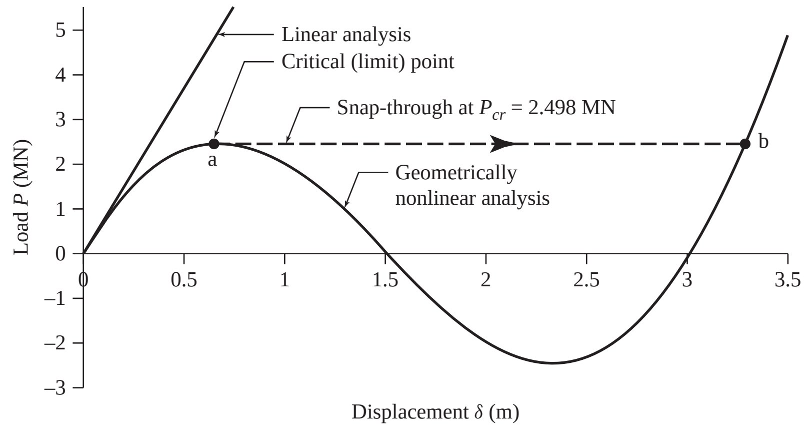


(a) Two-Member Truss

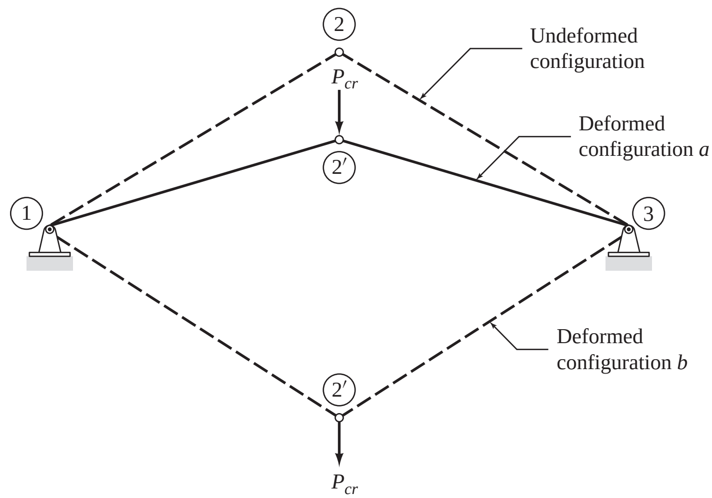
(b) Two-Member Truss: Linear Analysis

(c) Two-Member Truss: Geometrically Nonlinear Analysis

Fig. 10.1



(d) Response of Shallow Two-Member Truss



(e) Snap-Through of Two-Member Truss

Fig. 10.1 (*continued*)

In the geometrically nonlinear analysis, we allow the joint displacement δ to be arbitrarily large, and consider the truss to be in equilibrium in its deformed configuration as shown in Fig. 10.1(c). From this figure, we can see that the member lengths $\bar{L}$ and orientations $\bar{\theta}$, in the deformed configuration, can be expressed in terms of δ as

$$\bar{L} = \sqrt{(L\cos\theta)^2 + (L\sin\theta - \delta)^2} = L\sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta} \tag{10.7}$$

$$\sin\bar{\theta} = \frac{L\sin\theta - \delta}{\bar{L}} = \frac{\sin\theta - \left(\frac{\delta}{L}\right)}{\sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta}} \tag{10.8}$$

The member axial deformations and strains are now based on the actual deformed geometry of the members as

$$u = L - \bar{L} = L\left[1 - \sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta}\,\right] \tag{10.9}$$

$$\varepsilon = \frac{u}{L} = 1 - \sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta} \tag{10.10}$$

The free body diagram of joint 2 in the deformed configuration of the truss is shown in Fig. 10.1 (c), in which the member axial forces Q are inclined in the deformed member directions (i.e., at angles $\bar{\theta}$ with the horizontal). By considering the equilibrium of the joint in the vertical direction, we write the equilibrium equation

$$Q = \frac{P}{2\sin\bar{\theta}} = \frac{P\sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta}}{2\left[\sin\theta - \left(\frac{\delta}{L}\right)\right]} \tag{10.11}$$

which yields the expression for member axial stress as

$$\sigma = \frac{Q}{A} = \frac{P\sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta}}{2A\left[\sin\theta - \left(\frac{\delta}{L}\right)\right]} \tag{10.12}$$

Finally, by substituting the expressions for strain (Eq. (10.10)) and stress (Eq. (10.12)) into the stress-strain relationship $\sigma = E\varepsilon$ (Eq. (10.5)), we obtain the desired nonlinear (stiffness) relationship between the load P and the displacement δ of the truss:

$$\boxed{P = 2EA\left[\sin\theta - \left(\frac{\delta}{L}\right)\right]\left[\frac{1 - \sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta}}{\sqrt{1 + \left(\frac{\delta}{L}\right)^2 - 2\left(\frac{\delta}{L}\right)\sin\theta}}\right]} \tag{10.13}$$

By comparing the equations used to obtain the linear solution (Eqs. (10.1) through (10.6)) with those used to derive the geometrically nonlinear solution (Eqs. (10.7) through (10.13)), we notice two basic differences between the two types of analyses. The first difference is in the expressions of member axial deformation u in terms of the joint displacement δ (Eqs. (10.1) vs. (10.9)). In the linear analysis, δ is assumed to be so small that u can be expressed as the component of δ in the undeformed member direction, thereby yielding a linear relationship between u and δ (Eq. (10.1). In the geometrically nonlinear formulation, δ is allowed to be arbitrarily large and the relationship between u and δ is based on the exact geometry of the member's deformed configuration, thereby yielding a highly nonlinear relationship between u and δ (Eq. (10.9). The second basic distinction between the two types of analyses is in the way the equilibrium equations are established (Eqs. (10.3) vs. (10.11)). In linear analysis, the equilibrium equation is based on the undeformed geometry of the truss (thereby neglecting the joint displacement δ altogether). This assumption yields a direct linear relationship between the member axial force Q and the joint load P, which does not involve δ (Eq. (10.3)). In geometrically nonlinear analysis, however, since the equilibrium equation is based on the deformed configuration of the truss, the expression for Q not only contains P, but also involves nonlinear functions of δ (Eq. (10.11)). The reader is encouraged to verify that the linear solution (Eq. (10.6)) can be obtained by linearizing the geometrically nonlinear solution, that is, by expanding Eq. (10.13) via series expansion and retaining only the linear term of the series.

Geometrically nonlinear analysis provides important insight into the stability behavior of structures that is beyond the reach of linear analysis. Figure 10.1(d) shows the response of a typical shallow two-member truss as predicted by the linear and geometrically nonlinear analyses. These load-displacement plots are computed using the numerical values: $\theta = 30^{\circ}$, $L = 3\text{m}$, $E = 70$ GPa, and $A = 645.2\ \text{mm}^2$. It can be seen from this figure that the accuracy of the linear analysis gradually deteriorates as the magnitude of load P increases and the linear solution deviates from the exact geometrically nonlinear solution. With increasing load, the response becomes increasingly nonlinear as the truss's stiffness progressively decreases. This decrease in stiffness is characterized by a decrease in the slope of the tangent of the load-displacement curve. Note that at point a, where the curve reaches a peak, the slope of its tangent (called *tangent stiffness*) becomes zero, indicating that the structure's resistance to any further increase in load has vanished. Point a is referred to as a *critical* or *limit point*, because any further increase in load causes the truss to *snap-through* into an inverted configuration defined by point b on the response curve. The displacement of the truss at (or just prior to reaching) the critical point can be determined by setting to zero the derivative of Eq. (10.13) with respect to δ. This yields

$$\delta_{cr} = L\left[\sin\theta - \cos\theta\sqrt{\frac{1}{(\cos\theta)^{2/3}} - 1}\,\right] \tag{10.14}$$

The critical load P_{cr}, at which the *snap-through instability* occurs, can be calculated by substituting the value of δ_{cr} obtained from Eq. (10.14), for δ into Eq. (10.13).

We can see from Fig. 10.1(d) that the equilibrium configurations defined by the portion of the response curve between points a and b correspond to load levels below the critical level. Thus, unless the load magnitude can somehow be reduced after it has reached P_{cr}, the truss will snap-through from configuration a into the inverted configuration b (Fig. 10.1(e)).

It should be pointed out that the nonlinear response of the two-member truss (also known as the von Mises truss) considered herein has been examined by a number of researchers, and it is frequently used as a benchmark to validate the accuracy of computer programs for geometrically nonlinear structural analysis. It has been shown in references [16, 17] that while the shallow trusses (with $\theta \leq 69.295^{\circ}$) exhibit snap-through instability as discussed in the preceding paragraphs, the steep two-member trusses, with $\theta > 69.295^{\circ}$, experience bifurcation type of instability. As bifurcation instability occurs when the truss loses its stiffness in the horizontal direction, its detection requires analysis of a two degree-of-freedom model of the truss, instead of the single degree-of-freedom model used herein. A general formulation for geometrically nonlinear analysis of multi degree-of-freedom plane trusses is developed in the next section.

10.2 GEOMETRICALLY NONLINEAR ANALYSIS OF PLANE TRUSSES

In this section, we develop a general matrix stiffness method for geometrically nonlinear analysis of plane trusses [33, 45]. The process of developing the analytical models for nonlinear analysis (i.e., establishing a global coordinate system, numbering of joint and members, and identifying degrees of freedom and restrained coordinates) is the same as that for linear analysis of plane trusses (Chapter 3). However, the member local coordinate systems are now defined differently than in the case of linear analysis. Recall from Chapter 3 that in linear analysis, the local coordinate system is positioned in the *initial undeformed state* of the member, and it remains in that position regardless of where the member actually displaces due to the effect of external loads. In geometrically nonlinear analysis, it is more convenient to use a local coordinate system that is attached to, and displaces (translates and/or rotates) with, the member as the structure deforms. As shown in Fig. 10.2, *the origin of the local xyz coordinate system for a member is always located at the beginning,* b'*, of*

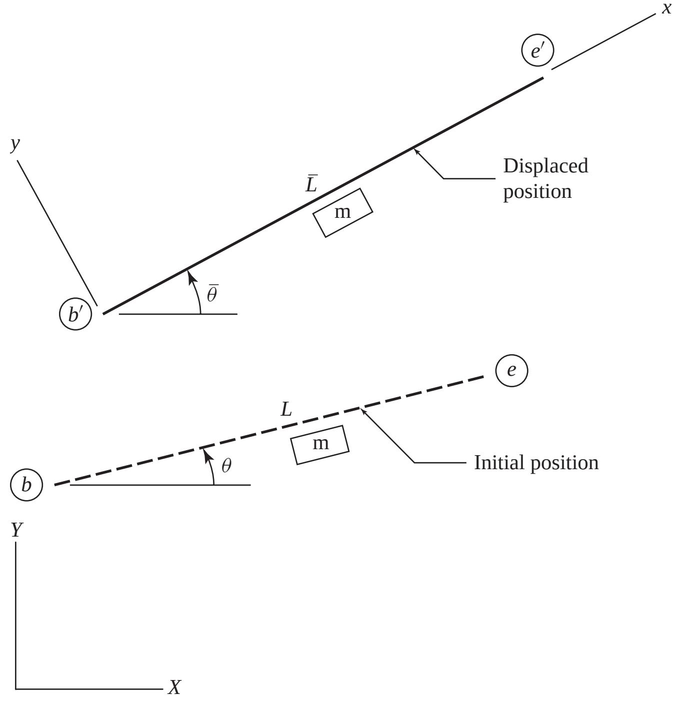


Fig. 10.2 *Corotational (or Eulerian) Local Coordinate System*

the member in its deformed state, with the x axis directed along the member's centroidal axis in the deformed state. The positive direction of the y axis is defined so that the coordinate system is right-handed, with the local z axis pointing in the positive direction of the global Z axis. This type of coordinate system, which continuously displaces with the member, is called an *Eulerian* or *corotational* coordinate system. The main advantage of using such a coordinate system is that it enables us to separate the member's axial deformation from its rigid body displacement, which is considered to be arbitrarily large in geometrically nonlinear analysis.

Member Force-Displacement Relations

To establish the member force-displacement relations, let us focus our attention on an arbitrary prismatic member m of a plane truss. When the truss is subjected to external loads, member m deforms and axial forces are induced at its ends. Figure 10.3 shows the displaced position of the member in its local coordinate system. Note that because of the modified definition of the local coordinate system, only one degree of freedom (that is the member axial deformation) is now needed to completely specify the displaced position of the member. The axial deformation u of the member can be expressed in terms of its initial and deformed lengths (L and $\bar{L}$, respectively) as

$$\boxed{u = L - \bar{L}} \tag{10.15}$$

in which, the axial deformation u is considered as positive when it corresponds to the shortening of the member's length, and negative when representing the elongation. Similarly, the member axial force Q is considered to be positive when compressive, and negative when tensile. To establish the relationship between the member's axial force Q and deformation u, we recall that the member's axial stress σ and axial strain ε are defined as

$$\sigma = \frac{Q}{A} \qquad \text{and} \qquad \varepsilon = \frac{u}{L} \tag{10.16}$$

and for linear elastic material, the stress-strain relationship is given by

$$\sigma = E\varepsilon \tag{10.17}$$

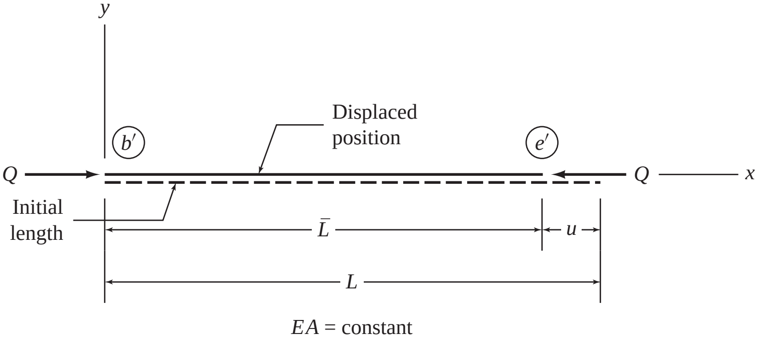


Fig. 10.3 *Member Axial Force and Deformation in the Local Coordinate System*

Substitution of Eqs. (10.16) into Eq. (10.17) yields the following force-displacement relation for the members of plane trusses in their local coordinate systems:

$$\boxed{Q = \left(\frac{EA}{L}\right) u} \tag{10.18}$$

Next, we consider the member force-displacement relations in the global coordinate system. Figures 10.4 (a) and (b) show the initial and displaced positions of an arbitrary member m of a plane truss. In Fig. 10.4(a), the member is depicted to be in equilibrium under the action of (local) axial forces Q; whereas in Fig. 10.4(b), the same member is shown to be in equilibrium under the action of an equivalent system of end forces $\mathbf{F}$ acting in the directions of the global X and Y coordinate axes. As indicated in Fig. 10.4(b), the global member end forces $\mathbf{F}$ and end displacements $\mathbf{v}$ are numbered in the same manner as in the case of linear analysis (Chapter 3), except that the forces $\mathbf{F}$ now act at the ends of the member in its deformed state.

Now, suppose that a member's global end displacements $\mathbf{v}$ (which may be arbitrarily large) are specified, and our objective is to find the corresponding end forces $\mathbf{F}$ so that the member is in equilibrium in its displaced position. If X_b, Y_b, and X_e, Y_e denote the global coordinates of the joints in their undeformed configurations, to which the member ends b and e, respectively are attached, then the initial (undeformed) length L of the member can be expressed (via Pythagorean theorem) as

$$L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2} \tag{10.19}$$

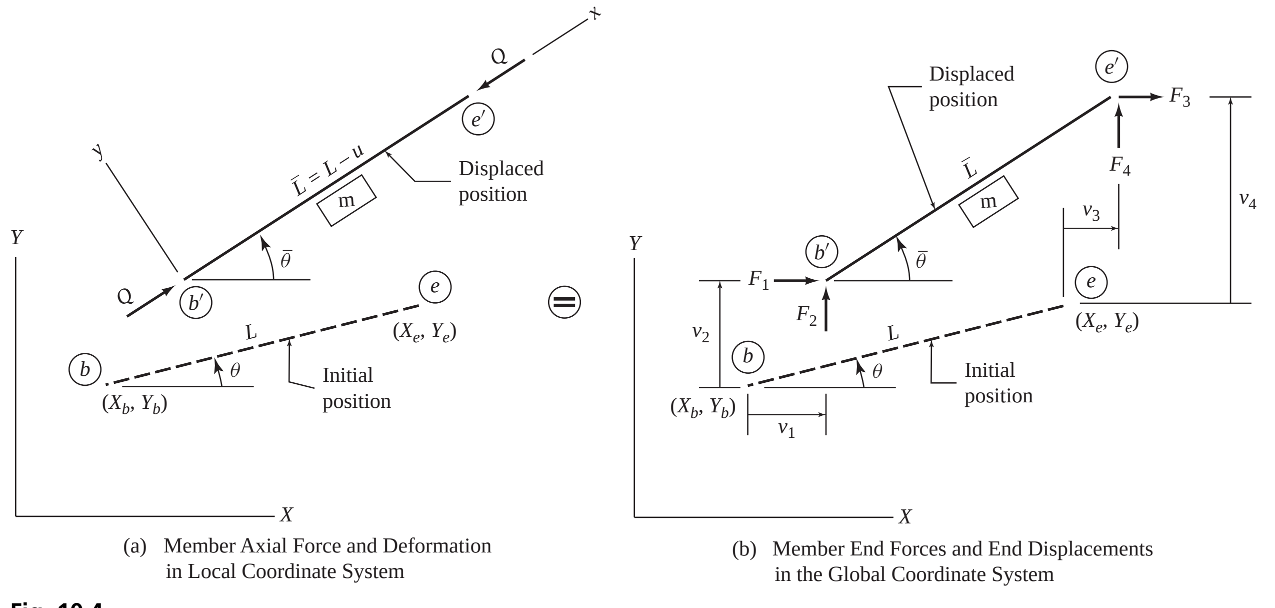


Fig. 10.4

Similarly, we can see from Fig. 10.4(b) that the deformed member length $\bar{L}$, and the direction cosines of the member in its displaced position, can be expressed in terms of the initial global coordinates and displacements **v** of the member's ends by the following relationships:

$$\boxed{\bar{L} = \sqrt{[(X_e + v_3) - (X_b + v_1)]^2 + [(Y_e + v_4) - (Y_b + v_2)]^2}} \tag{10.20}$$

$$\boxed{c_X = \cos\bar{\theta} = \frac{(X_e + v_3) - (X_b + v_1)}{\bar{L}}} \tag{10.21}$$

$$\boxed{c_Y = \sin\bar{\theta} = \frac{(Y_e + v_4) - (Y_b + v_2)}{\bar{L}}} \tag{10.22}$$

in which, $\bar{\theta}$ represents the angle measured counterclockwise from the positive direction of the global X axis to the positive direction of the local x axis of the member in its displaced position.

By comparing Figs. 10.4(a) and (b), we observe that at end b' of the member, the global forces F_1 and F_2 must be, respectively, equal to the components of (local) axial force Q in the directions of the global X and Y axes; that is,

$$F_1 = c_X Q \tag{10.23a}$$

$$F_2 = c_Y Q \tag{10.23b}$$

By using the same reasoning at end e', we express the global forces F_3 and F_4 in terms of Q as

$$F_3 = -c_X Q \tag{10.23c}$$

$$F_4 = -c_Y Q \tag{10.23d}$$

Equations 10.23(a) through (d) can be written in matrix form as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} c_X \\ c_Y \\ -c_X \\ -c_Y \end{bmatrix} Q \tag{10.24}$$

or, symbolically as

$$\boxed{\mathbf{F} = \mathbf{T}^T Q} \tag{10.25}$$

with the 1×4 transformation matrix **T** given by

$$\mathbf{T} = [\, c_X \quad c_Y \quad -c_X \quad -c_Y \,] \tag{10.26}$$

It should be recognized that the set of Eqs. (10.15), (10.18) through (10.22), and (10.24), does implicitly express **F** in terms of **v**, and therefore, is considered to represent the geometrically nonlinear force-displacement relations for members of plane trusses in the global coordinate system. Because of the highly nonlinear nature of some of the equations involved, it is quite cumbersome to express **F** explicitly in terms of **v**, as was previously done in the case of linear analysis. Note that if the global end displacements **v** of a member are known, its corresponding end forces **F** can be evaluated by first calculating L, $\bar{L}$, c_X, and c_Y using Eqs. (10.19) through (10.22); then evaluating the member axial deformation u and force Q, respectively, by applying Eqs. (10.15) and (10.18); and finally determining the member global end forces **F** via Eq. (10.24). It is important to realize that of all these equations, only one, that is, $Q = EAu/L$ (Eq. (10.18)), involves the material properties of the member. The remaining equations are essentially of a geometric character, and are *exact* in the sense that they are valid for arbitrarily large joint displacements.

The foregoing equations are also necessary and sufficient for establishing the geometrically nonlinear load-deformation relationships for the entire structure. The procedure for establishing such relations is essentially the same as in the case of linear analysis, and involves using member code numbers as illustrated by the following example.

EXAMPLE 10.1

By using geometrically nonlinear analysis, determine the joint loads **P** that cause the two-member truss to deform into the configuration shown in Fig. 10.5(a) on the next page.

SOLUTION

Joint Displacements: Using the analytical model of the truss shown in Fig. 10.5(b), we express the given deformed configuration in terms of its joint displacement vector,

$$\mathbf{d} = \begin{array}{c} \begin{bmatrix} 10 \\ -4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \text{ in.}$$

Member End Forces: The joint load vector **P**, corresponding to **d**, can be determined by performing the following operations for each member of the truss: (a) obtain the member's global end displacements **v** from **d** using the member's code numbers; (b) calculate the member's global end forces **F** using Eqs. (10.15), (10.18) through (10.22), and (10.24); and (c) store the elements of **F** into their proper positions in **P** and the support reaction vector **R**, using the member code numbers. Thus,

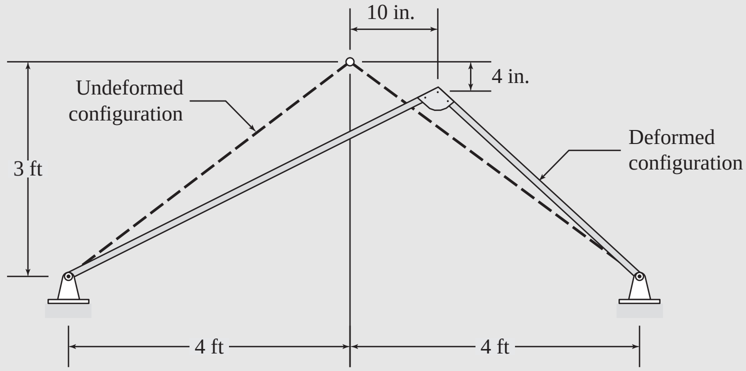
Member 1 $L = 60$ in., $X_1 = Y_1 = 0$, $X_2 = 48$ in., $Y_2 = 36$ in. By using the member code numbers 3, 4, 1, 2, we obtain

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 10 \\ -4 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{ in.}$$

By applying Eqs. (10.20) thru (10.22), we compute the length and direction cosines of the member in the displaced position to be

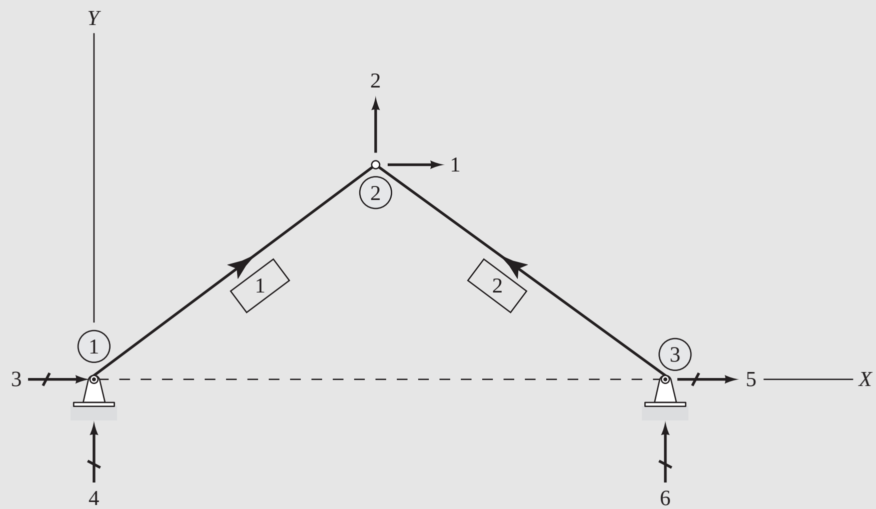
$$\bar{L} = \sqrt{[(48 + 10) - (0)]^2 + [(36 - 4) - (0)]^2} = 66.24198 \text{ in.}$$

$$c_X = \frac{(48 + 10) - (0)}{66.24198} = 0.8755777 \qquad c_Y = \frac{(36 - 4) - (0)}{66.24198} = 0.4830773$$



$EA =$ constant
$E \ = 10{,}000$ ksi
$A \ = 1$ in.^{2}

(a) Truss



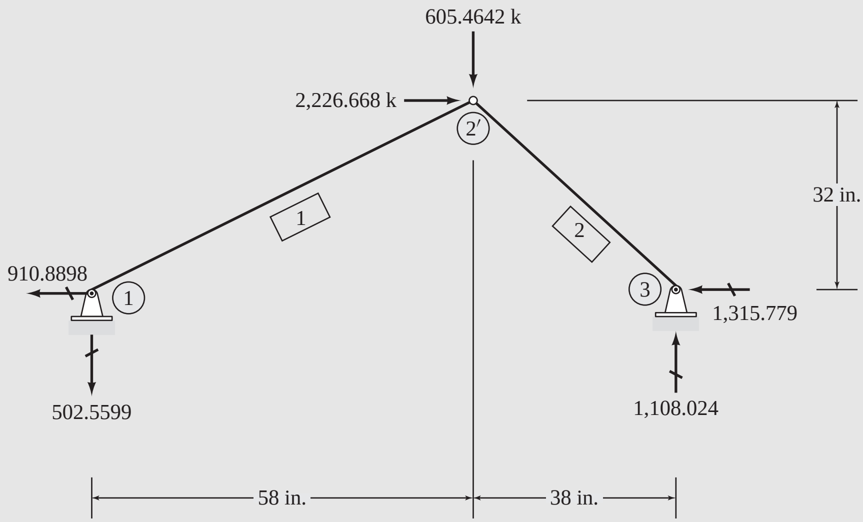
(b) Analytical Model

$$\mathbf{P} = \begin{array}{c} \left[\begin{array}{c} 910.8898 + 1{,}315.779 \\ 502.5599 - 1{,}108.024 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array} = \left[\begin{array}{r} 2{,}226.668 \\ -605.4642 \end{array}\right] \begin{array}{c} 1 \\ 2 \end{array} \text{ k}$$

$$\mathbf{R} = \left[\begin{array}{r} -910.8898 \\ -502.5599 \\ -1{,}315.779 \\ 1{,}108.024 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \text{ k}$$

(c) Joint Load and Support Reaction Vectors

Fig. 10.5



(d) Joint Loads and Support Reactions

Fig. 10.5 (*continued*)

Next, by using Eqs. (10.15) and (10.18), we calculate the member axial deformation and force as

$$u = L - \bar{L} = -6.24198 \text{ in.} \qquad Q = \left(\frac{EA}{L}\right) u = -1{,}040.33 \text{ k}$$

The member global end forces **F** can now be determined from Eq. (10.25):

$$\mathbf{F}_1 = \mathbf{T}^T Q = \begin{bmatrix} 0.8755777 \\ 0.4830773 \\ -0.8755777 \\ -0.4830773 \end{bmatrix} (-1{,}040.33) = \begin{array}{c} \begin{bmatrix} -910.8898 \\ -502.5599 \\ \hdashline 910.8898 \\ 502.5599 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{k}$$

The elements of $\mathbf{F}_1$ are stored in their proper positions in the 2×1 joint load vector **P** and the 4×1 reaction vector **R**, as shown in Fig. 10.5(c).

Member 2 $L = 60$ in., $X_3 = 96$ in., $Y_3 = 0$, $X_2 = 48$ in., $Y_2 = 36$ in.

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 10 \\ -4 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \text{ in.}$$

$$\bar{L} = \sqrt{[(48+10)-(96+0)]^2 + [(36-4)-(0)]^2} = 49.67897 \text{ in.}$$

$$c_X = \frac{(48+10)-(96+0)}{49.67897} = -0.7649112$$

$$c_Y = \frac{(36-4)-(0)}{49.67897} = 0.6441357$$

$$u = L - \bar{L} = 10.32103 \text{ in.} \qquad Q = \left(\frac{EA}{L}\right) u = 1{,}720.172 \text{ k}$$

$$\mathbf{F}_2 = \mathbf{T}^T Q = \begin{bmatrix} -0.7649112 \\ 0.6441357 \\ 0.7649112 \\ -0.6441357 \end{bmatrix} (1{,}720.172) = \begin{array}{c} \begin{bmatrix} -1{,}315.779 \\ 1{,}108.024 \\ \hdashline 1{,}315.779 \\ -1{,}108.024 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \text{ k}$$

Joint Loads and Support Reactions: The completed joint load vector **P** and the support reaction vector **R** are shown in Fig. 10.5(c), and these forces are depicted on a line diagram of the deformed configuration of the truss in Fig. 10.5(d). **Ans**

Equilibrium Check: Applying the equations of equilibrium to the free body of the truss in its deformed state (Fig. 10.5(d)), we obtain

$$+\rightarrow \sum F_X = 0 \qquad -910.8898 + 2{,}226.668 - 1{,}315.779 = -0.0008 \text{ k} \approx 0$$ **Checks**

$$+\uparrow \sum F_Y = 0 \qquad -502.5599 - 605.4642 + 1{,}108.024 = -0.0001 \text{ k} \approx 0$$ **Checks**

$$+\circlearrowleft \sum M_{①} = 0 \qquad -2{,}226.668(32) - 605.4642(58) + 1{,}108.024(96) = 0.0044 \text{ k-in.} \approx 0$$ **Checks**

Member Tangent Stiffness Matrix

As indicated by the foregoing example, when the deformed configuration **d** of a truss is known, the corresponding joint loads **P**, required to cause (and/or keep the structure in equilibrium in) that deformed configuration, can be determined by direct application of the nonlinear force-displacement relations derived in the preceding subsection. However, in most practical situations, it is the external loading that is specified, and the objective of the analysis is to determine the corresponding deformed configuration of the structure, thereby requiring the solution of a system of simultaneous nonlinear equations. The computational techniques commonly used for solving such systems of nonlinear equations are iterative in nature, and usually involve solving a linearized form of the structure's load-deformation relations repeatedly to move closer to the (yet unknown) exact nonlinear solution. Thus, before we discuss such a computational technique in a subsequent subsection, we develop the linearized form of the force-displacement relations for the planes truss members in the global coordinate system.

The member force-displacement relationships can be written in terms of differentials as

$$\boxed{\Delta\mathbf{F} = \mathbf{K_t}\,\Delta\mathbf{v}} \tag{10.27}$$

in which $\Delta\mathbf{F}$ and $\Delta\mathbf{v}$ denote increments of member global end forces **F** and end displacements **v**, respectively, and

$$\mathbf{K_t} = \left[\frac{\partial F_i}{\partial v_j}\right]; \text{ for i, j = 1 to 4} \tag{10.28}$$

is called the *member tangent stiffness matrix in the global coordinate system.* Equation (10.28) can be expanded into

$$\mathbf{K_t} = \begin{bmatrix} \dfrac{\partial F_1}{\partial v_1} & \dfrac{\partial F_1}{\partial v_2} & \dfrac{\partial F_1}{\partial v_3} & \dfrac{\partial F_1}{\partial v_4} \\ \dfrac{\partial F_2}{\partial v_1} & \dfrac{\partial F_2}{\partial v_2} & \dfrac{\partial F_2}{\partial v_3} & \dfrac{\partial F_2}{\partial v_4} \\ \dfrac{\partial F_3}{\partial v_1} & \dfrac{\partial F_3}{\partial v_2} & \dfrac{\partial F_3}{\partial v_3} & \dfrac{\partial F_3}{\partial v_4} \\ \dfrac{\partial F_4}{\partial v_1} & \dfrac{\partial F_4}{\partial v_2} & \dfrac{\partial F_4}{\partial v_3} & \dfrac{\partial F_4}{\partial v_4} \end{bmatrix} \tag{10.29}$$

To determine the explicit form of $\mathbf{K}_t$, we differentiate the expressions of F_1 through F_4 (Eqs. 10.23(a) through (d)) partially with respect to v_1 through v_4, respectively. Thus, by differentiating the expression of F_1 (Eq. 10.23(a)) partially with respect to v_1, we write

$$\frac{\partial F_1}{\partial v_1} = c_X \left(\frac{\partial Q}{\partial v_1}\right) + Q \left(\frac{\partial c_X}{\partial v_1}\right) \tag{10.30a}$$

To obtain $\partial Q/\partial v_1$, we substitute Eqs. (10.15) and (10.20), respectively, into Eq. (10.18), and differentiate the resulting equation partially with respect to v_1, thereby yielding

$$\frac{\partial Q}{\partial v_1} = \left(\frac{EA}{L}\right) c_X \tag{10.30b}$$

Similarly, by substituting Eq.(10.20) into Eq. (10.21), and differentiating the resulting equation partially with respect to v_1, we obtain

$$\frac{\partial c_X}{\partial v_1} = -\frac{c_Y^2}{\bar{L}} \tag{10.30c}$$

and finally, by substituting Eqs. (10.30b) and (10.30c) into Eq. (10.30a), we obtain the desired partial derivative as

$$\frac{\partial F_1}{\partial v_1} = \left(\frac{EA}{L}\right) c_X^2 - \left(\frac{Q}{\bar{L}}\right) c_Y^2 \tag{10.30d}$$

The remaining partial derivatives of F_i with respect to v_j can be derived in a similar manner. The explicit form of the member global tangent stiffness matrix thus obtained is [45, 46]

$$\mathbf{K_t} = \frac{EA}{L} \begin{bmatrix} c_X^2 & c_X c_Y & -c_X^2 & -c_X c_Y \\ c_X c_Y & c_Y^2 & -c_X c_Y & -c_Y^2 \\ -c_X^2 & -c_X c_Y & c_X^2 & c_X c_Y \\ -c_X c_Y & -c_Y^2 & c_X c_Y & c_Y^2 \end{bmatrix} + \frac{Q}{\bar{L}} \begin{bmatrix} -c_Y^2 & c_X c_Y & c_Y^2 & -c_X c_Y \\ c_X c_Y & -c_X^2 & -c_X c_Y & c_X^2 \\ c_Y^2 & -c_X c_Y & -c_Y^2 & c_X c_Y \\ -c_X c_Y & c_X^2 & c_X c_Y & -c_X^2 \end{bmatrix} \tag{10.31}$$

The reader should note that, in the initial (undeformed) configuration of the member (when $\bar{\theta} = \theta$ and $Q = 0$), the tangent stiffness matrix $\mathbf{K}_t$ reduces to the conventional stiffness matrix K derived previously for linear analysis of plane trusses in Chapter 3 (Eq. (3.73)).

Equation (10.31) is often written in compact form as

$$\boxed{\mathbf{K_t} = \left(\frac{EA}{L}\right)\mathbf{T}^T\mathbf{T} + Q\,\mathbf{g}} \tag{10.32}$$

in which the *geometric matrix* $\mathbf{g}$ is given by

$$\mathbf{g} = \frac{1}{\bar{L}}\begin{bmatrix} -c_Y^2 & c_X c_Y & c_Y^2 & -c_X c_Y \\ c_X c_Y & -c_X^2 & -c_X c_Y & c_X^2 \\ c_Y^2 & -c_X c_Y & -c_Y^2 & c_X c_Y \\ -c_X c_Y & c_X^2 & c_X c_Y & -c_X^2 \end{bmatrix} \tag{10.33}$$

Structure Load-Deformation Relations

The geometrically nonlinear structure load-deformation relations for plane trusses are expressed in the form of joint equilibrium equations:

$$\boxed{\mathbf{P} = \mathbf{f}(\mathbf{d})} \tag{10.34}$$

in which, $\mathbf{f}$ is referred to as the *internal joint force vector* of the structure. The vector $\mathbf{f}$ contains the resultants of internal member end forces at the locations, and in the directions of, the structure's degrees of freedom. As shown previously in Example 10.1, the resultant internal forces $\mathbf{f}$ are nonlinear functions of the structure's joint displacements $\mathbf{d}$. It should be noted that in Example 10.1, where the structure's deformed configuration $\mathbf{d}$ was known, no distinction was necessary between $\mathbf{P}$ and $\mathbf{f}$, and the former was assembled directly from the member end forces $\mathbf{F}$ via the member code numbers. However, when analyzing a structure for its unknown deformed configuration caused by a specified external loading, it becomes necessary to distinguish between the external loads $\mathbf{P}$, which remain constant throughout the iterative process, and the internal forces $\mathbf{f}$ that vary as the structure's assumed deformed configuration $\mathbf{d}$ is revised iteratively until $\mathbf{f}$ becomes sufficiently close to $\mathbf{P}$, that is, the structure's equilibrium equations (Eq. (10.34)) are satisfied within a prescribed tolerance.

The structure load-deformation relationships (Eq. (10.34)) can be written in terms of differentials as

$$\boxed{\Delta\mathbf{P} = \mathbf{S_t}\,\Delta\mathbf{d}} \tag{10.35}$$

in which $\Delta\mathbf{P}$ and $\Delta\mathbf{d}$ denote increments of external loads $\mathbf{P}$ and joint displacements $\mathbf{d}$, respectively, and

$$\mathbf{S_t} = \left[\frac{\partial f_i}{\partial d_j}\right]; \quad \text{for i, j = 1 to } NDOF \tag{10.36}$$

is called the *structure tangent stiffness matrix*. The structure matrix $\mathbf{S}_t$ can be conveniently assembled from the member global tangent stiffness matrices $\mathbf{K}_t$ using the member code numbers, as in the case of linear analysis.

Computational Technique—Newton-Raphson Method

Most computational techniques commonly used for nonlinear structural analysis are generally based on the classical Newton-Raphson iteration technique for root finding. Such an iterative method for geometrically nonlinear analysis of plane trusses is presented herein. The method of analysis is illustrated graphically for a single degree-of-freedom structure in Fig. 10.6.

Let us assume that our objective is to determine the deformed configuration (i.e., the joint displacements) $\mathbf{d}$ of a structure due to a given external loading $\mathbf{P}$. As shown in Fig. 10.6, we begin the process by performing the conventional linear analysis to determine the first approximate configuration $\mathbf{d}_1$ of the structure. Note that the linearized form of the nonlinear load-deformation relations (Eq. (10.35)) reduces to the conventional linear stiffness relations (Eq. (3.89))

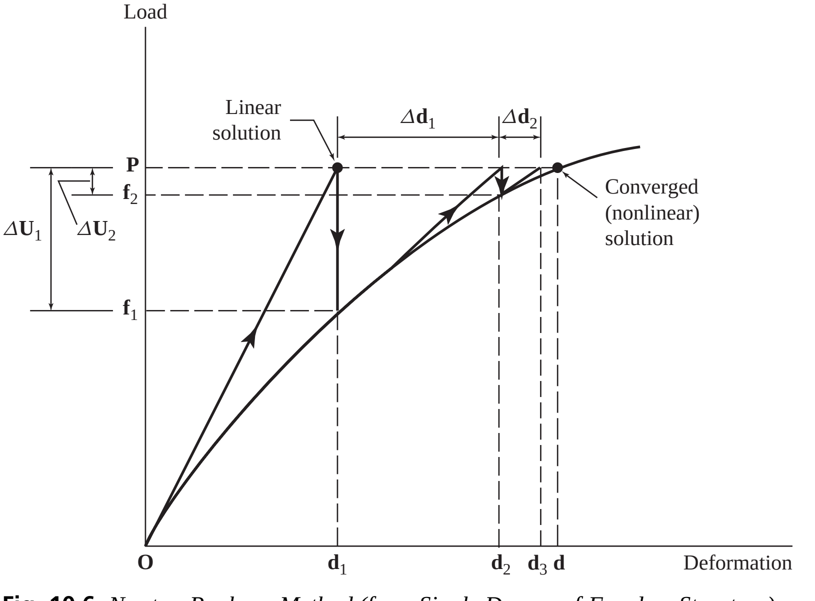


Fig. 10.6 *Newton-Raphson Method (for a Single Degree-of-Freedom Structure)*

when applied in the undeformed (initial) configuration of the structure, with $\Delta\mathbf{P} = \mathbf{P}$ and $\Delta\mathbf{d} = \mathbf{d}_1$; that is,

$$\mathbf{P} = \mathbf{S}_{t0}\,\mathbf{d}_1 = \mathbf{S}\mathbf{d}_1 \tag{10.37}$$

in which $\mathbf{S}_{t0}$ denotes the structure tangent stiffness matrix evaluated in the undeformed configuration (i.e., $\mathbf{d} = \mathbf{d}_0 = \mathbf{0}$).

It should be realized that the configuration $\mathbf{d}_1$, obtained by solving Eq. (10.37), represents an approximate configuration in the sense that the joint equilibrium equations (Eq. (10.34)) are not necessarily satisfied. To correct the approximate solution, we evaluate the structure's internal joint force vector, $\mathbf{f}_1 = \mathbf{f}(\mathbf{d}_1)$, corresponding to the configuration $\mathbf{d}_1$ and subtract it from the joint load vector $\mathbf{P}$ to calculate the *unbalanced joint force vector* for the structure

$$\Delta\mathbf{U}_1 = \mathbf{P} - \mathbf{f}_1 \tag{10.38}$$

The unbalanced joint forces $\Delta\mathbf{U}_1$ are now treated as a load increment and the correction vector $\Delta\mathbf{d}_1$ is obtained by applying the linearized incremental relationship (Eq. (10.35) as

$$\Delta\mathbf{U}_1 = \mathbf{S}_{t1}\,\Delta\mathbf{d}_1 \tag{10.39}$$

with $\mathbf{S}_{t1}$ now representing the structure tangent stiffness matrix evaluated in the configuration $\mathbf{d}_1$. A new approximate configuration $\mathbf{d}_2$ is then obtained by adding the correction vector $\Delta\mathbf{d}_1$ to the current configuration $\mathbf{d}_1$,

$$\mathbf{d}_2 = \mathbf{d}_1 + \Delta\mathbf{d}_1 \tag{10.40}$$

and the iteration is continued until the latest correction vector is sufficiently small.

Equations (10.38) through (10.40) refer to the first iteration cycle. For an *i*th iteration cycle, these equations can be expressed in recurrence form as:

$$\boxed{\Delta\mathbf{U}_i = \mathbf{P} - \mathbf{f}_i} \tag{10.41}$$

$$\boxed{\Delta\mathbf{U}_i = \mathbf{S}_{ti}\,\Delta\mathbf{d}_i} \tag{10.42}$$

$$\boxed{\mathbf{d}_{i+1} = \mathbf{d}_i + \Delta\mathbf{d}_i} \tag{10.43}$$

Various criteria can be used in deciding whether the iterative process has converged. In general, for structures exhibiting softening type of response, the convergence criteria based on the change in the structure's configuration between two consecutive iteration cycles do seem to yield reasonably accurate results. In the example presented in this chapter, we use a criterion based on a comparison of the changes, $\Delta\mathbf{d}_i$, in joint displacements to their cumulative

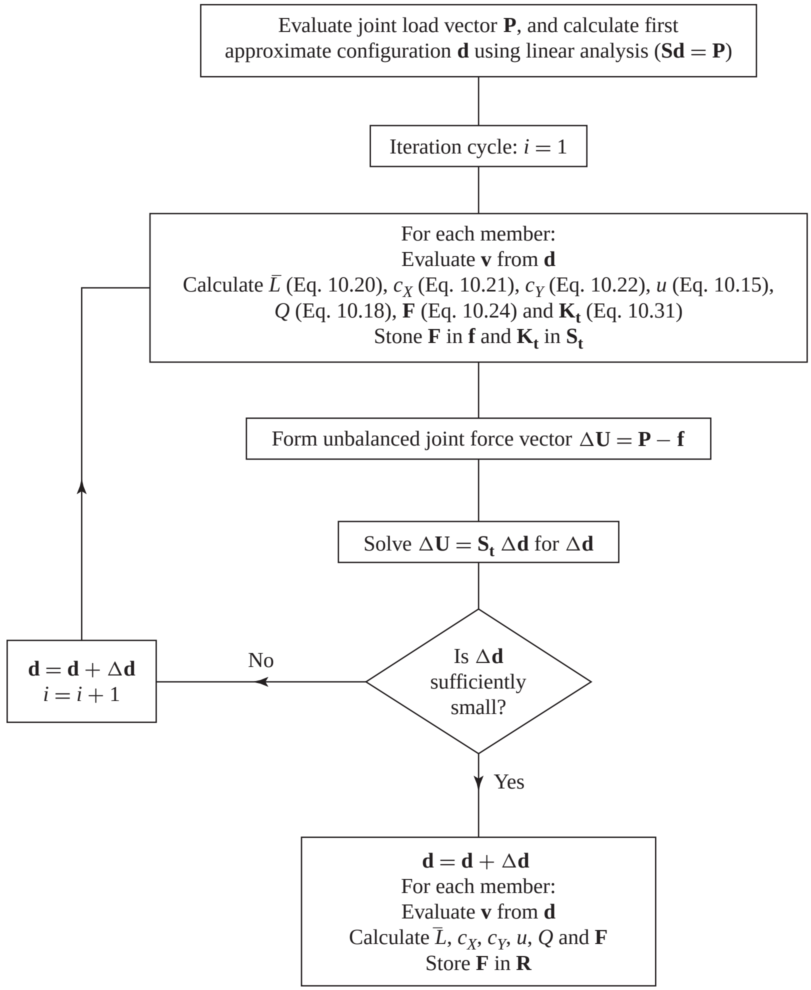


Fig. 10.7 *Procedure for Analysis*

values, $\mathbf{d}_i$, and consider the convergence to have occurred when the following inequality is satisfied

$$\boxed{\sqrt{\frac{\sum_{j=1}^{NDOF}\left(\Delta d_j\right)^2}{\sum_{j=1}^{NDOF}\left(d_j\right)^2}} \leq e} \tag{10.44}$$

in which the dimensionless quantity e represents a prescribed tolerance.

A block diagram summarizing the various steps of the method for geometrically nonlinear analysis of plane trusses is shown in Fig. 10.7. The method of analysis is illustrated by the following example.

EXAMPLE 10.2 Determine the joint displacements, member axial forces, and support reactions for the truss shown in Fig. 10.8(a) by geometrically nonlinear analysis. Use a displacement convergence tolerance of 0.1 percent.

SOLUTION *Analytical Model:* See Fig. 10.8(b). The truss has three degrees of freedom, numbered as 1, 2, and 3. The three restrained coordinates of the truss are identified by numbers 4, 5, and 6.

Linear Analysis: We begin by performing the conventional linear analysis of the truss subjected to the specified joint loads,

$$\mathbf{P} = \begin{bmatrix} 0 \\ -2{,}000 \\ 0 \end{bmatrix} \text{ kN} \tag{1}$$

The member global stiffness matrices can be evaluated by using either Eq. (3.73), or Eq. (10.31) with $\bar{\theta} = \theta$ and $Q = 0$. These are:

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccc} \quad 4 \quad & \quad 5 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{bmatrix} 5{,}781 & 4{,}335.7 & -5{,}781 & -4{,}335.7 \\ 4{,}335.7 & 3{,}251.8 & -4{,}335.7 & -3{,}251.8 \\ -5{,}781 & -4{,}335.7 & 5{,}781 & 4{,}335.7 \\ -4{,}335.7 & -3{,}251.8 & 4{,}335.7 & 3{,}251.8 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ kN/m}$$

$$\mathbf{K}_2 = \begin{array}{c} \begin{array}{cccc} \quad 3 \quad & \quad 6 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{bmatrix} 5{,}781 & -4{,}335.7 & -5{,}781 & 4{,}335.7 \\ -4{,}335.7 & 3{,}251.8 & 4{,}335.7 & -3{,}251.8 \\ -5{,}781 & 4{,}335.7 & 5{,}781 & -4{,}335.7 \\ 4{,}335.7 & -3{,}251.8 & -4{,}335.7 & 3{,}251.8 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ kN/m}$$

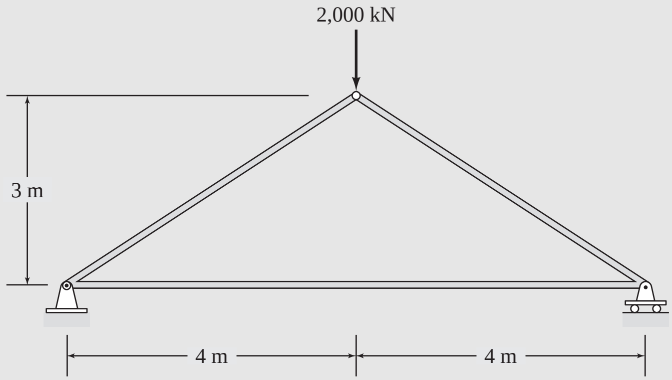
$$\mathbf{K}_3 = \begin{array}{c} \begin{array}{cccc} \quad 4 \quad & \quad 5 \quad & \quad 3 \quad & \quad 6 \quad \end{array} \\ \begin{bmatrix} 5{,}645.5 & 0 & -5{,}645.5 & 0 \\ 0 & 0 & 0 & 0 \\ -5{,}645.5 & 0 & \boxed{5{,}645.5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN/m}$$

The structure stiffness matrix thus obtained is

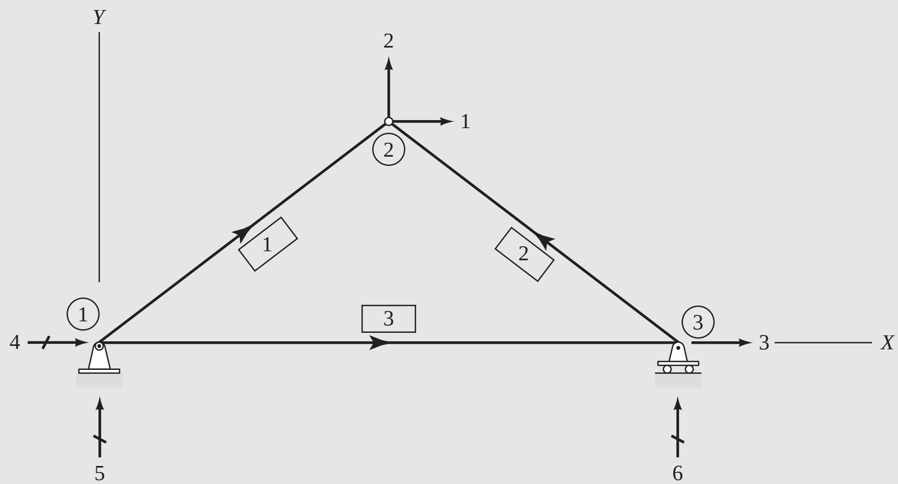
$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} 11{,}562 & 0 & -5{,}781 \\ 0 & 6{,}503.6 & 4{,}335.7 \\ -5{,}781 & 4{,}335.7 & 11{,}426 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

By solving the linear system of equations $\mathbf{P} = \mathbf{S}\,\mathbf{d}_1$, we determine the first approximation configuration to be

$$\mathbf{d}_1 = \begin{bmatrix} 0.11809 \\ -0.46497 \\ 0.23618 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ m}$$



(a) Truss



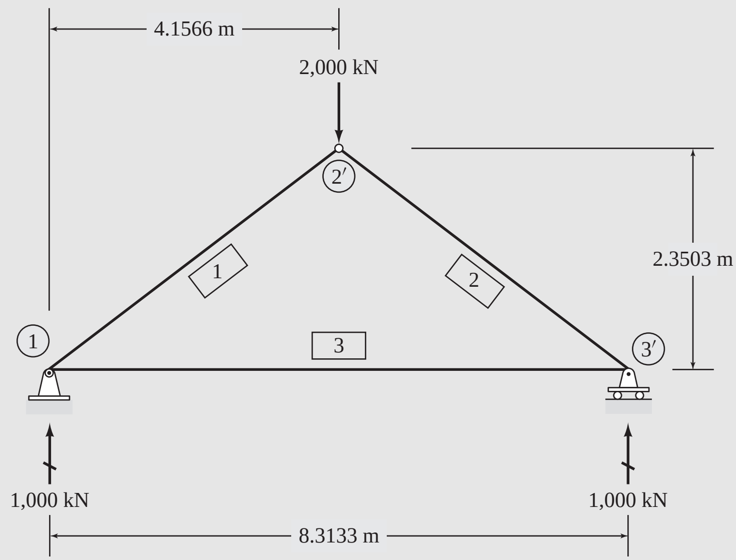
(b) Analytical Model

Fig. 10.8

Iteration Cycle 1: Next, for the current deformed configuration $\mathbf{d}_1$ of the structure, we evaluate its internal joint force vector $\mathbf{f}_1$ and the tangent stiffness matrix $\mathbf{S}_{\mathbf{t}1}$ by assembling the pertinent elements of the member $\mathbf{F}$ vectors and $\mathbf{K_t}$ matrices, respectively, as follows:

Member 1

$$\mathbf{v} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 0 \\ 0.11809 \\ -0.46497 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ m}$$



(c) Support Reactions

Fig. 10.8 (*continued*)

$L = 5$ m, $\bar{L} = 4.8358$ m, $c_X = 0.85158$, $c_Y = 0.52422$, $u = 0.16419$ m, $Q = 1{,}483.1$ kN

$$\mathbf{F} = \begin{array}{c} \left[\begin{array}{c} 1{,}263 \\ 777.49 \\ \hdashline -1{,}263 \\ -777.49 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \end{array} \text{kN} \qquad \mathbf{K}_t = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \left[\begin{array}{cc:cc} 6{,}466.2 & 4{,}169.3 & -6{,}466.2 & -4{,}169.3 \\ 4{,}169.3 & 2{,}259.9 & -4{,}169.3 & -2{,}259.9 \\ \hdashline -6{,}466.2 & -4{,}169.3 & 6{,}466.2 & 4{,}169.3 \\ -4{,}169.3 & -2{,}259.9 & 4{,}169.3 & 2{,}259.9 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{kN/m} \qquad \textbf{(2)}$$

Member 2

$$\mathbf{v} = \left[\begin{array}{c} 0.23618 \\ 0 \\ 0.11809 \\ -0.46497 \end{array}\right] \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{m}$$

L $= 5$ m, $\bar{L} = 4.8358$ m, $c_X = -0.85158$, $c_Y = 0.52422$, $u = 0.16419$ m, $Q = 1{,}483.1$ kN

$$\mathbf{F} = \left[\begin{array}{c} -1{,}263 \\ \hdashline 777.49 \\ \hdashline 1{,}263 \\ -777.49 \end{array}\right] \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{kN} \qquad \mathbf{K}_t = \begin{array}{c} \begin{array}{cccc} 3 & 6 & 1 & 2 \end{array} \\ \left[\begin{array}{c:c:cc} 6{,}466.2 & -4{,}169.3 & -6{,}466.2 & 4{,}169.3 \\ \hdashline -4{,}169.3 & 2{,}259.9 & 4{,}169.3 & -2{,}259.9 \\ \hdashline -6{,}466.2 & 4{,}169.3 & 6{,}466.2 & -4{,}169.3 \\ 4{,}169.3 & -2{,}259.9 & -4{,}169.3 & 2{,}259.9 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{kN/m} \qquad \textbf{(3)}$$

Member 3

$$\mathbf{v} = \begin{array}{c}\left[\begin{array}{c} 0 \\ 0 \\ 0.23618 \\ 0 \end{array}\right]\end{array}\begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ m}$$

$L = 8$ m, $\bar{L} = 8.2362$ m, $c_X = 1$, $c_Y = 0$, $u = -0.23618$ m, $Q = -1{,}333.3$ kN

$$\mathbf{F} = \left[\begin{array}{c} -1{,}333.3 \\ 0 \\ \boxed{1{,}333.3} \\ 0 \end{array}\right]\begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN} \qquad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 3 & 6 \end{array} \\ \left[\begin{array}{cccc} 5{,}645.5 & 0 & -5{,}645.5 & 0 \\ 0 & 161.89 & 0 & -161.89 \\ -5{,}645.5 & 0 & \boxed{5{,}645.5} & 0 \\ 0 & -161.89 & 0 & 161.89 \end{array}\right] \end{array}\begin{array}{c} \\ 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN/m} \qquad \textbf{(4)}$$

By assembling the pertinent elements of the member **F** vectors and $\mathbf{K_t}$ matrices given in Eqs. (2) through (4), we obtain

$$\mathbf{f}_1 = \left[\begin{array}{c} 0 \\ -1{,}555 \\ 70.322 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ kN} \qquad \mathbf{S_{t1}} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 12{,}932 & 0 & -6{,}466.2 \\ 0 & 4{,}519.7 & 4{,}169.3 \\ -6{,}466.2 & 4{,}169.3 & 12{,}112 \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

By subtracting $\mathbf{f}_1$ from **P** (Eq. 1), we compute the unbalanced joint force vector for the truss as

$$\Delta\mathbf{U}_1 = \mathbf{P} - \mathbf{f}_1 = \left[\begin{array}{c} 0 \\ -2{,}000 \\ 0 \end{array}\right] - \left[\begin{array}{c} 0 \\ -1{,}555 \\ 70.322 \end{array}\right] = \left[\begin{array}{c} 0 \\ -445.02 \\ -70.322 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ kN}$$

By solving the linearized system of equations $\Delta\mathbf{U}_1 = \mathbf{S}_{t1}\,\Delta\mathbf{d}_1$, we determine the displacement correction vector

$$\Delta\mathbf{d}_1 = \left[\begin{array}{c} 0.03380 \\ -0.16082 \\ 0.067599 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ m}$$

To determine whether or not the iteration has converged, we apply the convergence criterion (Eq. (10.44)) as

$$\sqrt{\frac{\sum_{j=1}^{3}\left(\Delta d_j\right)^2}{\sum_{j=1}^{3}\left(d_j\right)^2}} = \sqrt{\frac{(0.03380)^2 + (-0.16082)^2 + (0.067599)^2}{(0.11809)^2 + (-0.46497)^2 + (0.23618)^2}} = 0.33231 > e\ (= 0.001)$$

which indicates that the change in the structure's configuration is not sufficiently small, and therefore, another iteration is needed based on a new (second) approximate deformed configuration $\mathbf{d}_2$ obtained by adding the correction $\Delta\mathbf{d}_1$ to the previous (first) approximate configuration $\mathbf{d}_1$, that is,

$$\mathbf{d_2} = \mathbf{d_1} + \Delta\mathbf{d_1} = \begin{bmatrix} 0.11809 \\ -0.46497 \\ 0.23618 \end{bmatrix} + \begin{bmatrix} 0.03380 \\ -0.16082 \\ 0.067599 \end{bmatrix} = \begin{bmatrix} 0.15189 \\ -0.62579 \\ 0.30378 \end{bmatrix} \text{ m}$$

Iteration Cycle 2:

Member 1

$$\mathbf{v} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0.15189 \\ -0.62579 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ m}$$

$L = 5$ m, $\bar{L} = 4.7828$ m, $c_X = 0.86809$, $c_Y = 0.49641$, $u = 0.21721$ m, $Q = 1{,}962.1$ kN

$$\mathbf{F} = \begin{bmatrix} 1{,}703.2 \\ 973.98 \\ \hdashline -1{,}703.2 \\ -973.98 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ kN} \quad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \left[\begin{array}{cc:cc} 6{,}705.8 & 4{,}069.2 & -6{,}705.8 & -4{,}069.2 \\ 4{,}069.2 & 1{,}916.7 & -4{,}069.2 & -1{,}916.7 \\ \hdashline -6{,}705.8 & -4{,}069.2 & 6{,}705.8 & 4{,}069.2 \\ -4{,}069.2 & -1{,}916.7 & 4{,}069.2 & 1{,}916.7 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ kN/m}$$

Member 2

$$\mathbf{v} = \begin{bmatrix} 0.30378 \\ 0 \\ 0.15189 \\ -0.62579 \end{bmatrix} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ m}$$

$L = 5$ m, $\bar{L} = 4.7828$ m, $c_X = -0.86809$, $c_Y = 0.49641$, $u = 0.21721$ m, $Q = 1{,}962.1$ kN

$$\mathbf{F} = \begin{bmatrix} -1{,}703.2 \\ \hdashline 973.98 \\ \hdashline 1{,}703.2 \\ -973.98 \end{bmatrix} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ kN} \quad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 3 & 6 & 1 & 2 \end{array} \\ \left[\begin{array}{c:c:cc} 6{,}705.8 & -4{,}069.2 & -6{,}705.8 & 4{,}069.2 \\ \hdashline -4{,}069.2 & 1{,}916.7 & 4{,}069.2 & -1{,}916.7 \\ \hdashline -6{,}705.8 & 4{,}069.2 & 6{,}705.8 & -4{,}069.2 \\ 4{,}069.2 & -1{,}916.7 & -4{,}069.2 & 1{,}916.7 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ kN/m}$$

Member 3

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0.30378 \\ 0 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ m}$$

$L = 8$ m, $\bar{L} = 8.3038$ m, $c_X = 1$, $c_Y = 0$, $u = -0.30378$ m, $Q = -1{,}715$ kN

$$\mathbf{F} = \begin{array}{c} \begin{bmatrix} -1{,}715 \\ 0 \\ 1{,}715 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN} \qquad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 3 & 6 \end{array} \\ \begin{bmatrix} 5{,}645.5 & 0 & -5{,}645.5 & 0 \\ 0 & 206.53 & 0 & -206.53 \\ -5{,}645.5 & 0 & \boxed{5{,}645.5} & 0 \\ 0 & -206.53 & 0 & 206.53 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN/m}$$

Thus, the structure's internal joint force vector $\mathbf{f}_2$ and the tangent stiffness matrix $\mathbf{S}_{t2}$ are given by

$$\mathbf{f}_2 = \begin{bmatrix} 0 \\ -1{,}948 \\ 11.722 \end{bmatrix} \text{ kN} \qquad \mathbf{S}_{t2} = \begin{bmatrix} 13{,}412 & 0 & -6{,}705.8 \\ 0 & 3{,}833.4 & 4{,}069.2 \\ -6{,}705.8 & 4{,}069.2 & 12{,}351 \end{bmatrix} \text{ kN/m}$$

and the unbalanced joint force vector is obtained as

$$\Delta\mathbf{U}_2 = \mathbf{P} - \mathbf{f}_2 = \begin{bmatrix} 0 \\ -52.041 \\ -11.722 \end{bmatrix} \text{ kN}$$

Note that the magnitudes of the unbalanced forces are now significantly smaller than in the previous iteration cycle. By solving the linearized system of equations $\Delta\mathbf{U}_2 = \mathbf{S}_{t2}\,\Delta\mathbf{d}_2$, we determine the displacement correction vector

$$\Delta\mathbf{d}_2 = \begin{bmatrix} 0.0046508 \\ -0.023449 \\ 0.0093015 \end{bmatrix} \text{ m}$$

To check for convergence, we write

$$\sqrt{\frac{\sum_{j=1}^{3}\left(\Delta d_j\right)^2}{\sum_{j=1}^{3}\left(d_j\right)^2}} = \sqrt{\frac{(0.0046508)^2 + (-0.023449)^2 + (0.0093015)^2}{(0.15189)^2 + (-0.62579)^2 + (0.30378)^2}} = 0.036027 > e\ (= 0.001)$$

which indicates that, while the change in the structure's deformed configuration is now considerably smaller than in the previous iteration cycle, it is still not within the prescribed tolerance of 0.1 percent. Thus, we perform another (third) iteration based on a new approximate deformed configuration $\mathbf{d}_3$ of the structure, with

$$\mathbf{d}_3 = \mathbf{d}_2 + \Delta\mathbf{d}_2 = \begin{bmatrix} 0.15189 \\ -0.62579 \\ 0.30378 \end{bmatrix} + \begin{bmatrix} 0.00046508 \\ -0.023449 \\ 0.0093015 \end{bmatrix} = \begin{bmatrix} 0.15654 \\ -0.64924 \\ 0.31308 \end{bmatrix} \text{ m}$$

Iteration Cycle 3:

Member 1

$$\mathbf{v} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0.15654 \\ -0.64924 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ m}$$

$L = 5$ m, $\bar{L} = 4.7752$ m, $c_X = 0.87044$, $c_Y = 0.49228$, $u = 0.22476$ m, $Q = 2{,}030.3$ kN

$$\mathbf{F} = \begin{bmatrix} 1{,}767.2 \\ 999.45 \\ -1{,}767.2 \\ -999.45 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ kN} \qquad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 6{,}740.8 & 4{,}052.7 & -6{,}740.8 & -4{,}052.7 \\ 4{,}052.7 & 1{,}866.9 & -4{,}052.7 & -1{,}866.9 \\ -6{,}740.8 & -4{,}052.7 & 6{,}740.8 & 4{,}052.7 \\ -4{,}052.7 & -1{,}866.9 & 4{,}052.7 & 1{,}866.9 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ kN/m}$$

Member 2

$$\mathbf{v} = \begin{bmatrix} 0.31308 \\ 0 \\ 0.15654 \\ -0.64924 \end{bmatrix} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ m}$$

$L = 5$ m, $\bar{L} = 4.7752$ m, $c_X = -0.87044$, $c_Y = 0.49228$, $u = 0.22476$ m, $Q = 2{,}030.3$ kN

$$\mathbf{F} = \begin{bmatrix} -1{,}767.2 \\ 999.45 \\ 1{,}767.2 \\ -999.45 \end{bmatrix} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ kN} \qquad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 3 & 6 & 1 & 2 \end{array} \\ \begin{bmatrix} 6{,}740.8 & -4{,}052.7 & -6{,}740.8 & 4{,}052.7 \\ -4{,}052.7 & 1{,}866.9 & 4{,}052.7 & -1{,}866.9 \\ -6{,}740.8 & 4{,}052.7 & 6{,}740.8 & -4{,}052.7 \\ 4{,}052.7 & -1{,}866.9 & -4{,}052.7 & 1{,}866.9 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ kN/m}$$

Member 3

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0.31308 \\ 0 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ m}$$

$L = 8$ m, $\bar{L} = 8.3131$ m, $c_X = 1$, $c_Y = 0$, $u = -0.31308$ m, $Q = -1{,}767.5$ kN

$$\mathbf{F} = \begin{bmatrix} -1{,}767.5 \\ 0 \\ 1{,}767.5 \\ 0 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN} \qquad \mathbf{K_t} = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 3 & 6 \end{array} \\ \begin{bmatrix} 5{,}645.5 & 0 & -5{,}645.5 & 0 \\ 0 & 212.61 & 0 & -212.61 \\ -5{,}645.5 & 0 & \boxed{5{,}645.5} & 0 \\ 0 & -212.61 & 0 & 212.61 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN/m}$$

The structure's internal joint force vector $\mathbf{f}_3$ and the tangent stiffness matrix $\mathbf{S}_{t3}$ are given by

$$\mathbf{f}_3 = \begin{bmatrix} 0 \\ -1{,}998.9 \\ 0.27365 \end{bmatrix} \text{kN} \qquad \mathbf{S}_{t3} = \begin{bmatrix} 13{,}482 & 0 & -6{,}740.8 \\ 0 & 3{,}733.8 & 4{,}052.7 \\ -6{,}740.8 & 4{,}052.7 & 12{,}386 \end{bmatrix} \text{kN/m}$$

and the unbalanced joint force vector is

$$\Delta\mathbf{U}_3 = \mathbf{P} - \mathbf{f}_3 = \begin{bmatrix} 0 \\ -1.0925 \\ -0.27365 \end{bmatrix} \text{kN}$$

By solving the linearized system of equations $\Delta\mathbf{U}_3 = \mathbf{S}_{t3}\,\Delta\mathbf{d}_3$, we determine the displacement correction vector

$$\Delta\mathbf{d}_3 = \begin{bmatrix} 0.000098787 \\ -0.00050706 \\ 0.00019757 \end{bmatrix} \text{m}$$

To check for convergence, we write

$$\sqrt{\frac{\sum_{j=1}^{3}\left(\Delta d_j\right)^2}{\sum_{j=1}^{3}\left(d_j\right)^2}} = \sqrt{\frac{(0.000098787)^2 + (-0.00050706)^2 + (0.00019757)^2}{(0.15654)^2 + (-0.64924)^2 + (0.31308)^2}} = 0.00074985 < e \;(= 0.001)$$

which indicates that the change in the structure's deformed configuration $\Delta\mathbf{d}_3$ is now within the specified tolerance of 0.1 percent and, therefore, the iterative process has converged.

Results of Geometrically Nonlinear Analysis: The final deformed configuration of the truss is given by the joint displacement vector

$$\mathbf{d} = \mathbf{d}_3 + \Delta\mathbf{d}_3 = \begin{bmatrix} 0.15654 \\ -0.64924 \\ 0.31308 \end{bmatrix} + \begin{bmatrix} 0.000098787 \\ -0.00050706 \\ 0.00019757 \end{bmatrix} = \begin{bmatrix} 0.15664 \\ -0.64975 \\ 0.31327 \end{bmatrix} \text{m} \qquad \textbf{Ans}$$

Member 1

$$\mathbf{v} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0.15664 \\ -0.64975 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{m}$$

$$L = 5 \text{ m},\ \bar{L} = 4.7751 \text{ m},\ c_X = 0.87049,\ c_Y = 0.49219,\ u = 0.22493 \text{ m},\ Q = 2{,}031.7 \text{ kN}$$

$$Q_{a1} = 2{,}031.7 \text{ kN(C)} \qquad \textbf{Ans}$$

$$\mathbf{F} = \begin{array}{c} \left[\begin{array}{c} 1{,}768.6 \\ 1{,}000 \\ \hdashline -1{,}768.6 \\ -1{,}000 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \text{ k N} \qquad (5)$$

Member 2

$$\mathbf{v} = \left[\begin{array}{c} 0.31327 \\ 0 \\ 0.15664 \\ -0.64975 \end{array}\right] \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ m}$$

$$L = 5 \text{ m}, \bar{L} = 4.7751 \text{ m}, c_X = -0.87049, c_Y = 0.49219, u = 0.22493 \text{ m}, Q = 2{,}031.7 \text{ kN}$$

$$Q_{a2} = 2{,}031.7 \text{ kN (C)}$$ **Ans**

$$\mathbf{F} = \left[\begin{array}{c} -1{,}768.6 \\ \hdashline 1{,}000 \\ \hdashline 1{,}768.6 \\ -1{,}000 \end{array}\right] \begin{array}{c} 3 \\ 6 \\ 1 \\ 2 \end{array} \text{ kN} \qquad (6)$$

Member 3

$$\mathbf{v} = \left[\begin{array}{c} 0 \\ 0 \\ 0.31327 \\ 0 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ m}$$

$$L = 8 \text{ m}, \bar{L} = 8.3133 \text{ m}, c_X = 1, c_Y = 0, u = -0.31327 \text{ m}, Q = -1{,}768.6 \text{ kN}$$

$$Q_{a3} = 1{,}768.6 \text{ kN (T)}$$ **Ans**

$$\mathbf{F} = \left[\begin{array}{c} -1{,}768.6 \\ 0 \\ \hdashline 1{,}768.6 \\ \hdashline 0 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 3 \\ 6 \end{array} \text{ kN} \qquad (7)$$

Finally, the support reaction vector **R** is assembled from the elements of the member **F** vectors given in Eqs. (5) thru (7) as

$$\mathbf{R} = \left[\begin{array}{c} 1{,}768.6 - 1{,}768.6 \\ 1{,}000 + 0 \\ 1{,}000 + 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 1{,}000 \\ 1{,}000 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \text{ kN}$$ **Ans**

Equilibrium Check: Applying the equations of equilibrium to the free body of the truss in its deformed state (Fig. 10.8(c)), we obtain

$$+\rightarrow \sum F_X = 0$$ **Checks**

$$+\uparrow \sum F_Y = 0 \qquad 1{,}000 - 2{,}000 + 1{,}000 = 0$$ **Checks**

$$+\circlearrowleft \sum M_{①} = 0 \qquad -2{,}000\,(4.1566) + 1{,}000\,(8.3133) = 0.1 \text{ kN.m} \approx 0$$ **Checks**

SUMMARY

In this chapter, we have studied the basic concepts of the geometrically nonlinear analysis of plane trusses. A block diagram summarizing the various steps involved in the analysis is given in Fig. 10.7.

PROBLEMS

Section 10.1

10.1 Derive the relationships between load P and displacement δ of the truss shown in Fig. P10.1 by using: (a) the conventional linear theory, and (b) the geometrically nonlinear theory. Plot the P-δ equations using the numerical values: $\theta = 30^\circ$, $L = 3$ m, $E =$ 70 GPa, and $A = 645.2$ mm^2, in the range $0 \leq \delta/L \leq 0.5$, to compare the linear and nonlinear solutions.

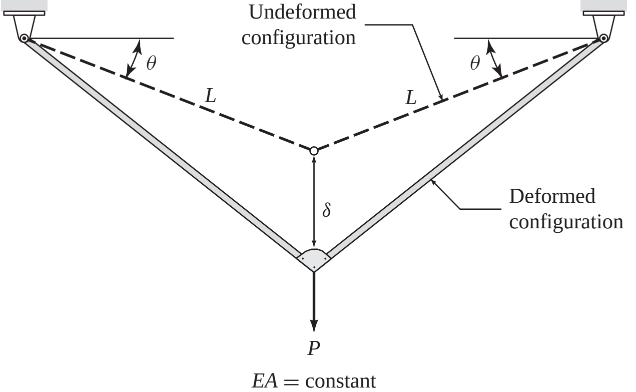


Fig. P10.1

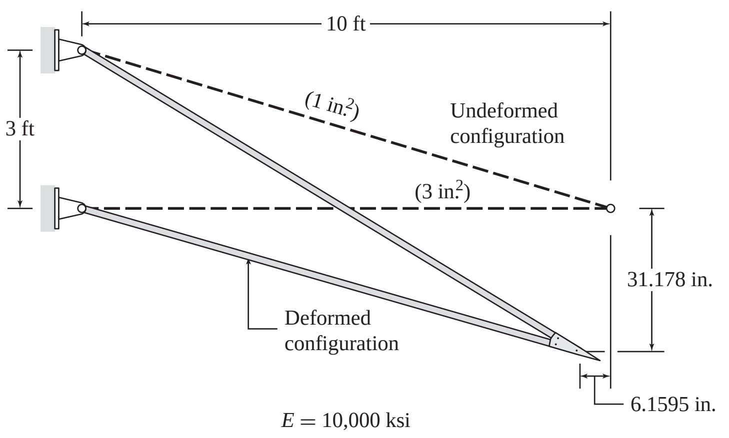


Fig. P10.2

Section 10.2

10.2 By using the geometrically nonlinear analysis, determine the joint loads **P** that cause the two-member truss to deform into the configuration shown in Fig. P10.2.

10.3 through 10.5 Determine the joint displacements, member axial forces, and support reactions for the trusses shown in Figs. P10.3 through P10.5 by geometrically nonlinear analysis. Use a displacement convergence tolerance of one percent.

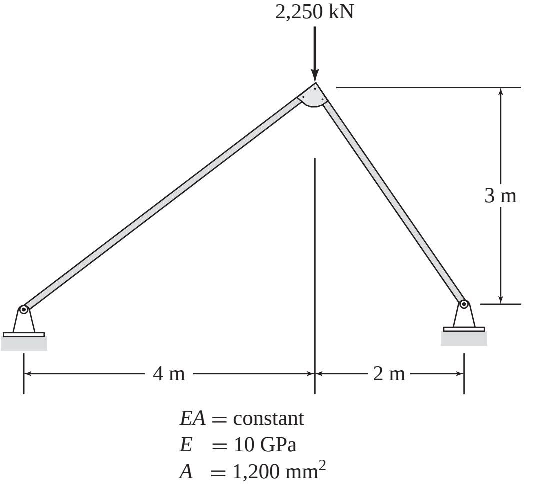


Fig. P10.3

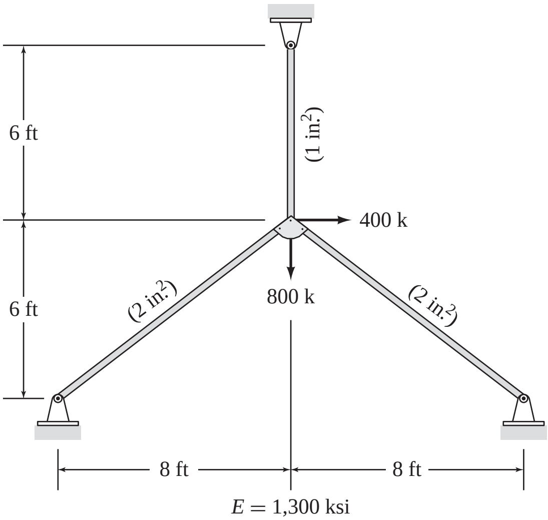


Fig. P10.4

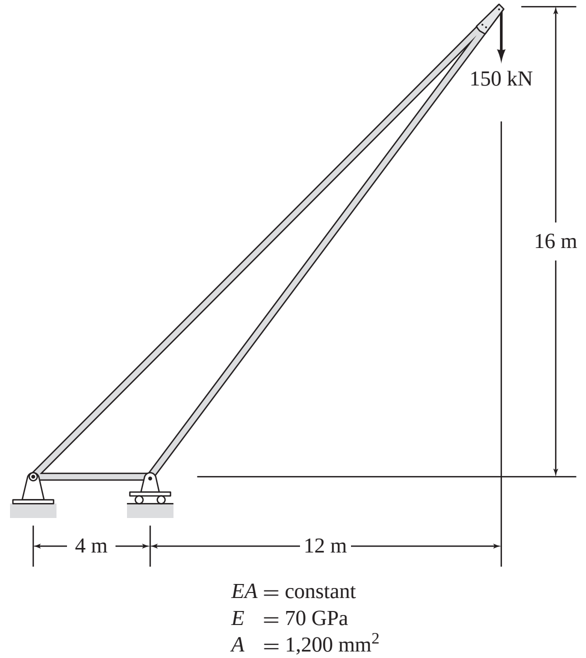


Fig. P10.5

10.6 Develop a computer program for geometrically nonlinear analysis of plane trusses. Use the program to analyze the trusses of Problems 10.3 through 10.5, and compare the computer-generated results to those obtained by hand calculations.