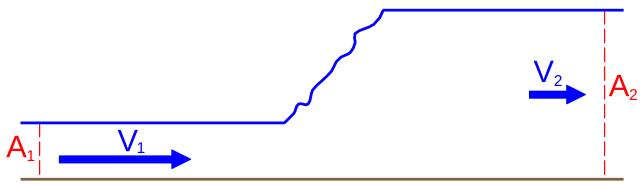
Rapidly-varied flow is a significant change in water depth over a short distance (a few times water depth). It occurs where there is a local disturbance to the balance between gravity and friction (e.g. at a weir, venturi, sluice, free overfall, sudden change in slope) or a mismatch between the depths imposed by upstream and downstream controls (hydraulic jump).

Often there is a flow transition between deep, slow flow (subcritical; Fr < 1) and shallow, fast flow (supercritical; Fr > 1).

The assumption that the flow varies rapidly over a relatively short distance means that bed friction is unimportant (because the work done is small). Thus, for a smooth transition (e.g. weir, venturi or sluice), the total head is usually assumed constant through this short region. For an abrupt transition (hydraulic jump) there may be significant head loss, but it is associated with high levels of turbulence in the jump, not bed friction.

Note that the hydrostatic pressure assumption can only be applied where near-parallel flow has been established, either side of the rapidly-varying-flow region.

2.1 Hydraulic Jump



A *hydraulic jump* is an abrupt change from a shallow, high-speed flow to a deep, low-speed flow of lower energy.

It occurs when a depth difference is imposed by upstream and downstream conditions. Rapid, shallow flow may be created by, for example, a steep spillway or sluice. A slower and deeper downstream flow may be controlled by a downstream weir or by a reduction in slope.

The triggering of a hydraulic jump at the base of a spillway is desirable to remove surplus kinetic energy, in order to reduce downstream erosion.

Across a hydraulic jump:
- **mass** is conserved;
- the **momentum** principle is satisfied;
- mechanical **energy** is lost (mostly as heat).

Assume, for simplicity:
- velocity uniform over upstream and downstream cross-sections;
- small slope (so that the downslope component of weight can be neglected);
- the length of the jump is short (so that bed friction can be neglected);
- wide or rectangular cross-section (but see the Examples for alternatives).

Continuity

The volume flow rate $Q = VA$ is the same at each section. Velocities can thus be related to cross-sectional area A (and hence to depth) by

$$V = \frac{Q}{A} \tag{1}$$

Momentum:

Consider a control volume encompassing the jump. By the momentum principle:

net pressure force = rate of change of momentum

$$\bar{p}_1 A_1 - \bar{p}_2 A_2 = \rho Q (V_2 - V_1)$$

Since streamlines are parallel there, pressures at inflow and outflow stations 1 and 2 are hydrostatic and the average pressure is the pressure at the centroid; i.e. $\bar{p} = \rho g \bar{d}$, where $\bar{d}$ is the depth of the centroid below the surface. Using this, and substituting for velocity,

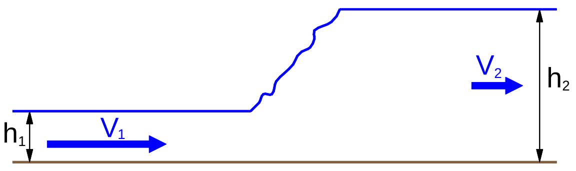
$$\rho g \bar{d}_1 A_1 - \rho g \bar{d}_2 A_2 = \rho Q^2 (\frac{1}{A_2} - \frac{1}{A_1}) \tag{2}$$

At this point we restrict ourselves to a rectangular or wide channel (but, for different shapes, see the Examples). With b the width of channel (or $b = 1$ unit for a "wide" channel):

$$\bar{d} = \tfrac{1}{2} h, \qquad A = hb, \qquad Q = qb$$

and the momentum principle reduces to

$$\tfrac{1}{2} \rho g h_1^2 b - \tfrac{1}{2} \rho g h_2^2 b = \rho q^2 b \; (\frac{1}{h_2} - \frac{1}{h_1})$$



Dividing by ρb:

$$\tfrac{1}{2} g (h_1^2 - h_2^2) = q^2 (\frac{1}{h_2} - \frac{1}{h_1})$$

$$\Rightarrow \qquad \tfrac{1}{2} g (h_1 - h_2)(h_1 + h_2) = q^2 (\frac{h_1 - h_2}{h_1 h_2})$$

Divide through by $g(h_1 - h_2)$ (non-zero by assumption) and then multiply by $h_1 h_2$:

$$\tfrac{1}{2} h_1 h_2 (h_1 + h_2) = \frac{q^2}{g} \tag{3}$$

Since we are looking for the depth ratio h_2/h_1, divide through by h_1^3:

$$\frac{1}{2} \frac{h_2}{h_1} (1 + \frac{h_2}{h_1}) = \frac{q^2}{g h_1^3}$$

Since $q = Vh$, the RHS is V_1^2/gh_1 or Fr_1^2. Hence,

$$\Rightarrow \qquad \frac{1}{2} \frac{h_2}{h_1} (1 + \frac{h_2}{h_1}) = \mathrm{Fr}_1^2 \tag{4}$$

$$\Rightarrow \qquad \left(\frac{h_2}{h_1}\right)^2 + \frac{h_2}{h_1} - 2\mathrm{Fr}_1^2 = 0 \tag{5}$$

This is a quadratic equation for the depth ratio h_2/h_1 and its positive root gives the downstream depth in terms of upstream quantities:

$$\boxed{h_2 = \frac{h_1}{2}(-1 + \sqrt{1 + 8\mathrm{Fr}_1^2}) \qquad (6)}$$

Notes.

(1) Indices 1 and 2 can be exchanged to write the *upstream* depth in terms of *downstream* quantities:

$$h_1 = \frac{h_2}{2}(-1 + \sqrt{1 + 8\mathrm{Fr}_2^2}) \qquad (7)$$

Thus, the depth formula, being dependent only on mass and momentum, doesn't care which of 1 and 2 refers to upstream or downstream conditions.

(2) The *head loss* in the jump is

$$H_1 - H_2 = z_{s1} - z_{s2} + \frac{V_1^2 - V_2^2}{2g}$$
$$= h_1 - h_2 + \frac{q^2}{2g}(\frac{1}{h_1^2} - \frac{1}{h_2^2})$$

Substituting for q^2/g from (3), then (after a lot of algebra, omitted here):

$$H_1 - H_2 = \frac{(h_2 - h_1)^3}{4h_1 h_2} \qquad (8)$$

Hence, for mechanical energy to be lost in the jump (H_1 bigger than H_2) we require $h_2 > h_1$; i.e., on energy grounds, a hydraulic jump will always go from shallow to deep in the direction of flow.

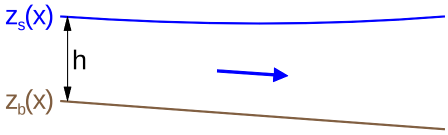
(3) Since $h_2/h_1 > 1$ and $h_1/h_2 < 1$ we have, from (4) and its equivalent with indices reversed:

$$\mathrm{Fr}_1 > 1 \qquad \text{and} \qquad \mathrm{Fr}_2 < 1$$

i.e. the upstream flow is supercritical and the downstream flow is subcritical.

(4) h_1 and h_2 are called *sequent depths*.

2.2 Specific Energy



Since the surface level $z_s = z_b + h$, the surface-elevation part of the total head can be subdivided into the bed elevation z_b and the depth of flow, h:

$$H \equiv z_s + \frac{V^2}{2g} \quad = z_b + h + \frac{V^2}{2g} \tag{9}$$

The *specific energy* E is the head *relative to the bed of the channel*; i.e.

$$\boxed{E = h + \frac{V^2}{2g}} \tag{10}$$

Hence,

$$H = z_b + E \tag{11}$$

If the bed is horizontal and we choose to measure vertical coordinate z from it, then we can take $z_b = 0$ and $H = E$. If, however, the bed varies in height then, if total head is constant,

$$\text{increase in } z_b \quad \Leftrightarrow \quad \text{decrease in } E$$

E is essentially the *flow energy* (in length units). It is rather like the kinetic energy of a particle rolling up a slope. For a particle, the total energy (H) is the sum of the potential energy (z_b in length units) and kinetic energy; in the fluid case the flow energy E also contains some potential energy associated with the finite depth h. In the particle analogy the particle cannot rise above a certain value of z_b because its kinetic energy cannot drop below zero. We shall see that the flow specific energy also cannot drop below a minimum value, although this is greater than zero.

2.2.1 Specific Energy in a Rectangular or Wide Channel

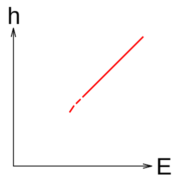
For a rectangular or wide channel we can work with quantities per unit width. As $V = q/h$:

$$E = h + \frac{q^2}{2gh^2} \tag{12}$$

The first part corresponds to potential energy and the second part to kinetic energy (both in length units: energy per unit weight).

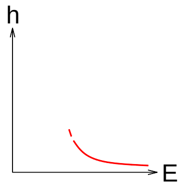
For very large h (deep, slow flow, dominated by potential energy),

$$E \approx h$$

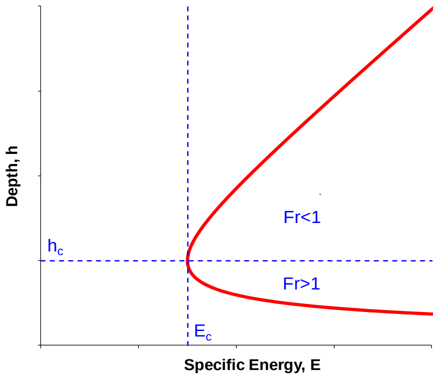


For very small h (shallow, fast flow, dominated by kinetic energy),

$$E \approx \frac{\text{constant}}{h^2}$$



For fixed discharge, the complete graph of E against h (putting independent variable h on the vertical axis because that is more natural for "depth") is shown below.



It is clear from the graph that E must have a minimum and that it is strictly positive. To find it, set $\mathrm{d}E/\mathrm{d}h = 0$; i.e.

$$E = h + \frac{q^2}{2gh^2}$$

$$\Rightarrow \qquad \frac{\mathrm{d}E}{\mathrm{d}h} = 1 - \frac{q^2}{gh^3}$$

Hence $\mathrm{d}E/\mathrm{d}h = 0$ when

$$\frac{q^2}{gh^3} = 1 \qquad (13)$$

$$\Rightarrow \qquad E = h + \frac{q^2}{2gh^2} \quad = h + \frac{1}{2}\left(\frac{q^2}{gh^3}\right)h \quad = h + \frac{1}{2}h \quad = \frac{3}{2}h$$

Hence, the specific-energy has a minimum E_c at a *critical depth* h_c, given by:

$$\boxed{h_c = \left(\frac{q^2}{g}\right)^{1/3} \qquad (14)}$$

$$\boxed{E_c = \frac{3}{2}h_c \qquad (15)}$$

The reason for the subscripts c (for "critical") is that $\mathrm{Fr} = 1$ at the minimum specific energy. This is readily shown for a rectangular or wide channel. Since $V = q/h$, we have:

$$\mathrm{Fr}^2 = \frac{V^2}{gh} \quad = \frac{q^2}{gh^3}$$

Hence, at the depth h where (13) holds:

$$\mathrm{Fr} = 1$$

We show in Section 2.2.4 that this is also true in an arbitrarily-shaped channel.

Since

$$\mathrm{Fr} = \frac{V}{\sqrt{gh}}$$

then, if h is larger than h_c then, to preserve volume flow rate, V must be smaller than V_c; both changes ensure that, for depths greater than the critical depth, $\mathrm{Fr} < 1$ (subcritical). Similarly, for depths smaller than the critical depth, $\mathrm{Fr} > 1$ (supercritical).

- For a given flow rate there is a (strictly positive) minimum specific energy, E_c, occurring at the critical depth where $\mathrm{Fr} = 1$.

- For any specific energy $E > E_c$ there are two possible depths with the same E and q:
  - a shallow ($h < h_c$), high-speed flow with $\mathrm{Fr} > 1$ (*supercritical*);
  - a deep ($h > h_c$), low-speed flow with $\mathrm{Fr} < 1$ (*subcritical*).

  These are called *alternate depths*.

2.2.2 Calculating the Alternate Depths

For a given specific energy E and discharge (per unit width) q, the alternate depths in a rectangular channel are the subcritical and supercritical solutions of

$$E = h + \frac{q^2}{2gh^2} \qquad (16)$$

This can, in principle, be rearranged as a cubic equation and solved directly (see Chanson's book). However, it is easily solved by iteration in a manner that deliberately isolates the deeper or shallower positive solution.

For the subcritical (deep, slow) solution the first term on the RHS of (16) dominates, so rearrange for iteration as:

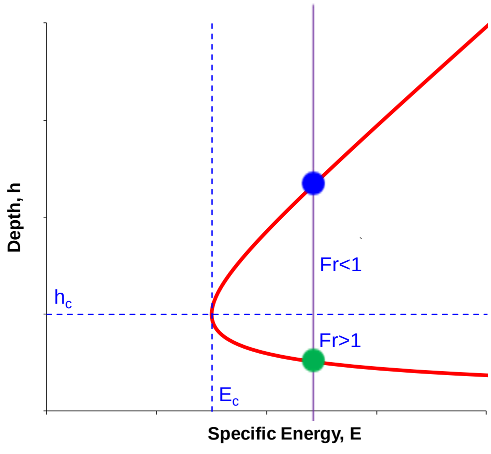
$$h = E - \frac{q^2}{2gh^2}$$

and start iterating from a subcritical depth (e.g. $h = E$).

For the supercritical (shallow, fast) solution the second term on the RHS of (16) dominates, so rearrange for iteration as:

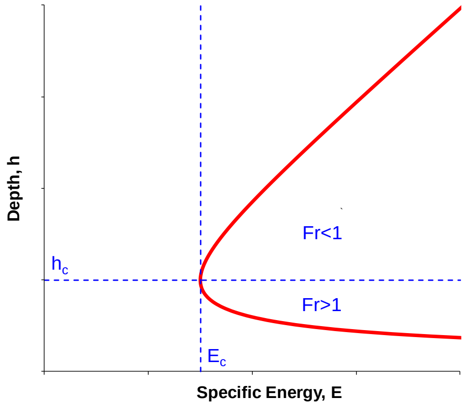
$$h = \frac{q}{\sqrt{2g(E-h)}}$$

and start iterating from a supercritical depth (e.g. $h = 0$).



Example.
A 3 m wide channel carries a total discharge of 12 m^3 s^{-1}. Calculate:
(a) the critical depth;
(b) the minimum specific energy;
(c) the alternate depths when $E = 4$ m.

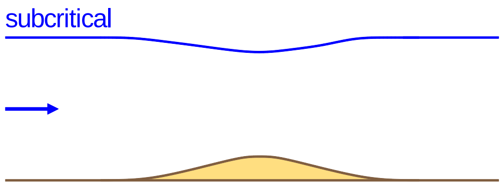
2.2.3 Flow Over a Bed Rise



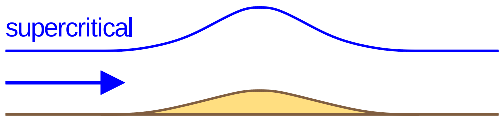
The total head is

$$H = z_b + E$$

Under the rapidly-varied-flow assumption, the total head is constant, so that, if the bed height z_b increases, the specific energy E must decrease by the same amount. Qualitative changes in specific energy E and water depth h can be determined simply from the *shape* of the $E-h$ graph.



- *Subcritical*:
  As E decreases, h decreases; i.e. water depth *decreases* over a bump.



- *Supercritical*:
  As E decreases, h increases; i.e. water depth *increases* over a bump.

(You should be able to work out from the specific-energy graph what happens to the depth of water if the bed of the channel is *depressed* rather than elevated.)

Strictly, we have shown in the subcritical case that the depth h decreases over a bump, but this does not necessarily imply that the actual water level z_s does likewise. However, it turns out that changes in actual water level ($\mathrm{d}z_s$) have the same sign as the changes in depth ($\mathrm{d}h$). This can be deduced by considering the total head:

$$\begin{aligned} H &= z_s + \frac{V^2}{2g} \\ &= z_s + \frac{q^2}{2gh^2} \end{aligned}$$

Considering differential changes:

$$\begin{aligned} \mathrm{d}H &= \mathrm{d}z_s - \frac{q^2}{gh^3}\mathrm{d}h \\ &= \mathrm{d}z_s - \mathrm{Fr}^2\mathrm{d}h \end{aligned}$$

Neglecting friction over short distances, total head is constant ($\mathrm{d}H = 0$), so that

$$\mathrm{d}z_s = \mathrm{Fr}^2\mathrm{d}h$$

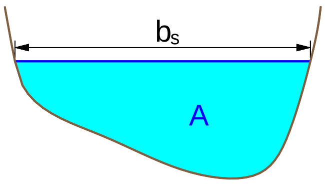
Hence:

(1)	at constant head, $\mathrm{d}z_s$ and $\mathrm{d}h$ have the same sign; i.e. *depth* increases/decreases if and only if the *water level* increases/decreases;
(2)	if the Froude number is very small ($\mathrm{Fr} \ll 1$) then surface displacement is negligible.

2.2.4 Specific Energy in a Non-Rectangular Channel

In this section we consider specific energy for a non-rectangular channel and, in particular, deduce that critical conditions ($\mathrm{Fr} = 1$) will occur at the minimum specific energy … provided that we use the *mean* depth $\bar{h}$ in the definition of the Froude number.

Let the cross-sectional area occupied by fluid be A and the surface width be b_s.



The total head is

$$H = z_s + \frac{V^2}{2g}$$

where $V = Q/A$. Hence,

$$H = z_b + E$$

where

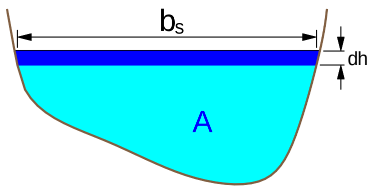
$$E = h + \frac{Q^2}{2gA^2}$$

(h is the depth at whichever point of the cross-section is used to determine the bed level z_b: usually the lowest point or *invert*.)

The specific energy has a minimum when $\mathrm{d}E/\mathrm{d}h = 0$. Now, by the chain rule,

$$\frac{\mathrm{d}E}{\mathrm{d}h} = 1 + \frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{Q^2}{2gA^2}\right) \times \frac{\mathrm{d}A}{\mathrm{d}h} \quad = 1 - \frac{Q^2}{gA^3}\frac{\mathrm{d}A}{\mathrm{d}h}$$

$$\Rightarrow \qquad \frac{Q^2}{gA^3}\frac{\mathrm{d}A}{\mathrm{d}h} = 1$$



Consider the area added when the depth is increased by $\mathrm{d}h$,

$$\mathrm{d}A = b_s\,\mathrm{d}h$$

Hence, at the minimum specific energy,

$$\frac{Q^2 b_s}{gA^3} = 1$$

Since $Q/A = V$ and $A/b_s = \bar{h}$ this gives

$$\frac{V^2}{g\bar{h}} = 1$$

Hence, minimum specific energy for a given discharge occurs at $\mathrm{Fr} = 1$, provided that we define

$$\mathrm{Fr} = \frac{V}{\sqrt{g\bar{h}}} \qquad (17)$$

This is the rationale for taking $\bar{h}$ as the length scale used to define the Froude number.

2.3 Critical-Flow Devices

A simple analysis is presented for 3 critical-flow devices:

- *broad-crested weir*;
- *venturi flume*;
- *sluice gate*;

and one additional critical-flow control:

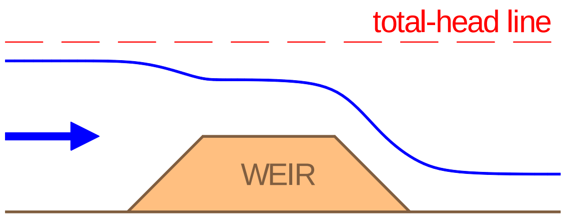
- *free overfall*.

In each case, under suitable conditions, the flow passes smoothly from subcritical to supercritical as it passes through the device. Since there is then a known relationship between flow depth and discharge these hydraulic structures can be used to:

(i) measure the flow rate;
(ii) provide a control point (i.e. boundary condition) for GVF calculations.

For a broad-crested weir or venturi flume, when critical conditions are established the specific energy – and hence the immediate upstream head – is fixed. This must be greater than or equal to the head in the absence of the device and hence the fluid must "back up"; i.e. the depth increases for some distance upstream. The flow is then said to be *controlled* or *choked* by the device.

In the analyses below it is assumed that changes take place over a length short enough for frictional losses to be negligible; i.e. *the total head is constant through the device*. In reality, departures from this are often accommodated by the use of discharge coefficients in formulae for discharge.

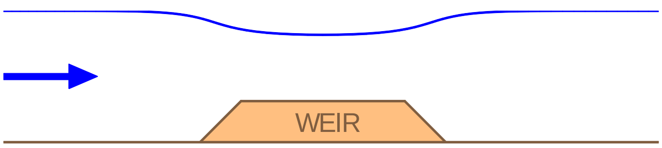


For simplicity, channels will be assumed to have rectangular (or wide) cross-sections.

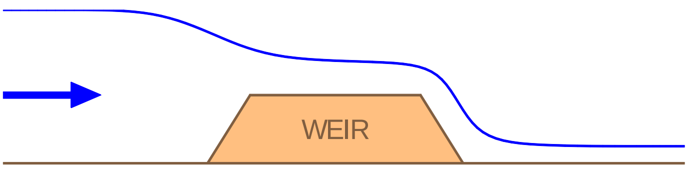
2.3.1 Broad-Crested Weir

Consider subcritical flow (with specific energy E_a, discharge per unit width q) approaching a region where the bed is raised by Δz_b. This region is sufficiently long for parallel flow to be established (hence "broad-crested"), but insufficiently long for significant frictional losses.

As total head ($H = z_b + E$) is constant, the specific energy is reduced over the weir (to $E_a - \Delta z_b$). If this still exceeds the minimum specific energy E_c for this discharge then the flow remains subcritical over the bump and resumes its original depth downstream.



If, however, the bed rise is sufficiently large then, as the specific energy cannot be less than E_c, the upstream flow must "back up", increasing the depth and specific energy immediately upstream of the weir.



In the latter case we have the following (writing $\Delta z_b = z_{\text{weir}}$):

- critical flow over the top of the weir with:

depth $$h_c = \left(\frac{q^2}{g}\right)^{1/3}$$

specific energy $$E_c = \frac{3}{2}h_c$$

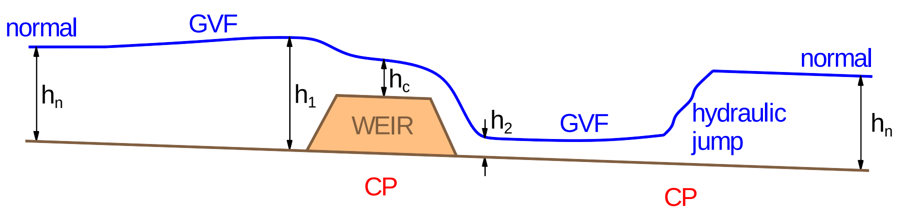
- smooth acceleration from subcritical to supercritical flow either side of the weir;
- total head immediately up or downstream of the weir is the same as that over the top:

$$H = z_{\text{weir}} + E_c$$

- the depths immediately up or downstream of the weir (where the bed level has returned to zero) can be found as the sub- and supercritical solutions, respectively, of

$$H = h + \frac{q^2}{2gh^2}$$

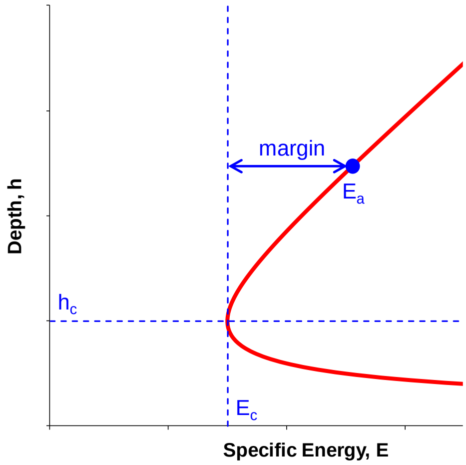
What happens further up- or downstream depends on other controls (if present), or normal flow if there are long fetches. An example for a long channel with subcritical normal flow is shown below. Upstream, the flow relaxes via GVF. Downstream, it jumps back to subcritical flow following a length of GVF. If any downstream controls are sufficiently far away then the flow jumps directly back to its "preferred" depth for the channel; i.e. normal depth. However, this cannot always be assumed: for shorter fetches, e.g. in the hydraulics laboratory flumes, the downstream depth will *not* be normal; the flumes are nowhere near long enough.



To establish whether the flow becomes critical over the weir, compare total head assuming critical conditions at the crest of the weir (H_c) with the total head available in the approach flow (H_a). (Often, but not always, this will be the head associated with normal flow).

In the approach flow find the specific energy E_a. If you are referring heights to the bed of the channel near the weir then this will be the same as the approach-flow total head H_a at the position of the weir.

At the weir find the critical depth h_c and minimum specific energy E_c. Then do one of the following.



(1) Find what the approach-flow specific energy would be reduced to following the bed rise:

$$E_a - z_{\text{weir}}$$

If this is less than the critical value E_c then the flow must become choked and a critical-flow transition will occur across the weir.

(2) Alternatively, find the total head associated with critical flow over the weir; i.e.

$$H_c = z_{\text{weir}} + E_c$$

This is the minimum head needed to pass this discharge over the weir. If it exceeds the head available in the approach flow ($H_a = E_a$) then critical conditions occur and a flow transition (sub- to supercritical flow) takes place across the weir.

Neglecting frictional losses, the total head H is constant across the device and equal to the larger of the head under critical conditions and the head in the approach flow. This head, together with the level of the bed and knowledge of whether the flow is subcritical or supercritical, will determine the depth at a specific location.

Example. (Exam 2020)

(a) Define:

(i) specific energy

(ii) Froude number

for open-channel flow. What is special about these quantities in critical conditions?

A long, wide channel has a slope of 1:1000, a Manning's n of 0.015 m$^{-1/3}$ s and a discharge of 3 m^3 s^{-1} per metre width.

(b) Calculate the normal and critical depths.

(c) In a region of the channel the bed is raised by a height of 0.8 m over a length sufficient for the flow to be parallel to the bed over this length. Determine the depths upstream, downstream and over the raised bed, ignoring frictional losses. Sketch the key features of the flow, indicating *all* hydraulic transitions caused by the bed rise.

(d) In the same channel, the bed is lowered by 0.8 m from its original level. Determine the depths upstream, downstream and over the lowered bed, ignoring frictional losses. Sketch the flow.

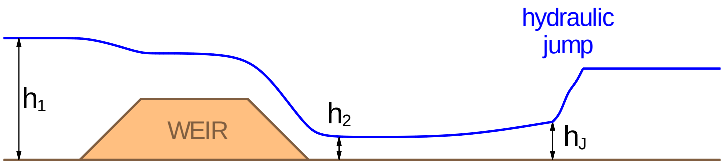
Supercritical flow downstream of the weir may or may not actually occur.

If the flow far downstream is subcritical then in between there must be a hydraulic jump. If conditions downstream of the jump are known (e.g. if normal flow) then the depth just upstream of the jump can be calculated from the hydraulic-jump sequent-depth relationship.

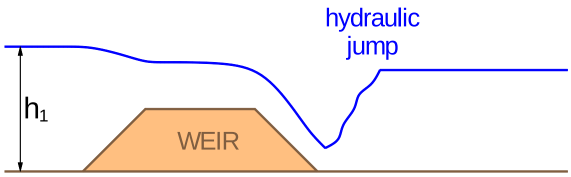
A region of supercritical GVF downstream of the weir will exist provided the hydraulic jump is not too close. The lectures on GVF will show that depth increases in supercritical flow on a mild slope (one for which the normal depth is subcritical). Hence, this will occur if and only if the supercritical depth just downstream of the weir is less than the depth upstream of the jump. Otherwise, the hydraulic jump will occur immediately at the downstream base of the weir, and there is no intervening region of supercritical GVF.

Denote the depth immediately downstream of the weir by h_2 and the sequent depth on the upstream side of the hydraulic jump by h_J. There are two possible cases:

(i) $h_2 < h_J$: region of supercritical GVF between the weir and the jump;



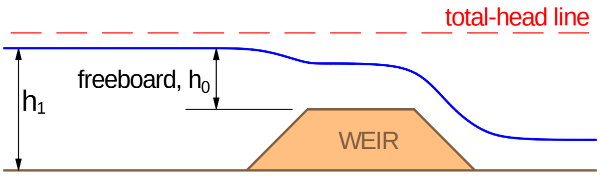
(ii) $h_2 \geq h_J$ jump occurs immediately downstream of the weir; no region of supercritical GVF (and the flow depth may never actually reach h_2).



It is therefore necessary to calculate and compare h_2 (the depth of any supercritical parallel flow just downstream of the weir) and h_J (the depth upstream of the jump, which is fixed by the hydraulic-jump relation, equation (7), and the depth downstream of the jump).

Example.
A long channel of rectangular cross-section with width 3.5 m and streamwise slope 1 in 800 carries a discharge of 15 m^3 s^{-1}. Manning's n may be taken as 0.016 m$^{-1/3}$ s. A broad-crested weir of height 0.7 m is constructed at the centre of the channel. Determine:
(a) the depth far upstream of the weir;
(b) the depth just upstream of the weir;
(c) whether or not a region of supercritical gradually-varied flow exists downstream of the weir.

<u>Measurement of Discharge</u>



If critical conditions are established over a weir there is a fixed relationship between head and flow rate, and the weir can be used for flow measurement.

Assuming no loss of head,

$$H_{\text{weir}} = H_{\text{upstream}}$$

$$H = z_s + \frac{V^2}{2g}$$

Because the upstream side is often a deep reservoir rather than a continuous channel (see the figure below) it is more common for this purpose to measure the vertical coordinate *z from the top of the weir*. Then, assuming critical flow over the crest of the weir:

$$\frac{3}{2}h_c = h_0 + \frac{V^2}{2g}$$

$$\Rightarrow \qquad \frac{3}{2}\left(\frac{q^2}{g}\right)^{1/3} = h_0 + \frac{q^2}{2gh_1^2}$$

where $h_0 = h_1 - z_{\text{weir}}$ is the *freeboard*; i.e. the upstream depth relative to the weir. (If you measure z from the bed of the channel instead then simply add z_{weir} to both sides.)

This can be rearranged to give an *implicit* equation for the discharge per unit width:

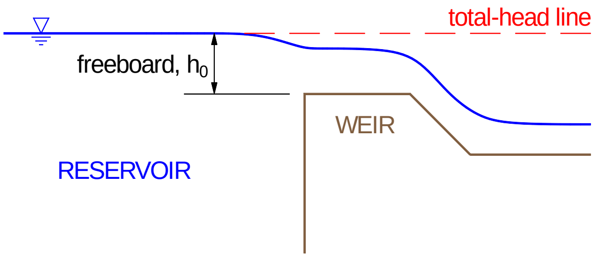
$$q = (2/3)^{3/2}\sqrt{g}(h_0 + \frac{q^2}{2gh_1^2})^{3/2}$$

Losses may be compensated for by a discharge coefficient c_d. Then, in metre-second units, the *total* discharge ($Q = qb$) is given by

$$Q = 1.705c_d b(h_0 + \frac{Q^2}{2gb^2h_1^2})^{3/2} \qquad (18)$$

This must be solved for Q by iteration (although the dynamic head on the RHS is usually small and is often neglected). A straightforward measurement of water level then allows the discharge in a channel to be gauged.

If the weir is discharging a deep reservoir rather than a channel then the upstream head is simply the still-water level and no iteration is necessary – see the example below.



Example. (Exam 2023, part)
A reservoir discharge is controlled by a weir of width 8 m and discharge coefficient 0.9.
(a) Calculate the flow rate over the weir when the freeboard is 0.65 m.
(b) Assuming negligible inflow and a constant plan area for the reservoir of 1.5 km^2, calculate the time in hours to reduce the level of the reservoir by 0.4 m.

2.3.2 Venturi Flume

In a duct or channel a region of contracted width is called a *venturi*.

As a channel narrows the discharge per unit width, $q = Q/b$, increases. However:

For a given specific energy there is a maximum discharge (per unit width), $q_{\max}$, occurring at the critical depth where $\text{Fr} = 1$.

Proof:
For a rectangular or wide channel we can work with quantities per unit width. As $V = q/h$:

$$E = h + \frac{V^2}{2gh^2} \quad = h + \frac{q^2}{2gh^2}$$

Rearranging for q^2:

$$q^2 = 2gh^2(E - h) \quad = 2g(Eh^2 - h^3)$$

The graph of q vs h for constant specific energy has the shape shown. From the graph it is clear that q must have a maximum. Since q^2 is largest when q is largest it is easier to maximise q^2 instead:

$$\frac{\mathrm{d}}{\mathrm{d}h}(q^2) = 2g(2Eh - 3h^2)$$

Setting $\mathrm{d}(q^2)/\mathrm{d}h = 0$ gives

$$2Eh - 3h^2 = 0$$

whence

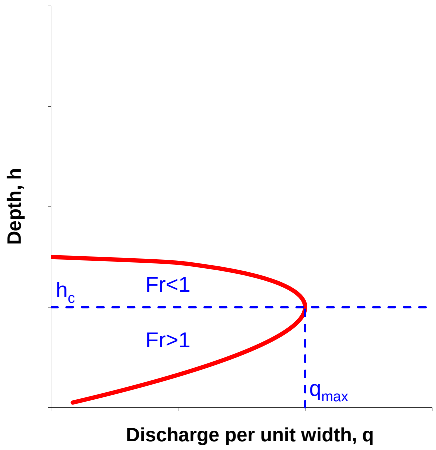
$$E = \frac{3}{2}h$$

Then, from the expression for q^2:

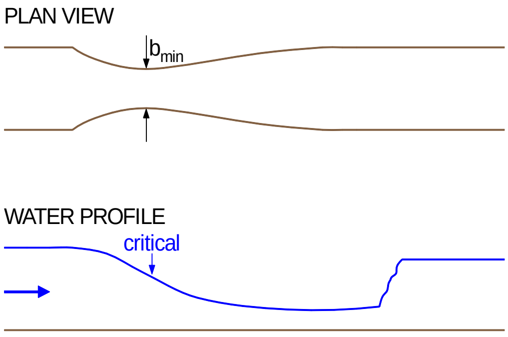
$$q^2 = gh^3$$

and hence

$$\text{Fr}^2 \equiv \frac{V^2}{gh} \quad = \frac{q^2}{gh^3} \quad = 1$$



Thus, at constant specific energy q cannot exceed $q_{\max}$. Where continuity insists that it does so, the flow becomes *choked* and critical conditions are maintained at the venturi throat by the flow backing up upstream to provide a greater depth and specific energy.



<u>If</u> critical conditions occur we have the following.

- There is smooth acceleration from sub- to supercritical flow through the throat.

- At the venturi throat:

  depth $h_c = (q_m^2/g)^{1/3}$ where $q_m = Q/b_{\text{min}}$

  specific energy $E_c = \frac{3}{2}h_c$

  Remember: q_m is not the same as in the main channel; the throat is narrower.

- The total head through the device is

  $H = H_c \quad = z_b + E_c$

  where z_b is the bed level (often, but not always, 0).

- The depths of parallel flow in the vicinity of the venturi can then be found as the sub- or supercritical solutions of

  $$E = h + \frac{Q^2}{2gb^2h^2}$$

  where b is the width at that particular location.

To establish whether critical conditions occur, then, just as for the weir, calculate the head H_c corresponding to critical conditions at the throat (the minimum energy required to pass this discharge) and compare with the head H_a in the approach flow (the energy available if approach-flow conditions were to occur all the way up to the venturi). If the approach-flow head is smaller than that corresponding to critical flow in the throat then the flow must back up to provide the extra energy and a critical-flow transition occurs. If the approach-flow head is larger than that required by critical flow in the throat then critical conditions do not occur and, for subcritical approach flow, the surface just dips and then returns to its original level.

As for the broad-crested weir the total head through the device is constant and equal to the larger of the approach-flow and critical heads.

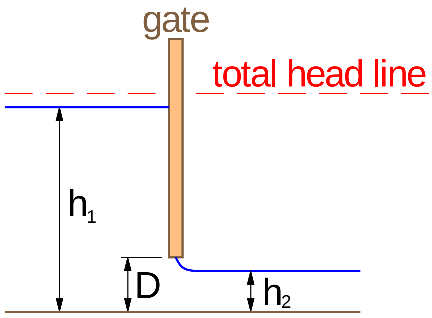
Provided critical flow is established at its throat, a venturi flume can, like a broad-crested weir, be used as a flowmeter.

Example.

A venturi flume is placed near the middle of a long rectangular channel with Manning's $n = 0.012\ \mathrm{m}^{-1/3}\ \mathrm{s}$. The channel has a width of 5 m, a discharge of 12.5 m^3 s^{-1} and a slope of 1:2500.

(a) Determine the critical depth and the normal depth in the main channel.
(b) Determine the venturi flume width which will just make the flow critical at the contraction.
(c) If the contraction width is 2 m find the depths just upstream, downstream and at the throat of the venturi flume (neglecting friction in this short section).
(d) Sketch the surface profile.

2.3.3 Sluice Gate



At the gate the flow passes smoothly through critical conditions from subcritical to supercritical flow. Neglecting frictional losses, the total head is the same on both sides:

$$H_1 = H_2$$

$$z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g}$$

Provided the gate is not lifted too high then, in a rectangular channel with $V = q/h$ and flat bed from which z is measured, depths h_1 and h_2 are the subcritical and supercritical solutions respectively, of

$$h_1 + \frac{q^2}{2gh_1^2} = h_2 + \frac{q^2}{2gh_2^2} \qquad (19)$$

(Note that, because of the hydrostatic assumption implicit in the expression for total head, h_2 is the depth where parallel flow has become established; i.e. at the vena contracta.)

Example.
The water depth upstream of a sluice gate is 0.8 m and the depth just downstream (at the vena contracta) is 0.2 m. Calculate:
(a) the discharge per unit width;
(b) the Froude numbers upstream and downstream.

Example.
A sluice gate controls the flow in a channel of width 2 m. If the discharge is 0.5 m^3 s^{-1} and the upstream water depth is 1.5 m, calculate the downstream depth and velocity.

In the general case, (19) can be rearranged for q and hence the total discharge ($Q = qb$):

$$Q = bh_2\sqrt{\frac{2gh_1}{1 + h_2/h_1}} \qquad (20)$$

In the "ideal" approximation, h_2 is approximated by gate opening D and $h_2 \ll h_1$, so that

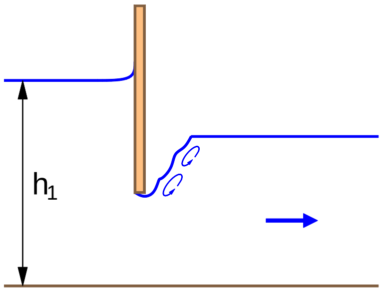
$$Q_{\text{ideal}} = bD\sqrt{2gh_1}$$

In reality, h_2 is smaller than the gate opening (typically, about 0.6 times), h_2/h_1 is small but not insignificant, and there are frictional losses. These modifications are all absorbed into a discharge coefficient c_d such that the *actual*, *measured* discharge can be written

$$Q = c_d bD\sqrt{2gh_1} \tag{21}$$

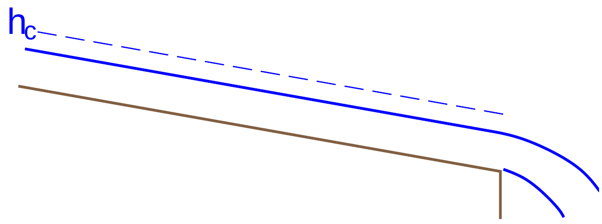
The gate opening D and either upstream total head H or depth h_1 control the discharge.

If the gate is opened too far, or if a downstream obstruction is too close, then the hydraulic jump occurs immediately and supercritical conditions cannot be attained. The flow on both sides is then subcritical, there is energy lost and the sluice gate is said to be *drowned*.

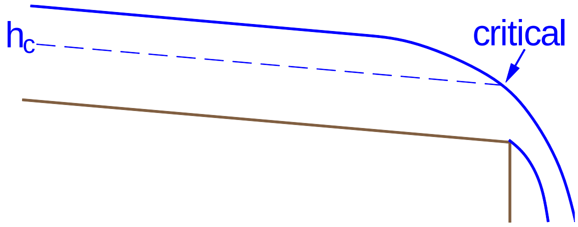


2.3.4 Free Overfall

If the approach flow is *supercritical* (Fr > 1) then there is upstream control and the supercritical flow simply continues over the overfall.



If the approach flow is *subcritical* (Fr < 1) then the flow accelerates smoothly through critical to supercritical flow a short (and usually neglected) distance upstream of the overfall.



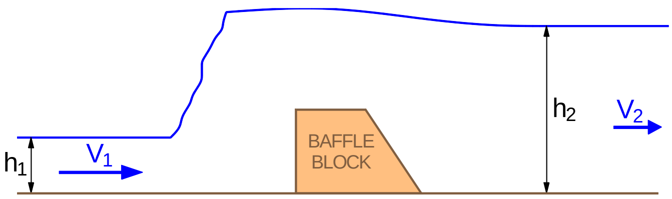
2.4 Forces On Objects

Obstacles (e.g. bridge piers, baffle blocks) placed in the flow provide a reactive force.

For *subcritical* approach flow, depth of flow is reduced over a bed rise. This may be enough to generate a critical-flow transition similar to that over a weir.

For *supercritical* approach flow, depth increases over a bed rise. If the flow has insufficient head then a hydraulic jump occurs to a subcritical depth, with overall loss of energy.

Baffle blocks are used in *stilling basins* to provoke a hydraulic jump in a controlled and precisely-located manner, so that the high-speed flow and/or the turbulent motions in the jump do not cause damaging erosion further downstream.



Forces may be determined using a control-volume analysis and the momentum principle.

Where pressure is hydrostatic, the magnitude of the pressure force is (for a rectangular channel of width b):

$$\text{(average pressure)} \times \text{area} = p(\text{centroid}) \times A \quad = \rho g(\tfrac{1}{2}h) \times hb \quad = \tfrac{1}{2}\rho g h^2 b$$

Then, from the steady-state momentum principle:

$$\text{force} = \text{rate of change of momentum}$$

$$-F + \tfrac{1}{2}\rho g h_1^2 b - \tfrac{1}{2}\rho g h_2^2 b = \rho Q(V_2 - V_1)$$

Hence,

$$F = (\rho Q V_1 + \tfrac{1}{2}\rho g h_1^2 b) - (\rho Q V_2 + \tfrac{1}{2}\rho g h_2^2 b) \qquad (22)$$

This can also be written

$$F = (M_1 + F_{p1}) - (M_2 + F_{p2}) \qquad (23)$$

where

$$M = \rho V^2 hb \quad = \text{momentum flux}$$

$$F_p = \tfrac{1}{2}\rho g h^2 b \quad = \text{hydrostatic pressure force}$$

The quantity $M + F_p$ (momentum flux + pressure force) is sometimes called *specific force*.

A hydraulic jump is just a special case of this analysis with $F = 0$; i.e. the specific force is constant:

$$M_1 + F_{p1} = M_2 + F_{p2}$$

This can be used to establish the jump relation in non-rectangular channels.

Example. (Exam 2018)
Water flows at 0.8 m^3 s^{-1} per metre width down a long, wide spillway of slope 1 in 30 onto a wide apron of slope 1 in 1000. Manning's roughness coefficient n = 0.014 m$^{-1/3}$ s on both slopes.

(a) Find the normal depths in both sections and show that normal flow is supercritical on the spillway and subcritical on the apron.

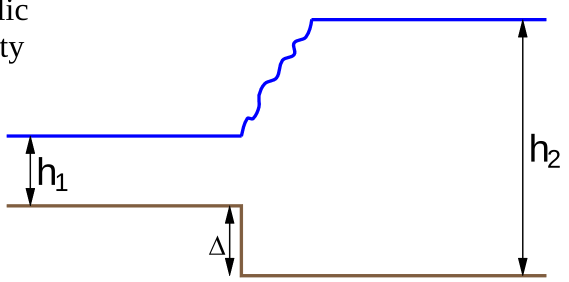
(b) Baffle blocks are placed a short distance downstream of the slope transition to provoke a hydraulic jump. Assuming that flow is normal on both the spillway and downstream of the hydraulic jump, calculate the force per metre width of channel that the blocks must impart.

(c) Find the head loss across the blocks.

A hydraulic jump may also be triggered by the drop in velocity associated with a *sudden expansion* – e.g. a downward step or an abrupt increase in width.

This can also be analysed by use of the momentum principle. An important approximation is that the *reaction force from downstream-facing expansion walls is approximated by a hydrostatic-pressure distribution*, as in the example below.

Example.
A downward step of height 0.5 m causes a hydraulic jump in a wide channel when the depth and velocity of the flow upstream are 0.5 m and 10 m s^{-1}, respectively.



(a) Find the downstream depth.

(b) Find the head lost in the jump.