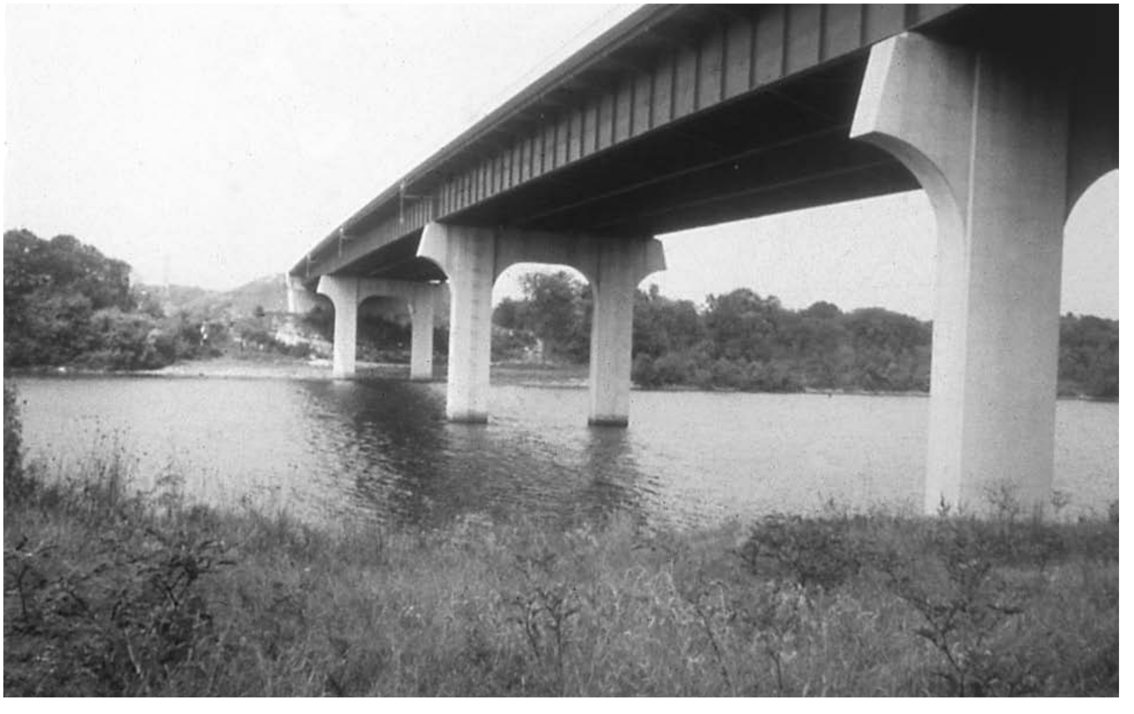
5 BEAMS



A Continuous Beam Bridge

(Photo courtesy of Bethlehem Steel Corporation)

The term "beam" is used herein to refer to *a long straight structure, which is supported and loaded in such a way that all the external forces and couples (including reactions) acting on it lie in a plane of symmetry of its cross-section, with all the forces perpendicular to its centroidal axis.* Under the action of external loads, beams are subjected only to bending moments and shear forces (but no axial forces).

In this chapter, we study the basic concepts of the analysis of beams by the matrix stiffness method, and develop a computer program for the analysis of beams based on the matrix stiffness formulation. As we proceed through the chapter, the reader will notice that, although the member stiffness relations for beams differ from those for plane trusses, the overall format of the method of analysis remains essentially the same—and many of the analysis steps developed in Chapter 3 for the case of plane trusses can be directly applied to beams. Therefore, the computer program developed in Chapter 4 for the analysis of plane trusses can be modified with relative ease for the analysis of beams.

We begin by discussing the preparation of analytical models of beams in Section 5.1, where the global and local coordinate systems and the degrees of freedom of beams are defined. Next, we derive the member stiffness relations in the local coordinate system in Section 5.2; and present the finite-element formulation of the member stiffness matrix, via the principle of virtual work, in Section 5.3. The derivation of the member fixed-end forces, due to external loads applied to members, is considered in Section 5.4; and the formation of the stiffness relations for the entire beam, by combining the member stiffness relations, is discussed in Section 5.5. The procedure for forming the structure fixed-joint force vectors, and the concept of equivalent joint loads, are introduced in Section 5.6; and a step-by-step procedure for the analysis of beams is presented in Section 5.7. Finally, a computer program for the analysis of beams is developed in Section 5.8.

5.1 ANALYTICAL MODEL

For analysis by the matrix stiffness method, *the continuous beam is modeled as a series of straight prismatic members connected at their ends to joints, so that the unknown external reactions act only at the joints*. Consider, for example, the two-span continuous beam shown in Fig. 5.1(a). Although the structure actually consists of a single continuous beam between the two fixed supports at the ends, for the purpose of analysis it is considered to be composed of three members (1, 2, and 3), rigidly connected at four joints (1 through 4), as shown in Fig. 5.1(b). Note that joint 2 has been introduced in the analytical model so that the vertical reaction at the roller support acts on a joint (instead of on a member), and joint 3 is used to subdivide the right span of the beam into two members, each with constant flexural rigidity (EI) along its length. This division of the beam into members and joints is necessary because the formulation of the stiffness method requires that the unknown external reactions act only at the joints (i.e., all the member loads be known in advance of analysis), and the member stiffness relationships used in the analysis (to be derived in the following sections) are valid for prismatic members only.

It is important to realize that because joints 1 through 4 (Fig. 5.1(b)) are modeled as rigid joints (i.e., the corresponding ends of the adjacent members are rigidly connected to the joints), they satisfy the continuity and restraint conditions of the actual structure (Fig. 5.1(a)). In other words, since the left end of member 1 and the right end of member 3 of the analytical model are rigidly

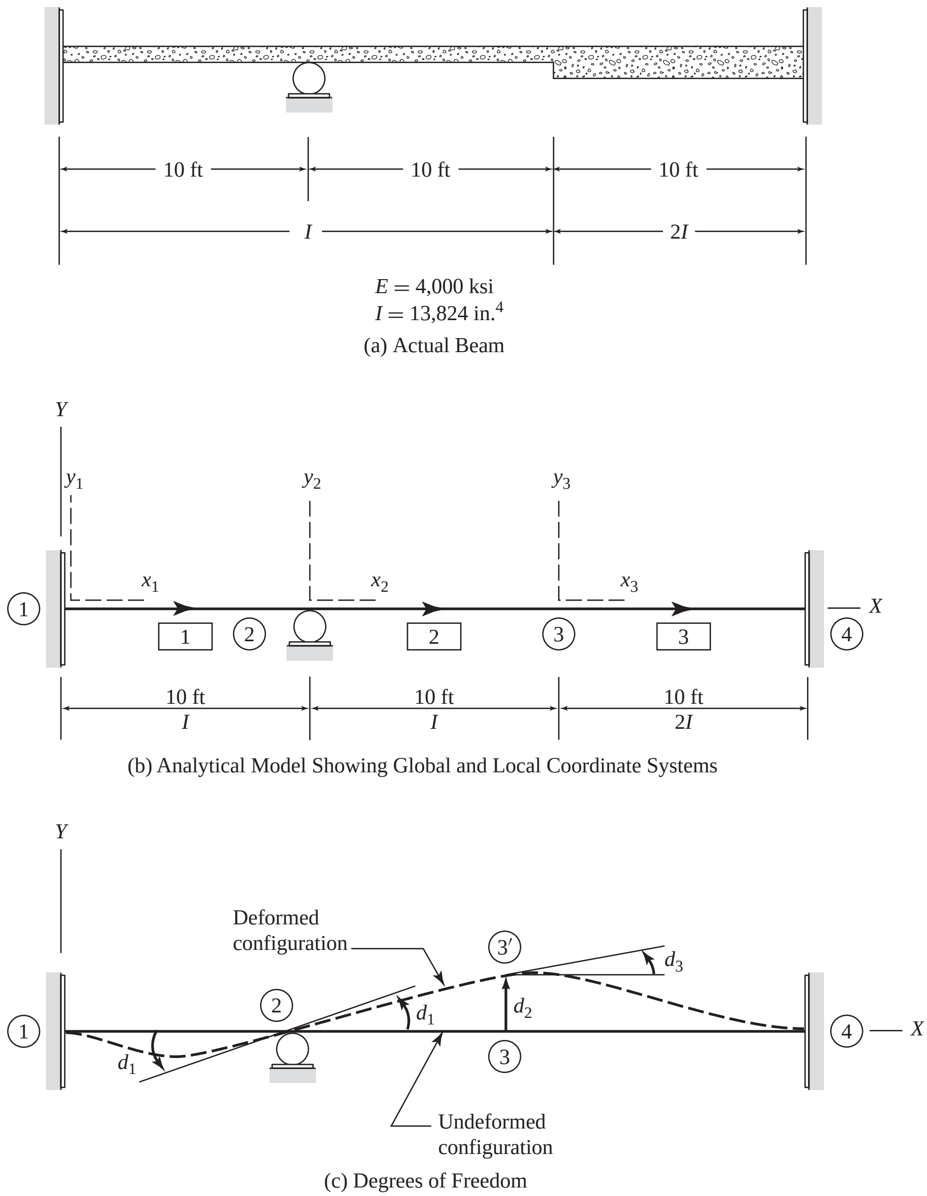
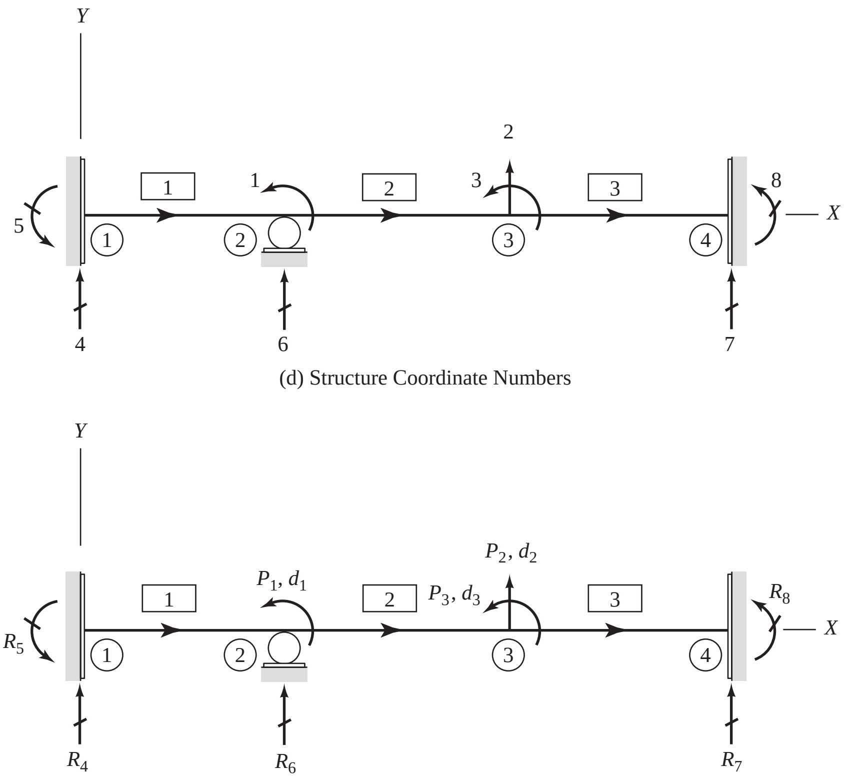


Fig. 5.1



(d) Structure Coordinate Numbers

(e) Degrees of Freedom, Joint Loads, and Support Reactions

Fig. 5.1 (*continued*)

connected to joints 1 and 4, respectively, which are in turn attached to the fixed supports, the displacements and rotations at the exterior ends of the members are 0; thereby satisfying the restraint conditions of the actual beam at the two ends. Similarly, as the right end of member 1 and the left end of member 2 (Fig. 5.1(b)) are connected to the rigid joint 2, which is attached to a roller support, the displacements at the foregoing ends of members 1 and 2 are 0, and the rotations at the two ends are equal. This indicates that the analytical model satisfies the restraint and continuity conditions of the actual beam at the location of joint 2. Finally, the right end of member 2 and the left end of member 3 (Fig. 5.1(b)) are rigidly connected to joint 3, to ensure that the continuity of both the displacement and the rotation is maintained at the location of joint 3 in the analytical model.

Global and Local Coordinate Systems

As discussed in Chapter 3, the overall geometry, as well as the loads and displacements (including rotations) at the joints of a structure are described with reference to a Cartesian global (*XYZ*) coordinate system. The particular orientation of the global coordinate system, used in this chapter, is as follows.

> *The global coordinate system used for the analysis of beams is a right-handed XYZ coordinate system, with the X axis oriented in the horizontal (positive to the right) direction, and coinciding with the centroidal axis of the beam in the undeformed state. The Y axis is oriented in the vertical (positive upward) direction, with all the external loads and reactions of the beam lying in the XY plane.*

Although not necessary, it is usually convenient to locate the origin of the global XY coordinate system at the leftmost joint of the beam, as shown in Fig. 5.1(b), so that the X coordinates of all the joints are positive. As will become apparent in Section 5.8, this definition of the global coordinate system simplifies the computer programming of beam analysis, because only one (X) coordinate is needed to specify the location of each joint of the structure.

As in the case of plane trusses (Chapter 3), a local (right-handed, xyz) coordinate system is defined for each member of the beam, to establish the relationships between member end forces and end displacements, in terms of member loads. Note that *the terms forces (or loads) and displacements are used in this text in the general sense to include moments and rotations, respectively.* The local coordinate system is defined as follows.

> *The origin of the local xyz coordinate system for a member is located at the left end (beginning) of the member in its undeformed state, with the x axis directed along its centroidal axis in the undeformed state, and the y axis oriented in the vertical (positive upward) direction.*

The local coordinate systems for the three members of the example continuous beam are depicted in Fig. 5.1(b). As this figure indicates, the local coordinate system of each member is oriented so that the positive directions of the local x and y axes are the same as the positive directions of the global X and Y axes, respectively.

The selection of the global and local coordinate systems, as specified in this section, considerably simplifies the analysis of continuous beams by eliminating the need for transformation of member end forces, end displacements, and stiffnesses, from the local to the global coordinate system and vice-versa.

Degrees of Freedom

The degrees of freedom (or free coordinates) of a beam are simply its unknown joint displacements (translations and rotations). Since the axial deformations of the beam are neglected, the translations of its joints in the global X direction are 0. Therefore, a joint of a beam can have up to two degrees of freedom, namely, a translation in the global Y direction (i.e., in the direction perpendicular to the beam's centroidal axis) and a rotation (about the global Z axis). Thus, the number of structure coordinates (i.e., free and/or restrained coordinates) at a joint of a beam equals 2, or $NCJT = 2$.

Let us consider the analytical model of the continuous beam as given in Fig. 5.1(b). The deformed shape of the beam, due to an arbitrary loading, is depicted in Fig. 5.1(c) using an exaggerated scale. From this figure, we can see that joint 1, which is attached to the fixed support, can neither translate nor rotate; therefore, it does not have any degrees of freedom. Since joint 2 of the beam is attached to the roller support, it can rotate, but not translate. Thus, joint 2 has only one degree of freedom, which is designated d_1 in the figure. As joint 3 is not attached to a support, two displacements—the translation d_2 in the Y direction, and the rotation d_3 about the Z axis—are needed to completely specify its deformed position $3'$. Thus, joint 3 has two degrees of freedom. Finally, joint 4, which is attached to the fixed support, can neither translate nor rotate; therefore, it does not have any degrees of freedom. Thus, the entire beam has a total of three degrees of freedom.

As indicated in Fig. 5.1(c), joint translations are considered positive when vertically upward, and joint rotations are considered positive when counterclockwise. All the joint displacements in Fig. 5.1(c) are shown in the positive sense. The $NDOF \times 1$ joint displacement vector $\mathbf{d}$ for the beam is written as

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Since the number of structure coordinates per joint equals 2 (i.e., $NCJT = 2$), the number of degrees of freedom, $NDOF$, of a beam can be obtained from Eq. (3.2) as

$$\boxed{\left.\begin{aligned} NCJT &= 2 \\ NDOF &= 2(NJ) - NR \end{aligned}\right\} \quad \text{for beams}} \tag{5.1}$$

in which, as in the case of plane trusses, NJ represents the number of joints of the beam, and NR denotes the number of joint displacements restrained by supports (or the number of restrained coordinates). Let us apply Eq. (5.1) to the analytical model of the beam in Fig. 5.1(b). The beam has four joints (i.e., $NJ = 4$); two joints, 1 and 4, are attached to the fixed supports that together restrain four joint displacements (namely, the translations in the Y direction and the rotations of joints 1 and 4). Furthermore, the roller support at joint 2 restrains one joint displacement, which is the translation of joint 2 in the Y direction. Thus, the total number of joint displacements that are restrained by all supports of the beam is 5 (i.e., $NR = 5$). Substitution of the numerical values of NJ and NR into Eq. (5.1) yields

$$NDOF = 2(4) - 5 = 3$$

which is the same as the number of degrees of freedom of the beam obtained previously. As in the case of plane trusses, the free and restrained coordinates of a beam are collectively referred to simply as the structure coordinates.

When analyzing a beam, it is not necessary to draw its deformed shape, as shown in Fig. 5.1(c), to identify the degrees of freedom. Instead, all the structure coordinates (i.e., degrees of freedom and restrained coordinates) are usually directly specified on the beam's line diagram by assigning numbers to the arrows drawn at the joints in the directions of the joint displacements, as shown in Fig. 5.1(d). In this figure, a slash (/) has been added to the arrows corresponding to the restrained coordinates to distinguish them from those representing the degrees of freedom.

The procedure for assigning numbers to the structure coordinates of beams is similar to that for the case of plane trusses, discussed in detail in Section 3.2. The degrees of freedom are numbered first, starting at the lowest-numbered joint, that has a degree of freedom, and proceeding sequentially to the highest-numbered joint. If a joint has two degrees of freedom, then the translation in the Y direction is numbered first, followed by the rotation. The first degree of freedom is assigned the number 1, and the last degree of freedom is assigned a number equal to $NDOF$.

After all the degrees of freedom of the beam have been numbered, its restrained coordinates are numbered beginning with a number equal to $NDOF + 1$. Starting at the lowest-numbered joint that is attached to a support, and proceeding sequentially to the highest-numbered joint, all of the restrained coordinates of the beam are numbered. If a joint has two restrained coordinates, then the coordinate in the Y direction (corresponding to the reaction force) is numbered first, followed by the rotation coordinate (corresponding to the reaction couple). The number assigned to the last restrained coordinate of the beam is always $2(NJ)$. The structure coordinate numbers for the example beam, obtained by applying the foregoing procedure, are given in Fig. 5.1(d).

Joint Load and Reaction Vectors

Unlike plane trusses, which are subjected only to joint loads, the external loads on beams may be applied at the joints as well as on the members. The external loads (i.e., forces and couples or moments) applied at the joints of a structure are referred to as the *joint loads*, whereas the external loads acting between the ends of the members of the structure are termed the *member loads*. In this section, we focus our attention only on the joint loads, with the member loads considered in subsequent sections. As discussed in Section 3.2, an external joint load can, in general, be applied to the beam at the location and in the direction of each of its degrees of freedom. For example, the beam of Fig. 5.1(b), with three degrees of freedom, can be subjected to a maximum of three joint loads, P_1 through P_3, as shown in Fig. 5.1(e). As indicated there, a load corresponding to a degree of freedom d_i is denoted symbolically by P_i. The 3×1 joint load vector $\mathbf{P}$ for the beam is written in the form

$$\mathbf{P} = \underset{NDOF \times 1}{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}}$$

As for the support reactions, when a beam is subjected to external joint and/or member loads, a reaction (force or moment) can develop at the location and in the direction of each of its restrained coordinates. For example, the beam of Fig. 5.1(b), which has five restrained coordinates, can develop up to five reactions, as shown in Fig. 5.1(e). As indicated in this figure, the reaction corresponding to the *i*th restrained coordinate is denoted symbolically by R_i. The 5×1 reaction vector **R** for the beam is expressed as

$$\mathbf{R} = \underset{NR \times 1}{\begin{bmatrix} R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix}}$$

EXAMPLE 5.1 Identify by numbers the degrees of freedom and restrained coordinates of the continuous beam with a cantilever overhang shown in Fig. 5.2(a). Also, form the beam's joint load vector **P**.

SOLUTION The beam has four degrees of freedom, which are identified by numbers 1 through 4 in Fig. 5.2(b). The four restrained coordinates of the beam are identified by numbers 5 through 8 in the same figure. **Ans**

By comparing Figs. 5.2(a) and (b), we can see that $P_1 = -50$ k-ft; $P_2 = 0$; $P_3 = -20$ k; and $P_4 = 0$. The negative signs assigned to the magnitudes of P_1 and P_3 indicate that these loads act in the clockwise and downward directions, respectively. Thus, the joint load vector can be expressed in the units of kips and feet, as

$$\mathbf{P} = \begin{bmatrix} -50 \\ 0 \\ -20 \\ 0 \end{bmatrix}$$ **Ans**

Alternative Approach: The analysis of beams with cantilever overhangs can be considerably expedited by realizing that the cantilever portions are statically determinate (in the sense that the shear and moment at a cantilever's end can be evaluated directly by applying the equilibrium equations to the free-body of the cantilever portion). Therefore, the cantilever portions can be removed from the beam, and only the remaining indeterminate part needs to be analyzed by the stiffness method. However, the end moments and the end forces exerted by the cantilevers on the remaining indeterminate part of the structure must be included in the stiffness analysis, as illustrated in the following paragraphs.

Since the beam of Fig. 5.2(a) has a cantilever member *CD*, we separate this statically determinate member from the rest of the beam, as shown in Fig. 5.2(c). The force S_{CD} and the moment M_{CD} at end *C* of the cantilever are then calculated by applying the equilibrium equations, as follows.

$$+\uparrow \sum F_Y = 0 \qquad S_{CD} - 20 = 0 \qquad S_{CD} = 20 \text{ k} \uparrow$$

$$+\circlearrowleft \sum M_C = 0 \qquad M_{CD} - 20(10) = 0 \qquad M_{CD} = 200 \text{ k-ft} \circlearrowleft$$

Next, the moment M_{CD} is applied as a joint load, in the clockwise (opposite) direction, at joint C of the indeterminate part *AC* of the beam, as shown in Fig. 5.2(c). Note that

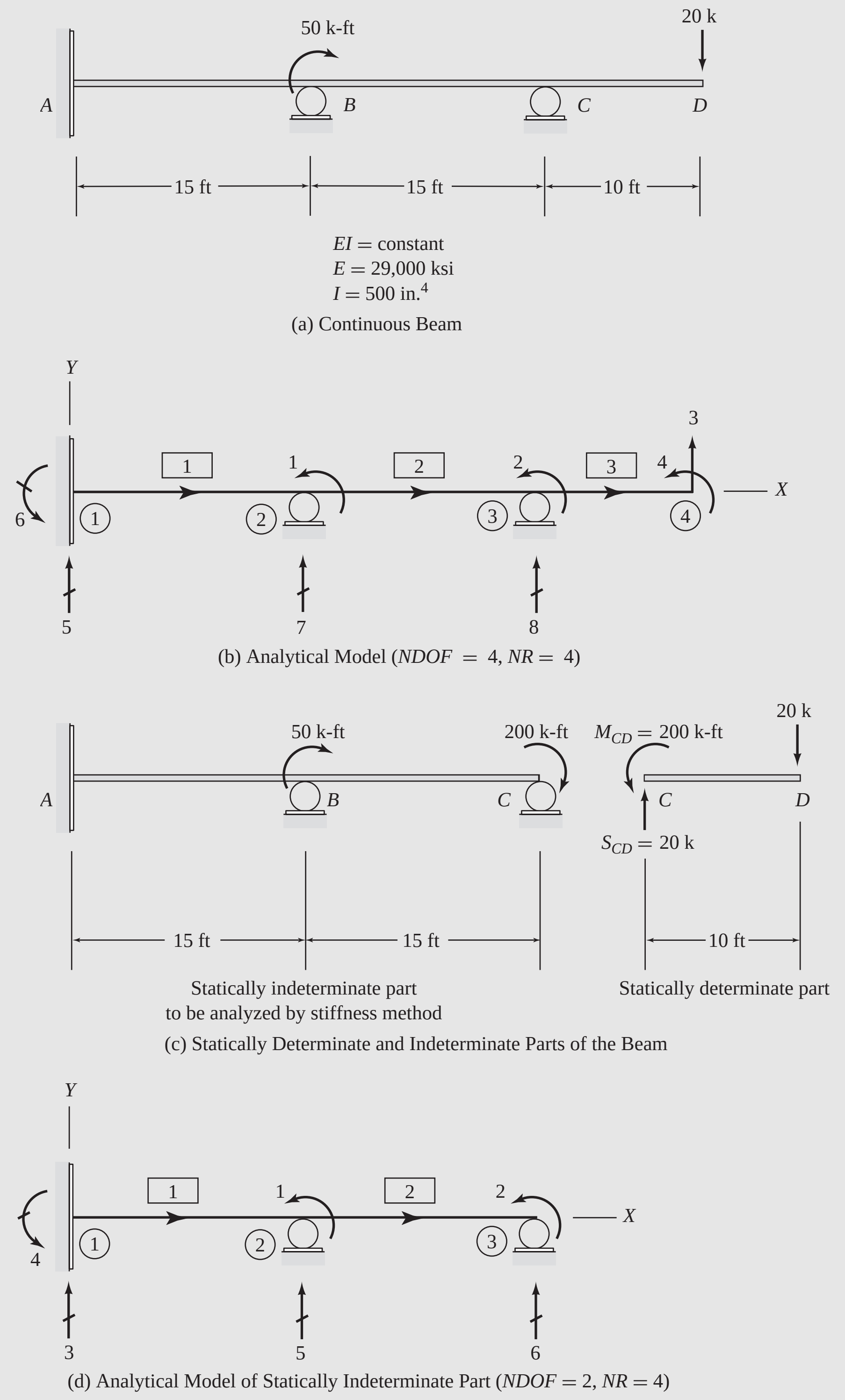

50 k-ft
20 k
A
B
C
D
15 ft
15 ft
10 ft
EI = constant
E = 29,000 ksi
I = 500 in.4
(a) Continuous Beam
Y
X
1
2
3
4
5
6
7
8
(b) Analytical Model (NDOF = 4, NR = 4)
50 k-ft
200 k-ft
MCD = 200 k-ft
20 k
A
B
C
C
D
SCD = 20 k
15 ft
15 ft
10 ft
Statically indeterminate part
to be analyzed by stiffness method
Statically determinate part
(c) Statically Determinate and Indeterminate Parts of the Beam
Y
X
1
2
3
4
5
6
(d) Analytical Model of Statically Indeterminate Part (NDOF = 2, NR = 4)

Fig. 5.2

the end force S_{CD} ($= 20$ k) need not be considered in the analysis of the indeterminate part because its only effect is to increase the reaction at support C by 20 k.

The analytical model of the indeterminate part of the beam is drawn in Fig. 5.2(d). Note that the number of degrees of freedom has now been reduced to only two, identified by numbers 1 and 2 in the figure. The number of restrained coordinates remains at four, and these coordinates are identified by numbers 3 through 6 in Fig. 5.2(d). By comparing the indeterminate part of the beam in Fig. 5.2(c) to its analytical model in Fig. 5.2(d), we obtain the joint load vector as

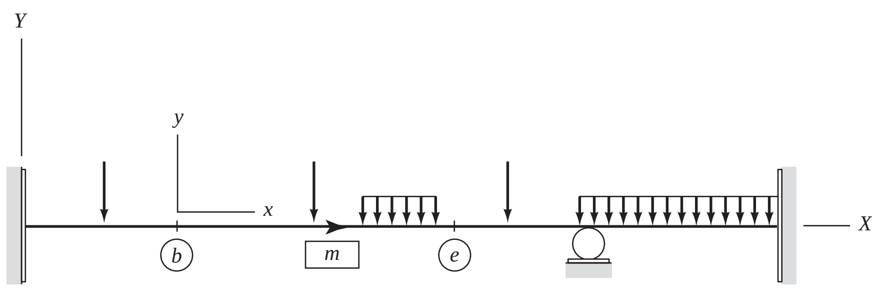
$$\mathbf{P} = \begin{bmatrix} -50 \\ -200 \end{bmatrix} \text{ k-ft} \qquad \textbf{Ans}$$

Once the analytical model of Fig. 5.2(d) has been analyzed by the stiffness method, the reaction force R_6 must be adjusted (i.e., increased by 20 k) to account for the end force S_{CD} being exerted by the cantilever CD on support C.

5.2 MEMBER STIFFNESS RELATIONS

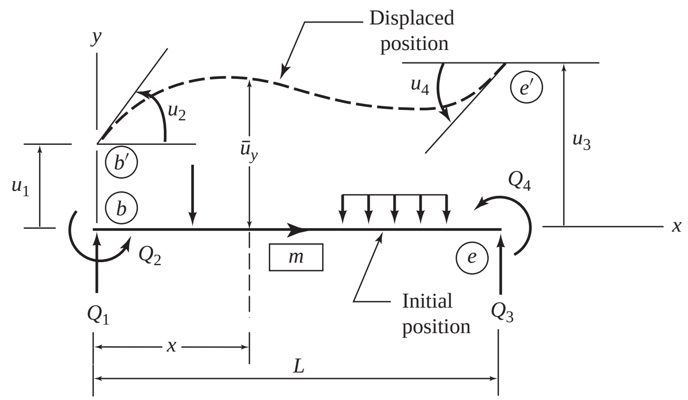
When a beam is subjected to external loads, internal moments and shears generally develop at the ends of its individual members. *The equations expressing the forces (including moments) at the end of a member as functions of the displacements (including rotations) of its ends, in terms of the external loads applied to the member, are referred to as the member stiffness relations.* Such member stiffness relations are necessary for establishing the stiffness relations for the entire beam, as discussed in Section 5.5. In this section, we derive the stiffness relations for the members of beams.

To develop the member stiffness relations, we focus our attention on an arbitrary prismatic member m of the continuous beam shown in Fig. 5.3(a). When the beam is subjected to external loads, member m deforms and internal shear forces and moments are induced at its ends. The initial and displaced positions of m are depicted in Fig. 5.3(b), in which L, E, and I denote the length, Young's modulus of elasticity, and moment of inertia, respectively, of the member. It can be seen from this figure that two displacements—translation in the y direction and rotation about the z axis—are necessary to completely specify



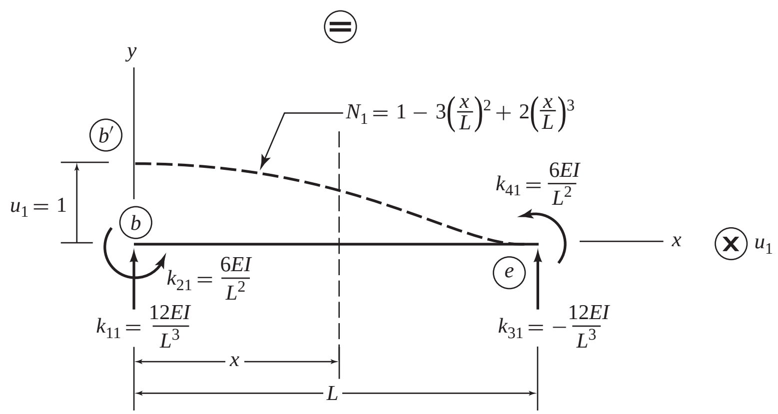
(a) Beam

Fig. 5.3

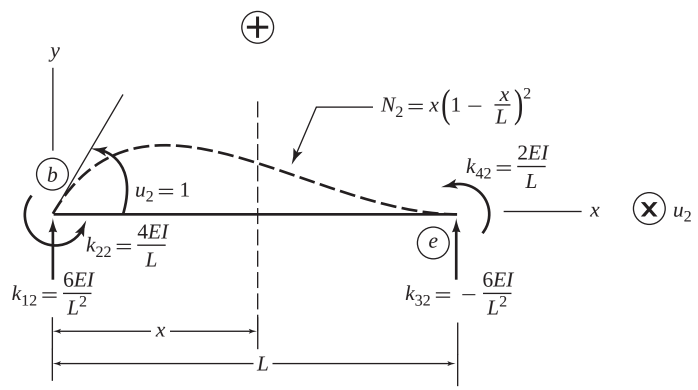


EI = constant

(b) Member Forces and Displacements in the Local Coordinate System



(c)



(d)

Fig. 5.3 (*continued*)

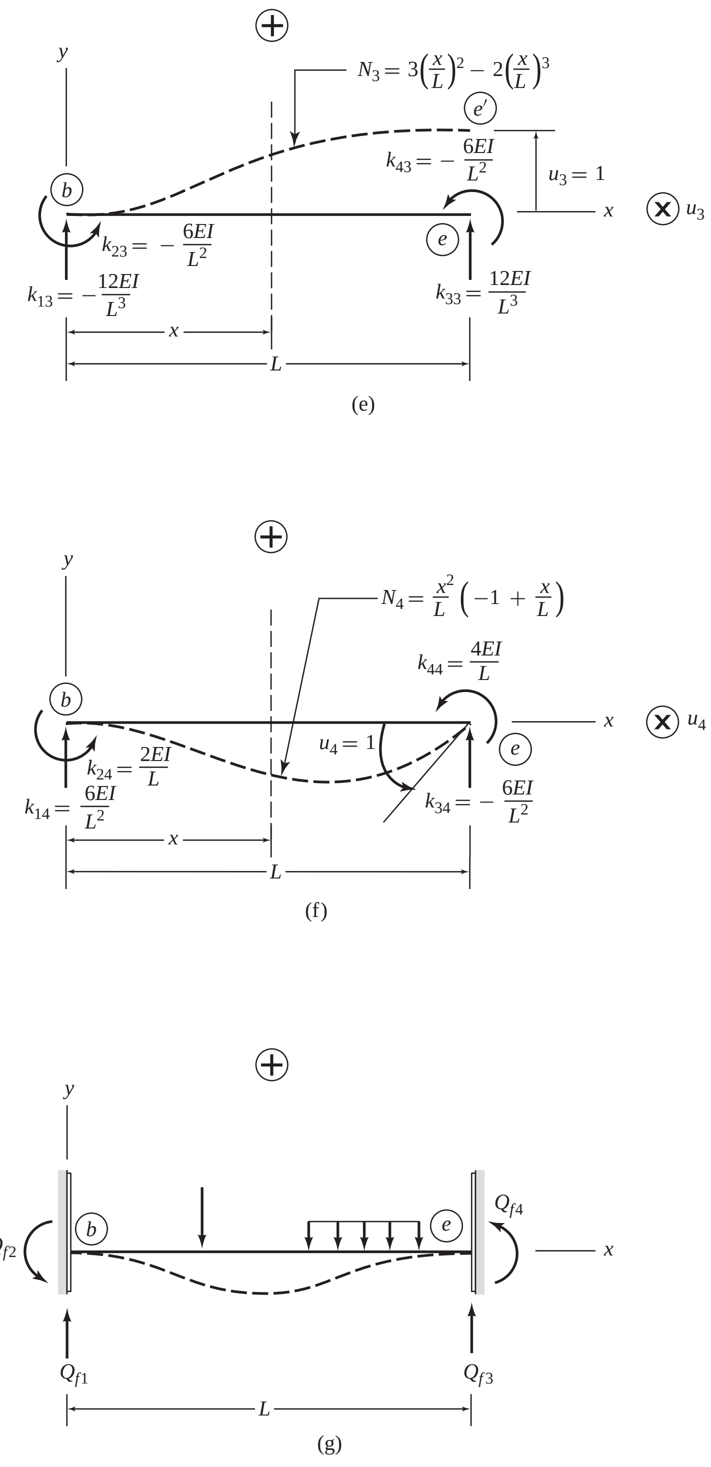


Fig. 5.3 (*continued*)

the displaced position of each end of the member. Thus, the member has a total of four end displacements or degrees of freedom. As Fig. 5.3(b) indicates, the member end displacements (including rotations) are denoted by u_1 through u_4, and the corresponding end forces (including moments) are denoted by Q_1 through Q_4. Note that the member end translations and forces are considered positive when vertically upward (i.e., in the positive direction of the local y axis), and the end rotations and moments are considered positive when counterclockwise. The numbering scheme used for identifying the member end displacements and forces is similar to that used previously for plane trusses in Chapter 3. As indicated in Fig. 5.3(b), *the member end displacements and forces are numbered by beginning at the left end b of the member, which is the origin of the local coordinate system, with the vertical translation and force numbered first, followed by the rotation and moment. The displacements and forces at the opposite end e of the member are then numbered in the same sequential order.*

The relationships between member end forces and end displacements can be conveniently established by subjecting the member, separately, to each of the four end displacements and external loads, as shown in Figs. 5.3(c) through (g); and by expressing the total member end forces as the algebraic sums of the end forces required to cause the individual end displacements and the forces caused by the external loads acting on the member with no end displacements. Thus, from Figs. 5.3(b) through (g), we can see that

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 + Q_{f1} \tag{5.2a}$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4 + Q_{f2} \tag{5.2b}$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4 + Q_{f3} \tag{5.2c}$$

$$Q_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4 + Q_{f4} \tag{5.2d}$$

in which, as defined in Chapter 3, *a stiffness coefficient k_{ij} represents the force at the location and in the direction of Q_i required, along with other end forces, to cause a unit value of displacement u_j, while all other end displacements are* 0, *and the member is not subjected to any external loading between its ends.* The last terms, Q_{fi} (with $i = 1$ to 4), on the right sides of Eqs. (5.2), represent *the forces that would develop at the member ends, due to external loads, if both ends of the member were fixed against translations and rotations* (see Fig. 5.3(g)). These forces are commonly referred to as the *member fixed-end forces* due to external loads. Equations (5.2) can be written in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} \tag{5.3}$$

or, symbolically, as

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f} \tag{5.4}$$

in which **Q** and **u** represent the member end force and member end displacement vectors, respectively, in the local coordinate system; **k** is the member stiffness matrix in the local coordinate system; and $\mathbf{Q}_f$ is called the *member fixed-end force vector in the local coordinate system*.

In the rest of this and the following section, we focus our attention on the derivation of the member stiffness matrix **k**. The fixed-end force vector $\mathbf{Q}_f$ is considered in detail in Section 5.4.

Derivation of Member Stiffness Matrix k

Various classical methods of structural analysis, such as the *method of consistent deformations* and the *slope-deflection equations*, can be used to determine the expressions for the stiffness coefficients k_{ij} in terms of member length and its flexural rigidity, *EI*. In the following, however, we derive such stiffness expressions by directly integrating the differential equation for beam deflection. This direct integration approach is not only relatively simple and straightforward, but it also yields member shape functions as a part of the solution. The shape functions are often used to establish the member mass matrices for the dynamic analysis of beams [34]; they also provide insight into the finite-element formulation of beam analysis (considered in the next section).

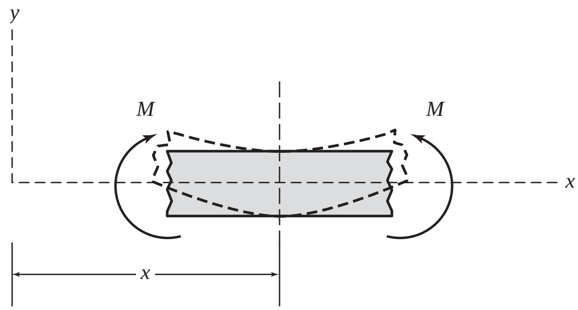
It may be recalled from a previous course on *mechanics of materials* that the differential equation for small-deflection bending of a beam, composed of linearly elastic homogenous material and loaded in a plane of symmetry of its cross-section, can be expressed as

$$\boxed{\frac{d^2\bar{u}_y}{dx^2} = \frac{M}{EI}} \qquad \textbf{(5.5)}$$

in which $\bar{u}_y$ represents the deflection of the beam's centroidal axis (which coincides with the neutral axis) in the y direction, at a distance x from the origin of the xy coordinate system as shown in Fig. 5.3(b); and M denotes the bending moment at the beam section at the same location, x. It is important to realize that the bending moment M is considered positive in accordance with the *beam sign convention*, which can be stated as follows (see Fig. 5.4).

The bending moment at a section of a beam is considered positive when the external force or couple tends to bend the beam concave upward (in the positive y direction), causing compression in the fibers above (in the positive y direction), and tension in the fibers below (in the negative y direction), the neutral axis of the beam at the section.

To obtain the expressions for the coefficients k_{i1} ($i = 1$ through 4) in the first column of the member stiffness matrix **k** (Eq. (5.3)), we subject the



Positive bending moment

Fig. 5.4 *Beam Sign Convention*

member to a unit value of the end displacement u_1 at end b, as shown in Fig. 5.3(c). Note that all other end displacements of the member are 0 (i.e., $u_2 = u_3 = u_4 = 0$), and the member is in equilibrium under the action of two end moments k_{21} and k_{41}, and two end shears k_{11} and k_{31}. To determine the equation for bending moment for the member, we pass a section at a distance x from end b, as shown in Fig. 5.3(c). Considering the free body to the left of this section, we obtain the bending moment M at the section as

$$M = -k_{21} + k_{11}x \tag{5.6}$$

Note that the bending moment due to the couple k_{21} is negative, in accordance with the *beam sign convention*, because of its tendency to bend the member concave downward, causing tension in the fibers above and compression in the fibers below the neutral axis. The bending moment $k_{11}x$ due to the end shear k_{11} is positive, however, in accordance with the *beam sign convention*.

Substitution of Eq. (5.6) into Eq. (5.5) yields

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}(-k_{21} + k_{11}x) \tag{5.7}$$

in which the flexural rigidity EI of the member is constant because the member is assumed to be prismatic. The equation for the slope θ of the member can be determined by integrating Eq. (5.7) as

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-k_{21}x + k_{11}\frac{x^2}{2}\right) + C_1 \tag{5.8}$$

in which C_1 denotes a constant of integration. By integrating Eq. (5.8), we obtain the equation for deflection as

$$\bar{u}_y = \frac{1}{EI}\left(-k_{21}\frac{x^2}{2} + k_{11}\frac{x^3}{6}\right) + C_1x + C_2 \tag{5.9}$$

in which C_2 is another constant of integration. The four unknowns in Eqs. (5.8) and (5.9)—that is, two constants of integration C_1 and C_2, and two stiffness coefficients k_{11} and k_{21}—can now be evaluated by applying the following four boundary conditions.

$$\begin{array}{lll} \text{At end } b, & x = 0, & \theta = 0 \\ & x = 0, & \bar{u}_y = 1 \\ \text{At end } e, & x = L, & \theta = 0 \\ & x = L, & \bar{u}_y = 0 \end{array}$$

By applying the first boundary condition—that is, by setting $x = 0$ and $\theta = 0$ in Eq. (5.8)—we obtain $C_1 = 0$. Next, by using the second boundary condition—that is, by setting $x = 0$ and $\bar{u}_y = 1$ in Eq. (5.9)—we obtain $C_2 = 1$. Thus, the equations for the slope and deflection of the member become

$$\theta = \frac{1}{EI}\left(-k_{21}x + k_{11}\frac{x^2}{2}\right) \tag{5.10}$$

$$\bar{u}_y = \frac{1}{EI}\left(-k_{21}\frac{x^2}{2} + k_{11}\frac{x^3}{6}\right) + 1 \tag{5.11}$$

We now apply the third boundary condition—that is, we set $x = L$ and $\theta = 0$ in Eq. (5.10)—to obtain

$$0 = \frac{1}{EI}\left(-k_{21}L + k_{11}\frac{L^2}{2}\right)$$

from which

$$k_{21} = k_{11}\frac{L}{2} \tag{5.12}$$

Next, we use the last boundary condition—that is, we set $x = L$ and $\bar{u}_y = 0$ in Eq. (5.11)—to obtain

$$0 = \frac{1}{EI}\left(-k_{21}\frac{L^2}{2} + k_{11}\frac{L^3}{6}\right) + 1$$

from which

$$k_{21} = \frac{2EI}{L^2} + k_{11}\frac{L}{3} \tag{5.13}$$

By substituting Eq. (5.12) into Eq. (5.13), we determine the expression for the stiffness coefficient k_{11}:

$$\boxed{k_{11} = \frac{12EI}{L^3}} \tag{5.14}$$

and the substitution of Eq. (5.14) into Eq. (5.12) yields

$$\boxed{k_{21} = \frac{6EI}{L^2}} \tag{5.15}$$

The remaining two stiffness coefficients, k_{31} and k_{41}, can now be determined by applying the equations of equilibrium to the free body of the member shown in Fig. 5.3(c). Thus,

$$+\uparrow \sum F_y = 0 \qquad \frac{12EI}{L^3} + k_{31} = 0$$

$$\boxed{k_{31} = -\frac{12EI}{L^3}} \tag{5.16}$$

$$+\circlearrowleft \sum M_e = 0 \qquad \frac{6EI}{L^2} - \frac{12EI}{L^3}(L) + k_{41} = 0$$

$$\boxed{k_{41} = \frac{6EI}{L^2}} \tag{5.17}$$

To determine the deflected shape of the member, we substitute the expressions for k_{11} (Eq. (5.14)) and k_{21} (Eq. (5.15)) into Eq. (5.11). This yields

$$\bar{u}_y = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \tag{5.18}$$

Since the foregoing equation describes the variation of $\bar{u}_y$ (i.e., the y displacement) along the member's length due to a unit value of the end displacement u_1, while all other end displacements are zero, it represents the member shape function N_1; that is,

$$\boxed{N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3} \tag{5.19}$$

The expressions for coefficients k_{i2} ($i = 1$ through 4) in the second column of the member stiffness matrix **k** (Eq. (5.3)) can be evaluated in a similar manner. We subject the member to a unit value of the end displacement u_2 at end b, as shown in Fig. 5.3(d). Note that all other member end displacements are 0 (i.e., $u_1 = u_3 = u_4 = 0$), and the member is in equilibrium under the action of two end moments k_{22} and k_{42}, and two end shears k_{12} and k_{32}. The equation for bending moment at a distance x from end b of the member can be written as

$$M = -k_{22} + k_{12}x \tag{5.20}$$

By substituting Eq. (5.20) into the differential equation for beam deflection (Eq. (5.5)), we obtain

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}\left(-k_{22} + k_{12}x\right) \tag{5.21}$$

By integrating Eq. (5.21) twice, we obtain the equations for the slope and deflection of the member as

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-k_{22}x + k_{12}\frac{x^2}{2}\right) + C_1 \tag{5.22}$$

$$\bar{u}_y = \frac{1}{EI}\left(-k_{22}\frac{x^2}{2} + k_{12}\frac{x^3}{6}\right) + C_1 x + C_2 \tag{5.23}$$

The four unknowns, C_1, C_2, k_{12} and k_{22}, in Eqs. (5.22) and (5.23) can now be evaluated by applying the boundary conditions, as follows.

$$\begin{array}{lll} \text{At end } b, & x = 0, & \theta = 1 \\ & x = 0, & \bar{u}_y = 0 \\ \text{At end } e, & x = L, & \theta = 0 \\ & x = L, & \bar{u}_y = 0 \end{array}$$

Application of the first boundary condition (i.e., $\theta = 1$ at $x = 0$) yields $C_1 = 1$; using the second boundary condition (i.e., $\bar{u}_y = 0$ at $x = 0$), we obtain $C_2 = 0$. By applying the third boundary condition (i.e., $\theta = 0$ at $x = L$), we obtain

$$0 = \frac{1}{EI}\left(-k_{22}L + k_{12}\frac{L^2}{2}\right) + 1$$

from which

$$k_{22} = \frac{EI}{L} + k_{12}\frac{L}{2} \tag{5.24}$$

and application of the last boundary condition (i.e., $\bar{u}_y = 0$ at $x = L$) yields

$$0 = \frac{1}{EI}\left(-k_{22}\frac{L^2}{2} + k_{12}\frac{L^3}{6}\right) + L$$

from which

$$k_{22} = \frac{2EI}{L} + k_{12}\frac{L}{3} \tag{5.25}$$

By substituting Eq. (5.24) into Eq. (5.25), we obtain the expression for the stiffness coefficient k_{12}:

$$\boxed{k_{12} = \frac{6EI}{L^2}} \tag{5.26}$$

and by substituting Eq. (5.26) into either Eq. (5.24) or Eq. (5.25), we obtain

$$\boxed{k_{22} = \frac{4EI}{L}} \tag{5.27}$$

To determine the two remaining stiffness coefficients, k_{32} and k_{42}, we apply the equilibrium equations to the free body of the member shown in Fig. 5.3(d):

$$+\uparrow \sum F_y = 0 \qquad \frac{6EI}{L^2} + k_{32} = 0$$

$$\boxed{k_{32} = -\frac{6EI}{L^2}} \tag{5.28}$$

$$+\circlearrowleft \sum M_e = 0 \qquad \frac{4EI}{L} - \frac{6EI}{L^2}(L) + k_{42} = 0$$

$$\boxed{k_{42} = \frac{2EI}{L}} \tag{5.29}$$

The shape function (i.e., deflected shape) of the member, due to a unit end displacement u_2, can now be obtained by substituting the expressions for k_{12} (Eq. (5.26)) and k_{22} (Eq. (5.27)) into Eq. (5.23), with $C_1 = 1$ and $C_2 = 0$. Thus,

$$\boxed{N_2 = x\left(1 - \frac{x}{L}\right)^2} \tag{5.30}$$

Next, we subject the member to a unit value of the end displacement u_3 at end e, as shown in Fig. 5.3(e), to determine the coefficients k_{i3} ($i = 1$ through 4) in the third column of the member stiffness matrix $\mathbf{k}$. The bending moment at a distance x from end b of the member is given by

$$M = -k_{23} + k_{13}x \tag{5.31}$$

Substitution of Eq. (5.31) into the beam deflection differential equation (Eq. (5.5)) yields

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}(-k_{23} + k_{13}x) \tag{5.32}$$

By integrating Eq. (5.32) twice, we obtain

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-k_{23}x + k_{13}\frac{x^2}{2}\right) + C_1 \tag{5.33}$$

$$\bar{u}_y = \frac{1}{EI}\left(-k_{23}\frac{x^2}{2} + k_{13}\frac{x^3}{6}\right) + C_1x + C_2 \tag{5.34}$$

The four unknowns, C_1, C_2, k_{13} and k_{23}, in Eqs. (5.33) and (5.34) are evaluated using the boundary conditions, as follows.

$$\begin{array}{lll} \text{At end } b, & x = 0, & \theta = 0 \\ & x = 0, & \bar{u}_y = 0 \\ \text{At end } e, & x = L, & \theta = 0 \\ & x = L, & \bar{u}_y = 1 \end{array}$$

Using the first two boundary conditions, we obtain $C_1 = C_2 = 0$. Application of the third boundary condition yields

$$0 = \frac{1}{EI}\left(-k_{23}L + k_{13}\frac{L^2}{2}\right)$$

from which

$$k_{23} = k_{13}\frac{L}{2} \tag{5.35}$$

and, using the last boundary condition, we obtain

$$1 = \frac{1}{EI}\left(-k_{23}\frac{L^2}{2} + k_{13}\frac{L^3}{6}\right)$$

from which

$$k_{23} = -\frac{2EI}{L^2} + k_{13}\frac{L}{3} \tag{5.36}$$

By substituting Eq. (5.35) into Eq. (5.36), we determine the stiffness coefficient k_{13} to be

$$\boxed{k_{13} = -\frac{12EI}{L^3}} \tag{5.37}$$

and the substitution of Eq. (5.37) into Eq. (5.35) yields

$$\boxed{k_{23} = -\frac{6EI}{L^2}} \tag{5.38}$$

The two remaining stiffness coefficients, k_{33} and k_{43}, are determined by considering the equilibrium of the free body of the member (Fig. 5.3(e)):

$$+\uparrow \sum F_y = 0 \qquad -\frac{12EI}{L^3} + k_{33} = 0$$

$$\boxed{k_{33} = \frac{12EI}{L^3}} \tag{5.39}$$

$$+\circlearrowleft\sum M_e = 0 \qquad -\frac{6EI}{L^2} + \frac{12EI}{L^3}(L) + k_{43} = 0$$

$$\boxed{k_{43} = -\frac{6EI}{L^2}} \tag{5.40}$$

and the shape function N_3 for the member is obtained by substituting Eqs. (5.37) and (5.38) into Eq. (5.34) with $C_1 = C_2 = 0$. Thus,

$$\boxed{N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3} \tag{5.41}$$

To determine the stiffness coefficients k_{i4} ($i = 1$ through 4) in the last (fourth) column of **k**, we subject the member to a unit value of the end displacement u_4 at end e, as shown in Fig. 5.3(f). The bending moment in the member is given by

$$M = -k_{24} + k_{14}x \tag{5.42}$$

Substitution of Eq. (5.42) into Eq. (5.5) yields

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}(-k_{24} + k_{14}x) \tag{5.43}$$

By integrating Eq. (5.43) twice, we obtain

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-k_{24}x + k_{14}\frac{x^2}{2}\right) + C_1 \tag{5.44}$$

$$\bar{u}_y = \frac{1}{EI}\left(-k_{24}\frac{x^2}{2} + k_{14}\frac{x^3}{6}\right) + C_1x + C_2 \tag{5.45}$$

To evaluate the four unknowns, C_1, C_2, k_{14} and k_{24}, in Eqs. (5.44) and (5.45), we use the boundary conditions, as follows.

$$\begin{array}{lll} \text{At end } b, & x = 0, & \theta = 0 \\ & x = 0, & \bar{u}_y = 0 \\ \text{At end } e, & x = L, & \theta = 1 \\ & x = L, & \bar{u}_y = 0 \end{array}$$

Application of the first two boundary conditions yields $C_1 = C_2 = 0$. Using the third boundary condition, we obtain

$$1 = \frac{1}{EI}\left(-k_{24}L + k_{14}\frac{L^2}{2}\right)$$

or

$$k_{24} = -\frac{EI}{L} + k_{14}\frac{L}{2} \tag{5.46}$$

and the use of the fourth boundary condition yields

$$0 = \frac{1}{EI}\left(-k_{24}\frac{L^2}{2} + k_{14}\frac{L^3}{6}\right)$$

from which

$$k_{24} = k_{14}\frac{L}{3} \qquad \textbf{(5.47)}$$

By substituting Eq. (5.47) into Eq. (5.46), we obtain the stiffness coefficient k_{14}:

$$\boxed{k_{14} = \frac{6EI}{L^2}} \qquad \textbf{(5.48)}$$

and by substituting Eq. (5.48) into Eq. (5.47), we obtain

$$\boxed{k_{24} = \frac{2EI}{L}} \qquad \textbf{(5.49)}$$

Next, we determine the remaining stiffness coefficients by considering the equilibrium of the free body of the member (Fig. 5.3(f)):

$$+\uparrow \sum F_y = 0 \qquad \frac{6EI}{L^2} + k_{34} = 0$$

$$\boxed{k_{34} = -\frac{6EI}{L^2}} \qquad \textbf{(5.50)}$$

$$+\circlearrowleft \sum M_e = 0 \qquad \frac{2EI}{L} - \frac{6EI}{L^2}(L) + k_{44} = 0$$

$$\boxed{k_{44} = \frac{4EI}{L}} \qquad \textbf{(5.51)}$$

To obtain the shape function N_4 of the beam, we substitute Eqs. (5.48) and (5.49) into Eq. (5.45), yielding

$$\boxed{N_4 = \frac{x^2}{L}\left(-1 + \frac{x}{L}\right)} \qquad \textbf{(5.52)}$$

Finally, by substituting the expressions for the stiffness coefficients (Eqs. (5.14–5.17), (5.26–5.29), (5.37–5.40), and (5.48–5.51)), into the matrix form of **k** given in Eq. (5.3), we obtain the following local stiffness matrix for the members of beams.

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{5.53}$$

Note that the stiffness matrix **k** is symmetric; that is, $k_{ij} = k_{ji}$.

EXAMPLE 5.2 Determine the stiffness matrices for the members of the beam shown in Fig. 5.5.

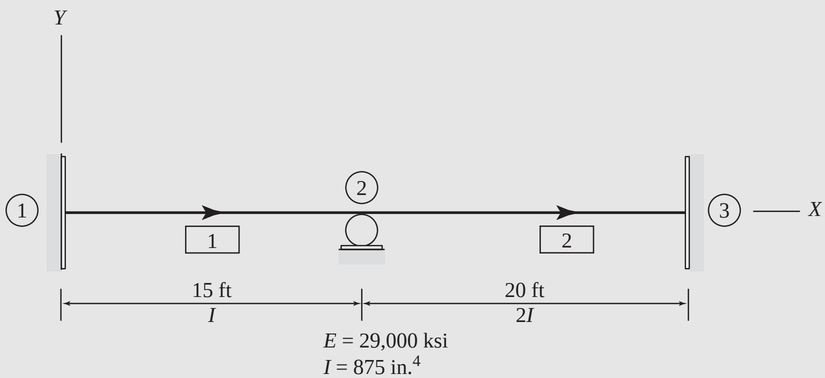


Fig. 5.5

SOLUTION **Member 1** $E = 29{,}000$ ksi, $I = 875$ in.^{4}, $L = 15$ ft $= 180$ in.

$$\frac{EI}{L^3} = \frac{29{,}000\,(875)}{(180)^3} = 4.351 \text{ k/in.}$$

Substitution in Eq. (5.53) yields

$$\mathbf{k}_1 = \begin{bmatrix} 52.212 & 4{,}699.1 & -52.212 & 4{,}699.1 \\ 4{,}699.1 & 563{,}889 & -4{,}699.1 & 281{,}944 \\ -52.212 & -4{,}699.1 & 52.212 & -4{,}699.1 \\ 4{,}699.1 & 281{,}944 & -4{,}699.1 & 563{,}889 \end{bmatrix}$$ **Ans**

Member 2 $E = 29{,}000$ ksi, $I = 1{,}750$ in.^{4}, $L = 20$ ft $= 240$ in.

$$\frac{EI}{L^3} = \frac{29{,}000\,(1{,}750)}{(240)^3} = 3.6712 \text{ k/in.}$$

Thus, from Eq. (5.53)

$$\mathbf{k}_2 = \begin{bmatrix} 44.054 & 5{,}286.5 & -44.054 & 5{,}286.5 \\ 5{,}286.5 & 845{,}833 & -5{,}286.5 & 422{,}917 \\ -44.054 & -5{,}286.5 & 44.054 & -5{,}286.5 \\ 5{,}286.5 & 422{,}917 & -5{,}286.5 & 845{,}833 \end{bmatrix}$$ **Ans**

5.3 FINITE-ELEMENT FORMULATION USING VIRTUAL WORK*

The member stiffness matrix **k**, as given by Eq. (5.53), is usually derived in the finite-element method by applying the principle of virtual work. The formulation involves essentially the same general steps that were outlined in Section 3.4 for the case of the members of plane trusses.

Displacement Function

Consider a prismatic member of a beam, subjected to end displacements u_1 through u_4, as shown in Fig. 5.6. Since the member displaces only in the y direction, only one displacement function $\bar{u}_y$ needs to be defined. In Fig. 5.6, the displacement function $\bar{u}_y$ is depicted as the displacement of an arbitrary point G located on the member's centroidal axis (which coincides with the neutral axis) at a distance x from the end b.

As discussed in Section 3.4, in the finite-element method, a displacement function is usually *assumed* in the form of a complete polynomial of such a degree that all of its coefficients can be evaluated from the available boundary conditions of the member. From Fig. 5.6, we realize that the boundary conditions for the member under consideration are as follows.

$$\text{At end } b, \qquad x = 0, \qquad \bar{u}_y = u_1 \tag{5.54a}$$

$$x = 0, \qquad \theta = \frac{d\bar{u}_y}{dx} = u_2 \tag{5.54b}$$

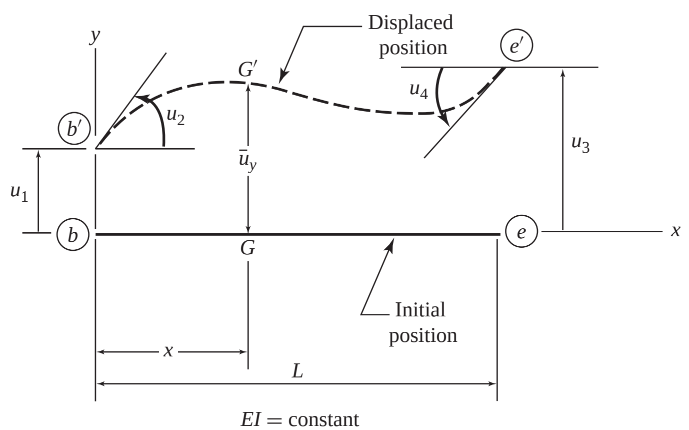


Fig. 5.6

*This section can be omitted without loss of continuity.

$$\text{At end } e, \qquad x = L, \qquad \bar{u}_y = u_3 \tag{5.54c}$$

$$x = L, \qquad \theta = \frac{d\bar{u}_y}{dx} = u_4 \tag{5.54d}$$

Since there are four boundary conditions, we can use a cubic polynomial (with four coefficients) for the displacement function $\bar{u}_y$, as

$$\bar{u}_y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{5.55}$$

in which a_0 through a_3 are the constants to be determined by applying the four boundary conditions specified in Eqs. (5.54). By differentiating Eq. (5.55) with respect to x, we obtain the equation for the slope of the member as

$$\theta = \frac{d\bar{u}_y}{dx} = a_1 + 2a_2 x + 3a_3 x^2 \tag{5.56}$$

Now, we apply the first boundary condition (Eq. (5.54a)) by setting $x = 0$ and $\bar{u}_y = u_1$ in Eq. (5.55). This yields

$$a_0 = u_1 \tag{5.57}$$

Similarly, using the second boundary condition—that is, by setting $x = 0$ and $\theta = u_2$ in Eq. (5.56)—we obtain

$$a_1 = u_2 \tag{5.58}$$

Next, we apply the third boundary condition, setting $x = L$ and $\bar{u}_y = u_3$ in Eq. (5.55). This yields

$$u_3 = a_0 + a_1 L + a_2 L^2 + a_3 L^3 \tag{5.59}$$

By substituting $a_0 = u_1$ (Eq. (5.57)) and $a_1 = u_2$ (Eq. (5.58)) into Eq. (5.59), we obtain

$$a_3 = \frac{1}{L^3}\left(-u_1 - u_2 L + u_3 - a_2 L^2\right) \tag{5.60}$$

To apply the fourth boundary condition (Eq. (5.54d)), we set $x = L$ and $\theta = u_4$ in Eq. (5.56). This yields

$$u_4 = a_1 + 2a_2 L + 3a_3 L^2 \tag{5.61}$$

By substituting Eqs. (5.57), (5.58), and (5.60) into Eq. (5.61), and solving the resulting equation for a_2, we obtain

$$a_2 = \frac{1}{L^2}\left(-3u_1 - 2u_2 L + 3u_3 - u_4 L\right) \tag{5.62}$$

and the backsubstitution of Eq. (5.62) into Eq. (5.60) yields

$$a_3 = \frac{1}{L^3}\left(2u_1 + u_2 L - 2u_3 + u_4 L\right) \tag{5.63}$$

Finally, by substituting Eqs. (5.57), (5.58), (5.62), and (5.63) into Eq. (5.55), we obtain the following expression for the displacement function $\bar{u}_y$, in terms of the end displacements u_1 through u_4.

$$\boxed{\begin{aligned}\bar{u}_y = &\left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right] u_1 + \left[x\left(1 - \frac{x}{L}\right)^2\right] u_2 \\ &+ \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] u_3 + \left[\frac{x^2}{L}\left(-1 + \frac{x}{L}\right)\right] u_4\end{aligned}} \tag{5.64}$$

Shape Functions

The displacement function $\bar{u}_y$, as given by Eq. (5.64), can alternatively be written as

$$\bar{u}_y = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 \tag{5.65}$$

with

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \tag{5.66a}$$

$$N_2 = x\left(1 - \frac{x}{L}\right)^2 \tag{5.66b}$$

$$N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \tag{5.66c}$$

$$N_4 = \frac{x^2}{L}\left(-1 + \frac{x}{L}\right) \tag{5.66d}$$

in which N_i ($i = 1$ through 4) are the member shape functions. A comparison of Eqs. (5.66a) through (5.66d) with Eqs. (5.19), (5.30), (5.41), and (5.52), respectively, indicates that the shape functions determined herein by assuming a cubic displacement function are identical to those obtained in Section 5.2 by exactly solving the differential equation for bending of beams. This is because a cubic polynomial represents the actual (or exact) solution of the governing differential equation (Eq. (5.5)), provided that the member is prismatic and it is not subjected to any external loading.

Equation (5.65) can be written in matrix form as

$$\bar{u}_y = [\, N_1 \quad N_2 \quad N_3 \quad N_4 \,] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{5.67}$$

or, symbolically, as

$$\boxed{\bar{u}_y = \mathbf{N}\mathbf{u}} \tag{5.68}$$

in which $\mathbf{N}$ is the member shape-function matrix.

Strain–Displacement Relationship

We recall from *mechanics of materials* that the normal (longitudinal) strain ε in a fiber of a member, located at a distance y above the neutral axis, can be expressed in terms of the displacement $\bar{u}_y$ of the member's neutral axis, by the relationship

$$\varepsilon = -y \frac{d^2 \bar{u}_y}{dx^2} \tag{5.69}$$

in which the minus sign indicates that the tensile strain is considered positive. By substituting Eq. (5.68) into Eq. (5.69), we write

$$\varepsilon = -y \frac{d^2}{dx^2} (\mathbf{Nu}) \tag{5.70}$$

Since the end-displacement vector **u** is not a function of x, it can be treated as a constant for the purpose of differentiation. Thus, Eq. (5.70) can be expressed as

$$\boxed{\varepsilon = \left(-y \frac{d^2 \mathbf{N}}{dx^2}\right) \mathbf{u} = \mathbf{Bu}} \tag{5.71}$$

To determine the member strain-displacement matrix **B**, we write

$$\mathbf{B} = -y \frac{d^2 \mathbf{N}}{dx^2} = -y \left[\frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right] \tag{5.72}$$

By differentiating twice the equations for the shape functions as given by Eqs. (5.66), and substituting the resulting expressions into Eq. (5.72), we obtain

$$\mathbf{B} = -\frac{y}{L^2} \left[6\left(-1 + 2\frac{x}{L}\right) \quad 2L\left(-2 + 3\frac{x}{L}\right) \quad 6\left(1 - 2\frac{x}{L}\right) \quad 2L\left(-1 + 3\frac{x}{L}\right) \right] \tag{5.73}$$

Stress–Displacement Relationship

To establish the relationship between the member normal stress and the end displacements, we substitute Eq. (5.71) into the stress–strain relation $\sigma = E\varepsilon$. This yields

$$\boxed{\sigma = E\mathbf{Bu}} \tag{5.74}$$

Member Stiffness Matrix, k

With both the member strain and stress expressed in terms of end displacements, we can now establish the member stiffness matrix **k** by applying the principle of virtual work for deformable bodies. Consider an arbitrary member of a beam in equilibrium under the action of end forces Q_1 through Q_4, as shown in Fig. 5.7. Note that the member is not subjected to any external loading between its ends; therefore, the fixed-end forces $\mathbf{Q}_f$ are 0.

Now, assume that the member is given small virtual end displacements δu_1 through δu_4, as shown in Fig. 5.7. The virtual external work done by the real member end forces Q_1 through Q_4 as they move through the corresponding virtual end displacements δu_1 through δu_4 is

$$\delta W_e = Q_1 \delta u_1 + Q_2 \delta u_2 + Q_3 \delta u_3 + Q_4 \delta u_4$$

which can be written in matrix form as

$$\delta W_e = \delta \mathbf{u}^T \mathbf{Q} \tag{5.75}$$

Substitution of Eq. (5.75) into the expression for the principle of virtual work for deformable bodies as given in Eq. (3.28) in Section 3.4, yields

$$\delta \mathbf{u}^T \mathbf{Q} = \int_V \delta \varepsilon^T \sigma \, dV \tag{5.76}$$

in which the right-hand side represents the virtual strain energy stored in the member. By substituting Eqs. (5.71) and (5.74) into Eq. (5.76), we obtain

$$\delta \mathbf{u}^T \mathbf{Q} = \int_V (\mathbf{B} \, \delta \mathbf{u})^T E \mathbf{B} \, dV \; \mathbf{u}$$

Since $(\mathbf{B} \, \delta \mathbf{u})^T = \delta \mathbf{u}^T \mathbf{B}^T$, the foregoing equation becomes

$$\delta \mathbf{u}^T \mathbf{Q} = \delta \mathbf{u}^T \int_V \mathbf{B}^T E \mathbf{B} \, dV \; \mathbf{u}$$

or

$$\delta \mathbf{u}^T \left(\mathbf{Q} - \int_V \mathbf{B}^T E \mathbf{B} \, dV \; \mathbf{u} \right) = 0$$

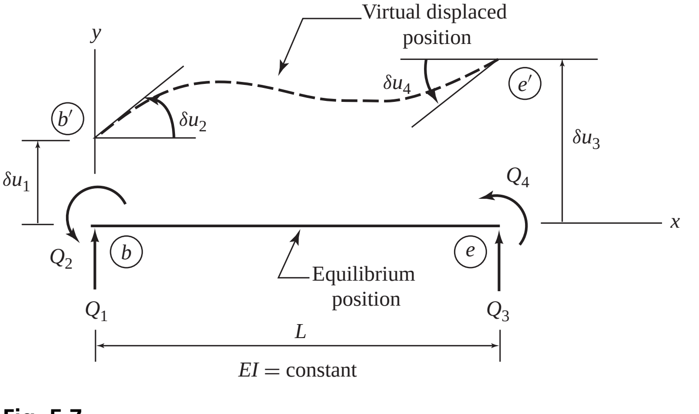


Fig. 5.7

As $\delta \mathbf{u}^T$ may be arbitrarily chosen and is not 0, the quantity in the parentheses must be 0; thus,

$$\mathbf{Q} = \left(\int_V \mathbf{B}^T E \mathbf{B} \, dV \right) \mathbf{u} = \mathbf{k}\mathbf{u} \tag{5.77}$$

with

$$\boxed{\mathbf{k} = \int_V \mathbf{B}^T E \mathbf{B} \, dV} \tag{5.78}$$

Note that the foregoing general form of **k** for beam members is the same as that obtained in Section 3.4 for the members of plane trusses (Eq. (3.55)). To explicitly determine the member stiffness matrix **k**, we substitute Eq. (5.73) for **B** into Eq. (5.78). This yields

$$\mathbf{k} = \frac{E}{L^4} \int_V y^2 \begin{bmatrix} 6\left(-1+2\dfrac{x}{L}\right) \\ 2L\left(-2+3\dfrac{x}{L}\right) \\ 6\left(1-2\dfrac{x}{L}\right) \\ 2L\left(-1+3\dfrac{x}{L}\right) \end{bmatrix} \left[6\left(-1+2\frac{x}{L}\right) \quad 2L\left(-2+3\frac{x}{L}\right) \quad 6\left(1-2\frac{x}{L}\right) \quad 2L\left(-1+3\frac{x}{L}\right) \right] dV \tag{5.79}$$

By substituting $dV = (dA)\,dx$ into Eq. (5.79), and realizing that $\int_A y^2 \, dA = I$, we obtain

$$\mathbf{k} = \frac{EI}{L^4} \int_0^L$$
$$\times \begin{bmatrix} 36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & -36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-1+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) \\ 12L\left(-2+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)\left(-1+3\frac{x}{L}\right) \\ -36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-1+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) \\ 12L\left(-1+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)\left(-1+3\frac{x}{L}\right) & 12L\left(-1+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 4L^2\left(-1+3\frac{x}{L}\right)^2 \end{bmatrix} dx \tag{5.80}$$

which, upon integration, becomes

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Note that the foregoing expression for **k** is identical to that derived in Section 5.2 (Eq. (5.53)) by directly integrating the differential equation for beam deflection and applying the equilibrium equations.

5.4 MEMBER FIXED-END FORCES DUE TO LOADS

It was shown in Section 5.2 that the stiffness relationships for a member of a beam can be written in matrix form (see Eq. (5.4)) as

$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

As the foregoing relationship indicates, the total forces **Q** that can develop at the ends of a member can be expressed as the sum of the forces **ku** due to the end displacements **u**, and the fixed-end forces $\mathbf{Q}_f$ that would develop at the member ends due to external loads if both member ends were fixed against translations and rotations.

In this section, we consider the derivation of the expressions for fixed-end forces due to external loads applied to the members of beams. To illustrate the procedure, consider a fixed member subjected to a concentrated load W, as shown in Fig. 5.8(a). As indicated in this figure, the fixed-end moments at the member ends b and e are denoted by FM_b and FM_e, respectively, whereas FS_b and FS_e denote the fixed-end shears at member ends b and e, respectively. Our objective is to determine expressions for the fixed-end moments and shears in terms of the magnitude and location of the load W; we will use the direct integration approach, along with the equations of equilibrium, for this purpose.

As the concentrated load W acts at point A of the member (Fig. 5.8(a)), the bending moment M cannot be expressed as a single continuous function of x over the entire length of the member. Therefore, we divide the member into two segments, bA and Ae; and we determine the following equations for bending moment in segments bA and Ae, respectively:

$$0 \le x \le l_1 \qquad M = -FM_b + FS_b x \tag{5.81}$$

$$l_1 \le x \le L \qquad M = -FM_b + FS_b x - W(x - l_1) \tag{5.82}$$

By substituting Eqs. (5.81) and (5.82) into the differential equation for beam deflection (Eq. (5.5)), we obtain, respectively,

$$0 \le x \le l_1 \qquad \frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}(-FM_b + FS_b x) \tag{5.83}$$

$$l_1 \le x \le L \qquad \frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}[-FM_b + FS_b x - W(x - l_1)] \tag{5.84}$$

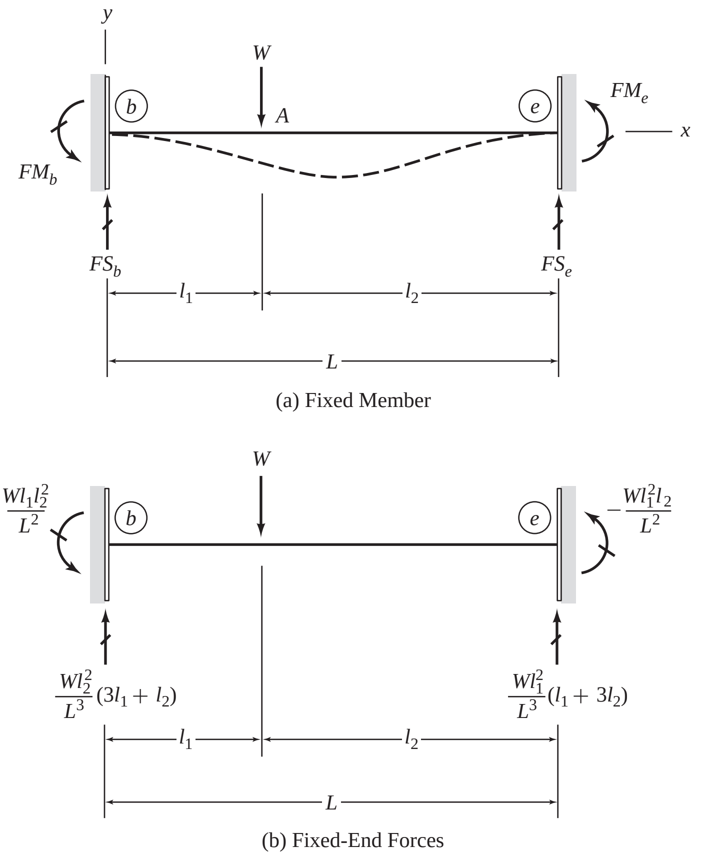


Fig. 5.8

By integrating Eq. (5.83) twice, we obtain the equations for the slope and deflection in segment bA of the member:

$$0 \leq x \leq l_1 \qquad \theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-FM_b x + FS_b \frac{x^2}{2}\right) + C_1 \tag{5.85}$$

$$0 \leq x \leq l_1 \qquad \bar{u}_y = \frac{1}{EI}\left(-FM_b \frac{x^2}{2} + FS_b \frac{x^3}{6}\right) + C_1 x + C_2 \tag{5.86}$$

Similarly, by integrating Eq. (5.84) twice, we obtain the equations for the slope and deflection in the segment Ae:

$$l_1 \leq x \leq L \qquad \theta = \frac{1}{EI}\left[-FM_b x + FS_b \frac{x^2}{2} - \frac{Wx}{2}(x - 2l_1)\right] + C_3 \tag{5.87}$$

$$l_1 \leq x \leq L \qquad \bar{u}_y = \frac{1}{EI}\left[-FM_b \frac{x^2}{2} + FS_b \frac{x^3}{6} - \frac{Wx^2}{6}(x - 3l_1)\right] + C_3 x + C_4 \tag{5.88}$$

Equations (5.85) through (5.88) indicate that the equations for the slope and deflection of the member contain a total of six unknowns; that is, four constants of integration, C_1 through C_4, and two fixed-end forces, FM_b and FS_b. These six unknowns can be evaluated by applying four boundary conditions (i.e., the slopes and deflections at the two fixed ends, b and e, must be 0), and two continuity conditions requiring that the slope and the deflection of the member's elastic curve be continuous at point A. By applying the two boundary conditions for the fixed end b (i.e., at $x = 0$, $\theta = \bar{u}_y = 0$) to Eqs. (5.85) and (5.86), we obtain

$$C_1 = C_2 = 0 \tag{5.89}$$

Next, to evaluate the constant C_3, we use the condition that the slope must be continuous at point A. This condition requires that the two slope equations (Eqs. (5.85) and (5.87)) yield the same slope θ_A at $x = l_1$. By setting $x = l_1$ in Eqs. (5.85) and (5.87), and equating the resulting expressions, we obtain

$$\frac{1}{EI}\left(-FM_b l_1 + FS_b \frac{l_1^2}{2}\right) = \frac{1}{EI}\left(-FM_b l_1 + FS_b \frac{l_1^2}{2} + \frac{Wl_1^2}{2}\right) + C_3$$

By solving for C_3, we determine that

$$C_3 = -\frac{Wl_1^2}{2EI} \tag{5.90}$$

In a similar manner, we evaluate the constant C_4 by applying the condition of continuity of deflection at point A. By setting $x = l_1$ in the two deflection equations (Eqs. (5.86) and (5.88)), and equating the resulting expressions, we obtain

$$\frac{1}{EI}\left(-FM_b \frac{l_1^2}{2} + FS_b \frac{l_1^3}{6}\right) = \frac{1}{EI}\left(-FM_b \frac{l_1^2}{2} + FS_b \frac{l_1^3}{6} + \frac{Wl_1^3}{3}\right) - \frac{Wl_1^3}{2EI} + C_4$$

from which

$$C_4 = \frac{Wl_1^3}{6EI} \tag{5.91}$$

With the four constants of integration known, we can now evaluate the two remaining unknowns, FS_b and FM_b, by applying the boundary conditions that the slope and deflection at the fixed end e must be 0 (i.e., at $x = L$, $\theta = \bar{u}_y = 0$). By setting $x = L$ in Eqs. (5.87) and (5.88), with $\theta = 0$ in Eq. (5.87) and $\bar{u}_y = 0$ in Eq. (5.88), we obtain

$$\frac{1}{EI}\left[-FM_b L + FS_b \frac{L^2}{2} - \frac{WL}{2}(L - 2l_1)\right] - \frac{Wl_1^2}{2EI} = 0 \tag{5.92}$$

$$\frac{1}{EI}\left[-FM_b \frac{L^2}{2} + FS_b \frac{L^3}{6} - \frac{WL^2}{6}(L - 3l_1)\right] - \frac{Wl_1^2 L}{2EI} + \frac{Wl_1^3}{6EI} = 0 \tag{5.93}$$

To solve Eqs. (5.92) and (5.93) for FS_b and FM_b, we rewrite Eq. (5.92) to express FM_b in terms of FS_b as

$$FM_b = FS_b \frac{L}{2} - \frac{W}{2}(L - 2l_1) - \frac{Wl_1^2}{2L} \tag{5.94}$$

By substituting Eq. (5.94) into Eq. (5.93), and solving the resulting equation for FS_b, we obtain

$$FS_b = \frac{W}{L^3}\left(L^3 - 3l_1^2 L + 2l_1^3\right)$$

Substitution of $L = l_1 + l_2$ into the numerator of the foregoing equation yields the expression for the fixed-end shear FS_b as

$$FS_b = \frac{Wl_2^2}{L^3}(3l_1 + l_2) \tag{5.95}$$

By back substituting Eq. (5.95) into Eq. (5.94), we obtain the expression for the fixed-end moment as

$$FM_b = \frac{Wl_1 l_2^2}{L^2} \tag{5.96}$$

Finally, the fixed-end forces, FS_e and FM_e, at the member end e, can be determined by applying the equations of equilibrium to the free body of the member (Fig. 5.8(a)). Thus,

$$+\uparrow \sum F_y = 0 \qquad \frac{Wl_2^2}{L^3}(3l_1 + l_2) - W + FS_e = 0$$

By substituting $L = l_1 + l_2$ into the numerator and solving for FS_e, we obtain

$$FS_e = \frac{Wl_1^2}{L^3}(l_1 + 3l_2) \tag{5.97}$$

and

$$+ \circlearrowleft \sum M_e = 0 \qquad \frac{Wl_1 l_2^2}{L^2} - \frac{Wl_2^2}{L^3}(3l_1 + l_2)L + Wl_2 + FM_e = 0$$

$$FM_e = -\frac{Wl_1^2 l_2}{L^2} \tag{5.98}$$

in which the negative answer for FM_e indicates that its actual sense is clockwise for the loading condition under consideration. Figure 5.8(b) depicts the four fixed-end forces that develop in a member of a beam subjected to a single concentrated load.

The expressions for fixed-end forces due to other types of loading conditions can be derived by using the direct integration approach as illustrated here, or by employing another classical method, such as the *method of consistent deformations*. The expressions for fixed-end forces due to some common types of member loads are given inside the front cover of this book for convenient reference.

Member Fixed-End Force Vector $\mathbf{Q}_f$

Once the fixed-end forces for a member have been evaluated, its fixed-end force vector $\mathbf{Q}_f$ can be generated by storing the fixed-end forces in their proper positions in a 4×1 vector. In accordance with the scheme for numbering member end forces adopted in Section 5.2, the fixed-end shear FS_b and the fixed-end moment FM_b, at the left end b of the member, must be stored in the first and second rows, respectively, of the $\mathbf{Q}_f$ vector; the fixed-end shear FS_e and the fixed-end moment FM_e, at the opposite member end e, are stored in the third and fourth rows, respectively, of the $\mathbf{Q}_f$ vector. Thus, the fixed-end force vector for a member of a beam (Fig 5.8(a)) is expressed as

$$\mathbf{Q}_f = \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} = \begin{bmatrix} FS_b \\ FM_b \\ FS_e \\ FM_e \end{bmatrix} \tag{5.99}$$

When storing numerical values or fixed-end force expressions in $\mathbf{Q}_f$, the appropriate sign convention for member end forces must be followed. In accordance with the sign convention adopted in Section 5.2, the fixed-end shears are considered positive when upward (i.e., in the positive direction of the local y axis); the fixed-end moments are considered positive when counterclockwise. For example, the fixed-end force vector for the beam member shown in Fig. 5.8(b) is given by

$$\mathbf{Q}_f = \begin{bmatrix} \dfrac{Wl_2^2}{L^3}(3l_1 + l_2) \\ \dfrac{Wl_1 l_2^2}{L^2} \\ \dfrac{Wl_1^2}{L^3}(l_1 + 3l_2) \\ -\dfrac{Wl_1^2 l_2}{L^2} \end{bmatrix}$$

EXAMPLE 5.3 Determine the fixed-end force vectors for the members of the two-span continuous beam shown in Fig. 5.9. Use the fixed-end force equations given inside the front cover.

SOLUTION **Member 1** By substituting $w = 2$ k/ft, $L = 30$ ft, and $l_1 = l_2 = 0$ into the fixed-end force expressions given for loading type 3, we obtain

$$FS_b = FS_e = \frac{2(30)}{2} = 30 \text{ k}$$

$$FM_b = \frac{2(30)^2}{12} = 150 \text{ k-ft}$$

$$FM_e = -\frac{2(30)^2}{12} = -150 \text{ k-ft}$$

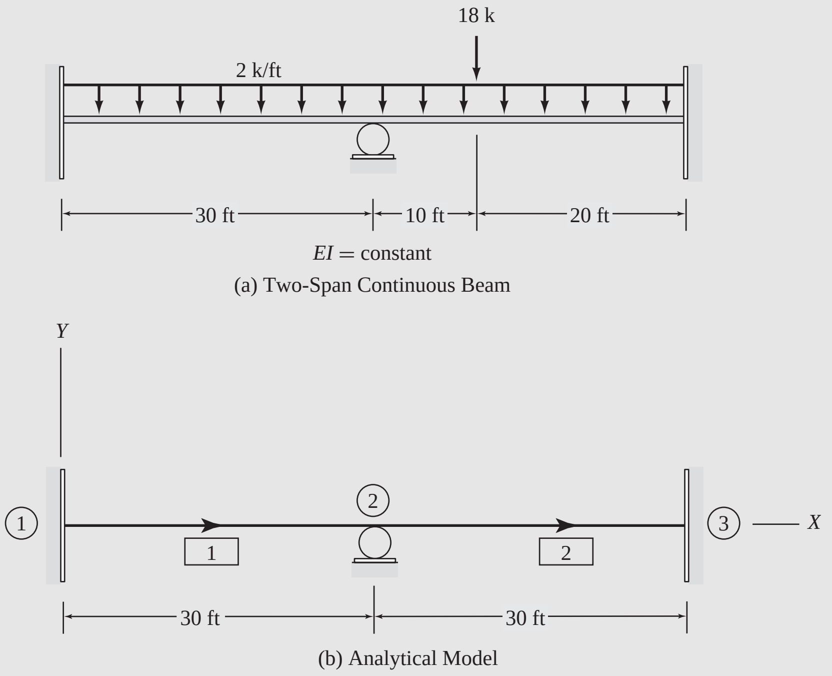


Fig. 5.9

By substituting these values of fixed-end forces into Eq. (5.99), we obtain the fixed-end force vector for member 1:

$$\mathbf{Q}_{f1} = \begin{bmatrix} 30 \\ 150 \\ 30 \\ -150 \end{bmatrix} \qquad \textbf{Ans}$$

Member 2 From Fig. 5.9(a), we can see that this member is subjected to two different loadings—a concentrated load $W = 18$ k with $l_1 = 10$ ft, $l_2 = 20$ ft, and $L = 30$ ft (load type 1), and a uniformly distributed load $w = 2$ k/ft with $l_1 = l_2 = 0$ and $L = 30$ ft (load type 3). The fixed-end forces for such a member, due to the combined effect of several loads, can be conveniently determined by superimposing (algebraically adding) the fixed-end forces due to each of the loads acting individually on the member. By using superposition, we determine the fixed-end forces for member 2 to be

$$FS_b = \frac{18(20)^2}{(30)^3}[3(10) + (20)] + \frac{2\,(30)}{2} = 43.333 \text{ k}$$

$$FM_b = \frac{18\,(10)\,(20)^2}{(30)^2} + \frac{2\,(30)^2}{12} = 230 \text{ k-ft}$$

$$FS_e = \frac{18(10)^2}{(30)^3}[10 + 3(20)] + \frac{2\,(30)}{2} = 34.667 \text{ k}$$

$$FM_e = -\frac{18\,(10)^2\,(20)}{(30)^2} - \frac{2\,(30)^2}{12} = -190 \text{ k-ft}$$

Thus, the fixed-end force vector for member 2 is

$$\mathbf{Q}_{f2} = \begin{bmatrix} 43.333 \\ 230 \\ 34.667 \\ -190 \end{bmatrix}$$ **Ans**

5.5 STRUCTURE STIFFNESS RELATIONS

The procedure for establishing the structure stiffness relations for beams is essentially the same as that for plane trusses discussed in Section 3.7. The procedure, called the *direct stiffness method*, involves: (a) expressing the joint loads **P** in terms of the member end forces **Q** by applying the joint equilibrium equations; (b) relating the joint displacements **d** to the member end displacements **u** by using the compatibility conditions that the member end displacements and rotations must be the same as the corresponding joint displacements and rotations; and (c) linking the joint displacements **d** to the joint loads **P** by means of the member force-displacement relations $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$.

Consider, for example, an arbitrary beam subjected to joint and member loads, as depicted in Fig. 5.10(a). The structure has three degrees of freedom, d_1 through d_3, as shown in Fig. 5.10(b). Our objective is to establish the structure stiffness relationships, which express the external loads as functions of the joint displacements **d**. The member end forces **Q** and end displacements **u** for the three members of the beam are given in Fig. 5.10(c), in which the superscript (i) denotes the member number.

By applying the equations of equilibrium $\sum F_Y = 0$ and $\sum M = 0$ to the free body of joint 2, and the equilibrium equation $\sum M = 0$ to the free body of joint 3, we obtain the following relationships between the external joint loads **P** and the internal member end forces **Q**.

$$P_1 = Q_3^{(1)} + Q_1^{(2)} \quad \textbf{(5.100a)}$$

$$P_2 = Q_4^{(1)} + Q_2^{(2)} \quad \textbf{(5.100b)}$$

$$P_3 = Q_4^{(2)} + Q_2^{(3)} \quad \textbf{(5.100c)}$$

Next, we determine compatibility conditions for the three members of the beam. Since the left end 1 of member 1 is connected to fixed support 1 (Fig. 5.10(b)), which can neither translate nor rotate, the displacements $u_1^{(1)}$ and $u_2^{(1)}$ of end 1 of the member (Fig. 5.10(c)) must be 0. Similarly, since end 2 of this member is connected to joint 2, the displacements $u_3^{(1)}$ and $u_4^{(1)}$ of end 2 must be the same as the displacements d_1 and d_2, respectively, of joint 2. Thus, the compatibility equations for member 1 are:

$$u_1^{(1)} = u_2^{(1)} = 0 \qquad u_3^{(1)} = d_1 \qquad u_4^{(1)} = d_2 \quad \textbf{(5.101)}$$

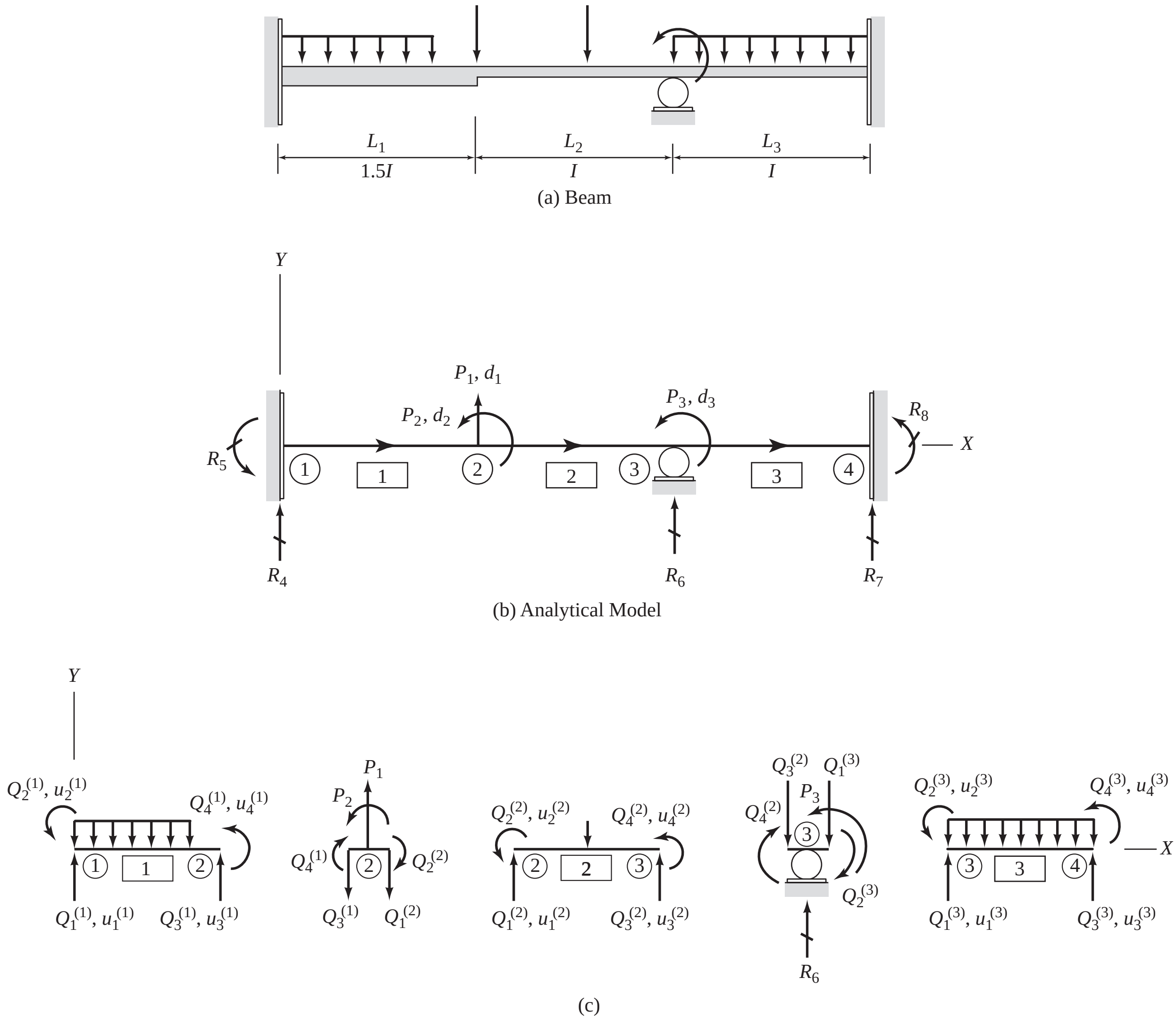


Fig. 5.10

In a similar manner, the compatibility equations for members 2 and 3, respectively, are given by

$$u_1^{(2)} = d_1 \qquad u_2^{(2)} = d_2 \qquad u_3^{(2)} = 0 \qquad u_4^{(2)} = d_3 \tag{5.102}$$

$$u_1^{(3)} = 0 \qquad u_2^{(3)} = d_3 \qquad u_3^{(3)} = u_4^{(3)} = 0 \tag{5.103}$$

The link between the joint equilibrium equations (Eqs. (5.100)) and the compatibility conditions (Eqs. (5.101) through (5.103)) is provided by the member stiffness relationship $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$ (Eq. (5.4)). To express the six member end forces that appear in Eqs. (5.100) in terms of the member end displacements, we will use the expanded form of the member stiffness relationship given in Eqs. (5.2). Thus, the end forces $Q_3^{(1)}$ and $Q_4^{(1)}$, of member 1, can be expressed in terms of the member end displacements as

$$Q_3^{(1)} = k_{31}^{(1)}u_1^{(1)} + k_{32}^{(1)}u_2^{(1)} + k_{33}^{(1)}u_3^{(1)} + k_{34}^{(1)}u_4^{(1)} + Q_{f3}^{(1)} \tag{5.104a}$$

$$Q_4^{(1)} = k_{41}^{(1)}u_1^{(1)} + k_{42}^{(1)}u_2^{(1)} + k_{43}^{(1)}u_3^{(1)} + k_{44}^{(1)}u_4^{(1)} + Q_{f4}^{(1)} \tag{5.104b}$$

Similarly, the end forces $Q_1^{(2)}$, $Q_2^{(2)}$, and $Q_4^{(2)}$, of member 2, are written as

$$Q_1^{(2)} = k_{11}^{(2)}u_1^{(2)} + k_{12}^{(2)}u_2^{(2)} + k_{13}^{(2)}u_3^{(2)} + k_{14}^{(2)}u_4^{(2)} + Q_{f1}^{(2)} \tag{5.105a}$$

$$Q_2^{(2)} = k_{21}^{(2)}u_1^{(2)} + k_{22}^{(2)}u_2^{(2)} + k_{23}^{(2)}u_3^{(2)} + k_{24}^{(2)}u_4^{(2)} + Q_{f2}^{(2)} \tag{5.105b}$$

$$Q_4^{(2)} = k_{41}^{(2)}u_1^{(2)} + k_{42}^{(2)}u_2^{(2)} + k_{43}^{(2)}u_3^{(2)} + k_{44}^{(2)}u_4^{(2)} + Q_{f4}^{(2)} \tag{5.105c}$$

and the end force $Q_2^{(3)}$, of member 3, is expressed as

$$Q_2^{(3)} = k_{21}^{(3)}u_1^{(3)} + k_{22}^{(3)}u_2^{(3)} + k_{23}^{(3)}u_3^{(3)} + k_{24}^{(3)}u_4^{(3)} + Q_{f2}^{(3)} \tag{5.106}$$

Next, we relate the joint displacements **d** to the member end forces **Q** by substituting the compatibility equations, Eqs. (5.101), (5.102), and (5.103), into the member force-displacement relations given by Eqs. (5.104), (5.105), and (5.106), respectively. Thus,

$$Q_3^{(1)} = k_{33}^{(1)}d_1 + k_{34}^{(1)}d_2 + Q_{f3}^{(1)} \tag{5.107a}$$

$$Q_4^{(1)} = k_{43}^{(1)}d_1 + k_{44}^{(1)}d_2 + Q_{f4}^{(1)} \tag{5.107b}$$

$$Q_1^{(2)} = k_{11}^{(2)}d_1 + k_{12}^{(2)}d_2 + k_{14}^{(2)}d_3 + Q_{f1}^{(2)} \tag{5.107c}$$

$$Q_2^{(2)} = k_{21}^{(2)}d_1 + k_{22}^{(2)}d_2 + k_{24}^{(2)}d_3 + Q_{f2}^{(2)} \tag{5.107d}$$

$$Q_4^{(2)} = k_{41}^{(2)}d_1 + k_{42}^{(2)}d_2 + k_{44}^{(2)}d_3 + Q_{f4}^{(2)} \tag{5.107e}$$

$$Q_2^{(3)} = k_{22}^{(3)}d_3 + Q_{f2}^{(3)} \tag{5.107f}$$

Finally, by substituting Eqs. (5.107) into the joint equilibrium equations (Eqs. (5.100)), we establish the desired structure stiffness relationships as

$$P_1 = \left(k_{33}^{(1)} + k_{11}^{(2)}\right)d_1 + \left(k_{34}^{(1)} + k_{12}^{(2)}\right)d_2 + k_{14}^{(2)}d_3 + \left(Q_{f3}^{(1)} + Q_{f1}^{(2)}\right) \tag{5.108a}$$

$$P_2 = \left(k_{43}^{(1)} + k_{21}^{(2)}\right)d_1 + \left(k_{44}^{(1)} + k_{22}^{(2)}\right)d_2 + k_{24}^{(2)}d_3 + \left(Q_{f4}^{(1)} + Q_{f2}^{(2)}\right) \tag{5.108b}$$

$$P_3 = k_{41}^{(2)}d_1 + k_{42}^{(2)}d_2 + \left(k_{44}^{(2)} + k_{22}^{(3)}\right)d_3 + \left(Q_{f4}^{(2)} + Q_{f2}^{(3)}\right) \tag{5.108c}$$

Equations (5.108) can be conveniently expressed in matrix form as

$$\mathbf{P} = \mathbf{Sd} + \mathbf{P}_f$$

or

$$\boxed{\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}} \tag{5.109}$$

in which

$$\mathbf{S} = \begin{bmatrix} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{14}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{44}^{(2)} + k_{22}^{(3)} \end{bmatrix} \tag{5.110}$$

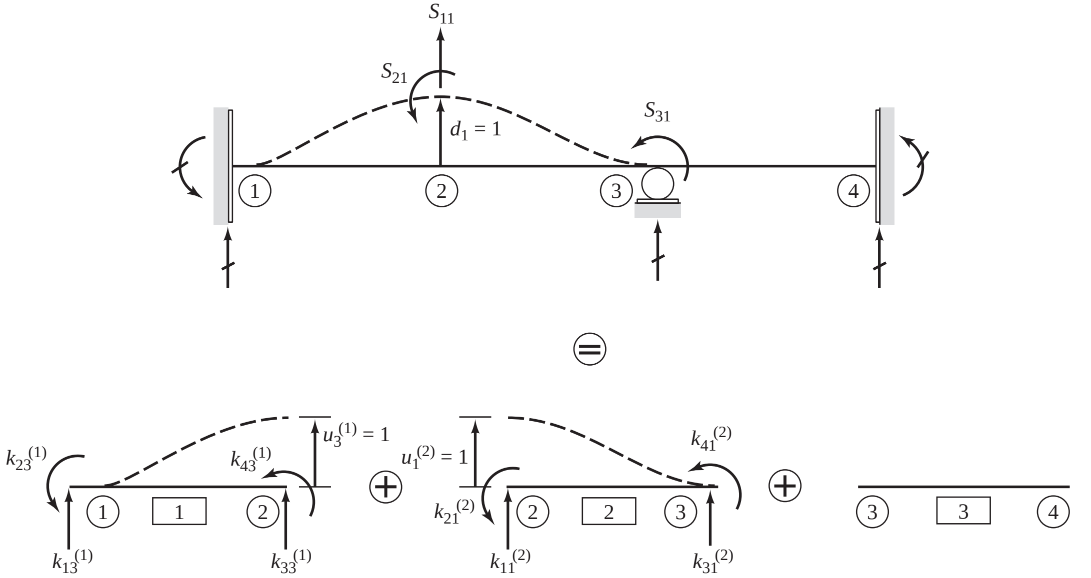
is the $NDOF \times NDOF$ structure stiffness matrix for the beam of Fig. 5.10(b), and

$$\mathbf{P}_f = \begin{bmatrix} Q_{f3}^{(1)} + Q_{f1}^{(2)} \\ Q_{f4}^{(1)} + Q_{f2}^{(2)} \\ Q_{f4}^{(2)} + Q_{f2}^{(3)} \end{bmatrix} \tag{5.111}$$

is the $NDOF \times 1$ *structure fixed-joint force vector*. The structure fixed-joint force vectors are further discussed in the following section. In the rest of this section, we focus our attention on the structure stiffness matrices.

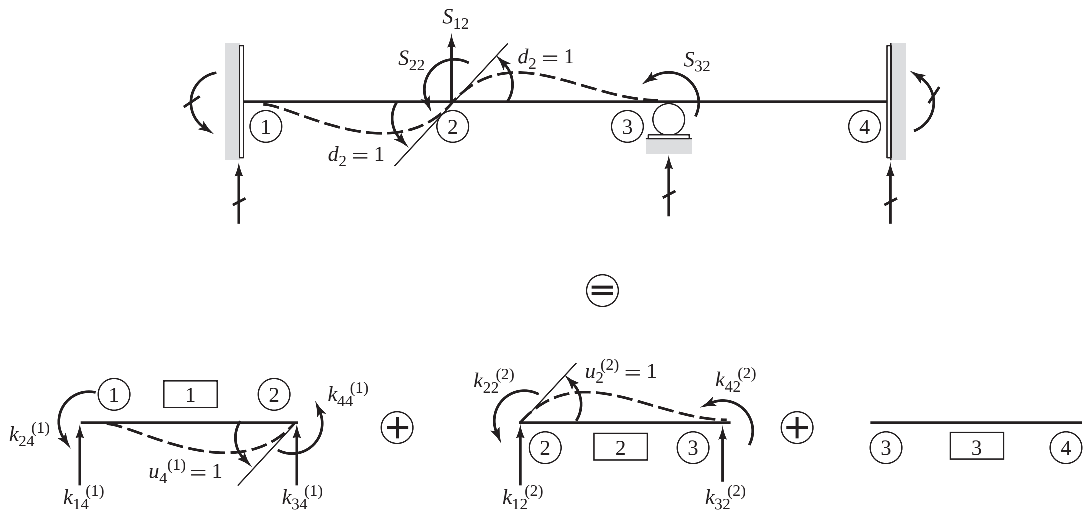
By examining Eq. (5.110), we realize that the structure stiffness matrix **S** of the beam of Fig. 5.10(b) is symmetric, because of the symmetric nature of the member stiffness matrices (i.e., $k_{ij} = k_{ji}$). (The structure stiffness matrices of all linear elastic structures are always symmetric.) As discussed in Chapter 3, a structure stiffness coefficient S_{ij} represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are 0, and the structure is not subjected to any external loads. We can use this definition to verify the **S** matrix (Eq. (5.110)) for the beam of Fig. 5.10.

In Figs. 5.11(a) through (c), the beam is subjected to the unit values of the three joint displacements d_1 through d_3, respectively. As depicted in Fig. 5.11(a),

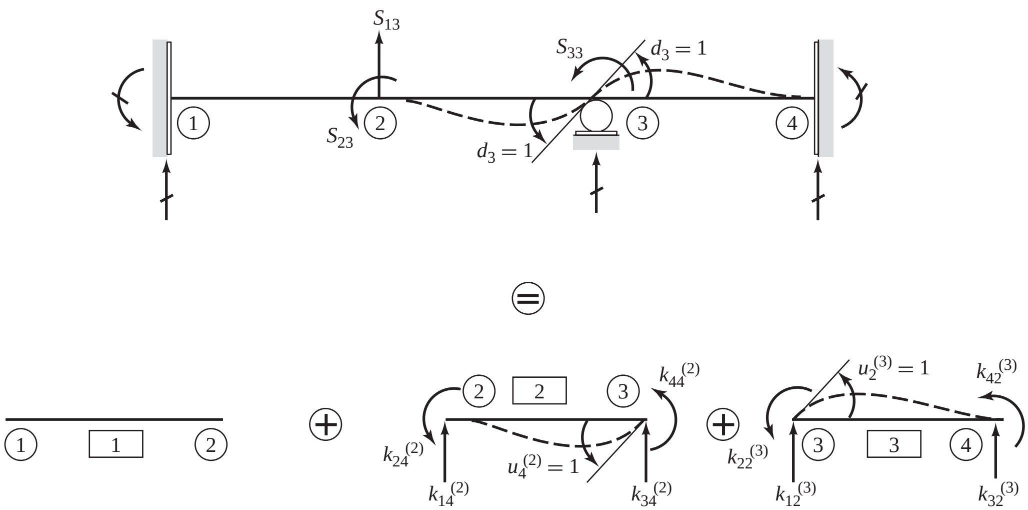


(a) First Column of **S** ($d_1 = 1$, $d_2 = d_3 = 0$)

Fig. 5.11



(b) Second Column of **S** ($d_2 = 1$, $d_1 = d_3 = 0$)



(c) Third Column of **S** ($d_3 = 1$, $d_1 = d_2 = 0$)

Fig. 5.11 (*continued*)

the joint displacement $d_1 = 1$ (with $d_2 = d_3 = 0$) induces unit displacements $u_3^{(1)} = 1$ at the right end of member 1 and $u_1^{(2)} = 1$ at the left end of member 2, while member 3 is not subjected to any displacements. The member stiffness coefficients (or end forces) necessary to cause the foregoing end displacements of the individual members are also shown in Fig. 5.11(a). (Recall that we derived the explicit expressions for member stiffness coefficients, in terms of E, I, and L of the member, in Section 5.2.) From the figure, we can see that the total vertical joint force S_{11} at joint 2, required to cause the joint displacement $d_1 = 1$ (with $d_2 = d_3 = 0$), must be equal to the algebraic sum of the vertical forces at the two member ends connected to this joint; that is,

$$S_{11} = k_{33}^{(1)} + k_{11}^{(2)} \tag{5.112a}$$

Similarly, the total joint moment S_{21} at joint 2 must be equal to the algebraic sum of the moments at the ends of members 1 and 2 connected to joint 2. Thus, (Fig. 5.11(a)),

$$S_{21} = k_{43}^{(1)} + k_{21}^{(2)} \tag{5.112b}$$

and the total joint moment S_{31} at joint 3 must equal the algebraic sum of the moments at the two member ends connected to the joint; that is,

$$S_{31} = k_{41}^{(2)} \tag{5.112c}$$

Note that the foregoing expressions for s_{i1} ($i = 1$ through 3) are identical to those listed in the first column of **S** in Eq. (5.110).

The second column of **S** can be verified in a similar manner. From Fig. 5.11(b), we can see that the joint rotation $d_2 = 1$ (with $d_1 = d_3 = 0$) induces unit rotations $u_4^{(1)} = 1$ at the right end of member 1, and $u_2^{(2)} = 1$ at the left end of member 2. The member stiffness coefficients associated with these end displacements are also shown in the figure. By comparing the joint forces with the member end forces, we obtain the expressions for the structure stiffness coefficients as

$$S_{12} = k_{34}^{(1)} + k_{12}^{(2)} \tag{5.112d}$$

$$S_{22} = k_{44}^{(1)} + k_{22}^{(2)} \tag{5.112e}$$

$$S_{32} = k_{42}^{(2)} \tag{5.112f}$$

which are the same as those in the second column of **S** in Eq. (5.110).

Similarly, by subjecting the beam to a unit rotation $d_3 = 1$ (with $d_1 = d_2 = 0$), as shown in Fig. 5.11(c), we obtain

$$S_{13} = k_{14}^{(2)} \tag{5.112g}$$

$$S_{23} = k_{24}^{(2)} \tag{5.112h}$$

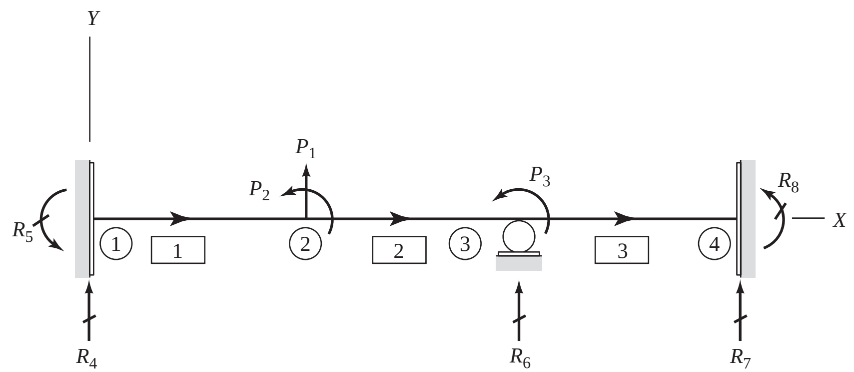
$$S_{33} = k_{44}^{(2)} + k_{22}^{(3)} \tag{5.112i}$$

The foregoing structure stiffness coefficients are identical to those listed in the third column of **S** in Eq. (5.110).

Assembly of the Structure Stiffness Matrix Using Member Code Numbers

Although the procedures discussed thus far for formulating **S** provide clearer insight into the basic concept of the structure stiffness matrix, it is more convenient from a computer programming viewpoint to directly form the structure stiffness matrix **S** by assembling the elements of the member stiffness matrices **k**. This technique, which is sometimes referred to as the *code number technique*, was described in detail in Section 3.7 for the case of plane trusses. The technique essentially involves storing the pertinent elements of the stiffness matrix **k** for each member of the beam, in the structure stiffness matrix **S**, by using the member code numbers. The code numbers for a member are simply the structure coordinate numbers at the location and in the direction of each of the member end displacements **u**, arranged in the order of the end displacements.

To illustrate this technique, consider again the three-member beam of Fig. 5.10. The analytical model of the beam is redrawn in Fig. 5.12(a), which shows its three degrees of freedom (numbered from 1 to 3) and five restrained coordinates (numbered from 4 to 8). In accordance with the notation for member end displacements adopted in Section 5.2, the first two end displacements of a member, u_1 and u_2, are always the vertical translation and rotation, respectively, at the left end (or beginning) of the member, whereas the last two end displacements, u_3 and u_4, are always the vertical translation and rotation, respectively, at the right end (or end) of the member. Thus, the first two code numbers for a member are always the structure coordinate numbers



Three degrees of freedom (1 through 3);
five restrained coordinates (4 through 8)

(a) Analytical Model

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \qquad \mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 6 & 3 \end{array} \\ \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 6 \\ 3 \end{array} \qquad \mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 6 & 3 & 7 & 8 \end{array} \\ \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} \\ k_{31}^{(3)} & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} \\ k_{41}^{(3)} & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)} \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 3 \\ 7 \\ 8 \end{array}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{14}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2/} & k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{44}^{(2)} + k_{22}^{(3)} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

(b) Assembling of Structure Stiffness Matrix **S**

Fig. 5.12

corresponding to the vertical translation and rotation, respectively, of the beginning joint; and the third and fourth member code numbers are always the structure coordinate numbers corresponding to the vertical translation and rotation, respectively, of the end joint.

From Fig. 5.12(a) we can see that, for member 1 of the beam, the beginning joint is 1 with restrained coordinates 4 and 5, and the end joint is 2 with degrees of freedom 1 and 2. Thus, the code numbers for member 1 are 4, 5, 1, 2. Similarly, the code numbers for member 2, for which the beginning and end joints are 2 and 3, respectively, are 1, 2, 6, 3. In a similar manner, the code numbers for member 3 are found to be 6, 3, 7, 8.

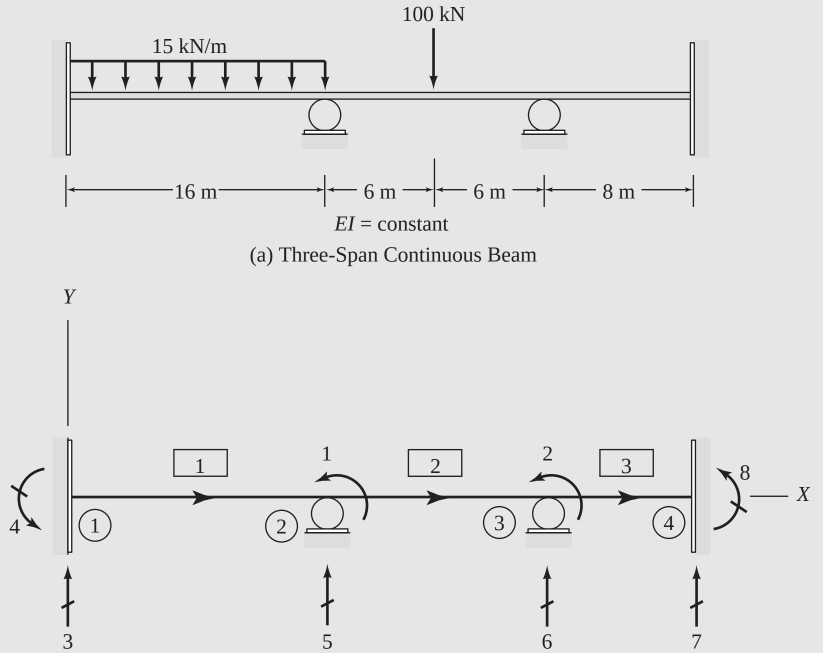
To establish the structure stiffness matrix **S**, we write the code numbers of each member on the right side and at the top of its stiffness matrix $\mathbf{k}_i$ ($i = 1, 2,$ or 3), as shown in Fig. 5.12(b). These code numbers now define the positions of the elements of the member stiffness matrices in the structure stiffness matrix **S**. In other words, the code numbers on the right side of a **k** matrix represent the row numbers of **S**; and the code numbers at the top represent the column numbers of **S**. Furthermore, since the number of rows and columns of **S** equal the number of degrees of freedom (*NDOF*) of the beam, only those elements of a **k** matrix for which both the row and the column code numbers are less than or equal to *NDOF* belong in the structure stiffness matrix **S**. The structure stiffness matrix **S** is obtained by algebraically adding the pertinent elements of the **k** matrices of all the members in their proper positions in the **S** matrix.

To assemble the **S** matrix for the beam of Fig. 5.12(a), we start by storing the pertinent elements of the stiffness matrix of member 1, $\mathbf{k}_1$, in the **S** matrix (see Fig. 5.12(b)). Thus, the element $k_{33}^{(1)}$ is stored in row 1 and column 1 of **S**, the element $k_{43}^{(1)}$ is stored in row 2 and column 1 of **S**, the element $k_{34}^{(1)}$ is stored in row 1 and column 2 of **S**, and the element $k_{44}^{(1)}$ is stored in row 2 and column 2 of **S**. It should be noted that since the beam has three degrees of freedom, only those elements of $\mathbf{k}_1$ whose row and column code numbers both are less than or equal to 3 are stored in **S**. The same procedure is then used to store the pertinent elements of $\mathbf{k}_2$ and $\mathbf{k}_3$, of members 2 and 3, respectively, in the **S** matrix. Note that when two or more member stiffness coefficients are stored in the same element of **S**, then the coefficients must be algebraically added. The completed structure stiffness matrix **S** for the beam is shown in Fig. 5.12(b), and is identical to the one derived previously (Eq. (5.110)) by substituting the member compatibility and stiffness relations into the joint equilibrium equations.

EXAMPLE 5.4 Determine the structure stiffness matrix for the three-span continuous beam shown in Fig. 5.13(a).

SOLUTION *Analytical Model:* The analytical model of the structure is shown in Fig. 5.13(b). The beam has two degrees of freedom—the rotations of joints 2 and 3—which are identified by the structure coordinate numbers 1 and 2 in the figure.

Structure Stiffness Matrix: To generate the 2×2 structure stiffness matrix **S**, we will determine, for each member, the stiffness matrix **k** and store its pertinent elements in their proper positions in **S** by using the member code numbers.



(a) Three-Span Continuous Beam

(b) Analytical Model

Fig. 5.13

Member 1 By substituting $L = 16$ m into Eq. (5.53), we obtain

$$\mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 1 \end{array} \\ \left[\begin{array}{cccc} 0.0029297 & 0.023438 & -0.0029297 & 0.023438 \\ 0.023438 & 0.25 & -0.023438 & 0.125 \\ -0.0029297 & -0.023438 & 0.0029297 & -0.023438 \\ 0.023438 & 0.125 & -0.023438 & \boxed{0.25} \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad \textbf{(1)}$$

From Fig. 5.13(b), we observe that the code numbers for this member are 3, 4, 5, 1. These numbers are written on the right side and at the top of $\mathbf{k}_1$ in Eq. (1), to indicate the rows and columns, respectively, of the structure stiffness matrix **S**, where the elements of $\mathbf{k}_1$ are to be stored. Thus, the element in row 4 and column 4 of $\mathbf{k}_1$ is stored in row 1 and column 1 of **S**, as

$$\mathbf{S} = EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 0.25 & 0 \\ 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad \textbf{(2)}$$

Note that the elements of $\mathbf{k}_1$ corresponding to the restrained coordinate numbers 3, 4, and 5 are disregarded.

Member 2 $L = 12$ m. By using Eq. (5.53),

$$\mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 2 \end{array} \\ \left[\begin{array}{cccc} 0.0069444 & 0.041667 & -0.0069444 & 0.041667 \\ 0.041667 & \boxed{0.33333} & -0.041667 & \boxed{0.16667} \\ -0.0069444 & -0.041667 & 0.0069444 & -0.041667 \\ 0.041667 & \boxed{0.16667} & -0.041667 & \boxed{0.33333} \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 1 \\ 6 \\ 2 \end{array} \qquad \textbf{(3)}$$

From Fig. 5.13(b), we can see that the code numbers for this member are 5, 1, 6, 2. These numbers are used to add the pertinent elements of $\mathbf{k}_2$ in their proper positions in the structure stiffness matrix **S** given in Eq. (2), which now becomes

$$\mathbf{S} = EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 0.25 + 0.33333 & 0.16667 \\ 0.16667 & 0.33333 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad (4)$$

Member 3 $L = 8$ m. Thus,

$$\mathbf{k}_3 = EI \begin{array}{c} \begin{array}{cccc} 6 & 2 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc} 0.023438 & 0.09375 & -0.023438 & 0.09375 \\ 0.09375 & \boxed{0.5} & -0.09375 & 0.25 \\ -0.023438 & -0.09375 & 0.023438 & -0.09375 \\ 0.09375 & 0.25 & -0.09375 & 0.5 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 2 \\ 7 \\ 8 \end{array} \qquad (5)$$

The code numbers for this member are 6, 2, 7, 8. Thus, the element in row 2 and column 2 of $\mathbf{k}_3$ is added in row 2 and column 2 of **S** in Eq. (4), as

$$\mathbf{S} = EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 0.25 + 0.33333 & 0.16667 \\ 0.16667 & 0.33333 + 0.5 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

Since the stiffnesses of all three members of the beam have now been stored in **S**, the structure stiffness matrix for the given beam is

$$\mathbf{S} = EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 0.58333 & 0.16667 \\ 0.16667 & 0.83333 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad \textsf{Ans}$$

Note that the structure stiffness matrix is symmetric.

5.6 STRUCTURE FIXED-JOINT FORCES AND EQUIVALENT JOINT LOADS

As discussed in the preceding section, the force–displacement relationships for an entire structure can be expressed in matrix form (see Eq. (5.109)) as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

in which $\mathbf{P}_f$ represents the structure fixed-joint force vector. It was also shown in the preceding section that by using the basic equations of equilibrium, compatibility, and member stiffness, the structure fixed-joint forces $\mathbf{P}_f$ can be expressed in terms of the member fixed-end forces $\mathbf{Q}_f$ (see Eq. (5.111)). In this section, we consider the physical interpretation of the structure fixed-joint forces; and discuss the formation of the $\mathbf{P}_f$ vector, by assembling the elements of the member $\mathbf{Q}_f$ vectors, using the member code numbers.

The concept of the structure fixed-joint forces $\mathbf{P}_f$ is analogous to that of the member fixed-end forces $\mathbf{Q}_f$. *The structure fixed-joint forces represent the reaction forces (and/or moments) that would develop at the locations and in the directions of the structure's degrees of freedom, due to the external member* *loads, if all the joints of the structure were fixed against translations and rotations.*

To develop some insight into the concept of structure fixed-joint forces, let us reconsider the beam of Fig. 5.10. The beam, subjected only to the member loads, is redrawn in Fig. 5.14(a), with its analytical model depicted in Fig. 5.14(b). Now, assume that joint 2, which is free to translate and rotate, is restrained against these displacements by an imaginary restraint (or clamp) applied to it, as shown in Fig. 5.14(c). Similarly, joint 3, which is free to rotate, is also restrained against rotation by means of an imaginary restraint (or clamp). When external loads are applied to the members of this hypothetical

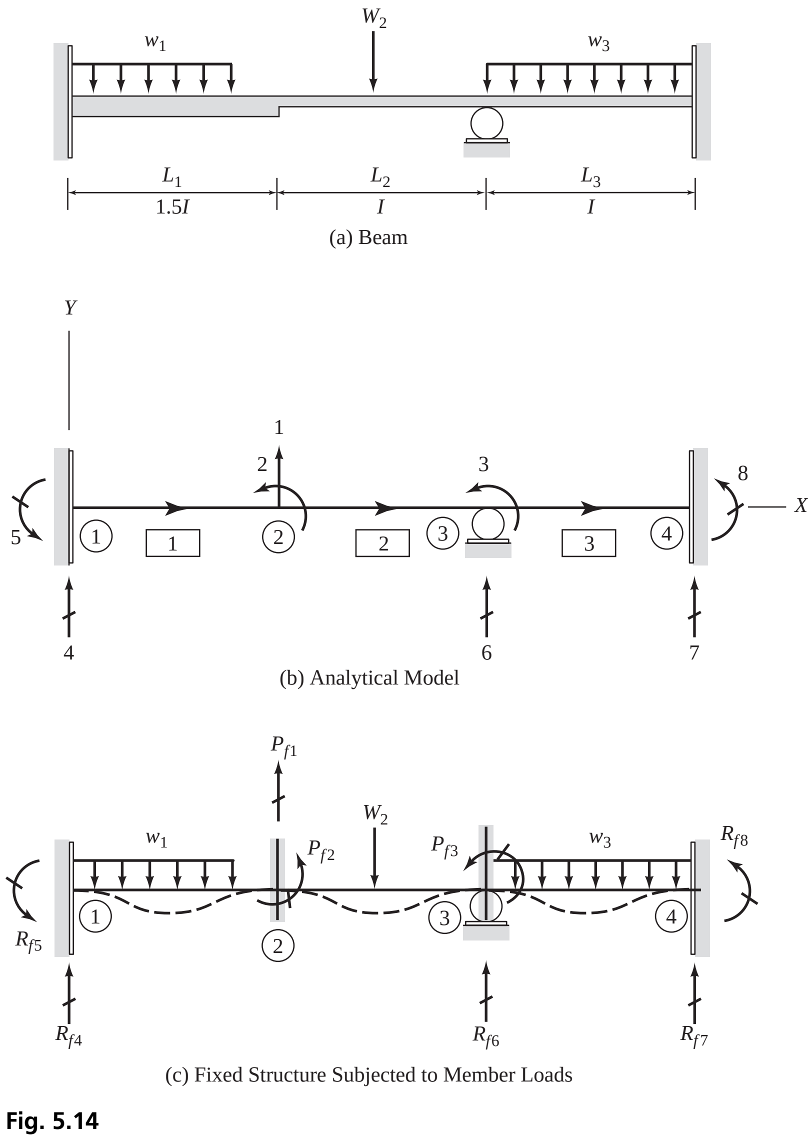
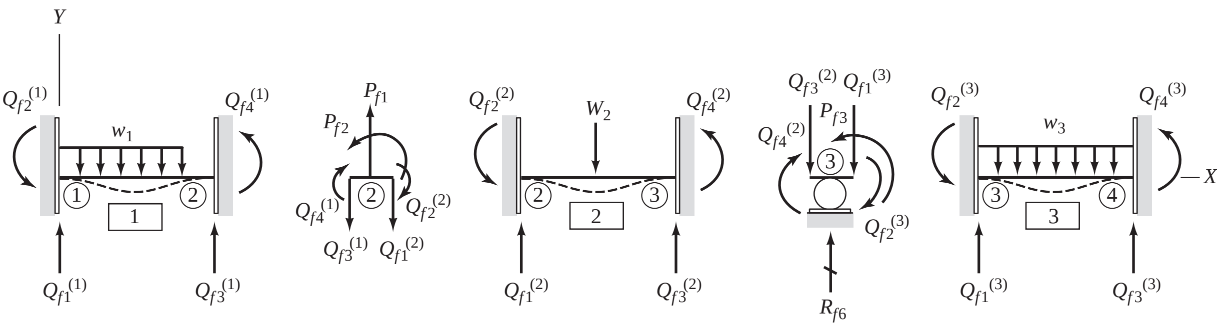


Fig. 5.14



(d) Member Fixed-End Forces

$$\mathbf{Q}_{f1} = \begin{array}{c} \left[\begin{array}{c} Q_{f1}^{(1)} \\ Q_{f2}^{(1)} \\ \hline Q_{f3}^{(1)} \\ Q_{f4}^{(1)} \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array}$$

$$\mathbf{P}_f = \left[\begin{array}{c} Q_{f3}^{(1)} + Q_{f1}^{(2)} \\ Q_{f4}^{(1)} + Q_{f2}^{(2)} \\ Q_{f4}^{(2)} + Q_{f2}^{(3)} \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \mathbf{Q}_{f2} = \left[\begin{array}{c} Q_{f1}^{(2)} \\ Q_{f2}^{(2)} \\ \hline Q_{f3}^{(2)} \\ \hline Q_{f4}^{(2)} \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 6 \\ 3 \end{array}$$

$$\mathbf{Q}_{f3} = \left[\begin{array}{c} Q_{f1}^{(3)} \\ \hline Q_{f2}^{(3)} \\ \hline Q_{f3}^{(3)} \\ Q_{f4}^{(3)} \end{array}\right] \begin{array}{c} 6 \\ 3 \\ 7 \\ 8 \end{array}$$

(e) Assembly of Structure Fixed-Joint Force Vector $\mathbf{P}_f$

Fig. 5.14 (*continued*)

completely fixed structure, reaction forces and moments develop at each of its joints. Note that, in Fig. 5.14(c), the reactions due to the imaginary restraints are denoted symbolically by P_{fi} ($i = 1$ through 3), whereas the reactions at the actual supports are denoted by R_{fi} ($i = 4$ through 8). The imaginary reactions P_{f1}, P_{f2}, and P_{f3}, which are at the locations and in the directions of the structure's three degrees of freedom 1, 2, and 3, respectively, are considered the structure fixed-joint forces due to member loads. Thus, the structure fixed-joint force vector, $\mathbf{P}_f$, for the beam, can be written as

$$\mathbf{P}_f = \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \end{bmatrix} \tag{5.113}$$

To relate the structure fixed-joint forces $\mathbf{P}_f$ to the member fixed-end forces $\mathbf{Q}_f$, we draw the free-body diagrams of the members and the interior joints of the hypothetical fixed beam, as shown in Fig. 5.14(d). In this figure, the superscript (i) denotes the member number. Note that, because all the joints of the beam are completely restrained, the member ends, which are rigidly connected to the joints, are also fixed against any displacements. Therefore, only the fixed-end forces due to member loads, $\mathbf{Q}_f$, can develop at the ends of the three members of the beam. By applying the equations of equilibrium $\sum F_Y = 0$ and $\sum M = 0$ to the free body of joint 2, and the equilibrium equation $\sum M = 0$ to the free body of joint 3, we obtain the following relationships between the structure fixed-joint forces and the member fixed-end forces.

$$P_{f1} = Q_{f3}^{(1)} + Q_{f1}^{(2)}$$

$$P_{f2} = Q_{f4}^{(1)} + Q_{f2}^{(2)}$$

$$P_{f3} = Q_{f4}^{(2)} + Q_{f2}^{(3)}$$

which can be expressed in vector form as

$$\mathbf{P}_f = \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \end{bmatrix} = \begin{bmatrix} Q_{f3}^{(1)} + Q_{f1}^{(2)} \\ Q_{f4}^{(1)} + Q_{f2}^{(2)} \\ Q_{f4}^{(2)} + Q_{f2}^{(3)} \end{bmatrix}$$

Note that the foregoing $\mathbf{P}_f$ vector is identical to that determined for the example beam in the preceding section (Eq. (5.111)).

Assembly of Structure Fixed-Joint Force Vector Using Member Code Numbers

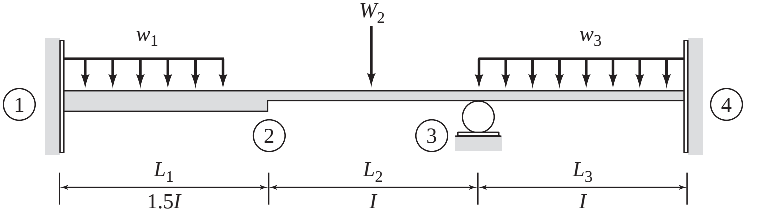
The structure fixed-joint force vector $\mathbf{P}_f$ can be conveniently assembled from the member fixed-end force vectors $\mathbf{Q}_f$, using the member code number technique. The technique is similar to that for forming the structure stiffness matrix $\mathbf{S}$, described in the preceding section. Essentially, the procedure involves storing the pertinent elements of the fixed-end force vector $\mathbf{Q}_f$ for each member of the beam in their proper positions in the structure fixed-joint force vector $\mathbf{P}_f$, using the member code numbers.

The foregoing procedure is illustrated for the example beam in Fig. 5.14(e). As shown there, the code numbers of each member are written on the right side of its fixed-end force vector $\mathbf{Q}_f$. Any member code number that is less than or equal to the number of degrees of freedom of the structure ($NDOF$), now identifies the row of $\mathbf{P}_f$ in which the corresponding member force is to be stored. Thus, as shown in Fig. 5.14(e), the third and fourth elements of $\mathbf{Q}_{f1}$, with code numbers 1 and 2, respectively, are stored in the first and second rows of $\mathbf{P}_f$. The same procedure is then repeated for members 2 and 3. Note that the completed $\mathbf{P}_f$ vector for the beam is identical to that obtained previously (Eq. (5.111)).

Equivalent Joint Loads

The negatives of the structure fixed-joint forces (i.e., $-\mathbf{P}_f$) are commonly known as the *equivalent joint loads*. This is because the structure fixed-joint forces, when applied to a structure with their directions reversed, cause the same joint displacements as the actual member loads.

The validity of the foregoing interpretation can be shown easily using the principle of superposition (Section 1.7), as illustrated in Fig. 5.15. Figure 5.15(a) shows a continuous beam subjected to arbitrary member loads. (This beam was considered previously, and its analytical model is given in Fig. 5.14(b).) In Fig. 5.15(b), joints 2 and 3 of the beam are fixed by imaginary restraints so that the translation and rotation of joint 2, and the rotation of joint 3, are 0. This hypothetical completely fixed beam is then subjected to member loads, causing the structure fixed-joint forces P_{f1}, P_{f2}, and P_{f3} to develop at the imaginary restraints, as shown in Fig. 5.15(b). Lastly, as shown in Fig. 5.15(c),



(a) Actual Beam Subjected to Member Loads

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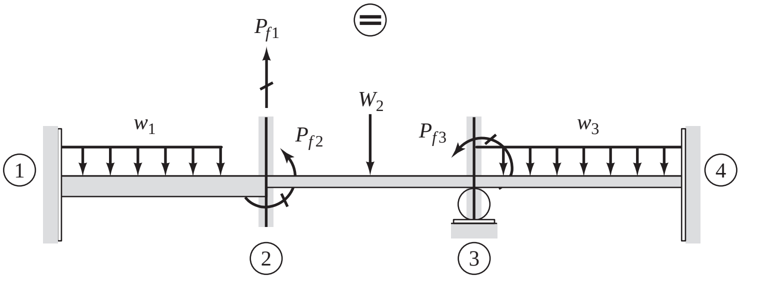

(b) Fixed Beam Subjected to Member Loads

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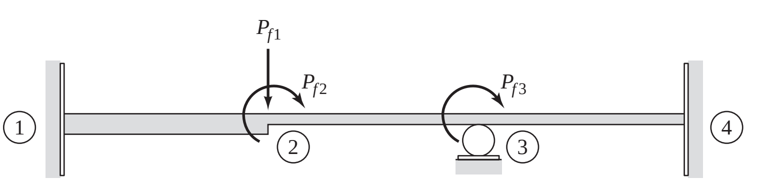


(c) Actual Beam Subjected to Equivalent Joint Loads

Fig. 5.15

the actual beam is subjected to external loads at its joints, which are equal in magnitude to P_{f1}, P_{f2}, and P_{f3}, but are reversed in direction.

By comparing Figs. 5.15(a), (b), and (c), we realize that the actual loading applied to the beam in Fig. 5.15(a) equals the algebraic sum of the loadings given in Figs. 5.15(b) and (c), because the reactive forces P_{f1}, P_{f2}, and P_{f3} in Fig. 5.15(b) cancel the corresponding applied loads in Fig. 5.15(c). Thus, according to the principle of superposition, any joint displacement of the actual beam due to the member loads (Fig. 5.15(a)) must equal the algebraic sum of the corresponding joint displacement of the fixed beam due to the member loads (Fig. 5.15(b)), and the corresponding joint displacement of the actual beam subjected to no member loads, but to the fixed-joint forces with their directions reversed. However, the joint displacements of the fixed beam (Fig. 5.15(b)) are 0. Therefore, the joint displacements of the beam due to the member loads (Fig. 5.15(a)) must be equal to the corresponding joint displacements of the beam due to the negatives of the fixed-joint forces (Fig. 5.15(c)). In other words, the negatives of the structure fixed-joint forces do indeed cause the same joint displacements of the beam as the actual member loads; and in that sense, such forces can be considered to be the equivalent joint loads. It is important to realize that this equivalency between the negative fixed-joint forces and the member loads is valid only for joint displacements, and cannot be generalized to member end forces and reactions, because such forces are generally not 0 in fixed structures subjected to member loads.

Based on the foregoing discussion of equivalent joint loads, we can define the *equivalent joint load vector* $\mathbf{P}_e$ for a structure as simply the negative of its fixed-joint force vector $\mathbf{P}_f$; that is,

$$\boxed{\mathbf{P}_e = -\mathbf{P}_f} \tag{5.114}$$

An alternative form of the structure stiffness relations, in terms of the equivalent joint loads, can now be obtained by substituting Eq. (5.114) into Eq. (5.109). This yields

$$\boxed{\mathbf{P} + \mathbf{P}_e = \mathbf{S}\mathbf{d}} \tag{5.115}$$

Once $\mathbf{S}$, $\mathbf{P}_f$ (or $\mathbf{P}_e$), and $\mathbf{P}$ have been evaluated, the structure stiffness relations (Eq. (5.109) or Eq. (5.115)), which now represent a system of simultaneous linear equations, can be solved for the unknown joint displacements $\mathbf{d}$. With $\mathbf{d}$ known, the end displacements $\mathbf{u}$ for each member can be determined by applying the compatibility equations defined by its code numbers; then the corresponding end forces $\mathbf{Q}$ can be computed by applying the member stiffness relations. Finally, the support reactions $\mathbf{R}$ are determined from the member end forces $\mathbf{Q}$, by considering the equilibrium of the support joints in the directions of the restrained coordinates via member code numbers, as discussed in Chapter 3 for the case of plane trusses.

EXAMPLE 5.5 Determine the fixed-joint force vector and the equivalent joint load vector for the propped-cantilever beam shown in Fig. 5.16(a).

SOLUTION *Analytical Model:* See Fig. 5.16(b).

Structure Fixed-Joint Force Vector: To generate the 3×1 structure fixed-joint force vector $\mathbf{P}_f$, we will, for each member: (a) determine the fixed-end force vector $\mathbf{Q}_f$, using the fixed-end force equations given inside the front cover; and (b) store the pertinent elements of $\mathbf{Q}_f$ in their proper positions in $\mathbf{P}_f$, using the member code numbers.

Member 1 By substituting $w = 30$ kN/m, $L = 9$ m, and $l_1 = l_2 = 0$ into the fixed-end force expressions for loading type 3, we obtain

$$FS_b = FS_e = \frac{30(9)}{2} = 135 \text{ kN}$$

$$FM_b = \frac{30(9)^2}{12} = 202.5 \text{ kN} \cdot \text{m}$$

$$FM_e = -\frac{30(9)^2}{12} = -202.5 \text{ kN} \cdot \text{m}$$

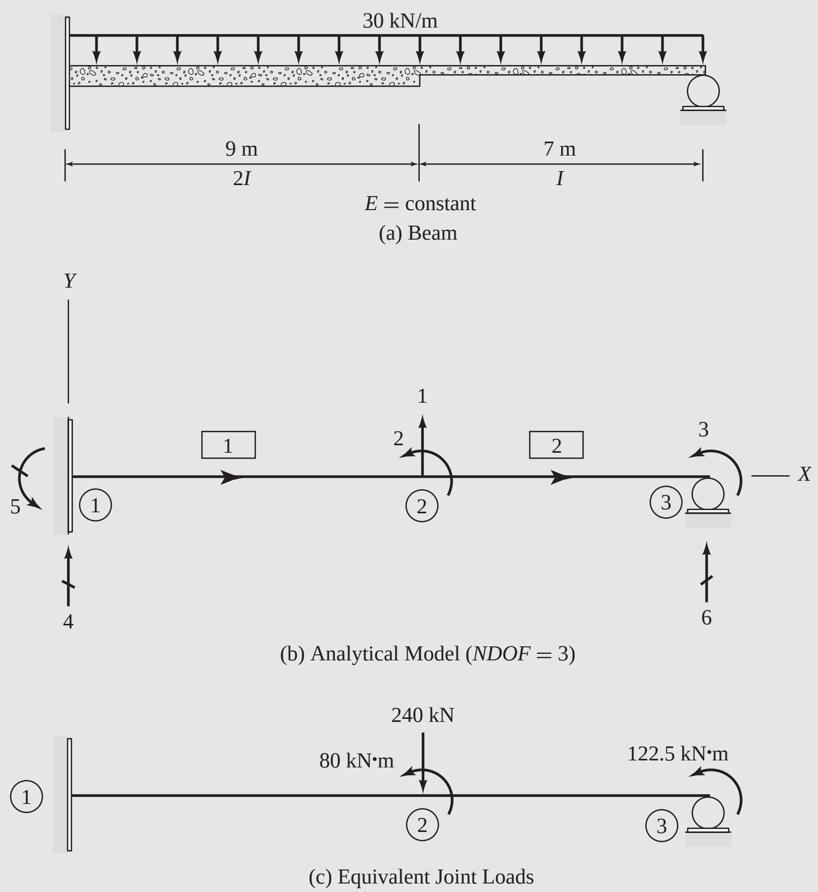


Fig. 5.16

Thus, the fixed-end force vector for member 1 is given by

$$\mathbf{Q}_{f1} = \begin{array}{c} \left[\begin{array}{c} 135 \\ 202.5 \\ \hdashline 135 \\ -202.5 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \tag{1}$$

From Fig. 5.16(b), we can see that the code numbers for member 1 are 4, 5, 1, 2. These numbers are written on the right side of $\mathbf{Q}_{f1}$ in Eq. (1) to indicate the rows of the structure fixed-joint vector $\mathbf{P}_f$, where the elements of $\mathbf{Q}_{f1}$ are to be stored. Thus, the elements in the third and fourth rows of $\mathbf{Q}_{f1}$ are stored in rows 1 and 2, respectively, of $\mathbf{P}_f$, as

$$\mathbf{P}_f = \left[\begin{array}{c} 135 \\ -202.5 \\ 0 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \tag{2}$$

Note that the elements of $\mathbf{Q}_{f1}$ corresponding to the restrained coordinate numbers 4 and 5 are disregarded.

Member 2 By substituting $w = 30$ kN/m, $L = 7$ m, and $l_1 = l_2 = 0$ into the fixed-end force expressions for loading type 3, we obtain

$$FS_b = FS_e = \frac{30(7)}{2} = 105 \text{ kN}$$

$$FM_b = \frac{30(7)^2}{12} = 122.5 \text{ kN} \cdot \text{m}$$

$$FM_e = -\frac{30(7)^2}{12} = -122.5 \text{ kN} \cdot \text{m}$$

Thus,

$$\mathbf{Q}_{f2} = \left[\begin{array}{c} 105 \\ 122.5 \\ \hdashline 105 \\ \hdashline -122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 6 \\ 3 \end{array} \tag{3}$$

From Fig. 5.16(b), we observe that the code numbers for this member are 1, 2, 6, 3. These numbers are used to add the pertinent elements of $\mathbf{Q}_{f2}$ in their proper positions in $\mathbf{P}_f$ given in Eq. (2), which now becomes

$$\mathbf{P}_f = \left[\begin{array}{c} 135 + 105 \\ -202.5 + 122.5 \\ -122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Since the fixed-end forces for both members of the beam have now been stored in $\mathbf{P}_f$, the structure fixed-joint force vector for the given beam is

$$\mathbf{P}_f = \left[\begin{array}{c} 240 \\ -80 \\ -122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \textbf{Ans}$$

Equivalent Joint Load Vector: By using Eq. (5.114), we obtain

$$\mathbf{P}_e = -\mathbf{P}_f = \left[\begin{array}{c} -240 \\ 80 \\ 122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \textbf{Ans}$$

These equivalent joint loads, when applied to the beam as shown in Fig. 5.16(c), cause the same joint displacements as the actual 30 kN/m uniformly distributed load given in Fig. 5.16(a).

5.7 PROCEDURE FOR ANALYSIS

Based on the concepts presented in the previous sections, we can develop the following step-by-step procedure for the analysis of beams by the matrix stiffness method. The reader should note that the overall format of this procedure is essentially the same as the procedure for analysis of plane trusses presented in Chapter 3.

1. Prepare an analytical model of the beam, as follows.

   a. Draw a line diagram of the beam, and identify each joint and member by a number. The origin of the global XY coordinate system is usually located at the farthest left joint, with the X and Y axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively. For each member, establish a local xy coordinate system, with the origin at the left end (beginning) of the member, and the x and y axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively.

   b. Number the degrees of freedom and restrained coordinates of the beam, as discussed in Section 5.1.

2. Evaluate the structure stiffness matrix $\mathbf{S}$ and fixed-joint force vector $\mathbf{P}_f$. The number of rows and columns of $\mathbf{S}$ must be equal to the number of degrees of freedom ($NDOF$) of the beam; the number of rows of $\mathbf{P}_f$ must equal $NDOF$. For each member of the structure, perform the following operations.

   a. Compute the member stiffness matrix $\mathbf{k}$ (Eq. (5.53)).

   b. If the member is subjected to external loads, then evaluate its fixed-end force vector $\mathbf{Q}_f$, using the expressions for fixed-end forces given inside the front cover.

   c. Identify the member code numbers and store the pertinent elements of $\mathbf{k}$ and $\mathbf{Q}_f$ in their proper positions in the structure stiffness matrix $\mathbf{S}$, and the fixed-joint force vector $\mathbf{P}_f$, respectively. The complete structure stiffness matrix $\mathbf{S}$, obtained by assembling the stiffness coefficients of all the members of the beam, must be symmetric.

3. If the beam is subjected to joint loads, then form the $NDOF \times 1$ joint load vector $\mathbf{P}$.

4. Determine the joint displacements $\mathbf{d}$. Substitute $\mathbf{P}$, $\mathbf{P}_f$, and $\mathbf{S}$ into the structure stiffness relations, $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$ (Eq. (5.109)), and solve the resulting system of simultaneous equations for the unknown joint displacements $\mathbf{d}$. To check that the simultaneous equations have been solved correctly, substitute the numerical values of the joint displacements $\mathbf{d}$ back into the structure stiffness relations, $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$.

If the solution is correct, then the stiffness relations should be satisfied. It should be noted that joint translations are considered positive when in the positive direction of the Y axis, and joint rotations are considered positive when counterclockwise.

5. Compute member end displacements and end forces, and support reactions. For each member of the beam, do the following.

a. Obtain member end displacements **u** from the joint displacements **d**, using the member code numbers.

b. Compute member end forces, using the relationship $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$ (Eq. (5.4)).

c. Using the member code numbers, store the pertinent elements of **Q** in their proper positions in the support reaction vector **R** (as discussed in Chapter 3).

6. Check the calculation of member end forces and support reactions by applying the equations of equilibrium, $\sum F_Y = 0$ and $\sum M = 0$, to the free body of the entire beam. If the calculations have been carried out correctly, then the equilibrium equations should be satisfied.

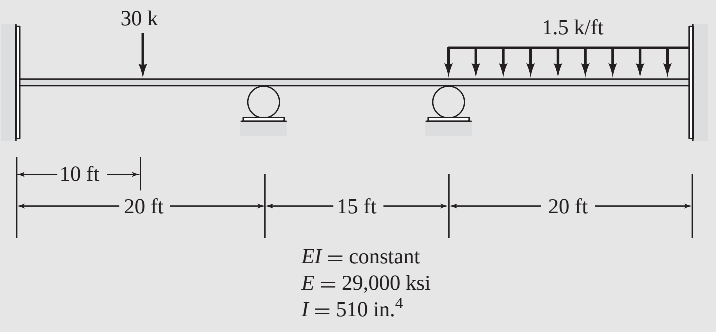
EXAMPLE 5.6 Determine the joint displacements, member end forces, and support reactions for the three-span continuous beam shown in Fig. 5.17(a), using the matrix stiffness method.

SOLUTION *Analytical Model:* See Fig. 5.17(b). The beam has two degrees of freedom—the rotations of joints 2 and 3—which are numbered 1 and 2, respectively. The six restrained coordinates of the beam are numbered 3 through 8.

Structure Stiffness Matrix and Fixed-Joint Force Vector:

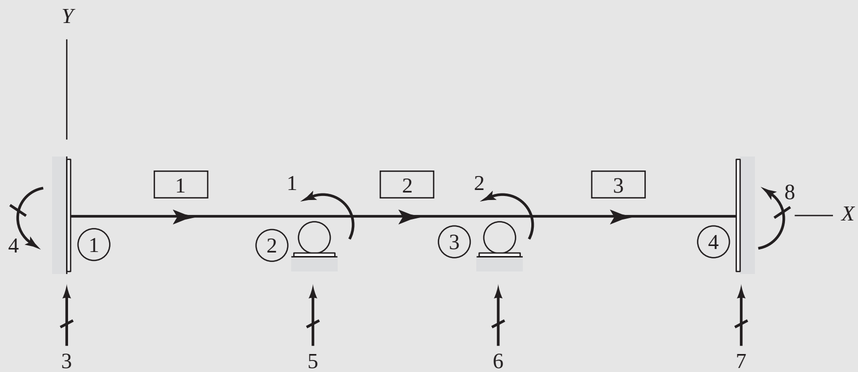
Member 1 By substituting $E = 29{,}000$ ksi, $I = 510$ in.^{4}, and $L = 240$ in. into Eq. (5.53), we obtain

$$\mathbf{k}_1 = \begin{array}{c} \begin{matrix} 3 & \quad 4 & \quad 5 & \quad 1 \end{matrix} \\ \begin{bmatrix} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{bmatrix} \end{array} \begin{matrix} 3 \\ 4 \\ 5 \\ 1 \end{matrix} \tag{1}$$



(a) Three-Span Continuous Beam

Fig. 5.17

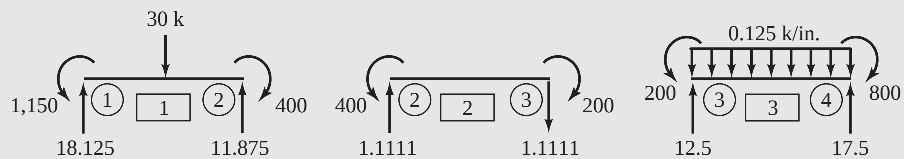


(b) Analytical Model

$$
\mathbf{S} = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cc} 1 & 2 \\ \left[\begin{array}{cc} 246,500 + 328,667 & 164,333 \\ 164,333 & 328,667 + 246,500 \end{array}\right] \end{array}
\begin{array}{c} \\ 1 \\ 2 \end{array}
=
\begin{array}{cc} 1 & 2 \\ \left[\begin{array}{cc} 575,167 & 164,333 \\ 164,333 & 575,167 \end{array}\right] \end{array}
\begin{array}{c} \\ 1 \\ 2 \end{array}
$$

$$
\mathbf{P}_f = \left[\begin{array}{c} -900 \\ 600 \end{array}\right] \begin{array}{c} 1 \\ 2 \end{array}
$$

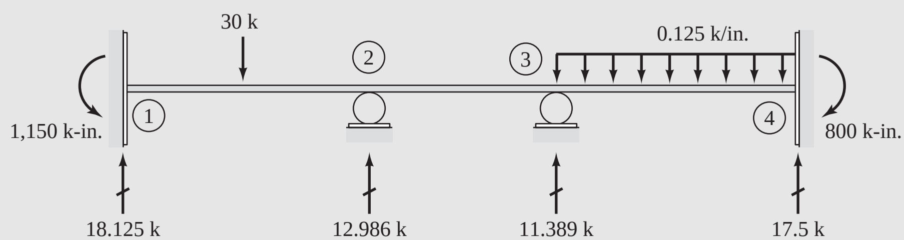
(c) Structure Stiffness Matrix and Fixed-Joint Force Vector



(d) Member End Forces

$$
\mathbf{R} = \left[\begin{array}{c} 18.125 \\ 1,150 \\ 11.875 + 1.1111 \\ -1.1111 + 12.5 \\ 17.5 \\ -800 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} = \left[\begin{array}{c} 18.125 \text{ k} \\ 1,150 \text{ k-in.} \\ 12.986 \text{ k} \\ 11.389 \text{ k} \\ 17.5 \text{ k} \\ -800 \text{ k-in.} \end{array}\right]
$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 5.17 (*continued*)

Using the fixed-end force equations given inside the front cover, we evaluate the fixed-end forces due to the 30 k concentrated load as

$$FS_b = \frac{30(120)^2}{(240)^3}[3(120) + 120] = 15 \text{ k}$$

$$FM_b = \frac{30(120)(120)^2}{(240)^2} = 900 \text{ k-in.}$$

$$FS_e = \frac{30(120)^2}{(240)^3}[120 + 3(120)] = 15 \text{ k}$$

$$FM_e = -\frac{30(120)^2(120)}{(240)^2} = -900 \text{ k-in.}$$

Thus, the fixed-end force vector for member 1 is

$$\mathbf{Q}_{f1} = \begin{array}{c} \begin{bmatrix} 15 \\ 900 \\ 15 \\ \hdashline -900 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad \textbf{(2)}$$

From Fig. 5.17(b), we observe that the code numbers for member 1 are 3, 4, 5, 1. Using these code numbers, the pertinent elements of $\mathbf{k}_1$ and $\mathbf{Q}_{f1}$ are stored in their proper positions in the 2×2 structure stiffness matrix $\mathbf{S}$ and the 2×1 structure fixed-joint force vector $\mathbf{P}_f$, respectively, as shown in Fig. 5.17(c).

Member 2 $E = 29{,}000$ ksi, $I = 510$ in.^{4}, and $L = 180$ in.

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 2 \end{array} \\ \begin{bmatrix} 30.432 & 2{,}738.9 & -30.432 & 2{,}738.9 \\ 2{,}738.9 & \boxed{328{,}667} & -2{,}738.9 & \boxed{164{,}333} \\ -30.432 & -2{,}738.9 & 30.432 & -2{,}738.9 \\ 2{,}738.9 & \boxed{164{,}333} & -2{,}738.9 & \boxed{328{,}667} \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 1 \\ 6 \\ 2 \end{array} \qquad \textbf{(3)}$$

Since this member is not subjected to any external loads, its fixed-end force vector is 0; that is,

$$\mathbf{Q}_{f2} = \mathbf{0} \qquad \textbf{(4)}$$

Using the code numbers 5, 1, 6, 2 for this member (see Fig. 5.17(b)), the relevant elements of $\mathbf{k}_2$ are stored into $\mathbf{S}$, as shown in Fig. 5.17(c).

Member 3 $E = 29{,}000$ ksi, $I = 510$ in.^{4}, and $L = 240$ in.

$$\mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 6 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & \boxed{246{,}500} & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 2 \\ 7 \\ 8 \end{array} \qquad \textbf{(5)}$$

The fixed-end forces due to the 0.125 k/in. (=1.5 k/ft) uniformly distributed load are

$$FS_b = \frac{0.125(240)}{2} = 15 \text{ k}$$

$$FM_b = \frac{0.125(240)^2}{12} = 600 \text{ k-in.}$$

$$FS_e = \frac{0.125(240)}{2} = 15 \text{ k}$$

$$FM_e = -\frac{0.125(240)^2}{12} = -600 \text{ k-in.}$$

Thus,

$$\mathbf{Q}_{f3} = \begin{array}{c} \left[\begin{array}{r} 15 \\ \hdashline 600 \\ \hdashline 15 \\ -600 \end{array}\right] \end{array} \begin{array}{l} 6 \\ 2 \\ 7 \\ 8 \end{array} \tag{6}$$

The relevant elements of $\mathbf{k}_3$ and $\mathbf{Q}_{f3}$ are stored in $\mathbf{S}$ and $\mathbf{P}_f$, respectively, using the member code numbers 6, 2, 7, 8.

The completed structure stiffness matrix $\mathbf{S}$ and structure fixed-joint force vector $\mathbf{P}_f$ are given in Fig. 5.17(c). Note that the $\mathbf{S}$ matrix is symmetric.

Joint Load Vector: Since no external loads (i.e., moments) are applied to the beam at joints 2 and 3, the joint load vector is 0; that is,

$$\mathbf{P} = \mathbf{0}$$

Joint Displacements: By substituting the numerical values of $\mathbf{P}$, $\mathbf{P}_f$, and $\mathbf{S}$ into Eq. (5.109), we write the stiffness relations for the entire continuous beam as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -900 \\ 600 \end{bmatrix} = \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

or

$$\begin{bmatrix} 900 \\ -600 \end{bmatrix} = \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving these equations simultaneously, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} 2.0284 \\ -1.6227 \end{bmatrix} \times 10^{-3} \text{ rad}$$ **Ans**

To check the foregoing solution, we substitute the numerical values of $\mathbf{d}$ back into the structure stiffness relationship to obtain

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d} = \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \begin{bmatrix} 2.0284 \\ -1.6227 \end{bmatrix} \times 10^{-3} = \begin{bmatrix} 900.01 \\ -599.99 \end{bmatrix}$$

Checks

Member End Displacements and End Forces:

Member 1 The member end displacements $\mathbf{u}$ can be obtained simply by comparing the member's degree of freedom numbers with its code numbers, as follows:

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{l} 3 \\ 4 \\ 5 \\ 1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.0284 \end{bmatrix} \times 10^{-3} \tag{7}$$

Note that the member code numbers (3, 4, 5, 1), when written on a side of $\mathbf{u}$ as shown in Eq. (7), define the compatibility equations for the member. Since the code numbers corresponding to u_1, u_2, and u_3 are the restrained coordinate numbers 3, 4, and 5, respectively, this indicates that $u_1 = u_2 = u_3 = 0$. Similarly, the code number 1 corresponding to u_4 indicates that $u_4 = d_1$. The foregoing compatibility equations can be easily verified by a visual inspection of the beam's line diagram, given in Fig. 5.17(b).

The member end forces can now be calculated, using the member stiffness relationship $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$ (Eq. (5.4)). Using $\mathbf{k}_1$ and $\mathbf{Q}_{f1}$ from Eqs. (1) and (2), respectively, we write

$$\mathbf{Q}_1 = \begin{bmatrix} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.0284 \end{bmatrix} \times 10^{-3}$$

$$+ \begin{bmatrix} 15 \\ 900 \\ 15 \\ -900 \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c} 18.125\text{ k} \\ 1{,}150\text{ k-in.} \\ 11.875\text{ k} \\ \hline -400\text{ k-in.} \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad \textbf{(8)} \quad \textbf{Ans}$$

The end forces for member 1 are shown in Fig. 5.17(d). We can check our calculation of end forces by applying the equilibrium equations, $\sum F_y = 0$ and $\sum M = 0$, to the free body of member 1 to ensure that it is in equilibrium. Thus,

$$+\uparrow \sum F_y = 0 \qquad 18.125 - 30 + 11.875 = 0 \qquad \textbf{Checks}$$

$$+ \circlearrowleft \sum M_{①} = 0 \qquad 1{,}150 - 30(120) - 400 + 11.875(240) = 0 \qquad \textbf{Checks}$$

Next, to generate the support reaction vector **R**, we write the member code numbers (3, 4, 5, 1) on the right side of $\mathbf{Q}_1$, as shown in Eq. (8), and store the pertinent elements of $\mathbf{Q}_1$ in their proper positions in **R** by matching the code numbers on the side of $\mathbf{Q}_1$ to the restrained coordinate numbers on the right side of **R** (see Fig. 5.17(e)).

Member 2 The member end displacements are given by

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} = \begin{bmatrix} 0 \\ d_1 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.0284 \\ 0 \\ -1.6227 \end{bmatrix} \times 10^{-3}$$

By using $\mathbf{k}_2$ from Eq. (3) and $\mathbf{Q}_{f2} = \mathbf{0}$, we compute member end forces as

$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

$$\mathbf{Q}_2 = \begin{bmatrix} 30.432 & 2{,}738.9 & -30.432 & 2{,}738.9 \\ 2{,}738.9 & 328{,}667 & -2{,}738.9 & 164{,}333 \\ -30.432 & -2{,}738.9 & 30.432 & -2{,}738.9 \\ 2{,}738.9 & 164{,}333 & -2{,}738.9 & 328{,}667 \end{bmatrix} \begin{bmatrix} 0 \\ 2.0284 \\ 0 \\ -1.6227 \end{bmatrix} \times 10^{-3}$$

$$= \left[\begin{array}{c} 1.1111\text{ k} \\ \hline 400\text{ k-in.} \\ \hline -1.1111\text{ k} \\ \hline -200\text{ k-in.} \end{array}\right] \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} \qquad \textbf{Ans}$$

The foregoing member end forces are shown in Fig. 5.17(d). To check our calculations, we apply the equations of equilibrium to the free body of member 2 as

$$+\uparrow \sum F_y = 0 \qquad 1.1111 - 1.1111 = 0 \qquad \textbf{Checks}$$

$$+ \circlearrowleft \sum M_{②} = 0 \qquad 400 - 200 - 1.1111(180) = 0.002 \approx 0 \qquad \textbf{Checks}$$

Next, we store the pertinent elements of $\mathbf{Q}_2$ in their proper positions in the reaction vector **R**, using the member code numbers (5, 1, 6, 2), as shown in Fig. 5.17(e).

Member 3

$$\mathbf{u}_3 = \begin{array}{c} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array} = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.6227 \\ 0 \\ 0 \end{bmatrix} \times 10^{-3}$$

By substituting $\mathbf{k}_3$ and $\mathbf{Q}_{f3}$ from Eqs. (5) and (6), respectively, into the member stiffness relationship $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$, we determine the end forces for member 3 to be

$$\mathbf{Q}_3 = \begin{bmatrix} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{bmatrix} \begin{bmatrix} 0 \\ -1.6227 \\ 0 \\ 0 \end{bmatrix} \times 10^{-3}$$

$$+ \begin{bmatrix} 15 \\ 600 \\ 15 \\ -600 \end{bmatrix} = \begin{bmatrix} 12.5\text{ k} \\ 200\text{ k-in.} \\ 17.5\text{ k} \\ -800\text{ k-in.} \end{bmatrix} \begin{array}{c} 6 \\ 2 \\ 7 \\ 8 \end{array}$$ **Ans**

These member end forces are shown in Fig. 5.17(d). To check our calculations, we apply the equilibrium equations:

$$+\uparrow \sum F_y = 0 \qquad 12.5 - 0.125(240) + 17.5 = 0$$ **Checks**

$$+\circlearrowleft \sum M_{③} = 0 \qquad 200 - 0.125(240)(120) - 800 + 17.5(240) = 0$$ **Checks**

Next, by using the code numbers (6, 2, 7, 8) for member 3, we store the relevant elements of $\mathbf{Q}_3$ in their proper positions in $\mathbf{R}$.

Support Reactions: The completed reaction vector $\mathbf{R}$ is shown in Fig. 5.17(e), and the support reactions are depicted on a line diagram of the structure in Fig. 5.17(f). **Ans**

Equilibrium Check: Finally, applying the equilibrium equations to the free body of the entire beam (Fig. 5.17(f)), we write

$$+\uparrow \sum F_y = 0$$

$$18.125 - 30 + 12.986 + 11.389 - 0.125(240) + 17.5 = 0$$ **Checks**

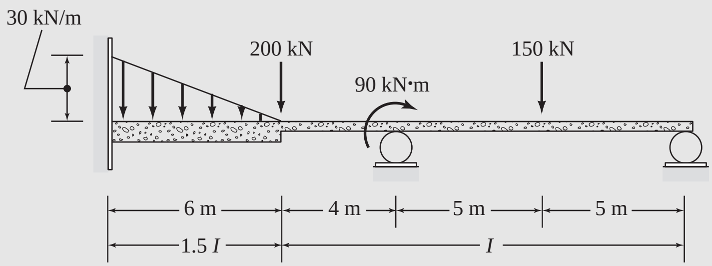
$$+\circlearrowleft \sum M_{①} = 0$$

$$1{,}150 - 30(120) + 12.986(240) + 11.389(420) - 0.125(240)(540)$$

$$+ 17.5(660) - 800 = 0.02 \approx 0$$ **Checks**

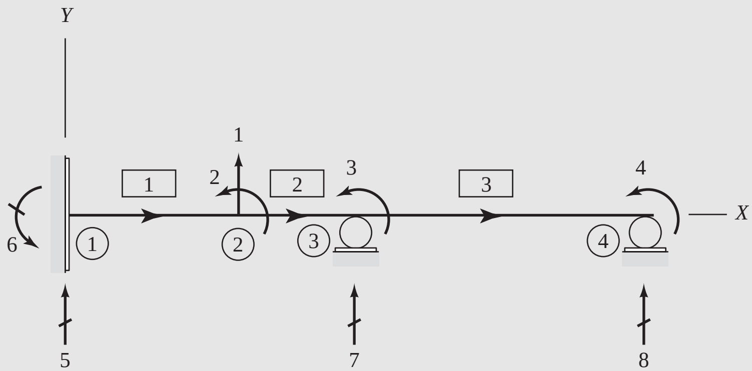
EXAMPLE 5.7 Determine the joint displacements, member end forces, and support reactions for the beam shown in Fig. 5.18(a), using the matrix stiffness method.

SOLUTION *Analytical Model:* See Fig. 5.18(b). The beam has four degrees of freedom (numbered 1 through 4) and four restrained coordinates (numbered 5 through 8).



E = constant
$E = 28$ GPa
$I = 5.8(10^9)$ mm^4

(a) Beam

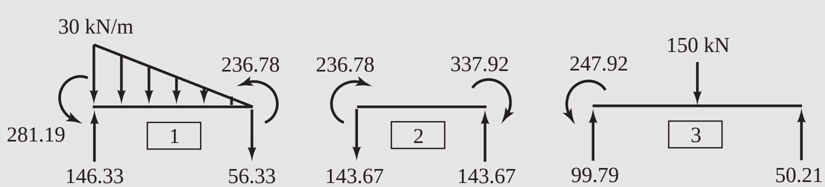


(b) Analytical Model

$$\mathbf{S} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} (13{,}533 + 30{,}450) & (-40{,}600 + 60{,}900) & 60{,}900 & 0 \\ (-40{,}600 + 60{,}900) & (162{,}400 + 162{,}400) & 81{,}200 & 0 \\ 60{,}900 & 81{,}200 & (162{,}400 + 64{,}960) & 32{,}480 \\ 0 & 0 & 32{,}480 & 64{,}960 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 43{,}983 & 20{,}300 & 60{,}900 & 0 \\ 20{,}300 & 324{,}800 & 81{,}200 & 0 \\ 60{,}900 & 81{,}200 & 227{,}360 & 32{,}480 \\ 0 & 0 & 32{,}480 & 64{,}960 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\mathbf{P}_f = \left[\begin{array}{c} 27 \\ -36 \\ 187.5 \\ -187.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

(c) Structure Stiffness Matrix and Fixed-Joint Force Vector

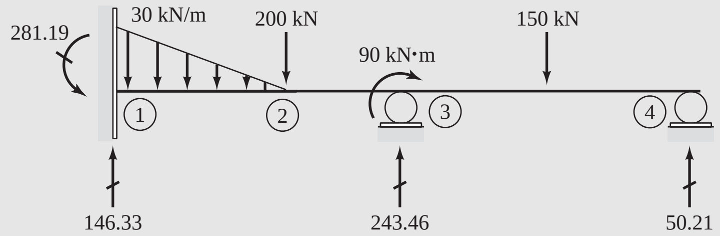


(d) Member End Forces

Fig. 5.18

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} 146.33 \\ 281.19 \\ 143.67 + 99.79 \\ 50.21 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} = \begin{bmatrix} 146.33 \text{ kN} \\ 281.19 \text{ kN} \cdot \text{m} \\ 243.46 \text{ kN} \\ 50.21 \text{ kN} \end{bmatrix}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 5.18 (*continued*)

Structure Stiffness Matrix and Fixed-Joint Force Vector:

Member 1 By substituting $E = 28(10^6)$ kN/m^2, $I = 8{,}700(10^{-6})$ m^4, and $L = 6$ m into Eq. (5.53), we write

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \left[\begin{array}{cc:cc} 13{,}533 & 40{,}600 & -13{,}533 & 40{,}600 \\ 40{,}600 & 162{,}400 & -40{,}600 & 81{,}200 \\ \hdashline -13{,}533 & -40{,}600 & 13{,}533 & -40{,}600 \\ 40{,}600 & 81{,}200 & -40{,}600 & 162{,}400 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array}$$

Using the fixed-end force expressions given inside the front cover, we obtain $FS_b = 63$ kN, $FM_b = 54$ kN $\cdot$ m, $FS_e = 27$ kN, and $FM_e = -36$ kN $\cdot$ m. Thus,

$$\mathbf{Q}_{f1} = \left[\begin{array}{c} 63 \\ 54 \\ \hdashline 27 \\ -36 \end{array}\right] \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array}$$

Using the code numbers (5, 6, 1, 2) for member 1, we store the pertinent elements of $\mathbf{k}_1$ and $\mathbf{Q}_{f1}$ in their proper positions in the $\mathbf{S}$ matrix and the $\mathbf{P}_f$ vector, respectively, as shown in Fig. 5.18(c).

Member 2 $E = 28(10^6)$ kN/m^2, $I = 5{,}800(10^{-6})$ m^4, and $L = 4$ m. Thus,

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 3 \end{array} \\ \left[\begin{array}{cc:c:c} 30{,}450 & 60{,}900 & -30{,}450 & 60{,}900 \\ 60{,}900 & 162{,}400 & -60{,}900 & 81{,}200 \\ \hdashline -30{,}450 & -60{,}900 & 30{,}450 & -60{,}900 \\ \hdashline 60{,}900 & 81{,}200 & -60{,}900 & 162{,}400 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 3 \end{array}$$

As this member is not subjected to any loads,

$$\mathbf{Q}_{f2} = \mathbf{0}$$

Using the member code numbers 1, 2, 7, 3, the relevant elements of $\mathbf{k}_2$ are stored in $\mathbf{S}$ in Fig. 5.18(c).

Member 3 $E = 28(10^6)$ kN/m^2, $I = 5{,}800(10^{-6})$ m^4, and $L = 10$ m.

$$\mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 7 & 3 & 8 & 4 \end{array} \\ \left[\begin{array}{cccc} 1{,}948.8 & 9{,}744 & -1{,}948.8 & 9{,}744 \\ 9{,}744 & 64{,}960 & -9{,}744 & 32{,}480 \\ -1{,}948.8 & -9{,}744 & 1{,}948.8 & -9{,}744 \\ 9{,}744 & 32{,}480 & -9{,}744 & 64{,}960 \end{array}\right] \end{array} \begin{array}{c} \\ 7 \\ 3 \\ 8 \\ 4 \end{array}$$

The fixed-end forces are determined to be $FS_b = 75$ kN, $FM_b = 187.5$ kN · m, $FS_e = 75$ kN, and $FM_e = -187.5$ kN · m. Thus,

$$\mathbf{Q}_{f3} = \begin{bmatrix} 75 \\ 187.5 \\ 75 \\ -187.5 \end{bmatrix} \begin{array}{c} 7 \\ 3 \\ 8 \\ 4 \end{array}$$

The relevant elements of $\mathbf{k}_3$ and $\mathbf{Q}_{f3}$ are stored in $\mathbf{S}$ and $\mathbf{P}_f$ respectively, using the member code numbers 7, 3, 8, 4. The completed structure stiffness matrix $\mathbf{S}$ and structure fixed-joint force vector $\mathbf{P}_f$ are given in Fig. 5.18(c).

Joint Load Vector: By comparing Figs. 5.18(a) and (b), we realize that $P_1 = -200$ kN, $P_2 = 0$, $P_3 = -90$ kN · m, and $P_4 = 0$. Thus, the joint load vector can be expressed as

$$\mathbf{P} = \begin{bmatrix} -200 \\ 0 \\ -90 \\ 0 \end{bmatrix}$$

Joint Displacements: The stiffness relations for the entire beam can be expressed as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$$

By substituting the numerical values of $\mathbf{P}$, $\mathbf{P}_f$, and $\mathbf{S}$, we obtain

$$\begin{bmatrix} -200 \\ 0 \\ -90 \\ 0 \end{bmatrix} - \begin{bmatrix} 27 \\ -36 \\ 187.5 \\ -187.5 \end{bmatrix} = \begin{bmatrix} 43{,}983 & 20{,}300 & 60{,}900 & 0 \\ 20{,}300 & 324{,}800 & 81{,}200 & 0 \\ 60{,}900 & 81{,}200 & 227{,}360 & 32{,}480 \\ 0 & 0 & 32{,}480 & 64{,}960 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

or

$$\begin{bmatrix} -227 \\ 36 \\ -277.5 \\ 187.5 \end{bmatrix} = \begin{bmatrix} 43{,}983 & 20{,}300 & 60{,}900 & 0 \\ 20{,}300 & 324{,}800 & 81{,}200 & 0 \\ 60{,}900 & 81{,}200 & 227{,}360 & 32{,}480 \\ 0 & 0 & 32{,}480 & 64{,}960 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

By solving the foregoing system of simultaneous equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} -4.4729 \text{ m} \\ 0.56143 \text{ rad} \\ -0.68415 \text{ rad} \\ 3.2285 \text{ rad} \end{bmatrix} \times 10^{-3}$$ **Ans**

Back substitution of the foregoing numerical values of $\mathbf{d}$ into the structure stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$ indicates that the solution of the simultaneous equations has indeed been carried out correctly.

Member End Displacements and End Forces:

Member 1

$$\mathbf{u}_1 = \begin{array}{c} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4.4729 \\ 0.56143 \end{bmatrix} \times 10^{-3}$$

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 146.33 \text{ kN} \\ 281.19 \text{ kN} \cdot \text{m} \\ -56.33 \text{ kN} \\ 236.78 \text{ kN} \cdot \text{m} \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array}$$ **Ans**

Member 2

$$\mathbf{u}_2 = \begin{bmatrix} -4.4729 \\ 0.56143 \\ 0 \\ -0.68415 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 7 \\ 3 \end{array} \times 10^{-3}; \quad \mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} -143.67 \text{ kN} \\ -236.78 \text{ kN} \cdot \text{m} \\ 143.67 \text{ kN} \\ -337.92 \text{ kN} \cdot \text{m} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 7 \\ 3 \end{array}$$ **Ans**

Member 3

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ -0.68415 \\ 0 \\ 3.2285 \end{bmatrix} \begin{array}{c} 7 \\ 3 \\ 8 \\ 4 \end{array} \times 10^{-3}; \quad \mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 + \mathbf{Q}_{f3} = \begin{bmatrix} 99.79 \text{ kN} \\ 247.92 \text{ kN} \cdot \text{m} \\ 50.21 \text{ kN} \\ 0 \end{bmatrix} \begin{array}{c} 7 \\ 3 \\ 8 \\ 4 \end{array}$$ **Ans**

The member end forces are shown in Fig. 5.18(d).

Support Reactions: The reaction vector **R**, as assembled from the appropriate elements of the member end-force vectors, is given in Fig. 5.18(e). Also, Fig. 5.18(f) depicts the support reactions on a line diagram of the structure. **Ans**

Equilibrium Check: The equilibrium equations check.

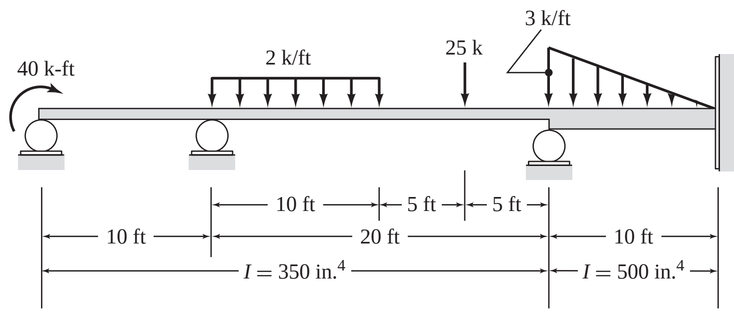
5.8 COMPUTER PROGRAM

In this section, we consider computer implementation of the procedure for the analysis of beams presented in this chapter. Because of the similarity in the methods for the analysis of beams and plane trusses, the overall format of the program for beam analysis remains the same as that for the analysis of plane trusses developed in Chapter 4. Therefore, many parts of the plane truss program can be copied and used, without any modifications, in the program for beam analysis. In the following, we discuss the development of an input module and consider programming of the analysis steps for beams.

Input Module

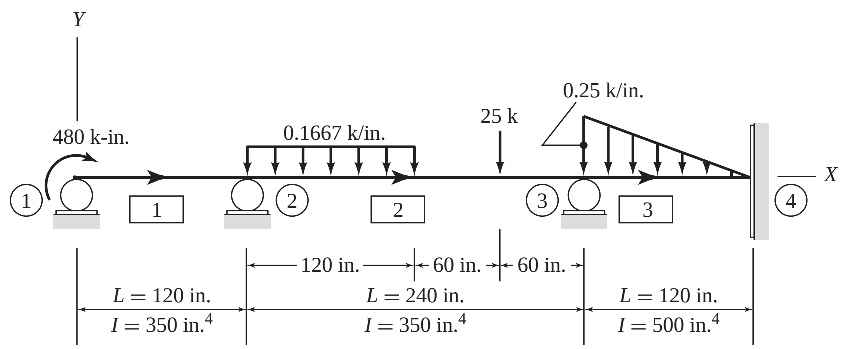
Joint Data The joint data consists of (a) the total number of joints (NJ) of the beam, and (b) the global X coordinate of each joint. (Recall that the global

XY coordinate system must be oriented so that the X axis coincides with the beam's centroidal axis.) The joint coordinates are stored in computer memory in the form of a *joint coordinate vector* **COORD** of the order $NJ \times 1$. Consider, for example, the continuous beam shown in Fig. 5.19(a), with its analytical model given in Fig. 5.19(b). As the beam has four joints, its **COORD** vector has four rows, with the X coordinate of a joint i stored in the ith row, as shown in Fig. 5.19(c). A flowchart for programming the reading



$E = 29{,}000$ ksi

(a) Actual Beam



$E = 29{,}000$ ksi

(b) Analytical Model (Units: Kips, Inches)

$$\mathbf{COORD} = \begin{array}{c} X \text{ Coordinate} \\ \begin{bmatrix} 0 \\ 120 \\ 360 \\ 480 \end{bmatrix} \\ NJ \times 1 \end{array} \begin{array}{l} \\ \leftarrow \text{Joint 1} \\ \leftarrow \text{Joint 2} \\ \leftarrow \text{Joint 3} \\ \leftarrow \text{Joint 4} \\ \\ \end{array}$$

(c) Joint Coordinate Vector

Joint number; Restraint in Y direction (0 = free, 1 = restrained); Rotational restraint (0 = free, 1 = restrained)

$$\mathbf{MSUP} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}_{NS \times (NCJT + 1)}$$

(d) Support Data Matrix

Fig. 5.19

$$\mathbf{EM} = [29000] \leftarrow \text{Material no. 1}$$

$$NMP \times 1$$

(e) Elastic Modulus Vector

$$\mathbf{CP} = \begin{array}{c} \text{Moment of inertia} \\ \begin{bmatrix} 350 \\ 500 \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow \text{Cross-section type no. 1} \\ \leftarrow \text{Cross-section type no. 2} \end{array}$$

$$NCP \times 1$$

(f) Cross-Sectional Property Vector

Beginning joint
End joint
Material no.
Cross-section type no.

$$\mathbf{MPRP} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix} \begin{array}{l} \leftarrow \text{Member 1} \\ \leftarrow \text{Member 2} \\ \leftarrow \text{Member 3} \end{array}$$

$$NM \times 4$$

(g) Member Data Matrix

$$\mathbf{JP} = \begin{array}{c} \text{Joint number} \\ \begin{bmatrix} 1 \end{bmatrix} \end{array} \qquad \mathbf{PJ} = \begin{array}{cc} \text{Force in } Y \text{ direction} & \text{Moment} \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & -480 \end{bmatrix}} \end{array}$$

$$NJL \times 1 \qquad NJL \times NCJT$$

(h) Joint Load Data Matrices

Member number
Load type number

$$\mathbf{MP} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{PM} = \begin{bmatrix} 0.1667 & 0 & 0 & 120 \\ 25 & 0 & 180 & 0 \\ 0.25 & 0 & 0 & 0 \end{bmatrix}$$

$$NML \times 2 \qquad NML \times 4$$

Column 1: W, M, w or w_1

Column 2: w_2 (if load type $= 4$); 0 (otherwise)

Column 3: l_1

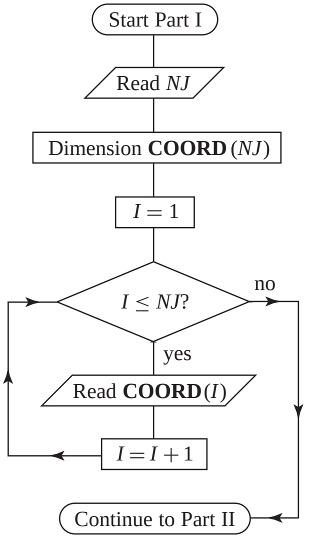
Column 4: l_2 (if load type $= 3$ or 4); 0 (otherwise)

(i) Member Load Data Matrices

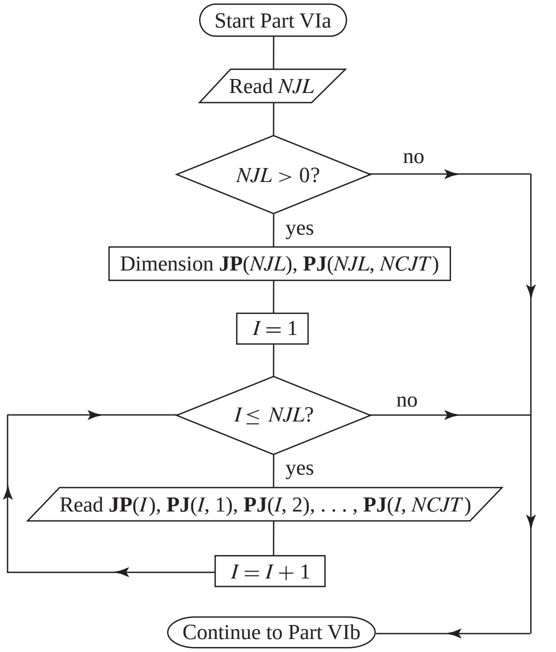
Fig. 5.19 (*continued*)

and storing of the joint data for beams is given in Fig. 5.20(a), and an example of the input data file for the beam of Fig. 5.19(b) is shown in Fig. 5.21 on page 229. Note that the first line of this data file contains the total number of joints of the beam (i.e., 4), with the next four lines containing the X coordinates of joints 1 through 4, respectively.

Support Data The support data consists of (a) the number of joints that are attached to supports (NS), and (b) the joint number, and the restraint code, for each support joint. Since the number of structure coordinates per joint of a beam equals 2 (i.e., $NCJT = 2$), a two-digit code is used to specify the restraints at a support joint. The first digit of the code represents the restraint condition at the joint in the global Y direction; it equals 0 if the joint is free to translate in the Y direction, or it equals 1 if the joint is restrained in the Y direction. Similarly, the second digit of the code represents the rotational restraint condition at the joint; a 0 indicates that the joint is free to rotate, and a 1 indicates that it is restrained against rotation. The restraint codes for the various types of supports for beams are given in Fig. 5.22 on page 230. Since the joints 1, 2, and 3 of the example beam (Fig. 5.19(b)) are attached to roller supports, their restraint codes are 1,0, indicating that these joints are restrained from translating in the Y direction, but are free to rotate. Similarly, the restraint code for joint 4, which is attached to a fixed support, is 1,1, because this joint can neither translate nor rotate. The support data for beams is stored in computer

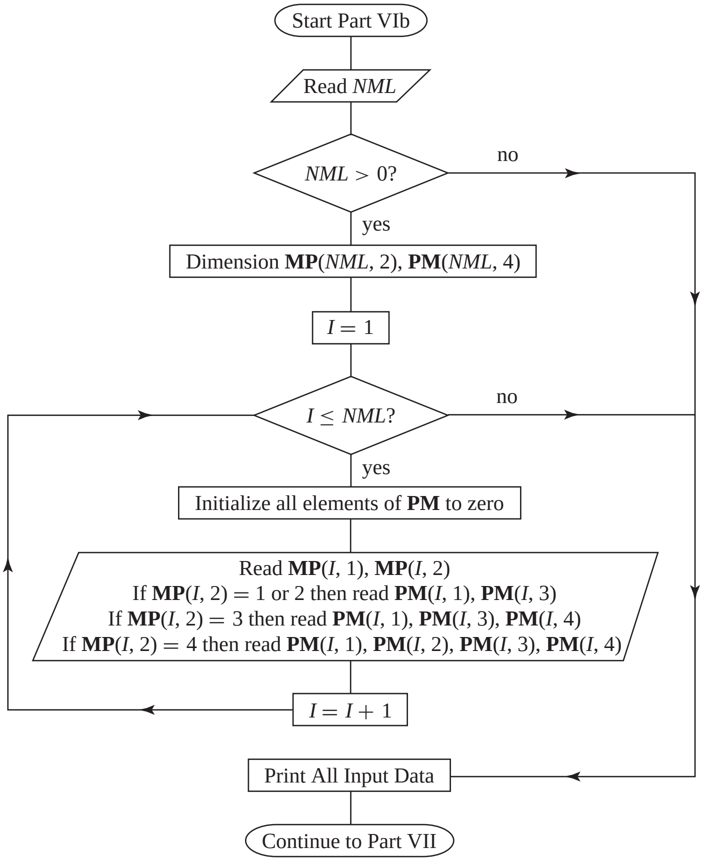


(a) Flowchart for Reading and Storing Joint Data for Beams



(b) Flowchart for Reading and Storing Joint Load Data

Fig. 5.20



(c) Flowchart for Reading and Storing Member Load Data

Fig. 5.20 (*continued*)

memory as an integer matrix **MSUP** of order $NS \times (NCJT + 1)$, as discussed in Chapter 4 for the case of plane trusses. Thus, for the beam of Fig. 5.19(b), which has four support joints, **MSUP** is a 4×3 matrix, as shown in Fig. 5.19(d). The computer code developed previously for Part II of the plane truss analysis program (see flowchart in Fig. 4.3(b)) can be copied and used in the beam analysis program for reading the support data, and storing it in computer memory. An example of how the support data for beams may appear in an input data file is given in Fig. 5.21.

Material Property Data The procedure for inputting material property data for beams is identical to that for the case of plane trusses, as described in Chapter 4. Thus, the computer code written for Part III of the plane truss program (see flowchart in Fig. 4.3(c)) can be used in the beam analysis program for inputting the material property data. The elastic modulus vector for the example beam of Fig. 5.19(b) is given in Fig. 5.19(e); Fig. 5.21 illustrates how this type of data may appear in an input data file.

Data	
4 0 120 360 480	Joint data
4 1, 1, 0 2, 1, 0 3, 1, 0 4, 1, 1	Support data
1 29000	Material property data
2 350 500	Cross-sectional property data
3 1, 2, 1, 1 2, 3, 1, 1 3, 4, 1, 2	Member data
1 1, 0, −480	Joint load data
3 2, 3, 0.1667, 0, 120 2, 1, 25, 180 3, 4, 0.25, 0, 0, 0	Member load data

Fig. 5.21 *An Example of an Input Data File*

Cross-Sectional Property Data The cross-sectional property data consists of (a) the number of different cross-section types used for the members of the beam (NCP), and (b) the moment of inertia (I) for each cross-section type. The moments of inertia are stored in computer memory in a cross-sectional property vector **CP** of order $NCP \times 1$, with the moment of inertia of cross-section i stored in the ith row of **CP**. For example, two types of member cross-sections are used for the beam of Fig. 5.19(b). We arbitrarily assign the numbers 1 and 2 to the cross-sections with the moments of inertia of 350 and 500 in.^{4}, respectively. Thus, the **CP** vector consists of two rows, with the moments of inertia of cross-section types 1 and 2 stored in rows 1 and 2, respectively, as shown in Fig. 5.19(f). The computer code developed in Part IV of the plane truss program (see flowchart in Fig. 4.3(d)) can be used for inputting cross-sectional property data for beams. An example of how this type of data may appear in an input data file is given in Fig. 5.21.

Member Data As in the case of plane trusses, the member data for beams consists of (a) the total number of members (NM) of the beam, and (b) for each member: the beginning joint number, the end joint number, the material number, and the cross-section type number. This member data is organized in

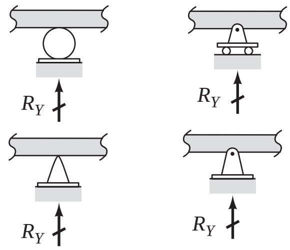
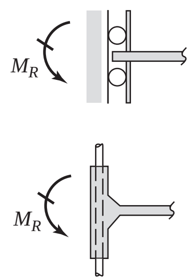
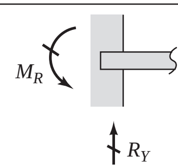
Type of Support		Restraint Code
Free joint (no support)		0, 0
Roller or hinge		1, 0
Support which prevents rotation, but not translation in *Y* direction; for example, a collar on a smooth shaft		0, 1
Fixed		1, 1

Fig. 5.22 *Restraint Codes for Beams*

computer memory in the form of an integer member data matrix, **MPRP**, of order $NM \times 4$, as discussed in Chapter 4. The computer code for Part V of the plane truss program (see flowchart in Fig. 4.3(e)) can be used for inputting member data for beams. The **MPRP** matrix for the example beam is shown in Fig. 5.19(g), with the corresponding input data file given in Fig. 5.21.

Joint Load Data The joint load data involves (a) the number of joints that are subjected to external loads (NJL), and (b) for each loaded joint, the joint number, and the magnitudes of the force in the global Y direction and the couple. As in the case of plane trusses (Chapter 4), the numbers of the loaded joints are stored in an integer vector **JP** of order $NJL \times 1$, with the corresponding force and couple being stored in the first and second columns, respectively, of a real matrix **PJ** of order $NJL \times NCJT$ (with $NCJT = 2$ for beams). The joint load matrices for the example beam of Fig. 5.19(b) are shown in Fig. 5.19(h). A flowchart for programming the input of joint load data is given in Fig. 5.20(b); and Fig. 5.21 shows the joint load data for the example beam in an input file that can be read by the program.

Member Load Data The member load data consists of (a) the total number of loads applied to the members of the beam (*NML*), and (b) for each member load: the member number, the load type, and the magnitude(s) and location(s) of the load. The four common types of member loads for beams are depicted as load types 1 through 4 inside the front cover of this book, along with the expressions for the corresponding member fixed-end forces. The total number of member loads, *NML*, represents the sum of the different loads acting on the individual members of the structure. From Fig. 5.19(b), we can see that member 1 of the example beam is not subjected to any loads, whereas member 2 is subjected to two loads—namely, a uniformly distributed load (type 3) and a concentrated load (type 1). Also, member 3 of the beam is subjected to one load—a linearly varying load (type 4). Thus, the beam is subjected to a total of three member loads; that is, $NML = 3$. For each member load, the member number and the load type are stored in the first and second columns, respectively, of an integer matrix **MP** of order $NML \times 2$, with the corresponding load magnitude(s) and location(s) being stored in a real matrix **PM** of order $NML \times 4$. With reference to the load types depicted inside the front cover: when the load type is 1 or 2, the magnitude of W or M is stored in the first column, and the distance l_1 is stored in the third column, of the **PM** matrix, with the elements of the second and fourth columns of **PM** left blank (or set equal to 0). In the case of load type 3, the magnitude of w is stored in the first column, and distances l_1 and l_2 are stored in the third and fourth columns, respectively, of the **PM** matrix, with the second column element left blank. When the load type is 4, the magnitudes of w_1 and w_2 are stored in the first and second columns, respectively, and the distances l_1 and l_2 are stored in the third and fourth columns, respectively, of the **PM** matrix. For example, as the beam of Fig. 5.19(b) is subjected to three member loads, its *member load-data matrices*, **MP** and **PM**, are of the orders 3×2 and 3×4, respectively, as shown in Fig. 5.19(i). The first rows of these matrices contain information about the first member load, which is arbitrarily chosen to be the uniformly distributed load acting on member 2. Thus, the first row of **MP** contains the member number, 2, and the load type, 3, stored in the first and second columns; and the first row of **PM** contains $w = 0.1667$ in column 1, 0 in column 2, $l_1 = 0$ in column 3, and $l_2 = 120$ in column 4. The information about the second member load—the concentrated load acting on member 2—is then stored in the second rows of **MP** and **PM**; with the member number 2 and the load type 1 stored in the first and second columns of **MP**, and $W = 25$ and $l_1 = 180$ stored in the first and third columns of **PM**. Similarly, the third member load—the linearly varying load on member 3—is defined in the third rows of **MP** and **PM**; with the member number 3 and the load type 4 stored in the first and second columns of **MP**, and $w_1 = 0.25$, $w_2 = 0$, $l_1 = 0$, and $l_2 = 0$ stored in columns 1 through 4, respectively, of **PM**, as shown in Fig. 5.19(i). It is important to realize that the member fixed-end force expressions given inside the front cover are based on the sign convention that the member loads W, w, w_1, and w_2 are positive when acting downward (i.e., in the negative direction of the member y axis), and the couple M is positive when clockwise. A flowchart for programming the input of member load data is given in Fig. 5.20(c); Fig. 5.21 shows the member load data in an input file that can be read by the program.

An example of a computer printout of the input data for the beam of Fig. 5.19 is given in Fig. 5.23.

```
**********************************
*        Computer Software       *
*              for               *
*  MATRIX ANALYSIS OF STRUCTURES *
*         Second Edition         *
*               by               *
*         Aslam Kassimali        *
**********************************

          ========================
          General Structural Data
          ========================

Project Title: Figure 5-19
Structure Type: Beam
Number of Joints: 4
Number of Members: 3
Number of Material Property Sets (E): 1
Number of Cross-Sectional Property Sets: 2

             =================
             Joint Coordinates
             =================

        Joint No.        X Coordinate
            1             0.0000E+00
            2             1.2000E+02
            3             3.6000E+02
            4             4.8000E+02

                 ========
                 Supports
                 ========

  Joint No.    Y Restraint    Rotational Restraint
     1             Yes                 No
     2             Yes                 No
     3             Yes                 No
     4             Yes                 Yes
```

Fig. 5.23 *A Sample Printout of Input Data*

```
Material Properties

Material      Modulus of         Co-efficient of
  No.       Elasticity (E)     Thermal Expansion
  1          2.9000E+04           0.0000E+00

Cross-Sectional Properties

Property No.        Moment of Inertia
     1                 3.5000E+02
     2                 5.0000E+02

Member Data

Member  Beginning   End   Material  Cross-Sectional
  No.     Joint    Joint    No.      Property No.
  1         1        2       1            1
  2         2        3       1            1
  3         3        4       1            2

Joint Loads

Joint No.        Y Force         Moment
    1          0.0000E+00     -4.8000E+02

Member Loads

                 Load
               Magnitude
               (W or M)
                  or          Load
Member  Load   Intensity   Intensity  Distance  Distance
 No.    Type   (w or w1)      w2         l1        l2
  2    Conc.   2.500E+1      ---      1.80E+2     ----
  2  Uniform   1.667E-1      ---      0.00E+0   1.20E+2
  3   Linear   2.500E-1   0.000E+0    0.00E+0   0.00E+0

************** End of Input Data **************
```

Fig. 5.23 (*continued*)

Analysis Module

Assignment of Structure Coordinate Numbers The process of programming the determination, for beams, of the number of degrees of freedom, *NDOF*, and the formation of the structure coordinate number vector, **NSC**, is identical to that for plane trusses. Thus, Parts VII and VIII of the plane truss program (as described by flowcharts in Figs. 4.8(a) and (b)) can be copied and used without any modifications in the program for the analysis of beams.

Generation of the Structure Stiffness Matrix and the Equivalent Joint Load Vector A flowchart for programming this part of our computer program is presented in Fig. 5.24. As the flowchart indicates, this part of the program

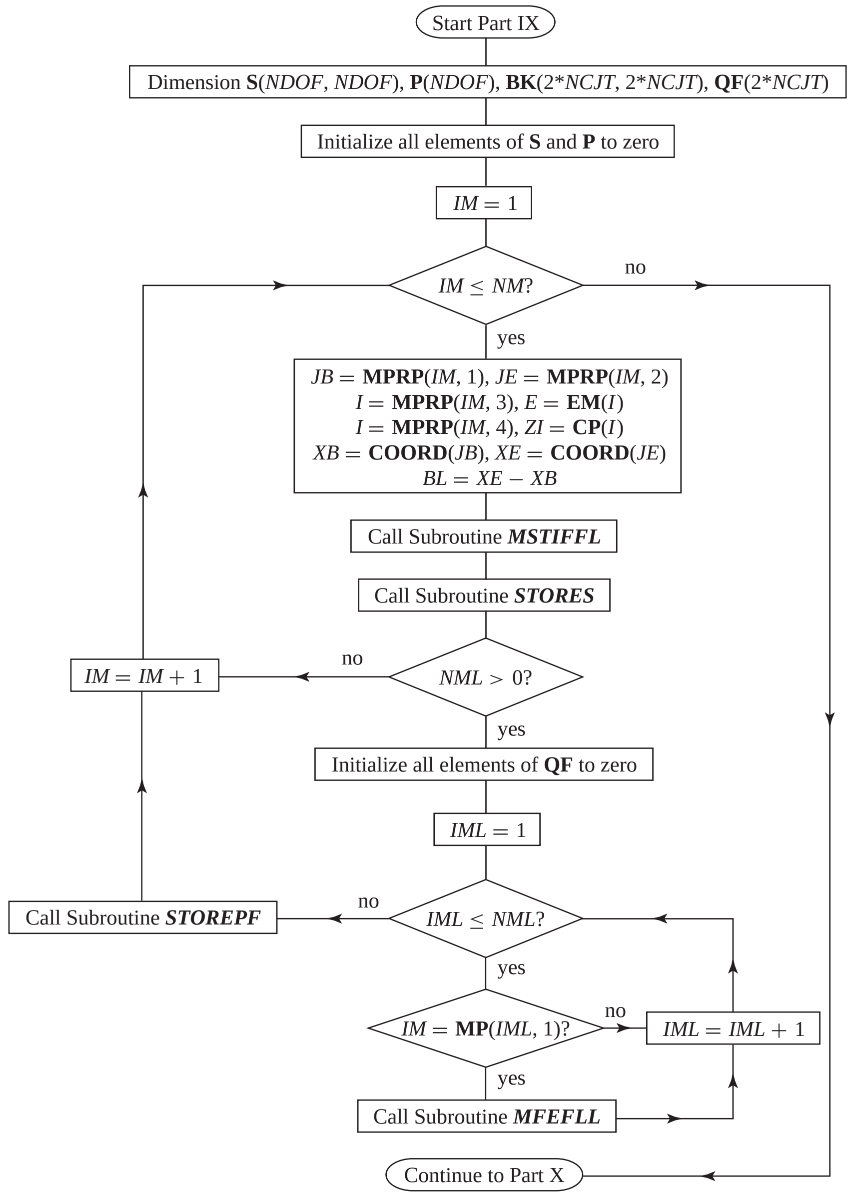


Fig. 5.24 *Flowchart for Generating Structure Stiffness Matrix and Equivalent Joint Load Vector for Beams*

begins by initializing all the elements of the **S** matrix, and a structure load vector **P** of order *NDOF* $\times$ 1, to 0. The assembly of the structure stiffness matrix, and the equivalent joint load vector due to member loads, is then carried out by using a *Do Loop*, in which the following operations are performed for each member of the beam: (a) For the member under consideration, *IM*, the program reads the modulus of elasticity *E* and the moment of inertia *ZI*, and calculates the member length *BL*. (b) Next, the program calls the subroutine ***MSTIFFL*** to form the member stiffness matrix **BK** (= **k**). As the flowchart in Fig. 5.25 indicates, this subroutine simply calculates the values of the various elements of the **BK** matrix, in accordance with Eq. (5.53). (c) The program then calls the subroutine ***STORES*** to store the pertinent elements of **BK** in their proper positions in the structure stiffness matrix **S**. A flowchart of this subroutine is given in Fig. 5.26 on the next page. By comparing the flowchart of the present ***STORES*** subroutine (Fig. 5.26) with that of the ***STORES*** subroutine of the plane truss program in Fig. 4.11, we can see that the two subroutines are identical, except that the present subroutine stores the elements of the member local stiffness matrix **BK** (instead of the global stiffness matrix **GK**) in **S**. (d) Returning our attention to Fig. 5.24, we can see that after the ***STORES*** subroutine has been executed, the program checks the first column of the member load data matrix **MP** to determine whether the member under consideration, *IM*, is subjected to any loads. If the member is subjected to loads, then the subroutine ***MFEFLL*** is called to form the member fixed-end force vector **QF** (= $\mathbf{Q}_f$). As the flowchart in Fig. 5.27 on page 237 indicates, this subroutine calculates the values of the member fixed-end forces, for load types 1 through 4, using the equations given inside

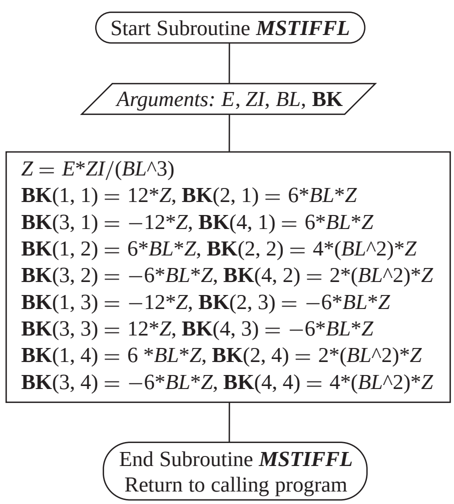


Fig. 5.25 *Flowchart of Subroutine **MSTIFFL** for Determining Member Stiffness Matrix for Beams*

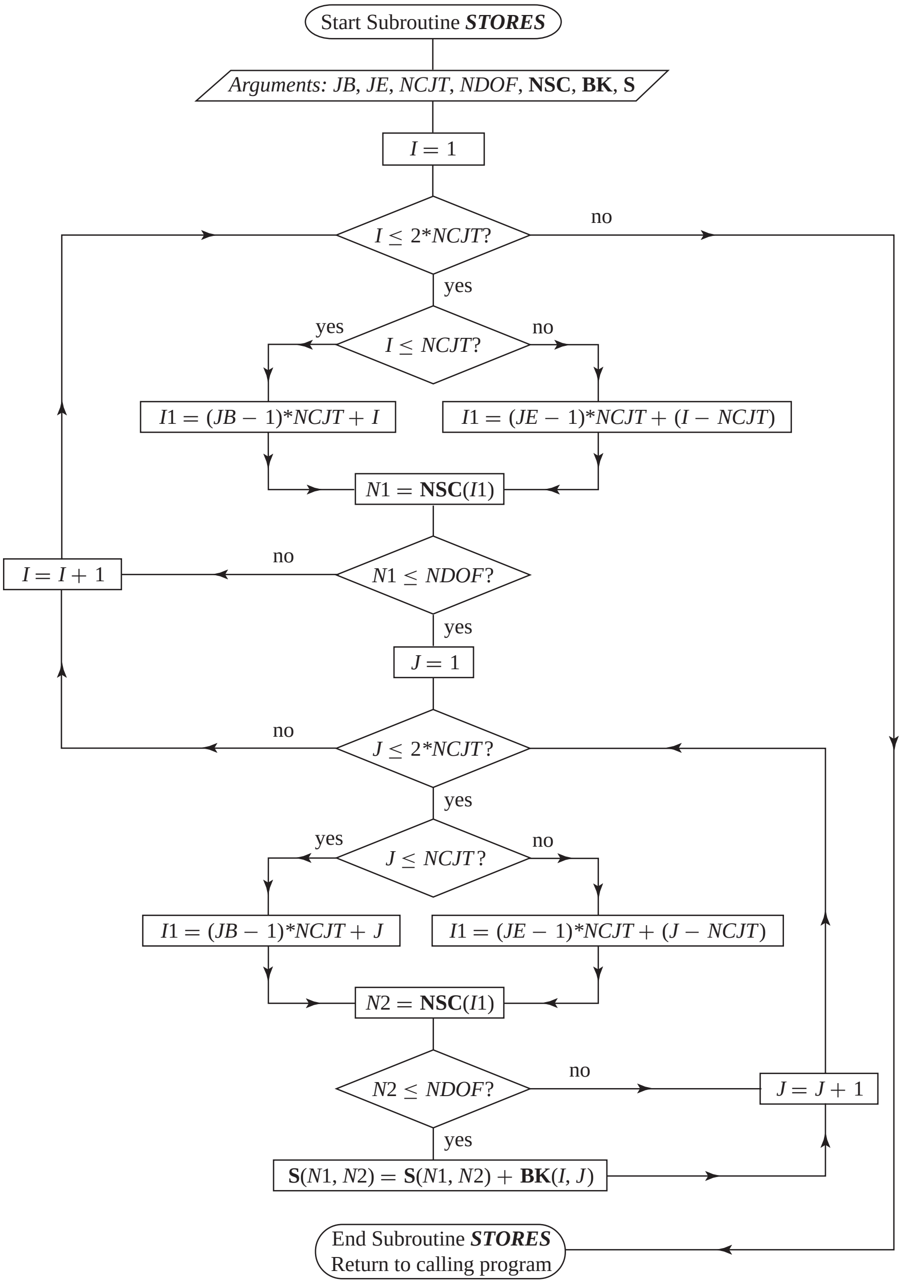


Fig. 5.26 *Flowchart of Subroutine **STORES** for Storing Member Stiffness Matrix in Structure Stiffness Matrix for Beams*

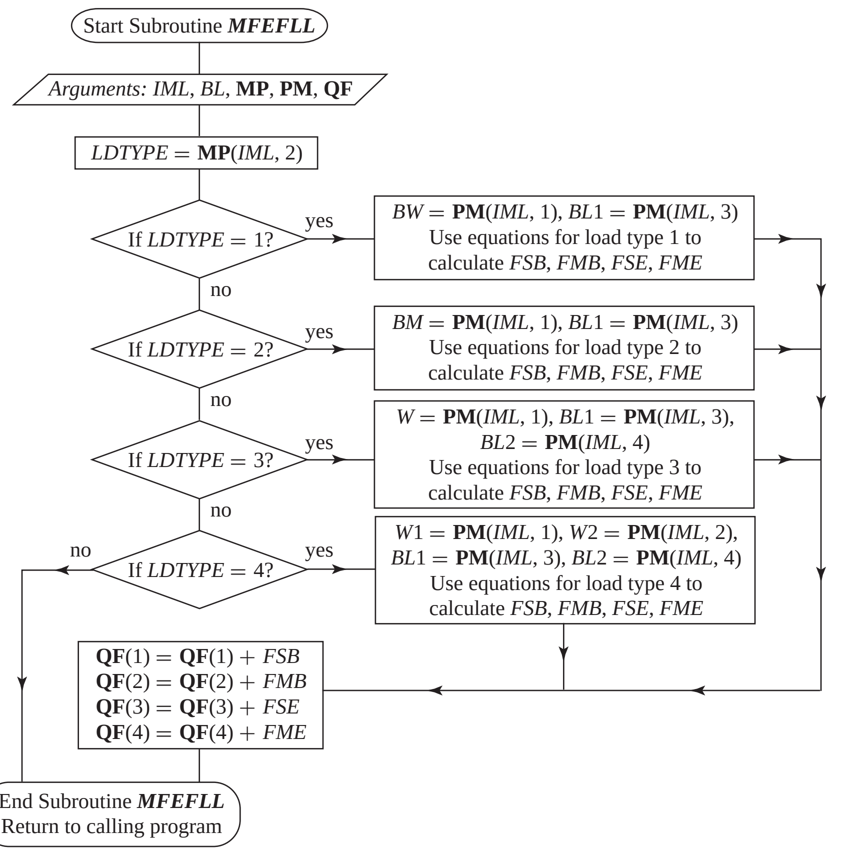


Fig. 5.27 *Flowchart of Subroutine* ***MFEFLL*** *for Determining Member Fixed-End Force Vector for Beams*

the front cover. (e) The program then calls the subroutine ***STOREPF*** to store the *negative* values of the pertinent elements of **QF** in their proper positions in the load vector **P**. A flowchart of this subroutine, which essentially consists of a *Do Loop*, is given in Fig. 5.28 on the next page. As shown in this flowchart, the subroutine reads, in order, for each of the member fixed-end forces, QF_I, the number of the corresponding structure coordinate, $N1$, from the **NSC** vector. If $N1$ is less than or equal to $NDOF$, then the value of QF_I is subtracted from the $N1$th row of the load vector **P**. From Fig. 5.24, we can see that when the foregoing operations have been performed for each member of the beam, the structure stiffness matrix **S** is completed, and the structure load vector **P** equals the equivalent joint load vector $\mathbf{P}_e$, or the negative of the structure fixed-joint force vector $\mathbf{P}_f$ (i.e., $\mathbf{P} = \mathbf{P}_e = -\mathbf{P}_f$).

Storage of the Joint Loads into the Structure Load Vector In this part of our computer program, the joint loads are added to the structure load vector **P**.

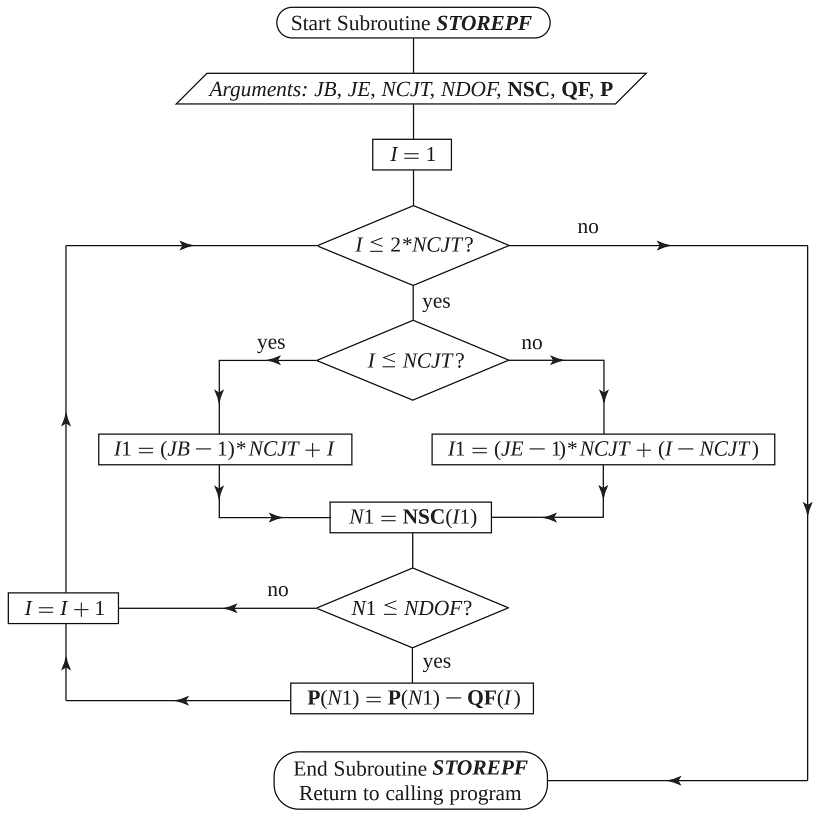


Fig. 5.28 *Flowchart of Subroutine **STOREPF** for Storing Member Fixed-End Force Vector in Structure Load Vector for Beams*

A flowchart for programming this process is shown in Fig. 5.29. This flowchart is the same as the previous flowchart (Fig. 4.12) for forming the joint load vector for plane trusses, except that the load vector **P** is not initialized to 0 in this part of the program (as it was previously), because it now contains the equivalent joint loads due to member loads.

Solution for Joint Displacements In this part, the program solves the system of simultaneous equations representing the beam's stiffness relationship, $\mathbf{Sd} = \mathbf{P}$, using Gauss–Jordan elimination. The programming of this process has been discussed previously (see the flowchart in Fig. 4.13), and it may be recalled that, upon completion of the Gauss–Jordan elimination process, the vector **P** contains the values of the joint displacements **d**. The computer code developed in Chapter 4 for Part XI of the plane truss program can be transported, without any alteration, into the beam analysis program for the calculation of joint displacements.

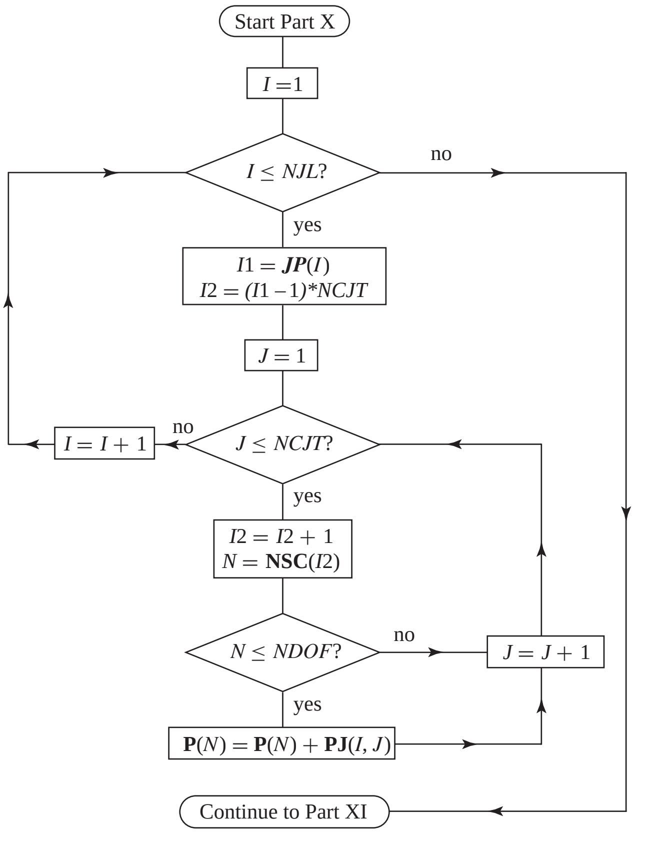


Fig. 5.29 *Flowchart for Storing Joint Loads in Structure Load Vector*

Calculation of Member Forces and Support Reactions The last part of our program involves the calculation of member forces and support reactions. A flowchart for programming this process is given in Fig. 5.30 on the next page. As this figure indicates, after initializing the reaction vector **R** to 0, the program uses a *Do Loop* to perform the following operations for each member of the beam: (a) For the member under consideration, *IM*, the program reads the modulus of elasticity *E* and the moment of inertia *ZI*, and calculates the member length *BL*. (b) Next, the program calls the subroutine ***MDISPL*** to obtain the member end displacements **U** (= **u**) from the joint displacements **P** (= **d**), using the member code numbers, as depicted by the flowchart in Fig. 5.31 on page 241. (c) The program then calls the subroutine ***MSTIFFL*** (Fig. 5.25) to form the member stiffness matrix **BK** (= **k**). (d) Returning our attention to Fig. 5.30, we can see that the program then initializes the **QF** vector to 0, and checks the first column of the member load matrix **MP** to determine if the member *IM* is subjected to any loads. If the member is subjected to loads, then the subroutine ***MFEFLL*** (Fig. 5.27) is used to form the fixed-end force vector **QF**. (e) Next, the program

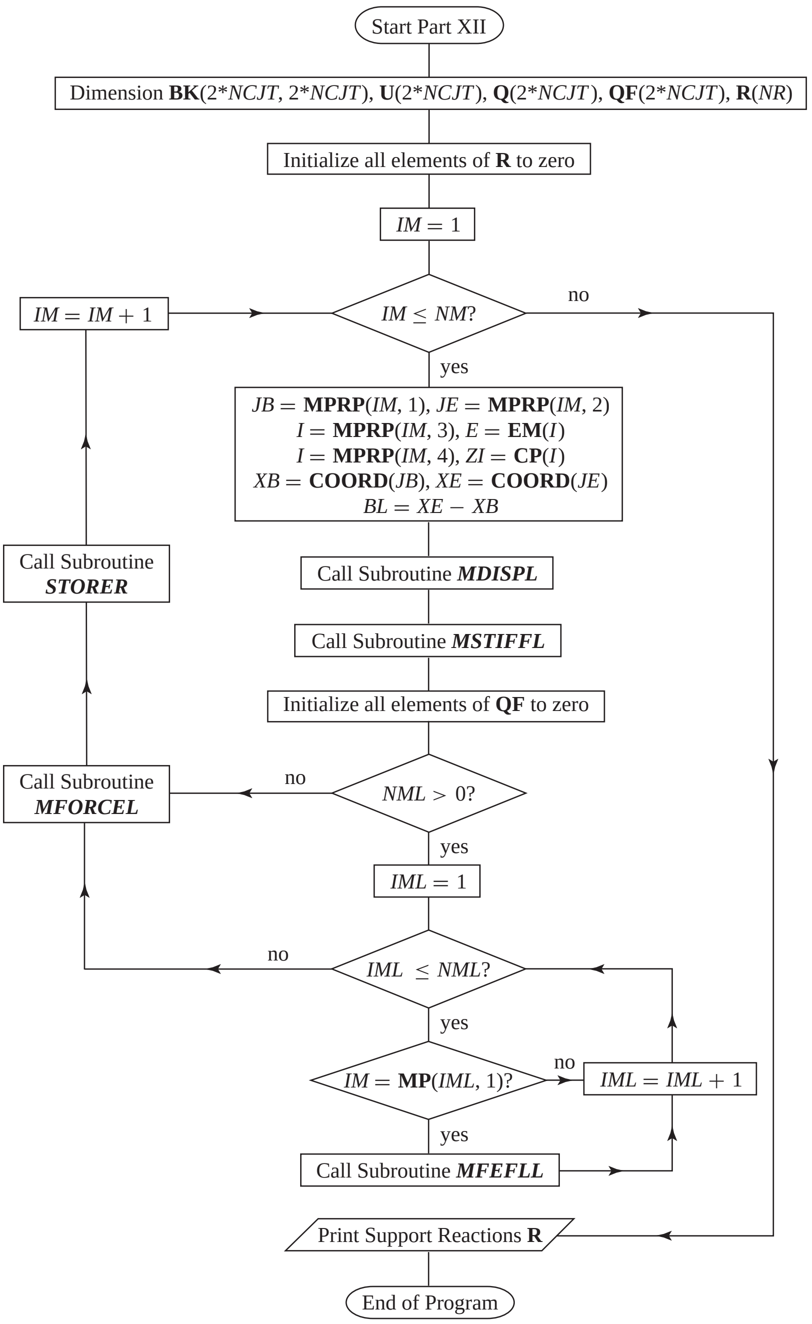


Fig. 5.30 *Flowchart for Determination of Member Forces and Support Reactions for Beams*

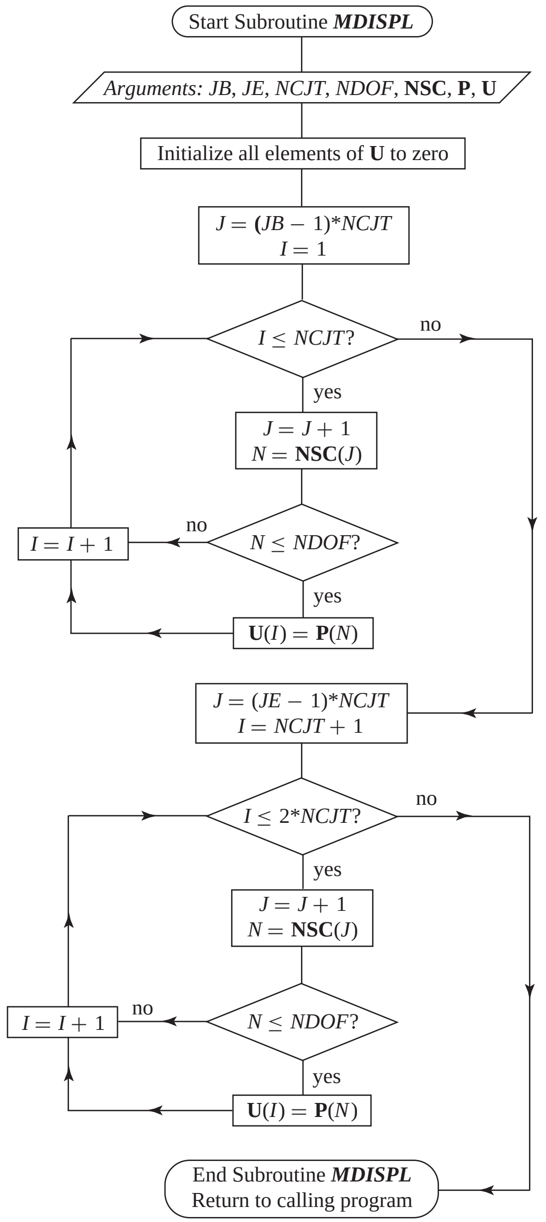


Fig. 5.31 *Flowchart of Subroutine **MDISPL** for Determining Member Displacement Vector for Beams*

calls the subroutine ***MFORCEL*** to evaluate the member end forces $\mathbf{Q}$, using the relationship $\mathbf{Q} = \mathbf{BK}\ \ \mathbf{U} + \mathbf{QF}$ (i.e., $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$, see Eq. (5.4)). A flowchart of this subroutine is shown in Fig. 5.32. (f) The program then stores the pertinent elements of $\mathbf{Q}$ in the support reaction vector $\mathbf{R}$, using the subroutine ***STORER***. The present ***STORER*** subroutine, whose flowchart is given in

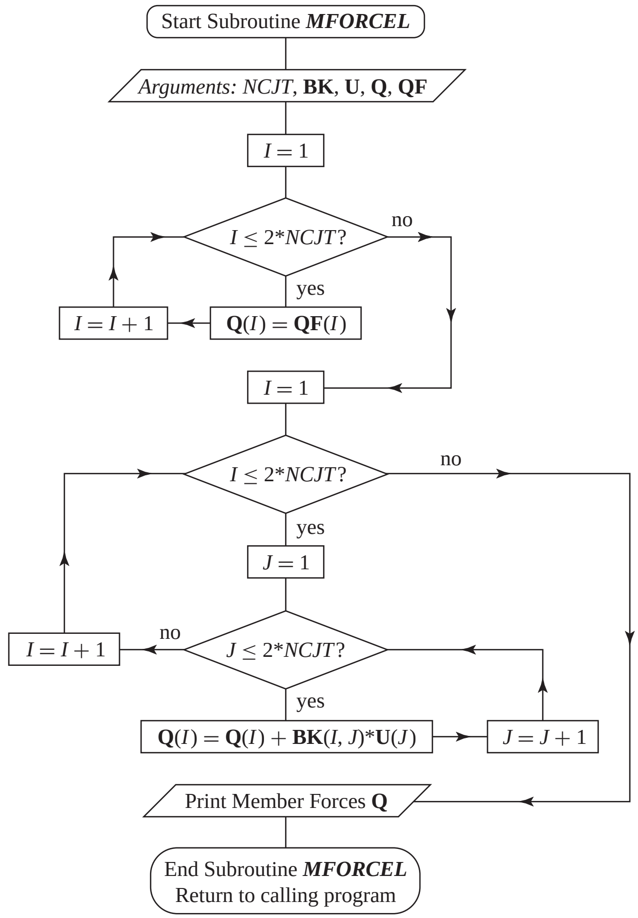


Fig. 5.32 *Flowchart of Subroutine **MFORCEL** for Determining Member Local Force Vector*

Fig. 5.33, is identical to the ***STORER*** subroutine of the plane truss program (Fig. 4.21), except that the present subroutine stores the elements of the member local force vector **Q** (instead of the global force vector **F**) in **R**. A sample computer printout, showing the results of the analysis of the example beam of Fig. 5.19, is given in Fig. 5.34.

Finally, the entire program for the analysis of beams is summarized in Table 5.1. As shown in this table, the program consists of a main program, divided into twelve parts, and seven subroutines. Brief descriptions of the various parts and subroutines of the program are also provided in Table 5.1 for quick reference. It should be noted that seven parts of the main program can be obtained from the plane truss computer program developed in Chapter 4. Furthermore, the computer code for many of the remaining parts of the main program, as well as the subroutines, can be conveniently developed by modifying the computer code written previously for the corresponding part or subroutine of the plane truss program.

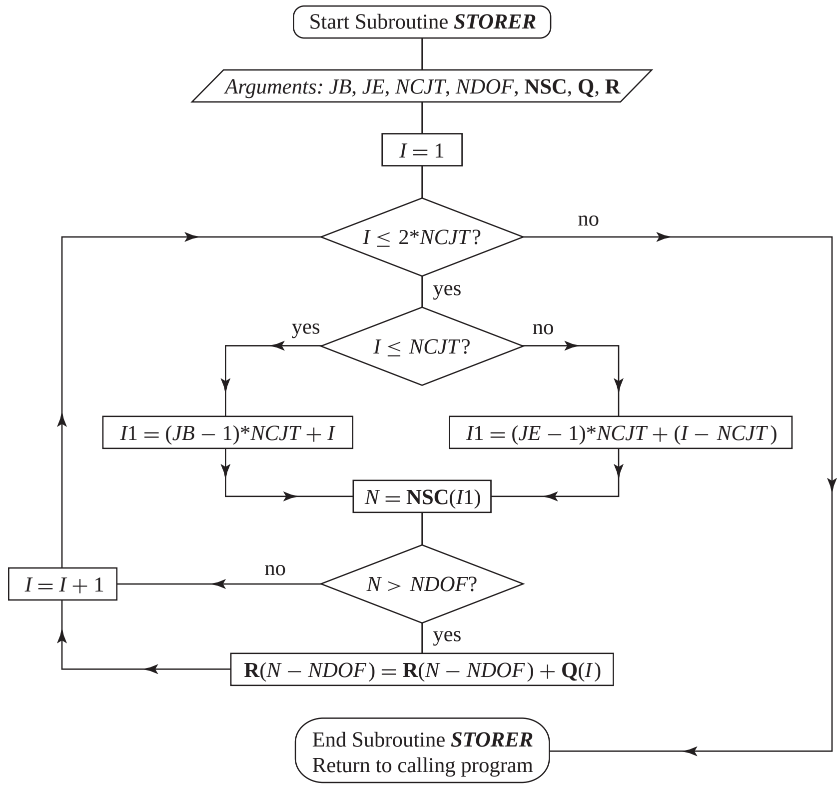


Fig. 5.33 *Flowchart of Subroutine **STORER** for Storing Member Forces in Support Reaction Vector for Beams*

```
***************************************************
*               Results of Analysis               *
***************************************************

                Joint Displacements

 Joint No.        Y Translation        Rotation (Rad)
     1             0.0000E+00          -5.5719E-04
     2             0.0000E+00          -1.7231E-03
     3             0.0000E+00           1.6238E-03
     4             0.0000E+00           0.0000E+00
```

Fig. 5.34 *A Sample Printout of Analysis Results*

```
Member End Forces in Local Coordinates

 Member       Joint     Shear Force      Moment
   1            1       -9.6435E+00   -4.8000E+02
                2        9.6435E+00   -6.7722E+02

   2            2        2.0055E+01    6.7722E+02
                3        2.4949E+01   -9.6485E+02

   3            3        2.0311E+01    9.6485E+02
                4       -5.3106E+00    2.7242E+02

                   Support Reactions

 Joint No.            Y Force             Moment
    1              -9.6435E+00         0.0000E+00
    2               2.9698E+01         0.0000E+00
    3               4.5260E+01         0.0000E+00
    4              -5.3106E+00         2.7242E+02
***************** End of Analysis *****************
```

Fig. 5.34 (*continued*)

Table 5.1 *Computer Program for Analysis of Beams*

Main program part	**Description**
I	Reads and stores joint data (Fig. 5.20(a))
II	Reads and stores support data (Fig. 4.3(b))
III	Reads and stores material properties (Fig. 4.3(c))
IV	Reads and stores cross-sectional properties (Fig. 4.3(d))
V	Reads and stores member data (Fig. 4.3(e))
VIa	Reads and stores joint loads (Fig. 5.20(b))
VIb	Reads and stores member loads (Fig. 5.20(c))
VII	Determines the number of degrees of freedom *NDOF* of the structure (Fig. 4.8(a))
VIII	Forms the structure coordinate number vector **NSC** (Fig. 4.8(b))
IX	Generates the structure stiffness matrix **S** and the structure load vector $\mathbf{P} = \mathbf{P}_e = -\mathbf{P}_f$ due to member loads (Fig. 5.24) Subroutines called: ***MSTIFFL***, ***STORES***, ***MFEFLL***, and ***STOREPF***
X	Stores joint loads in the structure load vector **P** (Fig. 5.29)
XI	Calculates the structure joint displacements by solving the stiffness relationship, $\mathbf{Sd} = \mathbf{P}$, using Gauss–Jordan elimination. The vector **P** now contains joint displacements (Fig. 4.13).

(*continued*)

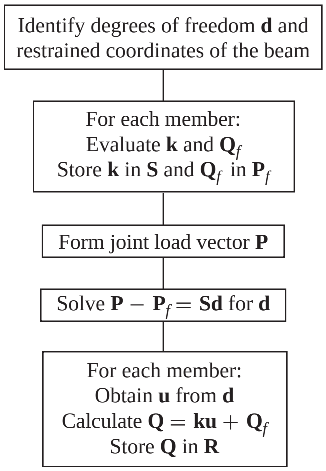


Fig. 5.35

Table 5.1 (*continued*)

Main program part	Description
XII	Determines the member end force vector **Q**, and the support reaction vector **R** (Fig. 5.30). Subroutines called: ***MDISPL***, ***MSTIFFL***, ***MFEFLL***, ***MFORCEL***, and ***STORER***
Subroutine	**Description**
MDISPL	Determines the member displacement vector **U** from the joint displacement vector **P** (Fig. 5.31)
MFEFLL	Calculates the member fixed-end force vector **QF** (Fig. 5.27)
MFORCEL	Evaluates the member local force vector $\mathbf{Q} = \mathbf{BK\ U} + \mathbf{QF}$ (Fig. 5.32)
MSTIFFL	Forms the member stiffness matrix **BK** (Fig. 5.25)
STOREPF	Stores the negative values of the pertinent elements of the member fixed-end force vector **QF** in the structure load vector **P** (Fig. 5.28)
STORER	Stores the pertinent elements of the member force vector **Q** in the reaction vector **R** (Fig. 5.33)
STORES	Stores the pertinent elements of the member stiffness matrix **BK** in the structure stiffness matrix **S** (Fig. 5.26)

SUMMARY

In this chapter, we have developed the matrix stiffness method for the analysis of beams. A block diagram summarizing the various steps of the analysis is presented in Fig. 5.35.

PROBLEMS

Section 5.1

5.1 through 5.4 Identify by numbers the degrees of freedom and restrained coordinates of the beams shown in Figs. P5.1 through P5.4. Also, form the joint load vector **P** for the beams.

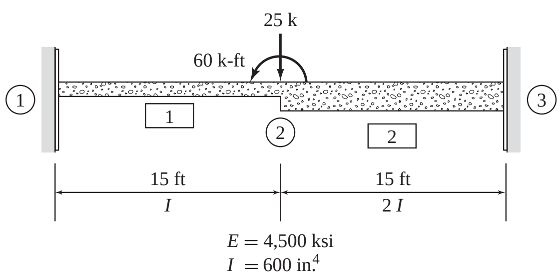


Fig. P5.1, P5.5, P5.19, P5.27

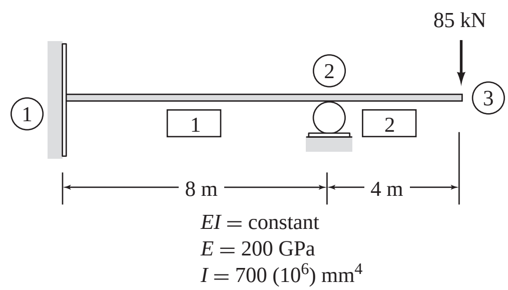


Fig. P5.2, P5.28

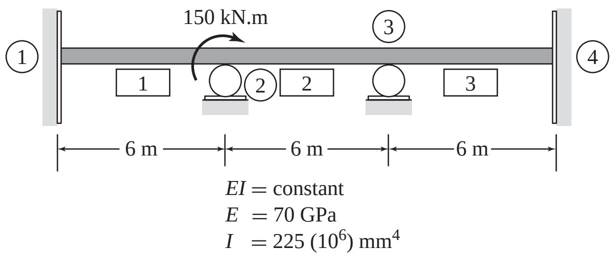


Fig. P5.3, P5.6, P5.20, P5.29

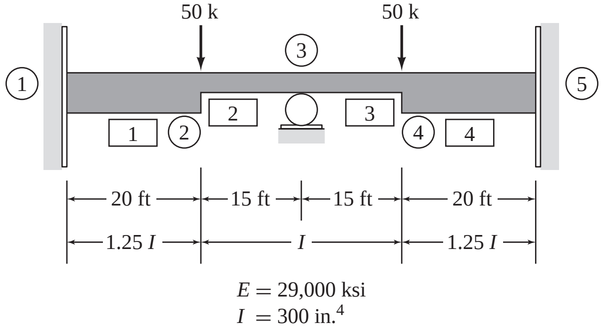


Fig. P5.4, P5.7, P5.21, P5.35

Section 5.2

5.5 through 5.8 Determine the stiffness matrices for the members of the beams shown in Figs. P5.5 through P5.8.

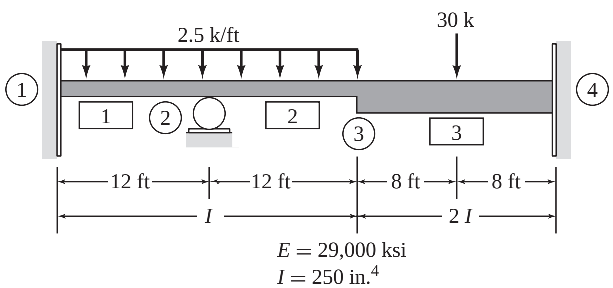


Fig. P5.8, P5.22, P5.23, P5.30

5.9 If the end displacements of member 1 of the beam shown in Fig. P5.9 are

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ -0.6667 \text{ in.} \\ -0.006667 \text{ rad} \end{bmatrix}$$

calculate the end forces for the member. Is the member in equilibrium under these forces?

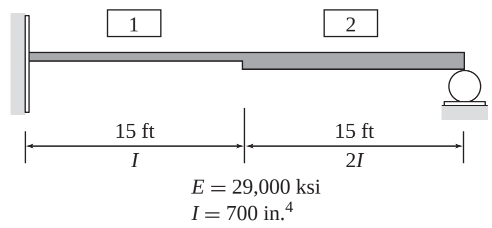


Fig. P5.9

5.10 If the end displacements of member 2 of the beam shown in Fig. P5.10 are

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0.08581 \text{ rad} \\ 0 \\ -0.08075 \text{ rad} \end{bmatrix}$$

calculate the end forces for the member. Is the member in equilibrium under these forces?

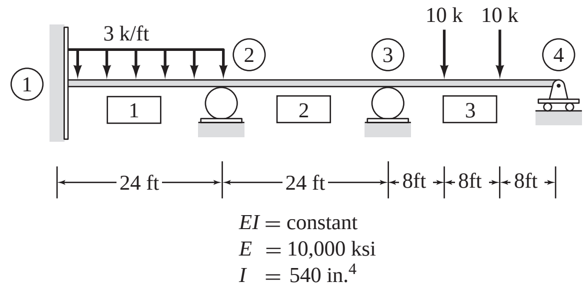


Fig. P5.10, P5.17, P5.24, P5.31

Section 5.4

5.11 through 5.14 Using the direct integration approach, derive the equations of fixed-end forces due to the member loads shown in Figs. P5.11 through P5.14. Check the results, using the fixed-end force expressions given inside the front cover.

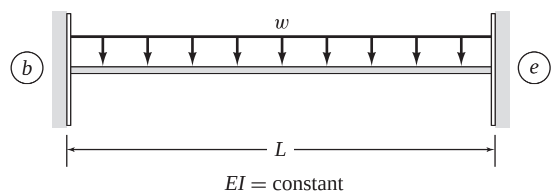


Fig. P5.11

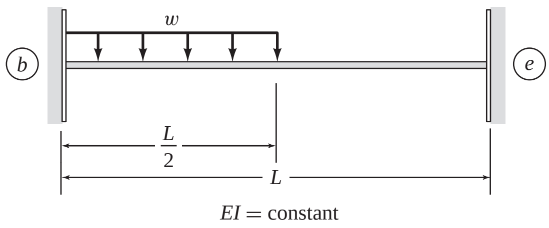


Fig. P5.12

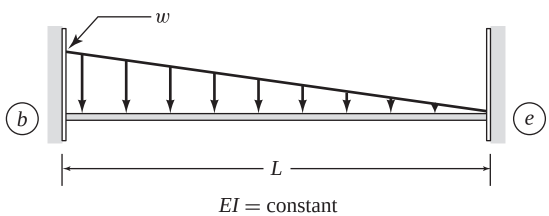


Fig. P5.13

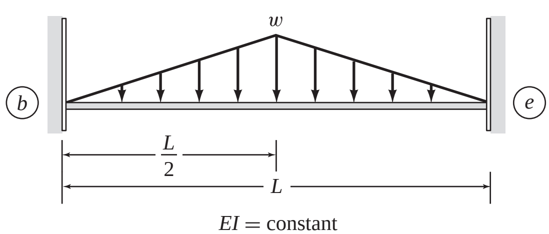


Fig. P5.14

5.15 and 5.16 Determine the fixed-end force vectors for the members of the beams shown in Figs. P5.15 and P5.16. Use the fixed-end force equations given inside the front cover.

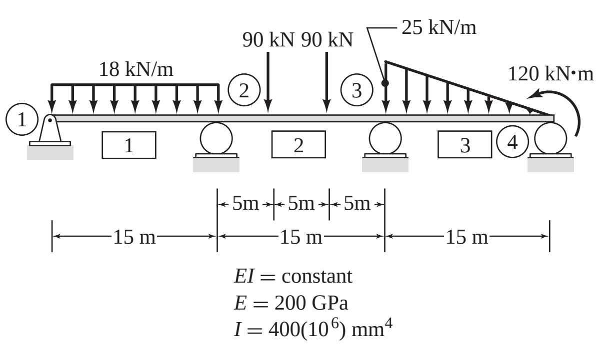


Fig. P5.15, P5.25, P5.33

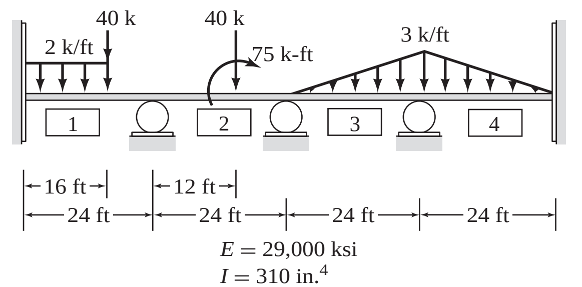


Fig. P5.16, P5.26, P5.32

5.17 If the end displacements of member 1 of the beam shown in Fig. P5.17 are

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.08581 \text{ rad} \end{bmatrix}$$

calculate the end forces for the member. Is the member in equilibrium under these forces?

5.18 If the end displacements of member 2 of the beam shown in Fig. P5.18 are

$$\mathbf{u}_2 = \begin{bmatrix} -0.02532 \text{ m} \\ -0.00434 \text{ rad} \\ -0.02532 \text{ m} \\ 0.00434 \text{ rad} \end{bmatrix}$$

calculate the end forces for the member. Is the member in equilibrium under these forces?

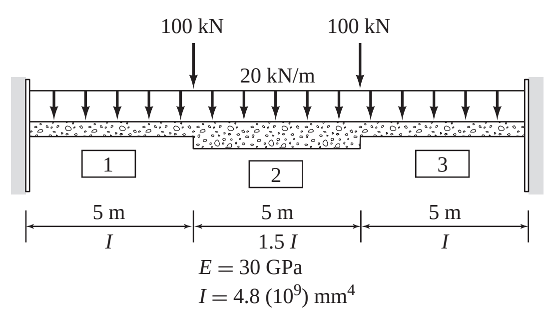


Fig. P5.18, P5.34

Section 5.5

5.19 through 5.22 Determine the structure stiffness matrices **S** for the beams shown in Figs. P5.19 through P5.22.

Section 5.6

5.23 through 5.26 Determine the fixed-joint force vectors and the equivalent joint load vectors for the beams shown in Figs. P5.23 through P5.26.

Section 5.7

5.27 through 5.35 Determine the joint displacements, member end forces, and support reactions for the beams shown in Figs. P5.27 through P5.35, using the matrix stiffness method. Check the hand-calculated results by using the computer program provided with this book, the publisher's website for this book (www.cengage.com/engineering), or by using any other general purpose structural analysis program available.

Section 5.8

5.36 Develop a general computer program for the analysis of beams by the matrix stiffness method. Use the program to analyze the beams of Problems 5.27 through 5.35, and compare the computer-generated results to those obtained by hand calculations.