# Appendix A
# The Gaussian Hypergeometric Function and Its Application

## A.1 Definition and Basic Relations of GHF

**(1) For** $|\ z\ | < 1$ :

The Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$, which is symbolized herein as GHF for simplicity, can be expressed as an infinite series (see, e.g., Abramowitz and Stegun 1972, Korn and Korn 1961, Kummer 1836, Pearson 1974, and Luke 1975) as

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \left[ \frac{(a)_k (b)_k}{(c)_k} \right] \frac{z^k}{k!}, \tag{A.1} $$

in which $a$, $b$, and $c$ are the function parameters and $z$ is the variable of the GHF. The parameters $a$, $b$, and $c$ are independent of $z$ and, in general, they may be complex. Such an infinite-series representation of the GHF in (A.1), is often referred to as the hypergeometric series; it is convergent for arbitrary $a$, $b$, and $c$ (provided that $c$ is neither a negative integer nor zero Seaborn (1991) and that $a$ or $b$ is not a negative integer (Whittaker and Watson, 1992) for real $-1 < z < 1$ (or $|\ z\ | < 1$), and for $z = \pm 1$ if $c > a + b$. The Pochhammer symbols $(a)_k$, $(b)_k$, and $(c)_k$ in the ascending factorial of $a$, $b$, and $c$ are respectively defined in terms of the gamma function, $\Gamma(n)$, as

$$ (a)_k := \frac{\Gamma(a + k)}{\Gamma(a)} = \Pi_{n=0}^{k-1}(a + n), \tag{A.2} $$

$$ (b)_k := \frac{\Gamma(b + k)}{\Gamma(b)} = \Pi_{n=0}^{k-1}(b + n), \tag{A.3} $$

$$ (c)_k := \frac{\Gamma(c + k)}{\Gamma(c)} = \Pi_{n=0}^{k-1}(c + n). \tag{A.4} $$

Therefore, GHF can be written in the following form as

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$$
{}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k, \tag{A.5}
$$

where the gamma function denoted by $\Gamma(n)$ for $n > 0$, is defined by

$$
\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt \quad (\text{for} \;\; n > 0). \tag{A.6}
$$

Using integration by parts, we will see that the gamma function satisfies the functional equation (i.e., the recurrence formula):

$$
\Gamma(n+1) = n\Gamma(n). \tag{A.7}
$$

If $n$ is a positive integer, $\Gamma(n+1) = n!$. For example, we can generalize the gamma function to $n < 0$ by using (A.7) in the form as

$$
\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad (\text{for} \;\; n < 0). \tag{A.8}
$$

$\Gamma(n)$ is convergent for $n > 0$ and $n < 0$ except when $n$ is a non-positive integer. Based on (A.6) and (A.8), we can plot the variation of $\Gamma(n)$ against $n$. Figure A.1 shows the gamma function $\Gamma(n)$ along part of the real axis. This figure was gererated by using a plot command of the Mathematica software. Some particular values of the gamma function are: $\Gamma(-1/2) = -2\sqrt{\pi}$, $\Gamma(0^+) = \infty$, $\Gamma(0^-) = -\infty$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(3/2) = \sqrt{\pi}/2$, $\Gamma(2) = 1$, and $\Gamma(5) = 4! = 24$.

As for the commutative rule of the GHF, it can be readily seen from the hypergeometric series in (A.1) that the parameters, $a$ and $b$, in the GHF are commutative; therefore,

$$
{}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z). \tag{A.9}
$$

Equation (A.9) shows a commutative rule in the GHF, connoting that the parameters, $b$ and $a$, in the GHF-based solution are independent of the order in the operation of multiplication within each term in the hypergeometric series (A.1), thus being commutative with each other. It merits attention to distinguish the expression of ${}_2F_1(a, b; c; z)$ in the Mathematica version 7.0.1 [2009] from that of ${}_2F_1(b, a; c; z)$ in its earlier versions, including version 7 [2008]. The reverse order of the two parameters, $b$ and $a$, in the expression of the GHF in the earlier versions actually accounts for the commutative rule of the GHF, as represented by (A.9). In truth, both GHFs in (A.9) expressed using the hypergeometric series, (A.1), are exactly identical.

**(2) For** $|\, z \,| > 1$ **:**

A survey of the literature reveals that there are two linearly independent solutions of the hypergeometric differential equation at each of the three singular points $z =$ 0, 1, and $\infty$ for a total of six special solutions, which are in fact fundamental to Kummer's (1836) 24 solutions. Besides one of two linearly independent solutions

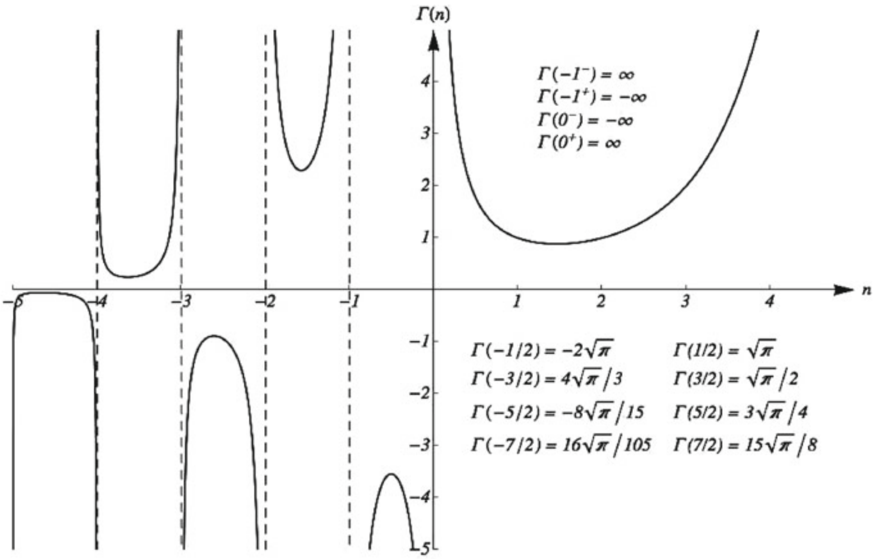



**Fig. A.1** The gamma function $\Gamma(n)$ along part of the real axis. $\Gamma(n)$ is convergent for $n > 0$ and $n < 0$ except $n$ = non-positive integers. The figure was generated by the plot command of the Mathematica software: Plot [Gamma[$x$], {$x$, –5, 5}, AxesStyle → Arrowheads [{0.03, 0.03}], PlotRange → {{–5, 5}, {–5, 5}}, BaseStyle → {FontWeight → "Bold", FontSize → 12}]

obtained around $z = 0$ is the GHF, as given in (A.1), we have a pair of linearly independent solutions around $z = \infty$ as follows: For $|\ z\ | > 1$ (i.e., covering both domains $z < -1$ and $z > 1$), if $(a - b)$ is not an integer,

$$(-z^{-1})^a {}_2F_1\,(\,a, a-c+1; a-b+1; z^{-1}\,) \\ = (-z^{-1})^a \sum_{k=0}^{\infty} \left[ \frac{(a)_k (a-c+1)_k}{(a-b+1)_k} \right] \frac{z^{-k}}{k!}, \tag{A.10}$$

$$(-z^{-1})^b {}_2F_1(b, b-c+1; -a+b+1; z^{-1}) \\ = (-z^{-1})^a \sum_{k=0}^{\infty} \left[ \frac{(b)_k (b-c+1)_k}{(-a+b+1)_k} \right] \frac{z^{-k}}{k!}. \tag{A.11}$$

Any three of the aforementioned six special solutions satisfy a relation, thus giving rise to twenty combinations thereof. Among them, there is a relation connecting one GHF in the domain of $|\ z\ | < 1$ to two GHF in the domain of $|\ z\ | > 1$ in the following way (Luke 1975):

$$
\begin{aligned}
&{}_2F_1(a, b; c; z) \\
&= \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z^{-1})^a\ {}_2F_1(a, a-c+1; a-b+1; z^{-1}) \\
&\quad + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z^{-1})^b\ {}_2F_1(b, b-c+1; -a+b+1; z^{-1}).
\end{aligned} \tag{A.12}
$$

## A.2 Specified GHF with the Parameters $a = 1$ and $c = b + 1$

For the special case used as in this book, $a$, $b$, and $c$ are fixed with specified relations, $a = 1$ and $c = b + 1$, Eq. (A.5) can be written as

$$
{}_2F_1(1, b; b+1, z) = b\sum_{k=0}^{\infty}\frac{z^k}{b+k}. \tag{A.13}
$$

The GHF used in the solutions of the gradually-varied-flow profiles in this book are all in the form of Eq. (A.13), as mentioned in Chap. 3. Since the first argument of the GHF of Eq. (A.13) is always unity, the second and third arguments differ in one unity, and the fourth argument is a variable only, we can express the GHF in a simpler expression as shown in the following equation for reducing the related equations to shorter expressions for facilitating the reading of the manuscript, as treated by Jan and Chen (2012):

$$
g(b, z) = {}_2F_1(1, b; b+1, z) = b\sum_{k=0}^{\infty}\frac{z^k}{b+k}. \tag{A.14}
$$

Therefore, for shortening expressions, we use $g(b, z)$ instead of $F(1, b; b + 1; z)$ to represent GHF when we treat the GHF-based GVF solutions.

## A.3 Some Examples of Specified GHF

Figure A.2 shows four examples of Gaussian hypergeometric functions, $g(2/5, u^{10/3})$ and $g(-3/5, u^{10/3})$ for $0 < u < 1$, and $g(3/5, u^{-10/3})$ and $g(-3/10, u^{-10/3})$ for $1 < u < 2$. Figure A.3 shows another examples of Gaussian hypergeometric functions, $g(1/N, u^N)$ and $g(-1/N, u^N)$ for $0 < u < 1$, and $g(1/N, u^{-N})$ and $g(-1/N, u^{-N})$ for $1 < u < 2$, with four different values of hydraulic exponent $N$ varying from 2 to 5. These figures show that $g(b, z) \to 1$ as $z \to 0$, $g(b, z) \to -\infty$ as $b$ is negative and $z \to 1$, and $g(b, z) \to \infty$ as $b$ is positive and $z \to 1$, in which $z$ $(= u^N$ or $u^{-N})$ is real variable of positive values. On the other hand, as $z$ $(= -u^N)$ is real variable of negative values, $g(b, z) \to$ a finite value as $z \to -1$ (i.e.,$u \to 1$), as

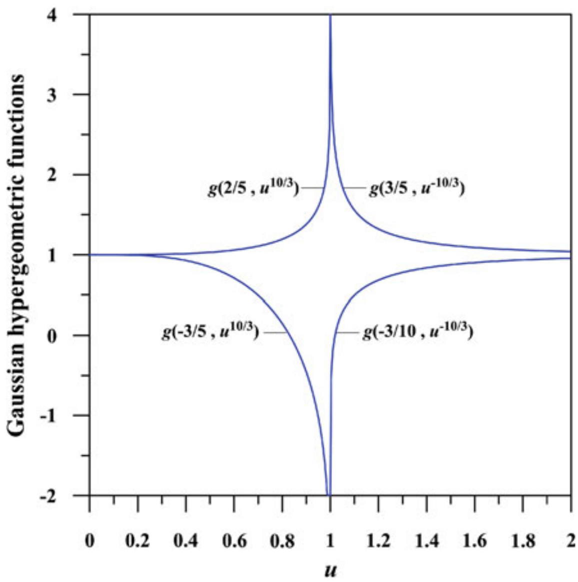



**Fig. A.2** Four examples of the GHF having positive variables. These GHF diverge at $u = 1$, and converge to unity as $u$ approachs either zero or the infinite. This type of GHF could be used in the GHF-based GVF solutions in sustaining channels

shown in Fig. A.4. Figure A.5 shows another examples of Gaussian hypergeometric functions, $g(1/N, -u^N)$ and $g(-1/N, -u^N)$ for $0 < u < 1$, and $g(1/N, -u^N)$ and $g(-1/N, -u^N)$ for $1 < u < 2$, with four different values of hydraulic exponent $N$ varying from 2 to 5. These figures show that $g(b, z) \to 1$ as $z \to 0$, and $g(b, z) \to$ a finite value as $z \to -1$ (i.e., $u \to 1$, as $z = -u^N$ or $z = -u^{-N}$). These kinds of Gaussian hypergeometric functions, $g(b, -u^N)$ and $g(b, -u^{-N})$, will be used to express the solutions of the surface profiles of gradually-varied flow in channels of an adverse slope (a negative slope). It means that there is no singularity for the surface slope of the profiles of gradually-varied flow in adverse channels.

## A.4 Integral Solutions in Terms of GHF

For example, the gradually-varied flow (GVF) in sustaining channels has a positive variable $z = u^N$, for the case of $0 \le u < 1$, the solution of the indefinite integral in the GVF equation can be expressed in terms of GHF as

$$\int \frac{u^{\phi}}{1 - u^N} du = \frac{u^{\phi+1}}{\phi + 1} g\left(\frac{\phi + 1}{N}, u^N\right) + \text{Const.}, \tag{A.15}$$

while, for the case of $u > 1$, letting $w = u^{-1}$, the solution of the indefinite integral in terms of GHF is written as

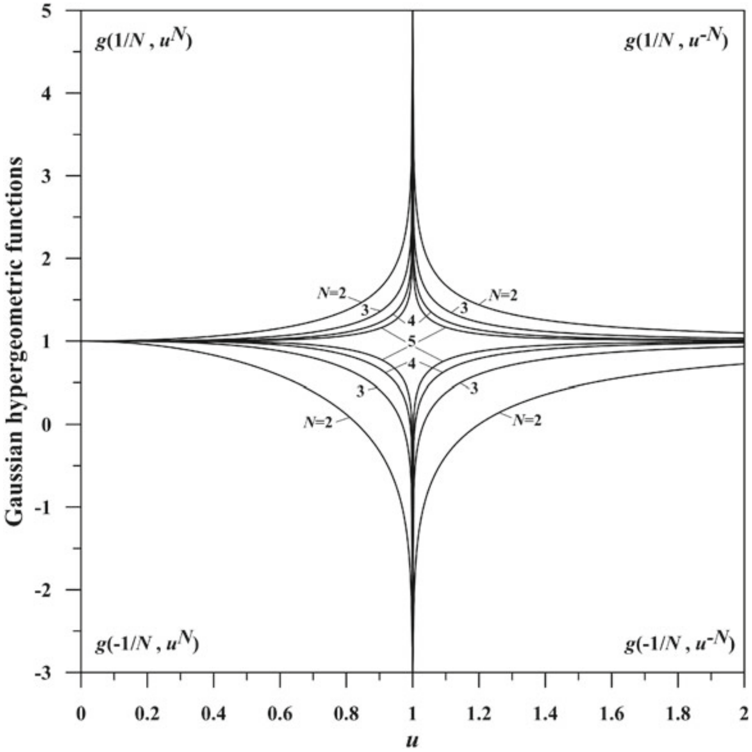
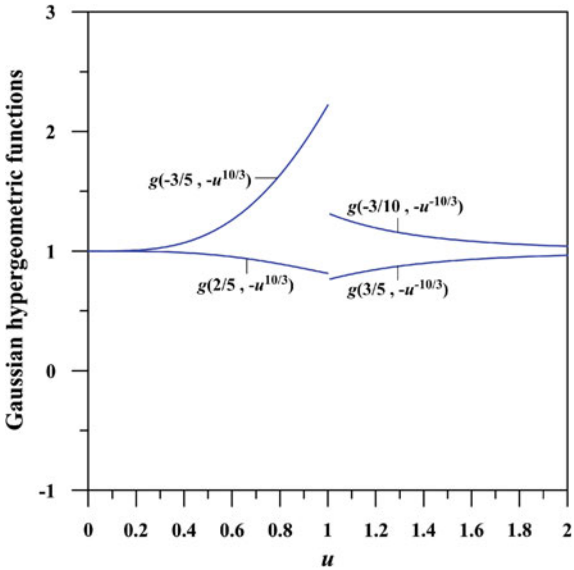



**Fig. A.3** Effect of the hydraulic exponent $N$ on the variation of GHF having positive variables. These GHF diverge at $u = 1$, and converge to unity as $u$ approaches either zero or the infinte

**Fig. A.4** Four examples of the GHF having negative variables. These GHF do not diverge at $u = 1$. This type of GHF can be used in the GHF-based GVF solutions in adverse channels



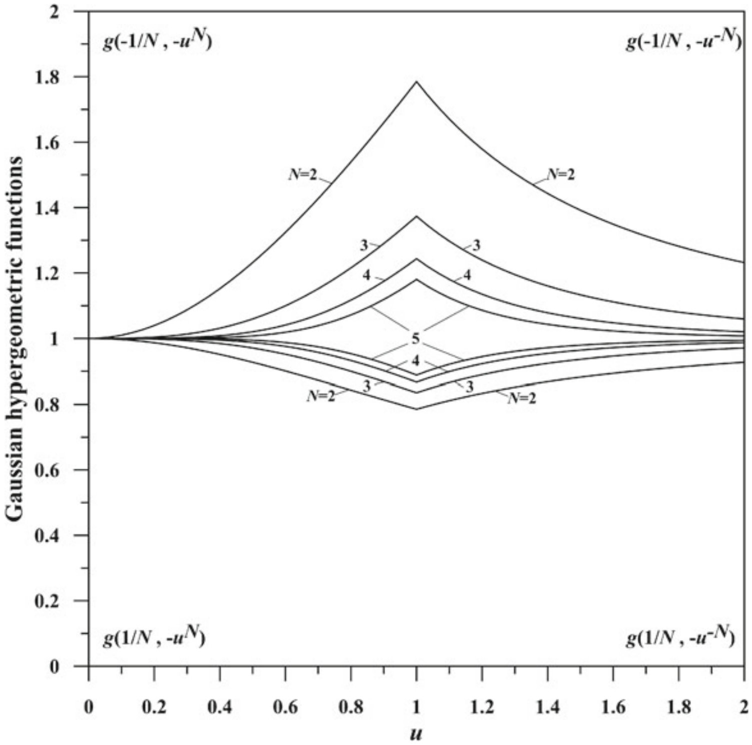



**Fig. A.5** Effect of the hydraulic exponent $N$ on the variation of GHF having negative variables. These GHF do not diverge at $u = 1$, and converge to unity as $u$ approaches either zero or the infinte

$$\int \frac{w^{\phi}}{1 - w^N} dw = \frac{w^{\phi+1}}{\phi + 1} g\left(\frac{\phi + 1}{N}, w^N\right) + \text{Const.}. \tag{A.16}$$

(A.16), on substitution of $w = u^{-1}$, yields

$$\frac{w^{(\phi+1)}}{\phi + 1} g\left(\frac{\phi + 1}{N}, w^N\right) = \frac{u^{-(\phi+1)}}{\phi + 1} g\left(\frac{\phi + 1}{N}, u^{-N}\right), \tag{A.17}$$

in which the parameters $\phi$ and $N$ are real numbers.

On the other hand, the gradually-varied flow (GVF) in adverse channels has a negative slope and negative varible $z = -u^N$, for the case of $0 \leq u < 1$, the solution of the indefinite integral in the GVF equation for flow in adverse channels can be expressed in terms of GHF as

$$\int \frac{u^{\phi}}{1 + u^N} du = \frac{u^{\phi+1}}{\phi + 1} g\left(\frac{\phi + 1}{N}, -u^N\right) + \text{Const.}, \tag{A.18}$$

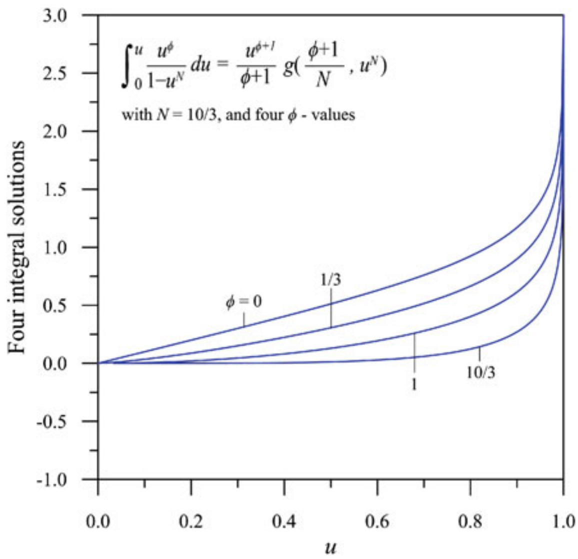



**Fig. A.6** Four examples of the integral solutions in terms of GHF for the specified case of $N = 10/3$, and $\phi = 0$, $1/3$, 1.0, and $10/3$, respectively

while, for the case of $u > 1$, letting $w = u^{-1}$, the solution of the indefinite integral in terms of GHF is written as

$$\int \frac{w^\phi}{1+w^N} dw = \frac{w^{\phi+1}}{\phi+1} g\left(\frac{\phi+1}{N}, -w^N\right) + \text{Const.}. \tag{A.19}$$

Equation (A.19), on substitution of $w = u^{-1}$, yields

$$\frac{w^{\phi+1}}{\phi+1} g\left(\frac{\phi+1}{N}, -w^N\right) = \frac{u^{-(\phi+1)}}{\phi+1} g\left(\frac{\phi+1}{N}, -u^{-N}\right). \tag{A.20}$$

Figure A.6 shows four examples of the integral solutions in terms of GHF for the specified case of $N = 10/3$, and $\phi = 0$, $1/3$, 1.0 and $10/3$, respectively. These integral solutions show that the results of integration diverge as $u \to 1$,

# Appendix B
# Proof of an Identity Relation Between Two Contiguous GHF

According to (A.1) or (A.5), the Gaussian hypergeometric function (GHF) can be written in the form

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)} \frac{\Gamma(b+k)}{\Gamma(b)} \frac{\Gamma(c+k)}{\Gamma(c)} \frac{z^k}{k!}. \tag{B.1} $$

If the parameters $a$, $b$, and $c$ in the GHF are fixed with specified relations, $a = 1$ and $c = b + 1$ as used in this book, the following functional identity relation between the two contiguous GHF can be generally established as

$$ {}_2F_1(1, b; b+1, z) = 1 + \frac{bz}{b+1} \, {}_2F_1(1, b+1; b+2, z). \tag{B.2} $$

By means of the induction method, we can validate (B.2) with the help of (A.1) through (A.4) as follows:

$$ \begin{aligned} & 1 + \frac{bz}{b+1} \, {}_2F_1(1, b+1; b+2; z) \\ &= 1 + \frac{bz}{b+1} \sum_{k=0}^{\infty} \left[ \frac{\Gamma(1+k)}{\Gamma(1)} \frac{\Gamma(b+1+k)}{\Gamma(b+1)} \frac{\Gamma(b+2)}{\Gamma(b+2+k)} \right] \frac{z^k}{k!} \\ &= 1 + \frac{b}{b+1} \sum_{k=0}^{\infty} \left[ \frac{\Gamma(1+k)}{\Gamma(1)} \frac{\Gamma(b+1+k)}{b\Gamma(b)} \frac{(b+1)\Gamma(b+1)}{\Gamma(b+2+k)} \right] \frac{z^{k+1}}{k!} \\ & \text{(letting } \ j = k+1) \\ &= 1 + \sum_{j=1}^{\infty} \left[ \frac{j\Gamma(j)}{\Gamma(1)} \frac{\Gamma(b+j)}{\Gamma(b)} \frac{\Gamma(b+1)}{\Gamma(b+1+j)} \right] \frac{z^j}{j(j-1)!} \end{aligned} \tag{B.3} $$

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$$= \sum_{j=0}^{\infty} \left[ \frac{\Gamma(1+j)}{\Gamma(1)} \frac{\Gamma(b+j)}{\Gamma(b)} \frac{\Gamma(b+1)}{\Gamma(b+1+j)} \right] \frac{z^j}{j!}$$
$$= {}_2F_1(1, b; b+1; z).$$

The functional identity relation between two contiguous GHF as proved in the derivation process of (B.3) is also called a recurrence formulas. Using a simplified expression of GHF, i.e., $g(b, z)$, to replace ${}_2F_1(1, b; b+1; z)$ as shown in (A.14) in Appendix A, we can rewrite the recurrence formulas into a shorter expression as done by Jan and Chen (2012, 2013).

$$g(b, z) = 1 + \frac{bz}{b+1} g(b+1, z). \tag{B.4}$$

The so-called recurrence formulas, such as (3.32), (3.33), (3.45), (3.46), (3.47) and (3.48) in Chap. 3, and (4.35), (4.36), (4.45), (4.46), (4.50), (4.51), (4.72), (4.73), (4.82), (4.83), (4.87) and (4.88) in Chap. 4, can be proved to possess the property of the functional identity relation between two contiguous GHF, as shown in (B.2) or (B.4). For convenience of discussion, we take the example of (3.33) in Chap. 3, listed here as

$$g\left(\frac{-M+1}{N}, u^N\right) = 1 + \frac{(-M+1)u^N}{N-M+1} g\left(\frac{N-M+1}{N}, u^N\right). \tag{B.5}$$

It can be readily identified from the two contiguous GHF in (3.33) in that the parameters and variable in the GHF on its left-hand side of the equal sign is constituted by $a = 1$, $b = (-M+1)/N$, $c = (N-M+1)/N$, and $z = u^N$, thus satisfying one of the two parameter relations, $a = 1$ and $c = b+1$, while the other GHF on its right-hand side of the equal sign has the other parameter relations, $a = 1$, $b = (N-M+1)/N$, and $c = (2N-M+1)/N$, thus meeting all the parameter relations specified on both sides of (B.2) or (B.4). For the specified case of $M = 3$ and $N = 10/3$, (B.5) [i.e., (3.33)] yields

$$g\left(\frac{-3}{5}, u^{10/3}\right) = 1 - \frac{3}{2} u^{10/3} g\left(\frac{2}{5}, u^{10/3}\right). \tag{B.6}$$

Similarly, we can prove that each of the above-mentioned recurrence formulas in Chaps. 3 and 4 has the same parameter relations in such two contiguous GHF, thus satisfying (B.4).

# Appendix C
# Obtain (3.43) Directly from (3.30) by Using the Transformation Relation (A.12), and Vice Versa

We start with the GHF-based solutions of (3.8) in Chap. 3 in the domains $\mid u \mid < 1$ and $\mid u \mid > 1$, as shown in (3.30) and (3.43), respectively,

$$x_* = u\left[1 - g\left(\frac{1}{N}, u^N\right)\right] + \frac{\lambda^M u^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, u^N\right) + \text{Const.}, \tag{3.30}$$

$$x_* = u\left[1 - \frac{u^{-N}}{N-1} g\left(\frac{N-1}{N}, u^{-N}\right)\right] + \frac{\lambda^M u^{-M+1}}{M-1} g\left(\frac{M-1}{N}, u^{-N}\right) + \text{Const.}. \tag{3.43}$$

Equation (A.12) in Appendix A has shown that there is a relation connecing one GHF in the domain of $\mid z \mid < 1$ to two GHF in the domain of $\mid z \mid > 1$ (Luke 1975). For the specified parameters $a = 1$ and $c = b + 1$, (A.12) connecting one GHF in the domain of $\mid z \mid < 1$ to two GHF in the domain of $\mid z \mid > 1$ can be simplified as

$$\begin{aligned} {}_2F_1(1, b; 1+b; z) &= \frac{\Gamma(b-1)\Gamma(b+1)}{\Gamma(b)\Gamma(b)} (-z^{-1})^1 \, {}_2F_1(1, 1-b; 2-b; z^{-1}) \\ &+ \frac{\Gamma(1-b)\Gamma(1+b)}{\Gamma(1)\Gamma(1)} (-z^{-1})^b \, {}_2F_1(b, 0; b; z^{-1}). \end{aligned} \tag{C.1}$$

Since $\Gamma(b) = (b-1)\Gamma(b-1)$, $\Gamma(b+1) = b\Gamma(b)$, and ${}_2F_1(b, 0; b; z^{-1}) = 1$, Eq. (C.1) yields

$${}_2F_1(1, b; 1+b; z) = \frac{b}{b-1}(-z^{-1}) \, {}_2F_1(1, 1-b; 2-b; z^{-1}) + \Gamma(1-b)\Gamma(1+b)(-z^{-1})^{-b}. \tag{C.2}$$

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For reducing the related equations to shorter expressions for facilitating the reading of the manuscript, we will use $g(b, z)$ instead of ${}_2F_1(1, b; b+1; z)$ in this book to represent GHF, namely, $g(b, z) = {}_2F_1(1, b; b+1; z)$. Therefore, the shorter expression of (C.2) is given as

$$g(b, z) = \frac{b}{b-1}(-z^{-1})g(1-b, z^{-1}) + \Gamma(1-b)\Gamma(1+b)(-z^{-1})^{-b}. \quad \text{(C.3)}$$

One may use (C.3) to transform the GHF-based solutions of (3.8) in the domain of $\mid u \mid < 1$ to their counterparts in the domain of $\mid u \mid > 1$ without recourse to the formulation of (3.14) [i.e., an alternative form of (3.8) with its variable, $w$, being expressed as $u^{-1}$], which can be solved for GVF profiles in the domain of $\mid u^{-1} \mid < 1$ (or $\mid u \mid > 1$). In this Appendix, we focus our attention on use of (C.3) to transform (3.30)– (3.43), thereby proving that we can obtain (3.43) directly from (3.30) rather than indirectly through (3.14).

Before using (C.3) to find the GHF-based solutions of (3.8) for the entire domain of $u$, namely $0 \le u < \infty$, excluding at $u = 1$ attention is focused on their generality in that the relation, (C.3), which originally connects one GHF in the domain of $\mid z \mid < 1$ to two GHF in the domain of $\mid z \mid > 1$, can also be applied in reverse, i.e., one GHF in the domain of $\mid z \mid > 1$ being transformed into two GHF in the domain of $\mid z \mid < 1$. In particular, for the specified parameter relations, $a = 1$ and $c = b + 1$, we obtain the reverse relation from (A.12) as follows: Substituting $(1 - b)$ for $b$ and $z$ for $z^{-1}$ into both sides of (C.3) and then rearranging its result through the relations $\Gamma(b) = \Gamma(1 + b)/b$ and $\Gamma(2 - b) = (1 - b)\Gamma(1 - b)$ yields

$$\begin{aligned} g(1-b, z^{-1}) = &\frac{b-1}{b}(-z)g(b, z) \\ &- \frac{b-1}{b}\Gamma(1+b)\Gamma(1-b)(-z)^{1-b}. \end{aligned} \quad \text{(C.4)}$$

In fact, we can derive (C.4) from (C.3) directly by means of transposition. The reversibility of (C.3) for the GHF to transform its domain of validity from $\mid z \mid < 1$ to $\mid z \mid > 1$, or vice versa, namely (C.4) for the GHF to switch its valid domain of $\mid z \mid$ in reverse makes such relations instrumental when the GHF-based solutions are used in the computation of GVF profiles. Such an exclusive property of the GHF in effect makes the GHF-based solutions of the dimensionless GVF Eq. (3.8) versatile in application. A few aspects of such relations proved in the GHF-based solutions are given below.

## C.1 Obtain (3.43) Directly from (3.30) by Using the Transformation Relation

For example, the relation connecting one GHF in the domain of $\mid z \mid < 1$ to two GHF in the domain of $\mid z \mid > 1$, as shown in (C.3), can be applied to transform the GHF-based solution of (3.8) in the domain of $\mid u \mid < 1$, namely (3.30), to the corresponding GHF-based solution of (3.8) in the domain of $\mid u \mid > 1$. Using (C.3), we can prove that (3.30) in the domain of $\mid u \mid < 1$ is transformed to (3.43) in the domain of $\mid u \mid > 1$, though (3.43) has been actually obtained from (3.14) rather than from (3.8). Because we have $b = 1/N$ and $b = (N - M + 1)/N$ in the first and second GHF with $z = u^N$ appearing on the right-hand side of (3.30), respectively, (C.3) on substitution of such respective $b$-and $z$-expressions yields the following two relations

$$g\left(\frac{1}{N}, u^N\right) = \frac{u^{-N}}{N-1} g\left(\frac{N-1}{N}, u^{-N}\right) + \Gamma\left(\frac{N-1}{N}\right)\Gamma\left(\frac{N+1}{N}\right)(-u^{-N})^{1/N}, \tag{C.5}$$

$$g\left(\frac{N-M+1}{N}, u^N\right) = \frac{(N-M+1)u^{-N}}{M-1} g\left(\frac{M-1}{N}, u^{-N}\right) + \Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}\right)(-u^{-N})^{(N-M+1)/N}. \tag{C.6}$$

Substituting the first and second GHF appearing on the right-hand side of (3.30) from (C.5) and (C.6), respectively, into (3.30) yields

$$\begin{aligned}
x_* &= u\left[1 - g\left(\frac{1}{N}, u^N\right)\right] + \frac{\lambda^M u^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, u^N\right) + \text{Const.} \\
&= u\left[1 - \frac{u^{-N}}{N-1} g\left(\frac{N-1}{N}, u^{-N}\right) - \Gamma\left(\frac{N-1}{N}\right)\Gamma\left(\frac{N+1}{N}\right)(-u^{-N})^{1/N}\right] \\
&\quad + \frac{\lambda^M u^{N-M+1}}{N-M+1}\left[\frac{(N-M+1)u^{-N}}{M-1} g\left(\frac{M-1}{N}, u^{-N}\right)\right. \\
&\quad \left. + \Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}(-u^{-N})^{(N-M+1)/N}\right)\right] + \text{Const.} \\
&= u\left[1 - \frac{u^{-N}}{N-1} g\left(\frac{N-1}{N}, u^{-N}\right)\right] + \frac{\lambda^M u^{-M+1}}{M-1} g\left(\frac{M-1}{N}, u^{-N}\right) \\
&\quad - (-1)^{1/N}\Gamma\left(\frac{N-1}{N}\right)\Gamma\left(\frac{N+1}{N}\right) \\
&\quad + \frac{\lambda^M (-1)^{(N-M+1)/N}}{N-M+1}\Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}\right) + \text{Const.}
\end{aligned}$$

$$= u\left[1 - \frac{u^{-N}}{N-1} g\left(\frac{N-1}{N}, u^{-N}\right)\right] + \frac{\lambda^M u^{-M+1}}{M-1} g\left(\frac{M-1}{N}, u^{-N}\right) + \text{Const}^*_{.}. \tag{C.7}$$

It merits attention to state that (C.7) is identical to (3.13) because "Const$^*_.$" in (C.7) that contains two constant terms appearing in (C.7) in addition to "Const." does not lose its generality in comparison. Consequently, one may conclude that (3.43) obtained from (3.14) is indeed the GHF-based solution of (3.8) in the domain of $\mid u \mid > 1$. This reconfirms the validity of (3.43) in the domain of $\mid u \mid > 1$ and the uniqueness of the role of (3.43) playing in the computation of GVF profiles in sustaining channels.

## C.2 Obtain (3.30) Directly from (3.43) by Using the Transformation Relation

The application of (C.4) in the reverse transformation of one GHF in the domain of $\mid z \mid > 1$ to two GHF in the domain of $\mid z \mid < 1$ is exemplified when (3.43) in the domain of $\mid u \mid > 1$ is transformed to (3.30) in the domain of $\mid u \mid < 1$. In this case, because we again have $b = 1/N$ and $b = (N - M - 1)/N$ in the first and second GHF with $z = u^{-N}$ appearing on the right-hand side of (3.43), respectively, (C.4) on substitution of such respective $b$- and $z$-expressions yields the following two relations

$$g\left(\frac{N-1}{N}, u^{-N}\right) = (N-1)u^N g\left(\frac{1}{N}, u^N\right) + (N-1)\Gamma\left(\frac{N+1}{N}\right)\Gamma\left(\frac{N-1}{N}\right)(-u^N)^{1-1/N}, \tag{C.8}$$

$$g\left(\frac{M-1}{N}, u^{-N}\right) = \frac{(M-1)u^N}{N-M+1} g\left(\frac{N-M+1}{N}, u^N\right) + \frac{(M-1)(-u^N)^{(M-1)/N}}{N-M+1}\Gamma\left(\frac{2N-M+1}{N}\right)\Gamma\left(\frac{M-1}{N}\right). \tag{C.9}$$

Substituting the first and second GHF appearing on the right-hand side of (3.43) from (C.8) and (C.9), respectively, into (3.43) yields

$$
\begin{aligned}
x_* &= u\left[1-\frac{u^{-N}}{N-1}g\left(\frac{N-1}{N},u^{-N}\right)\right]\\
&\quad+\frac{\lambda^M u^{-M+1}}{M-1}g\left(\frac{M-1}{N},u^{-N}\right)+\text{Const.}\\
&= u\left[1-g\left(\frac{1}{N},u^N\right)-u^{-N}\Gamma\left(\frac{N+1}{N}\right)\Gamma\left(\frac{N-1}{N}\right)(-u^N)^{1-1/N}\right]\\
&\quad+\frac{\lambda^M u^{-M+1}}{M-1}\left[\frac{(M-1)u^N}{N-M+1}g\left(\frac{N-M+1}{N},u^N\right)\right]\\
&\quad+\frac{\lambda^M u^{-M+1}}{M-1}\left[\frac{(M-1)(-u^N)^{(M-1)/N}}{N-M+1}\right.\\
&\qquad\qquad\left.\times\,\Gamma\left(\frac{2N-M+1}{N}\right)\Gamma\left(\frac{M-1}{N}\right)\right]+\text{Const.}\\
&= u\left[1-g\left(\frac{1}{N},u^N\right)\right]+\frac{\lambda^M u^{N-M+1}}{N-M+1}g\left(\frac{N-M+1}{N},u^N\right)\\
&\quad+(-1)^{-1/N}\Gamma\left(\frac{N+1}{N}\right)\Gamma\left(\frac{N-1}{N}\right)\\
&\quad+\frac{\lambda^M(-1)^{(M-1)/N}}{N-M+1}\Gamma\left(\frac{2N-M+1}{N}\right)\Gamma\left(\frac{M-1}{N}\right)+\text{Const.}\\
&= u\left[1-g\left(\frac{1}{N},u^N\right)\right]+\frac{\lambda^M u^{N-M+1}}{N-M+1}g\left(\frac{N-M+1}{N},u^N\right)+\text{Const}^*.
\end{aligned}
\tag{C.10}
$$

It is noted that (C.10) is identical to (3.30) because "Const*" in (C.10) that contains two constant terms appearing in (C.10) in addition to "Const." does not lose its generality in comparison.

Though we have proved that (3.30) can be transformed to (3.43), or vice versa, using a couple of the relations connecting one GHF in the domain of $|\ u\ | < 1$ to two GHFs in the domain of $|\ u\ | > 1$, or vice versa in the reverse transformation, we can by no means claim that (3.30) and (3.43) are identical because they are only valid and convergent in their respective domains of $u$. In other words, (3.30) is only valid in the domain of $|\ u\ | < 1$, whereas (3.43) is only valid in the domain of $|\ u\ | > 1$. Consequently, suffice it to say that the complete GHF-based solutions of (3.8) are constituted by a combination of (3.30) and (3.43).

# Appendix D
# ETF-Based Solutions of the First and Second Integrals in (3.12)

Let us start with (3.12) in Chap. 3 that is a direct integral solution of (3.8) for GVF in sustaining channels:

$$x_* = u - \int_0^u \frac{1}{1-u^N}du + \lambda^M \int_0^u \frac{u^{N-M}}{1-u^N}du + \text{Const.}. \tag{3.12}$$

With the help of the Mathematica software (Wolfram 1996) or the Maple software (Bernardin et al. 2011), the alternative analytical solutions of the first and second integrals appearing in the above equation for $N = 3$, $10/3$, $7/2$, $17/5$ and $11/3$ for fully rough flows in wide channels ($M = 3$) can be expressed in terms of elementary transcendental functions (ETF). The results are shown below.

## D.1 ETF-Based Solutions of the First Integral for Five $N$-Values

$$\begin{aligned}(1)\quad &\int_0^u \frac{1}{1-u^3}du \quad (\text{for } N=3)\\ &= \frac{1}{\sqrt{3}}\arctan\left[\frac{1+2u}{\sqrt{3}}\right] - \frac{1}{3}\ln[-1+u] + \frac{1}{6}\ln[1+u+u^2].\end{aligned} \tag{D.1}$$

$$\begin{aligned}(2)\quad &\int_0^u \frac{1}{1-u^{10/3}}du \quad (\text{for } N=10/3)\\ &= \frac{3}{40}\left\{-2\sqrt{2(5+\sqrt{5})}\,\arctan\left[\frac{1+\sqrt{5}-4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right]\right.\\ &\quad - 2\sqrt{10-2\sqrt{5}}\,\arctan\left[\frac{1-\sqrt{5}+4u^{1/3}}{\sqrt{2(5+\sqrt{5})}}\right]\end{aligned} \tag{D.2}$$



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$$- 2\sqrt{10-2\sqrt{5}}\arctan\left[\frac{-1+\sqrt{5}+4u^{1/3}}{\sqrt{2(5+\sqrt{5})}}\right]$$
$$+ 2\sqrt{10-2\sqrt{5}}\arctan\left[\frac{1+\sqrt{5}+4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right]$$
$$- 4\ln\left[-1+u^{1/3}\right] + 4\ln\left[1+u^{1/3}\right]$$
$$+ (1+\sqrt{5})\ln\left[1-\frac{1}{2}(-1+\sqrt{5})u^{1/3}+u^{2/3}\right]$$
$$- (1+\sqrt{5})\ln\left[1+\frac{1}{2}(-1+\sqrt{5})u^{1/3}+u^{2/3}\right]$$
$$+ (-1+\sqrt{5})\ln\left[1-\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right]$$
$$- (-1+\sqrt{5})\ln\left[1+\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right]\Bigg\}.$$

$$(3)\quad \int_0^u \frac{1}{1-u^{7/2}}du \quad (\text{for } N = 7/2) \tag{D.3}$$
$$= \frac{4}{7}\arctan\left[\sec[\frac{3\pi}{14}](\sqrt{u}-\sin[\frac{3\pi}{14}])\right]\cos[\frac{\pi}{14}]$$
$$- \frac{4}{7}\arctan\left[\csc[\frac{\pi}{7}](\sqrt{u}+\cos[\frac{\pi}{7}])\right]\cos[\frac{3\pi}{14}]$$
$$- \frac{2}{7}\ln[-1+\sqrt{u}] + \frac{2}{7}\cos[\frac{\pi}{7}]\ln\left[1+u+2\sqrt{u}\sin[\frac{\pi}{14}]\right]$$
$$+ \frac{2}{7}\sin[\frac{\pi}{14}]\ln\left[1+u-2\sqrt{u}\sin[\frac{3\pi}{14}]\right]$$
$$- \frac{4}{7}\arctan\left[\sec[\frac{\pi}{14}](\sqrt{u}+\sin[\frac{\pi}{7}])\right]\sin[\frac{\pi}{7}]$$
$$- \frac{2}{7}\sin[\frac{3\pi}{14}]\ln\left[1+u+2\sqrt{u}\cos[\frac{\pi}{7}]\right].$$

$$(4)\quad \int_0^u \frac{1}{1-u^{17/5}}du \quad (\text{for } N = 17/5) \tag{D.4}$$
$$= -\frac{10}{17}\arctan\left[\sec[\frac{7\pi}{34}](u^{1/5}+\sin[\frac{7\pi}{34}])\right]\cos[\frac{\pi}{34}]$$
$$+ \frac{10}{17}\arctan\left[(u^{1/5}-\cos[\frac{2\pi}{17}])\csc[\frac{2\pi}{17}]\right]\cos[\frac{3\pi}{34}]$$

$$
\begin{aligned}
&+\frac{10}{17}\arctan\left[\sec[\frac{\pi}{34}](u^{1/5}-\sin[\frac{\pi}{34}])\right]\cos[\frac{5\pi}{34}]\\
&+\frac{10}{17}\arctan\left[(u^{1/5}+\cos[\frac{\pi}{17}]\csc[\frac{\pi}{17}]\right]\cos[\frac{7\pi}{34}]\\
&-\frac{5}{17}\ln[-1+u^{1/5}]-\frac{5}{17}\cos[\frac{2\pi}{17}]\ln\left[1+u^{2/5}+2u^{1/5}\cos[\frac{3\pi}{17}]\right]\\
&+\frac{5}{17}\cos[\frac{3\pi}{17}]\ln\left[1+u^{2/5}-2u^{1/5}\cos[\frac{4\pi}{17}]\right]\\
&+\frac{5}{17}\cos[\frac{\pi}{17}]\ln\left[1+u^{2/5}+2u^{1/5}\sin[\frac{3\pi}{34}]\right]\\
&-\frac{5}{17}\cos[\frac{4\pi}{17}]\ln\left[1+u^{2/5}-2u^{1/5}\sin[\frac{5\pi}{34}]\right]\\
&-\frac{5}{17}\ln\left[1+u^{2/5}+2u^{1/5}\sin[\frac{7\pi}{34}]\right]\sin[\frac{\pi}{34}]\\
&+\frac{10}{17}\arctan\left[\sec[\frac{3\pi}{34}](u^{1/5}+\sin[\frac{3\pi}{34}])\right]\sin\left[\frac{\pi}{17}\right]\\
&+\frac{5}{17}\ln\left[1+u^{2/5}-2u^{1/5}\cos[\frac{2\pi}{17}]\right]\sin\left[\frac{3\pi}{34}\right]\\
&+\frac{10}{17}\arctan\left[(u^{1/5}+\cos[\frac{3\pi}{17}])\csc[\frac{3\pi}{17}]\right]\sin[\frac{2\pi}{17}]\\
&-\frac{5}{17}\ln\left[1+u^{2/5}-2u^{1/5}\sin[\frac{\pi}{34}]\right]\sin[\frac{5\pi}{34}]\\
&-\frac{10}{17}\arctan\left[(u^{1/5}-\cos[\frac{4\pi}{17}])\csc[\frac{4\pi}{17}]\right]\sin[\frac{3\pi}{17}]\\
&+\frac{5}{17}\ln\left[1+u^{2/5}+2u^{1/5}\cos[\frac{\pi}{17}]\right]\sin[\frac{7\pi}{34}]\\
&-\frac{10}{17}\arctan\left[\sec[\frac{5\pi}{34}](u^{1/5}-\sin[\frac{5\pi}{34}])\right]\sin[\frac{4\pi}{17}].
\end{aligned}
$$

$$
\text{(5)}\quad \int_0^u \frac{1}{1-u^{11/3}}du \quad (\text{for } N=11/3) \tag{D.5}
$$

$$
\begin{aligned}
&=\frac{6}{11}\arctan\left[(u^{1/3}-\cos[\frac{2\pi}{11}])\csc[\frac{2\pi}{11}]\right]\cos[\frac{\pi}{22}]\\
&-\frac{6}{11}\arctan\left[(u^{1/3}+\sin[\frac{\pi}{22}])\sec[\frac{\pi}{22}]\right]\cos[\frac{3\pi}{22}]\\
&+\frac{6}{11}\arctan\left[(u^{1/3}+\cos[\frac{\pi}{11}])\csc[\frac{\pi}{11}]\right]\cos[\frac{5\pi}{22}]\\
&-\frac{3}{11}\ln[-1+u^{1/3}]+\frac{3}{11}\cos[\frac{\pi}{11}]\ln\left[1+u^{2/3}-2u^{1/3}\sin[\frac{3\pi}{22}]\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{11}\cos[\frac{2\pi}{11}]\ln\left[1+u^{2/3}+2u^{1/3}\sin[\frac{5\pi}{22}]\right] \\
&+ \frac{3}{11}\sin[\frac{\pi}{22}]\ln\left[1+u^{2/3}-2u^{1/3}\cos[\frac{2\pi}{11}]\right] \\
&- \frac{6}{11}\arctan\left[(u^{1/3}-\sin[\frac{3\pi}{22}])\sec[\frac{3\pi}{22}]\right]\sin[\frac{\pi}{11}] \\
&- \frac{3}{11}\sin[\frac{3\pi}{22}]\ln\left[1+u^{2/3}+2u^{1/3}\sin[\frac{\pi}{22}]\right] \\
&+ \frac{6}{11}\arctan\left[(u^{1/3}+\sin[\frac{5\pi}{22}])\sec[\frac{5\pi}{22}]\right]\sin[\frac{2\pi}{11}] \\
&+ \frac{3}{11}\sin[\frac{5\pi}{22}]\ln\left[1+u^{2/3}+2u^{1/3}\cos[\frac{\pi}{11}]\right].
\end{aligned}
$$

## D.2 ETF-Based Solutions of the Second Integral for Five $N$-Values

The alternative analytical solutions of the second integral appearing in Eq. (3.12) for $N = 3, 10/3, 17/5, 7/2$ and $11/3$ for fully rough flows in wide channels ($M = 3$) can be expressed in terms of elementary transcendental functions (ETF), respectively.

(1) For $M = N = 3$

An equation for the ETF-based solutions of the second integral appearing in (3.12) for $M = N = 3$ is counted identically as (D.1) because the expressions of the first and second integrals appearing in (3.12) are identical if $M = N = 3$.

The remaining equations as shown below, i.e., (D.6) through (D.9), represent the ETF-based solutions of the second integral for $N$ =10/3, 17/5, 7/2 and 11/3, respectively.

$$
\begin{aligned}
(2)\ &\int_0^u \frac{u^{1/3}}{1-u^{10/3}}du \quad (\text{for } M = 3 \text{ and } N = 10/3) \qquad \text{(D.6)} \\
&= \frac{3}{40}\left\{-2\sqrt{10-2\sqrt{5}}\arctan\left[\frac{1+\sqrt{5}-4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right]\right. \\
&\quad - 2\sqrt{2(5+\sqrt{5})}\arctan\left[\frac{1-\sqrt{5}+4u^{1/3}}{\sqrt{2(5+\sqrt{5})}}\right] \\
&\quad + 2\sqrt{2(5+\sqrt{5})}\arctan\left[\frac{-1+\sqrt{5}+4u^{1/3}}{\sqrt{2(5+\sqrt{5})}}\right] \\
&\quad - 2\sqrt{10-2\sqrt{5}}\arctan\left[\frac{1+\sqrt{5}+4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-4\ln[-1+u^{1/3}]-4\ln[1+u^{1/3}]\\
&-(-1+\sqrt{5})\ln\left[1-\frac{1}{2}(-1+\sqrt{5})u^{1/3}+u^{2/3}\right]\\
&-(-1+\sqrt{5})\ln\left[1+\frac{1}{2}(-1+\sqrt{5})u^{1/3}+u^{2/3}\right]\\
&+(1+\sqrt{5})\ln\left[1-\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right]\\
&+(1+\sqrt{5})\ln\left[1+\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right]\bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
(3)\ &\int_0^u \frac{u^{1/2}}{1-u^{7/2}}du \quad (\text{for } M=3 \text{ and } N=7/2) \qquad \text{(D.7)}\\
&=\frac{4}{7}\arctan\left[\csc[\frac{\pi}{7}](\sqrt{u}+\cos[\frac{\pi}{7}])\right]\cos[\frac{\pi}{14}]\\
&-\frac{4}{7}\arctan\left[\sec[\frac{\pi}{14}](\sqrt{u}+\sin[\frac{\pi}{14}])\right]\cos[\frac{3\pi}{14}]\\
&-\frac{2}{7}\ln[-1+\sqrt{u}]+\frac{2}{7}\cos[\frac{\pi}{7}]\ln\left[1+u-2\sqrt{u}\sin[\frac{3\pi}{14}]\right]\\
&+\frac{2}{7}\sin[\frac{\pi}{14}]\ln\left[1+u+2\sqrt{u}\cos[\frac{\pi}{7}]\right]\\
&+\frac{4}{7}\arctan\left[\sec[\frac{3\pi}{14}](\sqrt{u}-\sin[\frac{3\pi}{14}])\right]\sin[\frac{\pi}{7}]\\
&-\frac{2}{7}\sin[\frac{3\pi}{14}]\ln\left[1+u+2\sqrt{u}\sin[\frac{\pi}{14}]\right].
\end{aligned}
$$

$$
\begin{aligned}
(4)\ &\int_0^u \frac{u^{2/5}}{1-u^{17/5}}du \quad (\text{for } M=3 \text{ and } N=17/5) \qquad \text{(D.8)}\\
&=\frac{10}{17}\arctan\left[\sec[\frac{5\pi}{34}](u^{1/5}-\sin[\frac{5\pi}{34}])\right]\cos[\frac{\pi}{34}]\\
&+\frac{10}{17}\arctan\left[(u^{1/5}+\cos[\frac{\pi}{17}])\csc[\frac{\pi}{17}]\right]\cos[\frac{3\pi}{34}]\\
&-\frac{10}{17}\arctan\left[\sec[\frac{\pi}{34}](u^{1/5}-\csc[\frac{4\pi}{17}])\right]\cos[\frac{4\pi}{17}]\\
&-\frac{10}{17}\arctan\left[(u^{1/5}-\sin[\frac{\pi}{34}])\sec[\frac{\pi}{34}]\right]\cos[\frac{7\pi}{34}]\\
&-\frac{5}{17}\ln[-1+u^{1/5}]+\frac{5}{17}\cos[\frac{3\pi}{17}]\ln\left[1+u^{2/5}-2u^{1/5}\cos[\frac{2\pi}{17}]\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{5}{17}\cos[\frac{4\pi}{17}]\ln\left[1+u^{2/5}+2u^{1/5}\cos[\frac{3\pi}{17}]\right]\\
&-\frac{5}{17}\cos[\frac{2\pi}{17}]\ln\left[1+u^{2/5}+2u^{1/5}\sin[\frac{3\pi}{34}]\right]\\
&+\frac{5}{17}\cos[\frac{\pi}{17}]\ln\left[1+u^{2/5}+2u^{1/5}\sin[\frac{7\pi}{34}]\right]\\
&-\frac{5}{17}\ln\left[1+u^{2/5}-2u^{1/5}\sin[\frac{5\pi}{34}]\right]\sin[\frac{\pi}{34}]\\
&+\frac{10}{17}\arctan\left[\sec[\frac{7\pi}{34}](u^{1/5}+\sin[\frac{7\pi}{34}])\right]\sin[\frac{\pi}{17}]\\
&+\frac{5}{17}\ln\left[1+u^{2/5}+2u^{1/5}\cos[\frac{\pi}{17}]\right]\sin[\frac{3\pi}{34}]\\
&+\frac{10}{17}\arctan\left[(u^{1/5}+\sin[\frac{3\pi}{34}])\sec[\frac{3\pi}{34}]\right]\sin[\frac{2\pi}{17}]\\
&-\frac{5}{17}\ln\left[1+u^{2/5}-2u^{1/5}\cos[\frac{4\pi}{17}]\right]\sin[\frac{5\pi}{34}]\\
&+\frac{10}{17}\arctan\left[(u^{1/5}-\cos[\frac{2\pi}{17}])\csc[\frac{2\pi}{17}]\right]\sin[\frac{3\pi}{17}]\\
&+\frac{5}{17}\ln[1+u^{2/5}-2u^{1/5}\sin[\frac{\pi}{34}]]\sin[\frac{7\pi}{34}]\\
&-\frac{10}{17}\arctan\left[\csc[\frac{3\pi}{17}](u^{1/5}+\cos[\frac{3\pi}{17}])\right]\sin[\frac{4\pi}{17}].
\end{aligned}
$$

$$
(5)\quad \int_0^u \frac{u^{2/3}}{1-u^{11/3}}du \quad (\text{for } M=3 \text{ and } N=11/3) \tag{D.9}
$$

$$
\begin{aligned}
&=\frac{6}{11}\arctan\left[(u^{1/3}+\cos[\frac{\pi}{11}])\csc[\frac{\pi}{11}]\right]\cos[\frac{\pi}{22}]\\
&-\frac{6}{11}\arctan\left[(u^{1/3}+\sin[\frac{5\pi}{22}])\sec[\frac{5\pi}{22}]\right]\cos[\frac{3\pi}{22}]\\
&+\frac{6}{11}\arctan\left[(u^{1/3}+\sin[\frac{\pi}{22}])\sec[\frac{\pi}{22}]\right]\cos[\frac{5\pi}{22}]\\
&-\frac{3}{11}\ln[-1+u^{1/3}]+\frac{3}{11}\cos[\frac{\pi}{11}]\ln\left[1+u^{2/3}-2u^{1/3}\cos[\frac{2\pi}{11}]\right]\\
&-\frac{3}{11}\cos[\frac{2\pi}{11}]\ln\left[1+u^{2/3}-2u^{1/3}\sin[\frac{3\pi}{22}]\right]\\
&+\frac{3}{11}\sin[\frac{\pi}{22}]\ln\left[1+u^{2/3}+2u^{1/3}\cos[\frac{\pi}{11}]\right]\\
&+\frac{6}{11}\arctan\left[(u^{1/3}-\cos[\frac{2\pi}{11}])\csc[\frac{2\pi}{11}]\right]\sin[\frac{\pi}{11}]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{11}\sin[\frac{3\pi}{22}]\ln\left[1+u^{2/3}+2u^{1/3}\sin[\frac{5\pi}{22}]\right]\\
&-\frac{6}{11}\arctan\left[(u^{1/3}-\sin[\frac{3\pi}{22}])\sec[\frac{3\pi}{22}]\right]\sin[\frac{2\pi}{11}]\\
&+\frac{3}{11}\sin[\frac{5\pi}{22}]\ln\left[1+u^{2/3}+2u^{1/3}\sin[\frac{\pi}{22}]\right]
\end{aligned}
$$

# Appendix E
# Proof of the Identity Between the GHF-Based Solution and the Bresse Solution

We start with the GHF-based solutions of (3.8) in Chap. 3 in the domains of $|u| < 1$ and $|u| > 1$, as shown in (3.30) and (3.43), respectively, as well as with its corresponding ETF-based solution for the specified condition $M = N = 3$ (usually named Bresse solution), as shown in (3.78),

$$x_* = u\left[1 - g\left(\frac{1}{N}, u^N\right)\right] + \frac{\lambda^M u^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, u^N\right) + \text{Const.}, \tag{3.30}$$

$$x_* = u\left[1 - \frac{u^{-N}}{N-1} g\left(\frac{N-1}{N}, u^{-N}\right)\right] + \frac{\lambda^M u^{-M+1}}{M-1} g\left(\frac{M-1}{N}, u^{-N}\right) + \text{Const.}, \tag{3.43}$$

$$x_* = u - \left[1 - \left(\frac{h_c}{h_n}\right)^3\right] \times \left[\frac{1}{6}\ \ln \frac{(u^2+u+1)}{(u-1)^2} + \frac{1}{\sqrt{3}}\ \arctan \frac{(2u+1)}{\sqrt{3}}\right] + \text{Const.}, \tag{3.78}$$

in which $\lambda = h_c/h_n$. To represent the complete GHF-based solution for $M = N = 3$, we opt to use a combination of (3.30) for $0 \leq u < 1$ and (3.43) for $u > 1$ as typical of the nine possible GHF-based solutions of (3.8) for $0 \leq u < \infty$ excluding at $u = 1$, as mentioned in Chap. 3. Therefore, (3.30) on substitution of $M = N = 3$ yields

$$x_* = u\left[1 - g\left(\frac{1}{3}, u^3\right)\right] + \lambda^3 u g\left(\frac{1}{3}, u^3\right) + \text{Const.}, \tag{E.1}$$

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which is valid for $0 \leq u < 1$. Likewise, (3.43) on substitution of $M = N = 3$ yields

$$x_* = u\left[1 - \frac{u^{-3}}{2} g\left(\frac{2}{3}, u^{-3}\right)\right] + \frac{\lambda^3 u^{-2}}{2} g\left(\frac{2}{3}, u^{-3}\right) + \text{Const.}, \tag{E.2}$$

which is valid for $u > 1$. Rearrangeing (E.1) and (E.2), we obtain the more concise expressions

$$x_* = u\left[1 - (1 - \lambda^3) g\left(\frac{1}{3}, u^3\right)\right] + \text{Const.}, \tag{E.3}$$

$$x_* = u - \frac{(1 - \lambda^3) u^{-2}}{2} g\left(\frac{2}{3}, u^{-3}\right) + \text{Const.}, \tag{E.4}$$

respectively. The GHF-based solutions for $M = N = 3$ consist of two parts, i.e., (E.3) for $0 \leq u < 1$ and (E.4) for $u > 1$, while the corresponding analytical solution of (3.8) obtained from the Mathematica software expressed in terms of the ETF for $M = N = 3$ is (3.78), which has been shown to be derivable from (3.12) upon substitution of (D.1) from Appendix D.

## E.1 Two Identity Relations Between the GHF-Based and ETF-Based Functions

Comparison of (E.3) for $0 \leq u < 1$ and (E.4) for $u > 1$ with (3.78) for $0 \leq u < \infty$, except $u = 1$, leads to the formulation of the following two identities:

$$u g\left(\frac{1}{3}, u^3\right) = \frac{1}{6} \ln\left(\frac{u^2 + u + 1}{(u - 1)^2}\right) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + \text{Const.}, \tag{E.5}$$

which is valid for $0 \leq u < 1$, and

$$\frac{u^{-2}}{2} g\left(\frac{2}{3}, u^{-3}\right) = \frac{1}{6} \ln\left(\frac{u^2 + u + 1}{(u - 1)^2}\right) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + \text{Const.}, \tag{E.6}$$

which is valid for $u > 1$. The constant of integration, “Const.” in (E.5) or (E.6) is determined using a boundary condition in the respective domain of $u$. We determine “Const.” in (E.5) and (E.6) subject to the boundary conditions $u = 0$ and $u = \infty$, respectively, as follows: Substituting $u = 0$ into (E.5) yields

$$C_0 = -\frac{1}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6\sqrt{3}}, \tag{E.7}$$

in which $\arctan(1/\sqrt{3}) = \pi/6$ because $\arctan(1/\sqrt{3})$ is the principal value of the angle whose tangent is $1/\sqrt{3}$ and is defined in the Mathematica software as $-\pi/2 \leq \arctan(1/\sqrt{3}) \leq \pi/2$. Likewise, substituting $u = \infty$ into (E.6) yields

$$C_\infty = -\frac{1}{\sqrt{3}}\arctan(\infty) = -\frac{\pi}{2\sqrt{3}}, \tag{E.8}$$

in which $\arctan(\infty) = \pi/2$ because $\arctan(\infty)$ is the principal value of the angle whose tangent is $\infty$ within the afore-defined range in the Mathematica software. This is attributed to the fact that the logarithmic function appearing in the first term on the right-hand side of (E.6) disappears by virtue of L'Hopital's rule, namely

$$\lim_{u\to\infty} \ln\left(\frac{u^2+u+1}{(u-1)^2}\right) = 0. \tag{E.9}$$

It follows that (E.5) and (E.6) on substitution of "Const." by $C_0$ from (E.7) and by $C_\infty$ from (E.8), respectively, yield

$$ug\left(\frac{1}{3}, u^3\right) = \frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{\pi}{6\sqrt{3}}, \tag{E.10}$$

$$\frac{u^{-2}}{2}g\left(\frac{2}{3}, u^{-3}\right) = \frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{\pi}{2\sqrt{3}}. \tag{E.11}$$

Equation (E.10) is an identity being valid for $0 \leq u < 1$ between the GHF and the corresponding logarithmic—inverse trigonometric functions, while (E.11) is another identity being valid for $u > 1$ between the GHF and the corresponding logarithmic—inverse trigonometric functions. It is noted that (E.10) and (E.11) are useful in the proof of the identity between the GHF-based solution for $M = N = 3$ and the Bresse solution. Two examples are given in the following to substantiate such identities.

## E.2 Proof of the Identity Between the GHF-Based and ETF-Based Solutions

For example, to plot the M2, M3, and S2 profiles, we can use either the GHF-based solution [i.e., (E.3)] or the ETF-based solution [i.e., (3.78)], and it can be shown that both solution equations should yield identical results. A boundary condition $(x_*, u) = (0, u_1)$ for plotting one of such profiles is arbitrarily selected at $x_* = 0$, where the flow depth, $h$, is assumed to be $h_1 = u_1 h_n$ ($u_1$ = any positive real number). Upon substitution of this boundary condition, the "Const." terms in (E.3) and (3.78) can, respectively, be expressed

$$C_1 = -u_1 + (1-\lambda^3)u_1 g\left(\frac{1}{3}, u_1^3\right), \tag{E.12}$$

$$C_2 = -u_1 + (1-\lambda^3)\left[\frac{1}{6}\ln\left(\frac{u_1^2+u_1+1}{(u_1-1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u_1+1}{\sqrt{3}}\right)\right]. \tag{E.13}$$

When we plot either (E.3) on substitution of "Const." by $C_1$ from (E.12) or (3.78) on substitution of "Const." by $C_2$ from (E.13), we find that both profiles so plotted must be identical because subtracting the right-hand expression of (3.78) from that of (E.3) or vice versa, the result can be proved to be zero. In other words, we have

$$\begin{aligned}&(1-\lambda^3)\left\{-ug\left(\frac{1}{3}, u^3\right) + \left[\frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right)\right]\right\}\\ &+(1-\lambda^3)\left\{u_1 g\left(\frac{1}{3}, u_1^3\right) - \left[\frac{1}{6}\ln\left(\frac{u_1^2+u_1+1}{(u_1-1)^2}\right)\right.\right.\\ &\qquad\qquad\left.\left.+\frac{1}{\sqrt{3}}\arctan\left(\frac{2u_1+1}{\sqrt{3}}\right)\right]\right\} = 0,\end{aligned} \tag{E.14}$$

because adding or subtracting the same constant, $-(1/\sqrt{3})\arctan(1/\sqrt{3})$, to or from the expressions appearing in the two square brackets of (E.14) can offset each other, but cause the expressions within the two braces to vanish by virtue of (E.7) and (E.10).

In another example, if the M1, S1, and S2 profiles are plotted, we can use either (E.4) or (3.78). Likewise, it can be readily shown that both equations should yield an identical result. A boundary condition $(x_*, u) = (0, u_2)$ for plotting one of such profiles is arbitrarily selected at $x_* = 0$, where the flow depth, $h$, is assumed to be $h_2 = u_2 h_n$ ($u_2$ = any positive real number). The "Const." in (E.4) and (3.78) can be respectively expressed from (E.4) and (3.78) upon substitution of this boundary condition as

$$C_3 = -u_2 + (1-\lambda^3)\frac{u_2^{-2}}{2} g\left(\frac{2}{3}, u_2^{-3}\right), \tag{E.15}$$

$$b4\, C_4 = -u_2 + (1-\lambda^3)\left[\frac{1}{6}\ln\left(\frac{u_2^2+u_2+1}{(u_2-1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u_2+1}{\sqrt{3}}\right)\right]. \tag{E.16}$$

When we plot either (E.4) on substitution of "Const." by $C_3$ from (E.15) or (3.78) on substitution of "Const." by $C_4$ from (E.16), we find that both profiles so plotted should be identical because subtracting the right-hand expression of (3.78) from that of (E.4) or vice versa the result can again be proved to be zero. In other words, we have

$$
\begin{aligned}
&(1-\lambda^3)\left\{-\frac{u^{-2}}{2}g\left(\frac{2}{3},u^{-3}\right)\right.\\
&\qquad\left.+\left[\frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right)+\frac{1}{\sqrt{3}}\ \arctan\left(\frac{2u+1}{\sqrt{3}}\right)\right]\right\}\\
&+(1-\lambda^3)\left\{\frac{u_2^{-2}}{2}g\left(\frac{2}{3},u_2^{-3}\right)\right.\\
&\qquad\left.-\left[\frac{1}{6}\ln\left(\frac{u_2^2+u_2+1}{(u_2-1)^2}\right)+\frac{1}{\sqrt{3}}\arctan\left(\frac{2u_2+1}{\sqrt{3}}\right)\right]\right\}=0.
\end{aligned}
\tag{E.17}
$$

because adding or subtracting the very same constant, $-(1/\sqrt{3})\arctan(\infty)$, to or from the expressions appearing in the two square brackets of (E.17) can offset each other, but cause the expressions within the two braces to vanish by virtue of (E.8) and (E.11).

## E.3 Alternative Forms of the Identity Relations and Bresse Solution

In Appendix E, we have analytically proved that the complete GHF-based solutions, i.e., a combination of (3.30) for $0 \le u < 1$ and (3.43) for $u > 1$, in a particular case of $M = N = 3$ is exactly identical to the so-called Bresse solution, i.e., (3.78), the simplest ETF-based solution of (3.8).

In addition, there is a relation between $\arctan z$ and $\operatorname{arccot} z$ for a variable $z$. That is to say,

$$
\arctan\left(\frac{2u+1}{\sqrt{3}}\right)+\operatorname{arccot}\left(\frac{2u+1}{\sqrt{3}}\right)=\frac{\pi}{2}.
\tag{E.18}
$$

Therefore, (E.10) for $0 \le u < 1$ and (E.11) for $u > 1$, can be written as (E.19) and (E.20), respectively.

$$
ug\left(\frac{1}{3},u^3\right)=\frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right)-\frac{1}{\sqrt{3}}\operatorname{arccot}\left(\frac{2u+1}{\sqrt{3}}\right)+\frac{\pi}{3\sqrt{3}},
\tag{E.19}
$$

$$
\frac{u^{-2}}{2}g\left(\frac{2}{3},u^{-3}\right)=\frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right)-\frac{1}{\sqrt{3}}\operatorname{arccot}\left(\frac{2u+1}{\sqrt{3}}\right)
\tag{E.20}
$$

and, the Bresse solution [i.e., (3.78)] can also be written in another form as

$$
x_* = u-(1-\lambda^3)\left[\frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right)-\frac{1}{\sqrt{3}}\operatorname{arccot}\left(\frac{2u+1}{\sqrt{3}}\right)\right]+\text{Const.}.
\tag{E.21}
$$

# Appendix F
# Obtain the GHF-Based Solution for | $\lambda v$ | > 1 from that for | $\lambda v$ | < 1, or Vice Versa

Let us start with the GHF-based solutions of (4.4) in the domains of $|\ \lambda v\ | < 1$ and $|\ \lambda v\ | > 1$ for GVF in sustaining channels, viz.,

$$x_{\sharp} = \frac{v^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, (\lambda v)^N\right) - \frac{v^{N+1}}{N+1} g\left(\frac{N+1}{N}, (\lambda v)^N\right) + \text{Const.}, \tag{4.32}$$

$$x_{\sharp} = \lambda^{-N} v g\left(-\frac{1}{N}, (\lambda v)^{-N}\right) + \frac{\lambda^{-N} v^{-M+1}}{M-1} g\left(\frac{M-1}{N}, (\lambda v)^{-N}\right) + \text{Const.} \tag{4.47}$$

and, the GHF-based solutions of (4.12) in the domains of $|\ \lambda v\ | < 1$ and $|\ \lambda v\ | > 1$ for GVF in adverse channels,

$$x_{\sharp} = \frac{v^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, -(\lambda v)^N\right) - \frac{v^{N+1}}{N+1} g\left(\frac{N+1}{N}, -(\lambda v)^N\right) + \text{Const.}, \tag{4.69}$$

$$x_{\sharp} = -\lambda^{-N} v g\left(-\frac{1}{N}, -(\lambda v)^{-N}\right) - \frac{\lambda^{-N} v^{-M+1}}{M-1} g\left(\frac{M-1}{N}, -(\lambda v)^{-N}\right) + \text{Const..} \tag{4.84}$$

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For specified parameter relations, $a = 1$ and $c = b + 1$, the relation connecting one GHF in the domain of $\mid z \mid < 1$ to two GHF in the domain of $\mid z \mid > 1$, as worked out in detail in Appendices A and C, can be written as

$$\begin{aligned} &{}_2F_1(1, b; 1 + b; z) \\ &\quad = \tfrac{b}{b-1}(-z^{-1})\ {}_2F_1(1, 1 - b; 2 - b; z^{-1}) + \Gamma(1 - b)\Gamma(1 + b)\left(-z^{-1}\right)^b . \end{aligned} \tag{F.1}$$

The second term on the right-hand side of (F.1) has been simplified from the second GHF of the relation in the domain of $\mid z \mid > 1$. For reducing the related equations to shorter expressions in order to facilitate the reading of the manuscript, we will use $g(b, z)$ instead of ${}_2F_1(1, b; b + 1; z)$ herein to represent the GHF when we treat the GHF-solutions. Therefore, the shorter expression of (F.1) is given as

$$g(b; z) = \frac{b}{b-1}(-z^{-1})g(1 - b, z^{-1}) + \Gamma(1 - b)\Gamma(1 + b)(-z^{-1})^b. \tag{F.2}$$

We exemplify (F.1) or (F.2) so formulated for the following two cases. The first case is to derive the GHF-based solution of (4.4) in the domain of $\mid \lambda v \mid > 1$ from its counterpart in the domain $\mid \lambda v \mid < 1$ for GVF in sustaining channels, and the second case is to derive the GHF-based solution of (4.12) in the domain of $\mid \lambda v \mid > 1$ from its counterpart in the domain $\mid \lambda v \mid < 1$ for GVF in adverse channels. In what follows, we work out both cases using (F.2) without recourse to the formulation of (4.13) and (4.14) [i.e., alternative forms of (4.4) and (4.12), respectively, with their variable, $w$, being expressed as $v^{-1}$], which can be solved for their respective GHF-based GVF profiles in the domain of $\mid \lambda v \mid > 1$.

## F.1 Obtain (4.47) Directly from (4.32) by Using the Transformation Relation

Because the first and second GHF on the right-hand side of (4.32) have the specified parameter relations, $b = (N + M - 1)/N$ and $b = (N + 1)/N$, respectively, (F.2) on substitution of such $b$-expressions to the first and second GHF on the right-hand side of (4.32) yield, respectively,

$$\begin{aligned} g\left(\frac{N-M+1}{N}, (\lambda v)^N\right) = &\frac{(N-M+1)(\lambda v)^{-N}}{M-1} g\left(\frac{M-1}{N}, (\lambda v)^{-N}\right) \\ &+ \Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}\right) \\ &- \left(-(\lambda v)^{-N}\right)^{(N-M+1)/N}, \end{aligned} \tag{F.3}$$

$$g\left(\frac{N+1}{N},(\lambda v)^N\right)=-\frac{(\lambda v)^{-N}}{N+1}g\left(\frac{-1}{N},(\lambda v)^{-N}\right)$$
$$+\Gamma\left(\frac{N-1}{N}\right)\Gamma\left(\frac{N+1}{N}\right)\left(-(\lambda v)^{-N}\right)^{1/N}. \tag{F.4}$$

Substituting (F.3) and (F.4), respectively, into the first and second GHF appearing on the right-hand side of (4.32), yields

$$x_\sharp=\lambda^{-N}vg\left(-\frac{1}{N},(\lambda v)^{-N}\right)+\frac{\lambda^{-N}v^{-M+1}}{M-1}g\left(\frac{M-1}{N},(\lambda v)^{-N}\right)$$
$$+\frac{(-1)^{(N+1)/N}\lambda^{-(N+1)}}{N+1}\Gamma\left(-\frac{1}{N}\right)\Gamma\left(\frac{2N+1}{N}\right)$$
$$+\frac{(-1)^{(N-M+1)/N}\lambda^{-(N-M+1)}}{N-M+1}\Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}\right)+\text{Const.}. \tag{F.5}$$

Comparison of (F.5) with (4.47) reveals that (F.5) is identical to (4.47) because two constant terms appearing in (F.5) in addition to “Const.” can be ignored without losing the general meaning of the constant term. Consequently, one may conclude that (4.47) obtained from (4.13) is indeed the GHF-based solution of (4.4) in the domain of $|\ \lambda v\ | > 1$. This reconfirms the validity of (4.47) in the domain of $|\ \lambda v\ | > 1$ and the uniqueness of a role of (4.47) playing in the computation of GVF profiles in sustaining channels.

## F.2 Obtain (4.84) Directly from (4.69) by Using the Transformation Relation

Likewise, in the second case, we use (F.2) to transform the GHF-based solution of (4.12) in the domain of $|\ \lambda v\ | < 1$, i.e., (4.69), to the corresponding GHF-based solution of (4.12) in the domain of $|\ (h_c/h_n)v\ | > 1$, i.e., (4.84) for GVF in adverse channels. In the following, we can prove that (4.69) for $|\ \lambda v\ | < 1$ is transformable to (4.84) for $|\ \lambda v\ | > 1$, which we have actually obtained from (4.14) [i.e., an alternative form of (4.12) with its variable, $w$, being expressed as $v^{-1}$] rather than through (4.12).

Again, we have $b=(N-M+1)/N$ and $b=(N+1)/N$ in the first and second GHF, respectively, with the variable $z=-(\lambda v)^N$ of both GHF appearing on the right-hand side of (4.69); therefore, (F.2) on substitution of such two respective $b$-expressions yield, respectively,

$$g\left(\frac{N-M+1}{N},-(\lambda v)^N\right)=\frac{(N-M+1)(\lambda v)^{-N}}{-M+1}g\left(\frac{M-1}{N},-(\lambda v)^{-N}\right) \tag{F.6}$$
$$+\Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}\right)(\lambda v)^{-(N-M+1)},$$

$$g\left(\frac{N+1}{N}, -(\lambda v)^N\right) = (\lambda v)^{-N}(N+1)g\left(\frac{-1}{N}, -(\lambda v)^{-N}\right) + \Gamma\left(\frac{-1}{N}\right)\Gamma\left(\frac{2N+1}{N}\right)(\lambda v)^{-(N+1)}. \tag{F.7}$$

Substituting (F.6) and (F.7) into the first and second GHF appearing on the right-hand side of (4.69), respectively, yields

$$\begin{aligned} x_\sharp = &-\lambda^{-N} v g\left(-\frac{1}{N}, -(\lambda v)^{-N}\right) - \frac{\lambda^{-N} v^{-M+1}}{M-1} g\left(\frac{M-1}{N}, -(\lambda v)^{-N}\right) \\ &+ \frac{\lambda^{-(N+1)}}{N+1}\Gamma\left(-\frac{1}{N}\right)\Gamma\left(\frac{2N+1}{N}\right) \\ &+ \frac{\lambda^{-(N-M+1)}}{N-M+1}\Gamma\left(\frac{M-1}{N}\right)\Gamma\left(\frac{2N-M+1}{N}\right) + \text{Const.}. \end{aligned} \tag{F.8}$$

Comparison of (F.8) with (4.84) reveals that (F.8) is identical to (4.84) because two constant terms appearing in (F.8) in addition to "Const." can be ignored without losing the gereral meaning of the constant term.

# Appendix G
# Asymptotic Reduction of the GHF-Based GVF Solutions for Flow in Sustaining and Adverse Channels to the Solutions for Flow in Horizontal Channels

The starting point is the GHF-based solutions of (4.4) in the domain $\mid \lambda v \mid < 1$ for GVF in sustaining channels:

$$x_{\sharp} = \frac{v^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, (\lambda v)^N\right) - \frac{v^{N+1}}{N+1} g\left(\frac{N+1}{N}, (\lambda v)^N\right) + \text{Const.} \tag{4.32}$$

and, the GHF-based solutions of (4.12) in the domain $0 \leq \lambda v < \infty$ for GVF in adverse channels:

$$x_{\sharp} = \frac{v^{N-M+1}}{N-M+1} g\left(\frac{N-M+1}{N}, -(\lambda v)^N\right) - \frac{v^{N+1}}{N+1} g\left(\frac{N+1}{N}, -(\lambda v)^N\right) + \text{Const.}. \tag{4.69}$$

To prove that GVF profiles on the horizontal slope can also be derived from the asymptotic reduction of their counterparts on the sustaining slope ($\theta > 0$) or on the adverse slope ($\theta < 0$) as $\theta \to 0$, we should simultaneously examine: firstly, whether the valid domain of $v$ for each GHF-based solution of (4.4) and (4.12) can reduce asymptotically to that of (4.11); and secondly, whether each GHF-based solution of (4.4) and (4.12) can by itself reduce asymptotically to (4.11) as $\lambda$ $(= h_c/h_n) \to 0$ (or $h_n \to \infty$). As shown in Table 5.1, the successful completion of one of the above two analyses without support from the other does not necessarily warrant the asymptotic reduction of GVF profiles on the sustaining or adverse slope to their counterparts on the horizontal slope as $\theta \to 0$. It appears simpler to examine the asymptotic reduction of the valid domain of $v$ for each GHF-based solution of (4.4) and (4.12) to that for (4.11) as $\lambda \to 0$ because we can readily grasp it without elaborating its proof. However, for clarity, our attention should be more focused below on the scrutiny of each GHF-based solution of (4.4) and (4.12) by checking the existence of

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its limit at $\lambda = 0$. This is in fact more critical than simply checking its valid domain of $v$, thus enabling us to determine which one of the GHF-based solutions of (4.4) and (4.12) can reduce asymptotically to (4.11) as $\lambda \to 0$.

The proof of the asymptotic reduction of each GHF-based solution of (4.4) and (4.12) to (4.11) as $\lambda \to 0$ (or $h_n \to \infty$) can be handled by substituting $\lambda = 0$ into every term in the hypergeometric series, as defined in (A.5) of Appendix A. For the convenience of discussion, we rewrite the definition of GHF [i.e., (A.5)] here as

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k, \tag{G.1} $$

in which $| z | < 1$. The expansion of the hypergeometric series is given as

$$ \begin{aligned} &{}_2F_1(a, b; c; z) \\ &\quad = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \left\{ \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)0!} + \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(c+1)1!} z + \ldots \right\} \\ &\quad = \left\{ 1 + \frac{ab}{c} z + \frac{ab(a+1)(b+1)}{c(c+1)2!} z^2 \ldots \right\}. \end{aligned} \tag{G.2} $$

Since $z^0 = 1$ and $0! = 1$, the first term of GHF in (G.1) is unity as shown in (G.2). The second or higher term ($k \geq 1$) of every GHF vary with the expressions of the parameters $a$, $b$, and $c$ as well as the form of the variable $z$ in ${}_2F_1(a, b; c; z)$. Since every GHF appearing in each GHF-based solution of (4.4) and (4.12) has the property of $a = 1$ and $c = b + 1$, the simpler expression $g(b, z)$ is used to replace ${}_2F_1(a, b; c; z)$ for shortening the related expressions,

$$ g(b, z) = {}_2F_1(1, b; b+1; z) = b \sum_{k=0}^{\infty} \frac{z^k}{b+k} = 1 + \frac{bz}{b+1} + \frac{bz^2}{b+2} + \frac{bz^3}{b+3} + \ldots. \tag{G.3} $$

As $\lambda\,(= h_c/h_n) \to 0$, we can express the asymptotic reduction of the first and second terms on the right-hand side of (4.32), respectively, as

$$ \begin{aligned} &\lim_{\lambda \to 0} \left\{ \frac{v^{N-M+1}}{N-M+1} g\left( \frac{N-M+1}{N}, (\lambda v)^N \right) \right\} \\ &\qquad = \lim_{\lambda \to 0} \left\{ \frac{v^{N-M+1}}{N-M+1} \left( 1 + \frac{(N-M+1)(\lambda v)^N}{2N-M+1} + \ldots \right) \right\} \\ &\qquad = \frac{v^{N-M+1}}{N-M+1}, \end{aligned} \tag{G.4} $$

$$\lim_{\lambda\to 0}\left\{-\frac{v^{N+1}}{N+1}g\left(\frac{N+1}{N},(\lambda v)^N\right)\right\}$$
$$=\lim_{\lambda\to 0}\left\{-\frac{v^{N+1}}{N+1}\left[1+\frac{(N+1)(\lambda v)^N}{2N+1}+\dots\right]\right\} \tag{G.5}$$
$$=-\frac{v^{N+1}}{N+1}.$$

Adding (G.4) and (G.5) yields exactly the right-hand expression of (4.11), thus proving that (4.32) reduces asymptotically to (4.11) as $\lambda$ ( $= h_c/h_n$ ) $\to 0$.

Clearly, if and only if (4.32) is used to describe M2 and M3 profiles on the M slope ($0 < h_c/h_n < 1$), it can be proved to have the limit at $\lambda$ ($= h_c/h_n$) $\to 0$, as shown in (G.4) and (G.5). In fact, as $h_c/h_n \to 0$, the valid domain of $v$ for M2 profiles reduces from $1 \le v < h_n/h_c$ to $1 \le v < \infty$, which is identical to that for the H2 profile, and left intact is the valid domain of $v$ for M3 profiles, i.e., $0 \le v \le 1$, which is identical to that for the H3 profile.

The asymptotic reduction of (4.69) to (3.11) as $h_c/h_n \to 0$ can be treated in the same way as that of (4.32) to (3.11). In the treatment, we pay attention to the sole difference in the expressions of the two corresponding GHF appearing in (4.32) and (4.69). Comparing them, we can readily see that it is the positive or negative sign of the variable in one set of the two GHF that differs from the other set of the two GHF; otherwise, the two corresponding GHF shown in (4.32) and (4.69) are identical. Following the same evaluation procedure which has resulted in (G.4) and (G.5), we can acquire that the first and second terms on the right-hand side of (4.69) reduce asymptotically to the following respective expressions as $h_c/h_n \to 0$.

As $\lambda$ ($= h_c/h_n$) $\to 0$, we can express the asymptotic reduction of the first and second terms on the right-hand side of (4.69), respectively, as

$$\lim_{\lambda\to 0}\left\{\frac{v^{N-M+1}}{N-M+1}g\left(\frac{N-M+1}{N},-(\lambda v)^N\right)\right\}$$
$$=\lim_{\lambda\to 0}\left\{\frac{v^{N-M+1}}{N-M+1}\left(1-\frac{(N-M+1)(\lambda v)^N}{2N-M+1}+\dots\right)\right\} \tag{G.6}$$
$$=\frac{v^{N-M+1}}{N-M+1},$$

$$\lim_{\lambda\to 0}\left\{-\frac{v^{N+1}}{N+1}g\left(\frac{N+1}{N},-(\lambda v)^N\right)\right\}$$
$$=\lim_{\lambda\to 0}\left\{-\frac{v^{N+1}}{N+1}\left[1-\frac{(N+1)(\lambda v)^N}{2N+1}+\dots\right]\right\} \tag{G.7}$$
$$=-\frac{v^{N+1}}{N+1}.$$

Adding (G.6) and (G.7) yields the right-hand expression of (4.11) too, thus proving that (4.69) also reduces asymptotically to (4.11) as $\lambda(= h_c/h_n) \to 0$. In fact, as $\lambda \to 0$, the valid domain of $v$ for A2 profiles reduces from $1 \le v < h_n/h_c$ to

$1 \leq v < \infty$, which is identical to that for the H2 profile, and left intact is the valid domain of $v$ for A3 profiles, i.e., $0 \leq v \leq 1$, which is identical to that for the H3 profile. These results are summarized in Table 5.1 in Chap. 5.

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