4 In a pavement slab of uniform thickness the maximum deformation occurs along the edge of the slab for both the four-wheel and six-wheel vehicles The deformation recorded by the gage on the diagonal showed 45 per cent for four-wheel and 48 per cent for six-wheel vehicles, of the deformation recorded in the edge of the slab

5 For the particular conditions of this test, a load passing along a pavement 9 inches from the edge produced approximately twice the fiber deformation in the edge of the pavement that was caused by the same load passing along a path 21 inches from the edge (The one stress was from 43 per cent to 51 per cent of the other in the various tests, with an average of 47 per cent for all tests)

COMPUTATION OF STRESSES IN CONCRETE ROADS

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One may obtain a computation of stresses in concrete roads by assuming the slab to act as a homogeneous isotropic elastic solid in equilibrium, and by assuming the reactions of the subgrade to be vertical only and to be proportional to the deflections of the slab With these assumptions introduced, the analysis is reduced to a problem of mathematical theory of elasticity

The reaction of the subgrade per unit of area at any given point will be expressed as a coefficient k times the deflection z at the point This coefficient is a measure of the stiffness of the subgrade, and may be stated in pounds per square inch of area per inch of deflection, that is, in lb /in^3 The coefficient k will be called the *modulus of subgrade reaction* It corresponds to the "modulus of elasticity of rail support" which has been used in recent investigations of stresses in railroad track [1] The modulus k is assumed to be constant at each point, independent of the deflections, and to be the same at all points within the area which is under consideration It is true that tests of bearing pressures on soils have indicated a modulus k which varies considerably depending upon the area over which the pressure is distributed [2]

[1] Progress report of the special committee to report on stresses in railroad track, Am Soc Civil Engineers, Trans , v 82, 1918, p 1191

[2] Tests dealing with this question have been reported by A T Goldbeck, Researches on the structural design of highways by the United States Bureau of Public Roads, Am Soc Civil Engineers, Trans , v 88, 1925, p 264, especially p 271, by A T Goldbeck and M J Bussard, The supporting value of soil as influenced by the bearing area, Public Roads, Jan 1925, and by A Bijls, in Génie Civil, v 82, 1923, p 490 According to these tests, in the case of a pressure which is distributed uniformly over an area, the modulus k would be approximately inversely proportional to the squareroot of the area This result is supported by theoretical considerations

Yet, so long as the loads are limited to a particular type, that of wheel loads on top of the pavement, it is reasonable to assume that some constant value of the modulus k, determined empirically, will lead to a sufficiently accurate analysis of the deflections and the stresses One finds an argument in favor of the assumption of a constant modulus k for a given stretch of road by examining the tables which are given below they show that an increase of k from 50 lb /in³ to 200 lb /in³, that is, an increase of the stiffness of the subgrade in the ratio of four to one, causes only minor changes of the important stresses Minor variations of k, therefore, can be of no great consequence, and an approximate single value of k should be sufficient for a quite accurate determination of the important stresses within a given stretch of the road The modulus k enters in the formula for the deflections of the pavements, and may be determined empirically, accordingly, for a given type of subgrade, by comparing the deflections found by tests of full-sized slabs with the deflections given by the formulas

It will be assumed for the time being that the thickness of the slab is uniform and is equal to h

A certain quantity which is a measure of the stiffness of the slab relative to that of the subgrade occurs repeatedly in the analysis It is of the nature of a linear dimension, like, for example, the radius of gyration It will be called the *radius of relative stiffness* It is denoted by l, and is expressed by the formula

$$l = \sqrt[4]{\frac{Eh^3}{12\,(1 - \mu^2)\,k}} \tag{1}$$

where E is the modulus of elasticity of the concrete, and μ is Poisson's ratio of lateral expansion to longitudinal shortening The stiffer the slab, and the less stiff the subgrade, the greater is l One may observe that l remains constant when E and k are multiplied by the same ratio Table I contains values of l for three different values of k and for different thickensses of the slab In computing this table as well as the three tables following, Poisson's ratio μ was assumed to be 0 15, this value agrees satisfactorily with the results of tests by A N Johnson [1] The values of l given in the table lie between 16 inches and 55 inches, about 36 inches may be considered to be a typical average

THREE CASES OF LOADING INVESTIGATED

Figure 1 shows three cases in which it is of particular interest to be able to compute the critical stresses In case I, a wheel load acts close to a rectangular corner of a large panel of the slab This load tends toward producing a corner break The critical stress is a tension at the top of the slab The resultant pressure is assumed to be on the

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[1] A N Johnson, Direct measurement of Poisson's ratio for concrete, Am Soc for Testing Materials, Proc , v 24, Part II, 1924, p 1024

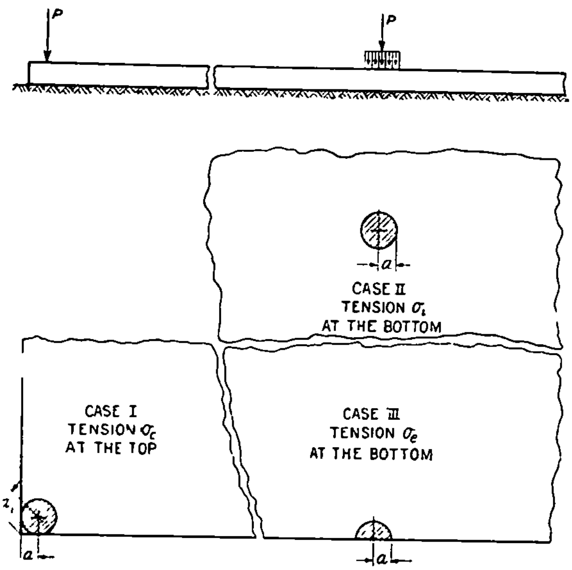

Figure 1—Three cases of loading Corresponding greatest stresses are given in Tables II, III, and IV

bisector of the right angle of the corner, at the small distance a from each of the two intersecting edges, the distance from the corner, accordingly, is $a_1 = a\sqrt{2}$ In case *II*, the wheel load is at a considerable

TABLE I

Values of the radius of relative stiffness, l, for different values of the slab thickness, h, and of the modulus of subgrade reaction, k, computed from equation (1)

$E = 3{,}000{,}000$ pounds per square inch $\mu = 0\ 15$

Thickness of slab in inches h	Radius of relative stiffness, l, in inches		
	$k = 50$ lb /in ^{3}	$k = 100$ lb /in ^{3}	$k = 200$ lb /in ^{3}
4	23 91	20 11	16 92
5	28 28	23 78	20 00
6	32 40	27 26	22 92
7	36 40	30 60	25 73
8	40 23	33 83	28 44
9	43 94	36 95	31 07
10	47 55	40 00	33 62
11	51 08	42 94	36 11
12	54 52	45 84	38 56

distance from the edges The pressure is assumed to be distributed uniformly over the area of a small circle with radius a The critical tension occurs at the bottom of the slab under the center of the circle In case *III*, the wheel load is at the edge, but at a considerable distance from any corner The pressure is assumed to be distributed uniformly over the area of a small semicircle with the center at the edge and with radius a The critical stress is a tension at the bottom under the center of the circle In each of the three cases the load mentioned is assumed for the time being to be the only load acting

For case I a computation which may be looked upon as a first approximation was proposed by A T Goldbeck Further emphasis was given to this method by Clifford Older [1] The load is treated as a force concentrated at the corner itself, that is, one assumes $a = a_1 = 0$ At small distances from the corner the influence of the reactions of the subgrade upon the stresses will be small compared with that due to the load The corner portion may be considered, therefore, to act as a cantilever of uniform strength At the distance x, measured diagonally from the corner along the bisector of the right angle of the corner, the bending moment is $-Px$ This bending moment may be assumed to be distributed uniformly over the cross-section, the width of which is $2x$ Thus one finds the bending moment per unit of width of cross-section equal to $-\frac{P}{2}$, and the tensile stress at the top equal to

$$\sigma = \frac{3P}{h^2} \tag{2}$$

Since the wheel load is distributed over the area of contact between the tire and the pavement, the distances a and a_1 can not be zero The greatest stress occurs, then, at some distances from the load This distance will be sufficiently large to make the reactions of the subgrade outside the critical section contribute a noticeable reduction of the numerical value of the bending moment

An improved approximation has been obtained in the following manner The origin of the horizontal rectangular coordinates x and y is taken at the corner, the axis of x bisecting the right angle of the corner By use of Ritz's method of successive approximation, which is based on the principle of minimum of energy,[1] the following approximate expression was found for the deflections in the neighborhood of the corner

$$z = \frac{P}{kl^2}\left(1\,1\,e^{-\frac{x}{l}} - \frac{a_1}{l}0\,88\,e^{-\frac{2x}{l}}\right) \tag{3}$$

Then the reactions of the subgrade will be expressed with sufficient exactness in terms of this function as kz One may compute, then, the total bending moment M^1 in the section $x = x_1$ due to the combined influence of the applied load and the reactions of the subgrade When x_1 is not too large, this bending moment will be approximately uniformly

[1] Clifford Older, Highway research in Illinois, Am Soc Civil Engineers, Trans, v 87, 1924, p 1180, especially p 1206

[1] W Ritz, Crelle's Journal, v 135, 1909, p 1

distributed over the width $2x_1$ of the cross-section. That is, the bending moment per unit of width becomes $M = \frac{M^1}{2x_1}$. The numerically greatest value of M was found, in this manner, to occur approximately at the distance

$$x_1 = 2\sqrt{a_1\, l} \tag{4}$$

and to be, approximately,

$$M = -\frac{P}{2}\left[1 - \left(\frac{a_1}{l}\right)^{0.6}\right] \tag{5}$$

Division by the section modulus per unit of width, $h^2/6$, leads to the corresponding greatest tensile stress

$$\sigma_c = \frac{3\,P}{h^2}\left[1 - \left(\frac{a_1}{l}\right)^{0.6}\right] \tag{6}$$

This stress may be stated also in the following form which is derived by substituting the value of l from equation (1)

$$\sigma_c = \frac{3\,P}{h^2}\left[1 - \left(\frac{E\,h^3}{12\,(1-\mu^2)\,k}\right)^{-0.15} a_1^{0.6}\right] \tag{7}$$

With $a_1 = 0$, the last two equations assume the simpler form of equation (2).

STRESS NOT GREATLY AFFECTED BY SUBGRADE CONDITION

Table II contains numerical values of the critical stress σ_c for $P = 10{,}000$ lb., $E = 3{,}000{,}000$ lb. per sq. in., and $\mu = 0.15$. The table shows the influence of three variables: the thickness h, the modulus k of subgrade reaction, and the distance a from the edges to the center of the load.

An inspection of the table shows the influence of the variation of the distance a to be appreciable, amounting easily to a reduction of more than 30 per cent as compared with the value found by the first approximation, with $a = 0$. The influence of the variation of the modulus k from 50 to 200 lb./in.^{3}, on the other hand, is not particularly large.

In case *II*, that of a wheel-load at a point of the interior, complications arise due to the fact that the load is concentrated within a rather small area. The theory of elasticity offers two types of theory of slabs: one theory may be called "ordinary theory of slabs," the other "special theory." The difference may be explained by an analogy with beams. In analysis of beams it is assumed ordinarily that a plane cross-section remains plane and perpendicular to the neutral surface during the bending. For beams of ordinary proportions, this assumption leads to satisfactory results, unless one is concerned with the local stresses in the immediate neighborhood of a concentrated load. In the latter case the assumption of the plane cross-section must be abandoned, and a special theory, which takes into account the deformations due to the vertical stresses, is required. In the ordinary theory of slabs it is assumed, correspondingly, that a straight line drawn through the slab perpendicular to the slab remains straight and perpendicular to the neutral surface With slabs of proportions as found in pavements, the theory based on these assumptions leads to a satisfactory determination of stresses at all points except in the immediate neighborhood of a concentrated load, and leads to a satisfactory determination of the deflections at all points At the point of application of a concentrated force this ordinary theory leads to a peak in the diagrams of bending moments, with infinite values at the point of the load itself (as indicated in Figures

TABLE II

Stresses in pounds per square inch computed from equation (7) for load condition as in Case I, Figure 1, for different values of h, k, and a

$P = 10{,}000$ pounds, $E = 3{,}000{,}000$ pounds per square inch, $\mu = 0\ 15$

Thickness of slab, h	Modulus of subgrade reaction, k	Stress in slab			
		$a = 0$	$a = 2$ in	$a = 4$ in	$a = 6$ in
Inches	*Lb /in* ^{3}	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*
6	50	833	641	541	461
	100	833	619	509	420
	200	833	596	474	375
7	50	612	480	412	357
	100	612	466	390	329
	200	612	450	366	298
8	50	469	373	325	285
	100	469	363	309	265
	200	469	352	291	242
9	50	370	299	262	233
	100	370	291	250	217
	200	370	282	237	201
10	50	300	245	216	193
	100	300	239	207	182
	200	300	232	197	169
11	50	248	204	182	164
	100	248	200	175	154
	200	248	194	167	144
12	50	208	173	155	140
	100	208	169	149	133
	200	208	165	143	124

5, 10, and 11) When the force is applied at the top of the slab, the tensile stresses at the bottom are not, in fact, infinite One may say then that the effect of the thickness of the slab is equivalent to a rounding off of the peak in the diagrams of moments In order to find out to what extent the diagrams are rounded off, it is necessary to abandon the assumption of the straight lines drawn through the slab remaining straight, as applying to the immediate neighborhood of the load, and a special theory is required This special theory rests on only two assumptions one is that Hooke's law applies, the constants being the modulus of elasticity E and Poisson's ratio μ, the other is that the material keeps its geometrical continuity at all points As in the case of beams, the ordinary theory is much simpler than the special theory, and is used, therefore, except in particular cases like the present one, which deals with local effects around a concentrated load

It is expedient to express the results of the special theory in terms of the ordinary theory in the following manner Let the load P be distributed uniformly over the area of the small circle with radius a The tensile stress produced by this load at the bottom of the slab under the center of the circle is denoted by σ_1 This stress is the critical stress

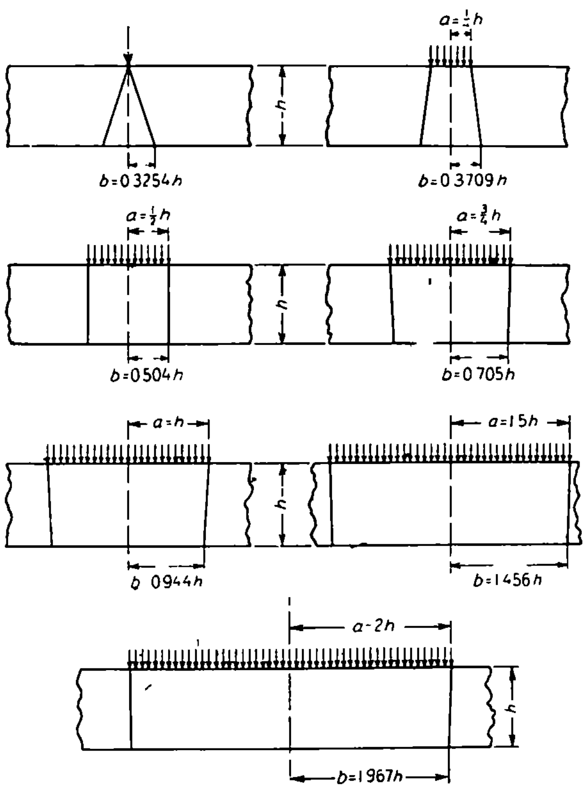

Figure 2—Cones of equivalent distribution of pressure

except when the radius a is so small that some of the vertical stresses near the top become more important, the latter exception need not be considered, however, in case of a wheel load which is applied through a rubber tire By use of the ordinary theory one may find the same stress at the same place by assuming the load to be distributed over the area of a circle with the same center, but with the radius b One finds that this equivalent radius b can be expressed with satisfactory approximation in terms of the true radius a and the thickness h only

In order to find the relation between h, a, and b, numerical computations were made in accordance with an analysis which is due to A Nadai[1] The center of the load P is assumed for the time being to be at the center of a circular slab The slab is supported at the edge in such a manner that the sum of the radial and tangential bending moments is zero at every point of the edge Computations according to Nádai's analysis, with the radius of the slab equal to $5h$ gave the results which are represented in Figure 2 in the manner of "cones of equivalent distribution" and in Figure 3 by a curve with coordinates a and b Approximately the same cones and the same curve are obtained for other radii of the slab, and the results may be applied generally to slabs of proportions such as are found in concrete pavements, with any kind of support which is not concentrated within a small area close to the load

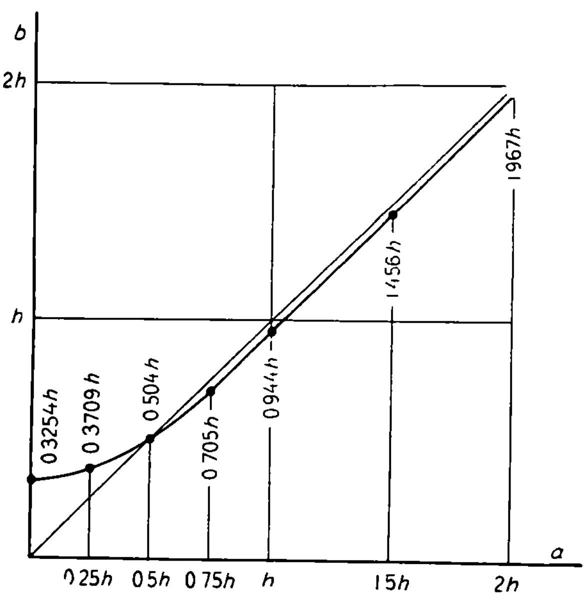

Figure 3—Relation between the true radius, a, the equivalent radius, b, and the thickness, h

[1] A Nádai, Die Biegungsbeanspruchung von Platten durch Einzelkrafte, Schweizerische Bauzeitung, v 76, 1920, p 257 and his book, Die elastischen Platten, (Berlin) 1925, p 308

One may notice that when a increases gradually from zero, b is at first larger than a, but when a passes a certain limit, b becomes smaller than a For the larger values of a, the ratio b/a converges toward unity, and the ordinary theory of slabs, accordingly, gives nearly the same results as the special theory

The curve in Figure 3 is found to lie close to a hyperbola, the equation of which may be written in the following form, which is suitable for numerical computations, and which may be used for values of a less than 1 724h

$$b=\sqrt{1\ 6\,a^2 \times h^2}-0\ 675h \tag{8}$$

For larger values of a, one may use $b=a$, that is, the ordinary theory may be used without corrections

By the ordinary theory one finds the following approximate expression for the critical stress

$$\sigma_i = \frac{3\,(1+\mu)\,P}{2\,\pi\,h^2}\left(\log_e \frac{l}{a} + 0\ 6159\right) \tag{9}$$

With $E=3{,}000{,}000$ lb per sq in and $\mu=0\ 15$, and with l substituted from equation (1), this formula takes the form

$$\sigma_i = 0\ 3162\frac{P}{h^2}\left(\log_{10}(h^3) - 4\log_{10} a - \log_{10} k + 6\ 478\right) \tag{10}$$

The correction to be made in this formula in order to make it agree with the special theory is merely to replace the true radius a by the equivalent radius b Thus one finds the following formula, which replaces equation (10) when a is less than 1 724h

$$\alpha_i = 0\ 3162\frac{P}{h^2}\left(\log_{10}(h^3) - 4\log_{10}\left(\sqrt{1\ 6\ a^2 + h^2} - 0\ \ 675\,h\right) - \log_{10} k + 6\ 478\right) \tag{11}$$

The stresses given in Table III have been computed in accordance with this formula for $P=10{,}000$ pounds Like Table II, this table shows the influence of three variables the thickness h, the modulus k of subgrade reaction, and a In Table III, as in Table II, one may notice the relatively greater influence of the variation of a as compared with the influence of the variation of k

In dealing with case *III*, that of a wheel load at the edge, it was assumed that an equivalent radius b may be introduced in the place of the true radius a in the same manner as in the preceding case, and by the same formula, that of equation (8) This assumption may be justified on the ground of the similarity in the two cases in the distribution of the energy due to vertical shearing stresses By introducing the equivalent radius b in the place of a in the formula for the tensile stress

σ_e along the bottom of the edge under the center of the circle, as obtained by the ordinary theory, one finds the following expression which, like the analogous equation (11), is based on $E = 3{,}000{,}000$ lb per sq in and $\mu = 0\ 15$

TABLE III

Stresses in pounds per square inch computed from equation (11) for load condition as in Case II, Figure 1, for different values of h, k, and a

$P = 10{,}000$ pounds, $E = 3{,}000{,}000$ pounds per square inch, $\mu = 0\ 15$

Thickness of slab h	Modulus of subgrade reaction k	Stress in slab				
		$a = 0$	$a = 2$ in	$a = 4$ in	$a = 6$ in	$a = 8$ in
Inches	*Lb /in* [3]	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*
4	50	1,231	1,058	848	693	588
	100	1,172	998	788	634	528
	200	1,112	939	729	574	469
5	50	763	694	580	487	415
	100	725	656	542	449	377
	200	687	617	504	411	339
6	50	523	487	421	361	313
	100	497	461	395	335	287
	200	470	435	368	308	260
7	50	380	360	319	279	245
	100	361	341	300	260	226
	200	341	321	280	240	206
8	50	288	276	250	222	197
	100	273	261	235	207	182
	200	258	246	220	192	167
9	50	226	218	200	180	162
	100	214	206	188	169	150
	200	202	194	177	157	138
10	50	181	176	164	149	136
	100	172	167	154	140	126
	200	162	157	145	130	116

$$\sigma_e = 0\ 572 \frac{P}{h^2}\left[\log_{10}(h^3) - 4\log_{10}(\sqrt{1\ 6\ a^2 + h^2} - 0\ 675\ h) - \log_{10} k + 5\ 767\right] \tag{12}$$

Stresses computed according to this formula are given in Table IV, again for $P = 10{,}000$ pounds The influence of the three variables h, k, and a is shown in the same manner as in the two preceding tables, and is seen to be of the same nature, the variation of a being of greater importance than that of k

TABLE IV

Stresses in pounds per square inch computed from equation (12) for load condition as in Case III, Figure 1, for different values of h, k, and a

P = 10,000 pounds, E = 3,000,000 pounds per squrae inch, μ = 0 15

Thickness of slab h	Modulus of subgrade reaction k	Stress in slab				
		$a = 0$	$a = 2$ in	$a = 4$ in	$a = 6$ in	$a = 8$ in
Inches	*Lb /in ^{3}*	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*	*Lbs per sq in*
6	50	833	769	649	541	453
	100	785	721	601	493	406
	200	738	673	553	445	358
7	50	604	568	494	422	360
	100	569	533	459	386	325
	200	534	498	424	351	290
8	50	457	436	388	337	293
	100	430	409	361	311	266
	200	404	382	334	284	239
9	50	358	344	312	276	243
	100	337	323	291	255	222
	200	315	301	269	233	200
10	50	287	278	256	230	204
	100	270	261	239	212	187
	200	253	244	221	195	170
11	50	235	229	213	194	174
	100	221	215	199	180	160
	200	207	201	185	165	146
12	50	196	192	180	165	150
	100	184	180	168	153	138
	200	172	168	156	142	126

BALANCED DESIGNS TESTED BY USE OF TABLES

From the three tables, for cases *I*, *II*, and *III*, one may obtain suggestions on the question of balanced design. Consider, for example, a pavement with the thicknesses 7 inches in the interior portion, and 9 inches at the edges. It may be assumed for the time being that the outer portions behave as a large slab with uniform thickness 9 inches. With the thickness diminishing slowly toward the interior, the stresses σ_c and σ_e would be somewhat larger than with constant thickness of 9 inches, but the correction needed for this reason is probably only small. For the time being only the one wheel load which is considered in each of the three tables will be taken into account. The influence of other wheel loads acting on the same panel, but at some distance, will be considered later; in any case it is found to be relatively small. With $P=10{,}000$ pounds, $k=50$ lb./in.^{3}, and $a=4$ inches, the three tables give the following value:

$$\sigma_c=262, \qquad \sigma_i=319, \qquad \sigma_e=312 \text{ lb. per sq. in.}$$

In comparing these stresses, their different characters should be considered. The stress σ_c at the corner acts presumably throughout the width of a whole cross-section, whereas σ_i and σ_e are localized within smaller regions. With equal tendency to rupture at the three places, σ_c, then, should be, probably, somewhat smaller than σ_i and σ_e. The stress σ_e is produced under the influence of a load which is distributed over an area only one-half of that assumed for σ_i. While the situation represented by the smaller area may occur when a wheel moves in over the edge of the pavement, it is reasonable, for the purpose of a comparative study of the tendency to rupture, to assume a larger radius of the semi-circle at the edge than for the full circle in the interior portion. With $a=6$ in., for example, at the edge, one finds the stress

$$\sigma_e=276 \text{ lb. per sq. in.}$$

In comparing this stress with σ_i, it should be observed that σ_i represents a state of equal stresses in all horizontal directions at the points, whereas σ_e is a one-directional stress. The elongations per unit of length are in the two cases $\sigma_i\,(1-\mu)/E$ and σ_e/E. It appears to be reasonable, therefore, for the purpose of comparison, to replace σ_i by an equivalent one-directional stress; if in this case the elongation is a direct measure of the tendency to rupture, this equivalent stress should be

$$\sigma'_i=\sigma_i(1-\mu)=319(1-0.15)=271 \text{ lb. per sq. in.}$$

The three values 262, 271, and 276 lb. per sq. in. point toward the conclusion that the assumed design is suitably balanced.

The suggestion has been made already that one may determine suitable values of k by comparing the deflections found by tests of full-sized slabs with those given by the formulas. The following formulas lend themselves to this purpose; they refer to the three cases shown in Fig. 1, in each case the load P is the only one acting.

Case I. Equation (3) gives the deflection at the corner

$$z_c = \left(1.1 - 0.88\,\frac{a_1}{l}\right)\frac{P}{k\,l^2} \tag{13}$$

Case II. The deflection under the center of the load differs only slightly from the following value which is accurate when $a=0$

$$z_i = \frac{P}{8\,k\,l^2} \tag{14}$$

Case III. The deflection at the point of application of a concentrated force P at the edge is approximately equal to

$$z_e = \frac{1}{\sqrt{6}}\left(1 + 0.4\,\mu\right)\frac{P}{k\,l^2} \tag{15}$$

that is, for $\mu=0.15$

$$z_e = 0.433\,\frac{P}{k\,l^2} \tag{16}$$

The quantity kl^2 occurring in each of these formulas may be expressed, according to equation (1), as

$$k\,l^2 = \sqrt{\frac{E\,h^3\,k}{12\,(1-\mu^2)}} \tag{17}$$

When experimental values of the deflections are at hand, one may determine the corresponding values of kl^2 by means of equations (13) to (16). Then equation (17) gives the value of k as

$$k = \frac{12\,(1-\mu^2)\,(k\,l^2)^2}{E\,h^3} \tag{18}$$

Figures 4 to 11 are diagrams of deflections and moments. The titles of these figures explain the nature of the diagrams. The deflections and bending moments have been computed by means of the ordinary theory of slabs. The diagrams, therefore, give information concerning deflections in general, and concerning bending moments except in the immediate neighborhood of the concentrated load which produces the bending moments.

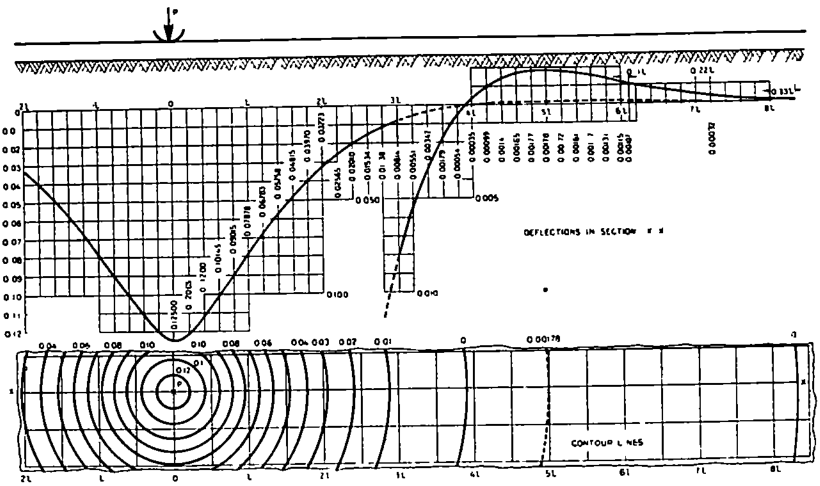

Figure 4—Deflections produced by a concentrated load which acts at a point of the interior at a considerable distance from the edges

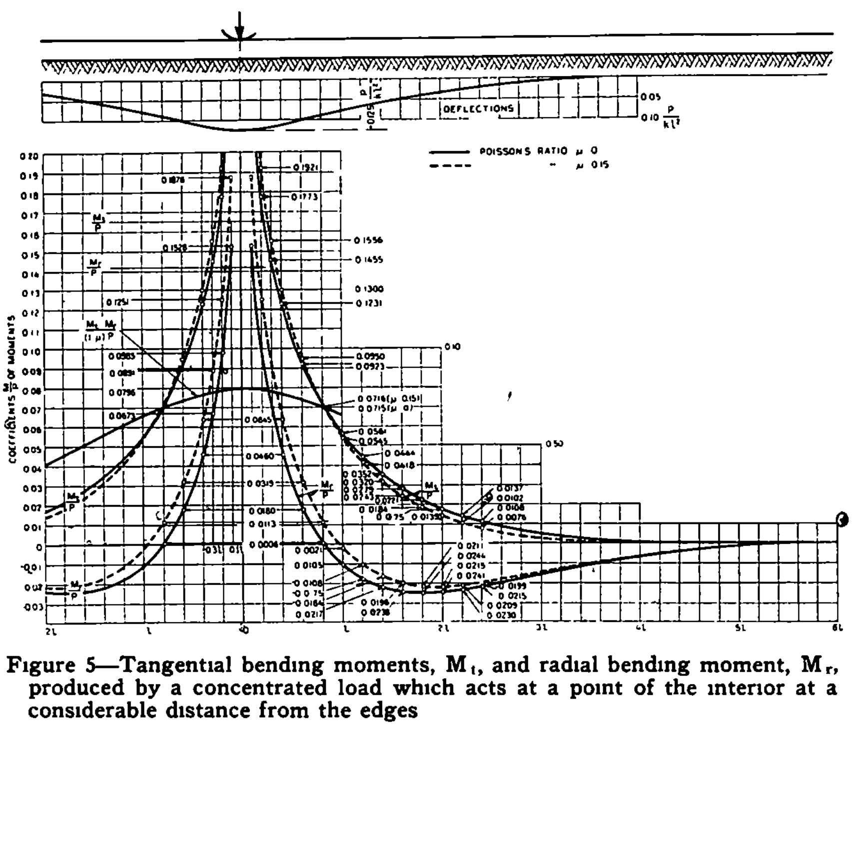

Figure 5—Tangential bending moments, M_t, and radial bending moment, M_r, produced by a concentrated load which acts at a point of the interior at a considerable distance from the edges

DETERMINATION OF DEFLECTIONS DUE TO MORE THAN ONE WHEEL

The diagrams in Figures 4 and 5 have been obtained by an analysis which rests essentially on that given by the physicist Hertz[1] in 1884

The diagrams in Figures 4 and 5 may be used in the following way, for the purpose of finding the resultant deflections and stresses due to the combined influence of two or four wheel loads each acting at a considerable distance from the edges of the slab

Let each load be 10,000 pounds, and let the horizontal rectangular coordinates of the centers of the four loads be as follows

Coordinate	Load No 1	Load No 2	Load No 3	Load No 4
$x =$	0	66 in	0	66 in
$y =$	0	0	66 in	66 in

Loads 1 and 2 alone may represent the two rear wheels of a four-wheel truck, and the four loads combined may represent the four rear wheels of a six-wheel truck

With $h=7$ in, $E=3{,}000{,}000$ lb per sq in, $\mu=0.15$, and $k=50$ lb /in ^{3}, one finds by equations (1) and (17) or by Table I

$$l=36.40 \text{ in}, \qquad kl^2=66{,}200 \text{ lb /in},$$

distances 1-2 and 1-3 66 in $=1.813l$, distance 1-4 $66\sqrt{2}=2.564l$

[1] H Hertz, Über das Gleichgewicht schwimmender elastischer Platten, Wiedemann's Annalen der Physik und Chemie, v 22, 1884, pp 449–455, also in his Gesammelte Werke, v 1, pp 288–294 Hertz dealt with the problem of a large swimming slab, for example, of ice, loaded by a single force A Foppl in his Technische Mechanik, v 5, 1907, pp 112–130, presented Hertz s theory in a modified, and in some ways simplified form, and he called attention to the applicability of this analysis to the problem of the slab on elastic support Hertz made use of Bessel functions in his analysis Since his analysis was published, the number of published numerical tables of Bessel functions has been increased Among the newer tables those representing Hankel's Bessel functions

$$H_0^{(1)}(x\sqrt{i}) \text{ and } H_1^{(1)}(x\sqrt{i})$$

are of especial interest for the present problem Tables of these functions may be found in the book of tables by E Jahnke and F Emde, Funktionentafeln mit Formeln und Kurven, 1909, pp 139 and 140 By means of these tables the numerical values given in Figures 4 and 5 were obtained by simple computations After these diagrams had been prepared, two papers have appeared in which the same functions are used for the purpose of analysis of slabs on elastic support One is by J J Koch, Berekening van vlakke platen, ondersteund in de hoekpunten van een willekeurig rooster, De Ingenieur, 1925, No 6, the other is by Ferdinand Schleicher, Über Kreisplatten auf elastischer Unterlage, Festschrift zur Hundertjahrfeier der Technischen Hochschule Karlsruhe, 1925

Then equation (14) as well as Fig 4, gives the following value of the deflection at point 1 due to load No 1

$$z_{1\ 1}=\frac{P}{8\,k\,l^2}-\frac{10{,}000}{8\times 66{,}200}=0\ 0189\ \text{in}$$

Furthermore, Fig 4 leads to the following value of the deflection at point 1 due to load No 2 alone

$$z_{1\ 2}=0\ 03921\,\frac{P}{k\,l^2}=0\ 03921\,\frac{10{,}000}{66{,}200}=0\ 0059\ \text{in}$$

Then, by superposition of the two deflections, one finds the deflection at point 1 due to the combined influence of the two rear wheels 1 and 2.

$$z_{1,\,(1,2)}=z_{1,\,1}+z_{1,\,2}=0\ 0248\ \text{in}$$

The deflection at point 1 due to load No 3 alone is

$$z_{1,\,3}=z_{1,\,2}=0\ 0059\ \text{in}$$

The deflection at point 1 due to load No 4 alone is, according to Fig 4,

$$z_{1\ 4}=0\ 01620\,\frac{P}{k\,l^2}=0\ 0024\ \text{in}$$

By superposition of the four deflections due to each separate load, one finds the resultant deflection due to the four loads

$$z_{1,\,(1,\,2,\,3,\,4)}=0\ 0331\ \text{in}$$

For the purpose of computing the state of stresses at the bottom of the slab under the center of load No 1 it will be assumed that load No 1 is distributed uniformly over the area of a circle with a radius $a=6$ inches The stresses due to load No 1 will be the same in all directions, and they are, according to Table 3

$$\sigma_x=\sigma_y=279\ \text{lb per sq in}$$

According to Fig 5, load No 2 produces a radial bending moment M_r, in this case in the direction of x, equal to

$$M_x=-0\ 0211P=-211\ \text{in lb per in (or }-211\text{ lb)},$$

and a tangential bending moment M_t, in this case in the direction of y, equal to

$$M_y=0\ 0181P=181\ \text{lb}$$

The corresponding stresses are found by dividing these bending moments by the section modulus per unit of width, that is, by $\frac{1}{6}h^2=8\ 167\ \text{in}^2$.

Thus one finds the stresses in the directions of x and y

$$\sigma_x = -\frac{211}{8\ 167} = -26 \text{ lb per sq in}$$

and

$$\sigma_y = \frac{181}{8\ 167} = 22 \text{ lb per sq in}$$

These stresses are principal stresses, that is, one is the maximum, the other the minimum stress, and there are no shearing stresses in the directions of x and y

For the case of the four-wheel truck, one finds, then, by super-position the following principal stresses due to the two rear wheels, loads No 1, and No 2, these principal stresses are in the directions of x and y

$$\sigma_x = 279 - 26 = 253 \text{ lb per sq in},$$
$$\sigma_y = 279 + 22 = 301 \text{ lb per sq in}$$

STRESSES DUE TO SIX-WHEEL TRUCK

In the case of the six-wheel truck the effects of loads No 3 and No 4 must be included Load No 3 contributes the same stresses at point 1 as does load No 2, only the indices x and y are to be interchanged Consequently the resultant stresses in the directions of x and y due to the combined influence of loads 1, 2, and 3 become

$$\sigma_x = \sigma_y = 279 - 26 + 22 = 275 \text{ lb per sq in}$$

These stresses, again, are principal stresses Since they are equal, the horizontal stresses will be the same in all directions, each stress being a principal stress

Let x^1, y^1 be a new system of horizontal rectangular coordinates with the axis of x^1 along the diagonal line from point 1 to point 4 Load No 4 produces a radial bending moment in the direction of x^1 and a tangential bending moment in the direction of y^1 According to Fig 5 these bending moments are

$$M_x{}^1 = -0\ 0186P = -186 \text{ lb} \quad \text{and} \quad M_y{}^1 = 0\ 0058P = 58 \text{ lb},$$

respectively The corresponding stresses are found, again, by dividing the bending moments by the section modulus per unit of width, that is, by 8 167 in ^{2}, and they are

$$\sigma_x{}^1 = -23 \text{ lb per sq in}, \text{ and } \sigma_y{}^1 = 7 \text{ lb per sq in}$$

These stresses are principal stresses The resultant principal stresses due to all four loads combined, therefore, are in the directions of x^1 and y^1, and have the values

$$\sigma_x{}^1 = 275 - 23 = 252 \text{ lb per sq in},$$
$$\sigma_y{}^1 = 275 + 7 = 282 \text{ lb per sq in}$$

One may draw the conclusion that the main part of the state of stresses at a given point is due to a wheel load right over the point In the case examined, the contribution due to the three additional rear wheels of the six-wheel truck is of less importance than that due to the one additional rear wheel of the four-wheel truck

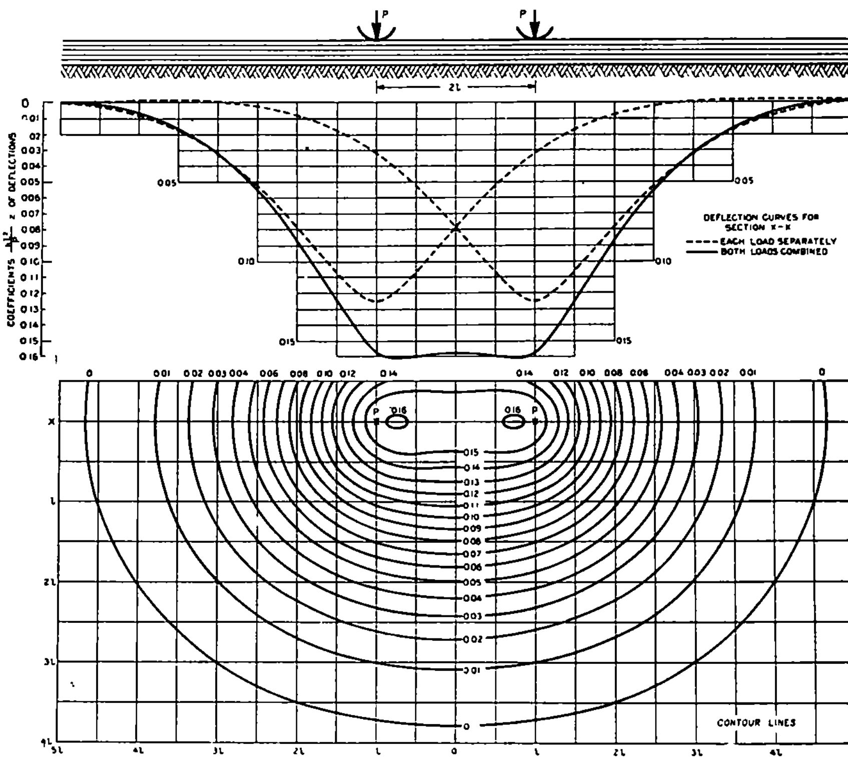

Figure 6—Deflections produced by two equal loads like the load in Figure 4, separated by a distance of 2l The deflections are found by superposition of two diagrams of the kind shown in Figure 4

Figures 6 and 7 show deflections due to two wheel loads combined Each of these diagrams was obtained by superposition of two diagrams such as shown in Figure 4

Figures 8 to 11 show effects of loads at the edge, but at a considerable distance from any corner [1]

By virtue of Maxwell's theorem of reciprocal deflections, the deflection at a point *B* of any slab due to a load *P* at the point *A* is the same as the deflection at *A* due to a load *P* at point *B* Figures 8 and 9 may

[1]The theory by which these diagrams were obtained may be found in a paper by the writer Om Beregning af Plader paa elastisk Underlag med særligt Hemblik paa Sporgsmaalet om Spændinger i Betonveje, Ingenioren (Copenhagen), v 32, 1923, pp 513–524 See also, A Nádai, Die elastischen Platten, (Berlin) 1925, p 186

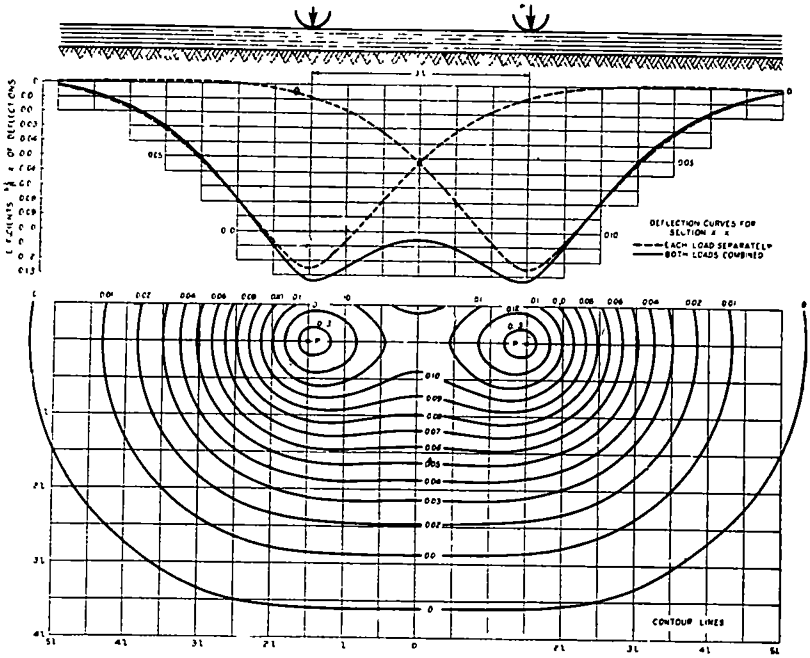

Figure 7—Deflections produced by two equal loads like the loads in Figure 4, separated by a distance of 3l

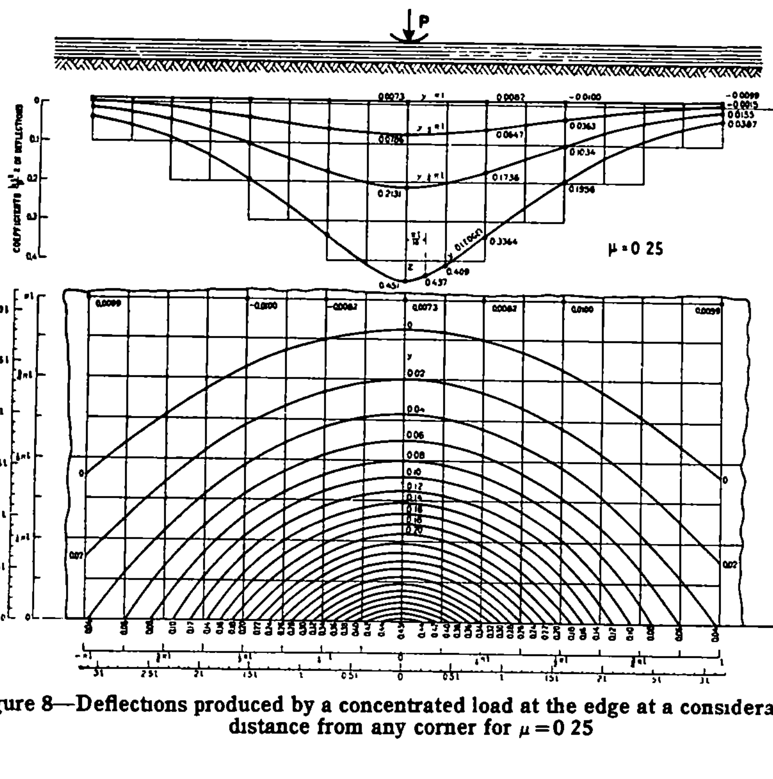

Figure 8—Deflections produced by a concentrated load at the edge at a considerable distance from any corner for $\mu = 0\ 25$

be interpreted, therefore, in a double manner first, as diagrams of deflections at any point B due to a load P at the particular point A at the edge, secondly, as influence diagrams, showing the deflection at the particular point A at the edge due to a load P at any point

From this reciprocity of deflections one may draw a further conclusion which may be applied to Figures 8 and 9, and which concerns the curve of deflections or elastic curve which is obtained by intersection of the deflected middle surface by a vertical plane Two lines L_A and L_B are drawn parallel to two opposite parallel edges of a slab Two equal loads are considered, one acting at a point A of the line L_A, the

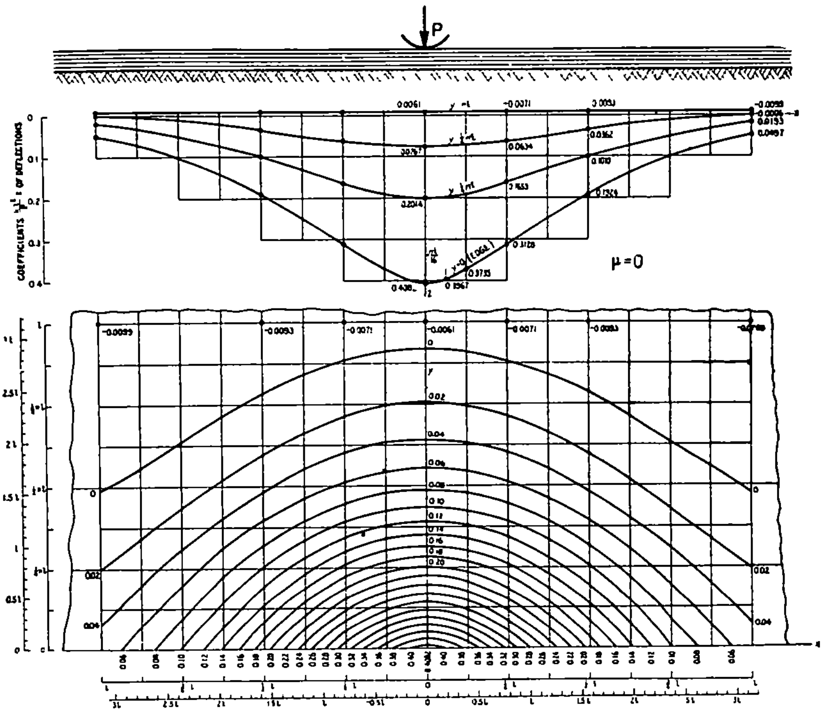

Figure 9—Deflections produced by a concentrated load at a considerable distance from any corner for $\mu = 0$

other acting at a point B of the line L_B The points A and B are assumed to be sufficiently far from the remaining two edges of the slab to permit the assumption of zero deformations at these edges Then one may conclude that the elastic curve produced along the line L_B under the influence of the load P at A has exactly the same shape as the elastic curve produced along the line L_A under the influence of the load P at point B In applying this conclusion to Figure 8 or Figure 9, let the line L_A be the edge shown in the drawing, and let the line L_B be at some distance from the edge By the direct use of the diagrams one obtains the elastic curve at any line L_B parallel to the edge, due to a load at the edge But one may interpret this curve as the elastic curve for the edge produced under the influence of a load at a point of the line L_B The curvature of the deflected middle surface at point A of the edge in the direction of the edge, produced by the load P at any point B at some distance from the edge, is the same, accordingly, as the

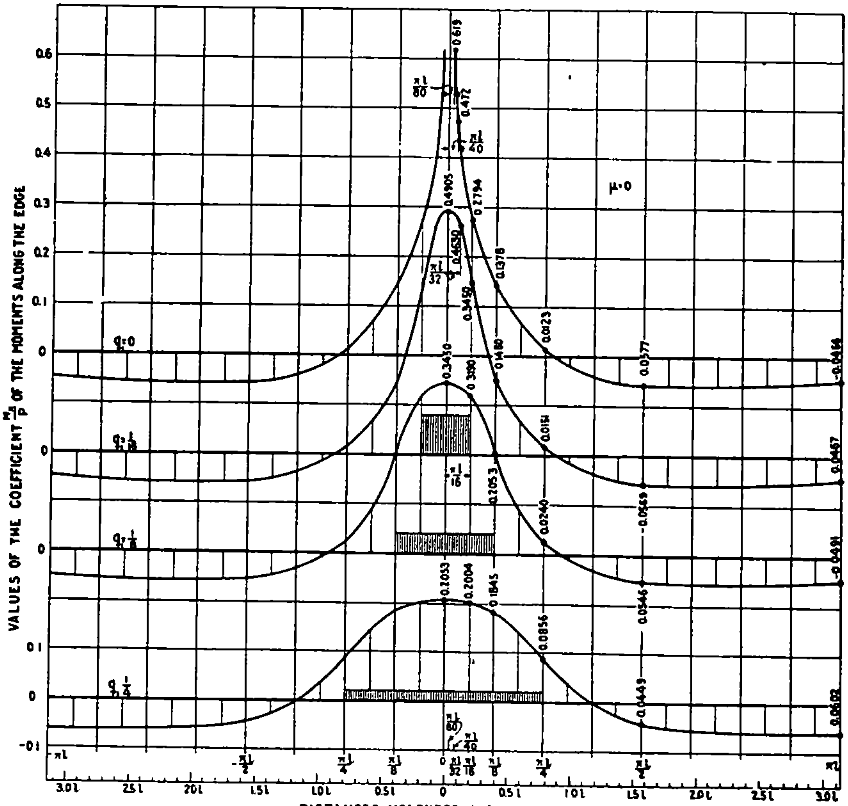

Figure 10—Bending moments along the edge for a load concentrated at a point of the edge (top diagram), and for loads distributed uniformly over lines of three lengths at the edge (lower three diagrams), $\mu = 0$

curvature of the deflected middle surface at point B in a direction parallel to the edge, as obtained in Figure 8 or Figure 9, due to the load P at the point A of the edge

Thus Figures 8 and 9 may be used in studying the stresses produced along the edge by a wheel load at some distance from the edge

The following use of the tables and diagrams is suggested Let it be assumed that a certain pavement has been proved by tests and experience to be satisfactory for a given type of traffic By the tables and diagrams one may compute, then, the corresponding critical stresses These stresses may be adopted for the time being as allowable working stresses With the stresses given, the tables and diagrams, through computations of the kind which has been shown, furnish answers to two questions what additional thicknesses are required if the wheel pressures are increased in a given manner, and, what may be saved in the thicknesses by eliminating some of the heaviest vehicles

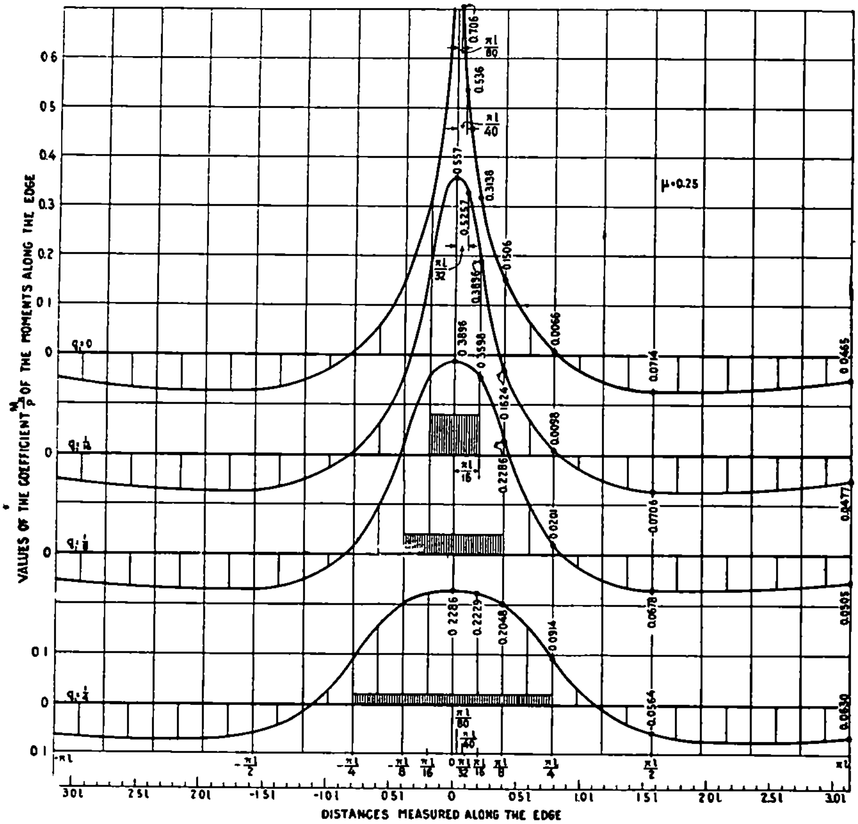

Figure 11—Bending moments along the edge as in Figure 10, but for $\mu = 0.25$

Professor T R Agg has called attention to the importance of having an answer to the latter question, when one attempts to apportion the cost of the pavement to the various kinds of traffic for which it is used

In using the tables and diagrams it should be kept in mind that the analysis is based on those assumptions which were stated at the beginning of this discussion By the nature of these assumptions certain influences were left out of consideration, especially the following (1) variations of temperature, and other causes for tendency to change of volume, (2) the gradual diminishing of the thickness from the edge toward the interior, (3) local soft or hard spots in the subgrade, (4) horizontal components of the reactions of the subgrade, and (5) the dynamic effect, expressed in terms of the inertia of the pavement and subgrade The horizontal components of the reactions of the subgrade, which are due to friction, may have a strengthening influence, especially at some distance from the edges, by causing a dome action in the pavement As to the dynamic effects, with known values of the maximum pressure developed between the tire and the pavement, the effect of the inertia of the pavement may possibly be expressed approximately in terms of an increased value of the modulus k These additional influences are suitable subjects for further analysis

REPORT ON EXPERIMENTS ON EXTENSIBILITY OF CONCRETE

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Two properties of materials are important—strength and toughness Available data are few resulting from measurements of the ability of concrete to withstand extension without the appearance of fissures These may range in magnitude (a) from those in the order of 0 0004 inch width seen only with a microscope or appearing as "water veins" or "water marks," as Feret termed them, when a skin-dried surface. breaks and capillary moisture comes from the interior through the fissures, to (b) larger fissures in the order of 0 0015-inch width, seen by the unaided eye, and (c) in the extreme to those large open cracks that occur when the elastic limit of reinforcing steel is exceeded In the class of microscopic fissures are those crazes that mar the appearance of architectural concrete or other concrete products Such crazes are not always evident to the unaided eye, but may be developed by a coating of light oil

The various fissures may be produced by load or by the action of temperature or moisture changes

The preservation of the integrity of the surface of exposed concrete is important In many cases surface cracks are the first indication of subsequent failure in concretes that have been made of defective materials, either cement or aggregate

We are increasingly required to compute expansions and contractions of structures, these movements are limited by extensibility

As has been said, the active agents may be tensions due to loads, or due to the working back and forth of the surface under temperature and moisture changes The latter express themselves most markedly when the surface of the concrete is of a richer composition than the interior, or when the surface is contracted by careless drying against a moist core Indeed, the falling off in strength of cement briquettes