1 INTRODUCTION

Beijing National Olympic Stadium—Bird's Nest
(Eastimages / Shutterstock)

Structural analysis, which is an integral part of any structural engineering project, *is the process of predicting the performance of a given structure under a prescribed loading condition.* The performance characteristics usually of interest in structural design are: (a) stresses or stress resultants (i.e., axial forces, shears, and bending moments); (b) deflections; and (c) support reactions. Thus, the analysis of a structure typically involves the determination of these quantities as caused by the given loads and/or other external effects (such as support displacements and temperature changes). This text is devoted to the analysis of *framed structures*—that is, structures composed of long straight members. Many commonly used structures such as beams, and plane and space trusses and rigid frames, are classified as framed structures (also referred to as *skeletal structures*).

In most design offices today, the analysis of framed structures is routinely performed on computers, using software based on the matrix methods of structural analysis. It is therefore essential that structural engineers understand the basic principles of matrix analysis, so that they can develop their own computer programs and/or properly use commercially available software—and appreciate the physical significance of the analytical results. The objective of this text is to present the theory and computer implementation of matrix methods for the analysis of framed structures in static equilibrium.

This chapter provides a general introduction to the subject of matrix computer analysis of structures. We start with a brief historical background in Section 1.1, followed by a discussion of how matrix methods differ from classical and finite-element methods of structural analysis (Section 1.2). Flexibility and stiffness methods of matrix analysis are described in Section 1.3; the six types of framed structures considered in this text (namely, plane trusses, beams, plane frames, space trusses, grids, and space frames) are discussed in Section 1.4; and the development of simplified models of structures for the purpose of analysis is considered in Section 1.5. The basic concepts of structural analysis necessary for formulating the matrix methods, as presented in this text, are reviewed in Section 1.6; and the roles and limitations of linear and nonlinear types of structural analysis are discussed in Section 1.7. Finally, we conclude the chapter with a brief note on the computer software that is provided on the publisher's website for this book (Section 1.8). (www.cengage.com/engineering)

1.1 HISTORICAL BACKGROUND

The theoretical foundation for matrix methods of structural analysis was laid by James C. Maxwell, who introduced the method of consistent deformations in 1864; and George A. Maney, who developed the slope-deflection method in 1915. These classical methods are considered to be the precursors of the matrix flexibility and stiffness methods, respectively. In the precomputer era, the main disadvantage of these earlier methods was that they required direct solution of simultaneous algebraic equations—a formidable task by hand calculations in cases of more than a few unknowns.

The invention of computers in the late 1940s revolutionized structural analysis. As computers could solve large systems of simultaneous equations, the analysis methods yielding solutions in that form were no longer at a disadvantage, but in fact were preferred, because simultaneous equations could be expressed in matrix form and conveniently programmed for solution on computers.

S. Levy is generally considered to have been the first to introduce the flexibility method in 1947, by generalizing the classical method of consistent deformations. Among the subsequent researchers who extended the flexibility method and expressed it in matrix form in the early 1950s were H. Falkenheimer, B. Langefors, and P. H. Denke. The matrix stiffness method was developed by R. K. Livesley in 1954. In the same year, J. H. Argyris and S. Kelsey presented a formulation of matrix methods based on energy principles. In 1956, M. T. Turner, R. W. Clough, H. C. Martin, and L. J. Topp derived stiffness matrices for the members of trusses and frames using the finite-element approach, and introduced the now popular *direct stiffness method* for generating the structure stiffness matrix. In the same year, Livesley presented a nonlinear formulation of the stiffness method for stability analysis of frames.

Since the mid-1950s, the development of matrix methods has continued at a tremendous pace, with research efforts in recent years directed mainly toward formulating procedures for the dynamic and nonlinear analysis of structures, and developing efficient computational techniques for analyzing large structures. Recent advances in these areas can be attributed to S. S. Archer, C. Birnstiel, R. H. Gallagher, J. Padlog, J. S. Przemieniecki, C. K. Wang, and E. L. Wilson, among others.

1.2 CLASSICAL, MATRIX, AND FINITE-ELEMENT METHODS OF STRUCTURAL ANALYSIS

Classical versus Matrix Methods

As we develop matrix methods in subsequent chapters of this book, readers who are familiar with classical methods of structural analysis will realize that both matrix and classical methods are based on the same fundamental principles—but that the fundamental relationships of equilibrium, compatibility, and member stiffness are now expressed in the form of matrix equations, so that the numerical computations can be efficiently performed on a computer.

Most classical methods were developed to analyze particular types of structures, and since they were intended for hand calculations, they often involve certain assumptions (that are unnecessary in matrix methods) to reduce the amount of computational effort required for analysis. The application of these methods usually requires an understanding on the part of the analyst of the structural behavior. Consider, for example, the moment-distribution method. This classical method can be used to analyze only beams and plane frames undergoing bending deformations. Deformations due to axial forces in the frames are ignored to reduce the number of independent joint translations. While this assumption significantly reduces the computational effort, it complicates the analysis by requiring the analyst to draw a deflected shape of the frame corresponding to each degree of freedom of sidesway (independent joint translation), to estimate the relative magnitudes of member fixed-end moments: a difficult task even in the case of a few degrees of freedom of sidesway if the frame has inclined members. Because of their specialized and intricate nature, classical methods are generally not considered suitable for computer programming.

In contrast to classical methods, matrix methods were specifically developed for computer implementation; they are *systematic* (so that they can be conveniently programmed), and *general* (in the sense that the same overall format of the analytical procedure can be applied to the various types of framed structures). It will become clear as we study matrix methods that, because of the latter characteristic, a computer program developed to analyze one type of structure (e.g., plane trusses) can be modified with relative ease to analyze another type of structure (e.g., space trusses or frames).

As the analysis of large and highly redundant structures by classical methods can be quite time consuming, matrix methods are commonly used. However, classical methods are still preferred by many engineers for analyzing smaller structures, because they provide a better insight into the behavior of structures. Classical methods may also be used for preliminary designs, for checking the results of computerized analyses, and for deriving the member force–displacement relations needed in the matrix analysis. Furthermore, a study of classical methods is considered to be essential for developing an understanding of structural behavior.

Matrix versus Finite Element Methods

Matrix methods can be used to analyze framed structures only. Finite-element analysis, which originated as an extension of matrix analysis to surface structures (e.g., plates and shells), has now developed to the extent that it can be applied to structures and solids of practically any shape or form. From a theoretical viewpoint, the basic difference between the two is that, in matrix methods, the member force–displacement relationships are based on the exact solutions of the underlying differential equations, whereas in finite-element methods, such relations are generally derived by work-energy principles from assumed displacement or stress functions.

Because of the approximate nature of its force–displacement relations, finite-element analysis generally yields approximate results. However, as will be shown in Chapters 3 and 5, in the case of linear analysis of framed structures composed of prismatic (uniform) members, both matrix and finite-element approaches yield identical results.

1.3 FLEXIBILITY AND STIFFNESS METHODS

Two different methods can be used for the matrix analysis of structures: the *flexibility* method, and the *stiffness* method. The flexibility method, which is also referred to as the *force* or *compatibility* method, is essentially a generalization in matrix form of the classical method of consistent deformations. In this approach, the primary unknowns are the redundant forces, which are calculated first by solving the structure's compatibility equations. Once the redundant forces are known, the displacements can be evaluated by applying the equations of equilibrium and the appropriate member force–displacement relations.

The stiffness method, which originated from the classical slope-deflection method, is also called the *displacement* or *equilibrium* method. In this approach, the primary unknowns are the joint displacements, which are determined first by solving the structure's equations of equilibrium. With the joint displacements known, the unknown forces are obtained through compatibility considerations and the member force–displacement relations.

Although either method can be used to analyze framed structures, the flexibility method is generally convenient for analyzing small structures with a few redundants. This method may also be used to establish member force-displacement relations needed to develop the stiffness method. The stiffness method is more systematic and can be implemented more easily on computers; therefore, it is preferred for the analysis of large and highly redundant structures. Most of the commercially available software for structural analysis is based on the stiffness method. In this text, we focus our attention mainly on the stiffness method, with emphasis on a particular version known as the *direct stiffness method*, which is currently used in professional practice. The fundamental concepts of the flexibility method are presented in Appendix B.

1.4 CLASSIFICATION OF FRAMED STRUCTURES

Framed structures are composed of straight members whose lengths are significantly larger than their cross-sectional dimensions. Common framed structures can be classified into six basic categories based on the arrangement of their members, and the types of primary stresses that may develop in their members under major design loads.

Plane Trusses

A *truss* is defined as an assemblage of straight members connected at their ends by flexible connections, and subjected to loads and reactions only at the joints (connections). The members of such an ideal truss develop only axial forces when the truss is loaded. In real trusses, such as those commonly used for supporting roofs and bridges, the members are connected by bolted or welded connections that are not perfectly flexible, and the dead weights of the members are distributed along their lengths. Because of these and other deviations from idealized conditions, truss members are subjected to some bending and shear. However, in most trusses, these secondary bending moments and shears are small in comparison to the primary axial forces, and are usually not considered in their designs. If large bending moments and shears are anticipated, then the truss should be treated as a rigid frame (discussed subsequently) for analysis and design.

If all the members of a truss as well as the applied loads lie in a single plane, the truss is classified as a *plane truss* (Fig. 1.1). The members of plane trusses are assumed to be connected by frictionless hinges. The analysis of plane trusses is considerably simpler than the analysis of space (or three-dimensional) trusses. Fortunately, many commonly used trusses, such as bridge and roof trusses, can be treated as plane trusses for analysis (Fig. 1.2).

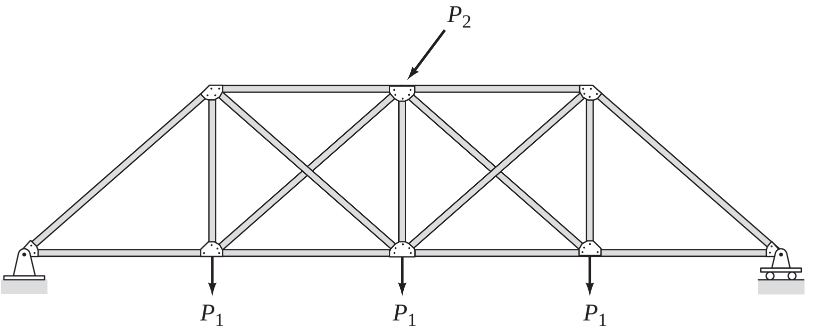


Fig. 1.1 *Plane Truss*

Fig. 1.2 *Roof Truss*
(Photo courtesy of Bethlehem Steel Corporation)

Beams

A *beam* is defined as a long straight structure that is loaded perpendicular to its longitudinal axis (Fig. 1.3). Loads are usually applied in a plane of symmetry of the beam's cross-section, causing its members to be subjected only to bending moments and shear forces.

Plane Frames

Frames, also referred to as *rigid frames,* are composed of straight members connected by rigid (moment resisting) and/or flexible connections (Fig. 1.4). Unlike trusses, which are subjected to external loads only at the joints, loads on frames may be applied on the joints as well as on the members.

If all the members of a frame and the applied loads lie in a single plane, the frame is called a *plane frame* (Fig. 1.5). The members of a plane frame are, in

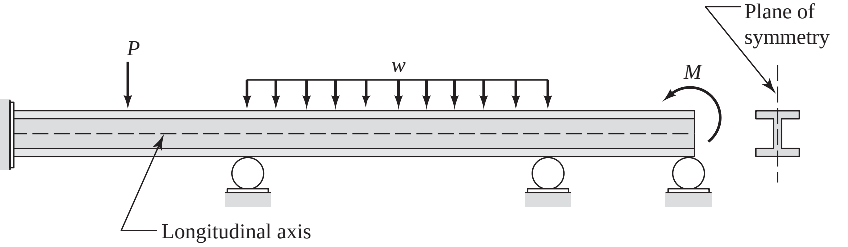


Fig. 1.3 *Beam*

Fig. 1.4 *Skeleton of a Structural Steel Frame Building*
(Joe Gough / Shutterstock)

general, subjected to bending moments, shears, and axial forces under the action of external loads. Many actual three-dimensional building frames can be subdivided into plane frames for analysis.

Space Trusses

Some trusses (such as lattice domes, transmission towers, and certain aerospace structures (Fig. 1.6)) cannot be treated as plane trusses because of the arrangement of their members or applied loading. Such trusses, referred to as *space trusses*, are analyzed as three-dimensional structures subjected to three-dimensional force systems. The members of space trusses are assumed to be connected by frictionless ball-and-socket joints, and the trusses are subjected to loads and reactions only at the joints. Like plane trusses, the members of space trusses develop only axial forces.

Grids

A *grid*, like a plane frame, is composed of straight members connected together by rigid and/or flexible connections to form a plane framework. The

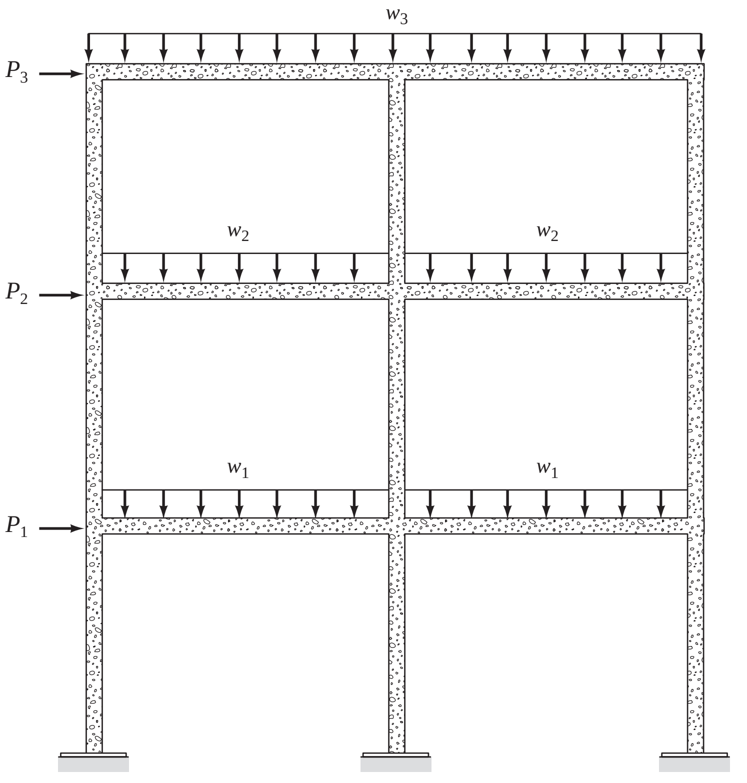


Fig. 1.5 *Plane Frame*

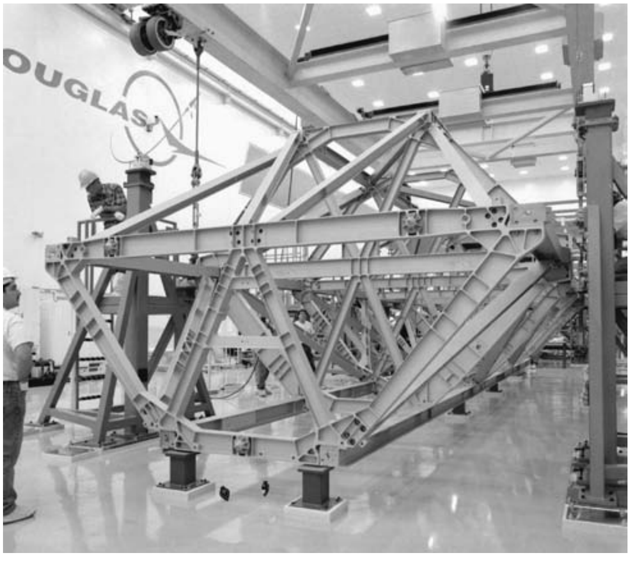


Fig. 1.6 *A Segment of the Integrated Truss Structure which Forms the Backbone of the International Space Station*

(Photo Courtesy of National Aeronautics and Space Administration 98-05165)

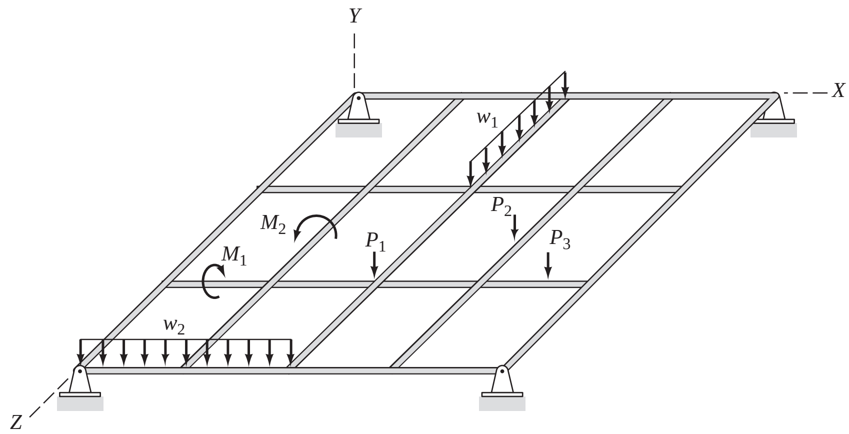


Fig. 1.7 *Grid*

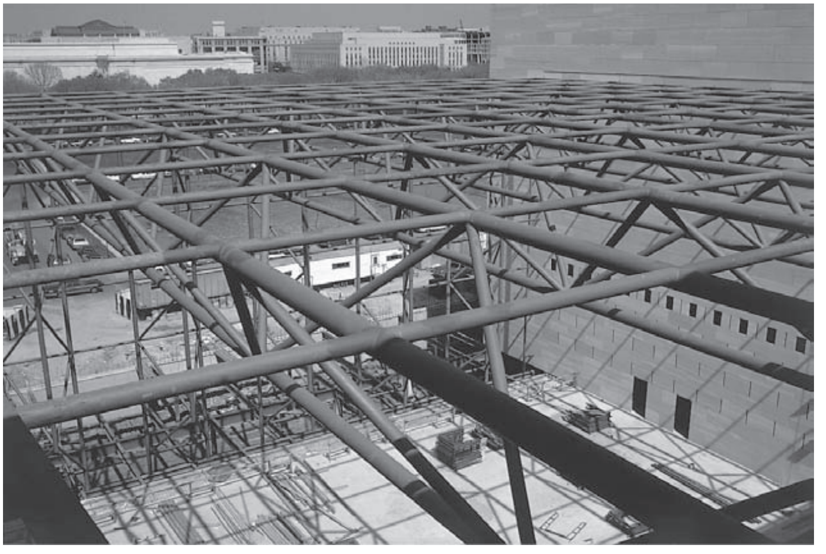


Fig. 1.8 *National Air and Space Museum, Washington, DC (under construction)*
(Photo courtesy of Bethlehem Steel Corporation)

main difference between the two types of structures is that plane frames are loaded in the plane of the structure, whereas the loads on grids are applied in the direction perpendicular to the structure's plane (Fig. 1.7). Members of grids may, therefore, be subjected to torsional moments, in addition to the bending moments and corresponding shears that cause the members to bend out of the plane of the structure. Grids are commonly used for supporting roofs covering large column-free areas in such structures as sports arenas, auditoriums, and aircraft hangars (Fig. 1.8).

Fig. 1.9 *Space Frame*
(© MNTravel / Alamy)

Space Frames

Space frames constitute the most general category of framed structures. Members of space frames may be arranged in any arbitrary directions, and connected by rigid and/or flexible connections. Loads in any directions may be applied on members as well as on joints. The members of a space frame may, in general, be subjected to bending moments about both principal axes, shears in both principal directions, torsional moments, and axial forces (Fig. 1.9).

1.5 ANALYTICAL MODELS

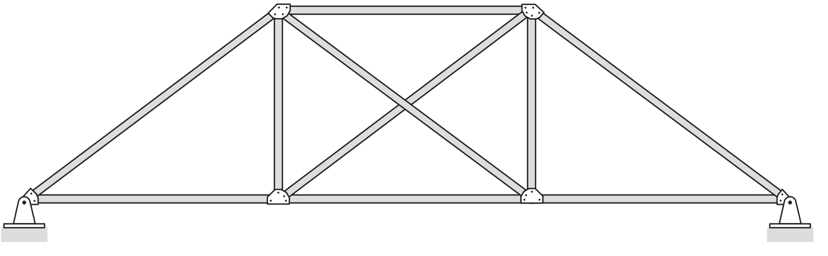
The first (and perhaps most important) step in the analysis of a structure is to develop its analytical model. An analytical model is an idealized representation of a real structure for the purpose of analysis. Its objective is to simplify the analysis of a complicated structure by discarding much of the detail (about connections, members, etc.) that is likely to have little effect on the structure's behavioral characteristics of interest, while representing, as accurately as practically possible, the desired characteristics. It is important to note that the structural response predicted from an analysis is valid only to the extent that the analytical model represents the actual structure. For framed structures, the establishment of analytical models generally involves consideration of issues such as whether the actual three-dimensional structure can be subdivided into plane structures for analysis, and whether to idealize the actual bolted or welded connections as hinged, rigid, or semirigid joints. Thus, the development of accurate analytical models requires not only a thorough understanding of structural behavior and methods of analysis, but also experience and knowledge of design and construction practices.

In matrix methods of analysis, a structure is modeled as an assemblage of straight members connected at their ends to joints. A *member* is defined as *a part of the structure for which the member force-displacement relationships to be used in the analysis are valid.* The member force-displacement relationships for the various types of framed structures will be derived in subsequent chapters. A *joint* is defined as *a structural part of infinitesimal size to which the ends of the members are connected*. In finite-element terminology, the members and joints of structures are generally referred to as *elements* and *nodes*, respectively.

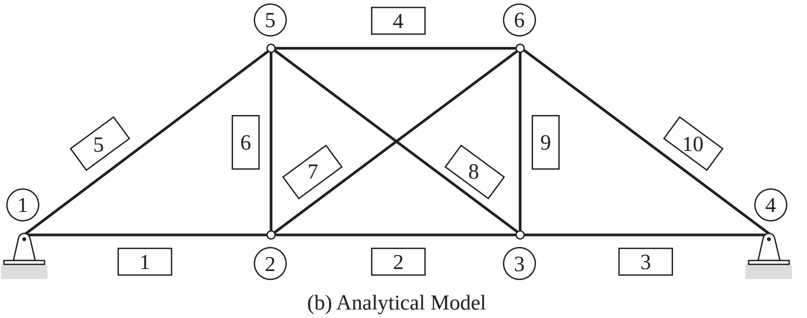
Supports for framed structures are commonly idealized as fixed supports, which do not allow any displacement; hinged supports, which allow rotation but prevent translation; or, roller or link supports, which prevent translation in only one direction. Other types of restraints, such as those which prevent rotation but permit translation in one or more directions, can also be considered in an analysis, as discussed in subsequent chapters.

Line Diagrams

The analytical model of a structure is represented by a *line diagram*, on which each member is depicted by a line coinciding with its centroidal axis. The member dimensions and the size of connections are not shown. Rigid joints are usually represented by points, and hinged joints by small circles, at the intersections of members. Each joint and member of the structure is identified by a number. For example, the analytical model of the plane truss of Fig. 1.10(a) is shown in Fig. 1.10(b), in which the joint numbers are enclosed within circles to distinguish them from the member numbers enclosed within rectangles.



(a) Plane Truss



(b) Analytical Model

Fig. 1.10

1.6 FUNDAMENTAL RELATIONSHIPS FOR STRUCTURAL ANALYSIS

Structural analysis, in general, involves the use of three types of relationships:

- Equilibrium equations,
- compatibility conditions, and
- constitutive relations.

Equilibrium Equations

A structure is considered to be in equilibrium if, initially at rest, it remains at rest when subjected to a system of forces and couples. If a structure is in equilibrium, then all of its members and joints must also be in equilibrium.

Recall from *statics* that for a plane (two-dimensional) structure lying in the *XY* plane and subjected to a coplanar system of forces and couples (Fig. 1.11), the necessary and sufficient conditions for equilibrium can be expressed in Cartesian (*XY*) coordinates as

$$\boxed{\sum F_X = 0 \qquad \sum F_Y = 0 \qquad \sum M = 0} \tag{1.1}$$

These equations are referred to as the *equations of equilibrium* for plane structures.

For a space (three-dimensional) structure subjected to a general three-dimensional system of forces and couples (Fig. 1.12), the equations of equilibrium are expressed as

$$\boxed{\begin{array}{lll} \sum F_X = 0 & \sum F_Y = 0 & \sum F_Z = 0 \\ \sum M_X = 0 & \sum M_Y = 0 & \sum M_Z = 0 \end{array}} \tag{1.2}$$

For a structure subjected to static loading, the equilibrium equations must be satisfied for the entire structure as well as for each of its members and joints. In structural analysis, equations of equilibrium are used to relate the forces (including couples) acting on the structure or one of its members or joints.

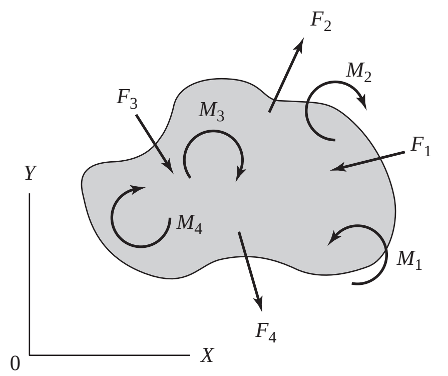


Fig. 1.11

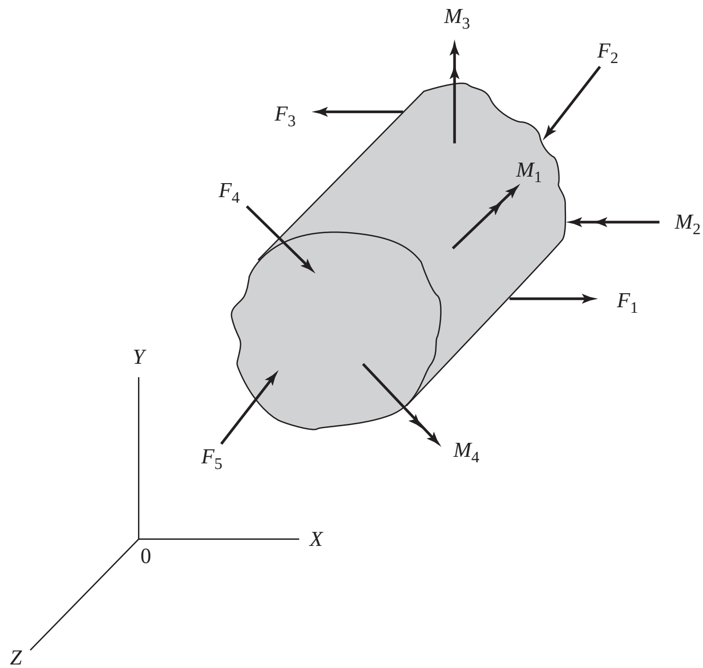


Fig. 1.12

Compatibility Conditions

The *compatibility conditions* relate the deformations of a structure so that its various parts (members, joints, and supports) fit together without any gaps or overlaps. These conditions (also referred to as the *continuity conditions*) ensure that the deformed shape of the structure is continuous (except at the locations of any internal hinges or rollers), and is consistent with the support conditions.

Consider, for example, the two-member plane frame shown in Fig. 1.13. The deformed shape of the frame due to an arbitrary loading is also depicted, using an exaggerated scale. When analyzing a structure, the compatibility conditions are used to relate member end displacements to joint displacements which, in turn, are related to the support conditions. For example, because joint 1 of the frame in Fig. 1.13 is attached to a roller support that cannot translate in the vertical direction, the vertical displacement of this joint must be zero. Similarly, because joint 3 is attached to a fixed support that can neither rotate nor translate in any direction, the rotation and the horizontal and vertical displacements of joint 3 must be zero.

The displacements of the ends of members are related to the joint displacements by the compatibility requirement that the displacements of a member's end must be the same as the displacements of the joint to which the member end is connected. Thus, as shown in Fig. 1.13, because joint 1 of the example frame displaces to the right by a distance d_1 and rotates clockwise by an angle θ_1, the left end of the horizontal member (member 1) that is attached to joint 1

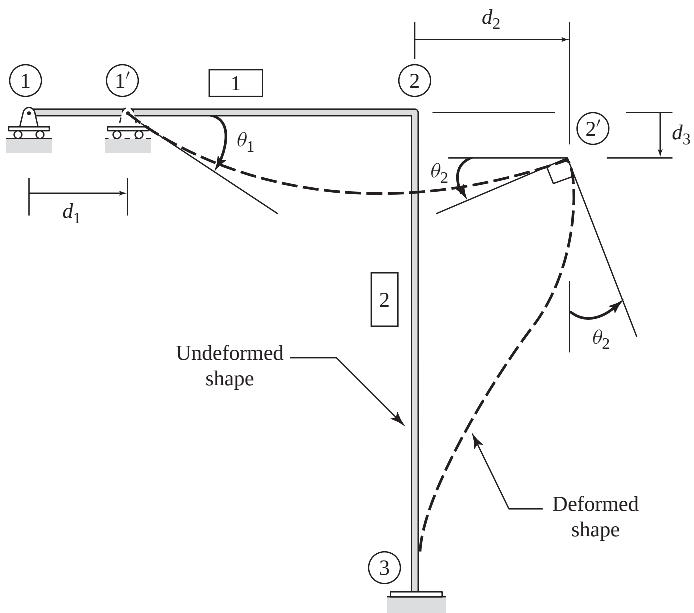


Fig. 1.13

must also translate to the right by distance d_1 and rotate clockwise by angle θ_1. Similarly, because the displacements of joint 2 consist of the translations d_2 to the right and d_3 downward and the counterclockwise rotation θ_2, the right end of the horizontal member and the top end of the vertical member that are connected to joint 2 must also undergo the same displacements (i.e., d_2, d_3, and θ_2). The bottom end of the vertical member, however, is not subjected to any displacements, because joint 3, to which this particular member end is attached, can neither rotate nor translate in any direction.

Finally, compatibility requires that the deflected shapes of the members of a structure be continuous (except at any internal hinges or rollers) and be consistent with the displacements at the corresponding ends of the members.

Constitutive Relations

The *constitutive relations* (also referred to as the *stress-strain relations*) describe the relationships between the stresses and strains of a structure in accordance with the stress-strain properties of the structural material. As discussed previously, the equilibrium equations provide relationships between the forces, whereas the compatibility conditions involve only deformations. The constitutive relations provide the link between the equilibrium equations and compatibility conditions that is necessary to establish the load-deformation relationships for a structure or a member.

In the analysis of framed structures, the basic stress-strain relations are first used, along with the member equilibrium and compatibility equations, to establish relationships between the forces and displacements at the ends of a member. The member force-displacement relations thus obtained are then treated as the constitutive relations for the entire structure, and are used to link the structure's equilibrium and compatibility equations, thereby yielding the load-deformation relationships for the entire structure. These load-deformation relations can then be solved to determine the deformations of the structure due to a given loading.

In the case of statically determinate structures, the equilibrium equations can be solved independently of the compatibility and constitutive relations to obtain the reactions and member forces. The deformations of the structure, if desired, can then be determined by employing the compatibility and constitutive relations. In the analysis of statically indeterminate structures, however, the equilibrium equations alone are not sufficient for determining the reactions and member forces. Therefore, it becomes necessary to satisfy simultaneously the three types of fundamental relationships (i.e., equilibrium, compatibility, and constitutive relations) to determine the structural response.

Matrix methods of structural analysis are usually formulated by direct application of the three fundamental relationships as described in general terms in the preceding paragraphs. (Details of the formulations are presented in subsequent chapters.) However, matrix methods can also be formulated by using work-energy principles that satisfy the three fundamental relationships indirectly. Work-energy principles are generally preferred in the formulation of finite-element methods, because they can be more conveniently applied to derive the approximate force-displacement relations for the elements of surface structures and solids.

The matrix methods presented in this text are formulated by the direct application of the equilibrium, compatibility, and constitutive relationships. However, to introduce readers to the finite-element method, and to familiarize them with the application of the work-energy principles, we also derive the member force-displacement relations for plane structures by a finite-element approach that involves a work-energy principle known as the *principle of virtual work*. In the following paragraphs, we review two statements of this principle pertaining to rigid bodies and deformable bodies, for future reference.

Principle of Virtual Work for Rigid Bodies

The *principle of virtual work for rigid bodies* (also known as the *principle of virtual displacements for rigid bodies*) can be stated as follows.

> *If a rigid body, which is in equilibrium under a system of forces (and couples), is subjected to any small virtual rigid-body displacement, the virtual work done by the external forces (and couples) is zero.*

In the foregoing statement, the term virtual simply means imaginary, not real. Consider, for example, the cantilever beam shown in Fig. 1.14(a). The free-body diagram of the beam is shown in Fig. 1.14(b), in which P_X, and P_Y are the components of the external load P in the X and Y directions, respectively, and R_1, R_2, and R_3 represent the reactions at the fixed support 1. Note that the beam is in equilibrium under the action of the forces P_X, P_Y, R_1, and R_2, and the couple R_3. Now, imagine that the beam is given an arbitrary, small virtual rigid-body displacement from its initial equilibrium position 1–2 to another position 1′–2′, as shown in Fig. 1.14(c). As this figure indicates, the total virtual

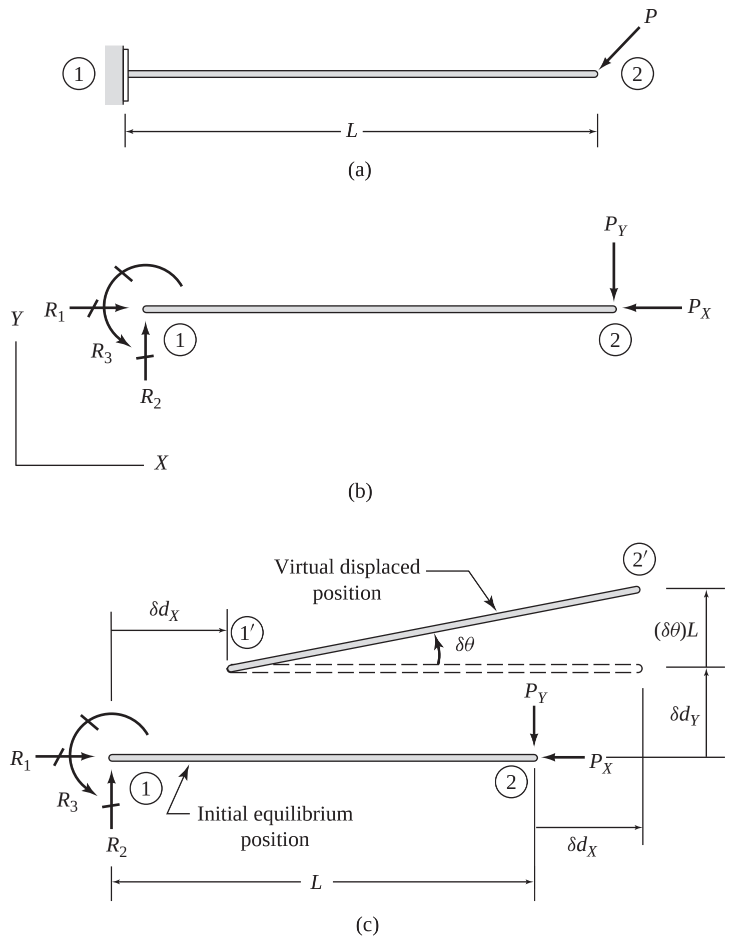


Fig. 1.14

displacement of the beam can be decomposed into rigid-body translations δd_X and δd_Y in the X and Y directions, respectively, and a rigid-body rotation $\delta\theta$ about point 1. Note that the symbol δ is used here to identify the virtual quantities. As the beam undergoes the virtual displacement from position 1–2 to position 1′–2′, the forces and the couple acting on it perform work, which is referred to as the *virtual work*. The total virtual work, δW_e, can be expressed as the algebraic sum of the virtual work δW_X and δW_Y, performed during translations in the X and Y directions, respectively, and the virtual work δW_R, done during the rotation; that is,

$$\delta W_e = \delta W_X + \delta W_Y + \delta W_R \tag{1.3}$$

During the virtual translation δd_X of the beam, the virtual work performed by the forces can be expressed as follows (Fig 1.14c).

$$\delta W_X = R_1 \delta d_X - P_X \delta d_X = (R_1 - P_X)\,\delta d_X = \left(\sum F_X\right)\delta d_X \tag{1.4}$$

Similarly, the virtual work done during the virtual translation δd_Y is given by

$$\delta W_Y = R_2 \delta d_Y - P_Y \delta d_Y = (R_2 - P_Y)\, \delta d_Y = \left(\sum F_Y\right) \delta d_Y \tag{1.5}$$

and the virtual work done by the forces and the couple during the small virtual rotation $\delta\theta$ can be expressed as follows (Fig. 1.14c).

$$\delta W_R = R_3 \delta\theta - P_Y\, (L \delta\theta) = (R_3 - P_Y L)\, \delta\theta = \left(\sum M_{1}\right) \delta\theta \tag{1.6}$$

The expression for the total virtual work can now be obtained by substituting Eqs. (1.4–1.6) into Eq. (1.3). Thus,

$$\delta W_e = \left(\sum F_X\right) \delta d_X + \left(\sum F_Y\right) \delta d_Y + \left(\sum M_{1}\right) \delta\theta \tag{1.7}$$

However, because the beam is in equilibrium, $\sum F_X = 0$, $\sum F_Y = 0$, and $\sum M_{1} = 0$; therefore, Eq. (1.7) becomes

$$\boxed{\delta W_e = 0} \tag{1.8}$$

which is the mathematical statement of the principle of virtual work for rigid bodies.

Principle of Virtual Work for Deformable Bodies

The *principle of virtual work for deformable bodies* (also called the *principle of virtual displacements for deformable bodies*) can be stated as follows.

> *If a deformable structure, which is in equilibrium under a system of forces (and couples), is subjected to any small virtual displacement consistent with the support and continuity conditions of the structure, then the virtual external work done by the real external forces (and couples) acting through the virtual external displacements (and rotations) is equal to the virtual strain energy stored in the structure.*

To demonstrate the validity of this principle, consider the two-member truss of Fig. 1.15(a), which is in equilibrium under the action of an external load P. The free-body diagram of joint 3 of the truss is shown in Fig. 1.15(b). Since joint 3 is in equilibrium, the external and internal forces acting on it must satisfy the following two equations of equilibrium:

$$\begin{aligned} &+\rightarrow \sum F_X = 0 \qquad && -F_1 \sin\theta_1 + F_2 \sin\theta_2 = 0 \\ &+\uparrow \sum F_Y = 0 \qquad && F_1 \cos\theta_1 + F_2 \cos\theta_2 - P = 0 \end{aligned} \tag{1.9}$$

in which F_1 and F_2 denote the internal (axial) forces in members 1 and 2, respectively; and θ_1 and θ_2 are, respectively, the angles of inclination of these members with respect to the vertical as shown in the figure.

Now, imagine that joint 3 is given a small virtual compatible displacement, δd, in the downward direction, as shown in Fig. 1.15(a). It should be noted that this virtual displacement is consistent with the support conditions of the truss in the sense that joints 1 and 2, which are attached to supports, are not displaced. Because the reaction forces at joints 1 and 2 do not perform any work,

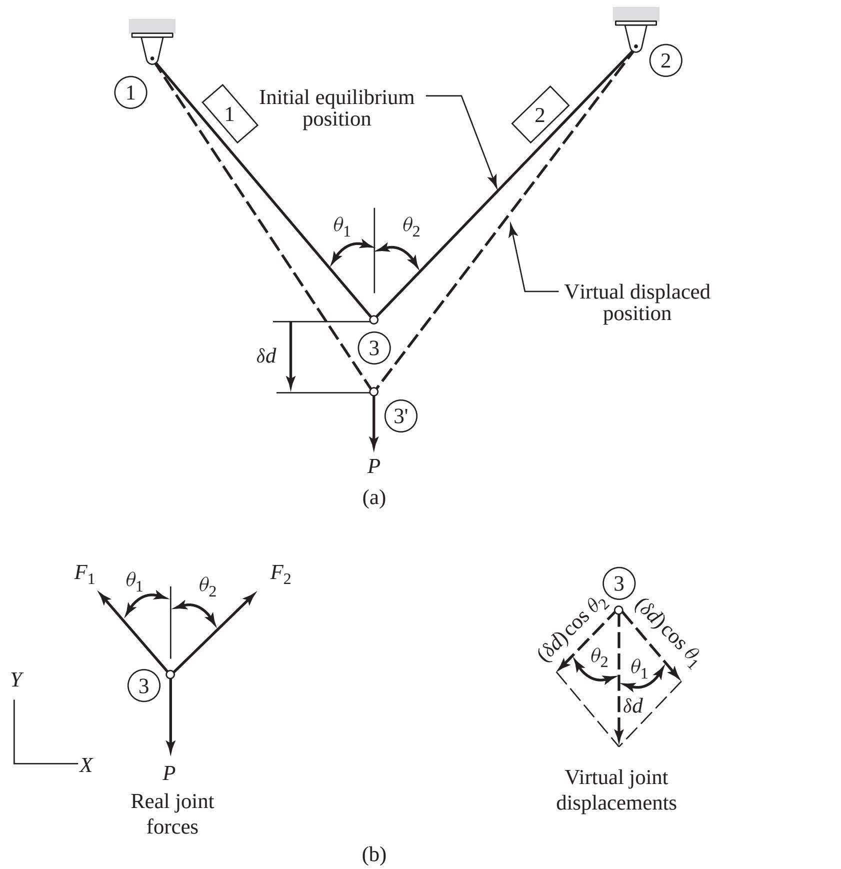


Fig. 1.15

the total virtual work for the truss, δW, is equal to the algebraic sum of the virtual work of the forces acting at joint 3. Thus, from Fig. 1.15(b),

$$\delta W = P\,\delta d - F_1(\delta d \cos\theta_1) - F_2(\delta d \cos\theta_2)$$

which can be rewritten as

$$\delta W = (P - F_1 \cos\theta_1 - F_2 \cos\theta_2)\,\delta d \tag{1.10}$$

As indicated by Eq. (1.9), the term in parentheses on the right-hand side of Eq. (1.10) is zero. Therefore, the total virtual work, δW, is zero. By substituting $\delta W = 0$ into Eq. (1.10) and rearranging terms, we write

$$P(\delta d) = F_1(\delta d \cos\theta_1) + F_2(\delta d \cos\theta_2) \tag{1.11}$$

in which the quantity on the left-hand side represents the virtual external work, δW_e, performed by the real external force P acting through the virtual external displacement δd. Furthermore, because the terms $(\delta d)\cos\theta_1$ and $(\delta d)\cos\theta_2$ are equal to the virtual internal displacements (elongations) of members 1 and 2, respectively, we can conclude that the right-hand side of Eq. (1.11) represents the virtual internal work, δW_{i}, done by the real internal forces acting through the corresponding virtual internal displacements; that is,

$$\delta W_{\mathrm{e}} = \delta W_{\mathrm{i}} \tag{1.12}$$

Realizing that the internal work is also referred to as the strain energy, U, we can express Eq. (1.12) as

$$\boxed{\delta W_{\mathrm{e}} = \delta U} \tag{1.13}$$

in which δU denotes the virtual strain energy. Note that Eq. (1.13) is the mathematical statement of the principle of virtual work for deformable bodies.

For computational purposes, it is usually convenient to express Eq. (1.13) in terms of the stresses and strains in the members of the structure. For that purpose, let us consider a differential element of a member of an arbitrary structure subjected to a general loading (Fig. 1.16). The element is in equilibrium under a general three-dimensional stress condition, due to the real forces acting on the structure. Now, as the structure is subjected to a virtual displacement, virtual strains develop in the element and the internal forces due to the real stresses perform virtual internal work as they move through the internal displacements caused by the virtual strains. For example, the virtual internal work done by the real force due to the stress σ_x as it moves through the virtual displacement caused by the virtual strain $\delta\varepsilon_x$ can be determined as follows.

$$\begin{aligned} \text{real force} &= \text{stress} \times \text{area} &&= \sigma_x\,(dy\,dz) \\ \text{virtual displacement} &= \text{strain} \times \text{length} &&= (\delta\varepsilon_x)\,dx \end{aligned}$$

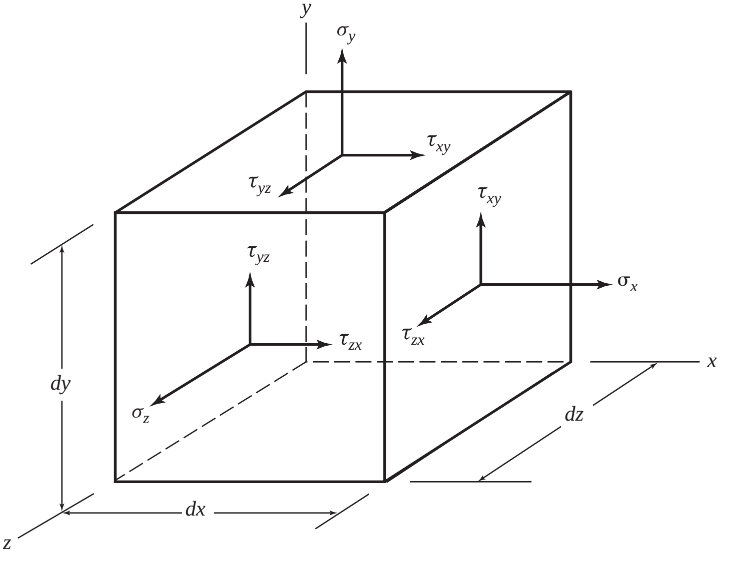


Fig. 1.16

Therefore,

$$\begin{aligned}\text{virtual internal work} &= \text{real force} \times \text{virtual displacement}\\ &= (\sigma_x\, dy\, dz)\,(\delta\varepsilon_x\, dx)\\ &= (\delta\varepsilon_x\, \sigma_x)\, dV\end{aligned}$$

in which $dV = dx\, dy\, dz$ is the volume of the differential element. Thus, the virtual internal work due to all six stress components is given by

$$\text{virtual internal work in element } dV$$
$$= (\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{yz}\tau_{yz} + \delta\gamma_{zx}\tau_{zx})\, dV \quad \textbf{(1.14)}$$

In Eq. (1.14), $\delta\varepsilon_x, \delta\varepsilon_y, \delta\varepsilon_z, \delta\gamma_{xy}, \delta\gamma_{yz},$ and $\delta\gamma_{zx}$ denote, respectively, the virtual strains corresponding to the real stresses $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz},$ and τ_{zx}, shown in Fig. 1.16.

The total virtual internal work, or the virtual strain energy stored in the entire structure, can be obtained by integrating Eq. (1.14) over the volume V of the structure. Thus,

$$\delta U = \int_V \left(\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{yz}\tau_{yz} + \delta\gamma_{zx}\tau_{zx}\right) dV \quad \textbf{(1.15)}$$

Finally, by substituting Eq. (1.15) into Eq. (1.13), we obtain the statement of the principle of virtual work for deformable bodies in terms of the stresses and strains of the structure.

$$\boxed{\delta W_e = \int_V \left(\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{yz}\tau_{yz} + \delta\gamma_{zx}\tau_{zx}\right) dV} \quad \textbf{(1.16)}$$

1.7 LINEAR VERSUS NONLINEAR ANALYSIS

In this text, we focus our attention mainly on *linear analysis* of structures. Linear analysis of structures is based on the following two fundamental assumptions:

1. The structures are composed of linearly elastic material; that is, the stress-strain relationship for the structural material follows Hooke's law.
2. The deformations of the structures are so small that the squares and higher powers of member slopes, (chord) rotations, and axial strains are negligible in comparison with unity, and the equations of equilibrium can be based on the undeformed geometry of the structure.

The reason for making these assumptions is to obtain linear relationships between applied loads and the resulting structural deformations. An important advantage of linear force-deformation relations is that the *principle of* *superposition* can be used in the analysis. This principle states essentially that *the combined effect of several loads acting simultaneously on a structure equals the algebraic sum of the effects of each load acting individually on the structure*.

Engineering structures are usually designed so that under service loads they undergo small deformations, with stresses within the initial linear portions of the stress-strain curves of their materials. Thus, linear analysis generally proves adequate for predicting the performance of most common types of structures under service loading conditions. However, at higher load levels, the accuracy of linear analysis generally deteriorates as the deformations of the structure increase and/or its material is strained beyond the yield point. Because of its inherent limitations, linear analysis cannot be used to predict the ultimate load capacities and instability characteristics (e.g., buckling loads) of structures.

With the recent introduction of design specifications based on the ultimate strengths of structures, the use of *nonlinear analysis* in structural design is increasing. In a nonlinear analysis, the restrictions of linear analysis are removed by formulating the equations of equilibrium on the deformed geometry of the structure that is not known in advance, and/or taking into account the effects of inelasticity of the structural material. The load-deformation relationships thus obtained for the structure are nonlinear, and are usually solved using iterative techniques. An introduction to this still-evolving field of nonlinear structural analysis is presented in Chapter 10.

1.8 SOFTWARE

Software for the analysis of framed structures using the matrix stiffness method is provided on the publisher's website for this book, www.cengage.com/engineering. The software can be used by readers to verify the correctness of various subroutines and programs that they will develop during the course of study of this text, as well as to check the answers to the problems given at the end of each chapter. A description of the software, and information on how to install and use it, is presented in Appendix A.

SUMMARY

In this chapter, we discussed the topics summarized in the following list.

1. *Structural analysis* is the prediction of the performance of a given structure under prescribed loads and/or other external effects.

2. Both matrix and classical methods of structural analysis are based on the same fundamental principles. However, classical methods were developed to analyze particular types of structures, whereas matrix methods are more general and systematic so that they can be conveniently programmed on computers.

3. Two different methods can be used for matrix analysis of structures; namely, the *flexibility* and *stiffness* methods. The stiffness method is more systematic and can be implemented more easily on computers, and is therefore currently preferred in professional practice.

4. *Framed structures* are composed of straight members whose lengths are significantly larger than their cross-sectional dimensions. Framed structures can be classified into six basic categories: plane trusses, beams, plane frames, space trusses, grids, and space frames.

5. An *analytical model* is a simplified (idealized) representation of a real structure for the purpose of analysis. Framed structures are modeled as assemblages of straight members connected at their ends to joints, and these analytical models are represented by *line diagrams*.

6. The analysis of structures involves three fundamental relationships: equilibrium equations, compatibility conditions, and constitutive relations.

7. The principle of virtual work for deformable bodies states that if a deformable structure, which is in equilibrium, is subjected to a small compatible virtual displacement, then the virtual external work is equal to the virtual strain energy stored in the structure.

8. Linear structural analysis is based on two fundamental assumptions: the stress-strain relationship for the structural material is linearly elastic, and the structure's deformations are so small that the equilibrium equations can be based on the undeformed geometry of the structure.