CHAPTER 8

Two-Dimensional Isoparametric Elements and Numerical Integration

8.1 INTRODUCTION

In Chapters 6 and 7, we have developed the constant-strain triangular (CST) element for stress analysis. In this chapter, we develop four-node and higher order *isoparametric* elements and apply them to stress analysis. These elements have proved effective on a wide variety of two- and three-dimensional problems in engineering. We discuss the two-dimensional four-node quadrilateral in detail. Development of higher order elements follows the same basic steps used in the four-node quadrilateral. The higher order elements can capture variations in stress near fillets, holes, etc. We can view the isoparametric family of elements in a unified manner due to the simple and versatile manner in which shape functions can be derived, followed by the generation of the element stiffness matrix using numerical integration.

8.2 THE FOUR-NODE QUADRILATERAL

Consider the general quadrilateral element shown in Fig. 8.1. The local nodes are numbered as 1, 2, 3, and 4 in a *counterclockwise* fashion as shown, and (x_i, y_i) are the coordinates of node i. The vector $\mathbf{q} = [q_1, q_2, \ldots, q_8]^{\mathrm{T}}$ denotes the element displacement vector. The displacement of an interior point P located at (x, y) is represented as $\mathbf{u} = [u(x, y), v(x, y)]^{\mathrm{T}}$.

Shape Functions

Following the steps in earlier chapters, we first develop the shape functions on a master element, shown in Fig. 8.2. The master element is defined in ξ-, η-coordinates (or *natural* coordinates) and is square-shaped. The Lagrange shape functions where $i = 1, 2, 3$, and 4, are defined such that N_i is equal to unity at node i and is zero at other nodes. In particular, consider the definition of N_i:

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned} \tag{8.1}$$

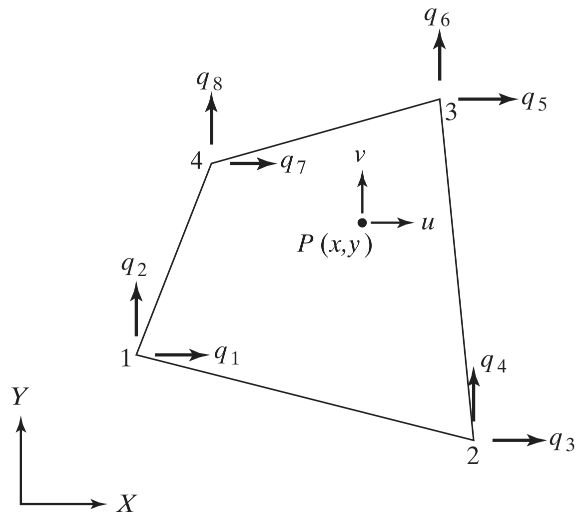


FIGURE 8.1 Four-node quadrilateral element.

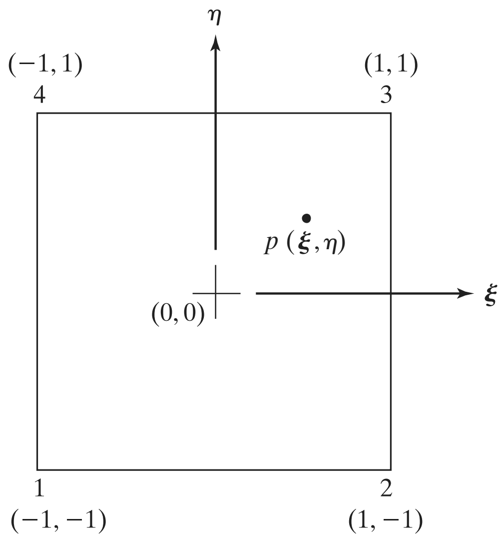


FIGURE 8.2 The quadrilateral element in ξ, η space (the master element).

Now, the requirement that $N_1 = 0$ at nodes 2, 3, and 4 is equivalent to requiring that $N_1 = 0$ along the edges $\xi = +1$ and $\eta = +1$ (Fig. 8.2). Thus, N_1 has to be of the form

$$N_1 = c(1 - \xi)\,(1 - \eta) \tag{8.2}$$

where c is some constant. The constant is determined from the condition $N_1 = 1$ at node 1. Since $\xi = -1,\ \eta = -1$ at node 1, we have

$$1 = c(2)(2) \tag{8.3}$$

which yields $c = 1/4$. Thus,

$$N_1 = \tfrac{1}{4}(1 - \xi)(1 - \eta) \tag{8.4}$$

All the four shape functions can be written as

$$
\begin{aligned}
N_1 &= \tfrac{1}{4}(1 - \xi)(1 - \eta) \\
N_2 &= \tfrac{1}{4}(1 + \xi)(1 - \eta) \\
N_3 &= \tfrac{1}{4}(1 + \xi)(1 + \eta) \\
N_4 &= \tfrac{1}{4}(1 - \xi)(1 + \eta)
\end{aligned}
\tag{8.5}
$$

While implementing in a computer program, the compact representation of Eq. 8.5 is useful

$$N_i = \tfrac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i) \tag{8.6}$$

where (ξ_i, η_i) are the coordinates of node i.

We now express the displacement field within the element in terms of the nodal values. Thus, if $\mathbf{u} = [u, v]^{\mathrm{T}}$ represents the displacement components of a point located at (ξ, η), and $\mathbf{q}$, of dimension (8×1), is the element displacement vector, then

$$
\begin{aligned}
u &= N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7 \\
v &= N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8
\end{aligned}
\tag{8.7a}
$$

which can be written in matrix form as

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{8.7b}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{8.8}$$

In the isoparametric formulation, we use the *same* shape functions N_i to also express the coordinates of a point within the element in terms of nodal coordinates. Thus,

$$
\begin{aligned}
x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \\
y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4
\end{aligned}
\tag{8.9}
$$

Subsequently, we will need to express the derivatives of a function in x-, y-coordinates in terms of its derivatives in ξ-, η-coordinates. This is done as follows: a function $f = f(x, y)$, according to Eq. 8.9, can be considered to be an implicit function of ξ and η as $f = f[x(\xi, \eta), y(\xi, \eta)]$. Using the chain rule of differentiation, we have

$$
\begin{aligned}
\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \\
\frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}
\end{aligned}
\tag{8.10}
$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} \tag{8.11}$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{8.12}$$

According to Eqs. 8.5 and 8.9, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right] \tag{8.13a}$$

$$= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{8.13b}$$

Equation 8.11 can be inverted as

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{8.14a}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{8.14b}$$

These expressions will be used in the derivation of the element stiffness matrix.

An additional result that will be needed is the relation

$$dx\, dy = \det \mathbf{J}\, d\xi\, d\eta \tag{8.15}$$

The proof of this result, found in many textbooks on calculus, is given in the appendix.

Element Stiffness Matrix

The stiffness matrix for the quadrilateral element can be derived from the strain energy in the body, given by

$$U = \int_V \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \tag{8.16}$$

or

$$U = \sum_e t_e \int_e \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dA \tag{8.17}$$

where t_e is the thickness of element e.

The strain–displacement relation is

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{8.18}$$

By considering $f \equiv u$ in Eq. 8.14b, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{8.19a}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{8.19b}$$

Equations 8.18 and 8.19a, b yield

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{8.20}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{8.21}$$

Now, using Eq. 8.7a, we get

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{Gq} \tag{8.22}$$

where

$$\mathbf{G} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \tag{8.23}$$

Equations 8.20 and 8.22 now yield

$$\boxed{\boldsymbol{\epsilon} = \mathbf{Bq}} \tag{8.24}$$

where

$$\mathbf{B} = \mathbf{AG} \tag{8.25}$$

The relation $\boldsymbol{\epsilon} = \mathbf{Bq}$ is the desired result. The strain in the element is expressed in terms of its nodal displacement. The stress is now given by

$$\boxed{\boldsymbol{\sigma} = \mathbf{DBq}} \tag{8.26}$$

where $\mathbf{D}$ is a (3×3) material matrix. The strain energy in Eq. 8.17 becomes

$$U = \sum_e \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det\mathbf{J}\,d\xi\,d\eta\right]\mathbf{q} \tag{8.27a}$$

$$= \sum_e \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{8.27b}$$

where

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det\mathbf{J}\,d\xi\,d\eta} \tag{8.28}$$

is the element stiffness matrix of dimension (8×8).

We note here that quantities $\mathbf{B}$ and $\det \mathbf{J}$ in the integral in Eq. 8.28 are involved functions of ξ and η, and so the integration has to be performed numerically. Methods of numerical integration are discussed subsequently.

Element Force Vectors

Body Force. A body force is the distributed force per unit volume that contributes to the global load vector $\mathbf{F}$. This contribution can be determined by considering the body force term in the potential energy expression

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\,dV \tag{8.29}$$

Using $\mathbf{u} = \mathbf{Nq}$, and treating the body force $\mathbf{f} = [f_x \;\; f_y]^{\mathrm{T}}$ as constant within each element, we get

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\,dV = \sum_e \mathbf{q}^{\mathrm{T}}\mathbf{f}^e \tag{8.30}$$

where the (8×1) element body force vector is given by

$$\mathbf{f}^e = t_e\left[\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^{\mathrm{T}}\det\mathbf{J}\,d\xi\,d\eta\right]\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \tag{8.31}$$

As with the stiffness matrix derived earlier, this body force vector has to be evaluated by numerical integration.

Traction Force. Assume that a constant traction force $\mathbf{T} = [T_x \;\; T_y]^{\mathrm{T}}$—a force per unit area—is applied on edge 2–3 of the quadrilateral element. Along this edge, we have $\xi = 1$. If we use the shape functions given in Eq. 8.5, this becomes $N_1 = N_4 = 0$, $N_2 = (1 - \eta)/2$, and $N_3 = (1 + \eta)/2$. Note that the shape functions are linear functions along the edges. Consequently, from the potential, the element traction load vector is readily given by

$$\mathbf{T}^e = \frac{t_e \ell_{2-3}}{2}[0 \quad 0 \quad T_x \quad T_y \quad T_x \quad T_y \quad 0 \quad 0]^{\mathrm{T}} \tag{8.32}$$

where $\ell_{2-3} =$ length of edge 2–3. For varying distributed loads, we may express T_x and T_y in terms of values of nodes 2 and 3 using shape functions. Numerical integration can be used in this case.

Finally, **point** loads are considered in the usual manner by having a structural node at that point and simply adding to the global load vector $\mathbf{F}$.

8.3 NUMERICAL INTEGRATION

Consider the problem of numerically evaluating a one-dimensional integral of the form

$$I = \int_{-1}^{1} f(\xi)\,d\xi \tag{8.33}$$

The *Gauss–Legendre quadrature* approach for evaluating I is given subsequently. This method has proved most useful in finite element work. Extension to integrals in two and three dimensions follows readily.

Consider the n-point approximation

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2) + \cdots + w_n f(\xi_n) \tag{8.34}$$

where $w_1, w_2, \ldots,$ and w_n are the **weights** and $\xi_1, \xi_2, \ldots,$ and ξ_n are the sampling points or **Gauss–Legendre points**. The idea behind Gauss–Legendre quadrature is to select the n Gauss points and n weights such that Eq. 8.34 provides an exact answer for polynomials $f(\xi)$ of as large a degree as possible. In other words, the idea is that if the n-point integration formula is exact for all polynomials up to as high a degree as possible, then the formula will work well even if f is not a polynomial. To get some intuition for the method, the one-point and two-point approximations are discussed in the sections that follow.

One-Point Formula. Consider the formula with $n = 1$ as

$$\int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) \tag{8.35}$$

Since there are two parameters, w_1 and ξ_1, we consider requiring the formula in Eq. 8.35 to be exact when $f(\xi)$ is a polynomial of order 1. Thus, if $f(\xi) = a_0 + a_1\xi$, then we require

$$\text{Error} = \int_{-1}^{1} (a_0 + a_1\xi)\, d\xi - w_1 f(\xi_1) = 0 \tag{8.36a}$$

$$\text{Error} = 2a_0 - w_1(a_0 + a_1\xi_1) = 0 \tag{8.36b}$$

or

$$\text{Error} = a_0(2 - w_1) - w_1 a_1 \xi_1 = 0 \tag{8.36c}$$

From Eq. 8.36c, we see that the error is zeroed if

$$w_1 = 2 \quad \xi_1 = 0 \tag{8.37}$$

For any general f, then, we have

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx 2f(0) \tag{8.38}$$

which is seen to be the familiar *midpoint rule* (Fig. 8.3).

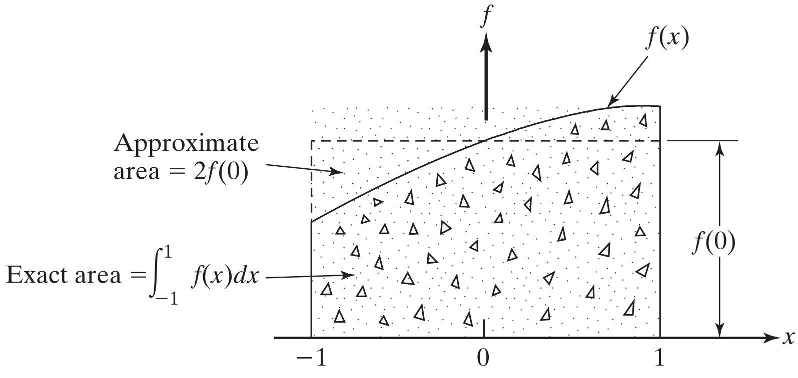


FIGURE 8.3 One-point Gauss quadrature.

Two-Point Formula. Consider the formula with $n = 2$ as

$$\int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2) \tag{8.39}$$

We have four parameters to choose: w_1, w_2, ξ_1, and ξ_2. We can therefore expect Eq. 8.39 to be exact for a cubic polynomial. Thus, choosing $f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$ yields

$$\text{Error} = \left[\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3)\, d\xi\right] - [w_1 f(\xi_1) + w_2 f(\xi_2)] \tag{8.40}$$

Requiring zero error yields

$$\begin{aligned} w_1 + w_2 &= 2 \\ w_1\xi_1 + w_2\xi_2 &= 0 \\ w_1\xi_1^2 + w_2\xi_2^2 &= \tfrac{2}{3} \\ w_1\xi_1^3 + w_2\xi_2^3 &= 0 \end{aligned} \tag{8.41}$$

These nonlinear equations have the unique solution:

$$w_1 = w_2 = 1 \quad -\xi_1 = \xi_2 = 1/\sqrt{3} = 0.5773502691\ldots \tag{8.42}$$

From this solution, we can conclude that n-point Gaussian quadrature will provide an exact answer if f is a polynomial of order $(2n - 1)$ or less. The Gauss–Legendre points are more easily obtained by recognizing that they are zeros of Legendre polynomials $P_k(x)$. Some Legendre polynomials and the zeros are given below.

$$P_1(x) = x \quad \text{zero at } x = 0$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad \text{zeros at } x = \pm\frac{1}{\sqrt{3}}$$

$$P_2(x) = \frac{1}{2}(5x^3 - 3x) \quad \text{zeros at } x = 0, x = \pm\sqrt{\frac{3}{5}}$$

$$\ldots$$

The Legendre polynomials may be generated using

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ (j+1)P_{j+1}(x) &= (2j+1)xP_j(x) - jP_{j-1}(x) \quad j = 1 \text{ to } n-1 \end{aligned} \tag{8.43}$$

Also the derivative $P_n'(x) = [dP_n(x)/dx]$ is given by the relation

$$(x^2 - 1)P_n'(x) = n[xP_n(x) - P_{n-1}(x)] \tag{8.44}$$

The kth zero can be obtained using the starting value $x_0 = \cos(\pi[k - (1/4)]/[n + (1/2)])$ and Newton's iteration

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{8.45}$$

For $i = 0, 1, 2, \ldots$ until convergence is reached.

The kth weight factor is then obtained using

$$w_k = \frac{2}{(1 - x_k^2)[P_n'(x_k)]^2} \tag{8.46}$$

These steps have been implemented in the javascript program *GaussLegendre.html* and the Excel program *GaussLegendre.xls*. These programs can be used to find zeros and weight factors for various values of n.

Another javascript program *GLInteg.html*, which can be used to calculate the integral of a one-variable function on an interval, has been included in the downloadable programs.

Table 8.1 gives the values of w_i and ξ_i for Gauss quadrature formulas of orders $n = 1$ through $n = 6$. Note that the Gauss points are located symmetrically with respect to the origin and that symmetrically placed points have the same weights. Moreover, the large number of digits given in Table 8.1 should be used in the calculations for accuracy (i.e., use double precision on the computer).

TABLE 8.1 Gauss Points and Weights for Gaussian Quadrature

$$\int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i)$$

No. of Points, n	Location, ξ_i	Weights, w_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	±0.7745966692	0.5555555556
	0.0	0.8888888889
4	±0.8611363116	0.3478548451
	±0.3399810436	0.6521451549
5	±0.9061798459	0.2369268851
	±0.5384693101	0.4786286705
	0.0	0.5688888889
6	±0.9324695142	0.1713244924
	±0.6612093865	0.3607615730
	±0.2386191861	0.4679139346

Example 8.1

Evaluate

$$I = \int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{(x + 2)} \right] dx$$

using one-point and two-point Gauss quadratures.

Solution For $n = 1$, we have $w_1 = 2$, $x_1 = 0$, and

$$\begin{aligned} I &\approx 2f(0) \\ &= 7.0 \end{aligned}$$

For $n = 2$, we have $w_1 = w_2 = 1$, $x_1 = -0.57735\ldots$, $x_2 = +0.57735\ldots$, and $I \approx 8.7857$. This may be compared with the exact solution

$$I_{\text{exact}} = 8.8165$$

■

The integral may be easily checked using the javascript program *GLInteg.html*.

Two-Dimensional Integrals

The extension of Gaussian quadrature to two-dimensional integrals of the form

$$I = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta)\, d\xi\, d\eta \tag{8.47}$$

follows readily, since

$$\begin{aligned} I &\approx \int_{-1}^{1} \left[\sum_{i=1}^{n} w_i f(\xi_i, \eta) \right] d\eta \\ &\approx \sum_{j=1}^{n} w_j \left[\sum_{i=1}^{n} w_i f(\xi_i, \eta_j) \right] \end{aligned}$$

or

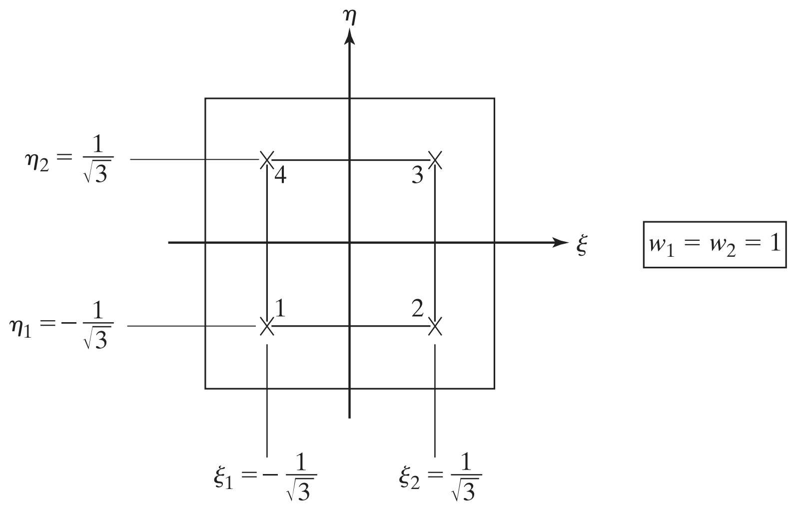
$$I \approx \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j f(\xi_i, \eta_j) \tag{8.48}$$

Stiffness Integration

To illustrate the use of Eq. 8.48, consider the element stiffness for a quadrilateral element

$$\mathbf{k}^e = t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \;\det \mathbf{J}\, d\xi\, d\eta$$

where $\mathbf{B}$ and $\det \mathbf{J}$ are functions of ξ and η. Note that this integral actually consists of the integral of each element in an (8×8) matrix. However, using the fact that $\mathbf{k}^e$ is symmetric, we do not need to integrate elements below the main diagonal.



$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx w_1^2 f(\xi_1,\eta_1) + w_2 w_1 f(\xi_2,\eta_1) + w_2^2 f(\xi_2,\eta_2) + w_1 w_2 f(\xi_1,\eta_2)$$

FIGURE 8.4 Gaussian quadrature in two dimensions using the 2×2 rule.

Let ϕ represent the *ij*th element in the integrand. That is, let

$$\phi(\xi, \eta) = t_e(\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J})_{ij} \tag{8.49}$$

Then, if we use a 2×2 rule, we get

$$k_{ij} \approx w_1^2\phi(\xi_1, \eta_1) + w_1 w_2 \phi(\xi_1, \eta_2) + w_2 w_1 \phi(\xi_2, \eta_1) + w_2^2 \phi(\eta_2, \eta_2) \tag{8.50a}$$

where $w_1 = w_2 = 1.0$, $\xi_1 = \eta_1 = -0.57735\ldots$, and $\xi_2 = \eta_2 = +0.57735\ldots$. The Gauss points for the two-point rule used above are shown in Fig. 8.4. Alternatively, if we label the Gauss points as 1, 2, 3, and 4, then k_{ij} in Eq. 8.50a can also be written as

$$k_{ij} = \sum_{\mathrm{IP}=1}^{4} W_{\mathrm{IP}} \phi_{\mathrm{IP}} \tag{8.50b}$$

where ϕ_{IP} is the value of ϕ and W_{IP} is the weight factor at integration point (IP). We note that $W_{\mathrm{IP}} = (1)(1) = 1$. Computer implementation is sometimes easier using Eq. 8.50b. We may readily follow the implementation of the previous integration procedure in program QUAD provided at the end of this chapter.

The evaluation of three-dimensional integrals is similar. For triangles, however, the weights and Gauss points are different, as discussed later in this chapter.

Stress Calculations

Unlike the CST element (Chapters 6 and 7), the stresses $\boldsymbol{\sigma} = \mathbf{DBq}$ in the quadrilateral element are not constant within the element; they are functions of ξ and η and consequently vary within the element. In practice, the stresses are evaluated at the Gauss points, which are also the points used for numerical evaluation of $\mathbf{k}^e$, where they are found to be accurate. For a quadrilateral with 2×2 integration, this gives four sets of stress values. For generating less data, one may evaluate stresses at one point per element, say, at $\xi = 0$ and $\eta = 0$. The latter approach is used in program QUAD.

Example 8.2

Consider a rectangular element as shown in Fig. E8.2. Assume plane stress condition, $E = 30 \times 10^6$ psi, $\nu = 0.3$, and, $\mathbf{q} = [0, 0, 0.002, 0.003, 0.006, 0.0032, 0, 0]^{\mathrm{T}}$ in. Evaluate $\mathbf{J}$, $\mathbf{B}$, and $\boldsymbol{\sigma}$ at $\xi = 0$ and $\eta = 0$.

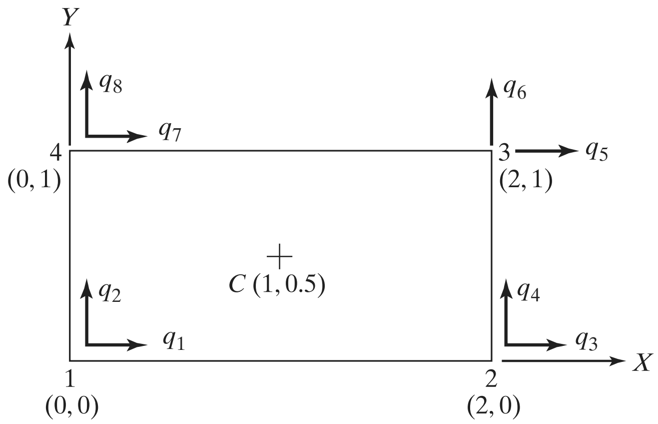


FIGURE E8.2

Solution Referring to Eq. 8.13a, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} 2(1-\eta) + 2(1+\eta) & (1+\eta) - (1+\eta) \\ -2(1+\xi) + 2(1+\xi) & (1+\xi) + (1-\xi) \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

For this rectangular element, we find that $\mathbf{J}$ is a constant matrix. From Eq. 8.21,

$$\mathbf{A} = \frac{1}{(1/2)}\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Evaluating $\mathbf{G}$ in Eq. 8.23 at $\xi = \eta = 0$, and using $\mathbf{B} = \mathbf{QG}$, we get

$$\mathbf{B}^0 = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

The stresses at $\xi = \eta = 0$ are now given by the product

$$\boldsymbol{\sigma}^0 = \mathbf{DB}^0\mathbf{q}$$

For the given data, we have

$$\mathbf{D} = \frac{30 \times 10^6}{(1 - 0.09)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.03 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Thus,

$$\boldsymbol{\sigma}^0 = [66920 \quad 23080 \quad 40960]^T \text{ psi}$$ ■

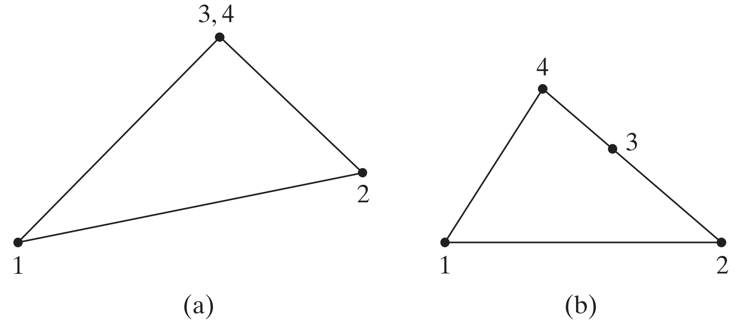


FIGURE 8.5 Degenerate four-node quadrilateral elements.

Comment on Degenerate Quadrilaterals. In some situations, we cannot avoid using degenerated quadrilaterals of the type shown in Fig. 8.5, where quadrilaterals degenerate into triangles. Numerical integration will permit the use of such elements, but the errors are higher than for regular elements. Standard codes normally permit the use of such elements.

8.4 HIGHER ORDER ELEMENTS

The concepts presented earlier for the four-node quadrilateral element can be readily extended to other, higher order, isoparametric elements. In the four-node quadrilateral element, the shape functions contain terms 1, ξ, η, and $\xi\eta$. In addition, the elements to be discussed here also contain terms such as $\xi^2\eta$ and $\xi\eta^2$, which generally provide greater accuracy. Only the shape functions **N** are given in Eq. 8.51. The generation of element stiffness follows the routine steps:

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{8.51a}$$

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q} \tag{8.51b}$$

$$\mathbf{k}^e = t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \; \det \mathbf{J} \, d\xi \, d\eta \tag{8.51c}$$

where $\mathbf{k}^e$ is evaluated using Gaussian quadrature.

Nine-Node Quadrilateral

The nine-node quadrilateral has been found to be very effective in finite element practice. The local node numbers for this element are shown in Fig. 8.6a. The square master element is shown in Fig. 8.6b. The shape functions are defined as follows:

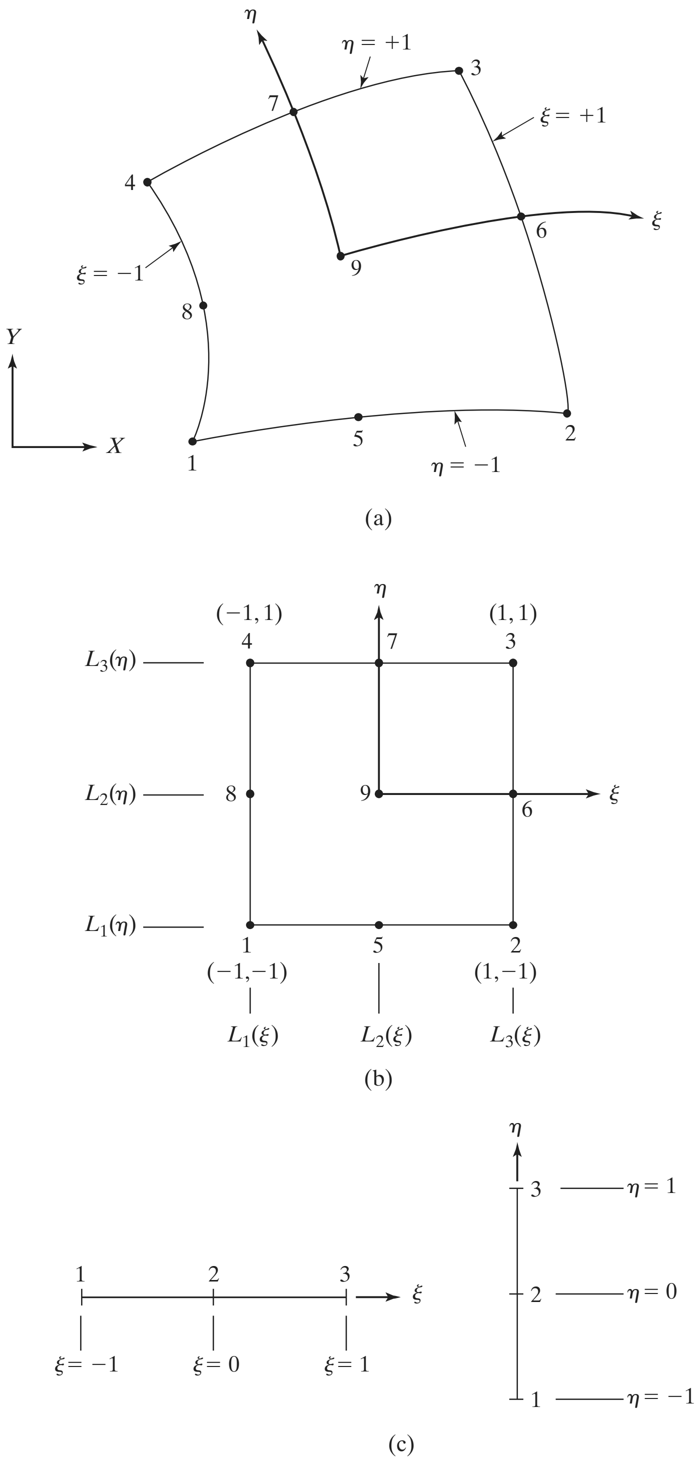


FIGURE 8.6 Nine-node quadrilateral in (a) x, y space and (b) ξ, η space. (c) Definition of general shape functions.

Consider, first, the ξ-axis alone as shown in Fig. 8.6c. The local node numbers 1, 2, and 3 on this axis correspond to locations $\xi = -1, 0$, and $+1$, respectively. At these nodes, we define the generic shape functions L_1, L_2, and L_3 as

$$\begin{aligned} L_i(\xi) &= 1 \quad \text{at node } i \\ &= 0 \quad \text{at other nodes} \end{aligned} \tag{8.52}$$

Now, consider L_1. Since $L_1 = 0$ at $\xi = 0$ and $\xi = +1$, we know that L_1 is of the form $L_1 = c\xi(1 - \xi)$. The constant c is obtained from $L_1 = 1$ at $\xi = -1$ as $c = -(1/2)$. Thus, $L_1 = (\xi) = -\xi(1 - \xi)/2$. L_2 and L_3 can be obtained by using similar arguments. We have

$$\begin{aligned} L_1(\xi) &= -\frac{\xi(1 - \xi)}{2} \\ L_2(\xi) &= (1 + \xi)(1 - \xi) \\ L_3(\xi) &= \frac{\xi(1 + \xi)}{2} \end{aligned} \tag{8.53a}$$

Similarly, generic shape functions can be defined along the η-axis (Fig. 8.6c) as

$$\begin{aligned} L_1(\eta) &= -\frac{\eta(1 - \eta)}{2} \\ L_2(\eta) &= (1 + \eta)(1 - \eta) \\ L_3(\eta) &= \frac{\eta(1 + \eta)}{2} \end{aligned} \tag{8.53b}$$

Referring back to the master element in Fig. 8.6b, we observe that every node has the coordinates $\xi = -1, 0$, or $+1$ and $\eta = -1, 0$, or $+1$. Thus, the *product rule* that follows yields the shape functions $N_1, N_2, \ldots, N_9$, as

$$\begin{array}{lll} N_1 = L_1(\xi)L_1(\eta) & N_5 = L_2(\xi)L_1(\eta) & N_2 = L_3(\xi)L_1(\eta) \\ N_8 = L_1(\xi)L_2(\eta) & N_9 = L_2(\xi)L_2(\eta) & N_6 = L_3(\xi)L_2(\eta) \\ N_4 = L_1(\xi)L_3(\eta) & N_7 = L_2(\xi)L_3(\eta) & N_3 = L_3(\xi)L_3(\eta) \end{array} \tag{8.54}$$

By the manner in which L_i are constructed, it can be readily verified that $N_i = 1$ at node i and $N_i = 0$ at other nodes, as desired.

As noted in the beginning of this section, the use of higher order terms in $\mathbf{N}$ leads to a higher order interpolation of the displacement field as given by $\mathbf{u} = \mathbf{Nq}$. In addition, since $x = \sum_i N_i x_i$ and $y = \sum_i N_i y_i$, it means that higher order terms can also be used to define geometry. Thus, the elements can have curved edges if desired. However, it is possible to define a *subparametric* element by using nine-node shape functions to interpolate displacement and using only four-node quadrilateral shape functions to define geometry.

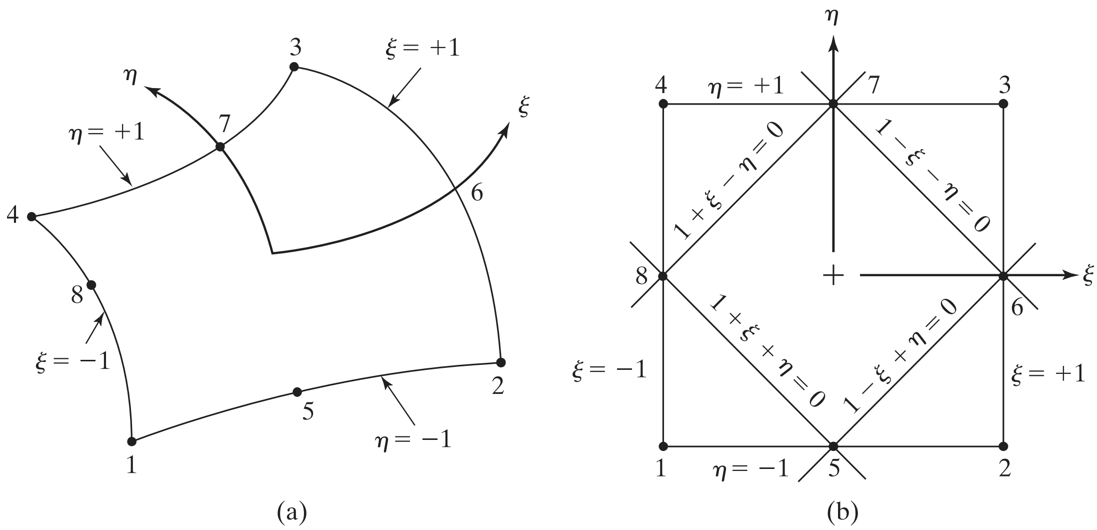


FIGURE 8.7 Eight-node quadrilateral in (a) x, y space and (b) ξ, η space.

Eight-Node Quadrilateral

This element belongs to the **serendipity** family of elements. The element consists of eight nodes (Fig. 8.7a), all of which are located on the boundary. Our task is to define shape functions N_i such that $N_i = 1$ at node i and 0 at all other nodes. In defining N_i, we refer to the master element shown in Fig. 8.7b. First, we define $N_1 - N_4$. For N_1, we note that $N_1 = 1$ at node 1 and 0 at other nodes. Thus, N_1 has to vanish along the lines $\xi = +1, \eta = +1$, and $\xi + \eta = -1$ (Fig. 8.7a). Consequently, N_1 is of the form

$$N_1 = c(1 - \xi)(1 - \eta)(1 + \xi + \eta) \tag{8.55}$$

At node 1, $N_1 = 1$ and $\xi = \eta = -1$. Thus, $c = -(1/4)$. We thus get

$$\begin{aligned}
N_1 &= -\frac{(1 - \xi)(1 - \eta)(1 + \xi + \eta)}{4} \\
N_2 &= -\frac{(1 + \xi)(1 - \eta)(1 - \xi + \eta)}{4} \\
N_3 &= -\frac{(1 + \xi)(1 + \eta)(1 - \xi - \eta)}{4} \\
N_4 &= -\frac{(1 - \xi)(1 + \eta)(1 + \xi - \eta)}{4}
\end{aligned} \tag{8.56}$$

Now, we define N_5, N_6, N_7, and N_8 at the midpoints. For N_5, we know that it vanishes along edges $\xi = +1, \eta = +1$, and $\xi = -1$. Consequently, it has to be of the form

$$N_5 = c(1 - \xi)(1 - \eta)(1 + \xi) \tag{8.57a}$$

$$= c(1 - \xi^2)(1 - \eta) \tag{8.57b}$$

The constant c in Eq. 8.57 is determined from the condition $N_5 = 1$ at node 5, or $N_5 = 1$ at $\xi = 0$, $\eta = -1$. Thus, $c = 1/2$ and

$$N_5 = \frac{(1 - \xi^2)(1 - \eta)}{2} \tag{8.57c}$$

We have

$$\begin{aligned} N_5 &= \frac{(1 - \xi^2)(1 - \eta)}{2} \\ N_6 &= \frac{(1 + \xi)(1 - \eta^2)}{2} \\ N_7 &= \frac{(1 - \xi^2)(1 + \eta)}{2} \\ N_8 &= \frac{(1 - \xi)(1 - \eta^2)}{2} \end{aligned} \tag{8.58}$$

Six-Node Triangle

The six-node triangle is shown in Figs. 8.8a and b. By referring to the master element in Fig. 8.8b, we can write the shape functions as

$$\begin{aligned} N_1 &= \xi(2\xi - 1) \qquad N_4 = 4\xi\eta \\ N_2 &= \eta(2\eta - 1) \qquad N_5 = 4\zeta\eta \\ N_3 &= \zeta(2\zeta - 1) \qquad N_6 = 4\xi\zeta \end{aligned} \tag{8.59}$$

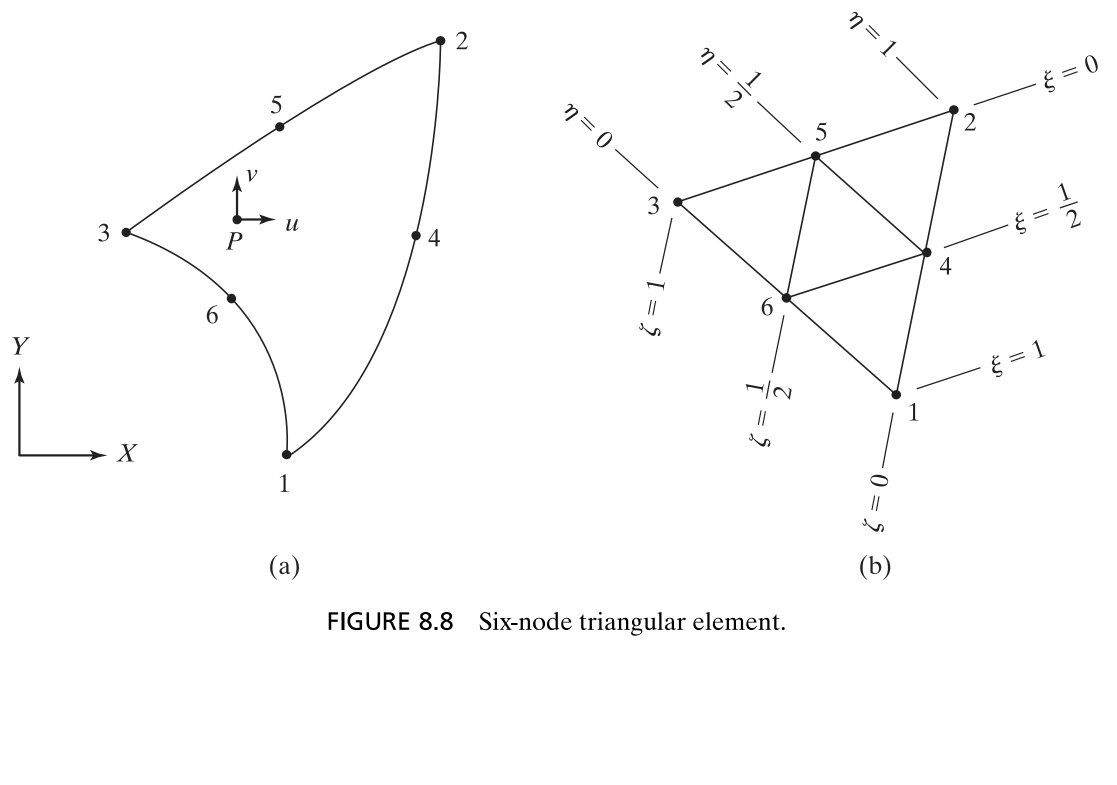


FIGURE 8.8 Six-node triangular element.

where $\zeta = 1 - \xi - \eta$. Because of terms ξ^2, η^2, etc., in the shape functions, this element is also called a *quadratic* triangle. The isoparametric representation is

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$
$$x = \sum N_i x_i \qquad y = \sum N_i y_i \tag{8.60}$$

The element stiffness, which has to be integrated numerically, is given by

$$\mathbf{k}^e = t_e \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J}\, d\xi\, d\eta \tag{8.61}$$

Integration on a Triangle—Symmetric Points

The Gauss points for a triangular region differ from the square region considered earlier. The simplest is the one-point rule at the centroid with weight $w_1 = 1/2$ and $\xi_1 = \eta_1 = \zeta_1 = 1/3$. Equation 8.61 then yields

$$\mathbf{k}^e \approx \tfrac{1}{2} t_e \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{D}}\ \overline{\mathbf{B}} \det \overline{\mathbf{J}} \tag{8.62}$$

where $\overline{\mathbf{B}}$ and $\overline{\mathbf{J}}$ are evaluated at the Gauss point. Other choices of weights and Gauss points are given in Table 8.2. The Gauss points given in Table 8.2 are arranged symmetrically within the triangle. Because of triangular symmetry, the Gauss points occur in groups or *multiplicity* of one, three, or six. For multiplicity of three, if ξ-, η-, and ζ-coordinates of a Gauss point are, for example, $[(2/3),(1/6),(1/6)]$, then the other two Gauss points are located at $[(1/6),(2/3),(1/6)]$, and $[(1/6),(1/6),(2/3)]$. Note that $\zeta = 1 - \xi - \eta$, as discussed in Chapter 6. For multiplicity of six, (ξ, η, ζ) in the table implies points given by all six permutations $(\xi, \eta, \zeta), (\xi, \zeta, \eta), (\eta, \xi, \zeta), (\eta, \zeta, \xi), (\zeta, \xi, \eta)$, and (ζ, η, ξ). When two coordinates are equal, it represents three points, and when all coordinates are equal it is one point (the centroid).

TABLE 8.2 Gauss Quadrature Formulas for a Triangle

$$\int_0^1 \int_0^{1-\xi} f(\xi,\eta)\, d\eta\, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i, \eta_i)$$

No. of Points, n	Weight, w_i	Multiplicity	ξ_i	η_i	ζ_i
One	1/2	1	1/3	1/3	1/3
Three	1/6	3	2/3	1/6	1/6
Three	1/6	3	1/2	1/2	0
Four	−(9/32)	1	1/3	1/3	1/3
	25/96	3	3/5	1/5	1/5
Six	1/12	6	0.6590276223	0.2319333685	0.1090390090

Integration on a Triangle—Degenerate Quadrilateral

We consider here the degenerate quadrilateral shown in Fig. 8.5a. A triangle 1–2–3 is defined as a quadrilateral 1–2–3–1, with two merging nodes. Four-point integration follows naturally. We provide here the proof of the general polynomial integration formula (Eq. 6.46) given by

$$\int_0^1 \int_0^{1-\xi} \xi^a \eta^b (1 - \xi - \eta)^c \, d\xi \, d\eta = \frac{a!b!c!}{(a + b + c + 2)}$$

Unit square mapped on to a master triangle is shown in Fig. 8.9. The shape functions are given by $N_1 = (1 - u)(1 - v)$, $N_2 = u(1 - v)$, $N_3 = uv$, and $N_4 = (1 - u)v$. The mapping is given by defining the triangle as a quadrilateral as stated above.

Node j	ξ_j	η_j
1	0	0
2	1	0
3	0	1
4	0	1

Now

$$\xi = \sum_{j=1}^{4} N_j \, \xi_j = u(1 - v)$$

$$\eta = \sum_{j=1}^{4} N_j \, \eta_j = v$$

$$1 - \xi - \eta = 1 - u(1 - v) - v = (1 - u)(1 - v)$$

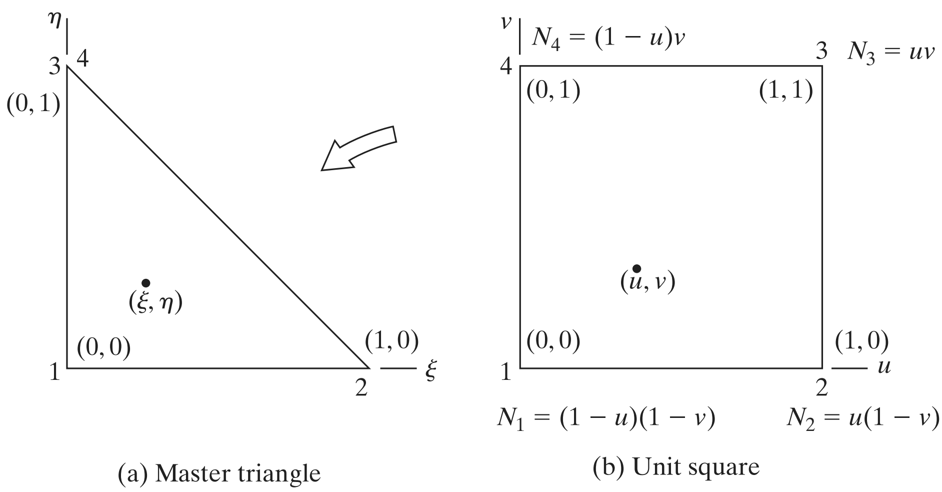


FIGURE 8.9 Triangle mapping.

The Jacobian of the transformation is

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \xi}{\partial u} & \dfrac{\partial \eta}{\partial u} \\ \dfrac{\partial \xi}{\partial v} & \dfrac{\partial \eta}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 - v & 0 \\ -u & 1 \end{bmatrix}$$

$$\det \mathbf{J} = 1 - v.$$

$$I = \int_0^1 \int_0^{1-\xi} \xi^a \eta^b (1 - \xi - \eta)^c \, d\xi \, d\eta = \int_0^1 \int_0^1 \xi^a \eta^b (1 - \xi - \eta)^c \det \mathbf{J} \, du \, dv$$

Introducing ξ, η, and det $\mathbf{J}$ expressions from above, the integral becomes

$$I = \int_0^1 \int_0^1 u^a (1 - v)^a \, v^b (1 - u)^c (1 - v)^c (1 - v) \, du \, dv$$

$$= \int_0^1 u^a (1 - u)^c \, du \int_0^1 v^b (1 - v)^{a+c+1} \, dv$$

These are complete beta integrals, giving

$$I = \frac{\Gamma(a + 1)\Gamma(c + 1)}{\Gamma(a + c + 2)} \frac{\Gamma(b + 1)\Gamma(a + c + 2)}{\Gamma(a + b + c + 3)}$$

Canceling the common terms, and noting that $\Gamma(x + 1) = x!$ for integer x, we get the right-hand side of Eq. 6.46. Gamma function and the factorial are defined in Chapter 6 following Eq. 6.46.

This derivation shows that triangle defined as a degenerate quadrilateral is valid for a general polynomial expression. It then ensures integration of other functions to within computational error.

Comment on Midside Node. In the higher order isoparametric elements discussed previously, we note the presence of midside nodes. The midside node should be as near as possible to the center of the side. The node should not be outside of $(1/4) < s/\ell < (3/4)$, as shown in Fig. 8.10. This condition ensures that det $\mathbf{J}$ does not attain a value of zero in the element.

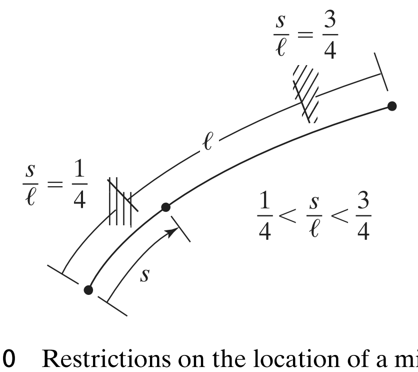


FIGURE 8.10 Restrictions on the location of a midside node.

Comment on Temperature Effect. Using the temperature strain defined in Eqs. 6.63 and 6.64 and following the derivation in Chapter 6, the nodal temperature load can be evaluated as

$$\boldsymbol{\theta}^e = t_e \int_A \int \mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0 \, dA = t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0 |\det \mathbf{J}| \, d\xi \, d\eta \tag{8.63}$$

This integral is performed using numerical integration.

8.5 FOUR-NODE QUADRILATERAL FOR AXISYMMETRIC PROBLEMS

The stiffness development for the four-node quadrilateral for axisymmetric problems follows steps similar to the quadrilateral element presented earlier. The x-, y-coordinates are replaced by r, z. The main difference occurs in the development of the **B** matrix, which relates the four strains to element nodal displacements. We partition the strain vector as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{\epsilon}} \\ \epsilon_\theta \end{bmatrix} \tag{8.64}$$

where $\overline{\boldsymbol{\epsilon}} = [\epsilon_r \epsilon_z \gamma_{rz}]^{\mathrm{T}}$.

Now in the relation $\boldsymbol{\epsilon} = \mathbf{Bq}$, we partition $\mathbf{B}$ as $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$ such that $\mathbf{B}_1$ is a 3×8 matrix relating $\overline{\boldsymbol{\epsilon}}$ and $\mathbf{q}$ by

$$\overline{\boldsymbol{\epsilon}} = \mathbf{B}_1\mathbf{q} \tag{8.65}$$

and $\mathbf{B}_2$ is a row vector 1×8 relating ϵ_θ and $\mathbf{q}$ by

$$\epsilon_\theta = \mathbf{B}_2\mathbf{q} \tag{8.66}$$

Noting that r, z replace x, y, it is clear that $\mathbf{B}_1$ is same as the 3×8 matrix given in Eq. 8.24 for the four-node quadrilateral. Since $\epsilon_\theta = u/r$ and $u = N_1q_1 + N_1q + N_3q_3 + N_4q_4$, $\mathbf{B}_2$ can be written as

$$\mathbf{B}_2 = \begin{bmatrix} \dfrac{N_1}{r} & 0 & \dfrac{N_2}{r} & 0 & \dfrac{N_3}{r} & 0 & \dfrac{N_4}{r} & 0 \end{bmatrix} \tag{8.67}$$

On introducing these changes, the element stiffness is then obtained by performing numerical integration on

$$\mathbf{k}^e = 2\pi \int_{-1}^{1} \int_{-1}^{1} r\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J} \, d\xi \, d\eta \tag{8.68}$$

The force terms (in Eqs. 8.31 and 8.32) are to be multiplied by the factor of 2π as in the axisymmetric triangle.

The axisymmetric quadrilateral element has been implemented in program AXIQUAD.

8.6 CONJUGATE GRADIENT IMPLEMENTATION OF THE QUADRILATERAL ELEMENT

The ideas of the conjugate gradient method have been presented in Chapter 2. The equations are reproduced here using the notation for displacements, force, and stiffness:

$$
\begin{aligned}
\mathbf{g}_0 &= \mathbf{K}\mathbf{Q}_0 - \mathbf{F}, \quad \mathbf{d}_0 = -\mathbf{g}_0 \\
\alpha_k &= \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{g}_k}{\mathbf{d}_k^{\mathrm{T}}\mathbf{K}\mathbf{d}_k} \\
\mathbf{Q}_{k+1} &= \mathbf{Q}_k + \alpha_k \mathbf{d}_k \\
\mathbf{g}_{k+1} &= \mathbf{g}_k + \alpha_k \mathbf{K}\mathbf{d}_k \\
\beta_k &= \frac{\mathbf{g}_{k+1}^{\mathrm{T}}\mathbf{g}_{k+1}}{\mathbf{g}_k^{\mathrm{T}}\mathbf{g}_k} \\
\mathbf{d}_{k+1} &= -\mathbf{g}_{k+1} + \beta_k \mathbf{d}_k
\end{aligned}
\tag{8.69}
$$

Here, $k = 0, 1, 2, \ldots$. The iterations are carried out until $\mathbf{g}_k^{\mathrm{T}}\mathbf{g}_k$ reaches a small value.

We state here the steps in its implementation in finite element analysis (FEA). The main difference in this implementation is that the stiffness of each element is first generated and stored in a three-dimensional array. The stiffness of an element can be recalled from this array without recalculating for the iterations carried out. We start with the initial displacements at $\mathbf{Q}_0 = \mathbf{0}$. In the evaluation of $\mathbf{g}_0$, the force modifications for the boundary conditions are implemented. The term $\mathbf{K}\mathbf{d}_k$ is evaluated directly using element stiffness values by using $\sum_e \mathbf{k}^e \mathbf{d}_k^e$. The conjugate gradient approach is implemented in QUADCG.

8.7 CONCLUDING REMARKS AND CONVERGENCE

The concept of a master element defined in ξ-, η-coordinates, the definition of shape functions for interpolating displacement and geometry, and use of numerical integration are all key ingredients of the isoparametric formulation. A wide variety of elements can be formulated in a unified manner. Though only stress analysis has been considered in this chapter, the elements can be applied to nonstructural problems quite readily.

As discussed earlier, smaller-sized elements are required in regions of stress concentration, where derivatives of **u** with respect to x and/or y are high. In practice, the "stress jumps" between adjacent elements is a measure of correctness of the solution. A question that naturally arises is whether decreasing the element size h to zero gives the mathematically exact solution. The *patch test* presented in Chapter 6 is also applicable to higher order elements. Some basic criteria for h-convergence, which are implied in the patch test, are summarized below.

(1) *Admissibility*: Boundary conditions must prevent any rigid body motion of the structure; in two dimensions, x-, y-translations and in-plane rotation must not be allowed. Further, since the energy functional involves first derivatives, continuity of displacement (C^0 continuity) must exist. Specifically, the displacements must be continuous across the element boundaries. In a four-node quadrilateral element, the shape functions are linear on any edge. For example, along edge 1–2 in Fig. 8.2, we have $\eta = -1$ which makes N_i in Eq. 8.5 all linear in ξ, making u and v from Eq. 8.7 linear in ξ too. Thus, the displacement u along edge 1–2 will be in linear form as $u = a + b\xi$, which involves two coefficients that are uniquely determined from q_1 and q_3 and satisfies the interelement boundary compatibility criterion. Generalizing, a quadratic shape function along the edge will be continuous provided there are three nodes on the edge. On the other hand, a quadratic function along an edge will not be compatible if there are four nodes on the common boundary.

(2) *Completeness*: The shape functions must satisfy completeness requirements in the coordinate system used. Specifically, $u = a + bx$ is complete for one-dimensional problems. In two dimensions, the linear polynomial $u = a + bx + cy$ and the quadratic $u = a + bx + cy + dx^2 + exy + fy^2$ are complete. However, $u = a + bx + cy + exy$ is incomplete in a quadratic sense.

The reader should note that the above discussion pertaining to mesh discretization error and its reduction to zero, in theory, is independent of modeling errors which have to do with choice of boundary conditions and loads. For example, a structural support may be chosen as pinned, fixed, or attached to a spring. Experiments may be necessary to make a decision, and lower/upper bounds can be generated to guide the engineering analysis. In industrial projects, making entries in a laboratory notebook regarding modeling and other aspects is very important for scientific and legal reasons.

Example 8.3

The problem in Example 6.9 (Fig. E6.9) is now solved using four-node quadrilateral elements using program QUAD. The loads, boundary conditions, and node locations are the same as in Fig. E6.9. The only difference is the modeling with 24 quadrilateral elements, as against 48 CST elements in Fig. E6.9. Again, MESHGEN has been used to create the mesh (Fig. E8.3a), and a text editor has been used to define the loads, boundary conditions, and material properties.

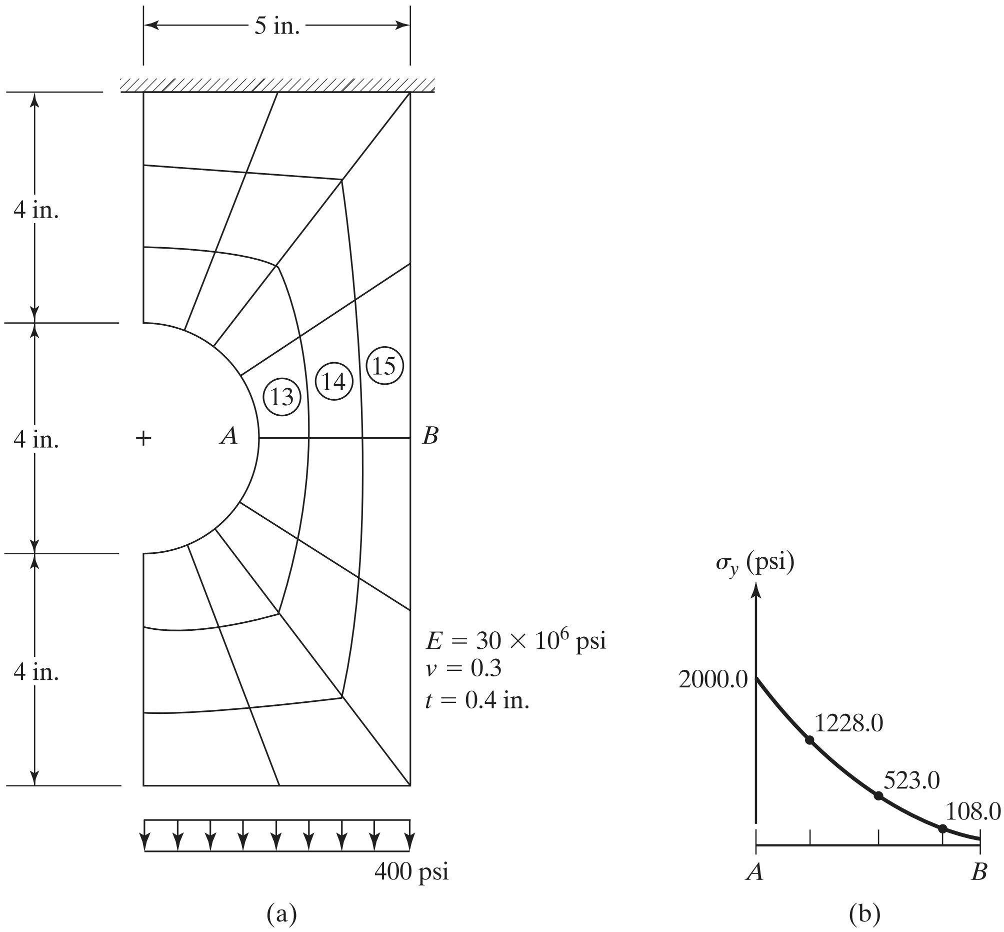


FIGURE E8.3

The stresses output by program QUAD correspond to the $(0, 0)$ location in the natural coordinate system (master element). Using this fact, we *extrapolate* the y-stresses in elements 13, 14, and 15 to obtain the maximum y-stress near the semicircular edge of the plate, as shown in Fig. E8.3b. ■

REFERENCES FOR CONVERGENCE

1. Irons, B. M., "Engineering application of numerical integration in stiffness method." *AIAA Journal* 14: 2035–2037 (1966).
2. Oden, J. T., "A general theory of finite elements." *International Journal for Numerical Methods in Engineering* 1: 205–221 and 247–259 (1969).
3. Strang, W.G. and G. Fix, *An analysis of the finite element method,* Prentice Hall, Englewood Cliffs, NJ (1973).
4. Tong, P. and T. H. H. Pian, "On the convergence of a finite element method in solving linear elastic problems." *International Journal of Solids and Structures* 3: 865–879 (1967).

Input Data/Output

```
INPUT TO QUAD
<< 2D STRESS ANALYSIS USING QUAD >>
EXAMPLE 8.4
NN     NE     NM      NDIM    NEN     NDN
9      4      1       2       4       2
ND     NL     NMPC
6      1      0
Node#  X      Y
1      0      0
2      0      15
3      0      30
4      30     0
5      30     15
6      30     30
7      60     0
8      60     15
9      60     30
Elem#  N1     N2      N3      N4      Material#     Thickness     TempRise
1      1      4       5       2       1             10            0
2      2      5       6       3       1             10            0
3      4      7       8       5       1             10            0
4      5      8       9       6       1             10            0
DOF#   Displ.
1      0
2      0
3      0
4      0
5      0
6      0
DOF#   Load
18     -10000
MAT#   E      Nu      Alpha
1      7.00E+04       0.33    1.20E-05
B1     i      B2      j       B3      (Multi-point constr. B1*Qi+B2*Qj=B3)
```

```
OUTPUT FROM QUAD
Program Quad - Plane Stress Analysis
EXAMPLE 8.4
Node# X-Displ        Y-Displ
1     -8.89837E-07   -2.83351E-07
2     1.77363E-08    1.50706E-07
3     8.72101E-07    -3.07839E-07
4     -0.088095167   -0.131050922
5     -0.001282636   -0.123052776
6     0.087963341    -0.126964356
7     -0.116924591   -0.365192248
8     0.000352218    -0.370143531
9     0.125124584    -0.386856887
```

```
Elem# Iteg1 Iteg2 Iteg3 Iteg4 <== vonMises Stresses
1     213.3629   160.2804    53.7790    141.1354
2     136.9611    48.5291   159.9454    208.3194
3     93.7355     58.8159    38.02357    91.4752
4     92.3071     69.3212    94.1831    120.1013
```

INPUT TO AXIQUAD

```
<< AXISYMMETRIC STRESS ANALYSIS USING AXIQUAD ELEMENT >>
EXAMPLE 6.4
NN     NE     NM      NDIM    NEN     NDN
6      2      1       2       4       2
ND     NL     NMPC
3      6      0
Node# X      Y
1      3      0
2      3      0.5
3      7.5    0
4      7.5    0.5
5      12     0
6      12     0.5
Elem# N1     N2      N3      N4      Material#      TempRise
1      1      3       4       2       1       0
2      3      5       6       4       1       0
DOF#  Displ.
2      0
6      0
10     0
DOF#  Load
1      3449
3      9580
5      23380
7      38711
9      32580
11     18780
MAT#  E              Nu      Alpha
1      3.00E+07      0.3     1.20E-05
B1     i      B2      j       B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

OUTPUT FROM AXIQUAD

```
Program AxiQuad - Stress Analysis
EXAMPLE 6.4
Node# R-Displ        Z-Displ
1      0.000829703   9.02758E-12
2      0.000828915   -5.42961E-05
3      0.000885462   -1.43252E-11
4      0.000887988   -2.52897E-05
5      0.000903563   5.29761E-12
6      0.000898857   -1.61269E-05
Elem#  Iteg1         Iteg2      Iteg3      Iteg4 <== vonMises Stresses
1       6271.290     3959.454   3964.174   6270.092
2       3094.231     2391.461   2390.800   3100.338
```

PROBLEMS

8.1. Figure P8.1 shows a four-node quadrilateral. The (x, y) coordinates of each node are given in the figure. The element displacement vector $\mathbf{q}$ is given as

$$\mathbf{q} = [1, \quad 0, \quad 0.15, \quad 0, \quad 0.2, \quad 0.35, \quad 0, \quad 0.08]^T$$

Find the following:

(a) The x-, y-coordinates of a point P whose location in the master element is given by $\xi = 1$ and $\eta = 1$

(b) The u, v displacements of the point P.

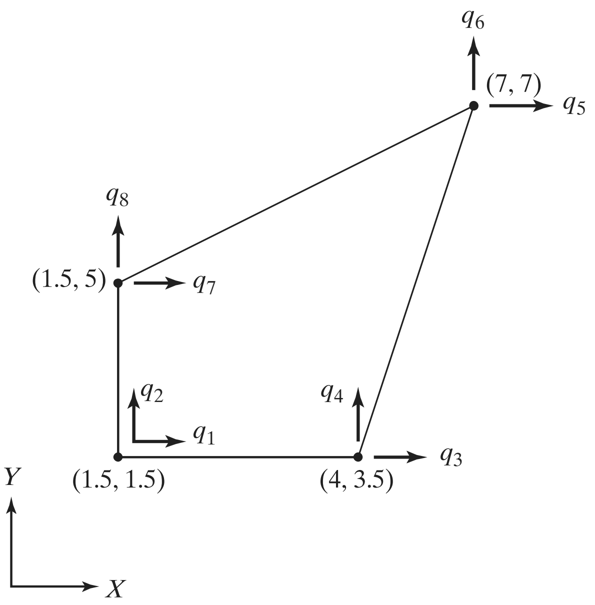


FIGURE P8.1

8.2. Using a 2×2 rule, evaluate the integral

$$\iint_A (x^2y + y^2)\, dx\, dy$$

by Gaussian quadrature, where A denotes the region shown in Fig. P8.1.

8.3. State whether the following statements are true or false:

(a) The shape functions are linear along an edge of a four-node quadrilateral element.

(b) For isoparametric elements, such as four-, eight-, and nine-node quadrilaterals, the point $\xi = 0, \eta = 0$ in the master element corresponds to the centroid of the element in x- and y-coordinates.

(c) The maximum stresses within an element occur at the Gauss points.

(d) The integral of a cubic polynomial can be performed exactly using two-point Gauss quadrature.

8.4. Solve Problem P6.17 with four-node quadrilaterals. Use program QUAD. (*Note*: Example input data set is given for 2×2 mesh division.)

8.5. A half-symmetry model of a culvert is shown in Fig. P8.5. The pavement load is a uniformly distributed load of 5000 N/m^2. Using program MESHGEN (discussed in Chapter 12), develop a finite element mesh with four-node quadrilateral elements. Using program QUAD determine the location and magnitude of maximum principal stress. First, try a mesh with about six elements and then compare results using about 18 elements.

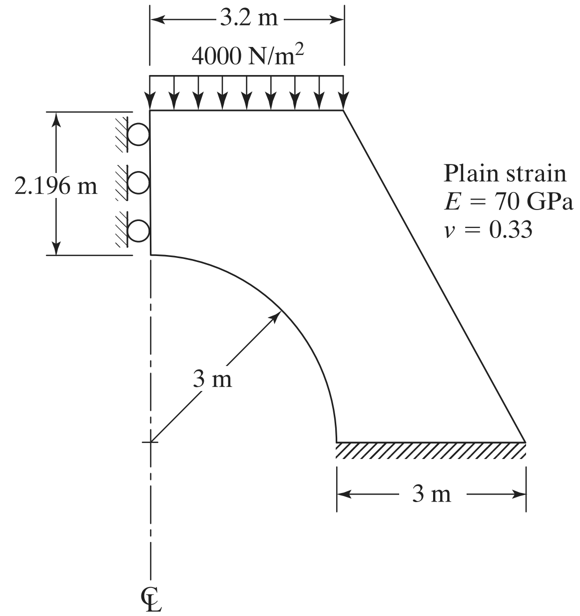


FIGURE P8.5

8.6. Solve Problem P6.18 using four-node quadrilateral elements (program QUAD). Compare the results with the solution obtained with CST elements. Use comparable-size meshes.

8.7. Solve Problem P6.19 using four-node quadrilaterals (program QUAD).

8.8. Solve Problem P6.22 using four-node quadrilaterals (program QUAD).

8.9. Program AXIQUAD is for axisymmetric stress analysis with four-node quadrilateral elements. Use that program to solve Example 7.2. Compare the results. [*Hint:* The first three rows of the **B** matrix are the same as for the plane stress problem in Eq. 8.25, and the last row can be obtained from $\epsilon_\theta = u/r$.]

8.10. This problem focuses on a concept used in program MESHGEN discussed in Chapter 12. An eight-node region is shown in Fig. P8.10a. The corresponding master element or *block* is shown in Fig. P8.10b. The block is divided into a grid of $3 \times 3 = 9$ smaller blocks of equal size, as indicated by dotted lines. Determine the corresponding x- and y-coordinates of all the 16 nodal points, and plot the nine subregions in Fig. P8.10a. Use the shape functions given in Eqs. 8.56 and 8.58.

8.11. Develop a computer program for the eight-node quadrilateral. Analyze the γ_{xy} cantilever beam shown in Fig. P8.11 with three finite elements. Compare results of x stress and centerline deflections with

(a) The six-element CST model

(b) Elementary beam theory.

8.12. Solve Problem 7.16 using axisymmetric quadrilateral elements (program AXIQUAD).

8.13. Answer the questions in brief:

(a) What is meant by the term "higher order elements?" ("Higher" than what?)

(b) How many independent material properties exist for an isotropic material?

(c) If a two-dimensional element for plane stress/plane strain is shaped as a six-node hexagon, what are the dimensions of **k** and **B**?

(d) Comment on what is meant by convergence in FEA. Search internet sources to learn h-convergence and p-convergence.

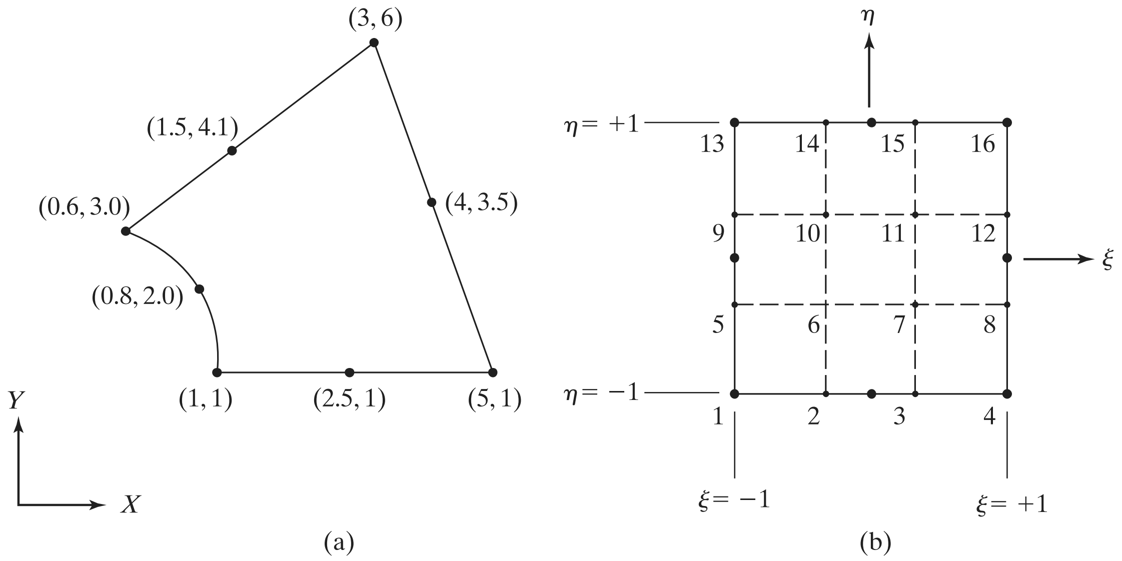


FIGURE P8.10

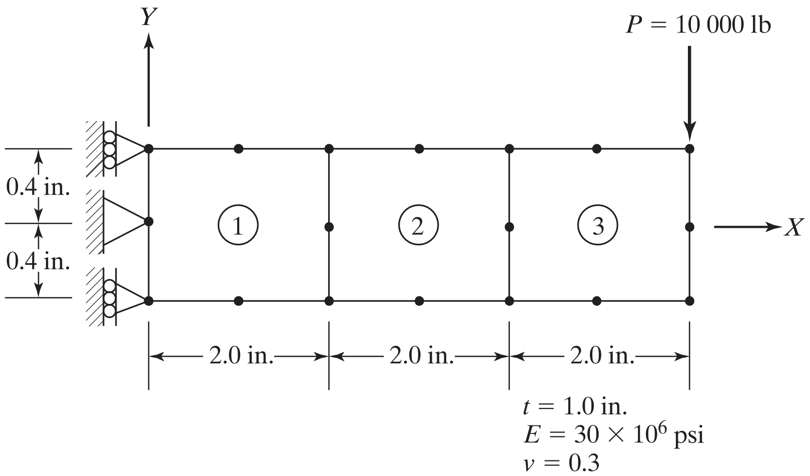


FIGURE P8.11

8.14. Indicate what is wrong with the model shown in Fig. P8.14 consisting of four quadrilateral elements. What do you suggest to set the model right?

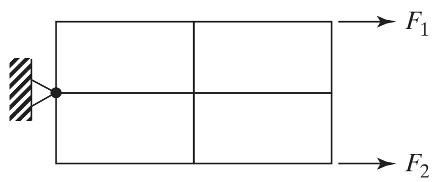


FIGURE P8.14

8.15. Answer the questions concisely and with justification:

(a) Are stresses constant within a four-node quadrilateral element?

(b) Are shape functions linear on the edge of a four-node quadrilateral element?

(c) Why do you need numerical integration?

(d) How many integration points are commonly used in computing matrices for the four-node quadrilateral element?

(e) A node k on a structure is connected by a *rigid link* to a fixed node j, as shown in Fig. P8.15. Assuming small deformations, write the corresponding boundary conditions (constraint equation) in the form of $\beta_1 Q_{kx} + \beta_2 Q_{ky} = \beta_0$. (*Hint*: Refer to discussion in Chapter 3.)

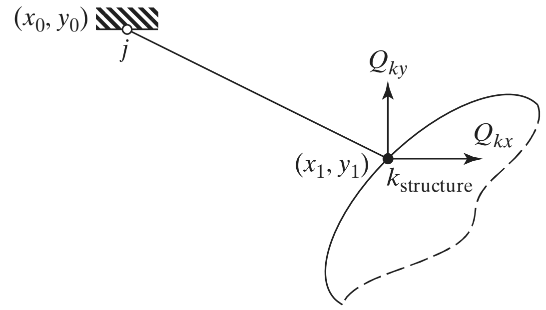


FIGURE P8.15

8.16. A four-node element shown in Fig. P8.16 undergoes deformation as shown in the figure. Determine expressions for ϵ_x, ϵ_y, and γ_{xy} in terms of x and y.

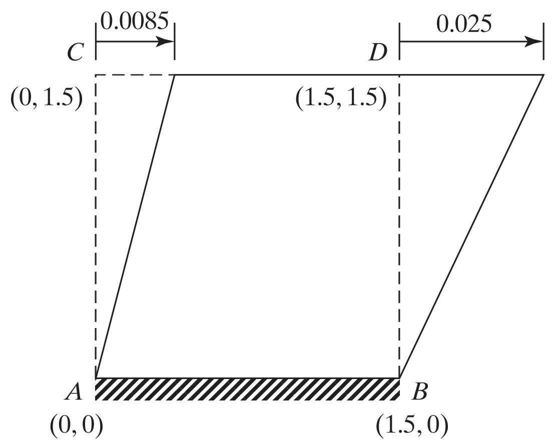


FIGURE P8.16

8.17. The potential energy of an element in terms of local displacements q_1 and q_2 is given by the expression

$$\Pi_e = 4q_1^2 + 8q_2^2 - 5q_1q_2 + 7q_1$$

Write down the expressions for element stiffness matrix **k** and element force **f**.

8.18. Answer the following questions concisely and with justification:

(a) What are the independent material constants for an *orthotropic* material in two dimensions?

(b) Comment on "modeling error" and "mesh-dependent error."

(c) The main advantage of an eight-node quadrilateral element over a four-node quadrilateral for plane elasticity problem is that the sides of the element can be curved. Comment.

(d) An element is subjected to hydrostatic stress, $\sigma_x = \sigma_y = \sigma_z = 0$, and all shear stresses are zero. What is the von Mises stress in the element?

(e) Does the structure shown in Fig. P8.18 exhibit rigid body motion?

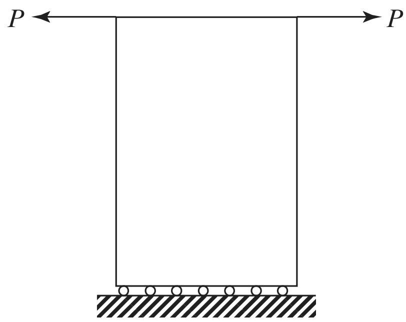


FIGURE P8.18

8.19. For the plane stress problem shown in Fig. P8.19, a coarse finite element mesh is shown. Give all the boundary conditions and loads for the mesh (similar to the BC and loads section of the data set).

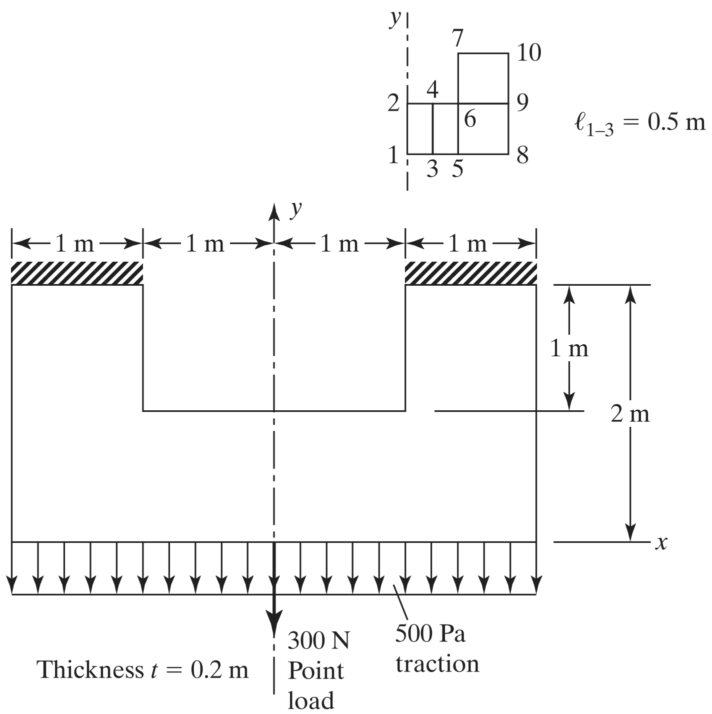


FIGURE P8.19

8.20. Which mesh, A or B, shown in Fig. P8.20, will give a better solution to the beam problem? Justify.

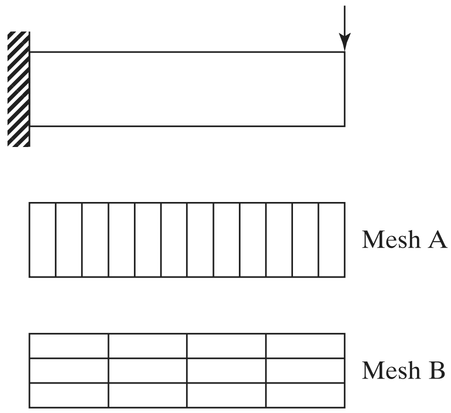


FIGURE P8.20

8.21. For the plane stress problem shown in Fig. P8.21, a simple model consisting of two elements 1-4-3-2 and 6-5-4-1 is shown. Loading consists of point load P along the axis of symmetry and symmetric-distributed load on the right edge. Give all the boundary conditions and equivalent point loads for the half-symmetry model using the dataset format for the computer program QUAD.

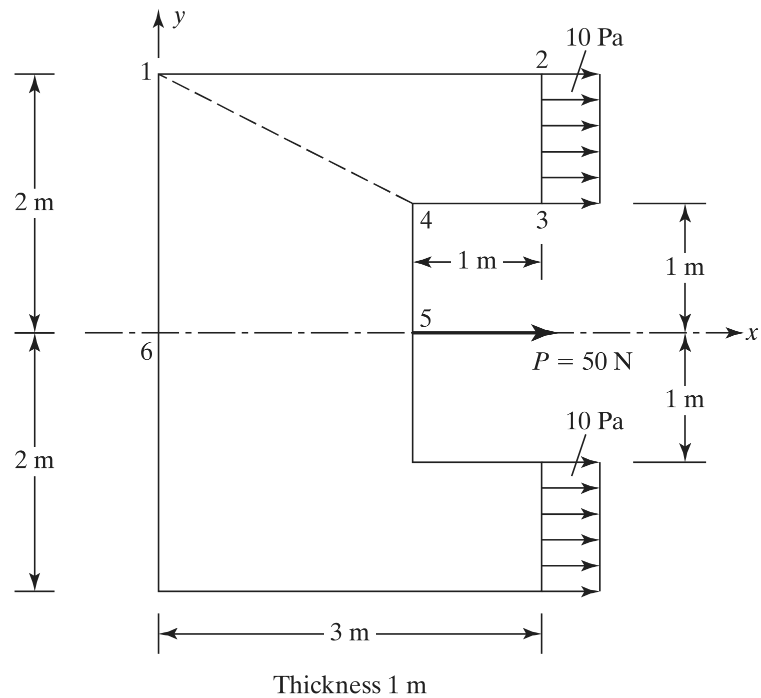


FIGURE P8.21

8.22. For the plane stress problem shown in Fig. P8.22, divide the symmetric part into four rectangular elements. Number the nodes and elements, and give all the boundary conditions and component loads in the dataset format for the computer program QUAD.

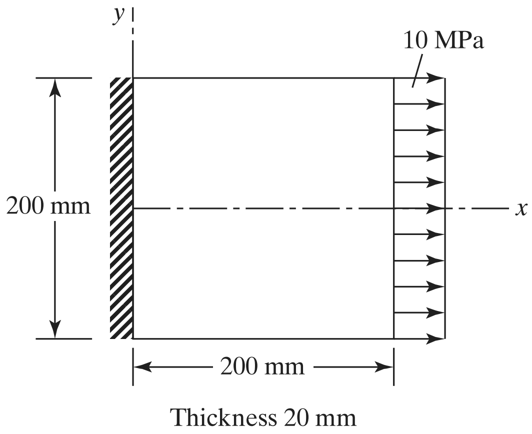


FIGURE P8.22

8.23. Consider a quadrilateral element whose nodes 1, 2, 3, and 4 in the counter clockwise (CCW) order are at coordinates (0, 0), (3, 0), (3, 1), and (0, 1), respectively. Thickness of the element t_e is 1. Evaluate the integral

$$I = \int_{\text{node 1}}^{\text{node 2}} N_1\, dS + \int_{\text{node 2}}^{\text{node 3}} N_1\, dS$$

using one-point Gauss–Legendre quadrature (i.e., one-point numerical integration). Note that $dS = t_e\, d\ell$.

8.24. Redo Example 8.3 with body force $\mathbf{f} = [f_x, f_y]^{\mathrm{T}}$ lb/in.^{3}.

8.25. Consider a four-node quadrilateral element whose edge 1–2 is loaded by triangularly varying traction as shown in Fig. P8.25. Derive the equivalent nodal forces acting at the nodes 1 and 2 using

(a) One-point numerical integration

(b) Two-point numerical integration.

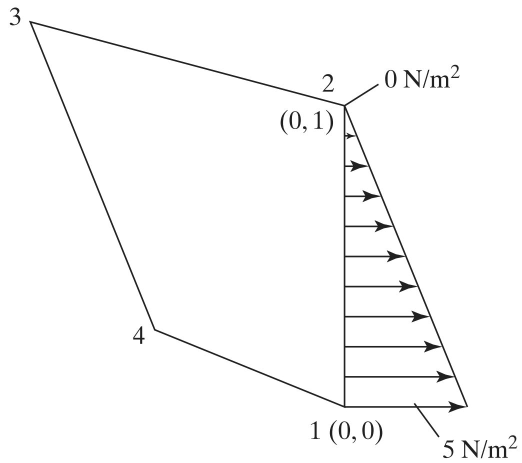


FIGURE P8.25

8.26. Evaluate $\int(\xi^2 + \xi\eta)\,dA$ over the CST (three-node triangular element) with node 1 at (0, 0), node 2 at (1, 0), node 3 at (0, 1) using one-point and three-point integration. Compare your values with the exact integration using the triangle integration formula for polynomials.

8.27. Perform the load patch test for the plane stress problem shown in Fig. P8.27.

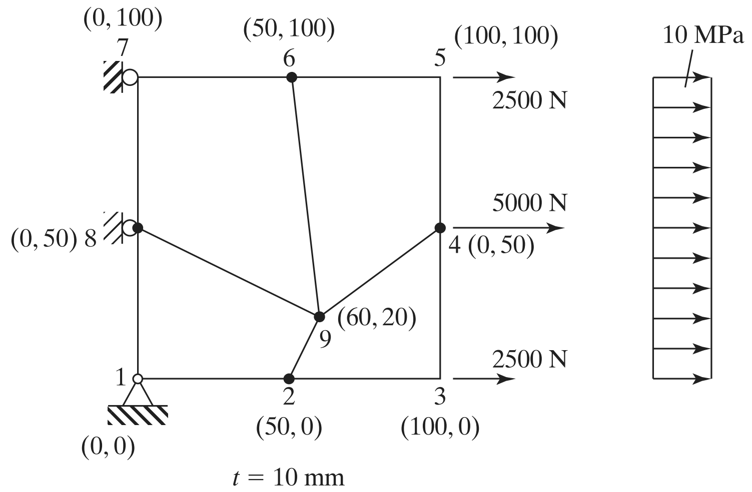


FIGURE P8.27

8.28. Perform the displacement patch test for the problem defined in Fig. 6.12a of Chapter 6 using degenerate quadrilateral elements 5-1-2-5, 5-2-3-5, 5-3-4-5, and 5-4-1-5.

8.29. Perform the load patch test for the problem defined in Fig. 6.12b of Chapter 6 using degenerate quadrilateral elements as defined in Problem 8.28.

8.30. The element in Fig. E8.2 is subjected to a body force $\mathbf{f} = [f_x \quad f_y]^{\mathrm{T}} = [x \quad 0]^{\mathrm{T}}$. Using thickness of the element as t, determine the equivalent point loads at the four nodes in the x directions using 2×2 Gauss–Legendre integration. Implement this in program QUAD or write a dedicated code in a computer language of your choice.

8.31. Figure P8.31 shows a traction load $T_x = 1\ \mathrm{N/m^2}$, $T_y = 0$ applied on the edge of an eight-node quadrilateral. Determine the equivalent nodal forces. Use thickness of the element of 1 m. [*Hint:* On the edge 2-6-3, $\xi = 1$ ($d\xi = 0$), and $dS = \sqrt{(dx)^2 + (dy)^2} = \left[\sqrt{(dx/d\eta)^2 + (dx/d\eta)^2}\right]d\eta$. Use two-point numerical integration.]

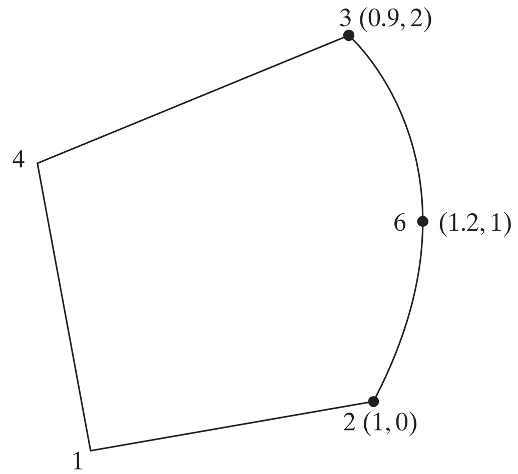


FIGURE P8.31

Program Listings

```
MAIN PROGRAM
'*************** PROGRAM QUAD **************
'*     2-D STRESS ANALYSIS USING 4-NODE    *
'* QUADRILATERAL ELEMENTS WITH TEMPERATURE *
'*    T.R.Chandrupatla and A.D.Belegundu   *
'*******************************************
Private Sub CommandButton1_Click()
     Call InputData
     Call Bandwidth
     Call Stiffness
     Call ModifyForBC
     Call BandSolver
     Call StressCalc
     Call ReactionCalc
     Call Output
End Sub
```

```
GLOBAL STIFFNESS
Private Sub Stiffness()
     ReDim S(NQ, NBW)
     '----- Global Stiffness Matrix -----
     For N = 1 To NE
     Call IntegPoints
     Call DMatrix(N)
     Call ElemStiffness(N)
     '----- <<< Stiffness Assembly same as other programs >>>
End Sub
```

```
INTEGRATION POINTS
Private Sub IntegPoints()
'------- Integration Points XNI() --------
     C = 0.57735026919
     XNI(1, 1) = -C: XNI(1, 2) = -C
     XNI(2, 1) = C: XNI(2, 2) = -C
     XNI(3, 1) = C: XNI(3, 2) = C
     XNI(4, 1) = -C: XNI(4, 2) = C
End Sub
```

```
D MATRIX
Private Sub DMatrix(N)
'-----      D() Matrix     -----
     '--- Material Properties
     MATN = MAT(N)
     E = PM(MATN, 1): PNU = PM(MATN, 2)
     AL = PM(MATN, 3)
     '--- D() Matrix
```

```
      If LC = 1 Then
         '--- Plane Stress
         C1 = E / (1 - PNU ^ 2): C2 = C1 * PNU
      Else
         '--- Plane Strain
         C = E / ((1 + PNU) * (1 - 2 * PNU))
         C1 = C * (1 - PNU): C2 = C * PNU
      End If
      C3 = 0.5 * E / (1 + PNU)
      D(1, 1) = C1: D(1, 2) = C2: D(1, 3) = 0
      D(2, 1) = C2: D(2, 2) = C1: D(2, 3) = 0
      D(3, 1) = 0: D(3, 2) = 0: D(3, 3) = C3
End Sub
```

ELEMENT STIFFNESS

```
Private Sub ElemStiffness(N)
'--------   Element Stiffness and Temperature Load       -----
      For I = 1 To 8
      For J = 1 To 8: SE(I, J) = 0: Next J: TL(I) = 0: Next I
      DTE = DT(N)
      '--- Weight Factor is ONE
      '--- Loop on Integration Points
      For IP = 1 To 4
         '--- Get DB Matrix at Integration Point IP
         Call DbMat(N, 1, IP)
         '--- Element Stiffness Matrix      SE
         For I = 1 To 8
            For J = 1 To 8
               C = 0
               For K = 1 To 3
                  C = C + B(K, I) * DB(K, J) * DJ * TH(N)
                  Next K
                  SE(I, J) = SE(I, J) + C
            Next J
         Next I
         '--- Determine Temperature Load TL
         AL = PM(MAT(N), 3)
         C = AL * DTE: If LC = 2 Then C = (1 + PNU) * C
         For I = 1 To 8
            TL(I) = TL(I) + TH(N) * DJ * C * (DB(1, I) + DB(2, I))
         Next I
      Next IP
End Sub
```

STRESS CALCULATIONS

```
Private Sub StressCalc()
      ReDim vonMisesStress(NE, 4), maxShearStress(NE, 4)
      '-----  Stress Calculations
      For N = 1 To NE
         Call DMatrix(N)
         For IP = 1 To 4
           '--- Get DB Matrix with Stress calculation
```

```
            '--- Von Mises Stress at Integration Point

            Call DbMat(N, 2, IP)
            C = 0: If LC = 2 Then C = PNU * (STR(1) + STR(2))
            C1 = (STR(1) - STR(2))^2 + (STR(2) - C)^2 + (C - STR(1))^2
            SV = Sqr(0.5 * C1 + 3 * STR(3) ^ 2)
            '--- Maximum Shear Stress R
            R = Sqr(0.25 * (STR(1) - STR(2))^2 + (STR(3))^2)
            maxShearStress(N, IP) = R
            vonMisesStress(N, IP) = SV
          Next IP
       Next N
End Sub
Private Sub DbMat(N, ISTR, IP)
```

```
'-------     DB()    MATRIX  ------
     XI = XNI(IP, 1): ETA = XNI(IP, 2)
     '--- Nodal Coordinates
     THICK = TH(N)
     N1 = NOC(N, 1): N2 = NOC(N, 2)
     N3 = NOC(N, 3): N4 = NOC(N, 4)
     X1 = X(N1, 1): Y1 = X(N1, 2)
     X2 = X(N2, 1): Y2 = X(N2, 2)
     X3 = X(N3, 1): Y3 = X(N3, 2)
     X4 = X(N4, 1): Y4 = X(N4, 2)
     '--- Formation of Jacobian    TJ
     TJ11 = ((1 - ETA) * (X2 - X1) + (1 + ETA) * (X3 - X4)) / 4
     TJ12 = ((1 - ETA) * (Y2 - Y1) + (1 + ETA) * (Y3 - Y4)) / 4
     TJ21 = ((1 - XI) * (X4 - X1) + (1 + XI) * (X3 - X2)) / 4
     TJ22 = ((1 - XI) * (Y4 - Y1) + (1 + XI) * (Y3 - Y2)) / 4
     '--- Determinant of the JACOBIAN
     DJ = TJ11 * TJ22 - TJ12 * TJ21
     '--- A(3,4) Matrix relates Strains to
     '--- Local Derivatives of u
     A(1, 1) = TJ22 / DJ: A(2, 1) = 0: A(3, 1) = -TJ21 / DJ
     A(1, 2) = -TJ12 / DJ: A(2, 2) = 0: A(3, 2) = TJ11 / DJ
     A(1, 3) = 0: A(2, 3) = -TJ21 / DJ: A(3, 3) = TJ22 / DJ
     A(1, 4) = 0: A(2, 4) = TJ11 / DJ: A(3, 4) = -TJ12 / DJ
     '--- G(4,8) Matrix relates Local Derivatives of u
     '--- to Local Nodal Displacements q(8)
     For I = 1 To 4: For J = 1 To 8
     G(I, J) = 0: Next J: Next I
     G(1, 1) = -(1 - ETA) / 4: G(2, 1) = -(1 - XI) / 4
     G(3, 2) = -(1 - ETA) / 4: G(4, 2) = -(1 - XI) / 4
     G(1, 3) = (1 - ETA) / 4: G(2, 3) = -(1 + XI) / 4
     G(3, 4) = (1 - ETA) / 4: G(4, 4) = -(1 + XI) / 4
     G(1, 5) = (1 + ETA) / 4: G(2, 5) = (1 + XI) / 4
     G(3, 6) = (1 + ETA) / 4: G(4, 6) = (1 + XI) / 4
     G(1, 7) = -(1 + ETA) / 4: G(2, 7) = (1 - XI) / 4
```

```
      G(3, 8) = -(1 + ETA) / 4: G(4, 8) = (1 - XI) / 4
      '--- B(3,8) Matrix Relates Strains to q
      For I = 1 To 3
         For J = 1 To 8
            C = 0
            For K = 1 To 4
               C = C + A(I, K) * G(K, J)
            Next K
            B(I, J) = C
         Next J
      Next I
      '--- DB(3,8) Matrix relates Stresses to q(8)
      For I = 1 To 3
         For J = 1 To 8
            C = 0
            For K = 1 To 3
               C = C + D(I, K) * B(K, J)
            Next K:
            DB(I, J) = C
         Next J
      Next I
      If ISTR = 2 Then
            '--- Stress Evaluation
            For I = 1 To NEN
               IIN = NDN * (NOC(N, I) - 1)
               II = NDN * (I - 1)
               For J = 1 To NDN
                  Q(II + J) = F(IIN + J)
               Next J
            Next I
            AL = PM(MAT(N), 3)
            C1 = AL * DT(N): If LC = 2 Then C1 = C1 * (1 + PNU)
            For I = 1 To 3
               C = 0
               For K = 1 To 8
                  C = C + DB(I, K) * Q(K)
               Next K
               STR(I) = C - C1 * (D(I, 1) + D(I, 2))
            Next I
      End If
  End Sub
```

CHAPTER 9

Three-Dimensional Problems in Stress Analysis

9.1 INTRODUCTION

Most engineering problems are three dimensional. So far, we have studied the possibilities of finite element analysis of simplified models, where rod elements, constant-strain triangles, axisymmetric elements, beams, and so on give reasonable results. In this chapter, we deal with the formulation of three-dimensional stress-analysis problems. The four-node tetrahedral element is discussed in detail. Problem modeling and brick elements are also discussed. In addition, frontal solution method is introduced.

We recall from the formulation given in Chapter 1 that

$$\mathbf{u} = [u \quad v \quad w]^{\mathrm{T}} \tag{9.1}$$

where u, v, and w are displacements in the x, y, and z directions, respectively. The stresses and strains are given by

$$\boldsymbol{\sigma} = [\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{xz} \quad \tau_{xy}]^{\mathrm{T}} \tag{9.2}$$

$$\boldsymbol{\epsilon} = [\epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}]^{\mathrm{T}} \tag{9.3}$$

The stress–strain relations are given by

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{9.4}$$

where $\mathbf{D}$ is a (6×6) symmetric matrix. For isotropic materials, $\mathbf{D}$ is given by Eq. 1.15.

The strain–displacement relations are given by

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial w}{\partial z} \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^{\mathrm{T}} \tag{9.5}$$

The body force and traction vectors are given by

$$\mathbf{f} = [f_x \quad f_y \quad f_z]^{\mathrm{T}} \tag{9.6}$$

$$\mathbf{T} = [T_x \quad T_y \quad T_z]^{\mathrm{T}} \tag{9.7}$$

The total potential and the Galerkin/virtual work form for three dimensions are given in Chapter 1.