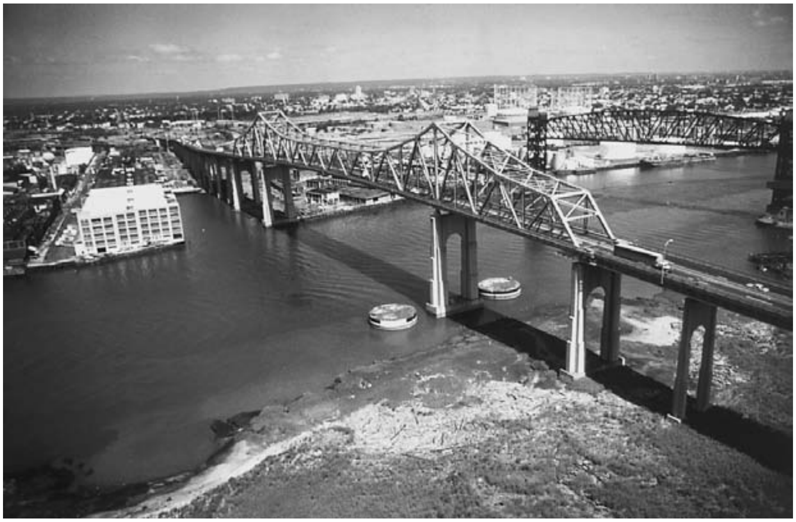
3

PLANE TRUSSES



Goethals Bridge, a Cantilever Truss Bridge between Staten Island, NY, and Elizabeth, NJ.

(Photo courtesy of Port Authority of New York and New Jersey)

A *plane truss* is defined as *a two-dimensional framework of straight prismatic members connected at their ends by frictionless hinged joints, and subjected to loads and reactions that act only at the joints and lie in the plane of the structure.* The members of a plane truss are subjected to axial compressive or tensile forces only.

The objective of this chapter is to develop the analysis of plane trusses based on the matrix stiffness method. This method of analysis is general, in the sense that it can be applied to statically determinate, as well as indeterminate, plane trusses of any size and shape. We begin the chapter with the definitions of the global and local coordinate systems to be used in the analysis. The concept of "degrees of freedom" is introduced in Section 3.2; and the member force–displacement relations are established in the local coordinate system, using the equilibrium equations and the principles of *mechanics of materials*, in Section 3.3. The finite-element formulation of member stiffness relations using the principle of virtual work is presented in Section 3.4; and transformation of member forces and displacements from a local to a global coordinate system, and vice versa, is considered in Section 3.5. Member stiffness relations in the global coordinate system are derived in Section 3.6; the formulation of the stiffness relations for the entire truss, by combining the member stiffness relations, is discussed in Section 3.7; and a step-by-step procedure for the analysis of plane trusses subjected to joint loads is developed in Section 3.8.

3.1 GLOBAL AND LOCAL COORDINATE SYSTEMS

In the matrix stiffness method, two types of coordinate systems are employed to specify the structural and loading data and to establish the necessary force–displacement relations. These are referred to as the *global* (or *structural*) and the *local* (or *member*) coordinate systems.

Global Coordinate System

The overall geometry and the load–deformation relationships for an entire structure are described with reference to a Cartesian or rectangular global coordinate system.

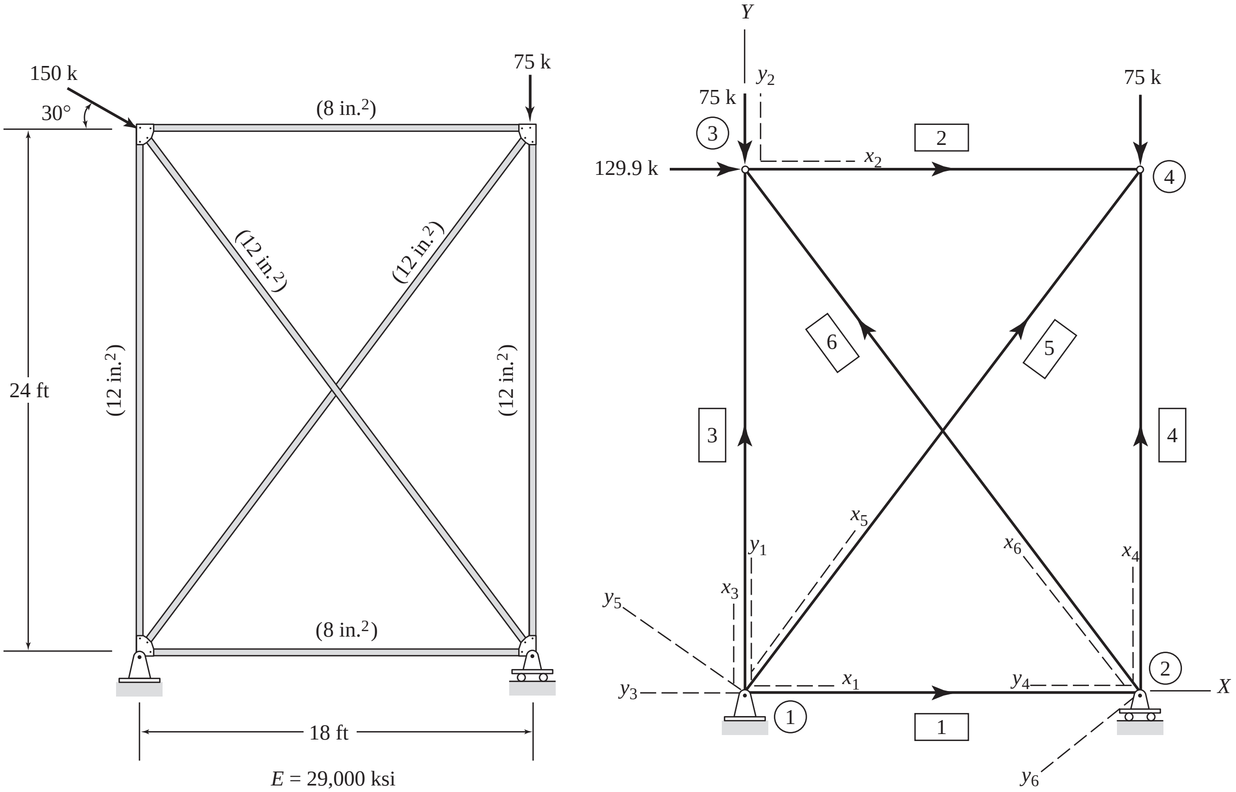
The global coordinate system used in this text is a right-handed XYZ coordinate system with the plane structure lying in the XY plane.

When analyzing a plane (two-dimensional) structure, the origin of the global XY coordinate system can be located at any point in the plane of the structure, with the X and Y axes oriented in any mutually perpendicular directions in the structure's plane. However, it is usually convenient to locate the origin at a lower left joint of the structure, with the X and Y axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively, so that the X and Y coordinates of most of the joints are positive.

Consider, for example, the truss shown in Fig. 3.1(a), which is composed of six members and four joints. Figure 3.1(b) shows the analytical model of the truss as represented by a line diagram, on which all the joints and members are identified by numbers that have been assigned arbitrarily. The global coordinate system chosen for analysis is usually drawn on the line diagram of the structure as shown in Fig. 3.1(b). Note that the origin of the global XY coordinate system is located at joint 1.

Local Coordinate System

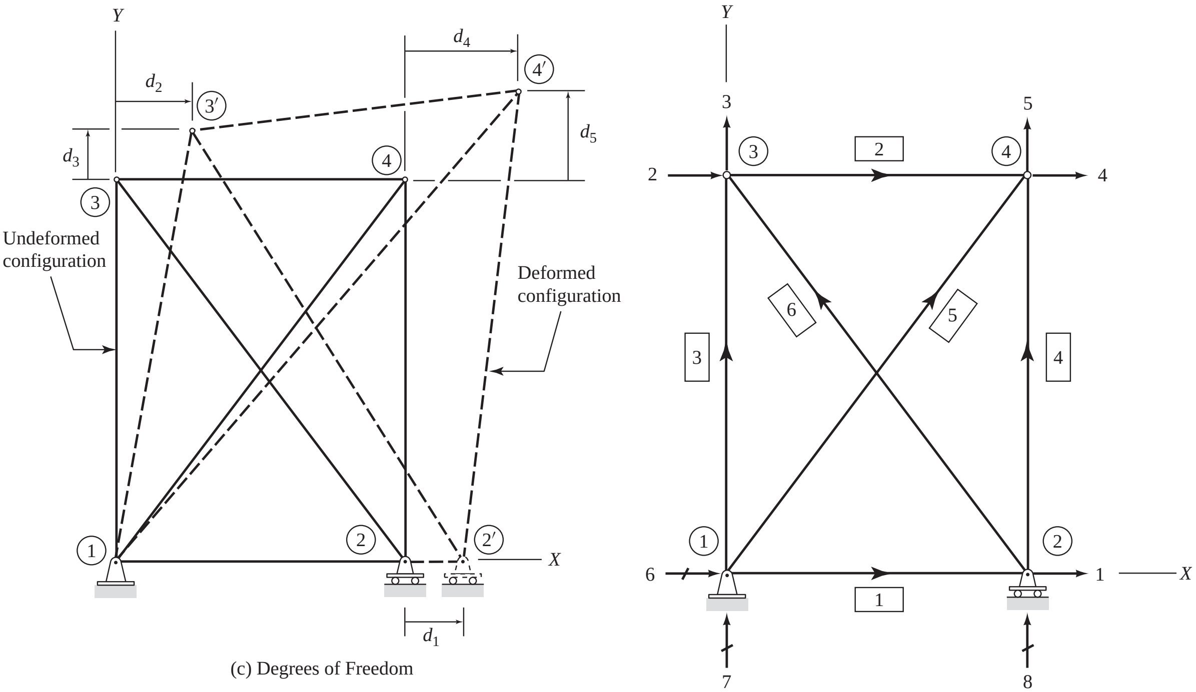
Since it is convenient to derive the basic member force–displacement relationships in terms of the forces and displacements in the directions along and perpendicular to members, a local coordinate system is defined for each member of the structure.



(a) Actual Truss

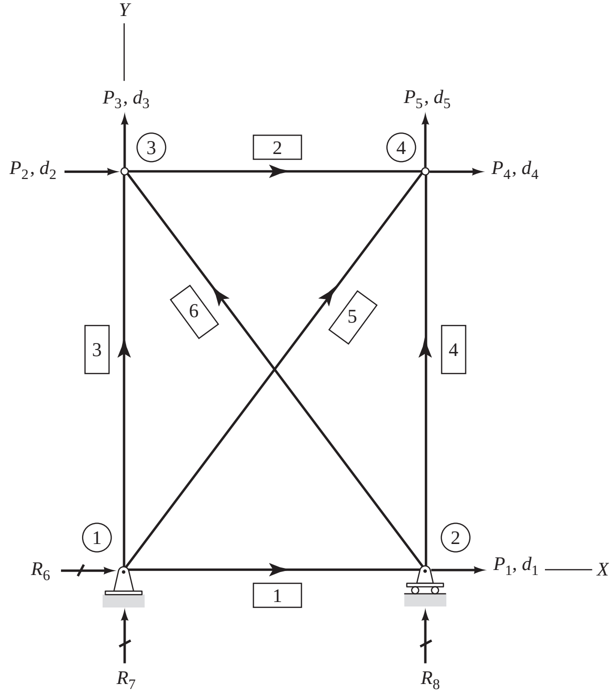
(b) Analytical Model Showing Global and Local Coordinate Systems

Fig. 3.1



(c) Degrees of Freedom

(d) Degrees of Freedom and Restrained Coordinate Numbers



(e) Degrees of Freedom, Joint Loads, and Reactions

Fig. 3.1 (*continued*)

The origin of the local xyz coordinate system for a member may be arbitrarily located at one of the ends of the member in its undeformed state, with the x axis directed along the member's centroidal axis in the undeformed state. The positive direction of the y axis is defined so that the coordinate system is right-handed, with the local z axis pointing in the positive direction of the global Z axis.

On the line diagram of the structure, the positive direction of the x axis for each member is indicated by drawing an arrow along each member, as shown in Fig. 3.1(b). For example, this figure shows the origin of the local coordinate system for member 1 located at its end connected to joint 1, with the x_1 axis directed from joint 1 to joint 2; the origin of the local coordinate system for member 4 located at its end connected to joint 2, with the x_4 axis directed from joint 2 to joint 4, etcetera. The joint to which the member end with the origin of the local coordinate system is connected is termed the *beginning joint* for the member, and the joint adjacent to the opposite end of the member is referred to as the *end joint*. For example, in Fig. 3.1(b), member 1 begins at joint 1 and ends at joint 2, member 4 begins at joint 2 and ends at joint 4, and so on. Once the local x axis is specified for a member, its y axis can be established by applying the right-hand rule. The local y axes thus obtained for all six members of the truss are depicted in Fig. 3.1(b). It can be seen that, for each member, if we curl the fingers of our right hand from the direction of the x axis toward the direction of the corresponding y axis, then the extended thumb points out of the plane of the page, which is the positive direction of the global Z axis.

3.2 DEGREES OF FREEDOM

The *degrees of freedom* of a structure, in general, are defined as *the independent joint displacements (translations and rotations) that are necessary to specify the deformed shape of the structure when subjected to an arbitrary loading.* Since the joints of trusses are assumed to be frictionless hinges, they are not subjected to moments and, therefore, their rotations are zero. Thus, only joint translations must be considered in establishing the degrees of freedom of trusses.

Consider again the plane truss of Fig. 3.1(a). The deformed shape of the truss, for an arbitrary loading, is depicted in Fig. 3.1(c) using an exaggerated scale. From this figure, we can see that joint 1, which is attached to the hinged support, cannot translate in any direction; therefore, it has no degrees of freedom. Because joint 2 is attached to the roller support, it can translate in the X direction, but not in the Y direction. Thus, joint 2 has only one degree of freedom, which is designated d_1 in the figure. As joint 3 is not attached to a support, two displacements (namely, the translations d_2 and d_3 in the X and Y directions, respectively) are needed to completely specify its deformed position $3'$. Thus, joint 3 has two degrees of freedom. Similarly, joint 4, which is also a *free* joint, has two degrees of freedom, designated d_4 and d_5. Thus, the entire truss has a total of five degrees of freedom. As shown in Fig. 3.1(c), the joint displacements are defined relative to the global coordinate system, and are considered to be positive when in the positive directions of the X and Y axes. Note that all the joint displacements are shown in the positive sense in Fig. 3.1(c). The five joint displacement of the truss can be collectively written in matrix form as

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}$$

in which **d** is called the *joint displacement vector,* with the number of rows equal to the number of degrees of freedom of the structure.

It is important to note that the five joint displacements d_1 through d_5 are necessary and sufficient to uniquely define the deformed shape of the truss under any arbitrary loading condition. Furthermore, the five joint displacements are independent, in the sense that each displacement can be varied arbitrarily and independently of the others.

As the foregoing example illustrates, the degrees of freedom of all types of framed structures, in general, are the same as the actual joint displacements. Thus, the number of degrees of freedom of a framed structure can be determined by subtracting the number of joint displacements restrained by supports from the total number of joint displacements of the unsupported structure; that is,

$$\begin{pmatrix} \text{number of} \\ \text{degrees of} \\ \text{freedom} \end{pmatrix} = \begin{pmatrix} \text{number of joint} \\ \text{displacements of} \\ \text{the unsupported} \\ \text{structure} \end{pmatrix} - \begin{pmatrix} \text{number of joint} \\ \text{displacements} \\ \text{restrained by} \\ \text{supports} \end{pmatrix} \qquad \textbf{(3.1)}$$

As the number of displacements of an unsupported structure equals the product of the number of degrees of freedom of a free joint of the structure and the total number of joints of the structure, we can express Eq. (3.1) as

$$NDOF = NCJT\,(NJ) - NR \qquad \textbf{(3.2)}$$

in which *NDOF* represents the number of degrees of freedom of the structure (sometimes referred to as the *degree of kinematic indeterminacy* of the structure); *NCJT* represents the number of degrees of freedom of a free joint (also called the number of structure coordinates per joint); *NJ* is the number of joints; and *NR* denotes the number of joint displacements restrained by supports.

Since a free joint of a plane truss has two degrees of freedom, which are translations in the X and Y directions, we can specialize Eq. (3.2) for the case of plane trusses:

$$\left.\begin{aligned} NCJT &= 2 \\ NDOF &= 2(NJ) - NR \end{aligned}\right\} \text{ for plane trusses} \qquad \textbf{(3.3)}$$

Let us apply Eq. (3.3) to the truss of Fig. 3.1(a). The truss has four joints (i.e., $NJ = 4$), and the hinged support at joint 1 restrains two joint displacements, namely, the translations of joint 1 in the X and Y directions; whereas the roller support at joint 2 restrains one joint displacement, which is the translation of joint 2 in the Y direction. Thus, the total number of joint displacements that are restrained by all supports of the truss is 3 (i.e., $NR = 3$). Substituting the numerical values of NJ and NR into Eq. (3.3), we obtain

$$NDOF = 2(4) - 3 = 5$$

which is the same as the number of degrees of freedom of the truss obtained previously.

The degrees of freedom (or joint displacements) of a structure are also termed the structure's *free coordinates;* the joint displacements restrained by supports are commonly called the *restrained coordinates* of the structure. The free and restrained coordinates are referred to collectively as simply the *structure coordinates*. It should be noted that each structure coordinate represents an unknown quantity to be determined by the analysis, with a free coordinate representing an unknown joint displacement, and a restrained coordinate representing an unknown support reaction. Realizing that $NCJT$ (i.e., the number of structure coordinates per joint) equals the number of unknown joint displacements and/or support reactions per joint of the structure, the total number of unknown joint displacements and reactions for a structure can be expressed as

$$\left(\begin{array}{l}\text{number of unknown}\\ \text{joint displacements}\\ \text{and support reactions}\end{array}\right) = NDOF + NR = NCJT(NJ)$$

Numbering of Degrees of Freedom and Restrained Coordinates

When analyzing a structure, it is not necessary to draw the structure's deformed shape, as shown in Fig. 3.1(c), to identify its degrees of freedom. Instead, the degrees of freedom can be directly specified on the line diagram of the structure by assigning numbers to the arrows drawn at the joints in the directions of the joint displacements, as shown in Fig. 3.1(d). The restrained coordinates are identified in a similar manner. However, the arrows representing the restrained coordinates are usually drawn with a slash ($\not\rightarrow$) to distinguish them from the arrows identifying the degrees of freedom.

The degrees of freedom of a plane truss are numbered starting at the lowest-numbered joint that has a degree of freedom, and proceeding sequentially to the highest-numbered joint. In the case of more than one degree of freedom at a joint, the translation in the X direction is numbered first, followed by the translation in the Y direction. The first degree of freedom is assigned the number one, and the last degree of freedom is assigned a number equal to $NDOF$.

Once all the degrees of freedom of the structure have been numbered, we number the restrained coordinates in a similar manner, but begin with a number equal to $NDOF + 1$. We start at the lowest-numbered joint that is attached to a support, and proceed sequentially to the highest-numbered joint. In the case of more than one restrained coordinate at a joint, the coordinate in the

X direction is numbered first, followed by the coordinate in the Y direction. Note that this procedure will always result in the last restrained coordinate of the structure being assigned a number equal to $2(NJ)$.

The degrees of freedom and the restrained coordinates of the truss in Fig. 3.1(d) have been numbered using the foregoing procedure. We start numbering the degrees of freedom by examining joint 1. Since the displacements of joint 1 in both the X and Y directions are restrained, this joint does not have any degrees of freedom; therefore, at this point, we do not assign any numbers to the two arrows representing its restrained coordinates, and move on to the next joint. Focusing our attention on joint 2, we realize that this joint is free to displace in the X direction, but not in the Y direction. Therefore, we assign the number 1 to the horizontal arrow indicating that the X displacement of joint 2 will be denoted by d_1. Note that, at this point, we do not assign any number to the vertical arrow at joint 2, and change our focus to the next joint. Joint 3 is free to displace in both the X and Y directions; we number the X displacement first by assigning the number 2 to the horizontal arrow, and then number the Y displacement by assigning the number 3 to the vertical arrow. This indicates that the X and Y displacements of joint 3 will be denoted by d_2 and d_3, respectively. Next, we focus our attention on joint 4, which is also free to displace in both the X and Y directions; we assign numbers 4 and 5, respectively, to its displacements in the X and Y directions, as shown in Fig. 3.1(d). Again, the arrow that is numbered 4 indicates the location and direction of the joint displacement d_4; the arrow numbered 5 shows the location and direction of d_5.

Having numbered all the degrees of freedom of the truss, we now return to joint 1, and start numbering the restrained coordinates of the structure. As previously discussed, joint 1 has two restrained coordinates; we first assign the number 6 (i.e., $NDOF + 1 = 5 + 1 = 6$) to the X coordinate (horizontal arrow), and then assign the number 7 to the Y coordinate (vertical arrow). Finally, we consider joint 2, and assign the number 8 to the vertical arrow representing the restrained coordinate in the Y direction at that joint. We realize that the displacements corresponding to the restrained coordinates 6 through 8 are zero (i.e., $d_6 = d_7 = d_8 = 0$). However, we use these restrained coordinate numbers to specify the reactions at supports of the structure, as discussed subsequently in this section.

Joint Load Vector

External loads applied to the joints of trusses are specified as force components in the global X and Y directions. These load components are considered positive when acting in the positive directions of the X and Y axes, and vice versa. Any loads initially given in inclined directions are resolved into their X and Y components, before proceeding with an analysis. For example, the 150 k inclined load acting on a joint of the truss in Fig. 3.1(a) is resolved into its rectangular components as

$$\text{load component in } X \text{ direction} = 150 \cos 30^\circ = 129.9 \text{ k}\rightarrow$$

$$\text{load component in } Y \text{ direction} = 150 \sin 30^\circ = 75 \text{ k}\downarrow$$

These components are applied at joint 3 of the line diagram of the truss shown in Fig. 3.1(b).

In general, a load can be applied to a structure at the location and in the direction of each of its degrees of freedom. For example, a five-degree-of-freedom truss can be subjected to a maximum of five loads, P_1 through P_5, as shown in Fig. 3.1(e). As indicated there, the numbers assigned to the degrees of freedom are also used to identify the joint loads. In other words, a load corresponding to a degree of freedom d_i is denoted by the symbol P_i. The five joint loads of the truss can be collectively written in matrix form as

$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 129.9 \\ -75 \\ 0 \\ -75 \end{bmatrix} \text{ k}$$

in which **P** is called the *joint load vector* of the truss. The numerical form of **P** is obtained by comparing Figs. 3.1(b) and 3.1(e). This comparison shows that: $P_1 = 0$; $P_2 = 129.9$ k; $P_3 = -75$ k; $P_4 = 0$; and $P_5 = -75$ k. The negative signs assigned to the magnitudes of P_3 and P_5 indicate that these loads act in the negative Y (i.e., downward) direction. The numerical values of P_1 through P_5 are then stored in the appropriate rows of the joint load vector **P**, as shown in the foregoing equation. It should be noted that the number of rows of **P** equals the number of degrees of freedom (*NDOF*) of the structure.

Reaction Vector

A support that prevents the translation of a joint of a structure in a particular direction exerts a reaction force on the joint in that direction. Thus, when a truss is subjected to external loads, a reaction force component can develop at the location and in the direction of each of its restrained coordinates. For example, a truss with three restrained coordinates can develop up to three reactions, as shown in Fig. 3.1(e). As indicated there, the numbers assigned to the restrained coordinates are used to identify the support reactions. In other words, a reaction corresponding to an *i*th restrained coordinate is denoted by the symbol R_i. The three support reactions of the truss can be collectively expressed in matrix form as

$$\mathbf{R} = \begin{bmatrix} R_6 \\ R_7 \\ R_8 \end{bmatrix}$$

in which **R** is referred to as the *reaction vector* of the structure. Note that the number of rows of **R** equals the number of restrained coordinates (*NR*) of the structure.

The procedure presented in this section for numerically identifying the degrees of freedom, joint loads, and reactions of a structure considerably simplifies the task of programming an analysis on a computer, as will become apparent in Chapter 4.

EXAMPLE 3.1

Identify numerically the degrees of freedom and restrained coordinates of the tower truss shown in Fig. 3.2(a). Also, form the joint load vector **P** for the truss.

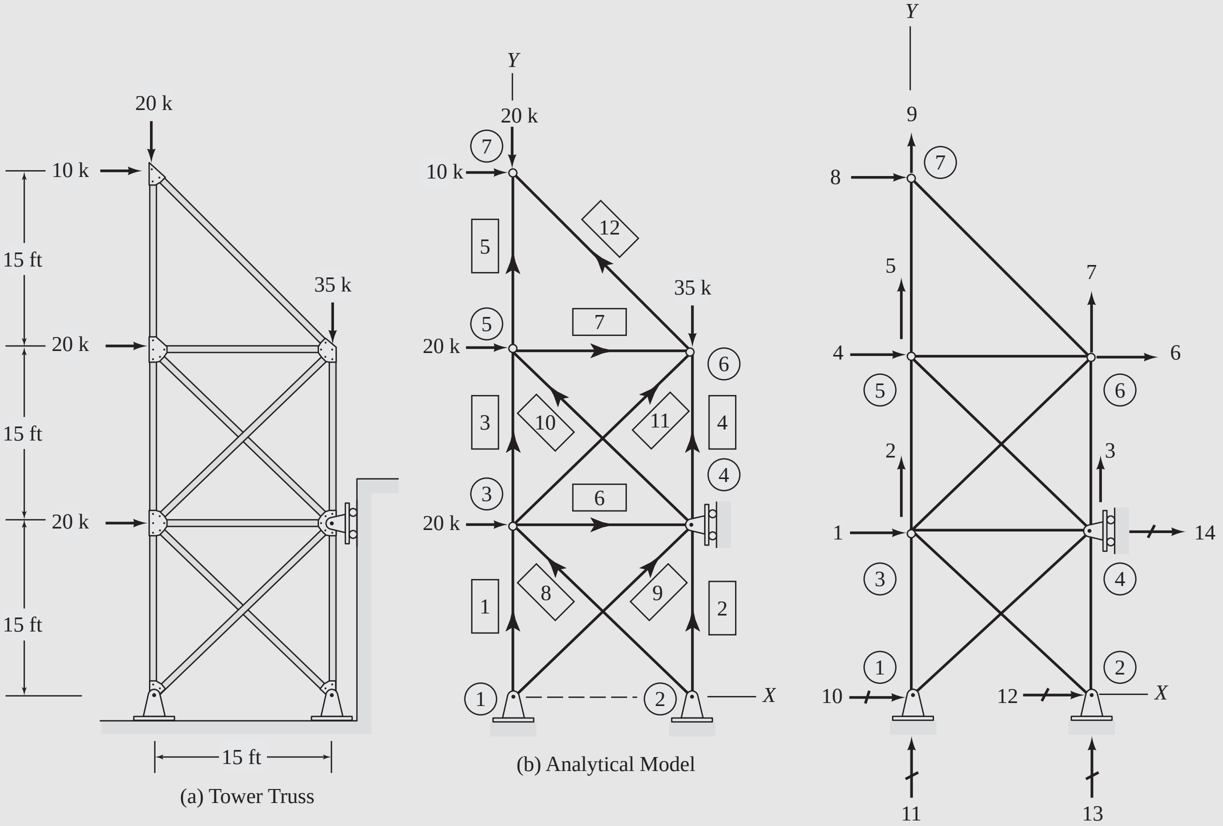
SOLUTION

The truss has nine degrees of freedom, which are identified by the numbers 1 through 9 in Fig. 3.2(c). The five restrained coordinates of the truss are identified by the numbers 10 through 14 in the same figure. **Ans**

By comparing Figs. 3.2(b) and (c), we express the joint load vector as

$$\mathbf{P} = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 20 \\ 0 \\ 0 \\ -35 \\ 10 \\ -20 \end{bmatrix} \text{ k}$$

Ans



(a) Tower Truss

(b) Analytical Model

(c) Degrees of Freedom and Restrained Coordinates ($NDOF = 9$, $NR = 5$)

Fig. 3.2

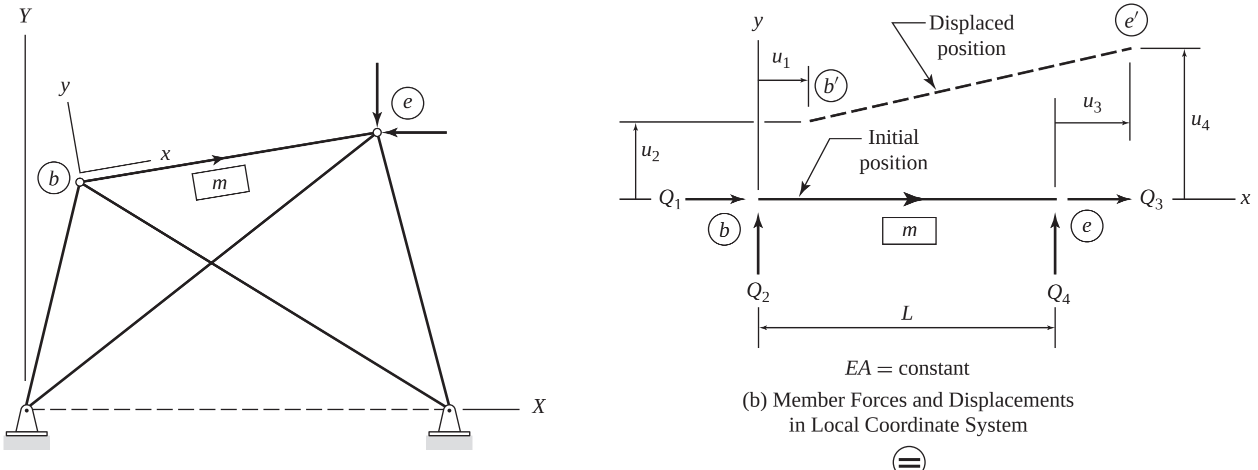
3.3 MEMBER STIFFNESS RELATIONS IN THE LOCAL COORDINATE SYSTEM

In the stiffness method of analysis, the joint displacements, **d**, of a structure due to an external loading, **P**, are determined by solving a system of simultaneous equations, expressed in the form

$$\mathbf{P} = \mathbf{S}\mathbf{d} \tag{3.4}$$

in which **S** is called the *structure stiffness matrix*. It will be shown subsequently that the stiffness matrix for the entire structure, **S**, is formed by assembling the stiffness matrices for its individual members. *The stiffness matrix for a member expresses the forces at the ends of the member as functions of the displacements of those ends*. In this section, we derive the stiffness matrix for the members of plane trusses in the local coordinate system.

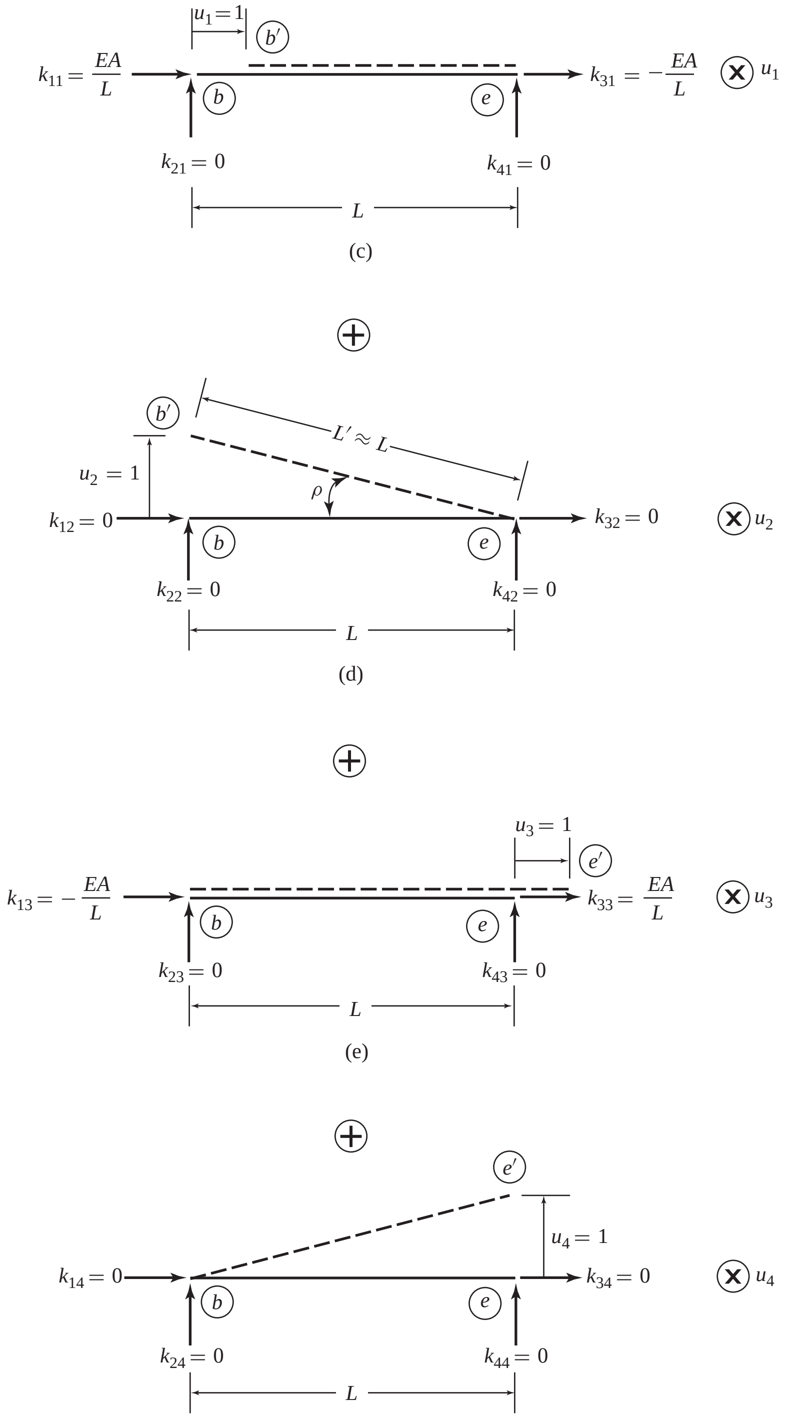
To establish the member stiffness relations, let us focus our attention on an arbitrary member m of the plane truss shown in Fig. 3.3(a). When the truss is subjected to external loads, m deforms and internal forces are induced at its ends. The initial and displaced positions of m are shown in Fig. 3.3(b), where L, E, and A denote, respectively, the length, Young's modulus of elasticity, and the cross-sectional area of m. The member is prismatic in the sense that its axial rigidity, EA, is constant. As Fig. 3.3(b) indicates, two displacements—translations in the x and y directions—are needed to completely specify the displaced position of each end of m. Thus, m has a total of four end displacements or degrees of freedom. As shown in Fig. 3.3(b), the member end displacements are denoted by u_1 through u_4, and the corresponding member end forces are denoted by Q_1 through Q_4. Note that these end displacements and forces are defined relative to the local coordinate system of the member, and are considered positive when in the positive directions of the local x and y axes. As indicated in



(a) Plane Truss

(b) Member Forces and Displacements in Local Coordinate System

Fig. 3.3



(c)

(d)

(e)

(f)

Fig. 3.3 (*continued*)

Fig. 3.3(b), the numbering scheme used to identify the member end displacements and forces is as follows:

> *Member end displacements and forces are numbered by beginning at the end of the member designated "b", where the origin of the local coordinate system is located, with the translation and force in the x direction numbered first, followed by the translation and force in the y direction. The displacements and forces at the opposite end of the member, designated "e," are then numbered in the same sequential order.*

It should be remembered that our objective here is to determine the relationships between member end forces and end displacements. Such relationships can be conveniently established by subjecting the member, separately, to each of the four end displacements as shown in Figs. 3.3(c) through (f); and by expressing the total member end forces as the algebraic sums of the end forces required to cause the individual end displacements. Thus, from Figs. 3.3(b) through (f), we can see that

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 \tag{3.5a}$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4 \tag{3.5b}$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4 \tag{3.5c}$$

$$Q_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4 \tag{3.5d}$$

in which k_{ij} *represents the force at the location and in the direction of* Q_i *required, along with other end forces, to cause a unit value of displacement* u_j*, while all other end displacements are zero*. These forces per unit displacement are called *stiffness coefficients*. It should be noted that a double-subscript notation is used for stiffness coefficients, with the first subscript identifying the force and the second subscript identifying the displacement.

By using the definition of matrix multiplication, Eqs. (3.5) can be expressed in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.6}$$

or, symbolically, as

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{u}} \tag{3.7}$$

in which **Q** and **u** are the member end force and member end displacement vectors, respectively, in the local coordinate system; and **k** is called the *member stiffness matrix in the local coordinate system*.

The stiffness coefficients k_{ij} can be evaluated by subjecting the member, separately, to unit values of each of the four end displacements. The member end forces required to cause the individual unit displacements are then determined by applying the equations of equilibrium, and by using the principles of *mechanics of materials*. The member end forces thus obtained represent the stiffness coefficients for the member.

Let us determine the stiffness coefficients corresponding to a unit value of the displacement u_1 at end b of m, as shown in Fig. 3.3(c). Note that all other displacements of m are zero (i.e., $u_2 = u_3 = u_4 = 0$). Since m is in equilibrium, the end forces k_{11}, k_{21}, k_{31}, and k_{41} acting on it must satisfy the three equilibrium equations: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$. Applying the equations of equilibrium, we write

$$+\rightarrow \sum F_x = 0 \qquad k_{11} + k_{31} = 0$$

$$k_{31} = -k_{11} \tag{3.8}$$

$$+\uparrow \sum F_y = 0 \qquad k_{21} + k_{41} = 0 \tag{3.9}$$

$$+\circlearrowleft \sum M_e = 0 \qquad -k_{21}(L) = 0$$

Since L is not zero, k_{21} must be zero; that is

$$k_{21} = 0 \tag{3.10}$$

By substituting Eq. (3.10) into Eq. (3.9), we obtain

$$k_{41} = 0 \tag{3.11}$$

Equations (3.8), (3.10), and (3.11) indicate that m is in equilibrium under the action of two axial forces, of equal magnitude but with opposite senses, applied at its ends. Furthermore, since the displacement $u_1 = 1$ results in the shortening of the member's length, the two axial forces causing this displacement must be compressive; that is, k_{11} must act in the positive direction of the local x axis, and k_{31} (with a magnitude equal to k_{11}) must act in the negative direction of the x axis.

To relate the axial force k_{11} to the unit axial deformation ($u_1 = 1$) of m, we use the principles of the *mechanics of materials*. Recall that in a prismatic member subjected to axial tension or compression, the normal stress σ is given by

$$\sigma = \frac{\text{axial force}}{\text{cross-sectional area}} = \frac{k_{11}}{A} \tag{3.12}$$

and the normal strain, ε, is expressed as

$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{1}{L} \tag{3.13}$$

For linearly elastic materials, the stress–strain relationship is given by Hooke's law as

$$\sigma = E\varepsilon \tag{3.14}$$

Substitution of Eqs. (3.12) and (3.13) into Eq. (3.14) yields

$$\frac{k_{11}}{A} = E\left(\frac{1}{L}\right)$$

from which we obtain the expression for the stiffness coefficient k_{11},

$$k_{11} = \frac{EA}{L} \tag{3.15}$$

The expression for k_{31} can now be obtained from Eq. (3.8) as

$$k_{31} = -k_{11} = -\frac{EA}{L} \tag{3.16}$$

in which the negative sign indicates that this force acts in the negative x direction. Figure 3.3(c) shows the expressions for the four stiffness coefficients required to cause the end displacement $u_1 = 1$ of m.

By using a similar approach, it can be shown that the stiffness coefficients required to cause the axial displacement $u_3 = 1$ at end e of m are as follows (Fig. 3.3e).

$$k_{13} = -\frac{EA}{L} \qquad k_{23} = 0 \qquad k_{33} = \frac{EA}{L} \qquad k_{43} = 0 \tag{3.17}$$

The deformed shape of m due to a unit value of displacement u_2, while all other displacements are zero, is shown in Fig. 3.3(d). Applying the equilibrium equations, we write

$$\begin{aligned} &+\rightarrow \sum F_x = 0 \qquad && k_{12} + k_{32} = 0 \\ & && k_{32} = -k_{12} && \textbf{(3.18)} \\ &+\uparrow \sum F_y = 0 && k_{22} + k_{42} = 0 && \textbf{(3.19)} \\ &+\circlearrowleft \sum M_e = 0 && -k_{22}(L) = 0 \end{aligned}$$

from which we obtain

$$k_{22} = 0 \tag{3.20}$$

Substitution of Eq. (3.20) into Eq. (3.19) yields

$$k_{42} = 0 \tag{3.21}$$

Thus, the forces k_{22} and k_{42}, which act perpendicular to the longitudinal axis of m, are both zero.

As for the axial forces k_{12} and k_{32}, Eq. (3.18) indicates that they must be of equal magnitude but with opposite senses. From Fig. 3.3(d), we can see that the deformed length of the member, L', can be expressed in terms of its undeformed length L as

$$L' = \frac{L}{\cos \rho} \tag{3.22}$$

in which the angle ρ denotes the rotation of the member due to the end displacement $u_2 = 1$. Since the displacements are assumed to be small, $\cos \rho \approx 1$ and Eq. (3.22) reduces to

$$L' \approx L \tag{3.23}$$

which can be rewritten as

$$L' - L \approx 0 \tag{3.24}$$

As Eq. (3.24) indicates, the change in the length of *m* (or its axial deformation) is negligibly small and, therefore, no axial forces develop at the ends of *m*; that is,

$$k_{12} = k_{32} = 0 \tag{3.25}$$

Thus, as shown in Fig. 3.3(d), no end forces are required to produce the displacement $u_2 = 1$ of *m*.

Similarly, the stiffness coefficients required to cause the small end displacement $u_4 = 1$, in the direction perpendicular to the longitudinal axis of *m*, are also all zero, as shown in Fig. 3.3(f). Thus,

$$k_{14} = k_{24} = k_{34} = k_{44} = 0 \tag{3.26}$$

By substituting the foregoing values of the stiffness coefficients into Eq. (3.6), we obtain the following stiffness matrix for the members of plane trusses in their local coordinate systems.

$$\mathbf{k} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.27}$$

Note that the *i*th column of the member stiffness matrix **k** consists of the end forces required to cause a unit value of the end displacement u_i, while all other displacements are zero. For example, the third column of **k** consists of the four end forces required to cause the displacement $u_3 = 1$, as shown in Fig. 3.3(e), and so on. The units of the stiffness coefficients are expressed in terms of force divided by length (e.g., k/in or kN/m). When evaluating a stiffness matrix for analysis, it is important to use a consistent set of units. For example, if we wish to use the units of kips and feet, then the modulus of elasticity (E) must be expressed in k/ft^2, area of cross-section (A) in ft^2, and the member length (L) in ft.

From Eq. (3.27), we can see that the stiffness matrix **k** is symmetric; that is, $k_{ij} = k_{ji}$. As shown in Section 3.4, the stiffness matrices for linear elastic structures are always symmetric.

EXAMPLE 3.2

Determine the local stiffness matrices for the members of the truss shown in Fig. 3.4.

SOLUTION

Members 1 and 2 $E = 29{,}000$ ksi, $A = 8$ in.^{2}, $L = 18$ ft $= 216$ in.

$$\frac{EA}{L} = \frac{29{,}000(8)}{216} = 1{,}074.1 \text{ k/in.}$$

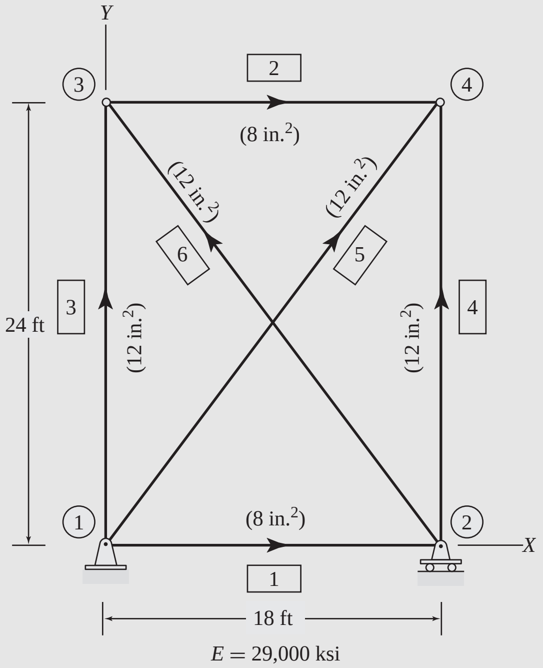


Fig. 3.4

Substitution into Eq. (3.27) yields

$$\mathbf{k}_1 = \mathbf{k}_2 = \begin{bmatrix} 1{,}074.1 & 0 & -1{,}074.1 & 0 \\ 0 & 0 & 0 & 0 \\ -1{,}074.1 & 0 & 1{,}074.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.} \qquad \textbf{Ans}$$

Members 3 and 4 $E = 29{,}000$ ksi, $A = 12$ in.^{2}, $L = 24$ ft $= 288$ in.

$$\frac{EA}{L} = \frac{29{,}000(12)}{288} = 1{,}208.3 \text{ k/in.}$$

Thus, from Eq. (3.27),

$$\mathbf{k}_3 = \mathbf{k}_4 = \begin{bmatrix} 1{,}208.3 & 0 & -1{,}208.3 & 0 \\ 0 & 0 & 0 & 0 \\ -1{,}208.3 & 0 & 1{,}208.3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.} \qquad \textbf{Ans}$$

Members 5 and 6 $E = 29{,}000$ ksi, $A = 12$ in.^{2},

$$L = \sqrt{(18)^2 + (24)^2} = 30 \text{ ft} = 360 \text{ in.}$$

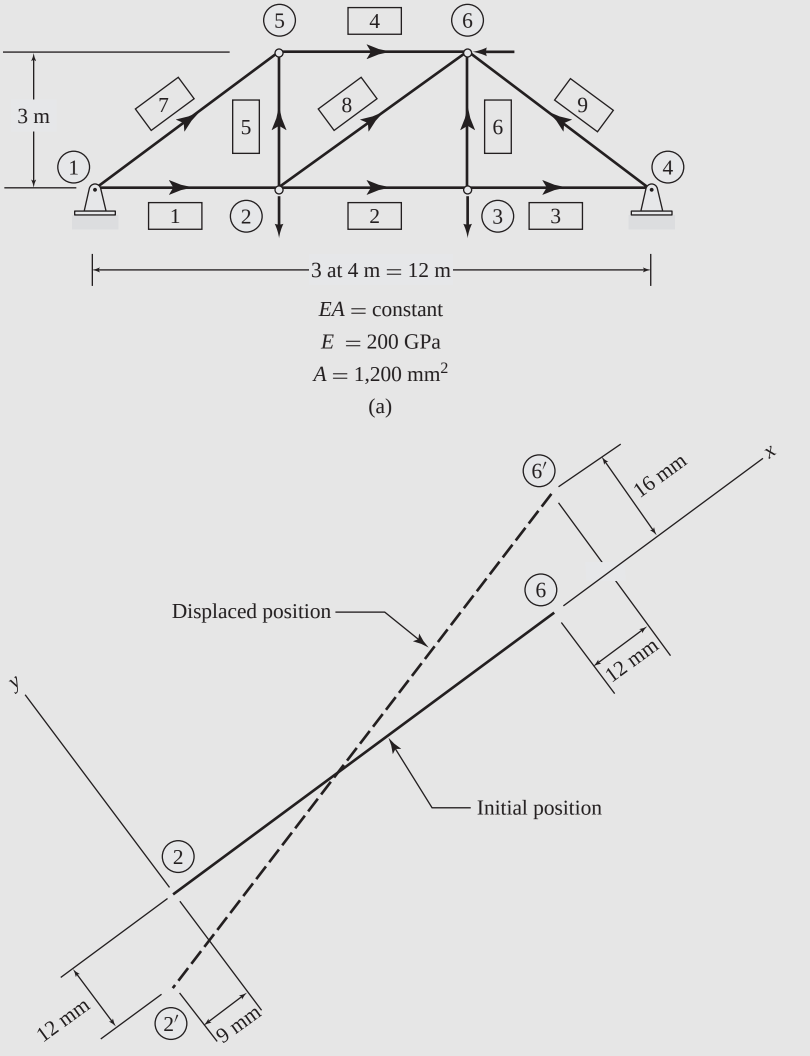
$$\frac{EA}{L} = \frac{29{,}000(12)}{360} = 966.67 \text{ k/in.}$$

Thus,

$$\mathbf{k}_5 = \mathbf{k}_6 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{k/in.}$$

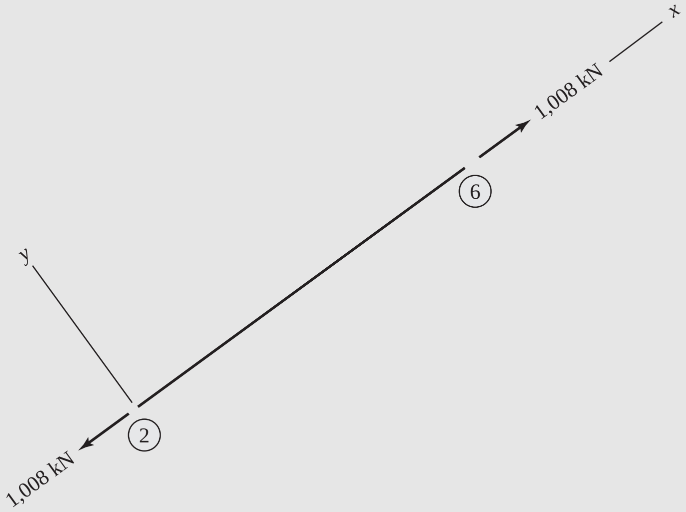
Ans

EXAMPLE 3.3 The displaced position of member 8 of the truss in Fig. 3.5(a) is given in Fig. 3.5(b). Calculate the axial force in this member.



(b) Displaced Position of Member 8

Fig. 3.5



(c) Member End Forces in Local Coordinate System

Fig. 3.5 (*continued*)

SOLUTION *Member Stiffness Matrix in the Local Coordinate System:* From Fig. 3.5(a), we can see that $E = 200$ GPa $= 200(10^6)$ kN/m^2; $A = 1{,}200$ mm$^2 = 0.0012$ m^2; and $L = \sqrt{(4)^2 + (3)^2} = 5$ m. Thus,

$$\frac{EA}{L} = \frac{200(10^6)(0.0012)}{5} = 48{,}000 \text{ kN/m}$$

From Eq. (3.27), we obtain

$$\mathbf{k}_8 = 48{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{kN/m}$$

Member End Displacements in the Local Coordinate System: From Fig. 3.5(b), we can see that the beginning end, 2, of the member displaces 9 mm in the negative x direction and 12 mm in the negative y direction. Thus, $u_1 = -9$ mm $= -0.009$ m and $u_2 = -12$ mm $= -0.012$ m. Similarly, the opposite end, 6, of the member displaces 12 mm and 16 mm, respectively, in the x and y directions; that is, $u_3 = 12$ mm $= 0.012$ m and $u_4 = 16$ mm $= 0.016$ m. Thus, the member end displacement vector in the local coordinate system is given by

$$\mathbf{u}_8 = \begin{bmatrix} -0.009 \\ -0.012 \\ 0.012 \\ 0.016 \end{bmatrix} \text{m}$$

Member End Forces in the Local Coordinate System: We calculate the member end forces by applying Eq. (3.7). Thus,

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

$$\mathbf{Q}_8 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = 48{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.009 \\ -0.012 \\ 0.012 \\ 0.016 \end{bmatrix} = \begin{bmatrix} -1{,}008 \\ 0 \\ 1{,}008 \\ 0 \end{bmatrix} \text{kN}$$

The member end forces, **Q**, are depicted on the free-body diagram of the member in Fig. 3.5(c), from which we can see that, since the end force Q_1 is negative, but Q_3 is positive, member 8 is subjected to a tensile axial force, Q_a, of magnitude 1,008 kN; that is,

$Q_{a8} = 1{,}008$ kN (T) **Ans**

3.4 FINITE-ELEMENT FORMULATION USING VIRTUAL WORK*

In this section, we present an alternate formulation of the member stiffness matrix **k** in the local coordinate system. This approach, which is commonly used in the finite-element method, essentially involves expressing the strains and stresses at points within the member in terms of its end displacements **u**, and applying the principle of virtual work for deformable bodies as delineated by Eq. (1.16) in Section 1.6. Before proceeding with the derivation of **k**, let us rewrite Eq. (1.16) in a more convenient matrix form as

$$\delta W_e = \int_V \delta\varepsilon^T \sigma \, dV \tag{3.28}$$

in which δW_e denotes virtual external work; V represents member volume; and $\delta\varepsilon$ and σ denote, respectively, the virtual strain and real stress vectors, which for a general three-dimensional stress condition can be expressed as follows (see Fig. 1.16).

$$\delta\varepsilon = \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\varepsilon_z \\ \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \tag{3.29}$$

Displacement Functions

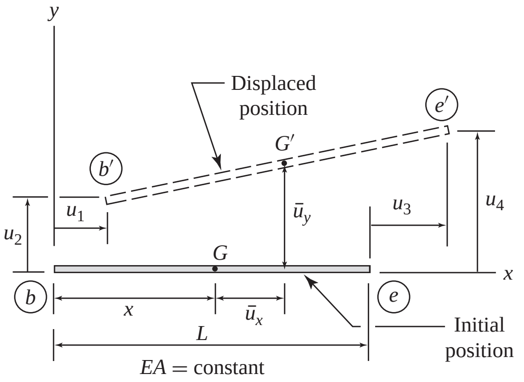
In the finite-element method, member stiffness relations are based on *assumed* variations of displacements within members. Such displacement variations are

---

*This section can be omitted without loss of continuity.

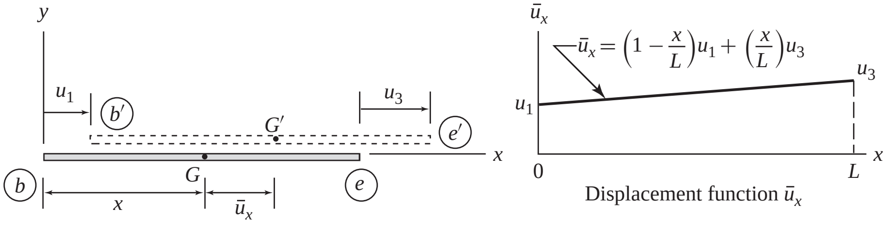
referred to as the *displacement* or *interpolation functions.* A displacement function describes *the variation of a displacement component along the centroidal axis of a member in terms of its end displacements.*

Consider a prismatic member of a plane truss, subjected to end displacements u_1 through u_4, as shown in Fig. 3.6(a). Since the member displaces in both the x and y directions, we need to define two displacement functions,



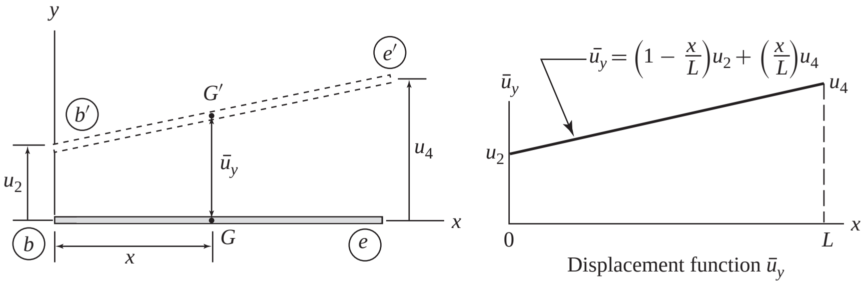
(a) Member Displacements in Local Coordinate System

⊜



(b) Member Displacements in x Direction

⊕



(c) Member Displacements in y Direction

Fig. 3.6

$\bar{u}_x$ and $\bar{u}_y$, for the displacements in the x and y directions, respectively. In Fig. 3.6(a), the displacement functions $\bar{u}_x$ and $\bar{u}_y$ are depicted as the displacements of an arbitrary point G located on the member's centroidal axis at a distance x from end b (left end).

The total displacement of the member (due to u_1 through u_4) can be decomposed into the displacements in the x and y directions, as shown in Figs. 3.6(b) and (c), respectively. Note that Fig. 3.6(b) shows the member subjected to the two end displacements, u_1 and u_3, in the x direction (with $u_2 = u_4 = 0$); Fig. 3.6(c) depicts the displacement of the member due to the two end displacements, u_2 and u_4, in the y direction (with $u_1 = u_3 = 0$).

The displacement functions assumed in the finite-element method are usually in the form of *complete* polynomials,

$$\bar{u}(x) = \sum_{i=0}^{n} a_i x^i \qquad \text{with } a_i \neq 0 \tag{3.30}$$

in which n is the degree of the polynomial. The polynomial used for a particular displacement function should be of such a degree that all of its coefficients can be evaluated by applying the available boundary conditions; that is,

$$n = n_{\mathrm{bc}} - 1 \tag{3.31}$$

with n_{bc} = number of boundary conditions.

Thus, the displacement function $\bar{u}_x$ for the truss member (Fig. 3.6b) is assumed in the form of a linear polynomial as

$$\bar{u}_x = a_0 + a_1 x \tag{3.32}$$

in which a_0 and a_1 are the constants that can be determined by applying the following two boundary conditions:

$$\text{at } x = 0 \qquad \bar{u}_x = u_1$$

$$\text{at } x = L \qquad \bar{u}_x = u_3$$

By applying the first boundary condition—that is, by setting $x = 0$ and $\bar{u}_x = u_1$ in Eq. (3.32)—we obtain

$$a_0 = u_1 \tag{3.33}$$

Next, by using the second boundary condition—that is, by setting $x = L$ and $\bar{u}_x = u_3$—we obtain

$$u_3 = u_1 + a_1 L$$

from which follows

$$a_1 = \frac{u_3 - u_1}{L} \tag{3.34}$$

By substituting Eqs. (3.33) and (3.34) into Eq. (3.32), we obtain the expression for $\bar{u}_x$,

$$\bar{u}_x = u_1 + \left(\frac{u_3 - u_1}{L}\right) x$$

or

$$\boxed{\bar{u}_x = \left(1 - \frac{x}{L}\right)u_1 + \left(\frac{x}{L}\right)u_3} \tag{3.35}$$

The displacement function $\bar{u}_y$, for the member displacement in the y direction (Fig. 3.6(c)), can be determined in a similar manner; that is, using a linear polynomial and evaluating its coefficients by applying the boundary conditions. The result is

$$\boxed{\bar{u}_y = \left(1 - \frac{x}{L}\right)u_2 + \left(\frac{x}{L}\right)u_4} \tag{3.36}$$

The plots of the displacement functions $\bar{u}_x$ and $\bar{u}_y$ are shown in Figs. 3.6(b) and (c), respectively.

It is important to realize that the displacement functions as given by Eqs. (3.35) and (3.36) have been *assumed,* as is usually done in the finite-element method. There is no guarantee that an assumed displacement function defines the actual displacements of the member, except at its ends. In general, the displacement functions used in the finite-element method only approximate the actual displacements within members (or elements), because they represent approximate solutions of the underlying differential equations. For this reason, the finite-element method is generally considered to be an approximate method of analysis. However, for the prismatic members of framed structures, the displacement functions in the form of complete polynomials do happen to describe exactly the actual member displacements and, therefore, such functions yield exact member stiffness matrices for prismatic members.

From Fig. 3.6(c), we observe that the graph of the displacement function $\bar{u}_y$ exactly matches the displaced shape of the member's centroidal axis due to the end displacements u_2 and u_4. As this displaced shape defines the actual displacements in the y direction of all points along the member's length, we can conclude that the function $\bar{u}_y$, as given by Eq. (3.36), is exact.

To demonstrate that Eq. (3.35) describes the actual displacements in the x direction of all points along the truss member's centroidal axis, consider again the member subjected to end displacements, u_1 and u_3, in the x direction as shown in Fig. 3.7(a). Since the member is subjected to forces only at its ends, the axial force, Q_a, is constant throughout the member's length. Thus, the axial stress, σ, at point G of the member is

$$\sigma = \frac{Q_a}{A} \tag{3.37}$$

in which A represents the cross-sectional area of the member at point G. Note that the axial stress is distributed uniformly over the cross-sectional area A. By substituting the linear stress–strain relationship $\varepsilon = \sigma/E$ into Eq. (3.37), we obtain the strain at point G as

$$\varepsilon = \frac{Q_{\mathrm{a}}}{EA} = \text{constant} = a_1 \tag{3.38}$$

in which a_1 is a constant. As this equation indicates, since the member is prismatic (i.e., $EA =$ constant), the axial strain is constant throughout the member length.

To relate the strain ε to the displacement $\bar{u}_x$, we focus our attention on the differential element GH of length dx (Fig. 3.7(a)). The undeformed and deformed positions of the element are shown in Fig. 3.7(b), in which $\bar{u}_x$ and $\bar{u}_x + d\bar{u}_x$ denote, respectively, the displacements of the ends G and H of the element in the x direction. From this figure, we can see that

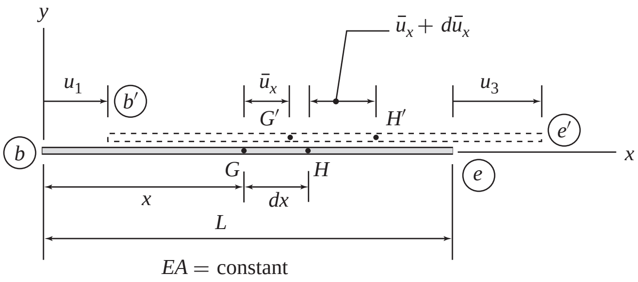
$$\begin{aligned}\text{deformed length of element} &= dx + (\bar{u}_x + d\bar{u}_x) - \bar{u}_x \\ &= dx + d\bar{u}_x\end{aligned}$$

Therefore, the strain in the element is given by

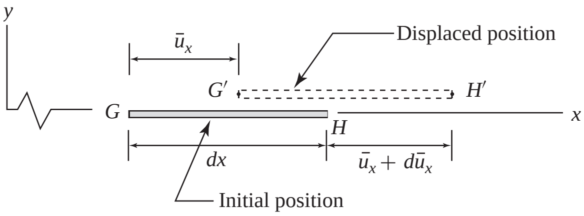
$$\varepsilon = \frac{\text{deformed length} - \text{initial length}}{\text{initial length}} = \frac{(dx + d\bar{u}_x) - dx}{dx}$$

or

$$\boxed{\varepsilon = \frac{d\bar{u}_x}{dx}} \tag{3.39}$$



(a)



(b)

Fig. 3.7

By equating the two expressions for strain as given by Eqs. (3.38) and (3.39), we obtain

$$\frac{d\bar{u}_x}{dx} = a_1 \tag{3.40}$$

which can be rewritten as

$$d\bar{u}_x = a_1 dx \tag{3.41}$$

By integrating Eq. (3.41), we obtain

$$\bar{u}_x = a_1 x + a_0 \tag{3.42}$$

in which a_0 is the constant of integration. Note that Eq. (3.42), obtained by integrating the actual strain–displacement relationship, indicates that the linear polynomial form assumed for $\bar{u}_x$ in Eq. (3.32) was indeed correct. Furthermore, if we evaluate the two constants in Eq. (3.42) by applying the boundary conditions, we obtain an equation which is identical to Eq. (3.35), indicating that our assumed displacement function $\bar{u}_x$ (as given by Eq. (3.35)) does indeed describe the actual member displacements in the x direction.

Shape Functions

The displacement functions, as given by Eqs. (3.35) and (3.36), can be expressed alternatively as

$$\bar{u}_x = N_1 u_1 + N_3 u_3 \tag{3.43a}$$

$$\bar{u}_y = N_2 u_2 + N_4 u_4 \tag{3.43b}$$

with

$$N_1 = N_2 = 1 - \frac{x}{L} \tag{3.44a}$$

$$N_3 = N_4 = \frac{x}{L} \tag{3.44b}$$

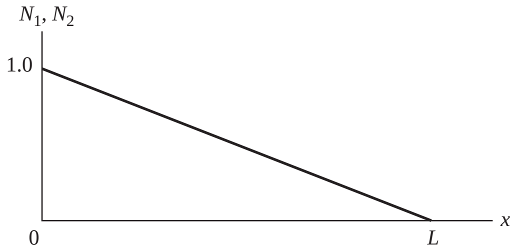
in which N_i (with $i = 1,4$) are called the *shape functions*. The plots of the four shape functions for a plane truss member are given in Fig. 3.8. We can see from this figure that *a shape function N_i describes the displacement variation along a member's centroidal axis due to a unit value of the end displacement u_i, while all other end displacements are zero.*

Equations (3.43) can be written in matrix form as

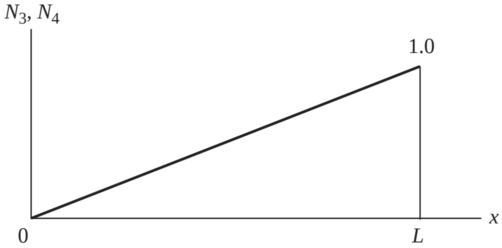
$$\begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.45}$$

or, symbolically, as

$$\boxed{\bar{\mathbf{u}} = \mathbf{N}\mathbf{u}} \tag{3.46}$$



(a) Shape Functions N_1 ($u_1 = 1$, $u_2 = u_3 = u_4 = 0$) and N_2 ($u_2 = 1$, $u_1 = u_3 = u_4 = 0$)



(b) Shape Functions N_3 ($u_3 = 1$, $u_1 = u_2 = u_4 = 0$) and N_4 ($u_4 = 1$, $u_1 = u_2 = u_3 = 0$)

Fig. 3.8 *Shape Functions for Plane Truss Member*

in which $\bar{\mathbf{u}}$ is the *member displacement function vector,* and $\mathbf{N}$ is called the *member shape function matrix*.

Strain–Displacement Relationship

As discussed previously, the relationship between the axial strain, ε, and the displacement, $\bar{u}_x$, is given by $\varepsilon = d\bar{u}_x/dx$ (see Eq. (3.39)). This strain–displacement relationship can be expressed in matrix form as

$$\varepsilon = \begin{bmatrix} \dfrac{d}{dx} & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} = \mathbf{D}\bar{\mathbf{u}} \tag{3.47}$$

in which $\mathbf{D}$ is known as the *differential operator matrix*. To relate the strain, ε, to the member end displacements, $\mathbf{u}$, we substitute Eq. (3.46) into Eq. (3.47) to obtain

$$\varepsilon = \mathbf{D}(\mathbf{N}\mathbf{u}) \tag{3.48}$$

Since the end displacement vector $\mathbf{u}$ does not depend on x, it can be treated as a constant in the differentiation indicated by Eq. (3.48). In other words, the differentiation applies to $\mathbf{N}$, but not to $\mathbf{u}$. Thus, Eq. (3.48) can be rewritten as

$$\boxed{\varepsilon = (\mathbf{D}\mathbf{N})\mathbf{u} = \mathbf{B}\mathbf{u}} \tag{3.49}$$

in which, $\mathbf{B} = \mathbf{DN}$ is called the *member strain–displacement matrix*. To determine $\mathbf{B}$, we write

$$\mathbf{B} = \mathbf{DN} = \begin{bmatrix} \dfrac{d}{dx} & 0 \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix}$$

By multiplying matrices $\mathbf{D}$ and $\mathbf{N}$,

$$\mathbf{B} = \begin{bmatrix} \dfrac{dN_1}{dx} & 0 & \dfrac{dN_3}{dx} & 0 \end{bmatrix}$$

Next, we substitute the expressions for N_1 and N_3 from Eqs. (3.44) into the preceding equation to obtain

$$\mathbf{B} = \begin{bmatrix} \dfrac{d}{dx}\left(1 - \dfrac{x}{L}\right) & 0 & \dfrac{d}{dx}\left(\dfrac{x}{L}\right) & 0 \end{bmatrix}$$

Finally, by performing the necessary differentiations, we determine the strain–displacement matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} -\dfrac{1}{L} & 0 & \dfrac{1}{L} & 0 \end{bmatrix} = \frac{1}{L}\,[-1 \quad 0 \quad 1 \quad 0] \qquad \textbf{(3.50)}$$

Stress–Displacement Relationship

The relationship between member axial stress and member end displacements can now be established by substituting Eq. (3.49) into the stress–strain relationship, $\sigma = E\varepsilon$. Thus,

$$\boxed{\sigma = E\mathbf{Bu}} \qquad \textbf{(3.51)}$$

Member Stiffness Matrix, k

With both member strain and stress expressed in terms of end displacements, we can now establish the relationship between member end forces $\mathbf{Q}$ and end displacements $\mathbf{u}$, by applying the principle of virtual work for deformable bodies. Consider an arbitrary member of a plane truss that is in equilibrium under the action of end forces Q_1 through Q_4. Assume that the member is given small virtual end displacements δu_1 through δu_4, as shown in Fig. 3.9. The virtual

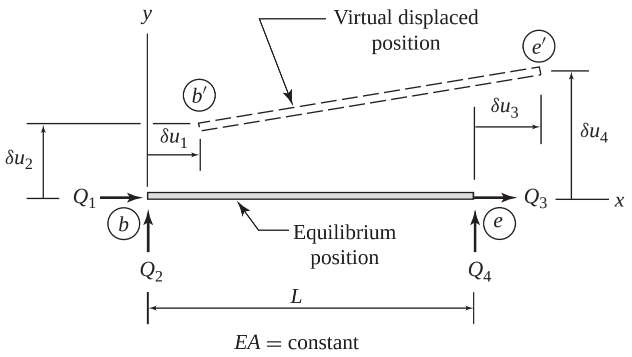


Fig. 3.9

external work done by the real member end forces Q_1 through Q_4 as they move through the corresponding virtual end displacements δu_1 through δu_4 is

$$\delta W_{\mathrm{e}} = Q_1\,\delta u_1 + Q_2\,\delta u_2 + Q_3\,\delta u_3 + Q_4\,\delta u_4$$

which can be expressed in matrix form as

$$\delta W_{\mathrm{e}} = [\,\delta u_1 \quad \delta u_2 \quad \delta u_3 \quad \delta u_4\,]\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

or

$$\delta W_{\mathrm{e}} = \delta\mathbf{u}^T\mathbf{Q} \tag{3.52}$$

By substituting Eq. (3.52) into the expression for the principle of virtual work for deformable bodies, as given by Eq. (3.28), we obtain

$$\delta\mathbf{u}^T\mathbf{Q} = \int_V \delta\varepsilon^T\sigma\,dV \tag{3.53}$$

Recall that the right-hand side of Eq. (3.53) represents the virtual strain energy stored in the member. Substitution of Eqs. (3.49) and (3.51) into Eq. (3.53) yields

$$\delta\mathbf{u}^T\mathbf{Q} = \int_V (\mathbf{B}\,\delta\mathbf{u})^T E\mathbf{B}\,dV\,\mathbf{u}$$

Since $(\mathbf{B}\,\delta\mathbf{u})^T = \delta\mathbf{u}^T\mathbf{B}^T$, we can write the preceding equation as

$$\delta\mathbf{u}^T\mathbf{Q} = \delta\mathbf{u}^T\int_V \mathbf{B}^T E\mathbf{B}\,dV\,\mathbf{u}$$

or

$$\delta\mathbf{u}^T\left(\mathbf{Q} - \int_V \mathbf{B}^T E\mathbf{B}\,dV\,\mathbf{u}\right) = 0$$

Since $\delta\mathbf{u}^T$ can be arbitrarily chosen and is not zero, the quantity in the parentheses must be zero. Thus,

$$\mathbf{Q} = \left(\int_V \mathbf{B}^T E\mathbf{B}\,dV\right)\mathbf{u} = \mathbf{k}\mathbf{u} \tag{3.54}$$

in which

$$\boxed{\mathbf{k} = \int_V \mathbf{B}^T E\mathbf{B}\,dV} \tag{3.55}$$

is the member stiffness matrix in the local coordinate system. To determine the explicit form of **k**, we substitute Eq. (3.50) for **B** into Eq. (3.55) to obtain

$$\mathbf{k} = \frac{E}{L^2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}[\,-1 \quad 0 \quad 1 \quad 0\,]\int_V dV = \frac{E}{L^2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\int_V dV$$

Since $\int_V dV = V = AL$, the member stiffness matrix, **k**, becomes

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the preceding expression for **k** is identical to that derived in Section 3.3 (Eq. (3.27)) using the direct equilibrium approach.

Symmetry of the Member Stiffness Matrix

The expression for the stiffness matrix **k** as given by Eq. (3.55) is general, in the sense that the stiffness matrices for members of other types of framed structures, as well as for elements of surface structures and solids, can also be expressed in the integral form of this equation. We can deduce from Eq. (3.55) that *for linear elastic structures, the member stiffness matrices are symmetric*.

Transposing both sides of Eq. (3.55), we write

$$\mathbf{k}^T = \int_V (\mathbf{B}^T\mathbf{E}\mathbf{B})^T dV$$

Now, recall from Section 2.3 that the transpose of a product of matrices equals the product of the transposed matrices in reverse order; that is, $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$. Thus, the preceding equation becomes

$$\mathbf{k}^T = \int_V \mathbf{B}^T\mathbf{E}^T(\mathbf{B}^T)^T dV$$

For linear elastic structures, **E** is either a scalar (in the case of framed structures), or a symmetric matrix (for surface structures and solids). Therefore, $\mathbf{E}^T = \mathbf{E}$. Furthermore, by realizing that $(\mathbf{B}^T)^T = \mathbf{B}$, we can express the preceding equation as

$$\mathbf{k}^T = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,dV \tag{3.56}$$

Finally, a comparison of Eqs. (3.55) and (3.56) yields

$$\boxed{\mathbf{k}^T = \mathbf{k}} \tag{3.57}$$

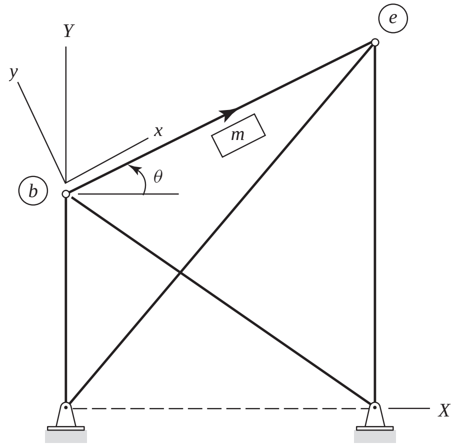
which shows that **k** is a symmetric matrix.

3.5 COORDINATE TRANSFORMATIONS

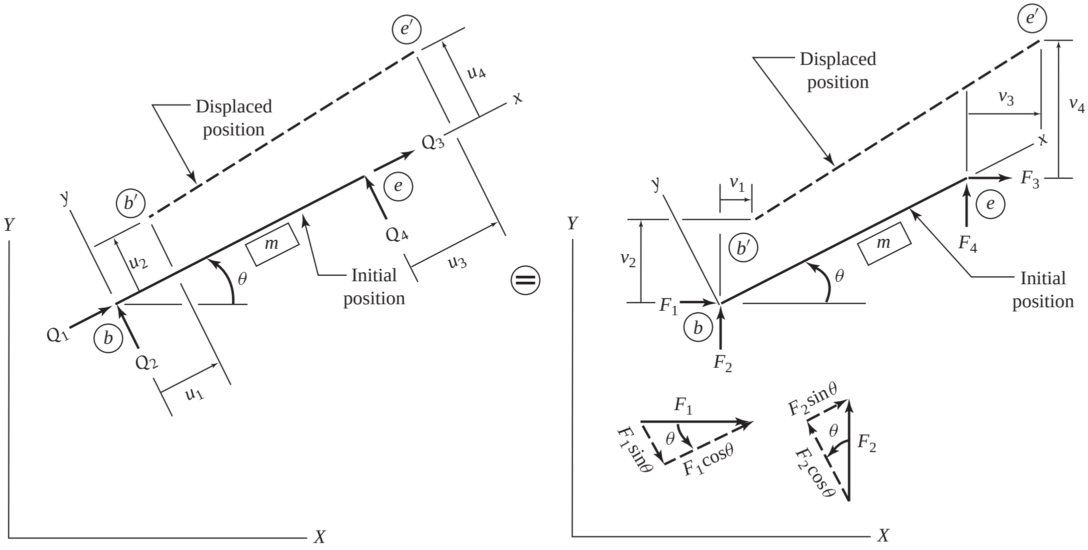
When members of a structure are oriented in different directions, it becomes necessary to transform the stiffness relations for each member from its local coordinate system to a single global coordinate system selected for the entire structure. The member stiffness relations as expressed in the global coordinate system are then combined to establish the stiffness relations for the whole structure. In this section, we consider the transformation of member end forces and end displacements from local to global coordinate systems, and vice versa, for members of plane trusses. The transformation of the stiffness matrices is discussed in Section 3.6.

Transformation from Global to Local Coordinate Systems

Consider an arbitrary member *m* of a plane truss (Fig. 3.10(a)). As shown in this figure, the orientation of *m* relative to the global *XY* coordinate system is defined by an angle θ, measured counterclockwise from the positive direction of the global *X* axis to the positive direction of the local *x* axis. Recall that the stiffness matrix **k** derived in the preceding sections relates member end forces **Q** and end displacement **u** described with reference to the local *xy* coordinate system of the member, as shown in Fig. 3.10(b).



(a) Truss



(b) Member End Forces and End Displacements in the Local Coordinate System

(c) Member End Forces and End Displacements in the Global Coordinate System

Fig. 3.10

Now, suppose that the member end forces and end displacements are specified with reference to the global *XY* coordinate system (Fig. 3.10(c)), and we wish to determine the equivalent system of end forces and end displacements, in the local *xy* coordinates, which has the same effect on *m*. As indicated in Fig. 3.10(c), the member end forces in the global coordinate system are denoted by F_1 through F_4, and the corresponding end displacements are denoted by v_1 through v_4. These global member end forces and end displacements are numbered beginning at member end *b*, with the force and translation in the *X* direction numbered first, followed by the force and translation in the *Y* direction. The forces and displacements at the member's opposite end *e* are then numbered in the same sequential order.

By comparing Figs. 3.10(b) and (c), we observe that at end *b* of *m*, the local force Q_1 must be equal to the algebraic sum of the components of the global forces F_1 and F_2 in the direction of the local *x* axis; that is,

$$Q_1 = F_1 \cos\theta + F_2 \sin\theta \tag{3.58a}$$

Similarly, the local force Q_2 equals the algebraic sum of the components of F_1 and F_2 in the direction of the local *y* axis. Thus,

$$Q_2 = -F_1 \sin\theta + F_2 \cos\theta \tag{3.58b}$$

By using a similar reasoning at end *e*, we express the local forces in terms of the global forces as

$$Q_3 = F_3 \cos\theta + F_4 \sin\theta \tag{3.58c}$$

$$Q_4 = -F_3 \sin\theta + F_4 \cos\theta \tag{3.58d}$$

Equations 3.58(a) through (d) can be written in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \tag{3.59}$$

or, symbolically, as

$$\mathbf{Q} = \mathbf{T}\mathbf{F} \tag{3.60}$$

with

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{3.61}$$

in which **T** is referred to as the *transformation matrix*. The direction cosines of the member, necessary for the evaluation of **T**, can be conveniently determined by using the following relationships:

$$\cos\theta = \frac{X_e - X_b}{L} = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}} \tag{3.62a}$$

$$\sin\theta = \frac{Y_e - Y_b}{L} = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}} \tag{3.62b}$$

in which X_b and Y_b denote the global coordinates of the beginning joint b for the member, and X_e and Y_e represent the global coordinates of the end joint e.

The member end displacements, like end forces, are vectors, which are defined in the same directions as the corresponding forces. Therefore, the transformation matrix **T** (Eq. (3.61)), developed for transforming end forces, can also be used to transform member end displacements from the global to local coordinate system; that is,

$$\boxed{\mathbf{u} = \mathbf{T}\mathbf{v}} \tag{3.63}$$

Transformation from Local to Global Coordinate Systems

Next, let us consider the transformation of member end forces and end displacements from local to global coordinate systems. A comparison of Figs. 3.10(b) and (c) indicates that at end b of m, the global force F_1 must be equal to the algebraic sum of the components of the local forces Q_1 and Q_2 in the direction of the global X axis; that is,

$$F_1 = Q_1 \cos\theta - Q_2 \sin\theta \tag{3.64a}$$

In a similar manner, the global force F_2 equals the algebraic sum of the components of Q_1 and Q_2 in the direction of the global Y axis. Thus,

$$F_2 = Q_1 \sin\theta + Q_2 \cos\theta \tag{3.64b}$$

By using a similar reasoning at end e, we express the global forces in terms of the local forces as

$$F_3 = Q_3 \cos\theta - Q_4 \sin\theta \tag{3.64c}$$

$$F_4 = Q_3 \sin\theta + Q_4 \cos\theta \tag{3.64d}$$

We can write Eqs. 3.64(a) through (d) in matrix form as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \tag{3.65}$$

By comparing Eqs. (3.59) and (3.65), we observe that the transformation matrix in Eq. (3.65), which transforms the forces from the local to the global coordinate system, is the transpose of the transformation matrix **T** in Eq. (3.59), which transforms the forces from the global to the local coordinate system. Therefore, Eq. (3.65) can be expressed as

$$\boxed{\mathbf{F} = \mathbf{T}^T\,\mathbf{Q}} \tag{3.66}$$

Furthermore, a comparison of Eqs. (3.60) and (3.66) indicates that the inverse of the transformation matrix must be equal to its transpose; that is,

$$\mathbf{T}^{-1} = \mathbf{T}^T \tag{3.67}$$

which indicates that the transformation matrix **T** is orthogonal.

As discussed previously, because the member end displacements are also vectors, which are defined in the same directions as the corresponding forces, the matrix $\mathbf{T}^T$ also defines the transformation of member end displacements from the local to the global coordinate system; that is,

$$\mathbf{v} = \mathbf{T}^T\mathbf{u} \tag{3.68}$$

EXAMPLE 3.4 Determine the transformation matrices for the members of the truss shown in Fig. 3.11.

SOLUTION **Member 1** From Fig. 3.11, we can see that joint 1 is the beginning joint and joint 2 is the end joint for member 1. By applying Eqs. (3.62), we determine

$$\cos\theta = \frac{X_2 - X_1}{L} = \frac{6-0}{6} = 1$$

$$\sin\theta = \frac{Y_2 - Y_1}{L} = \frac{0-0}{6} = 0$$

The transformation matrix for member 1 can now be obtained by using Eq. (3.61)

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$ **Ans**

As the preceding result indicates, for any member with the positive directions of its local x and y axes oriented in the positive directions of the global X and Y axes, respectively, the transformation matrix always equals a unit matrix, $\mathbf{I}$.

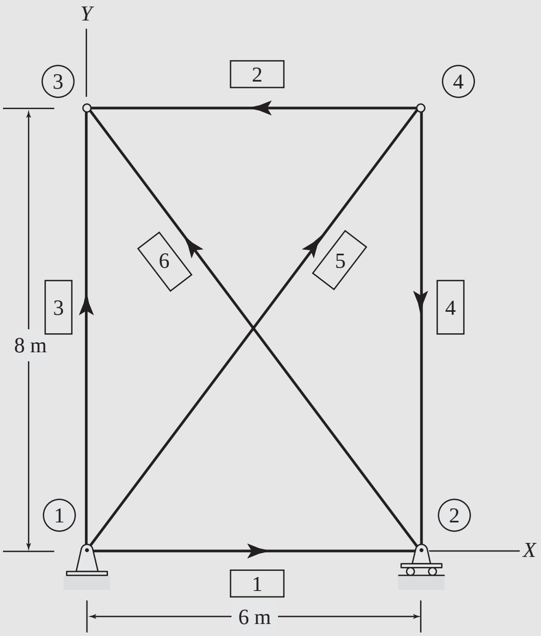


Fig. 3.11

Member 2

$$\cos\theta = \frac{X_3 - X_4}{L} = \frac{0-6}{6} = -1$$

$$\sin\theta = \frac{Y_3 - Y_4}{L} = \frac{8-8}{6} = 0$$

Thus, from Eq. (3.61)

$$\mathbf{T}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ **Ans**

Member 3

$$\cos\theta = \frac{X_3 - X_1}{L} = \frac{0-0}{8} = 0$$

$$\sin\theta = \frac{Y_3 - Y_1}{L} = \frac{8-0}{8} = 1$$

Thus,

$$\mathbf{T}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$ **Ans**

Member 4

$$\cos\theta = \frac{X_2 - X_4}{L} = \frac{6-6}{8} = 0$$

$$\sin\theta = \frac{Y_2 - Y_4}{L} = \frac{0-8}{8} = -1$$

Thus,

$$\mathbf{T}_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ **Ans**

Member 5

$$L = \sqrt{(X_4 - X_1)^2 + (Y_4 - Y_1)^2} = \sqrt{(6-0)^2 + (8-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{6-0}{10} = 0.6$$

$$\sin\theta = \frac{Y_4 - Y_1}{L} = \frac{8-0}{10} = 0.8$$

$$\mathbf{T}_5 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$ **Ans**

Member 6

$$L = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2} = \sqrt{(0-6)^2 + (8-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_3 - X_2}{L} = \frac{0-6}{10} = -0.6$$

$$\sin\theta = \frac{Y_3 - Y_2}{L} = \frac{8-0}{10} = 0.8$$

$$\mathbf{T}_6 = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix}$$

Ans

EXAMPLE 3.5

For the truss shown in Fig. 3.12(a), the end displacements of member 2 in the global coordinate system are (Fig. 3.12(b)):

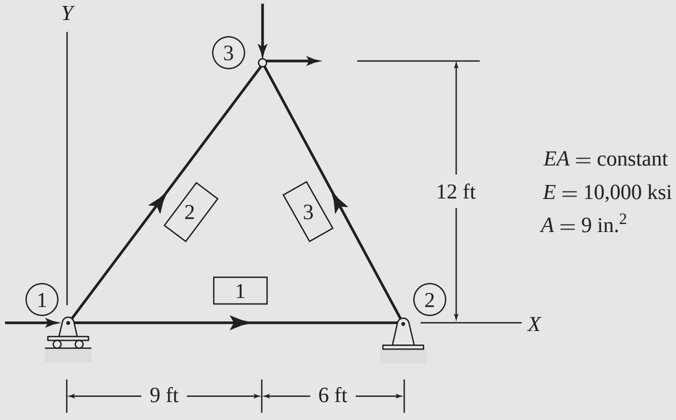
$$\mathbf{v}_2 = \begin{bmatrix} 0.75 \\ 0 \\ 1.5 \\ -2 \end{bmatrix} \text{ in.}$$

Calculate the end forces for this member in the global coordinate system. Is the member in equilibrium under these forces?

SOLUTION

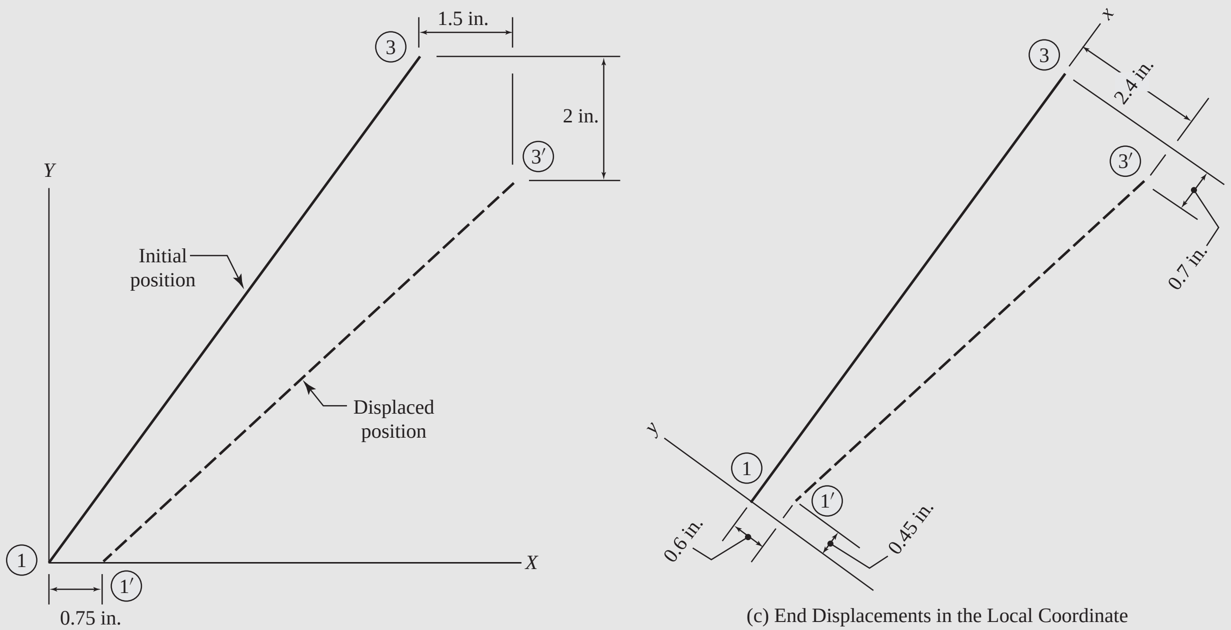
Member Stiffness Matrix in the Local Coordinate System: $E = 10{,}000$ ksi, $A = 9$ in.^{2}, $L = \sqrt{(9)^2 + (12)^2} = 15$ ft $= 180$ in.

$$\frac{EA}{L} = \frac{10{,}000(9)}{180} = 500 \text{ k/in.}$$



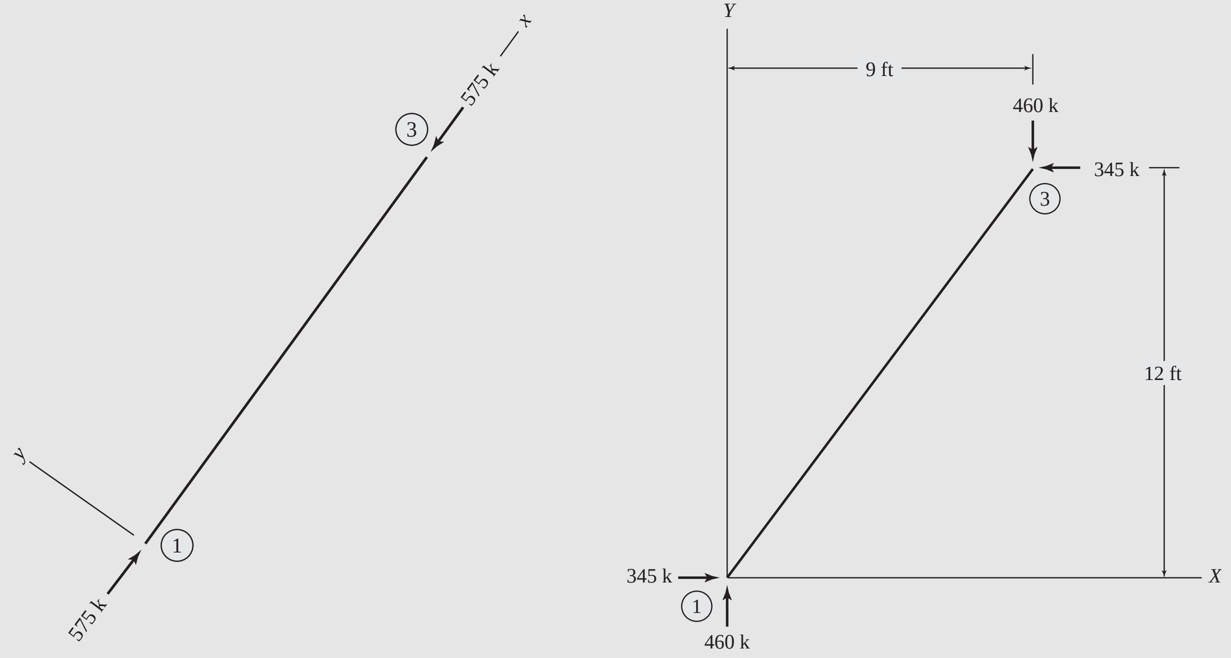
(a) Truss

Fig. 3.12



(b) End Displacements in the Global Coordinate System, **v**, for Member 2

(c) End Displacements in the Local Coordinate System, **u**, for Member 2



(d) End Forces in the Local Coordinate System, **Q**, for Member 2

(e) End Forces in the Global Coordinate System, **F**, for Member 2

Fig. 3.12 (*continued*)

Thus, from Eq. (3.27),

$$\mathbf{k}_2 = 500\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.}$$

Transformation Matrix: From Fig. 3.12(a), we can see that joint 1 is the beginning joint and joint 3 is the end joint for member 2. By applying Eqs. (3.62), we determine

$$\cos\theta = \frac{X_3 - X_1}{L} = \frac{9-0}{15} = 0.6$$

$$\sin\theta = \frac{Y_3 - Y_1}{L} = \frac{12-0}{15} = 0.8$$

The transformation matrix for member 2 can now be evaluated by using Eq. (3.61):

$$\mathbf{T}_2 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

Member End Displacements in the Local Coordinate System: To determine the member global end forces, first we calculate member end displacements in the local coordinate system by using the relationship $\mathbf{u} = \mathbf{Tv}$ (Eq. (3.63)). Thus,

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}\begin{bmatrix} 0.75 \\ 0 \\ 1.5 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.45 \\ -0.6 \\ -0.7 \\ -2.4 \end{bmatrix} \text{ in.}$$

These end displacements are depicted in Fig. 3.12(c).

Member End Forces in the Local Coordinate System: Next, by using the expression $\mathbf{Q} = \mathbf{ku}$ (Eq. (3.7)), we compute the member local end forces as

$$\mathbf{Q}_2 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = 500\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.45 \\ -0.6 \\ -0.7 \\ -2.4 \end{bmatrix} = \begin{bmatrix} 575 \\ 0 \\ -575 \\ 0 \end{bmatrix} \text{ k}$$

Note that, as shown in Fig. 3.12(d), the member is in compression with an axial force of magnitude 575 k.

Member End Forces in the Global Coordinate System: Finally, we determine the desired member end forces by applying the relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ as given in Eq. (3.66). Thus,

$$\mathbf{F}_2 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix}\begin{bmatrix} 575 \\ 0 \\ -575 \\ 0 \end{bmatrix} = \begin{bmatrix} 345 \\ 460 \\ -345 \\ -460 \end{bmatrix} \text{ k} \qquad \textbf{Ans}$$

The member end forces in the global coordinate system are shown in Fig. 3.12(e).

Equilibrium Check: To check whether or not the member is in equilibrium, we apply the three equations of equilibrium, as follows.

$+\rightarrow \sum F_X = 0 \qquad 345 - 345 = 0$ **Checks**

$+\uparrow \sum F_Y = 0 \qquad 460 - 460 = 0$ **Checks**

$+\circlearrowleft \sum M_{①} = 0 \qquad 345(12) - 460(9) = 0$ **Checks**

Therefore, the member is in equilibrium. **Ans**

3.6 MEMBER STIFFNESS RELATIONS IN THE GLOBAL COORDINATE SYSTEM

By using the member stiffness relations in the local coordinate system from Sections 3.3 and 3.4, and the transformation relations from Section 3.5, we can now establish the stiffness relations for members in the global coordinate system.

First, we substitute the local stiffness relations $\mathbf{Q} = \mathbf{ku}$ (Eq. (3.7)) into the force transformation relations $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (3.66)) to obtain

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q} = \mathbf{T}^T\mathbf{ku} \tag{3.69}$$

Then, by substituting the displacement transformation relations $\mathbf{u} = \mathbf{Tv}$ (Eq. (3.63)) into Eq. (3.69), we determine that the desired relationship between the member end forces $\mathbf{F}$ and end displacements $\mathbf{v}$, in the global coordinate system, is

$$\mathbf{F} = \mathbf{T}^T\mathbf{kTv} \tag{3.70}$$

Equation (3.70) can be conveniently expressed as

$$\boxed{\mathbf{F} = \mathbf{Kv}} \tag{3.71}$$

with

$$\boxed{\mathbf{K} = \mathbf{T}^T\mathbf{kT}} \tag{3.72}$$

in which the matrix $\mathbf{K}$ is called the *member stiffness matrix in the global coordinate system*. The explicit form of $\mathbf{K}$ can be determined by substituting Eqs. (3.27) and (3.61) into Eq. (3.72), as

$$\mathbf{K} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

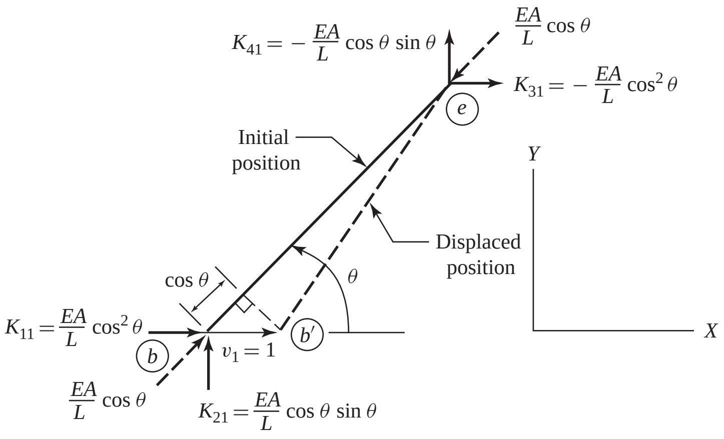
Performing the matrix multiplications, we obtain

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \tag{3.73}$$

Note that, like the member local stiffness matrix, *the member global stiffness matrix,* **K**, *is symmetric*. The physical interpretation of the member global stiffness matrix **K** is similar to that of the member local stiffness matrix; that is, *a stiffness coefficient K_{ij} represents the force at the location and in the direction of F_i required, along with other end forces, to cause a unit value of displacement v_j, while all other end displacements are zero*. Thus, the *j*th column of matrix **K** consists of the end forces in the global coordinate system required to cause a unit value of the end displacement v_j, while all other end displacements are zero.

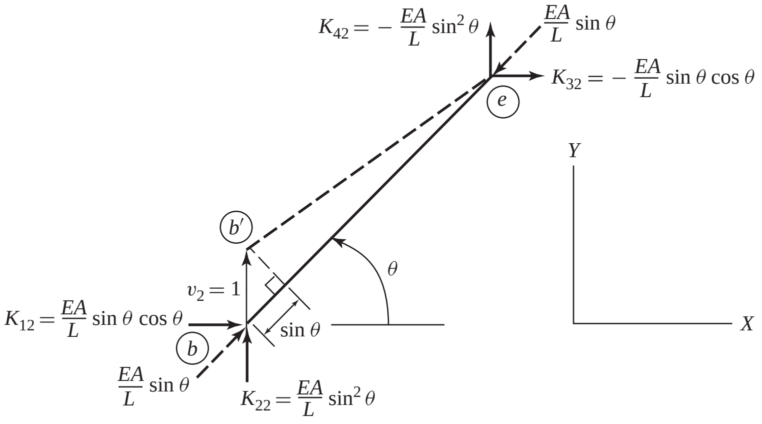
As the preceding interpretation indicates, the member global stiffness matrix **K** can alternately be derived by subjecting an inclined truss member, separately, to unit values of each of the four end displacements in the global coordinate system as shown in Fig. 3.13, and by evaluating the end forces in the global coordinate system required to cause the individual unit displacements. Let us verify the expression for **K** given in Eq. (3.73), using this alternative approach. Consider a prismatic plane truss member inclined at an angle θ relative to the global *X* axis, as shown in Fig. 3.13(a). When end *b* of the member is given a unit displacement $v_1 = 1$, while the other end displacements are held at zero, the member shortens and an axial compressive force develops in it. In the case of small displacements (as assumed herein), the axial deformation u_a of the member due to v_1 is equal to the component of $v_1 = 1$ in the undeformed direction of the member; that is (Fig. 3.13(a)),

$$u_a = v_1\cos\theta = 1\cos\theta = \cos\theta$$

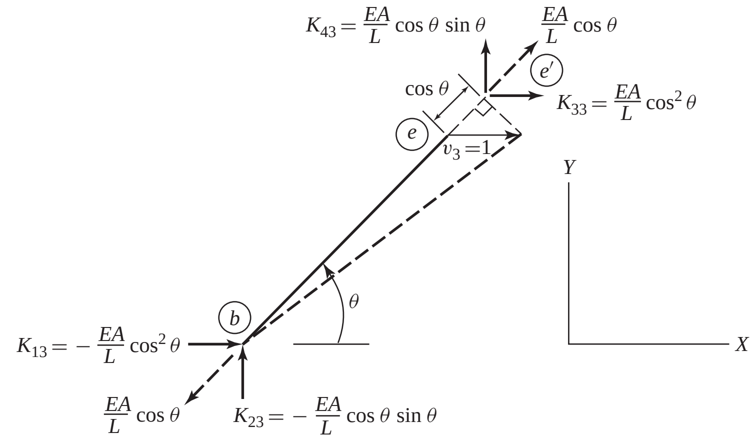


(a) First Column of **K** ($v_1 = 1$, $v_2 = v_3 = v_4 = 0$)

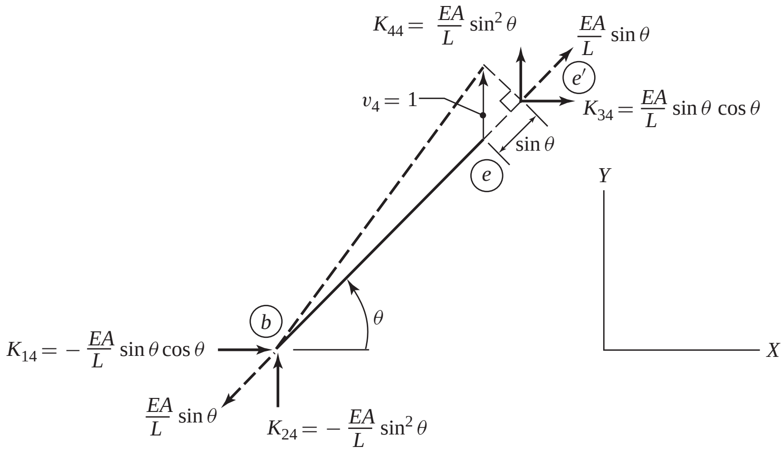
Fig. 3.13



(b) Second Column of **K** ($v_2 = 1$, $v_1 = v_3 = v_4 = 0$)



(c) Third Column of **K** ($v_3 = 1$, $v_1 = v_2 = v_4 = 0$)



(d) Fourth Column of **K** ($v_4 = 1$, $v_1 = v_2 = v_3 = 0$)

Fig. 3.13 (*continued*)

The axial compressive force Q_a in the member caused by the axial deformation u_a can be expressed as

$$Q_a = \left(\frac{EA}{L}\right) u_a = \left(\frac{EA}{L}\right) \cos\theta$$

From Fig. 3.13(a), we can see that the stiffness coefficients must be equal to the components of the member axial force Q_a in the directions of the global X and Y axes. Thus, at end b,

$$K_{11} = Q_a \cos\theta = \left(\frac{EA}{L}\right) \cos^2\theta \qquad \textbf{(3.74a)}$$

$$K_{21} = Q_a \sin\theta = \left(\frac{EA}{L}\right) \cos\theta \sin\theta \qquad \textbf{(3.74b)}$$

Similarly, at end e,

$$K_{31} = -Q_a \cos\theta = -\left(\frac{EA}{L}\right) \cos^2\theta \qquad \textbf{(3.74c)}$$

$$K_{41} = -Q_a \sin\theta = -\left(\frac{EA}{L}\right) \cos\theta \sin\theta \qquad \textbf{(3.74d)}$$

in which the negative signs for K_{31} and K_{41} indicate that these forces act in the negative directions of the X and Y axes, respectively. Note that the member must be in equilibrium under the action of the four end forces, K_{11}, K_{21}, K_{31}, and K_{41}. Also, note that the expressions for these stiffness coefficients (Eqs. (3.74)) are identical to those given in the first column of the **K** matrix in Eq. (3.73).

The stiffness coefficients corresponding to the unit values of the remaining end displacements v_2, v_3, and v_4 can be evaluated in a similar manner, and are given in Figs. 3.13 (b) through (d), respectively. As expected, these stiffness coefficients are the same as those previously obtained by transforming the stiffness relations from the local to the global coordinate system (Eq. (3.73)).

EXAMPLE 3.6

Solve Example 3.5 by using the member stiffness relationship in the global coordinate system, $\mathbf{F} = \mathbf{K}\mathbf{v}$.

SOLUTION *Member Stiffness Matrix in the Global Coordinate System:* It was shown in Example 3.5 that for member 2,

$$\frac{EA}{L} = 500 \text{ k/in.}, \cos\theta = 0.6, \sin\theta = 0.8$$

Thus, from Eq. (3.73):

$$\mathbf{K}_2 = \begin{bmatrix} 180 & 240 & -180 & -240 \\ 240 & 320 & -240 & -320 \\ -180 & -240 & 180 & 240 \\ -240 & -320 & 240 & 320 \end{bmatrix} \text{k/in.}$$

Member End Forces in the Global Coordinate System: By applying the relationship $\mathbf{F} = \mathbf{K}\mathbf{v}$ as given in Eq. (3.71), we obtain

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 180 & 240 & -180 & -240 \\ 240 & 320 & -240 & -320 \\ -180 & -240 & 180 & 240 \\ -240 & -320 & 240 & 320 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0 \\ 1.5 \\ -2 \end{bmatrix} = \begin{bmatrix} 345 \\ 460 \\ -345 \\ -460 \end{bmatrix} \text{k}$$

Ans

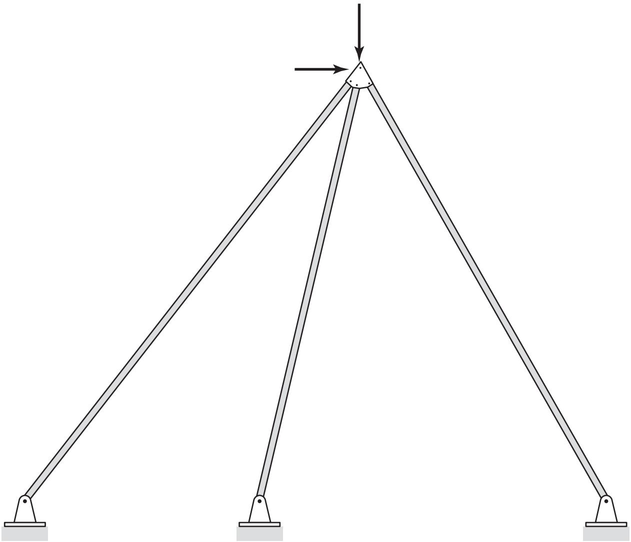
Equilibrium check: See Example 3.5.

3.7 STRUCTURE STIFFNESS RELATIONS

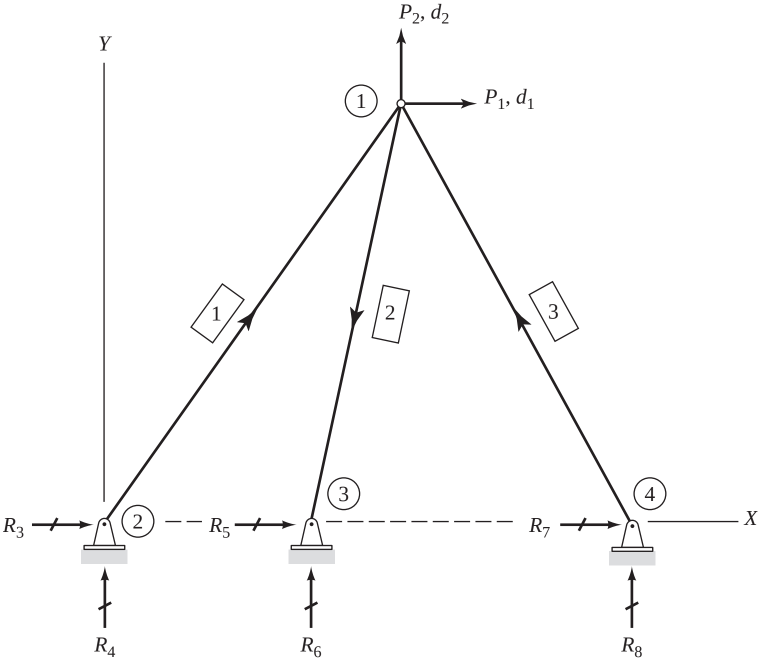
Having determined the member force–displacement relationships in the global coordinate system, we are now ready to establish the stiffness relations for the entire structure. The structure stiffness relations express the external loads **P** acting at the joints of the structure, as functions of the joint displacements **d**. Such relationships can be established as follows:

1. The joint loads **P** are first expressed in terms of the member end forces in the global coordinate system, **F**, by applying the equations of equilibrium for the joints of the structure.
2. The joint displacements **d** are then related to the member end displacements in the global coordinate system, **v**, by using the compatibility conditions that the displacements of the member ends must be the same as the corresponding joint displacements.
3. Next, the compatibility equations are substituted into the member force–displacement relations, $\mathbf{F} = \mathbf{K}\mathbf{v}$, to express the member global end forces **F** in terms of the joint displacements **d**. The **F**–**d** relations thus obtained are then substituted into the joint equilibrium equations to establish the desired structure stiffness relationships between the joint loads **P** and the joint displacements **d**.

Consider, for example, an arbitrary plane truss as shown in Fig. 3.14(a). The analytical model of the truss is given in Fig. 3.14(b), which indicates that the structure has two degrees of freedom, d_1 and d_2. The joint loads corresponding to these degrees of freedom are designated P_1 and P_2, respectively. The global end forces **F** and end displacements **v** for the three members of the truss are shown in Fig. 3.14(c), in which the superscript (i) denotes the member number. Note that for members 1 and 3, the bottom joints (i.e., joints 2 and 4, respectively) have been defined as the beginning joints; whereas, for member 2, the top joint 1 is the beginning joint. As stated previously, our objective is to express the joint loads **P** as functions of the joint displacement **d**.

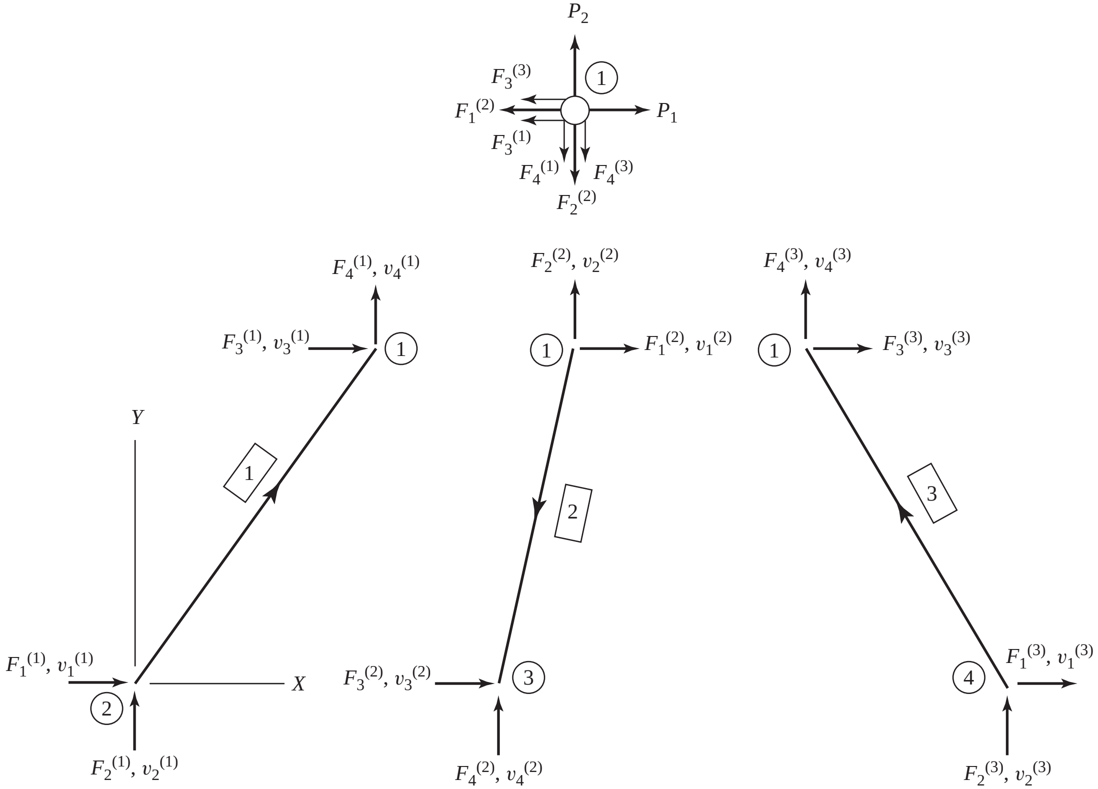


(a) Truss



(b) Analytical Model

Fig. 3.14



(c)

Fig. 3.14 (*continued*)

Equilibrium Equations

To relate the external joint loads **P** to the internal member end forces **F**, we apply the two equations of equilibrium, $\sum F_X = 0$ and $\sum F_Y = 0$, to the free body of joint 1 shown in Fig. 3.14(c). This yields the equilibrium equations,

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)} \tag{3.75a}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)} \tag{3.75b}$$

Compatibility Equations

By comparing Figs. 3.14(b) and (c), we observe that since the lower end 2 of member 1 is connected to the hinged support 2, which cannot translate in any direction, the two displacements of end 2 of the member must be zero. Similarly, since end 1 of this member is connected to joint 1, the displacements of end 1 must be the same as the displacements of joint 1. Thus, the compatibility conditions for member 1 are

$$v_1^{(1)} = v_2^{(1)} = 0 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2 \tag{3.76}$$

In a similar manner, the compatibility conditions for members 2 and 3, respectively, are found to be

$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = v_4^{(2)} = 0 \tag{3.77}$$

$$v_1^{(3)} = v_2^{(3)} = 0 \qquad v_3^{(3)} = d_1 \qquad v_4^{(3)} = d_2 \tag{3.78}$$

Member Stiffness Relations

Of the two types of relationships established thus far, the equilibrium equations (Eqs. (3.75)) express joint loads in terms of member end forces, whereas the compatibility equations (Eqs. (3.76) through (3.78)) relate joint displacements to member end displacements. Now, we will link the two types of relationships by employing the member stiffness relationship in the global coordinate system derived in the preceding section.

We can write the member global stiffness relation $\mathbf{F} = \mathbf{K}\mathbf{v}$ (Eq. (3.71)) in expanded form for member 1 as

$$\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{bmatrix} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \\ v_4^{(1)} \end{bmatrix} \tag{3.79}$$

from which we obtain the expressions for forces at end 1 of the member:

$$F_3^{(1)} = K_{31}^{(1)} v_1^{(1)} + K_{32}^{(1)} v_2^{(1)} + K_{33}^{(1)} v_3^{(1)} + K_{34}^{(1)} v_4^{(1)} \tag{3.80a}$$

$$F_4^{(1)} = K_{41}^{(1)} v_1^{(1)} + K_{42}^{(1)} v_2^{(1)} + K_{43}^{(1)} v_3^{(1)} + K_{44}^{(1)} v_4^{(1)} \tag{3.80b}$$

In a similar manner, we write the stiffness relations for member 2 as

$$\begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{bmatrix} = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \\ v_4^{(2)} \end{bmatrix} \tag{3.81}$$

from which we obtain the forces at end 1 of the member:

$$F_1^{(2)} = K_{11}^{(2)} v_1^{(2)} + K_{12}^{(2)} v_2^{(2)} + K_{13}^{(2)} v_3^{(2)} + K_{14}^{(2)} v_4^{(2)} \tag{3.82a}$$

$$F_2^{(2)} = K_{21}^{(2)} v_1^{(2)} + K_{22}^{(2)} v_2^{(2)} + K_{23}^{(2)} v_3^{(2)} + K_{24}^{(2)} v_4^{(2)} \tag{3.82b}$$

Similarly, for member 3, the stiffness relations are written as

$$\begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{bmatrix} = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \begin{bmatrix} v_1^{(3)} \\ v_2^{(3)} \\ v_3^{(3)} \\ v_4^{(3)} \end{bmatrix} \tag{3.83}$$

and the forces at end 1 of the member are given by

$$F_3^{(3)} = K_{31}^{(3)} v_1^{(3)} + K_{32}^{(3)} v_2^{(3)} + K_{33}^{(3)} v_3^{(3)} + K_{34}^{(3)} v_4^{(3)} \tag{3.84a}$$

$$F_4^{(3)} = K_{41}^{(3)} v_1^{(3)} + K_{42}^{(3)} v_2^{(3)} + K_{43}^{(3)} v_3^{(3)} + K_{44}^{(3)} v_4^{(3)} \tag{3.84b}$$

Note that Eqs. (3.80), (3.82), and (3.84) express the six member end forces that appear in the joint equilibrium equations (Eqs. (3.75)), in terms of member end displacements.

To relate the joint displacements **d** to the member end forces **F**, we substitute the compatibility equations into the foregoing member force–displacement relations. Thus, by substituting the compatibility equations for member 1 (Eqs. (3.76)) into its force–displacement relations as given by Eqs. (3.80), we express the member end forces $\mathbf{F}^{(1)}$ in terms of the joint displacements **d** as

$$F_3^{(1)} = K_{33}^{(1)} d_1 + K_{34}^{(1)} d_2 \tag{3.85a}$$

$$F_4^{(1)} = K_{43}^{(1)} d_1 + K_{44}^{(1)} d_2 \tag{3.85b}$$

In a similar manner, for member 2, by substituting Eqs. (3.77) into Eqs. (3.82), we obtain

$$F_1^{(2)} = K_{11}^{(2)} d_1 + K_{12}^{(2)} d_2 \tag{3.86a}$$

$$F_2^{(2)} = K_{21}^{(2)} d_1 + K_{22}^{(2)} d_2 \tag{3.86b}$$

Similarly, for member 3, substitution of Eqs. (3.78) into Eqs. (3.84) yields

$$F_3^{(3)} = K_{33}^{(3)} d_1 + K_{34}^{(3)} d_2 \tag{3.87a}$$

$$F_4^{(3)} = K_{43}^{(3)} d_1 + K_{44}^{(3)} d_2 \tag{3.87b}$$

Structure Stiffness Relations

Finally, by substituting Eqs. (3.85) through (3.87) into the joint equilibrium equations (Eqs. (3.75)), we establish the desired relationships between the joint loads **P** and the joint displacements **d** of the truss:

$$P_1 = \left(K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}\right) d_1 + \left(K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)}\right) d_2 \tag{3.88a}$$

$$P_2 = \left(K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)}\right) d_1 + \left(K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}\right) d_2 \tag{3.88b}$$

Equations (3.88) can be conveniently expressed in condensed matrix form as

$$\boxed{\mathbf{P} = \mathbf{S}\mathbf{d}} \tag{3.89}$$

in which

$$\mathbf{S} = \begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix} \tag{3.90}$$

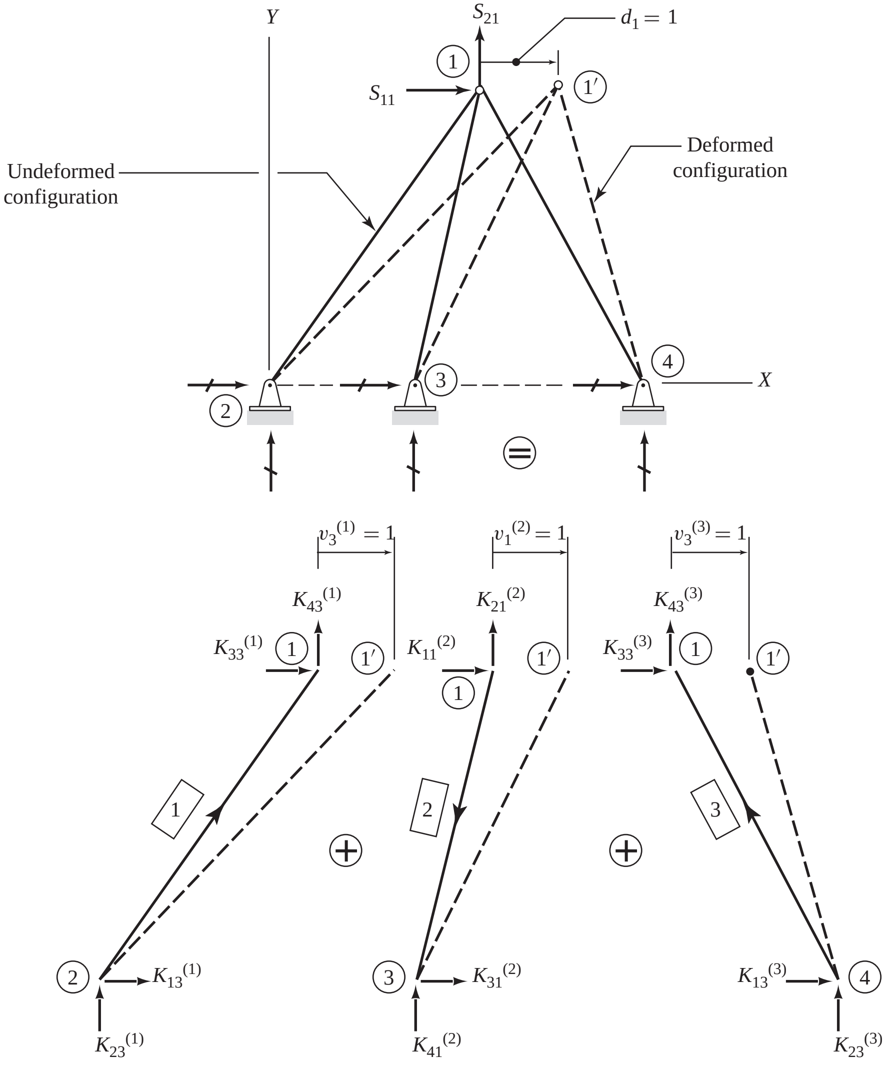
The matrix **S**, which is a square matrix with the number of rows and columns equal to the degrees of freedom (*NDOF*), is called the *structure stiffness matrix*. The preceding method of determining the structure stiffness relationships by combining the member stiffness relations is commonly referred to as the *direct stiffness method* [48].

Like member stiffness matrices, *structure stiffness matrices of linear elastic structures are always symmetric*. Note that in Eq. (3.90) the two off-diagonal elements of **S** are equal to each other, because $K_{34}^{(1)} = K_{43}^{(1)}$, $K_{12}^{(2)} = K_{21}^{(2)}$, and $K_{34}^{(3)} = K_{43}^{(3)}$; thereby making **S** a symmetric matrix.

Physical Interpretation of Structure Stiffness Matrix

The structure stiffness matrix **S** can be interpreted in a manner analogous to the member stiffness matrix. *A structure stiffness coefficient S_{ij} represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are zero.* Thus, the jth column of the structure stiffness matrix **S** consists of the joint loads required, at the locations and in the directions of all the degrees of freedom of the structure, to cause a unit value of the displacement d_j while all other displacements are zero. This interpretation of the structure stiffness matrix indicates that such a matrix can, alternatively, be determined by subjecting the structure, separately, to unit values of each of its joint displacements, and by evaluating the joint loads required to cause the individual displacements.

To illustrate this approach, consider again the three-member truss of Fig. 3.14. To determine its structure stiffness matrix **S**, we subject the truss to the joint displacements $d_1 = 1$ (with $d_2 = 0$), and $d_2 = 1$ (with $d_1 = 0$), as shown in Figs. 3.15(a) and (b), respectively. As depicted in Fig. 3.15(a), the stiffness coefficients S_{11} and S_{21} (elements of the first column of **S**) represent the horizontal and vertical forces at joint 1 required to cause a unit displacement of the joint in the horizontal direction ($d_1 = 1$), while holding it in place vertically ($d_2 = 0$). The unit horizontal displacement of joint 1 induces unit displacements, in the same direction, at the top ends of the three members connected to the joint. The member stiffness coefficients (or end forces) necessary to cause these unit end displacements of the individual members are shown in Fig. 3.15(a). Note that these stiffness coefficients are labeled in accordance with the notation for member end forces adopted in Section 3.5. (Also, recall



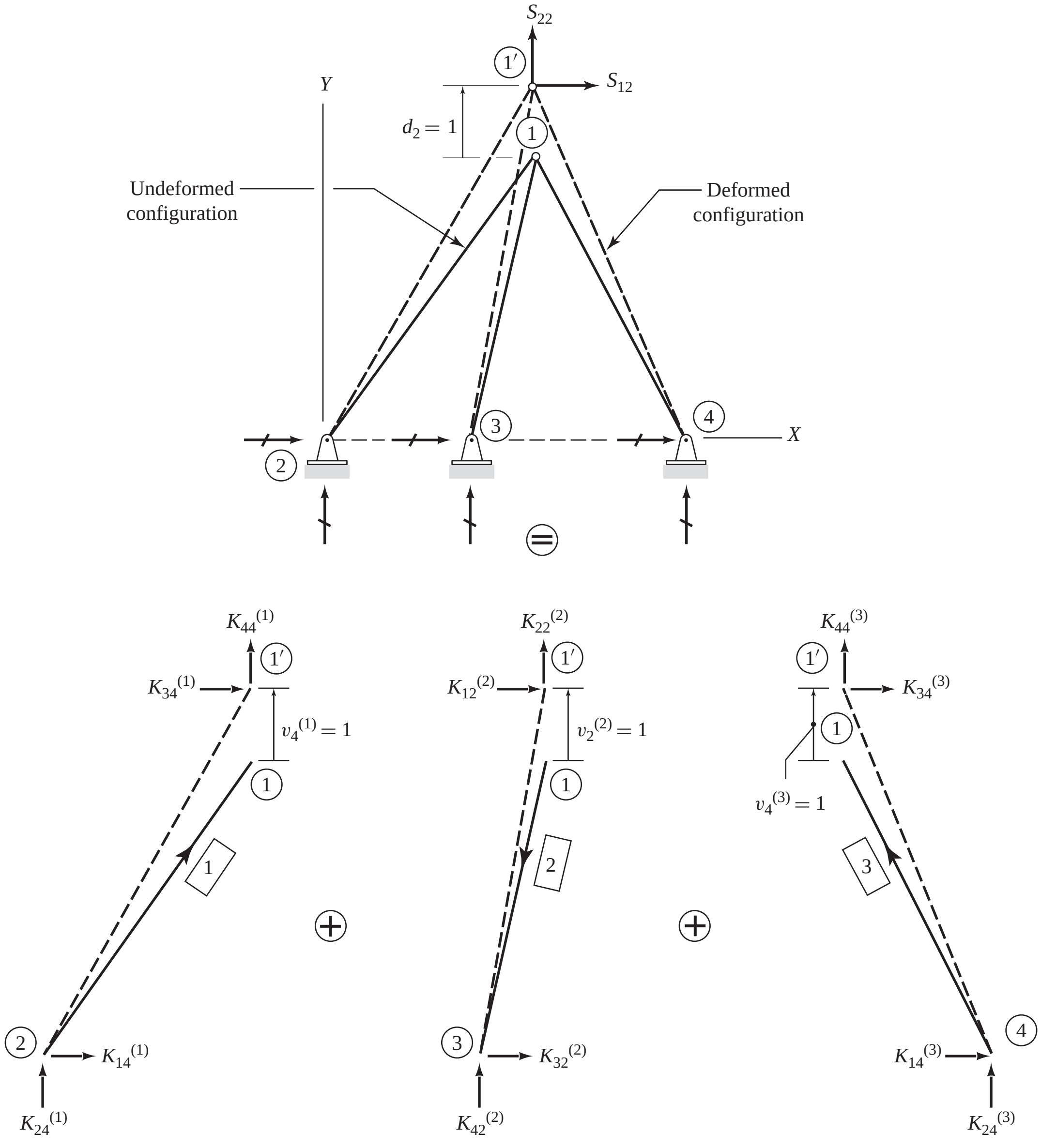
(a) First Column of **S** ($d_1 = 1$, $d_2 = 0$)

Fig. 3.15

that the explicit expressions for member stiffness coefficients, in terms of E, A, L, and θ of a member, were derived in Section 3.6.)

From Fig. 3.15(a), we realize that the total horizontal force S_{11} at joint 1, required to cause the joint displacement $d_1 = 1$ (with $d_2 = 0$), must be equal to the algebraic sum of the horizontal forces at the top ends of the three members connected to the joint; that is,

$$S_{11} = K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} \tag{3.91a}$$



(b) Second Column of **S** ($d_2 = 1$, $d_1 = 0$)

Fig. 3.15 (*continued*)

Similarly, the total vertical force S_{21} at joint 1 must be equal to the algebraic sum of the vertical forces at the top ends of all the members connected to the joint. Thus (Fig. 3.15a),

$$S_{21} = K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} \tag{3.91b}$$

Note that the expressions for S_{11} and S_{21}, as given in Eqs. 3.91(a) and (b), are identical to those listed in the first column of the **S** matrix in Eq. (3.90).

The stiffness coefficients in the second column of the **S** matrix can be determined in a similar manner. As depicted in Fig. 3.15(b), the structure stiffness coefficients S_{12} and S_{22} represent the horizontal and vertical forces at joint 1 required to cause a unit displacement of the joint in the vertical direction (d_2), while holding it in place horizontally ($d_1 = 0$). The joint displacement $d_2 = 1$ induces unit vertical displacements at the top ends of the three members; these, in turn, cause the forces (member stiffness coefficients) to develop at the ends of the members. From Fig. 3.15(b), we can see that the stiffness coefficient S_{12} of joint 1, in the horizontal direction, must be equal to the algebraic sum of the member stiffness coefficients, in the same direction, at the top ends of all the members connected to the joint; that is,

$$S_{12} = K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \tag{3.91c}$$

Similarly, the structure stiffness coefficient S_{22}, in the vertical direction, equals the algebraic sum of the vertical member stiffness coefficients at the top ends of the three members connected to joint 1. Thus (Fig. 3.15b),

$$S_{22} = K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \tag{3.91d}$$

Again, the expressions for S_{12} and S_{22}, as given in Eqs. 3.91(c) and (d), are the same as those listed in the second column of the **S** matrix in Eq. (3.90).

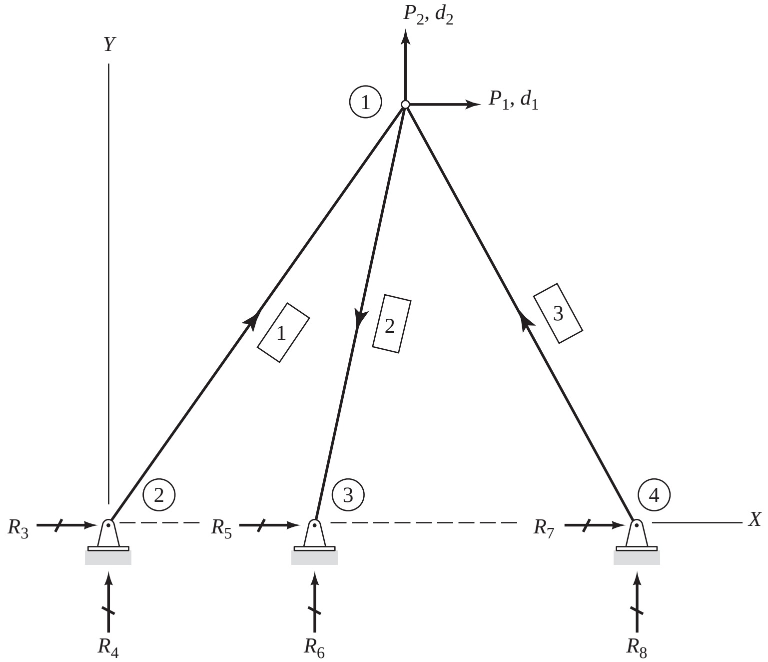
Assembly of the Structure Stiffness Matrix Using Member Code Numbers

In the preceding paragraphs of this section, we have studied two procedures for determining the structure stiffness matrix **S**. Although a study of the foregoing procedures is essential for developing an understanding of the concept of the stiffness of multiple-degrees-of-freedom structures, these procedures cannot be implemented easily on computers and, therefore, are seldom used in practice.

From Eqs. (3.91), we observe that the structure stiffness coefficient of a joint in a direction equals the algebraic sum of the member stiffness coefficients, in that direction, at all the member ends connected to the joint. This fact indicates that the structure stiffness matrix **S** can be formulated directly by adding the elements of the member stiffness matrices into their proper positions in the structure matrix. This technique of directly forming a structure stiffness matrix by assembling the elements of the member global stiffness matrices can be programmed conveniently on computers. The technique was introduced by S. S. Tezcan in 1963 [44], and is sometimes referred to as the *code number technique*.

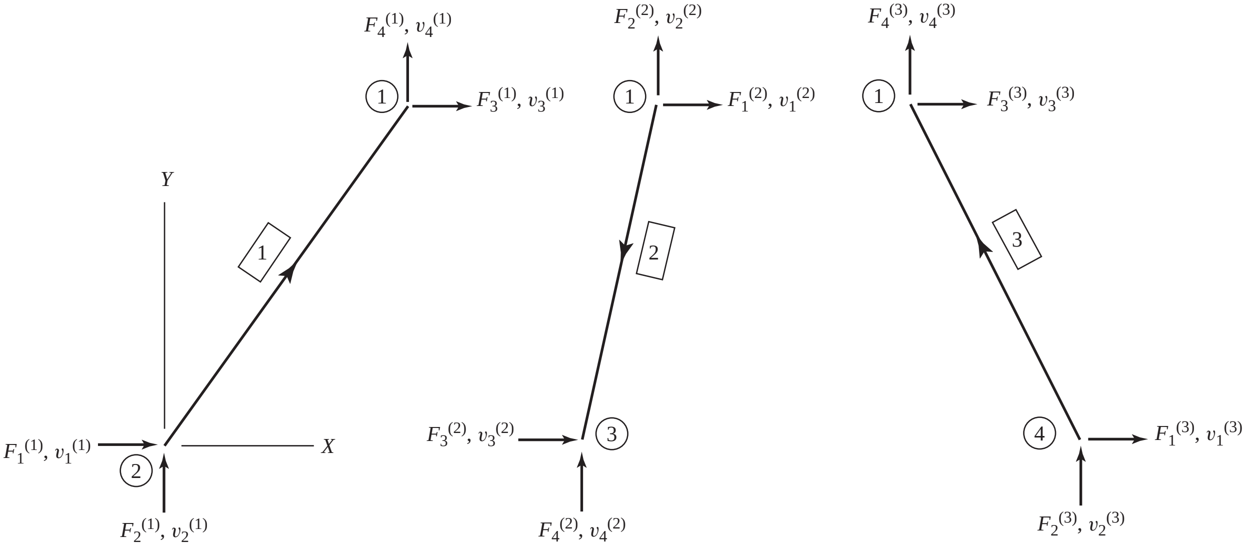
To illustrate this technique, consider again the three-member truss of Fig. 3.14. The analytical model of the truss is redrawn in Fig. 3.16(a), which shows that the structure has two degrees of freedom (numbered 1 and 2), and six restrained coordinates (numbered from 3 to 8). The stiffness matrices in the global coordinate system for members 1, 2, and 3 of the truss are designated $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$, respectively (Fig. 3.16(c)). Our objective is to form the structure stiffness matrix **S** by assembling the elements of $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$.

To determine the positions of the elements of a member matrix **K** in the structure matrix **S**, we identify the number of the structure's degree of freedom or restrained coordinate, at the location and in the direction of each of the



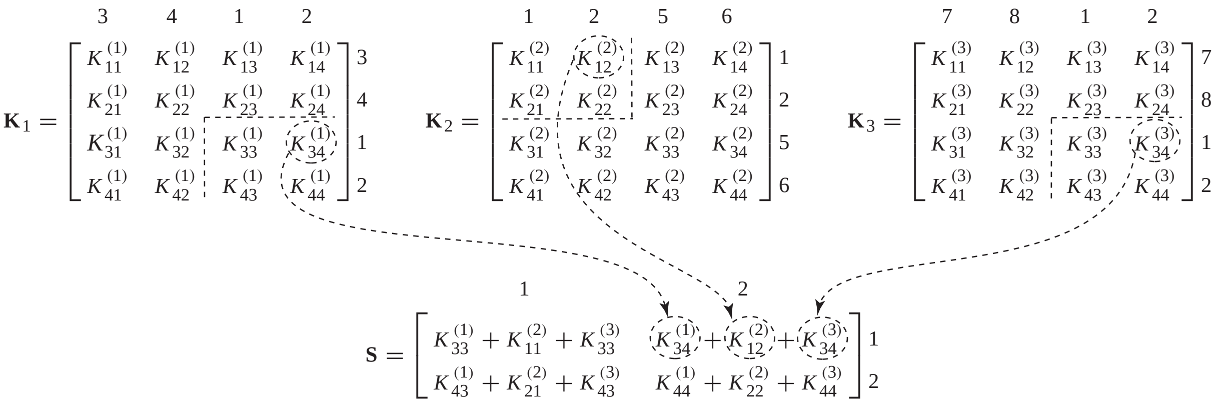
Two degrees of freedom (1 and 2);
six restrained coordinates (3 through 8)

(a) Analytical Model



(b) Member End Forces and End Displacements in
the Global Coordinate System

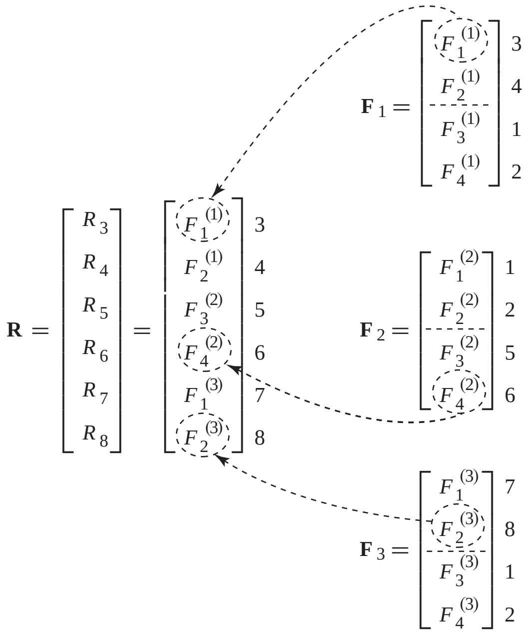
Fig. 3.16



$$\mathbf{K}_1 = \begin{array}{cccc|c} 3 & 4 & 1 & 2 & \\ \hline K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 3 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 4 \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} & 1 \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} & 2 \end{array} \qquad \mathbf{K}_2 = \begin{array}{cccc|c} 1 & 2 & 5 & 6 & \\ \hline K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} & 1 \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} & 2 \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} & 5 \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} & 6 \end{array} \qquad \mathbf{K}_3 = \begin{array}{cccc|c} 7 & 8 & 1 & 2 & \\ \hline K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} & 7 \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} & 8 \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} & 1 \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} & 2 \end{array}$$

$$\mathbf{S} = \begin{array}{cc|c} 1 & 2 & \\ \hline K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} & 1 \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} & 2 \end{array}$$

(c) Assembling of Structure Stiffness Matrix **S**



$$\mathbf{F}_1 = \begin{array}{c|c} F_1^{(1)} & 3 \\ F_2^{(1)} & 4 \\ F_3^{(1)} & 1 \\ F_4^{(1)} & 2 \end{array} \qquad \mathbf{F}_2 = \begin{array}{c|c} F_1^{(2)} & 1 \\ F_2^{(2)} & 2 \\ F_3^{(2)} & 5 \\ F_4^{(2)} & 6 \end{array} \qquad \mathbf{F}_3 = \begin{array}{c|c} F_1^{(3)} & 7 \\ F_2^{(3)} & 8 \\ F_3^{(3)} & 1 \\ F_4^{(3)} & 2 \end{array}$$

$$\mathbf{R} = \begin{bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix} = \begin{array}{c|c} F_1^{(1)} & 3 \\ F_2^{(1)} & 4 \\ F_3^{(2)} & 5 \\ F_4^{(2)} & 6 \\ F_1^{(3)} & 7 \\ F_2^{(3)} & 8 \end{array}$$

(d) Assembly of Support Reaction Vector **R**

Fig. 3.16 (*continued*)

member's global end displacements, **v**. Such structure degrees of freedom and restrained coordinate numbers for a member, when arranged in the same order as the member's end displacements, are referred to as the member's *code numbers*. In accordance with the notation for member end displacements adopted in Section 3.5, the first two end displacements, v_1 and v_2, are always specified in the X and Y directions, respectively, at the beginning of the member; and the last two end displacements, v_3 and v_4, are always in the X and Y directions, respectively, at the end of the member. Therefore, the first two code numbers for a member are always the numbers of the structure degrees of freedom and/or restrained coordinates in the X and Y directions, respectively, at the beginning joint for the member; and the third and fourth member code numbers are always the numbers of the structure degrees of freedom and/or restrained coordinates in the *X* and *Y* directions, respectively, at the end joint for the member.

From Fig. 3.16(a), we can see that for member 1 of the truss, the beginning and the end joints are 2 and 1, respectively. At the beginning joint 2, the restrained coordinate numbers are 3 and 4 in the *X* and *Y* directions, respectively; whereas, at the end joint 1, the structure degree of freedom numbers, in the *X* and *Y* directions, are 1 and 2, respectively. Thus, the code numbers for member 1 are 3, 4, 1, 2. Similarly, since the beginning and end joints for member 2 are 1 and 3, respectively, the code numbers for this member are 1, 2, 5, 6. In a similar manner, the code numbers for member 3 are found to be 7, 8, 1, 2. The code numbers for the three members of the truss can be verified by comparing the member global end displacements shown in Fig. 3.16(b) with the structure degrees of freedom and restrained coordinates given in Fig. 3.16(a).

The code numbers for a member define the compatibility equations for the member. For example, the code numbers 3, 4, 1, 2 imply the following compatibility equations for member 1:

$$v_1^{(1)} = d_3 \qquad v_2^{(1)} = d_4 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2$$

Since the displacements corresponding to the restrained coordinates 3 and 4 are zero (i.e., $d_3 = d_4 = 0$), the compatibility equations for member 1 become

$$v_1^{(1)} = v_2^{(1)} = 0 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2$$

which are identical to those given in Eqs. (3.76).

The member code numbers can also be used to formulate the joint equilibrium equations for a structure (such as those given in Eqs. (3.75)). The equilibrium equation corresponding to an *i*th degree of freedom (or restrained coordinate) can be obtained by equating the joint load P_i (or the reaction R_i) to the algebraic sum of the member end forces, with the code number *i*, of all the members of the structure. For example, to obtain the equilibrium equations for the truss of Fig. 3.16(a), we write the code numbers for its three members by the side of their respective end force vectors, as

$$\mathbf{F}_1 = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} \qquad \mathbf{F}_2 = \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} \qquad \mathbf{F}_3 = \begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix} \tag{3.92}$$

From Eq. (3.92), we can see that the member end forces with the code number 1 are: $F_3^{(1)}$ of member 1, $F_1^{(2)}$ of member 2, and $F_3^{(3)}$ of member 3. Thus, the equilibrium equation corresponding to degree of freedom 1 is given by

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

which is identical to Eq. 3.75(a). Similarly, the equilibrium equation corresponding to degree of freedom 2 can be obtained by equating P_2 to the sum of the end forces, with code number 2, of the three members. Thus, from Eq. (3.92)

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

which is the same as Eq. (3.75(b)).

To establish the structure stiffness matrix **S**, we write the code numbers of each member on the right side and at the top of its stiffness matrix **K**, as shown in Fig. 3.16(c). These code numbers now define the positions of the elements of the member stiffness matrices in the structure stiffness matrix **S**. In other words, the code numbers on the right side of a matrix **K** represent the row numbers of the **S** matrix, and the code numbers at the top represent the column numbers of **S**. Furthermore, since the number of rows and columns of **S** equal the number of degrees of freedom (*NDOF*) of the structure, only those elements of a **K** matrix with both row and column code numbers less than or equal to *NDOF* belong in **S**. For example, since the truss of Fig. 3.16(a) has two degrees of freedom, only the bottom-right quarters of the member matrices $\mathbf{K}_1$ and $\mathbf{K}_3$, and the top-left quarter of $\mathbf{K}_2$, belong in **S** (see Fig. 3.16(c)).

The structure stiffness matrix **S** is established by algebraically adding the pertinent elements of the **K** matrices of all the members, in their proper positions, in the **S** matrix. For example, to assemble **S** for the truss of Fig. 3.16(a), we start by storing the pertinent elements of $\mathbf{K}_1$ in **S** (see Fig. 3.16(c)). Thus, the element $K_{33}^{(1)}$ is stored in row 1 and column 1 of **S**, the element $K_{43}^{(1)}$ is stored in row 2 and column 1 of **S**, the element $K_{34}^{(1)}$ is stored in row 1 and column 2 of **S** (see Fig. 3.16(c)), and the element $K_{44}^{(1)}$ is stored in row 2 and column 2 of **S**. Note that only those elements of $\mathbf{K}_1$ whose row and column code numbers are either 1 or 2 are stored in **S**. The same procedure is then repeated for members 2 and 3. When two or more member stiffness coefficients are stored in the same position in **S**, then the coefficients must be algebraically added. The completed structure stiffness matrix **S** for the truss is shown in Fig. 3.16(c). Note that this matrix is identical to the one determined previously by substituting the member compatibility equations and stiffness relations into the joint equilibrium equations (Eq. (3.90)).

Once **S** has been determined, the structure stiffness relations, $\mathbf{P} = \mathbf{Sd}$ (Eq. (3.89)), which now represent a system of simultaneous linear algebraic equations, can be solved for the unknown joint displacements **d**. With **d** known, the end displacements **v** for each member can be obtained by applying the compatibility equations defined by its code numbers; then the corresponding end displacements **u** and end forces **Q** and **F** can be computed by using the member's transformation and stiffness relations. Finally, the support reactions **R** can be determined from the member end forces **F**, by considering the equilibrium of the support joints in the directions of the restrained coordinates, as discussed in the following paragraphs.

Assembly of the Support Reaction Vector Using Member Code Numbers

The support reactions **R** of a structure can be expressed in terms of the member global end forces **F**, using the equilibrium requirement that the reaction in a direction at a joint must be equal to the algebraic sum of all the forces, in that direction, at all the member ends connected to the joint. Because the code numbers of a member specify the locations and directions of its global end forces with respect to the structure's degrees of freedom and/or restrained coordinates, the reaction corresponding to a restrained coordinate can be evaluated by algebraically summing those elements of the **F** vectors of all the members whose code numbers are the same as the restrained coordinate.

As the foregoing discussion suggests, the reaction vector **R** can be assembled from the member end force vectors **F**, using a procedure similar to that for forming the structure stiffness matrix. To determine the reactions, we write the restrained coordinate numbers ($NDOF + 1$ through $2(NJ)$) on the right side of vector **R**, as shown in Fig. 3.16(d). Next, the code numbers of each member are written on the right side of its end force vector **F** (Fig. 3.16(d)). Any member code number that is greater than the number of degrees of freedom of the structure ($NDOF$) now represents the restrained coordinate number of the row of **R** in which the corresponding member force is to be stored. The reaction vector **R** is obtained by algebraically adding the pertinent elements of the **F** vectors of all the members in their proper positions in **R**.

For example, to assemble **R** for the truss of Fig. 3.16(a), we begin by storing the pertinent elements of $\mathbf{F}_1$ in **R**. Thus, as shown in Fig. 3.16(d), the element $F_1^{(1)}$ with code number 3 is stored in row 1 of **R**, which has the restrained coordinate number 3 by its side. Similarly, the element $F_2^{(1)}$ (with code number 4) is stored in row 2 (with restrained coordinate number 4) of **R**. Note that only those elements of $\mathbf{F}_1$ whose code numbers are greater than 2 ($= NDOF$) are stored in **R**. The same procedure is then repeated for members 2 and 3. The completed support reaction vector **R** for the truss is shown in Fig. 3.16(d).

EXAMPLE 3.7

Determine the structure stiffness matrix for the truss shown in Fig. 3.17(a).

SOLUTION

Analytical Model: The analytical model of the truss is shown in Fig. 3.17(b). The structure has three degrees of freedom—the translation in the X direction of joint 1, and the translations in the X and Y directions of joint 4. These degrees of freedom are identified by numbers 1 through 3; and the five restrained coordinates of the truss are identified by numbers 4 through 8, as shown in Fig. 3.17(b).

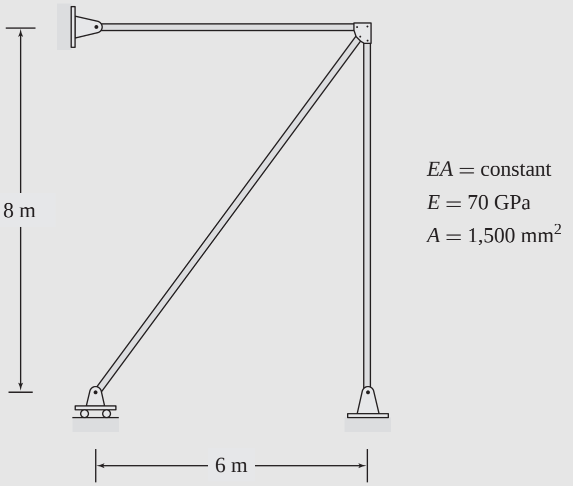
Structure Stiffness Matrix: To generate the 3×3 structure stiffness matrix **S**, we will determine, for each member, the global stiffness matrix **K** and store its pertinent elements in their proper positions in **S** by using the member's code numbers.

Member 1 $L = 6$ m, $\cos\theta = 1$, $\sin\theta = 0$

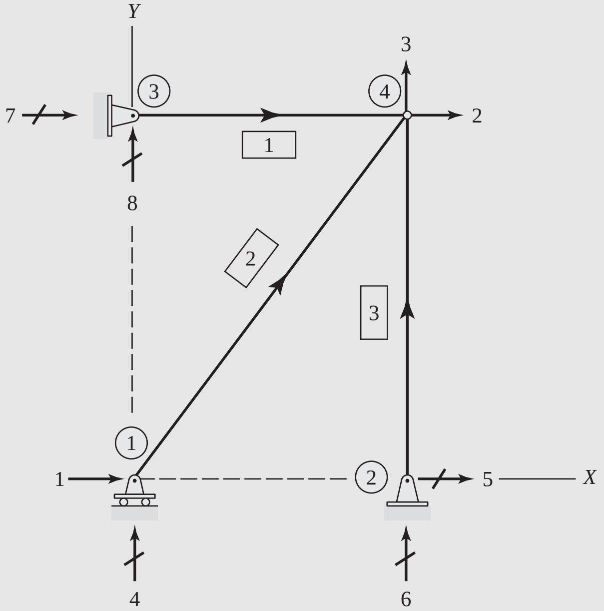
$$\frac{EA}{L} = \frac{70(10^6)\,(0.0015)}{6} = 17{,}500 \text{ kN/m}$$

The member stiffness matrix in global coordinates can now be evaluated by using Eq. (3.73).

$$\mathbf{K}_1 = \begin{array}{cc} \begin{array}{cccc} 7 & 8 & 2 & 3 \end{array} & \\ \left[\begin{array}{cc:cc} 17{,}500 & 0 & -17{,}500 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline -17{,}500 & 0 & 17{,}500 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} 7 \\ 8 \\ 2 \\ 3 \end{array} \end{array} \text{kN/m} \qquad (1)$$



(a) Truss



(b) Analytical Model

Fig. 3.17

From Fig. 3.17(b), we observe that joint 3 has been selected as the beginning joint, and joint 4 as the end joint, for member 1. Thus, the code numbers for this member are 7, 8, 2, 3. These numbers are written on the right side and at the top of $\mathbf{K}_1$ (see Eq. (1)) to indicate the rows and columns, respectively, of the structure stiffness matrix $\mathbf{S}$, where the elements of $\mathbf{K}_1$ must be stored. Note that the elements of $\mathbf{K}_1$ that correspond to the restrained coordinate numbers 7 and 8 are simply disregarded. Thus, the element in row 3 and column 3 of $\mathbf{K}_1$ is stored in row 2 and column 2 of $\mathbf{S}$, as

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 17{,}500 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \qquad \textbf{(2)}$$

Member 2 As shown in Fig. 3.17(b), joint 1 is the beginning joint, and joint 4 is the end joint, for member 2. By applying Eqs. (3.62), we determine

$$L = \sqrt{(X_4 - X_1)^2 + (Y_4 - Y_1)^2} = \sqrt{(6-0)^2 + (8-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{6-0}{10} = 0.6$$

$$\sin\theta = \frac{Y_4 - Y_1}{L} = \frac{8-0}{10} = 0.8$$

$$\frac{EA}{L} = \frac{70(10^6)(0.0015)}{10} = 10{,}500 \text{ kN/m}$$

By using the expression for $\mathbf{K}$ given in Eq. (3.73), we obtain

$$\mathbf{K}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 4 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 3{,}780 & 5{,}040 & -3{,}780 & -5{,}040 \\ 5{,}040 & 6{,}720 & -5{,}040 & -6{,}720 \\ -3{,}780 & -5{,}040 & 3{,}780 & 5{,}040 \\ -5{,}040 & -6{,}720 & 5{,}040 & 6{,}720 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 4 \\ 2 \\ 3 \end{array} \text{ kN/m} \qquad \textbf{(3)}$$

From Fig. 3.17(b), we can see that the code numbers for this member are 1, 4, 2, 3. These numbers are used to add the pertinent elements of $\mathbf{K}_2$ in their proper positions in $\mathbf{S}$, as given in Eq. (2). Thus, $\mathbf{S}$ now becomes

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3{,}780 & -3{,}780 & -5{,}040 \\ -3{,}780 & 17{,}500 + 3{,}780 & 5{,}040 \\ -5{,}040 & 5{,}040 & 6{,}720 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \qquad \textbf{(4)}$$

Member 3 $L = 8$ m, $\cos\theta = 0$, $\sin\theta = 1$

$$\frac{EA}{L} = \frac{70(10^6)(0.0015)}{8} = 13{,}125 \text{ kN/m}$$

By using Eq. (3.73),

$$\mathbf{K}_3 = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 13{,}125 & 0 & -13{,}125 \\ 0 & 0 & 0 & 0 \\ 0 & -13{,}125 & 0 & 13{,}125 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 2 \\ 3 \end{array} \text{ kN/m} \qquad \textbf{(5)}$$

The code numbers for this member are 5, 6, 2, 3. By using these code numbers, the pertinent elements of $\mathbf{K}_3$ are added in $\mathbf{S}$ (as given in Eq. (4)), yielding

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3{,}780 & -3{,}780 & -5{,}040 \\ -3{,}780 & 17{,}500 + 3{,}780 & 5{,}040 \\ -5{,}040 & 5{,}040 & 6{,}720 + 13{,}125 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

Since the stiffnesses of all three members of the truss have now been stored in **S**, the structure stiffness matrix for the given truss is

$$\mathbf{S} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} \quad 1 & \quad 2 & \quad 3 \end{array} \\ \left[\begin{array}{rrr} 3{,}780 & -3{,}780 & -5{,}040 \\ -3{,}780 & 21{,}280 & 5{,}040 \\ -5{,}040 & 5{,}040 & 19{,}845 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

Ans

Note that the structure stiffness matrix **S**, obtained by assembling the stiffness coefficients of the three members, is symmetric.

3.8 PROCEDURE FOR ANALYSIS

Based on the discussion presented in the previous sections, the following step-by-step procedure can be developed for the analysis of plane trusses subjected to joint loads.

1. Prepare an analytical model of the truss as follows.
   - **a.** Draw a line diagram of the structure, on which each joint and member is identified by a number.
   - **b.** Establish a global *XY* coordinate system, with the *X* and *Y* axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively. It is usually convenient to locate the origin of the global coordinate system at a lower left joint of the structure, so that the *X* and *Y* coordinates of most of the joints are positive.
   - **c.** For each member, establish a local *xy* coordinate system by selecting one of the joints at its ends as the beginning joint and the other as the end joint. On the structure's line diagram, indicate the positive direction of the local *x* axis for each member by drawing an arrow along the member pointing toward its end joint. For horizontal members, the coordinate transformations can be avoided by selecting the joint at the member's left end as the beginning joint.
   - **d.** Identify the degrees of freedom (or joint displacements) and the restrained coordinates of the structure. These quantities are specified on the line diagram by assigning numbers to the arrows drawn at the joints in the *X* and *Y* directions. The degrees of freedom are numbered first, starting at the lowest-numbered joint and proceeding sequentially to the highest. In the case of more than one degree of freedom at a joint, the *X*-displacement is numbered first, followed by the *Y*-displacement. After all the degrees of freedom have been numbered, the restrained coordinates are numbered, beginning with a number equal to $NDOF + 1$. Starting at the lowest-numbered joint and proceeding sequentially to the highest, all of the restrained coordinates of the structure are numbered. In the case of more than one restrained coordinate at a joint, the *X*-coordinate is numbered first, followed by the *Y*-coordinate.
2. Evaluate the structure stiffness matrix **S**. The number of rows and columns of **S** must be equal to the degrees of freedom (*NDOF*) of the

structure. For each member of the truss, perform the following operations.

- **a.** Calculate its length and direction cosines. (The expressions for cos θ and sin θ are given in Eqs. (3.62).)
- **b.** Compute the member stiffness matrix in the global coordinate system, **K**, using Eq. (3.73).
- **c.** Identify its code numbers, and store the pertinent elements of **K** in their proper positions in **S**, using the procedure described in Section 3.7.

The complete structure stiffness matrix, obtained by assembling the stiffness coefficients of all the members of the truss, must be a symmetric matrix.

3. Form the $NDOF \times 1$ joint load vector **P**.
4. Determine the joint displacements **d**. Substitute **P** and **S** into the structure stiffness relations, $\mathbf{P} = \mathbf{Sd}$ (Eq. (3.89)), and solve the resulting system of simultaneous equations for the unknown joint displacements **d**. To check that the solution of simultaneous equations has been carried out correctly, substitute the numerical values of **d** back into the structure stiffness relations, $\mathbf{P} = \mathbf{Sd}$. If the solution is correct, then the stiffness relations should be satisfied. Note that joint displacements are considered positive when in the positive directions of the global *X* and *Y* axes; similarly, the displacements are negative in the negative directions.
5. Compute member end displacements and end forces, and support reactions. For each member of the truss, do the following.
   - **a.** Obtain member end displacements in the global coordinate system, **v**, from the joint displacements, **d**, using the member's code numbers.
   - **b.** Calculate the member's transformation matrix **T** by using Eq. (3.61), and determine member end displacements in the local coordinate system, **u**, using the transformation relationship $\mathbf{u} = \mathbf{Tv}$ (Eq. (3.63)). For horizontal members with local *x* axis positive to the right (i.e., in the same direction as the global *X* axis), member end displacements in the global and local coordinate systems are the same; that is, $\mathbf{u} = \mathbf{v}$. Member axial deformation, u_a, if desired, can be obtained from the relationship $u_a = u_1 - u_3$, in which u_1 and u_3 are the first and third elements, respectively, of vector **u**. A positive value of u_a indicates shortening (or contraction) of the member in the axial direction, and a negative value indicates elongation.
   - **c.** Determine the member stiffness matrix in the local coordinate system, **k**, using Eq. (3.27); then calculate member end forces in the local coordinate system by using the stiffness relationship $\mathbf{Q} = \mathbf{ku}$ (Eq. (3.7)). The member axial force, Q_a, equals the first element, Q_1, of the vector **Q** (i.e., $Q_a = Q_1$); a positive value of Q_a indicates that the axial force is compressive, and a negative value indicates that the axial force is tensile.
   - **d.** Compute member end forces in the global coordinate system, **F**, by using the transformation relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (3.66)). For horizontal members with the local *x* axis positive to the right, the member

end forces in the local and global coordinate systems are the same; that is, **F** = **Q**.

e. By using member code numbers, store the pertinent elements of **F** in their proper positions in the support reaction vector **R**, as discussed in Section 3.7.

6. To check the calculation of member end forces and support reactions, apply the three equations of equilibrium ($\sum F_X = 0$, $\sum F_Y = 0$, and $\sum M = 0$) to the free body of the entire truss. If the calculations have been carried out correctly, then the equilibrium equations should be satisfied.

Instead of following steps 5c and d, the member end forces can be determined alternatively by first evaluating the global forces **F**, using the global stiffness relationship $\mathbf{F} = \mathbf{K}\mathbf{v}$ (Eq. (3.71)), and then obtaining the local forces **Q** from the transformation relationship $\mathbf{Q} = \mathbf{T}\mathbf{F}$ (Eq. (3.60)).

EXAMPLE 3.8

Determine the joint displacements, member axial forces, and support reactions for the truss shown in Fig. 3.18(a) by the matrix stiffness method.

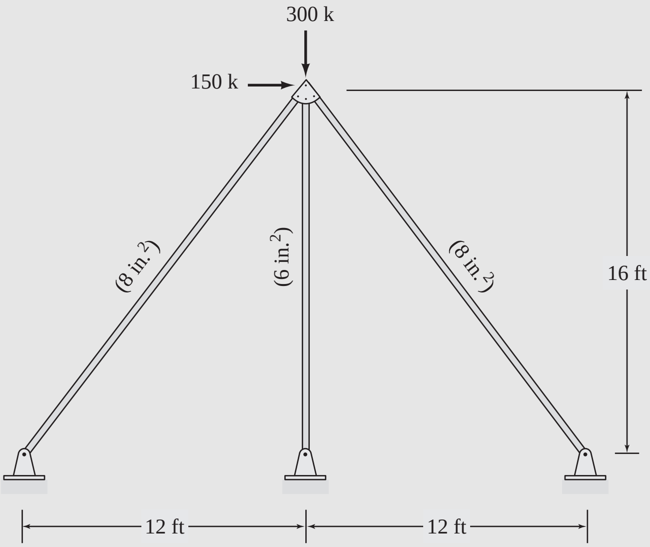
SOLUTION

Analytical Model: The analytical model of the truss is shown in Fig. 3.18(b). The truss has two degrees of freedom, which are the translations of joint 1 in the *X* and *Y* directions. These are numbered as 1 and 2, respectively. The six restrained coordinates of the truss are identified by numbers 3 through 8.

Structure Stiffness Matrix:

Member 1 As shown in Fig. 3.18(b), we have selected joint 2 as the beginning joint, and joint 1 as the end joint, for member 1. By applying Eqs. (3.62), we determine

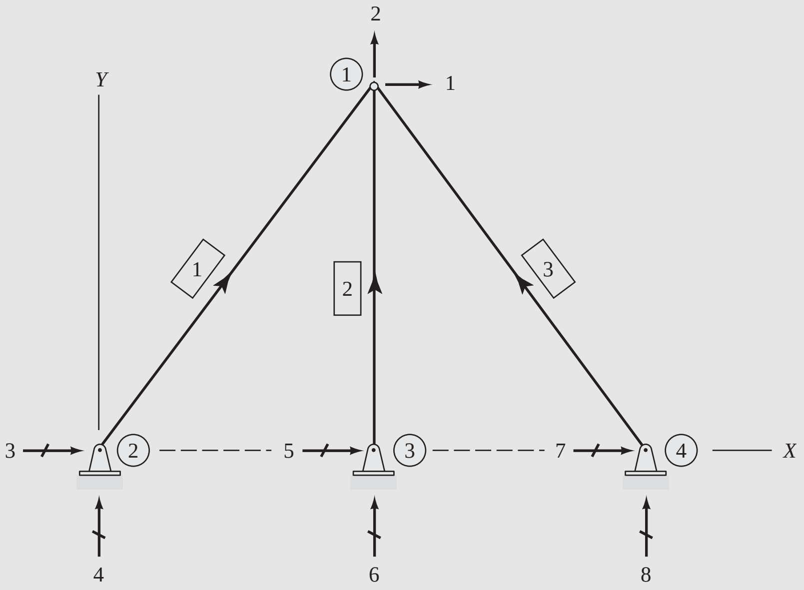
$$L = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2} = \sqrt{(12 - 0)^2 + (16 - 0)^2} = 20 \text{ ft}$$



E = 29,000 ksi

(a) Truss

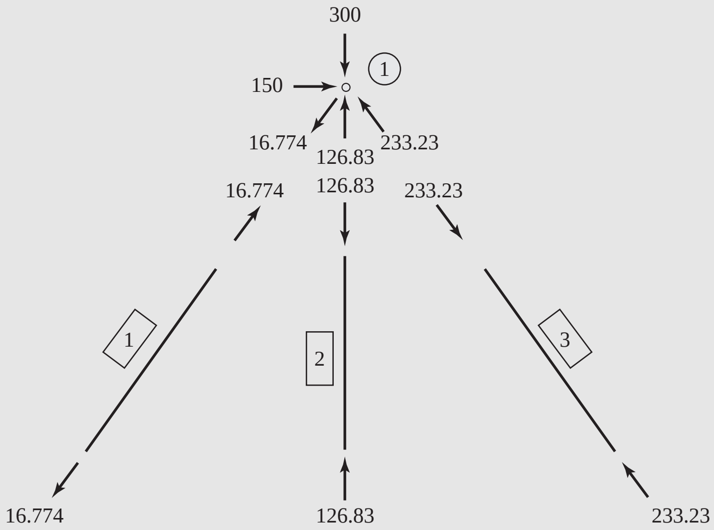
Fig. 3.18



(b) Analytical Model

$$\mathbf{S}=\begin{array}{c}\begin{array}{cc}1 & \qquad\qquad\qquad 2\end{array}\\ \left[\begin{array}{cc}(348+\ 0+348) & (464+\ 0-464)\\ (464+\ 0-464) & (618.67+\ 906.25+618.67)\end{array}\right]\begin{array}{c}1\\2\end{array}\end{array}=\begin{array}{c}\begin{array}{cc}1 & \qquad 2\end{array}\\ \left[\begin{array}{cc}696 & 0\\ 0 & 2{,}143.6\end{array}\right]\begin{array}{c}1\\2\end{array}\end{array}\ \text{k/in.}$$

(c) Structure Stiffness Matrix

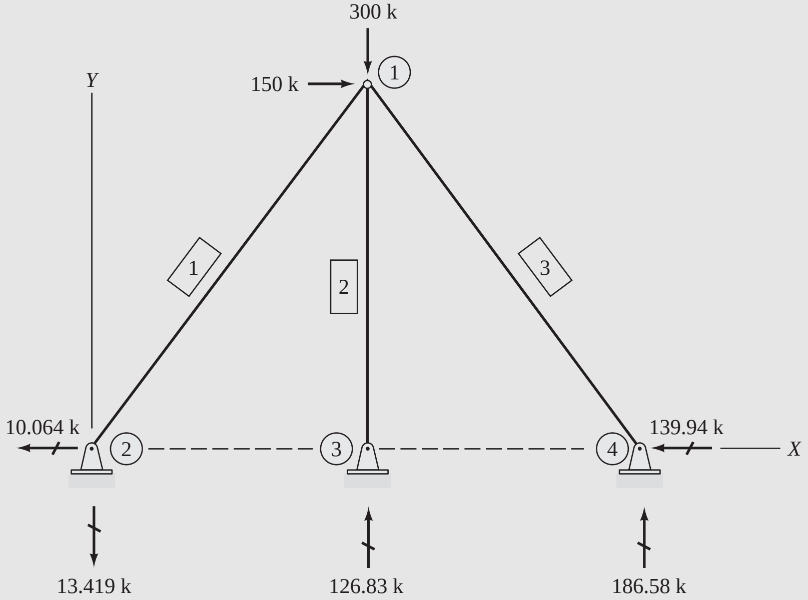


(d) Member End Forces in Local Coordinate Systems

Fig. 3.18 (*continued*)

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} -10.064 \\ -13.419 \\ 0 \\ 126.83 \\ -139.94 \\ 186.58 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \text{ k}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 3.18 (*continued*)

$$\cos\theta = \frac{X_1 - X_2}{L} = \frac{12 - 0}{20} = 0.6$$

$$\sin\theta = \frac{Y_1 - Y_2}{L} = \frac{16 - 0}{20} = 0.8$$

Using the units of kips and inches, we evaluate the member's global stiffness matrix (Eq. (3.73)) as

$$\mathbf{K}_1 = \frac{(29{,}000)(8)}{(20)(12)} \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

or

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \left[\begin{array}{cc|cc} 348 & 464 & -348 & -464 \\ 464 & 618.67 & -464 & -618.67 \\ \hline -348 & -464 & 348 & 464 \\ -464 & -618.67 & 464 & 618.67 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \text{ k/in.} \qquad \textbf{(1)}$$

From Fig. 3.18(b), we observe that the code numbers for member 1 are 3, 4, 1, 2. These numbers are written on the right side and at the top of $\mathbf{K}_1$ (see Eq. (1)) to indicate the rows and columns, respectively, of the structure stiffness matrix **S**, in which the elements of $\mathbf{K}_1$ must be stored. Note that the elements of $\mathbf{K}_1$, which correspond to the restrained coordinate numbers 3 and 4, are simply ignored. Thus, the element in row 3 and column 3 of $\mathbf{K}_1$ is stored in row 1 and column 1 of **S**, as shown in Fig. 3.18(c); and the element in row 4 and column 3 of $\mathbf{K}_1$ is stored in row 2 and column 1 of **S**. The remaining elements of $\mathbf{K}_1$ are stored in **S** in a similar manner, as shown in Fig. 3.18(c).

Member 2 From Fig. 3.18(b), we can see that joint 3 is the beginning joint, and joint 1 is the end joint, for member 2. Applying Eqs. (3.62), we write

$$L = \sqrt{(X_1 - X_3)^2 + (Y_1 - Y_3)^2} = \sqrt{(12-12)^2 + (16-0)^2} = 16 \text{ ft}$$

$$\cos\theta = \frac{X_1 - X_3}{L} = \frac{12-12}{16} = 0$$

$$\sin\theta = \frac{Y_1 - Y_3}{L} = \frac{16-0}{16} = 1$$

Thus, using Eq. (3.73),

$$\mathbf{K}_2 = \begin{array}{cc} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} & \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 906.25 & 0 & -906.25 \\ 0 & 0 & 0 & 0 \\ 0 & -906.25 & 0 & 906.25 \end{array}\right] & \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \end{array} \text{ k/in.}$$

From Fig. 3.18(b), we can see that the code numbers for this member are 5, 6, 1, 2. These numbers are used to store the pertinent elements of $\mathbf{K}_2$ in their proper positions in **S**, as shown in Fig. 3.18(c).

Member 3 It can be seen from Fig. 3.18(b) that joint 4 is the beginning joint, and joint 1 is the end joint, for member 3. Thus,

$$L = \sqrt{(X_1 - X_4)^2 + (Y_1 - Y_4)^2} = \sqrt{(12-24)^2 + (16-0)^2} = 20 \text{ ft}$$

$$\cos\theta = \frac{X_1 - X_4}{L} = \frac{12-24}{20} = -0.6$$

$$\sin\theta = \frac{Y_1 - Y_4}{L} = \frac{16-0}{20} = 0.8$$

Using Eq. (3.73),

$$\mathbf{K}_3 = \begin{array}{cc} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} & \\ \left[\begin{array}{cccc} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{array}\right] & \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \end{array} \text{ k/in.}$$

The code numbers for this member are 7, 8, 1, 2. Using these numbers, the pertinent elements of $\mathbf{K}_3$ are stored in **S**, as shown in Fig. 3.18(c).

The complete structure stiffness matrix **S**, obtained by assembling the stiffness coefficients of the three members of the truss, is given in Fig. 3.18(c). Note that **S** is symmetric.

Joint Load Vector: By comparing Figs. 3.18(a) and (b), we realize that

$$P_1 = 150 \text{ k} \qquad P_2 = -300 \text{ k}$$

Thus, the joint load vector is

$$\mathbf{P} = \begin{bmatrix} 150 \\ -300 \end{bmatrix} \text{ k} \tag{2}$$

Joint Displacements: By substituting **P** and **S** into the structure stiffness relationship given by Eq. (3.89), we write

$$\mathbf{P} = \mathbf{Sd}$$

$$\begin{bmatrix} 150 \\ -300 \end{bmatrix} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Solving these equations, we determine the joint displacements:

$$d_1 = 0.21552 \text{ in.} \qquad d_2 = -0.13995 \text{ in.}$$

or

$$\mathbf{d} = \begin{bmatrix} 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.} \qquad \textbf{Ans}$$

To check that the solution of equations has been carried out correctly, we substitute the numerical values of joint displacements back into the structure stiffness relationship to obtain

$$\mathbf{P} = \mathbf{Sd} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \begin{bmatrix} 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 150 \\ -300 \end{bmatrix} \qquad \textbf{Checks}$$

which is the same as the load vector **P** given in Eq. (2), thereby indicating that the calculated joint displacements do indeed satisfy the structure stiffness relations.

Member End Displacements and End Forces:

Member 1 The member end displacements in the global coordinate system can be obtained simply by comparing the member's global degree of freedom numbers with its code numbers, as follows:

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.} \tag{3}$$

Note that the code numbers for the member (3, 4, 1, 2) are written on the right side of **v**, as shown in Eq. (3). Because the code numbers corresponding to v_1 and v_2 are the restrained coordinate numbers 3 and 4, this indicates that $v_1 = v_2 = 0$. Similarly, the code numbers 1 and 2 corresponding to v_3 and v_4, respectively, indicate that $v_3 = d_1$ and $v_4 = d_2$. It should be clear that these compatibility equations could have been established alternatively by a simple visual inspection of the line diagram of the structure depicted in Fig. 3.18(b). However, as will be shown in Chapter 4, the use of the member code numbers enables us to conveniently program this procedure on a computer.

To determine the member end displacements in the local coordinate system, we first evaluate its transformation matrix **T** as defined in Eq. (3.61):

$$\mathbf{T}_1 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

The member local end displacements can now be calculated, using the relationship $\mathbf{u} = \mathbf{T}\mathbf{v}$ (Eq. (3.63)), as

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.017352 \\ -0.25639 \end{bmatrix} \text{ in.}$$

Before we can calculate the member end forces in the local coordinate system, we need to determine its local stiffness matrix **k**, using Eq. (3.27):

$$\mathbf{k}_1 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.}$$

Now, using Eq. (3.7), we compute the member local end forces as

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

$$\mathbf{Q}_1 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.017352 \\ -0.25639 \end{bmatrix} = \begin{bmatrix} -16.774 \\ 0 \\ 16.774 \\ 0 \end{bmatrix} \text{ k}$$

The member axial force is equal to the first element of the vector $\mathbf{Q}_1$; that is,

$$Q_{a1} = -16.774 \text{ k}$$

in which the negative sign indicates that the axial force is tensile, or

$$Q_{a1} = 16.774 \text{ k (T)}$$ **Ans**

This member axial force can be verified by visually examining the free-body diagram of the member subjected to the local end forces, as shown in Fig. 3.18(d).

By applying Eq. (3.66), we determine the member end forces in the global coordinate system:

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q}$$

$$\mathbf{F}_1 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -16.774 \\ 0 \\ 16.774 \\ 0 \end{bmatrix} = \left[\begin{array}{c} -10.064 \\ -13.419 \\ \hdashline 10.064 \\ 13.419 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{ k} \tag{4}$$

Next, we write the member code numbers (3, 4, 1, 2) on the right side of $\mathbf{F}_1$ (see Eq. (4)), and store the pertinent elements of $\mathbf{F}_1$ in their proper positions in the reaction vector **R** by matching the code numbers (on the side of $\mathbf{F}_1$) to the restrained coordinate numbers written on the right side of **R** (see Fig. 3.18(e)). Thus, the element in row 1 of $\mathbf{F}_1$ (with code number 3) is stored in row 1 of **R** (with restrained coordinate number 3); and the element in row 2 of $\mathbf{F}_1$ (with code number 4) is stored in row 2 of **R** (with restrained coordinate number 4), as shown in Fig. 3.18(e). Note that the elements in rows 3 and 4 of $\mathbf{F}_1$, with code numbers corresponding to degrees of freedom 1 and 2 of the structure, are simply disregarded.

Member 2 The member end displacements in the global coordinate system are given by

$$\mathbf{v}_2 = \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\right]\begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} = \left[\begin{array}{c} 0 \\ 0 \\ d_1 \\ d_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{array}\right] \text{ in.}$$

The member end displacements in the local coordinate system can now be determined by using the relationship $\mathbf{u} = \mathbf{T}\mathbf{v}$ (Eq. (3.63)), with $\mathbf{T}$ as defined in Eq. (3.61):

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.13995 \\ -0.21552 \end{bmatrix} \text{ in.}$$

Using Eq. (3.7), we compute member end forces in the local coordinate system:

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

$$\mathbf{Q}_2 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = 906.25 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.13995 \\ -0.21552 \end{bmatrix} = \begin{bmatrix} 126.83 \\ 0 \\ -126.83 \\ 0 \end{bmatrix} \text{ k}$$

from which we obtain the member axial force (see also Fig. 3.18(d)):

$Q_{a2} = 126.83$ k (C) **Ans**

Using the relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (3.66)), we calculate the member global end forces to be

$$\mathbf{F}_2 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 126.83 \\ 0 \\ -126.83 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 126.83 \\ \hdashline 0 \\ -126.83 \end{array}\right]\begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \text{ k}$$

The pertinent elements of $\mathbf{F}_2$ are now stored in their proper positions in the reaction vector $\mathbf{R}$, by using member code numbers (5, 6, 1, 2), as shown in Fig. 3.18(e).

Member 3 The member global end displacements are

$$\mathbf{v}_3 = \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\right]\begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} = \left[\begin{array}{c} 0 \\ 0 \\ d_1 \\ d_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{array}\right] \text{ in.}$$

As in the case of members 1 and 2, we can determine the end forces $\mathbf{Q}_3$ and $\mathbf{F}_3$ for member 3 by using the relationships $\mathbf{u} = \mathbf{T}\mathbf{v}$, $\mathbf{Q} = \mathbf{k}\mathbf{u}$, and $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$, in sequence. However, such member forces can also be obtained by applying sequentially the global stiffness relationship $\mathbf{F} = \mathbf{K}\mathbf{v}$ (Eq. (3.71)) and the transformation relation $\mathbf{Q} = \mathbf{T}\mathbf{F}$ (Eq. (3.60)). Let us apply this alternative approach to determine the end forces for member 3.

Applying Eq. (3.71), we compute the member end forces in the global coordinate system:

$$\mathbf{F} = \mathbf{K}\mathbf{v}$$

$$\mathbf{F}_3 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix}$$

$$= \left[\begin{array}{r} -139.94 \\ 186.58 \\ \hdashline 139.94 \\ -186.58 \end{array}\right] \begin{array}{l} 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k}$$

Using the member code numbers (7, 8, 1, 2), the pertinent elements of $\mathbf{F}_3$ are stored in the reaction vector **R**, as shown in Fig. 3.18(e).

The member end forces in the local coordinate system can now be obtained by using the transformation relationship $\mathbf{Q} = \mathbf{TF}$ (Eq. (3.60)), with **T** as defined in Eq. (3.61).

$$\mathbf{Q}_3 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} -139.94 \\ 186.58 \\ 139.94 \\ -186.58 \end{bmatrix} = \begin{bmatrix} 233.23 \\ 0 \\ -233.23 \\ 0 \end{bmatrix} \text{ k}$$

from which the member axial force is found to be (see also Fig. 3.18(d))

$$Q_{a3} = 233.23 \text{ k (C)}$$ **Ans**

Support Reactions: The completed reaction vector **R** is shown in Fig. 3.18(e), and the support reactions are depicted on a line diagram of the truss in Fig. 3.18(f). **Ans**

Equilibrium Check: Applying the equations of equilibrium to the free body of the entire truss (Fig. 3.18(f)), we obtain

$$+\rightarrow \sum F_X = 0 \qquad 150 - 10.064 - 139.94 = -0.004 \approx 0$$ **Checks**

$$+\uparrow \sum F_Y = 0 \qquad -300 - 13.419 + 126.83 + 186.58 = -0.009 \approx 0$$ **Checks**

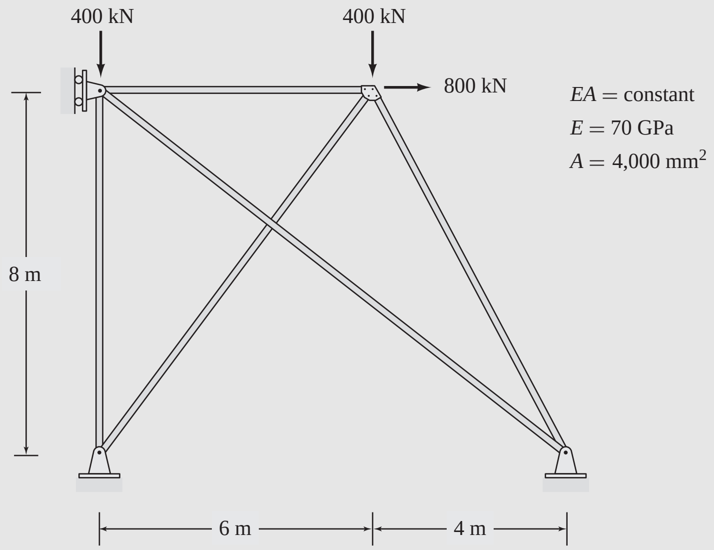
$$+\circlearrowleft \sum M_{①} = 0 \qquad -10.064(16) + 13.419(12) - 139.94(16) + 186.58(12) = -0.076 \text{ k-ft} \approx 0$$ **Checks**

EXAMPLE 3.9

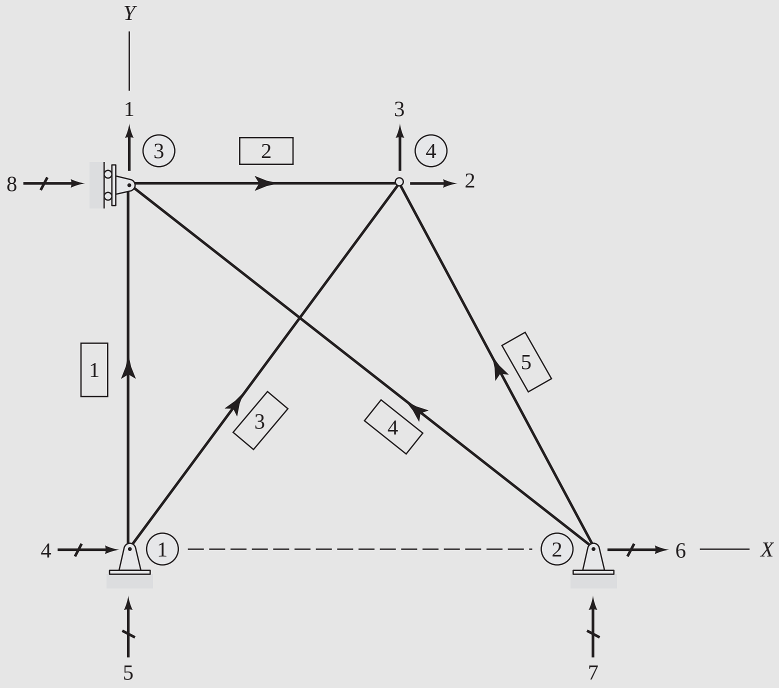
Determine the joint displacements, member axial forces, and support reactions for the truss shown in Fig. 3.19(a), using the matrix stiffness method.

SOLUTION

Analytical Model: From the analytical model of the truss shown in Fig. 3.19(b), we observe that the structure has three degrees of freedom (numbered 1, 2, and 3), and five restrained coordinates (numbered 4 through 8). Note that for horizontal member 2, the left end joint 3 is chosen as the beginning joint, so that the positive directions of local axes are the same as the global axes. Thus, no coordinate transformations are necessary for this member; that is, the member stiffness relations in the local and global coordinate systems are the same.



(a) Truss



(b) Analytical Model

Fig. 3.19

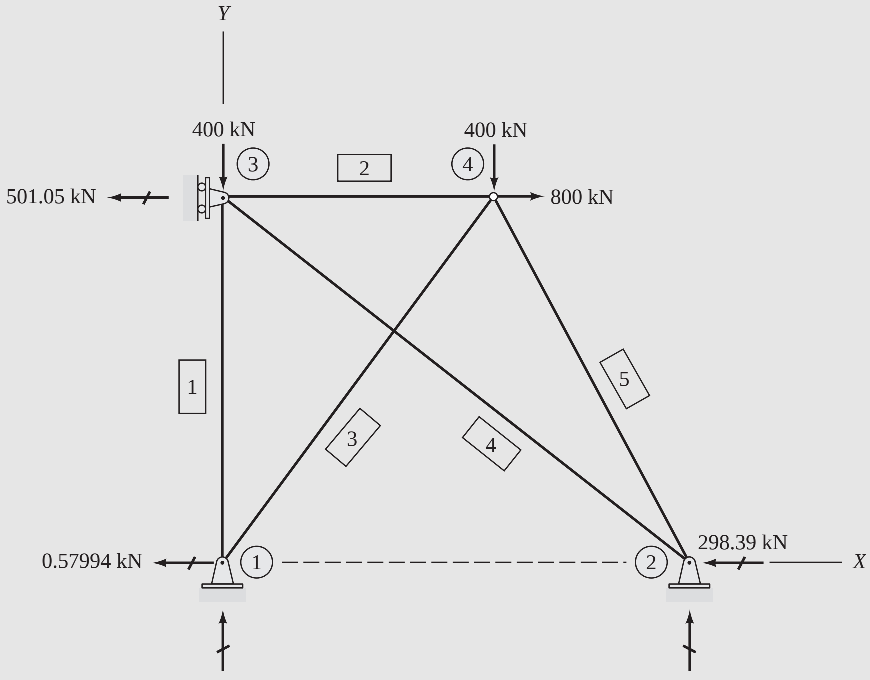
$$\mathbf{S} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left[\begin{array}{ccc} 35{,}000 + 8{,}533 & 0 & 0 \\ 0 & 46{,}667 + 10{,}080 + 6{,}260.9 & 13{,}440 - 12{,}522 \\ 0 & 13{,}440 - 12{,}522 & 17{,}920 + 25{,}043 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$= \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left[\begin{array}{ccc} 43{,}533 & 0 & 0 \\ 0 & 63{,}008 & 918 \\ 0 & 918 & 42{,}963 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \text{ kN/m}$$

(c) Structure Stiffness Matrix

$$\mathbf{R} = \left[\begin{array}{c} -0.57994 \\ 321.59 - 0.77325 \\ -98.008 - 200.38 \\ 78.407 + 400.76 \\ -599.06 + 98.008 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} = \left[\begin{array}{c} -0.57994 \\ 320.82 \\ -298.39 \\ 479.17 \\ -501.05 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \text{ kN}$$

(d) Support Reaction Vector



(e) Support Reactions

Fig. 3.19 (*continued*)

Structure Stiffness Matrix:

Member 1 Using Eqs. (3.62), we write

$$L = \sqrt{(X_3 - X_1)^2 + (Y_3 - Y_1)^2} = \sqrt{(0-0)^2 + (8-0)^2} = 8\,\text{m}$$

$$\cos\theta = \frac{X_3 - X_1}{L} = \frac{0-0}{8} = 0$$

$$\sin\theta = \frac{Y_3 - Y_1}{L} = \frac{8-0}{8} = 1$$

Using the units of kN and meters, we obtain the member global stiffness matrix (Eq. (3.73)):

$$\mathbf{K}_1 = \frac{70(10^6)(0.004)}{8}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{array}{c@{}c} \begin{array}{cccc} 4 & 5 & 8 & 1 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 35{,}000 & 0 & -35{,}000 \\ 0 & 0 & 0 & 0 \\ 0 & -35{,}000 & 0 & 35{,}000 \end{bmatrix} & \begin{array}{c} 4 \\ 5 \\ 8 \\ 1 \end{array} \end{array} \text{kN/m}$$

From Fig. 3.19(b), we observe that the code numbers for member 1 are 4, 5, 8, 1. These numbers are written on the right side and at the top of $\mathbf{K}_1$, and the pertinent elements of $\mathbf{K}_1$ are stored in their proper positions in the structure stiffness matrix **S**, as shown in Fig. 3.19(c).

Member 2 As discussed, no coordinate transformations are needed for this horizontal member; that is, $\mathbf{T}_2 = \mathbf{I}$, and $\mathbf{K}_2 = \mathbf{k}_2$. Substituting $E = 70(10^6)$ kN/m^2, $A = 0.004$ m^2, and $L = 6$ m into Eq. (3.27), we obtain

$$\mathbf{K}_2 = \mathbf{k}_2 = \begin{array}{c@{}c} \begin{array}{cccc} 8 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 46{,}667 & 0 & -46{,}667 & 0 \\ 0 & 0 & 0 & 0 \\ -46{,}667 & 0 & 46{,}667 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 8 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \text{kN/m}$$

From Fig. 3.19(b), we can see that the code numbers for member 2 are 8, 1, 2, 3. These numbers are used to store the appropriate elements of $\mathbf{K}_2$ in **S**, as shown in Fig. 3.19(c).

Member 3

$$L = \sqrt{(X_4 - X_1)^2 + (Y_4 - Y_1)^2} = \sqrt{(6-0)^2 + (8-10)^2} = 10\text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{6-0}{10} = 0.6$$

$$\sin\theta = \frac{Y_4 - Y_1}{L} = \frac{8-0}{10} = 0.8$$

$$\mathbf{K}_3 = \begin{array}{c@{}c} \begin{array}{cccc} 4 & 5 & 2 & 3 \end{array} & \\ \begin{bmatrix} 10{,}080 & 13{,}440 & -10{,}080 & -13{,}440 \\ 13{,}440 & 17{,}920 & -13{,}440 & -17{,}920 \\ -10{,}080 & -13{,}440 & 10{,}080 & 13{,}440 \\ -13{,}440 & -17{,}920 & 13{,}440 & 17{,}920 \end{bmatrix} & \begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} \end{array} \text{kN/m}$$

Using the code numbers (4, 5, 2, 3) of member 3, the relevant elements of $\mathbf{K}_3$ are stored in **S**, as shown in Fig. 3.19(c).

Member 4

$$L = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2} = \sqrt{(0-10)^2 + (8-0)^2} = 12.806 \text{ m}$$

$$\cos\theta = \frac{X_3 - X_2}{L} = \frac{0-10}{12.806} = -0.78088$$

$$\sin\theta = \frac{Y_3 - Y_2}{L} = \frac{8-0}{12.806} = 0.62471$$

$$\mathbf{K}_4 = \begin{array}{c} \begin{array}{cccc} 6 & 7 & 8 & 1 \end{array} \\ \left[\begin{array}{rrrr} 13{,}333 & -10{,}666 & -13{,}333 & 10{,}666 \\ -10{,}666 & 8{,}533 & 10{,}666 & -8{,}533 \\ -13{,}333 & 10{,}666 & 13{,}333 & -10{,}666 \\ 10{,}666 & -8{,}533 & -10{,}666 & 8{,}533 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 1 \end{array} \text{ kN/m}$$

The member code numbers are 6, 7, 8, 1. Thus, the element in row 4 and column 4 of $\mathbf{K}_4$ is stored in row 1 and column 1 of **S**, as shown in Fig. 3.19(c).

Member 5

$$L = \sqrt{(X_4 - X_2)^2 + (Y_4 - Y_2)^2} = \sqrt{(6-10)^2 + (8-0)^2} = 8.9443 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_2}{L} = \frac{6-10}{8.9443} = -0.44721$$

$$\sin\theta = \frac{Y_4 - Y_2}{L} = \frac{8-0}{8.9443} = 0.89442$$

$$\mathbf{K}_5 = \begin{array}{c} \begin{array}{cccc} 6 & 7 & 2 & 3 \end{array} \\ \left[\begin{array}{rrrr} 6{,}260.9 & -12{,}522 & -6{,}260.9 & 12{,}522 \\ -12{,}522 & 25{,}043 & 12{,}522 & -25{,}043 \\ -6{,}260.9 & 12{,}522 & 6{,}260.9 & -12{,}522 \\ 12{,}522 & -25{,}043 & -12{,}522 & 25{,}043 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

The code numbers for member 5 are 6, 7, 2, 3. These numbers are used to store the pertinent elements of $\mathbf{K}_5$ in **S**.

The completed structure stiffness matrix **S** is given in Fig. 3.19(c).

Joint Load Vector: By comparing Figs. 3.19(a) and (b), we obtain

$$\mathbf{P} = \begin{bmatrix} -400 \\ 800 \\ -400 \end{bmatrix} \text{ kN}$$

Joint Displacements: The structure stiffness relationship (Eq. (3.89)) can now be written as

$$\mathbf{P} = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} -400 \\ 800 \\ -400 \end{bmatrix} = \begin{bmatrix} 43{,}533 & 0 & 0 \\ 0 & 63{,}008 & 918 \\ 0 & 918 & 42{,}963 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Solving these equations simultaneously, we determine the joint displacements.

$$\mathbf{d} = \begin{bmatrix} -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} \text{ m} = \begin{bmatrix} -9.1884 \\ 12.837 \\ -9.5846 \end{bmatrix} \text{ mm}$$ **Ans**

To check our solution, the numerical values of **d** are back-substituted into the structure stiffness relation $\mathbf{P} = \mathbf{Sd}$ to obtain

$$\mathbf{P} = \mathbf{Sd} = \begin{bmatrix} 43{,}533 & 0 & 0 \\ 0 & 63{,}008 & 918 \\ 0 & 918 & 42{,}963 \end{bmatrix} \begin{bmatrix} -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} = \begin{bmatrix} -400 \\ 800.04 \approx 800 \\ -400 \end{bmatrix}$$

Checks

Member End Displacements and End Forces:

Member 1 The global end displacements of member 1 are obtained by comparing its global degree-of-freedom numbers with its code numbers. Thus,

$$\mathbf{v}_1 = \begin{array}{c|c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 8 \\ 1 \end{matrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix} \text{m}$$

To determine its local end displacements, we apply the relationship $\mathbf{u} = \mathbf{Tv}$ (Eq. (3.63)), with **T** as given in Eq. (3.61):

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.0091884 \\ 0 \end{bmatrix} \text{m}$$

Next, we compute the end forces in the local coordinate system by using the relationship $\mathbf{Q} = \mathbf{ku}$ (Eq. (3.7)), with **k** as defined in Eq. (3.27). Thus,

$$\mathbf{Q}_1 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = 35{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.0091884 \\ 0 \end{bmatrix} = \begin{bmatrix} 321.59 \\ 0 \\ -321.59 \\ 0 \end{bmatrix} \text{kN}$$

Therefore, the member axial force, which equals the first element of the vector $\mathbf{Q}_1$, is

$$Q_{a1} = 321.59 \text{ kN (C)}$$

Ans

The global end forces can now be obtained by using the relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (3.66)):

$$\mathbf{F}_1 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 321.59 \\ 0 \\ -321.59 \\ 0 \end{bmatrix} = \begin{array}{c|c} \begin{bmatrix} 0 \\ 321.59 \\ 0 \\ -321.59 \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 8 \\ 1 \end{matrix} \end{array} \text{kN}$$

Using the code numbers (4, 5, 8, 1), the elements of $\mathbf{F}_1$ corresponding to the restrained coordinates (4 through 8) are stored in their proper positions in **R**, as shown in Fig. 3.19(d).

Member 2

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{array}{c|c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} & \begin{matrix} 8 \\ 1 \\ 2 \\ 3 \end{matrix} \end{array} = \begin{bmatrix} 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} \text{m}$$

Using the relationship $\mathbf{Q} = \mathbf{ku}$ (Eq. (3.7)), we determine the member end forces:

$$\mathbf{F}_2 = \mathbf{Q}_2 = 46{,}667 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} = \left[\begin{array}{c} -599.06 \\ 0 \\ 599.06 \\ 0 \end{array}\right]\begin{array}{c} 8 \\ 1 \\ 2 \\ 3 \end{array} \text{kN}$$

from which the member axial force is found.

$Q_{a2} = -599.06 \text{ kN} = 599.06 \text{ kN (T)}$ **Ans**

The element in the first row of $\mathbf{F}_2$ (with code number 8) is stored in the fifth row of $\mathbf{R}$ (with restrained coordinate number 8), as shown in Fig. 3.19(d).

Member 3

$$\mathbf{v}_3 = \left[\begin{array}{c} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{array}\right]\begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} \text{m}$$

Using Eq. (3.63),

$$\mathbf{u} = \mathbf{Tv}$$

$$\mathbf{u}_3 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.00003452 \\ -0.01602 \end{bmatrix} \text{m}$$

Applying Eq. (3.7),

$$\mathbf{Q} = \mathbf{ku}$$

$$\mathbf{Q}_3 = 28{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.00003452 \\ -0.01602 \end{bmatrix} = \begin{bmatrix} -0.96656 \\ 0 \\ 0.96656 \\ 0 \end{bmatrix} \text{kN}$$

from which,

$Q_{a3} = -0.96656 \text{ kN} = 0.96656 \text{ kN (T)}$ **Ans**

From Eq. (3.66), we obtain

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q}$$

$$\mathbf{F}_3 = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -0.96656 \\ 0 \\ 0.96656 \\ 0 \end{bmatrix} = \left[\begin{array}{c} -0.57994 \\ -0.77325 \\ 0.57994 \\ 0.77325 \end{array}\right]\begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} \text{kN}$$

The pertinent elements of $\mathbf{F}_3$ are stored in $\mathbf{R}$, using the member code numbers (4, 5, 2, 3), as shown in Fig. 3.19(d).

Member 4

$$\mathbf{v}_4 = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix} \end{array}\begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \end{array} \text{ m}$$

$$\mathbf{u} = \mathbf{T}\mathbf{v}$$

$$\mathbf{u}_4 = \begin{bmatrix} -0.78088 & 0.62471 & 0 & 0 \\ -0.62471 & -0.78088 & 0 & 0 \\ 0 & 0 & -0.78088 & 0.62471 \\ 0 & 0 & -0.62471 & -0.78088 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -0.0057401 \\ 0.007175 \end{bmatrix} \text{ m}$$

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

$$\mathbf{Q}_4 = 21{,}865 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.0057401 \\ 0.007175 \end{bmatrix} = \begin{bmatrix} 125.51 \\ 0 \\ -125.51 \\ 0 \end{bmatrix} \text{ kN}$$

from which,

$$Q_{a4} = 125.51 \text{ kN (C)} \qquad \textbf{Ans}$$

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q}$$

$$\mathbf{F}_4 = \begin{bmatrix} -0.78088 & -0.62471 & 0 & 0 \\ 0.62471 & -0.78088 & 0 & 0 \\ 0 & 0 & -0.78088 & -0.62471 \\ 0 & 0 & 0.62471 & -0.78088 \end{bmatrix} \begin{bmatrix} 125.51 \\ 0 \\ -125.51 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -98.008 \\ 78.407 \\ 98.008 \\ -78.407 \end{bmatrix}\begin{array}{c} 6 \\ 7 \\ 8 \\ 1 \end{array} \text{ kN}$$

The relevant elements of $\mathbf{F}_4$ are stored in $\mathbf{R}$, as shown in Fig. 3.19(d).

Member 5

$$\mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix}\begin{array}{c} 6 \\ 7 \\ 2 \\ 3 \end{array} \text{ m}$$

$$\mathbf{u} = \mathbf{T}\mathbf{v}$$

$$\mathbf{u}_5 = \begin{bmatrix} -0.44721 & 0.89442 & 0 & 0 \\ -0.89442 & -0.44721 & 0 & 0 \\ 0 & 0 & -0.44721 & 0.89442 \\ 0 & 0 & -0.89442 & -0.44721 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -0.014313 \\ -0.0071953 \end{bmatrix} \text{m}$$

$$\mathbf{Q} = \mathbf{k}\mathbf{u}$$

$$\mathbf{Q}_5 = 31{,}305 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.014313 \\ -0.0071953 \end{bmatrix} = \begin{bmatrix} 448.07 \\ 0 \\ -448.07 \\ 0 \end{bmatrix} \text{kN}$$

Thus,

$$Q_{a5} = 448.07 \text{ kN (C)}$$ **Ans**

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q}$$

$$\mathbf{F}_5 = \begin{bmatrix} -0.44721 & -0.89442 & 0 & 0 \\ 0.89442 & -0.44721 & 0 & 0 \\ 0 & 0 & -0.44721 & -0.89442 \\ 0 & 0 & 0.89442 & -0.44721 \end{bmatrix} \begin{bmatrix} 448.07 \\ 0 \\ -448.07 \\ 0 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} -200.38 \\ 400.76 \\ 200.38 \\ -400.76 \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 7 \\ 2 \\ 3 \end{array} \text{kN}$$

The pertinent elements of $\mathbf{F}_5$ are stored in **R**, as shown in Fig. 3.19(d).

Support Reactions: The completed reaction vector **R** is given in Fig. 3.19(d), and the support reactions are shown on a line diagram of the structure in Fig. 3.19(e). **Ans**

Equilibrium Check: Considering the equilibrium of the entire truss, we write (Fig. 3.19(e)),

$$+\rightarrow \sum F_X = 0 \quad -0.57994 - 298.39 - 501.05 + 800 = -0.02 \text{ kN} \approx 0$$ **Checks**

$$+\uparrow \sum F_Y = 0 \quad 320.82 + 479.17 - 400 - 400 = -0.01 \text{ kN} \approx 0$$ **Checks**

$$+\circlearrowleft \sum M_{①} = 0 \quad 479.17(10) + 501.05(8) - 800(8) - 400(6) = 0.1 \text{kN}\cdot\text{m} \approx 0$$ **Checks**

SUMMARY

In this chapter, we have studied the basic concepts of the analysis of plane trusses based on the matrix stiffness method. A block diagram that summarizes the various steps involved in this analysis is presented in Fig. 3.20.

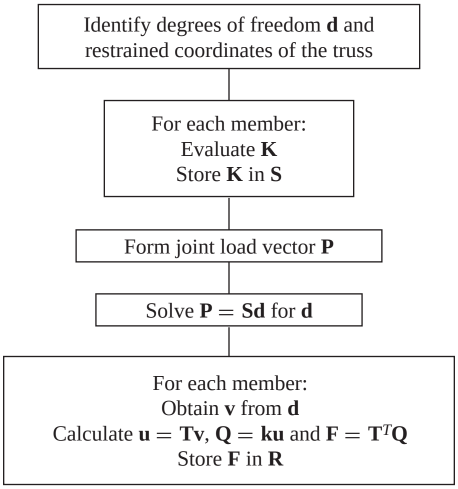


Fig. 3.20

PROBLEMS

Section 3.2

3.1 through 3.3 Identify by numbers the degrees of freedom and restrained coordinates of the trusses shown in Figs. P3.1–P3.3. Also, form the joint load vector **P** for the trusses.

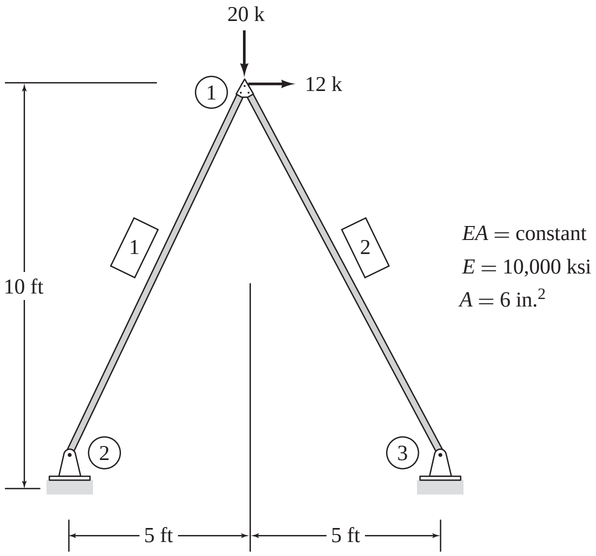


Fig. P3.1, P3.17

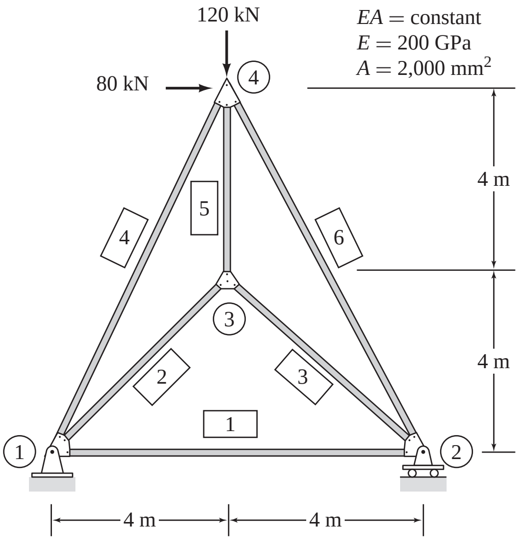


Fig. P3.2, P3.23

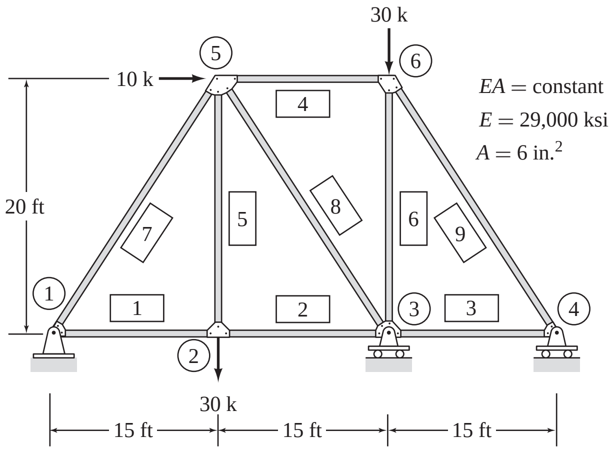


Fig. P3.3, P3.25

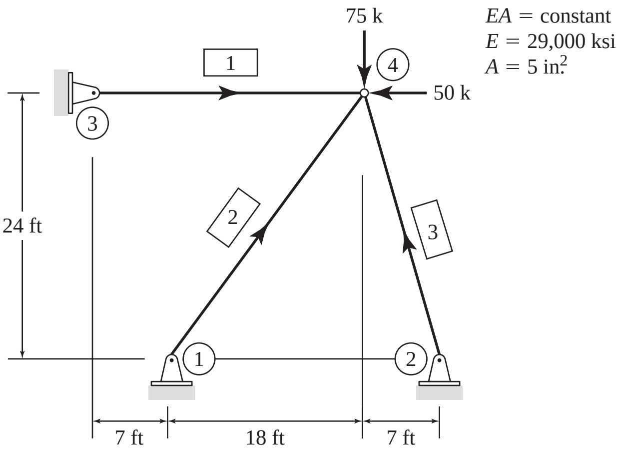


Fig. P3.5, P3.9, P3.15, P3.19

Section 3.3

3.4 and 3.5 Determine the local stiffness matrix **k** for each member of the trusses shown in Figs. P3.4 and P3.5.

3.6 If end displacements in the local coordinate system for member 5 of the truss shown in Fig. P3.6 are

$$\mathbf{u}_5 = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.75 \\ 1.25 \end{bmatrix} \text{ in.}$$

calculate the axial force in the member.

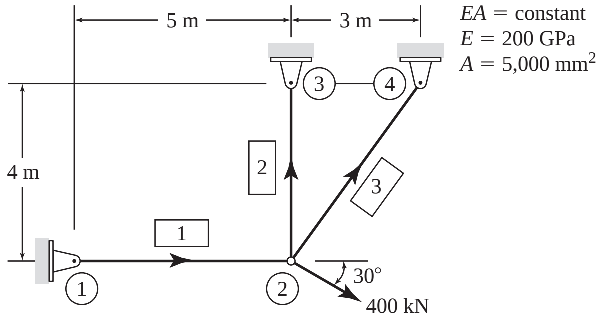


Fig. P3.4, P3.8, P3.14, P3.18

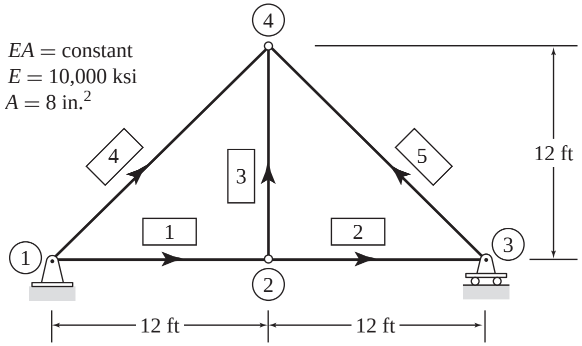


Fig. P3.6, P3.10, P3.12

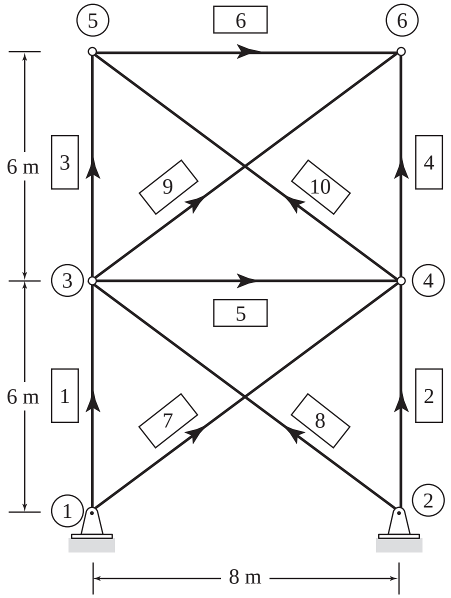
3.7 If end displacements in the local coordinate system for member 9 of the truss shown in Fig. P3.7 are

$$\mathbf{u}_9 = \begin{bmatrix} 17.6 \\ 3.2 \\ 33 \\ 6 \end{bmatrix} \text{ mm}$$

calculate the axial force in the member.

Section 3.5

3.8 and 3.9 Determine the transformation matrix **T** for each member of the trusses shown in Figs. P3.8 and P3.9.



$EA =$ constant
$E = 200$ GPa
$A = 5{,}000$ mm^2

Fig. P3.7, P3.11, P3.13

3.10 If the end displacements in the global coordinate system for member 5 of the truss shown in Fig. P3.10 are

$$\mathbf{v}_5 = \begin{bmatrix} 0.5 \\ 0 \\ 0.25 \\ -1 \end{bmatrix} \text{ in.}$$

calculate the end forces for the member in the global coordinate system. Is the member in equilibrium under these forces?

3.11 If the end displacements in the global coordinate system for member 9 of the truss shown in Fig. P3.11 are

$$\mathbf{v}_9 = \begin{bmatrix} 16 \\ -8 \\ 30 \\ -15 \end{bmatrix} \text{ mm}$$

calculate the end forces for the member in the global coordinate system. Is the member in equilibrium under these forces?

Section 3.6

3.12 Solve Problem 3.10, using the member stiffness relationship in the global coordinate system, $\mathbf{F} = \mathbf{K}\mathbf{v}$.

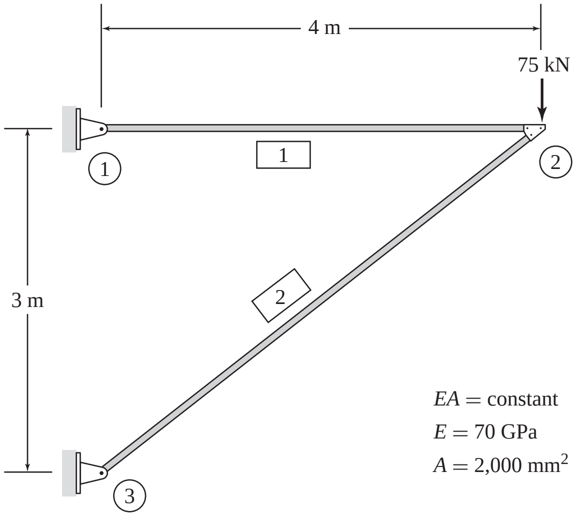
3.13 Solve Problem 3.11, using the member stiffness relationship in the global coordinate system, $\mathbf{F} = \mathbf{K}\mathbf{v}$.

Section 3.7

3.14 and 3.15 Determine the structure stiffness matrices **S** for the trusses shown in Figs. P3.14 and P3.15.

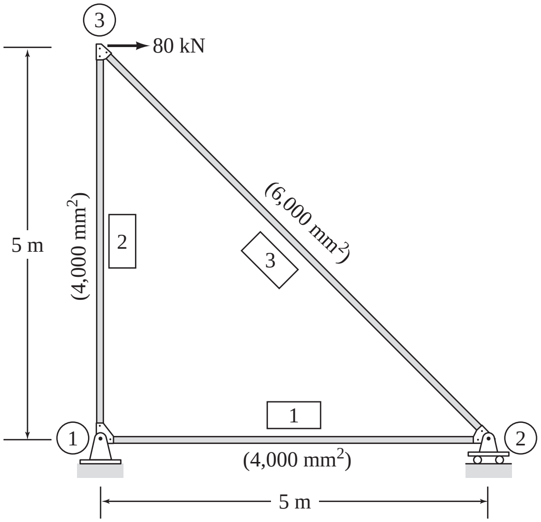
Section 3.8

3.16 through 3.25 Determine the joint displacements, member axial forces, and support reactions for the trusses shown in Figs. P3.16 through P3.25, using the matrix stiffness method. Check the hand-calculated results by using the computer program on the publisher's website for this book (www.cengage.com/engineering), or by using any other general purpose structural analysis program available.



$EA =$ constant
$E = 70$ GPa
$A = 2{,}000$ mm^2

Fig. P3.16



$E = 200$ GPa

Fig. P3.20

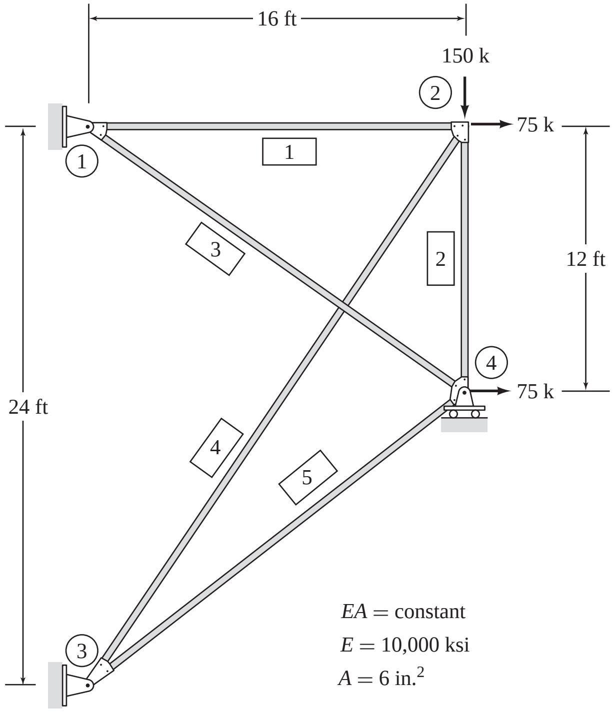


Fig. P3.21

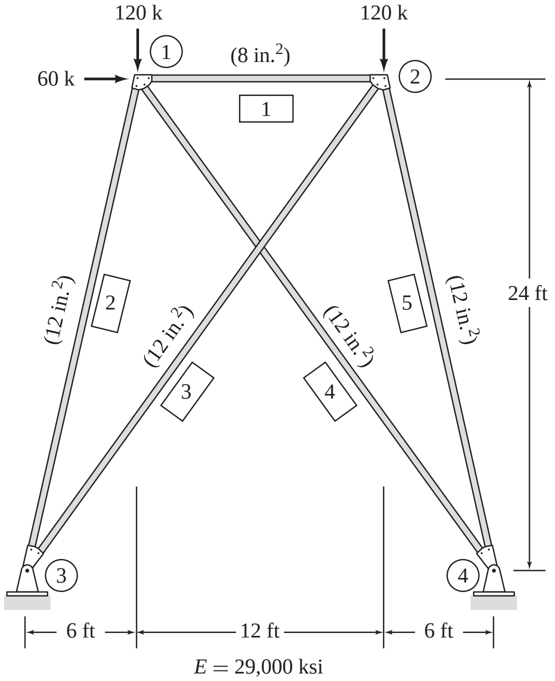


Fig. P3.22

3.26 and 3.27 Using a structural analysis computer program, determine the joint displacements, member axial forces, and support reactions for the Fink roof truss and the Baltimore bridge truss shown in Figs. P3.26 and P3.27, respectively. Verify the computer-generated results by manually checking the equilibrium equations for the entire truss, and for its joints numbered 5,10 and 15.

3.28 and 3.29 Using a structural analysis computer program, determine the largest value of the load parameter P that can be applied to the trusses shown in Figs. P3.28 and P3.29 without causing yielding and buckling of any of the members.

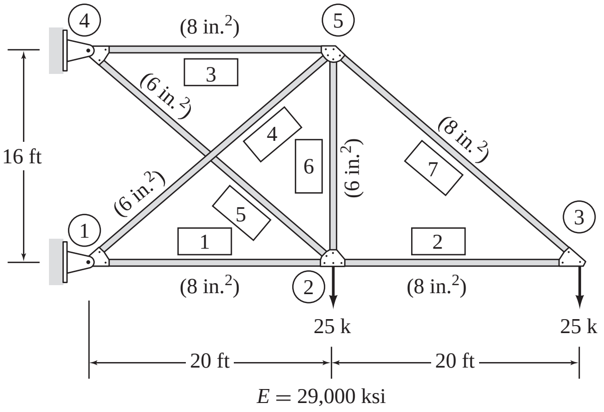
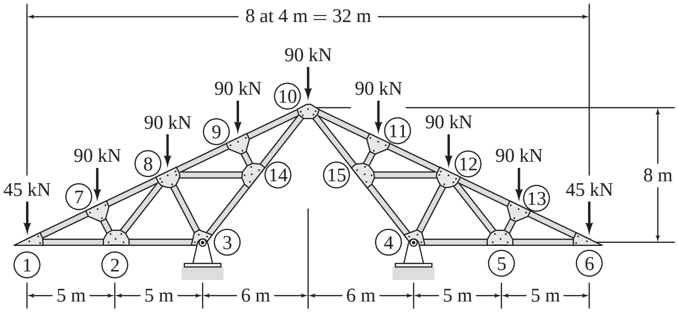
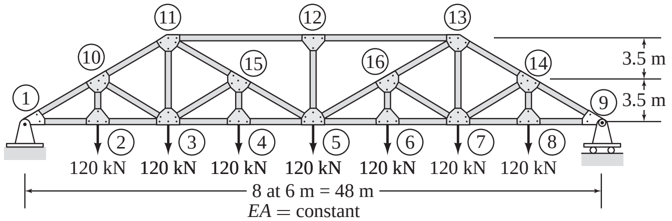


Fig. P3.24



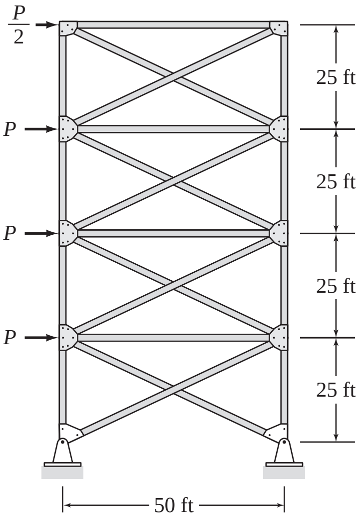
EA = constant
E = 70 GPa
A = 10,000 mm^2

Fig. P3.26



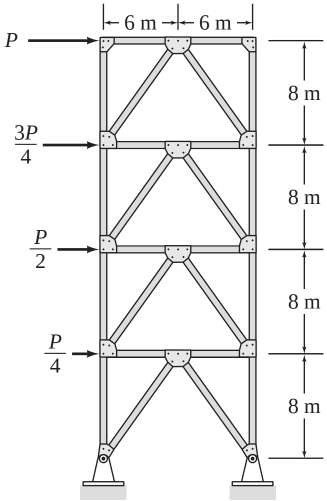
EA = constant
E = 200 GPa
A = 20,000 mm^2

Fig. P3.27



E, A, I, σ_y = constant
E = 29,000 ksi A = 14.7 in.^{2}
Moment of Inertia: I = 400 in.^{4}
Yield Stress: σ_y = 36 ksi

Fig. P3.28



E, A, I, σ_y = constant
E = 200 GPa A = 14,600 mm^2
Moment of Inertia: $I = 462(10^6)$ mm^4
Yield Stress: σ_y = 250 MPa

Fig. P3.29