Appendix C

NUMERICAL INTEGRATION

C.1 1D NUMERICAL INTEGRATION

Let us assume that the integral of a function $f(x)$ in the interval [-1,1] is required, i.e.

$$I = \int_{-1}^{+1} f(\xi)\, d\xi \tag{C.1}$$

The Gauss integration rule, or Gauss *quadrature*, expresses the value of the above integral as a sum the function values at a number of known points multiplied by prescribed weights. For a quadrature of order q

$$I \simeq I_q = \sum_{i=1}^{q} f(\xi_i) W_i \tag{C.2}$$

where W_i is the weight corresponding to the ith sampling point located at $\xi = \xi_i$ and q the number of sampling points. A *Gauss quadrature of qth order integrates exactly a polynomial function of degree* $2q - 1$ [Ral]. The error in the computation of the integral is of the order $0(\triangle^{2q})$, where $\triangle$ is the spacing between the sampling points. Table C.1 shows the coordinates of the sampling points and their weights for the first eight 1D Gauss quadratures.

Note that the sampling points are all located within the normalized domain [-1,1]. This is useful for computing the element integrals expressed in terms of the natural coordinate ξ. The Gauss quadrature requires the minimum number of sampling points to achieve a prescribed error in the computation of an integral. Thus, it minimizes the number of times the integrand function is computed. The reader can find the details in [Dem,PFTV,Ral,WR].

E. Oñate, *Structural Analysis with the Finite Element Method. Linear Statics: Volume 2: Beams, Plates and Shells*, Lecture Notes on Numerical Methods in Engineering and Sciences, DOI 10.1007/978-1-4020-8743-1,

q	ξ_q	W_q
1	0.0	2.0
2	±0.5773502692	1.0
3	±0.774596697 0.0	0.5555555556 0.8888888889
4	±0.8611363116 ±0.3399810436	0.3478548451 0.6521451549
5	±0.9061798459 ±0.5384693101 0.0	0.2369268851 0.4786286705 0.5688888889
6	±0.9324695142 ±0.6612093865 ±0.2386191861	0.1713244924 0.3607615730 0.4679139346
7	±0.9491079123 ±0.7415311856 ±0.4058451514 0.0	0.1294849662 0.2797053915 0.3818300505 0.4179591837
8	±0.9602898565 ±0.7966664774 ±0.5255324099 ±0.1834346425	0.1012285363 0.2223810345 0.3137066459 0.3626837834

Table C.1 Coordinates and weights for 1D Gauss quadratures

C.2 NUMERICAL INTEGRATION IN 2D

C.2.1 Numerical integration in quadrilateral domains

The integral of a term $g(\xi,\eta)$ over the normalized isoparametric quadrilateral domain can be evaluated using a 2D Gauss quadrature by

$$\int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{+1} d\xi\left[\sum_{q=1}^{n_q} g(\xi,\eta_q)W_q\right] = \sum_{p=1}^{n_p}\sum_{q=1}^{n_q} g(\xi_p,\eta_q)W_pW_q \tag{C.3}$$

where n_p and n_q are the number of integration points along each natural coordinate ξ and η respectively; ξ_p and η_q are the natural coordinates of the pth integration point and W_p, W_q are the corresponding weights.

The coordinates and weights for each natural direction are directly deduced from those given in Table C.1 for the 1D case. Let us recall that a 1D quadrature of qth order integrates *exactly* a polynomial of degree $q \leq 2n-1$. Figure C.1 shows the more usual quadratures for quadrilateral elements.

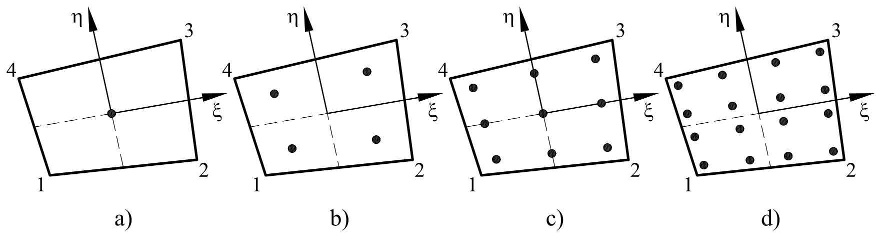

Fig. C.1 Gauss quadratures over quadrilateral elements, a) 1×1, b) 2×2, c) 3×3, d) 4×4 integration points

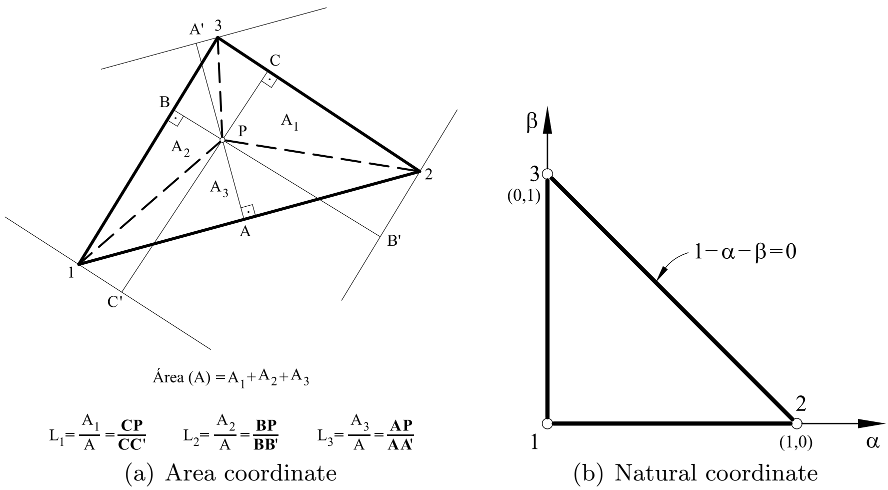

Fig. C.2 Triangular element. (a) Area coordinates. (b) Natural coordinates

C.2.2 Numerical integration over triangles

The Gauss quadrature for triangles is written as

$$\int_0^1 \int_0^{1-L_3} f(L_1, L_2, L_3)\, dL_2\, dL_3 = \sum_{p=1}^{n_p} f(L_{1_p}, L_{2_p}, L_{3_p})\, W_p \qquad \text{(C.4)}$$

where n_p is the number of integration points: $L_{1_p}, L_{2_p}, L_{3_p}$ and W_p are the area coordinates (Figure C.2a) and the corresponding weights for the pth integration point [On4].

Figure C.3 shows the more usual coordinates and weights. The term “accuracy” in the figure refers to the highest degree polynomial which is exactly integrated by each quadrature. Figure C.3 is also of direct application for computing the integrals defined in terms of the natural coordinates for triangles α and β (Figure C.2b) defined as $L_2 = \alpha, L_3 = \beta$ and $L_1 = 1 - \alpha - \beta$ [On4].

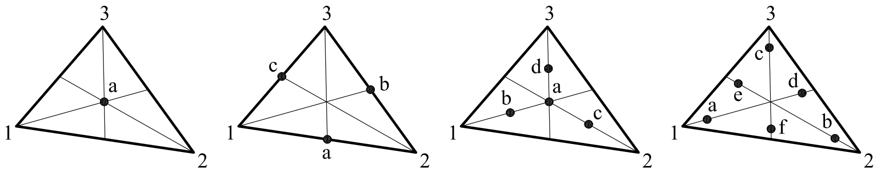

Figure	n	Accuracy	Points	L_1	L_2	L_3	W_i
(a)	1	Linear	a	1/3	1/3	1/3	1/2
(b)	3	Quadratic	a	1/2	1/2	0	1/6
			b	0	1/2	1/2	1/6
			c	1/2	0	1/2	1/6
(c)	4	Cubic	a	1/3	1/3	1/3	γ_1
			b	0.6	0.2	0.2	γ_2
			c	0.2	0.6	0.2	γ_2
			d	0.2	0.2	0.6	γ_2
(d)	6	Quartic	a	α_1	β_1	β_1	γ_3
			b	β_1	α_1	β_1	γ_3
			c	β_1	β_1	α_1	γ_3
			d	α_2	β_2	β_2	γ_4
			e	β_2	α_2	β_2	γ_4
			f	β_2	β_2	α_2	γ_4

$$\alpha_1 = 0.8168475730\ ; \quad \beta_1 = 0.0915762135\ ; \quad \gamma_1 = -\frac{27}{96}\ ; \quad 2\gamma_3 = 0.1099517437$$

$$\alpha_2 = 0.1081030182\ ; \quad \beta_2 = 0.4459484909\ ; \quad \gamma_2 = \frac{25}{96}\ ; \quad 2\gamma_4 = 0.2233815897$$

Fig. C.3 Coordinates and weights for the Gauss quadrature in triangular elements

The weights in Figure C.3 are normalized so that their sum is 1/2. In many references this value is changed to the unity and this requires the sum of Eq.(D.4) to be multiplied by 1/2 so that the element area is correctly computed in those cases [On4].

The quadrature of Figure C.2 can be extended for tetrahedral elements. For details see [On4].

C.3 NUMERICAL INTEGRATION OVER HEXAEDRA

Let us consider the integration of a function $f(x,y,z)$ over a hexahedral isoparametric element. The following transformations are required

$$\iiint_{V^{(e)}} f(x,y,z)\,dx\,dy\,dz = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)\,\left|\mathbf{J}^{(e)}\right|\,d\xi\,d\eta\,d\zeta =$$

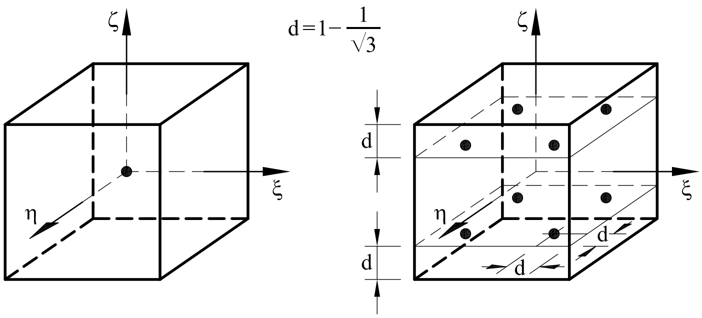

Fig. C.4 Gauss quadratures of $1\times1\times1$ and $2\times2\times2$ points in hexahedral elements

$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \tag{C.5}$$

Gauss quadrature over the normalized cubic domain leads to

$$\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta = \int_{-1}^{+1}\int_{-1}^{+1}\sum_{p=1}^{n_p} W_p\; g(\xi_p,\eta,\zeta)\, d\eta\, d\zeta =$$
$$= \int_{-1}^{+1}\sum_{q=1}^{n_q}\sum_{p=1}^{n_p} W_pW_q g(\xi_p,\eta_q,\zeta)\, d\zeta = \sum_{r=1}^{n_r}\sum_{q=1}^{n_q}\sum_{p=1}^{n_p} W_pW_qW_r g(\xi_p,\eta_q,\zeta_r) \tag{C.6}$$

where n_p, n_q and n_r are the integration points via the ξ, η, ζ directions, respectively, ξ_p, η_q, ζ_r are the coordinates of the integration point (p,q,r) and W_p, W_q, W_r are the weights for each natural direction.

The local coordinates and weights for each quadrature are deduced from Table D.3 for the 1D case. We recall that a qth order quadrature integrates exactly a 1D polynomial of degree $2q-1$. This rule helps us to identify the number of integration points in each natural direction. Figure C.4 shows the sampling points for the $1\times1\times1$ and $2\times2\times2$ quadratures.