6 PLANE FRAMES

Beekman Tower, New York
(Estormiz, Wikimedia Commons)

A plane frame is defined as *a two-dimensional assemblage of straight members connected together by rigid and/or hinged connections, and subjected to loads and reactions that lie in the plane of the structure.* Under the action of external loads, the members of a plane frame may be subjected to axial forces like the members of plane trusses, as well as bending moments and shears like the members of beams. Therefore, the stiffness relations for plane frame members can be conveniently obtained by combining the stiffness relations for plane truss and beam members.

The members of frames are usually connected by rigid connections, although hinged connections are sometimes used. In this chapter, we develop the analysis of rigidly connected plane frames based on the matrix stiffness method. The modifications in the method of analysis necessary to account for the presence of any hinged connections in the frame are considered in Chapter 7.

We begin, in Section 6.1, with a discussion of the process of developing an analytical model of the frame. We establish the force–displacement relations for the members of plane frames in their local coordinate systems in Section 6.2, where we also consider derivation of the member fixed-end axial forces due to external loads applied to the members. The transformation of member forces and displacements from a local to a global coordinate system, and vice versa, is considered in Section 6.3; and the member stiffness relations in the global coordinate system are developed in Section 6.4. The stiffness relations for the entire frame are formulated in Section 6.5, where the process of forming the structure fixed-joint force vectors, due to member loads, is also discussed. We then develop a step-by-step procedure for the analysis of plane frames in Section 6.6; finally, in Section 6.7, we cover the computer implementation of the procedure for analysis of plane frames.

6.1 ANALYTICAL MODEL

The process of dividing plane frames into members and joints, for the purpose of analysis, is the same as that for beams (Chapter 5); that is, *a plane frame is divided into members and joints so that:* (a) *all of the members are straight and prismatic, and* (b) *all the external reactions act only at the joints*. Consider, for example, the frame shown in Fig. 6.1(a). The analytical model of the frame is depicted in Fig. 6.1(b), which shows that, for the purpose of analysis, the frame is considered to be composed of four members and five joints. Note that because the member stiffness relationships to be used in the analysis are valid for prismatic members only, the left column of the frame has been subdivided into two members, each with constant cross-sectional properties (i.e., cross-sectional area and moment of inertia) along its length.

Global and Local Coordinate Systems

The global and local coordinate systems for plane frames are established in a manner similar to that for plane trusses (Chapter 3). The global coordinate system used for plane frames is a right-handed *XYZ* coordinate system with the frame lying in the *XY* plane, as shown in Fig. 6.1(b). It is usually convenient to

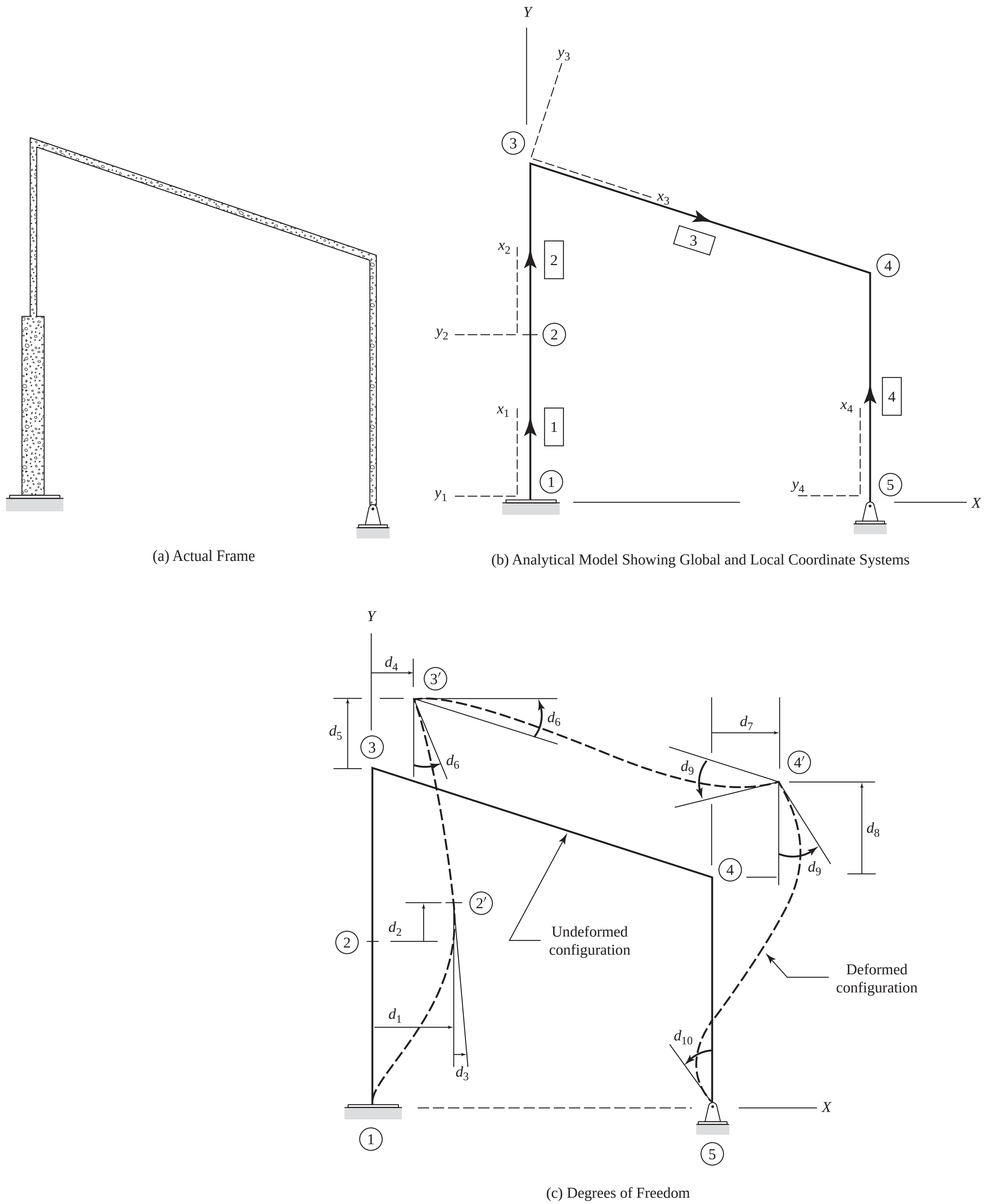
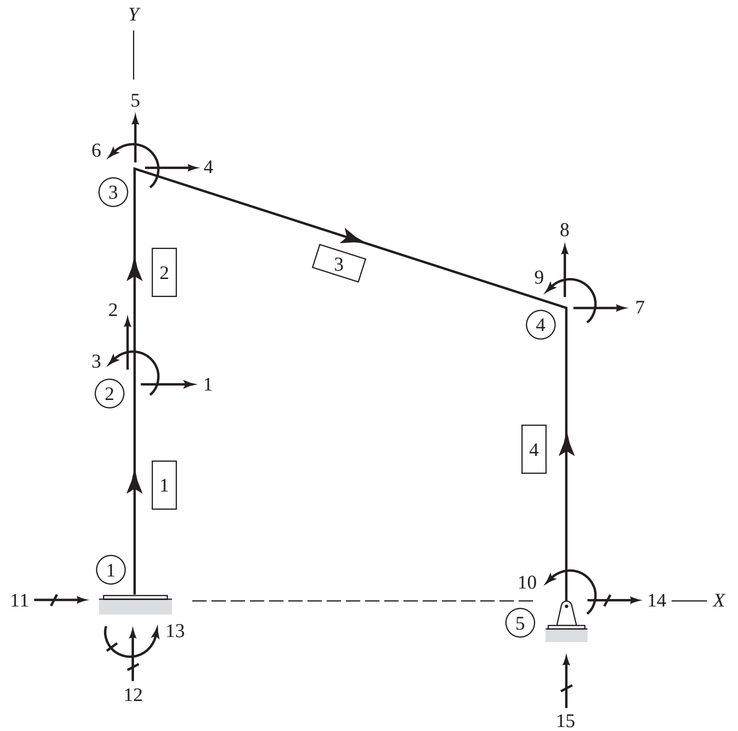

Y
y_3
3
x_3
3
x_2
2
4
y_2
2
x_1
1
x_4
4
1
y_1
y_4
5
X
(a) Actual Frame
(b) Analytical Model Showing Global and Local Coordinate Systems
Y
d_4
3′
d_6
d_5
3
d_6
d_7
d_9
4′
d_8
4
d_9
2′
d_2
2
Undeformed configuration
Deformed configuration
d_1
d_{10}
d_3
X
1
5
(c) Degrees of Freedom

Fig. 6.1



(d) Structure Coordinate Numbers
$(NDOF = 10, NR = 5)$

Fig. 6.1 (*continued*)

locate the origin of the global coordinate system at a lower left joint of the frame with the X and Y axes oriented in the horizontal (positive to the right) and the vertical (positive upward) directions, respectively (see Fig. 6.1(b)).

For each member of the frame, a local xyz coordinate system is established, with its origin at an end of the member and the x axis directed along the member's centroidal axis in the undeformed state. The positive direction of the y axis is defined so that the local coordinate system is right-handed, with the local z axis pointing in the positive direction of the global Z axis. The member end at which the origin of the local coordinate system is located can be chosen arbitrarily, and is usually considered to be the *beginning* of the member; the opposite member end is simply referred to as the *end* of the member. The local coordinate systems selected for the four members of the example frame are depicted in Fig. 6.1(b). As indicated in this figure, the member local coordinate systems can be conveniently shown on the line diagram of the structure by drawing an arrow on each member in the positive direction of its x axis.

Degrees of Freedom and Restrained Coordinates

The degrees of freedom of a plane frame are simply the unknown displacements (translations and rotations) of its joints. Since an unsupported joint of a plane frame can translate in any direction in the XY plane and rotate about the Z axis, three displacements—the translations in the X and Y directions and the rotation about the Z axis—are needed to completely specify its deformed position. Thus, a free joint of a plane frame has three degrees of freedom, and three structure coordinates (i.e., free and/or restrained coordinates) need to be defined at each joint, for the purpose of analysis (i.e., $NCJT = 3$).

Let us examine the degrees of freedom of the analytical model of the example frame given in Fig. 6.1(b). The deformed shape of the frame, due to an arbitrary loading, is depicted in Fig. 6.1(c), using an exaggerated scale. From this figure, we can see that joint 1, which is attached to a fixed support, can neither translate nor rotate; therefore, it does not have any degrees of freedom. Since joint 2 is not attached to any support, it is free to translate as well as rotate, and three displacements—the translations d_1 and d_2 in the X and Y directions, respectively, and the rotation d_3—are needed to completely specify its deformed position $2'$. Thus, joint 2 has three degrees of freedom. Similarly, joints 3 and 4, which are also free joints, have three degrees of freedom each. The displacements of joint 3 are designated d_4, d_5, and d_6; the degrees of freedom of joint 4 are designated d_7, d_8, and d_9. Finally, joint 5, which is attached to a hinged support, can rotate, but it cannot translate; therefore, it has only one degree of freedom, designated d_{10}. Thus, the entire frame has a total of ten degrees of freedom. All the joint displacements are shown in Fig. 6.1(c) in the positive sense. As indicated in this figure, the joint translations are considered positive when in the positive directions of the X and Y axes and joint rotations considered positive when counterclockwise. The $NDOF \times 1$ joint displacement vector $\mathbf{d}$ for this frame is written as

$$\mathbf{d} = \underset{10 \times 1}{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_9 \\ d_{10} \end{bmatrix}}$$

As discussed in Section 3.2, the number of degrees of freedom, $NDOF$, of a framed structure, in general, can be determined by subtracting the number of joint displacements restrained by supports, NR, from the total number of joint displacements of the unsupported structure (which equals $NCJT \times NJ$). Since $NCJT$ equals 3 for plane frames, the number of degrees of freedom of such structures can be expressed as (see Eq. (3.2))

$$\boxed{\left.\begin{aligned} NCJT &= 3 \\ NDOF &= 3(NJ) - NR \end{aligned}\right\} \quad \text{for plane frames}} \tag{6.1}$$

From Fig. 6.1(b), we can see that the example frame has five joints (i.e., $NJ = 5$); of these, joint 1 is attached to a fixed support that restrains three joint displacements, and joint 5 is attached to a hinged support that restrains two joint displacements. Thus, the total number of joint displacements that are restrained by all supports of the frame equals 5 (i.e., $NR = 5$). Substitution of $NJ = 5$ and $NR = 5$ into Eq. (6.1) yields the number of degrees of freedom of the frame:

$$NDOF = 3(5) - 5 = 10$$

which is the same as the number of degrees of freedom of the frame obtained previously.

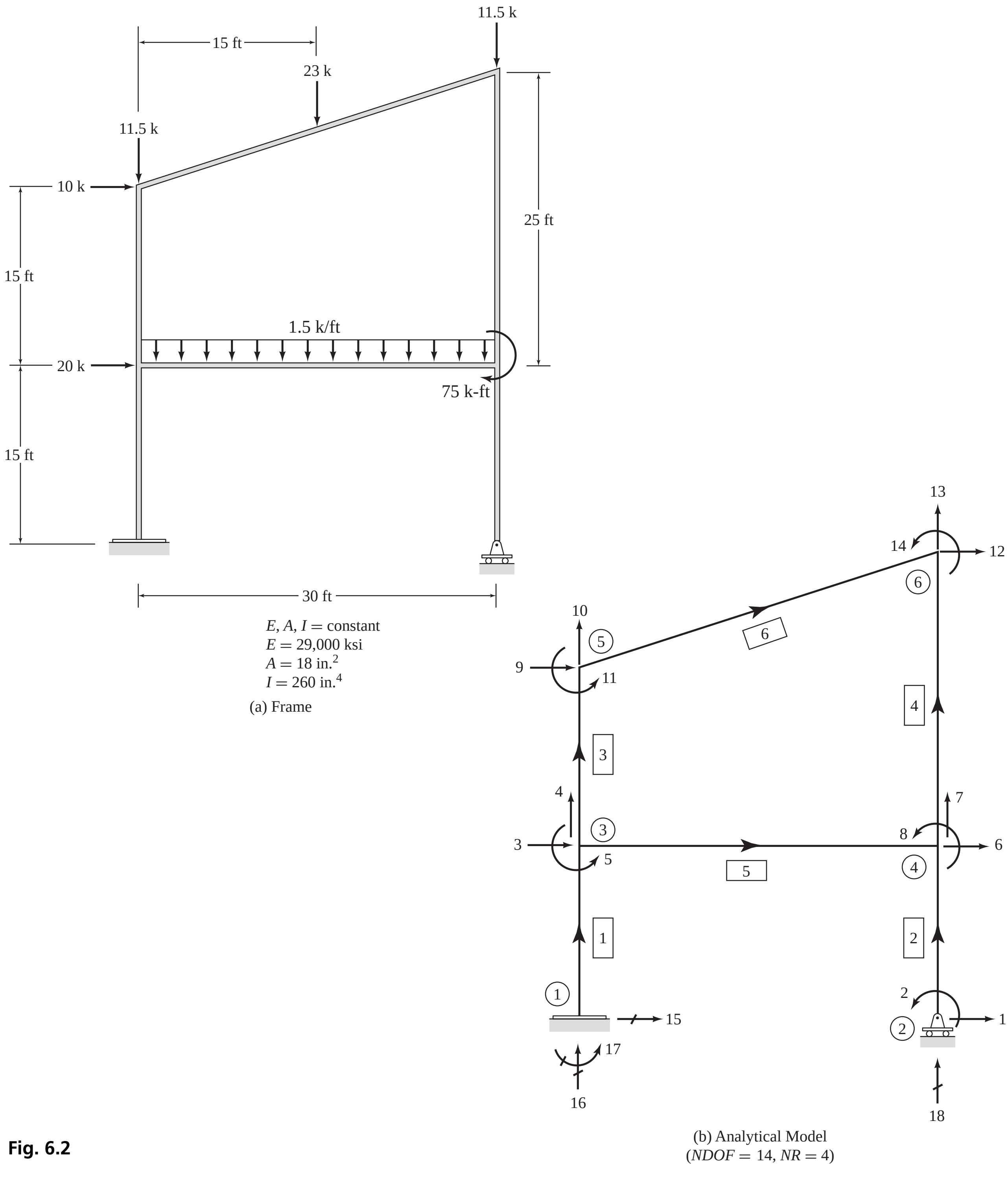
As in the case of plane trusses and beams, the structure coordinates of a plane frame are usually specified on the frame's line diagram by assigning numbers to the arrows drawn at the joints in the directions of the joint displacements, with a slash (/) added to the arrows representing the restrained coordinates to distinguish them from the degrees of freedom, as shown in Fig. 6.1(d). The procedure for assigning numbers to the structure coordinates of a plane frame is analogous to that for plane trusses and beams. The degrees of freedom of the frame are numbered first by beginning at the lowest-numbered joint with a degree of freedom, and proceeding sequentially to the highest-numbered joint. If a joint has more than one degree of freedom, then the translation in the X direction is numbered first, followed by the translation in the Y direction, and then the rotation. The first degree of freedom is assigned the number one, and the last degree of freedom is assigned the number equal to $NDOF$. After all the degrees of freedom have been numbered, the restrained coordinates of the frame are numbered in the same manner as the degrees of freedom, but starting with the number equal to $NDOF + 1$ and ending with the number equal to $3(NJ)$. The structure coordinate numbers for the example frame, obtained by applying this procedure, are given in Fig. 6.1(d).

EXAMPLE 6.1 Identify by numbers the degrees of freedom and restrained coordinates of the frame shown in Fig. 6.2(a). Also, form the joint load vector **P** for the frame.

SOLUTION *Degrees of Freedom and Restrained Coordinates:* See Fig. 6.2(b). **Ans**

Joint Load Vector: Units are kips and feet.

$$\mathbf{P} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 0 \\ 0 \\ 0 \\ 0 \\ -75 \\ 10 \\ -11.5 \\ 0 \\ 0 \\ -11.5 \\ 0 \end{bmatrix} \qquad \textbf{Ans}$$

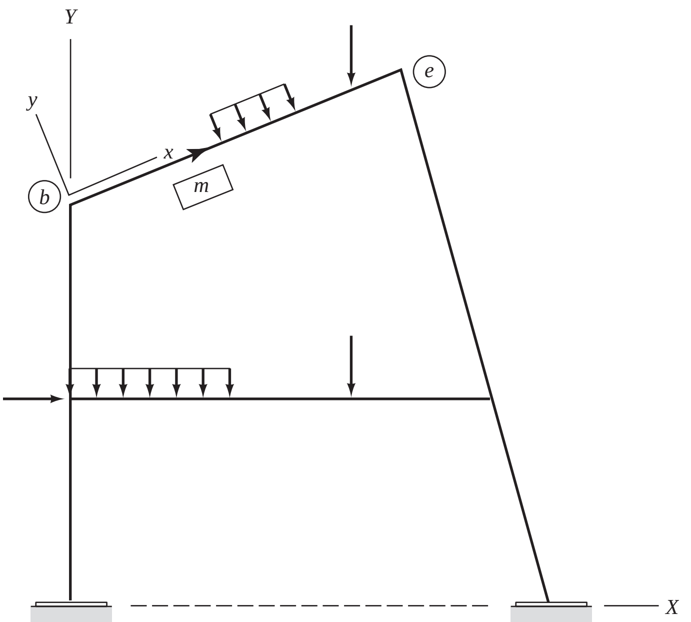


15 ft
11.5 k
23 k
11.5 k
10 k
25 ft
15 ft
1.5 k/ft
20 k
75 k-ft
15 ft
30 ft
E, A, I = constant
E = 29,000 ksi
A = 18 in.²
I = 260 in.⁴
(a) Frame
13
14
12
6
10
5
6
9
11
4
3
4
7
3
3
8
6
5
5
4
1
2
2
1
15
2
1
17
16
18
(b) Analytical Model
(NDOF = 14, NR = 4)

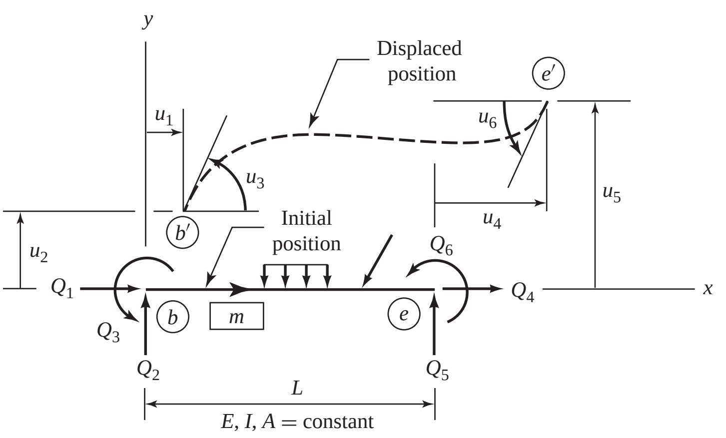
Fig. 6.2

6.2 MEMBER STIFFNESS RELATIONS IN THE LOCAL COORDINATE SYSTEM

Consider an arbitrary prismatic member *m* of the plane frame shown in Fig. 6.3(a). When the frame is subjected to external loads, member *m* deforms and internal axial forces, shears, and moments are induced at its ends. The initial and displaced positions of the member are shown in Fig. 6.3(b), from



(a) Frame



(b) Member Forces and Displacements in the Local Coordinate System

Fig. 6.3

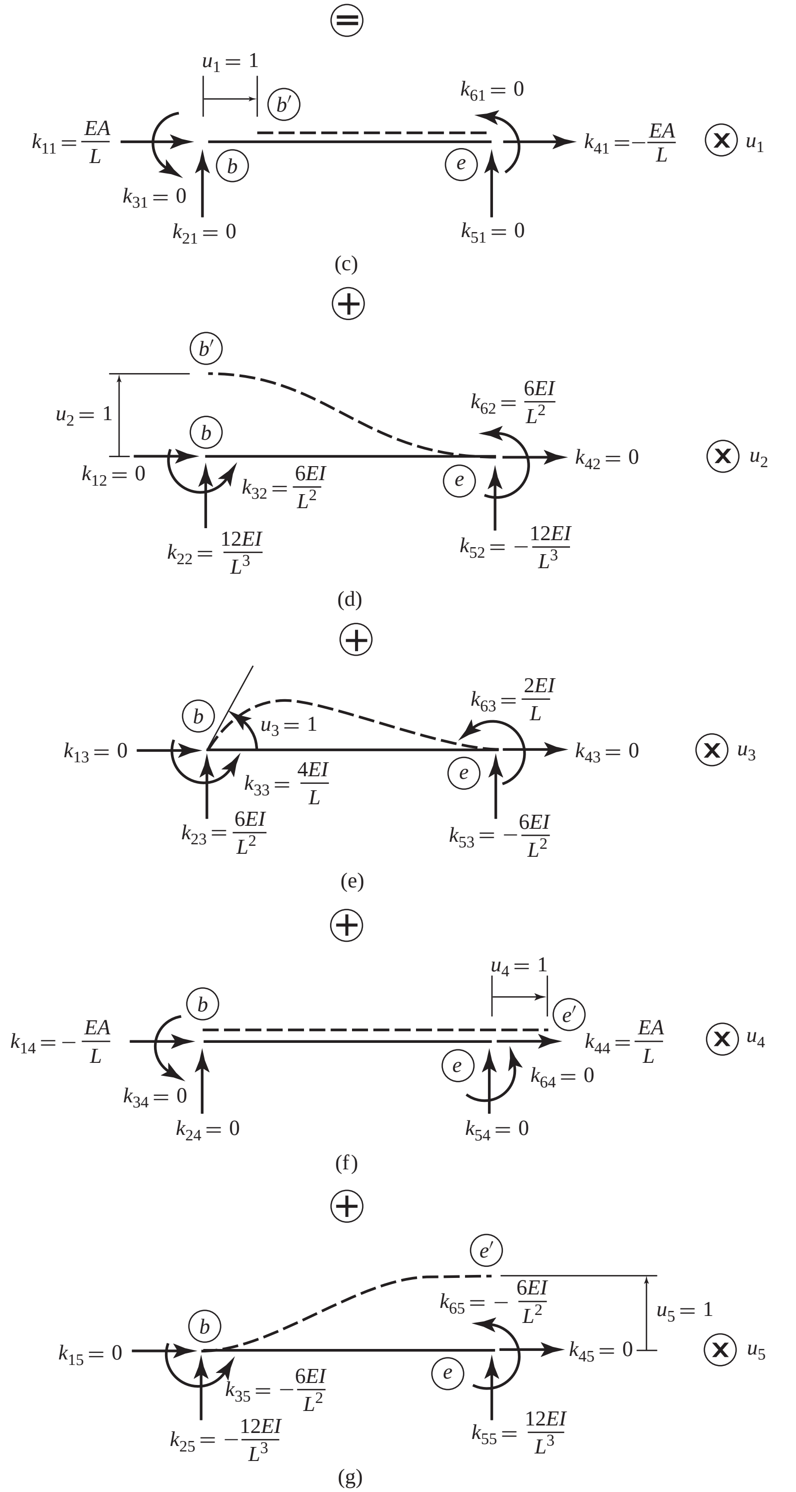


Fig. 6.3 (*continued*)

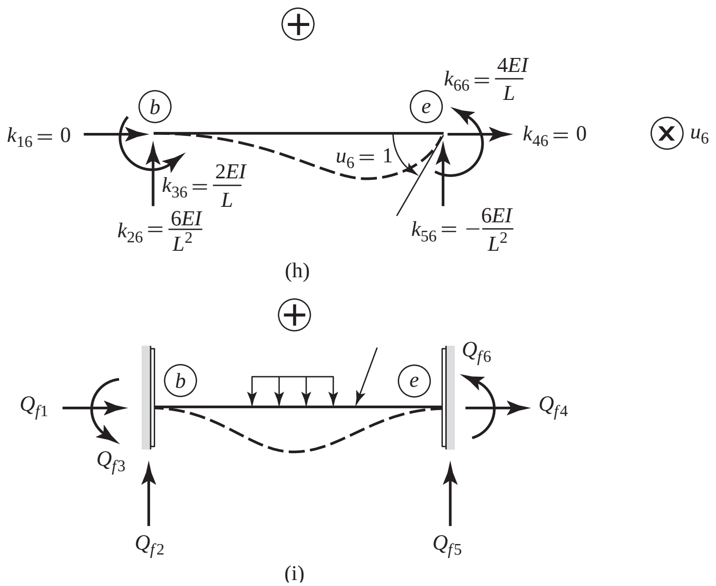


Fig. 6.3 (*continued*)

which we can see that three displacements—translations in the x and y directions and rotation about the z axis—are needed to completely specify the displaced position of each end of the member. Thus, the member has a total of six degrees of freedom. As indicated in Fig. 6.3(b), the six member end displacements are denoted by u_1 through u_6, and the corresponding member end forces are denoted by Q_1 through Q_6. Note that the member end displacements $\mathbf{u}$ and end forces $\mathbf{Q}$ are defined relative to the local coordinate system of the member, with translations and forces in the positive directions of the local x and y axes considered positive, and counterclockwise rotations and moments considered positive. As shown in Fig. 6.3(b), *a member's local end displacements and end forces are numbered by beginning at its end b, with the translation and force in the x direction numbered first, followed by the translation and force in the y direction, and then the rotation and moment. The displacements and forces at the member's opposite end e are then numbered in the same sequential order.*

The relationships between the end forces $\mathbf{Q}$ and the end displacements $\mathbf{u}$, for the members of plane frames, can be established by essentially the same process as used previously for the case of beams (Section 5.2). The process involves subjecting the member, separately, to each of the six end displacements as shown in Fig. 6.3(c) through (h), and to the external loading with no end displacements (i.e., with both member ends completely fixed against translations and rotations), as shown in Fig. 6.3(i). The total member end forces due to the combined effect of the six end displacements, and the external loading, can now be expressed as

$$Q_i = \sum_{j=1}^{6} \left(k_{ij} u_j\right) + Q_{fi} \qquad i = 1, 2, \ldots, 6 \tag{6.2}$$

in which the stiffness coefficient k_{ij} represents the force corresponding to Q_i due to a unit value of the displacement u_j, and Q_{fi} denotes the fixed-end force corresponding to Q_i due to the external loads acting on the member. Equation (6.2) can be expressed in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{bmatrix} \tag{6.3}$$

or, symbolically, as

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f} \tag{6.4}$$

in which $\mathbf{Q}$ and $\mathbf{u}$ denote the 6×1 member end-force and member end-displacement vectors, respectively, in the local coordinate system; $\mathbf{k}$ represents the 6×6 member local stiffness matrix; and $\mathbf{Q}_f$ is the 6×1 member fixed-end force vector in the local coordinate system.

Member Local Stiffness Matrix k

The explicit form of the local stiffness matrix $\mathbf{k}$ (in terms of E, A, I, and L) for the members of plane frames can be conveniently developed by using the expressions for the member stiffness coefficients of trusses and beams derived in Chapters 3 and 5, respectively.

To obtain the first column of $\mathbf{k}$, we subject the member to a unit end displacement $u_1 = 1$ (with $u_2 = u_3 = u_4 = u_5 = u_6 = 0$), as shown in Fig. 6.3(c). The expressions for the member axial forces required to cause this unit axial deformation were derived in Section 3.3, and are given in Fig. 3.3(c). By comparing Figs. 6.3(c) and 3.3(c), we obtain the stiffness coefficients for the plane frame member, due to end displacement $u_1 = 1$, as

$$k_{11} = \frac{EA}{L}, \qquad k_{41} = -\frac{EA}{L}, \qquad k_{21} = k_{31} = k_{51} = k_{61} = 0 \tag{6.5a}$$

Note that the imposition of end displacement $u_1 = 1$ does not cause the member to bend; therefore, no moments or shears develop at the ends of the member.

Similarly, the fourth column of $\mathbf{k}$ can be determined by comparing Fig. 6.3(f) to Fig. 3.3(e), which yields

$$k_{14} = -\frac{EA}{L}, \qquad k_{44} = \frac{EA}{L}, \qquad k_{24} = k_{34} = k_{54} = k_{64} = 0 \tag{6.5b}$$

To determine the second column of $\mathbf{k}$, the member is subjected to a unit end displacement $u_2 = 1$ (with $u_1 = u_3 = u_4 = u_5 = u_6 = 0$), as shown in 6.3(d).

The expressions for the member end shears and moments required to cause this deflected shape were derived in Section 5.2, and are given in Fig. 5.3(c). By comparing Figs. 6.3(d) and 5.3(c), we obtain the stiffness coefficients for the plane frame member, due to $u_2 = 1$, as

$$k_{22} = \frac{12EI}{L^3}, \qquad k_{32} = \frac{6EI}{L^2}, \qquad k_{52} = -\frac{12EI}{L^3}, \qquad k_{62} = \frac{6EI}{L^2},$$
$$k_{12} = k_{42} = 0 \tag{6.5c}$$

The third, fifth, and sixth columns of **k** can be developed in a similar manner, by comparing Figs. 6.3(e), (g), and (h) to Figs. 5.3(d), (e), and (f), respectively. This process yields

$$k_{23} = \frac{6EI}{L^2}, \qquad k_{33} = \frac{4EI}{L}, \qquad k_{53} = -\frac{6EI}{L^2}, \qquad k_{63} = \frac{2EI}{L},$$
$$k_{13} = k_{43} = 0 \tag{6.5d}$$

$$k_{25} = -\frac{12EI}{L^3}, \qquad k_{35} = -\frac{6EI}{L^2}, \qquad k_{55} = \frac{12EI}{L^3},$$
$$k_{65} = -\frac{6EI}{L^2}, \qquad k_{15} = k_{45} = 0 \tag{6.5e}$$

and

$$k_{26} = \frac{6EI}{L^2}, \qquad k_{36} = \frac{2EI}{L}, \qquad k_{56} = -\frac{6EI}{L^2}, \qquad k_{66} = \frac{4EI}{L},$$
$$k_{16} = k_{46} = 0 \tag{6.5f}$$

Finally, by substituting Eqs. (6.5) into the appropriate columns of **k** given in Eq. (6.3), we can express the local stiffness matrix for the members of plane frames as

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \tag{6.6}$$

Member Local Fixed-End Force Vector $\mathbf{Q}_f$

Unlike the members of beams, which are loaded only perpendicular to their longitudinal axes, the members of plane frames can be subjected to loads oriented in any direction in the plane of the structure. Before proceeding with the calculation of the fixed-end forces for a plane frame member, any loads acting on it in inclined directions are resolved into their components in the directions

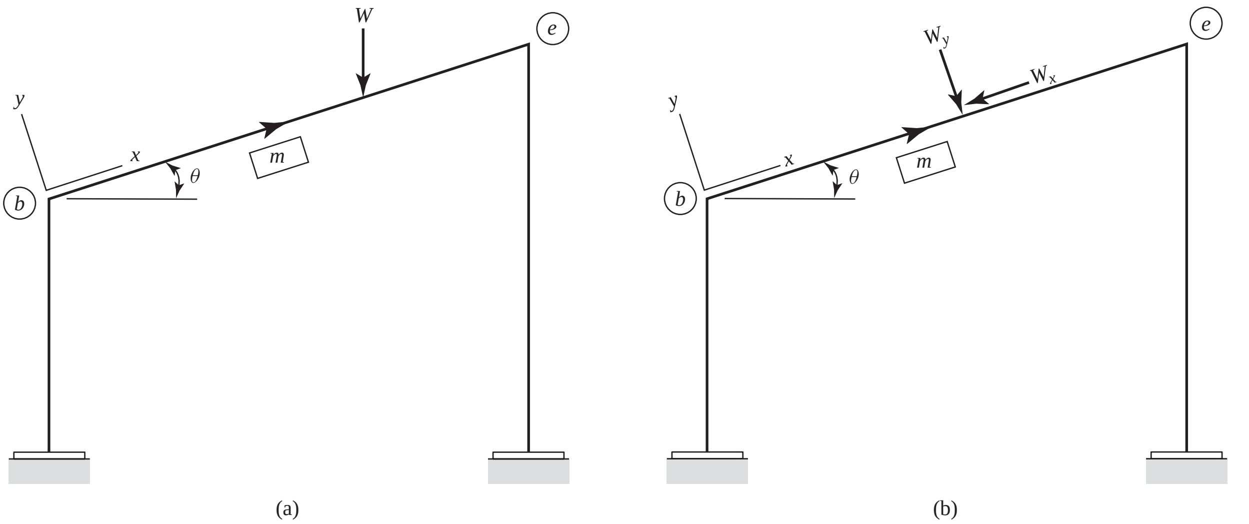


Fig. 6.4

of the local x and y axes of the member. For example, the vertical load W acting on the inclined member m of the frame of Fig. 6.4(a) is resolved into its rectangular components in the local x and y directions of the member m as

$$W_x = W \sin\theta \qquad \text{and} \qquad W_y = W \cos\theta$$

as shown in Fig. 6.4(b).

After all the loads acting on a member have been resolved into components parallel and perpendicular to the longitudinal axis of the member (i.e., in the local x and y directions, respectively), the fixed-end shears (FS_b and FS_e) and moments (FM_b and FM_e) due to the perpendicular loading and any couples can be calculated by using the fixed-end force equations for loading types 1 through 4 (given inside the front cover). The procedure for deriving these fixed-end shear and moment equations was discussed in Section 5.4.

The expressions for the member fixed-end axial forces, due to two common types of member axial loadings, are also given inside the front cover (see loading types 5 and 6). Such expressions can be conveniently determined by integrating the differential equation for the member axial deformation. This approach is illustrated in the following paragraphs, with loading type 6 taken as an example.

Consider a fixed member of a plane frame, subjected to a uniformly distributed axial load w over a part of its length, as shown in Fig. 6.5(a). As indicated there, the fixed-end axial forces at the member ends b and e are denoted by FA_b and FA_e, respectively. To develop the differential equation for axial deformation of an elastic member, we recall from Section 3.4 that the relationship between the axial strain ε_a and the axial displacement $\bar{u}_x$, of the centroidal axis of a member, is given by (see Eq. (3.39))

$$\varepsilon_a = \frac{d\bar{u}_x}{dx}$$

Substitution of this strain-displacement equation into Hooke's law yields

$$\sigma_a = E\varepsilon_a = E\frac{d\bar{u}_x}{dx}$$

in which σ_a represents the axial stress. To relate the axial displacement $\bar{u}_x$ to the axial force Q_a acting at the cross-section, we multiply both sides of the preceding equation by the cross-sectional area A to obtain

$$Q_a = \sigma_a A = EA\frac{d\bar{u}_x}{dx}$$

or

$$\boxed{\frac{d\bar{u}_x}{dx} = \frac{Q_a}{EA}} \qquad \textbf{(6.7)}$$

Equation (6.7) represents the differential equation for axial deformation of a member composed of linearly elastic homogeneous material. In this equation, $\bar{u}_x$ denotes the displacement of the member's centroidal axis in the x direction, at a distance x from the origin b of the local xy coordinate system of the member (Fig. 6.5(a)); Q_a represents the axial force at the member cross-section at the same location, x. Furthermore, Eq. (6.7) is based on the sign convention that the axial force Q_a is considered positive when causing tension at the member cross-section. The total axial deformation of a member can be obtained by multiplying both sides of Eq. (6.7) by dx and integrating the resulting equation

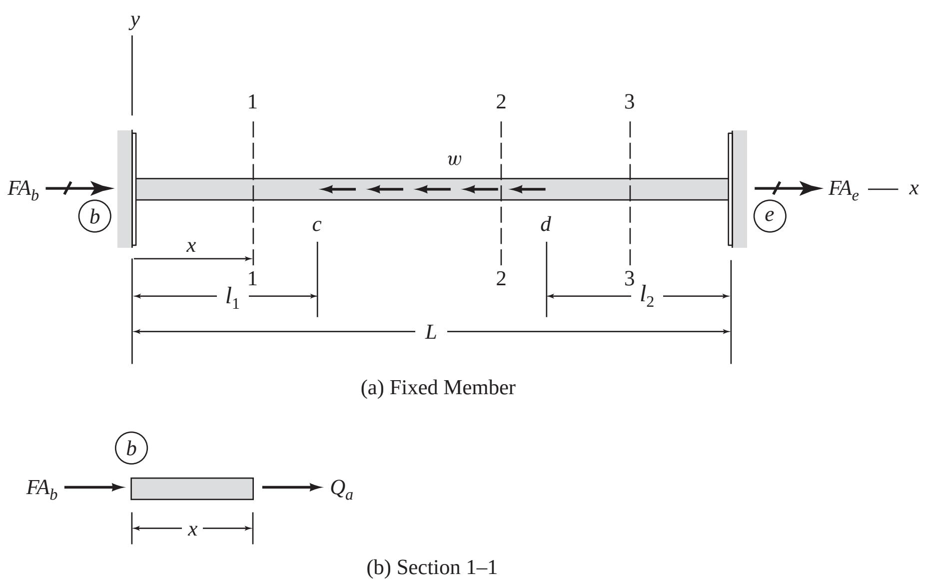


Fig. 6.5

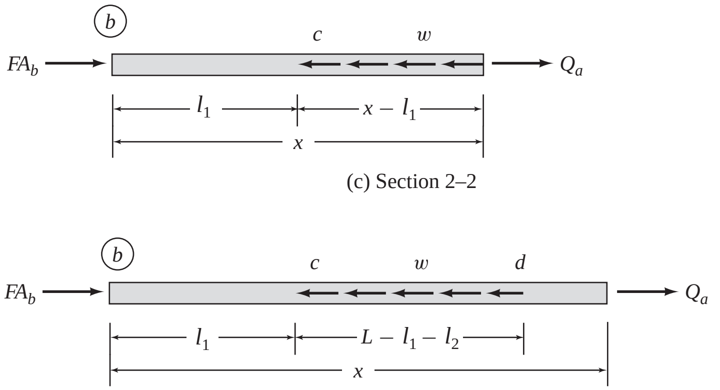


Fig. 6.5 (*continued*)

over the length L of the member:

$$\bar{u}_{xe}(\text{at } x = L) = \int_0^L \frac{Q_a}{EA}\, dx \tag{6.8}$$

Realizing that EA is constant for prismatic members, the axial deformation of such members can be expressed as

$$\bar{u}_{xe} = \frac{1}{EA}\int_0^L Q_a\, dx \tag{6.9}$$

To obtain the expressions for the fixed-end axial forces FA_b and FA_e for the member shown in Fig. 6.5(a), we first determine the equations for axial force Q_a in terms of one of the unknowns, FA_b. Since the uniformly distributed load w is applied over member portion cd (Fig. 6.5(a)), the axial force Q_a cannot be expressed as a single continuous function over the entire length of the member. Therefore, we divide the member into three segments, bc, cd, and de, and determine the equations for axial force in these segments by passing sections 1–1, 2–2, and 3–3, respectively, through the member, as shown in Fig. 6.5(a). By considering the equilibrium of the free body of the member to the left of section 1–1 (Fig. 6.5(b)), we determine the axial force Q_a at section 1–1 to be

$$\overset{+}{\rightarrow} \sum F_x = 0 \qquad FA_b + Q_a = 0 \qquad Q_a = -FA_b$$

Thus, the equation of the axial force in segment bc can be expressed as

$$0 \le x \le l_1 \qquad Q_a = -FA_b \tag{6.10a}$$

Similarly, by considering the free bodies of the member to the left of sections 2–2 and 3–3 (Fig. 6.5(c) and (d)), we obtain the equations of the axial force in segments cd and de, respectively, as

$$l_1 \le x \le L - l_2 \qquad Q_a = -FA_b + w(x - l_1) \tag{6.10b}$$

$$L - l_2 \le x \le L \qquad Q_a = -FA_b + w(L - l_1 - l_2) \tag{6.10c}$$

Next, by substituting Eqs. (6.10) into Eq. (6.9), we write

$$\bar{u}_{xe} = \frac{1}{EA}\left[\int_0^{l_1} -FA_b\,dx + \int_{l_1}^{L-l_2}\{-FA_b + w(x-l_1)\}\,dx \right.$$
$$\left. + \int_{L-l_2}^{L}\{-FA_b + w(L-l_1-l_2)\}\,dx\right]$$

By integrating and simplifying the right-hand side of the foregoing equation, we obtain the axial deformation of the member as

$$\bar{u}_{xe} = \frac{1}{EA}\left[-FA_bL + \frac{w}{2}(L-l_1-l_2)(L-l_1+l_2)\right] \quad \textbf{(6.11)}$$

The expression for FA_b can now be determined by using the compatibility condition that, because both ends b and e of the member are attached to fixed supports, the axial deformation of the member must be 0. Thus, by substituting $\bar{u}_{xe} = 0$ into Eq. (6.11), we write

$$\bar{u}_{xe} = \frac{1}{EA}\left[-FA_bL + \frac{w}{2}(L-l_1-l_2)(L-l_1+l_2)\right] = 0 \quad \textbf{(6.12)}$$

Solving Eq. (6.12) for FA_b, we obtain

$$FA_b = \frac{w}{2L}(L-l_1-l_2)(L-l_1+l_2) \quad \textbf{(6.13)}$$

With the fixed-end axial force FA_b known, we can now determine the remaining fixed-end axial force FA_e by applying the equation of equilibrium $\sum F_x = 0$ to the free body of the entire member. Thus (see Fig. 6.5(a)),

$$\overset{+}{\rightarrow} \sum F_x = 0 \qquad FA_b - w(L-l_1-l_2) + FA_e = 0$$

Substituting Eq. (6.13) into the foregoing equation, and simplifying the result, we obtain the expression for FA_e:

$$FA_e = \frac{w}{2L}(L-l_1-l_2)(L+l_1-l_2) \quad \textbf{(6.14)}$$

The expressions for fixed-end axial forces due to other types of axial loadings can be derived in a similar manner, using the integration approach illustrated here.

Once the fixed-end axial and shear forces and moments for a member have been evaluated, its fixed-end force vector $\mathbf{Q}_f$ can be generated by storing the fixed-end forces and moments in their proper positions in a 6×1 vector, as follows.

$$\mathbf{Q}_f = \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{bmatrix} = \begin{bmatrix} FA_b \\ FS_b \\ FM_b \\ FA_e \\ FS_e \\ FM_e \end{bmatrix} \quad \textbf{(6.15)}$$

The sign convention for member local fixed-end forces, $\mathbf{Q}_f$, is the same as that for the member end forces in the local coordinate system, $\mathbf{Q}$. Thus, the member local fixed-end axial forces and shears are considered positive when in the positive directions of the member's local x and y axes, and the local fixed-end moments are considered positive when counterclockwise. However, the member loads are commonly defined to be positive in the directions *opposite* to those for the local fixed-end forces. In other words, the member axial and perpendicular loads are considered positive when in the *negative* directions of the member's local x and y axes, respectively, and the external couples applied to the members are considered positive when clockwise. The expressions for the member fixed-end forces (including moments) given inside the front cover of this text are based on this sign convention, in which all the fixed-end forces and member loads (including couples) are shown in the positive sense.

EXAMPLE 6.2

The displaced position of member 2, of the frame of Fig. 6.6(a), is given in Fig. 6.6(b). Calculate the end forces for this member in the local coordinate system. Is the member in equilibrium under these forces?

SOLUTION

Member Local Stiffness Matrix: From Fig. 6.6(a), we can see that, for member 2, $E = 29{,}000$ ksi, $A = 28.2$ in.^{2}, $I = 833$ in.^{4}, and $L = \sqrt{(16)^2 + (12)^2} = 20$ ft $=$ 240 in. By substituting the numerical values of E, A, I, and L into Eq. (6.6), we obtain the following local stiffness matrix for member 2, in units of kips and inches.

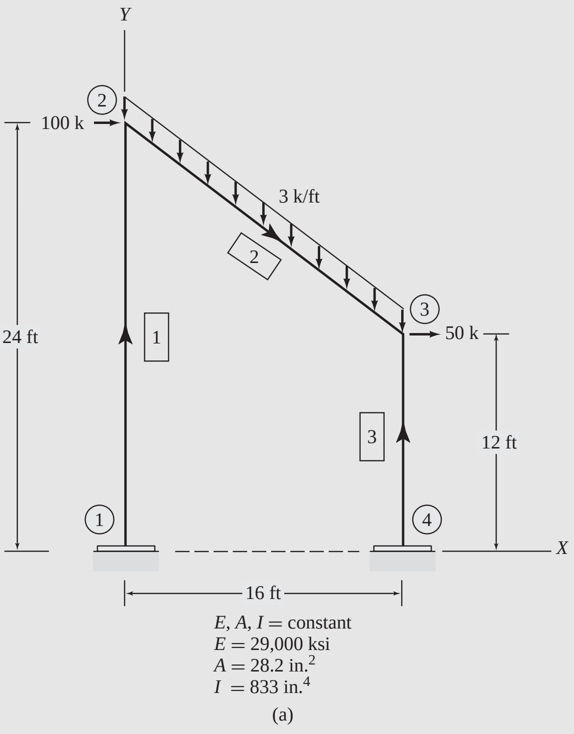
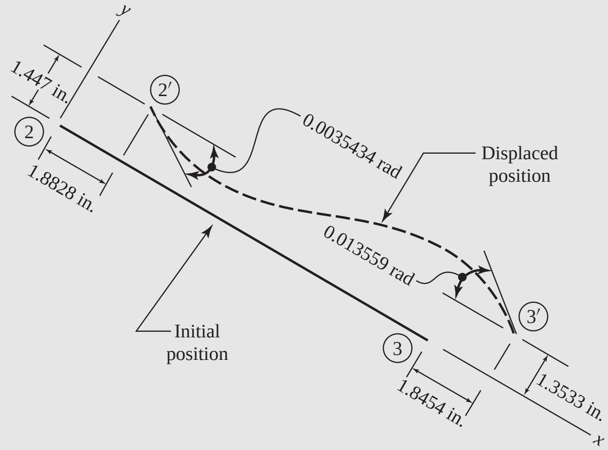
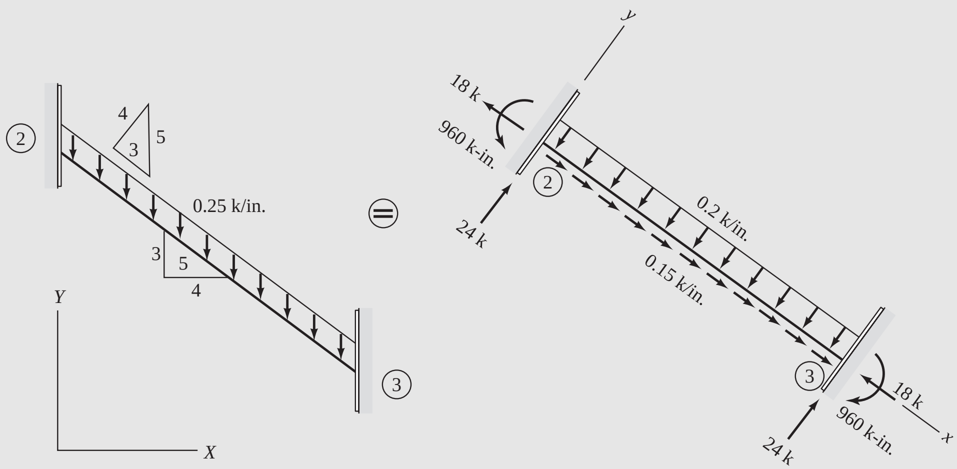


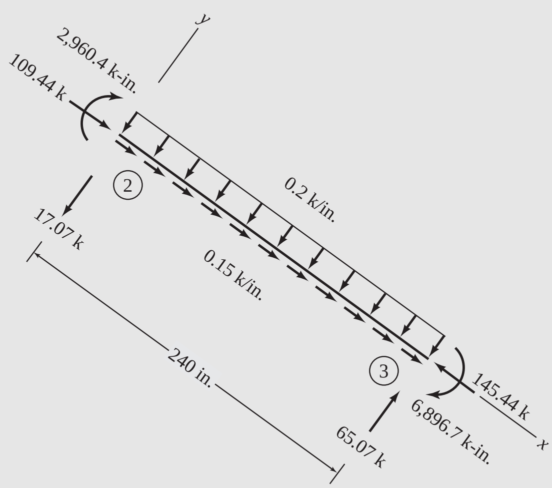
Fig. 6.6



(b) Displaced Position of Member 2



(c) Local Fixed-end Forces for Member 2



(d) Local End Forces for Member 2

Fig. 6.6 (*continued*)

$$\mathbf{k}_2 = \begin{bmatrix} 3{,}407.5 & 0 & 0 & -3{,}407.5 & 0 & 0 \\ 0 & 20.97 & 2{,}516.4 & 0 & -20.97 & 2{,}516.4 \\ 0 & 2{,}516.4 & 402{,}620 & 0 & -2{,}516.4 & 201{,}310 \\ -3{,}407.5 & 0 & 0 & 3{,}407.5 & 0 & 0 \\ 0 & -20.97 & -2{,}516.4 & 0 & 20.97 & -2{,}516.4 \\ 0 & 2{,}516.4 & 201{,}310 & 0 & -2{,}516.4 & 402{,}620 \end{bmatrix} \quad \textbf{(1)}$$

Member Local End Displacements: See Fig. 6.6(b).

$$\mathbf{u}_2 = \begin{bmatrix} 1.8828 \\ 1.4470 \\ -0.0035434 \\ 1.8454 \\ 1.3533 \\ -0.013559 \end{bmatrix} \quad \textbf{(2)}$$

Note that the values of u_3 and u_6 are negative, because both member ends rotate in the clockwise direction.

Member Local Fixed-end Force Vector: As the 0.25 k/in. (= 3 k/ft) uniformly distributed load, applied to the member, acts in the vertical direction, it is necessary to resolve it into components parallel and perpendicular to the member. The components of the vertical distributed load in the local x and y directions are (see Fig. 6.6(c)):

$$w_x = -\frac{3}{5}(0.25) = -0.15 \text{ k/in.}$$

$$w_y = \frac{4}{5}(0.25) = 0.2 \text{ k/in.}$$

in which, in accordance with the sign convention for member loads discussed previously, a negative sign is assigned to the magnitude of w_x because it acts in the positive direction of the local x axis.

The local fixed-end forces can now be evaluated, using the expressions given inside the front cover. By substituting $w = -0.15$ k/in., $L = 240$ in., and $l_1 = l_2 = 0$ into the expressions for the fixed-end axial forces given for loading type 6, we obtain

$$FA_b = FA_e = \frac{-0.15(240)}{2} = -18 \text{ k}$$

Similarly, substitution of $w = 0.2$ k/in., $L = 240$ in., and $l_1 = l_2 = 0$ into the expressions for the fixed-end shears and moments given for loading type 3 yields

$$FS_b = FS_e = \frac{0.2(240)}{2} = 24 \text{ k}$$

$$FM_b = -FM_e = \frac{0.2(240)^2}{12} = 960 \text{ k-in.}$$

These fixed-end forces for member 2 are shown in Fig. 6.6(c). The local fixed-end force vector for the member is given by

$$\mathbf{Q}_{f2} = \begin{bmatrix} -18 \\ 24 \\ 960 \\ -18 \\ 24 \\ -960 \end{bmatrix} \quad \textbf{(3)}$$

Member Local End Forces: The local end forces for member 2 can now be determined by substituting the numerical forms of $\boldsymbol{k}_2$, $\boldsymbol{u}_2$, and $\boldsymbol{Q}_{f2}$ (Eqs. (1), (2), and (3), respectively), into Eq. (6.4), and performing the required matrix multiplication and addition. This yields

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} 109.44 \text{ k} \\ -17.07 \text{ k} \\ -2{,}960.4 \text{ k-in.} \\ -145.44 \text{ k} \\ 65.07 \text{ k} \\ -6{,}896.7 \text{ k-in.} \end{bmatrix}$$ **Ans**

These member end forces are depicted in Fig. 6.6(d).

Equilibrium Check: To check whether the member is in equilibrium, we apply the three equations of equilibrium to the free body of the member shown in Fig. 6.6(d). Thus,

$+\searrow \sum F_x = 0 \quad 109.44 + 0.15(240) - 145.44 = 0$ **Checks**

$+\nearrow \sum F_y = 0 \quad -17.07 - 0.2(240) + 65.07 = 0$ **Checks**

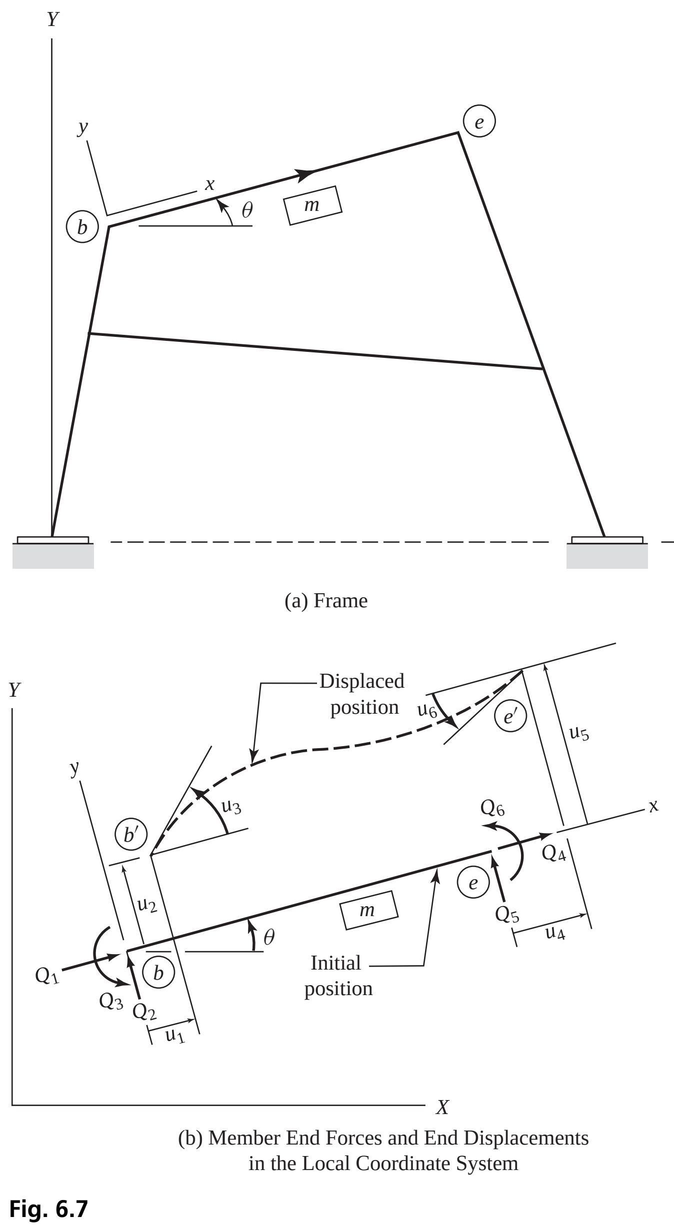
$+\circlearrowleft \sum M_{②} = 0 \quad -2{,}960.4 - 0.2(240)(120) - 6{,}896.7 + 65.07(240) = -0.3 \cong 0$ **Checks**

Therefore, the member is in equilibrium. **Ans**

6.3 COORDINATE TRANSFORMATIONS

Unlike beams, whose members all are oriented in the same direction, plane frames usually contain members oriented in various directions in the plane of the structure. Therefore, it becomes necessary to transform the stiffness relations of the members of a plane frame from their local coordinate systems to the global coordinate system before they can be combined to establish the stiffness relations for the entire frame. In this section, we extend the transformation relationships developed in Section 3.5 for plane truss members to include end moments and rotations, so that they can be used for the members of plane frames. The revised transformation relations thus obtained are then used in Section 6.4 to develop the member stiffness relations in the global coordinate system for plane frames.

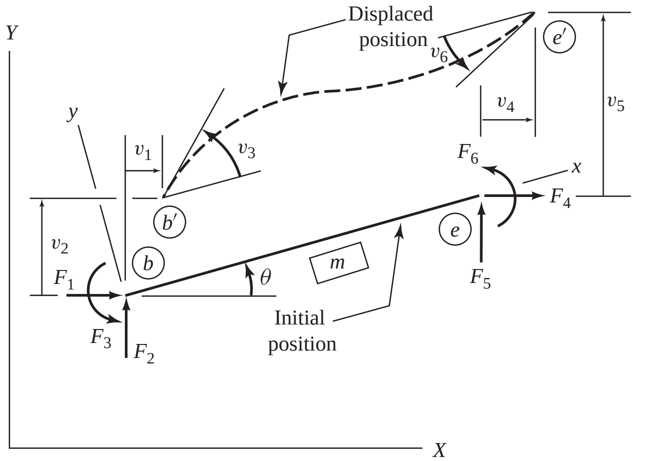
Consider an arbitrary member m of a plane frame, as shown in Fig. 6.7(a). The orientation of the member with respect to the global XY coordinate system is defined by an angle θ, measured counterclockwise from the positive direction of the global X axis to the positive direction of the local x axis, as shown in Fig. 6.7(a). When the frame is subjected to external loads, member m deforms, and internal forces and moments develop at its ends. The displaced position of member m, due to an arbitrary loading applied to the frame, is shown in Figs. 6.7(b) and (c). In Fig. 6.7(b), the member end displacements, $\mathbf{u}$, and end forces, $\mathbf{Q}$, are measured relative to the local xy coordinate system of the member; whereas, in Fig. 6.7(c), the member end displacements, **v**, and end forces, **F**, are defined with respect to the global *XY* coordinate system of the frame. The local and global systems of member end displacements and forces are *equivalent*, in the sense that both systems cause the same translations and rotations of the member ends *b* and *e*, and produce the same state of strain and stress in the member. As shown in Fig. 6.7(c), the global member end forces, **F**, and end displacements, **v**, are numbered by beginning at member end *b*, with the force and translation in the *X* direction numbered first, followed by the



(a) Frame

(b) Member End Forces and End Displacements in the Local Coordinate System

Fig. 6.7



(c) Member End Forces and End Displacements
in the Global Coordinate System

Fig. 6.7 (*continued*)

force and translation in the Y direction, and then the moment and rotation. The forces and displacements at the member's opposite end e are then numbered in the same sequential order.

Now, suppose that the member's global end forces and end displacements are specified, and we wish to determine the corresponding end forces and end displacements in the local coordinate system of the member. As discussed in Section 3.5, the local forces Q_1 and Q_2 must be equal to the algebraic sums of the components of the global forces F_1 and F_2 in the directions of the local x and y axes, respectively; that is,

$$Q_1 = F_1 \cos\theta + F_2 \sin\theta \tag{6.16a}$$

$$Q_2 = -F_1 \sin\theta + F_2 \cos\theta \tag{6.16b}$$

Note that Eqs. (6.16a and b) are identical to Eqs. (3.58a and b), respectively, derived previously for the case of plane truss members.

As for the relationship between the local end moment Q_3 and the global end moment F_3—because the local z axis and the global Z axis are oriented in the same direction (i.e., directed out of the plane of the page), the local moment Q_3 must be equal to the global moment F_3. Thus,

$$Q_3 = F_3 \tag{6.16c}$$

Using a similar reasoning at end e of the member, we express the local forces in terms of the global forces as

$$Q_4 = F_4 \cos\theta + F_5 \sin\theta \tag{6.16d}$$

$$Q_5 = -F_4 \sin\theta + F_5 \cos\theta \tag{6.16e}$$

$$Q_6 = F_6 \tag{6.16f}$$

We can write Eqs. (6.16a through f) in matrix form as

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} \tag{6.17}$$

or, symbolically, as

$$\mathbf{Q} = \mathbf{T}\mathbf{F} \tag{6.18}$$

in which the transformation matrix **T** is given by

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.19}$$

The direction cosines ($\cos\theta$ and $\sin\theta$) of the plane frame members can be evaluated using Eqs. (3.62a and b), given in Section 3.5.

Because member end displacements, like end forces, are vectors, which are defined in the same directions as the corresponding forces, the transformation matrix **T** (Eq. (6.19)) can also be used to transform member end displacements from the global to the local coordinate system; that is,

$$\boxed{\mathbf{u} = \mathbf{T}\mathbf{v}} \tag{6.20}$$

Next, we consider the transformation of member end forces and end displacements from the local to the global coordinate system. Returning our attention to Figs. 6.7(b) and (c), we realize that at end b of the member, the global forces F_1 and F_2 must be equal to the algebraic sums of the components of the local forces Q_1 and Q_2 in the directions of the global X and Y axes, respectively; that is,

$$F_1 = Q_1 \cos\theta - Q_2 \sin\theta \tag{6.21a}$$

$$F_2 = Q_1 \sin\theta + Q_2 \cos\theta \tag{6.21b}$$

and, as discussed previously, the global moment F_3 equals the local moment Q_3, or

$$F_3 = Q_3 \tag{6.21c}$$

In a similar manner, the global forces at end e of the member can be expressed in terms of the local forces as

$$F_4 = Q_4 \cos\theta - Q_5 \sin\theta \tag{6.21d}$$

$$F_5 = Q_4 \sin\theta + Q_5 \cos\theta \tag{6.21e}$$

$$F_6 = Q_6 \tag{6.21f}$$

We can write Eqs. (6.21a through f) in matrix form as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} \tag{6.22}$$

By comparing Eq. (6.22) to Eq. (6.17), we realize that the transformation matrix in Eq. (6.22), which transforms the forces from the local to the global coordinate system, is the transpose of the transformation matrix **T** in Eq. (6.17), which transforms the forces from the global to the local coordinate system. Therefore, Eq. (6.22) can be written as

$$\boxed{\mathbf{F} = \mathbf{T}^T\mathbf{Q}} \tag{6.23}$$

Also, a comparison of Eqs. (6.18) and (6.23) indicates that the inverse of **T** equals its transpose; that is,

$$\mathbf{T}^{-1} = \mathbf{T}^T \tag{6.24}$$

which indicates that the transformation matrix **T** is orthogonal.

As discussed previously, because the member end displacements are also vectors defined in the directions of their corresponding forces, the matrix $\mathbf{T}^T$ also defines the transformation of member end displacements from the local to the global coordinate system; that is,

$$\mathbf{v} = \mathbf{T}^T\,\mathbf{u} \tag{6.25}$$

By comparing the transformation matrix **T** derived herein for plane frame members (Eq. (6.19)) with the one developed in Section 3.5 for plane truss members (Eq. (3.61)), we observe that the **T** matrix for plane trusses can be obtained by deleting the third and sixth columns and the third and sixth rows from the **T** matrix for plane frame members. This is because there are no moments and rotations induced at the ends of plane truss members, which are subjected to axial forces only.

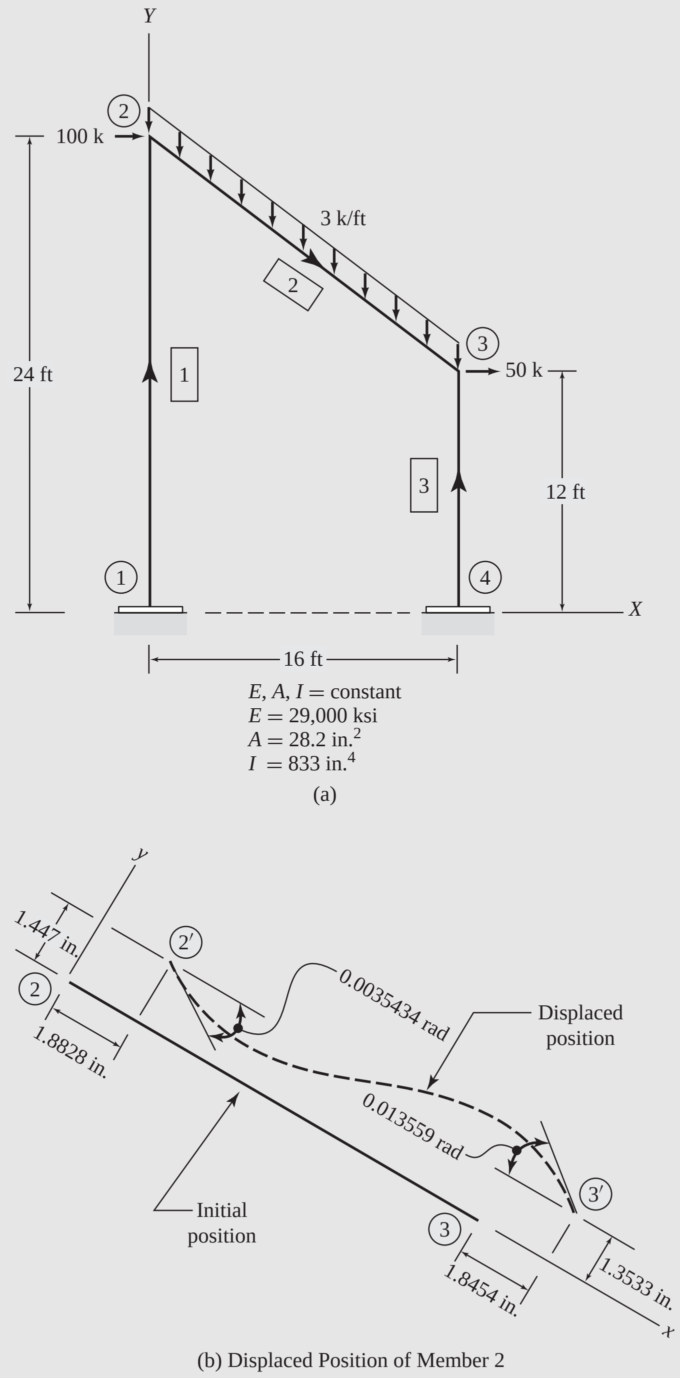
EXAMPLE 6.3

The displaced position of member 2, of the frame of Fig. 6.8(a), is given in Fig. 6.8(b). Calculate the end displacements and end forces for this member in the global coordinate system. Is the member in equilibrium under the global end forces?

SOLUTION

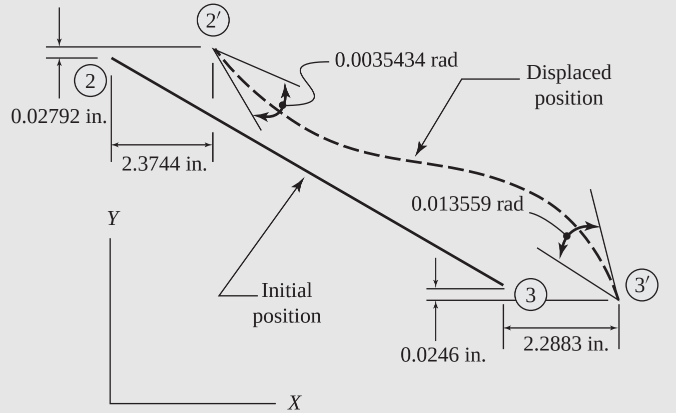
Member Local End Displacements and Forces: In Example 6.2, we obtained the local end displacement and force vectors for the member under consideration as

$$\mathbf{u}_2 = \begin{bmatrix} 1.8828 \text{ in.} \\ 1.4470 \text{ in.} \\ -0.0035434 \text{ rad} \\ 1.8454 \text{ in.} \\ 1.3533 \text{ in.} \\ -0.013559 \text{ rad} \end{bmatrix} \tag{1}$$

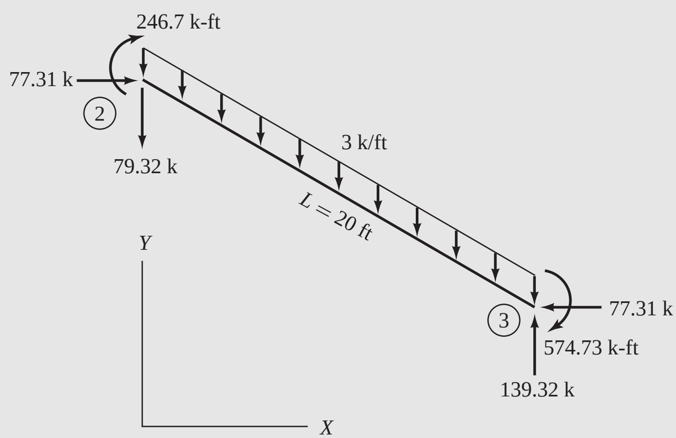


Y
100 k
3 k/ft
2
1
3
24 ft
50 k
12 ft
X
16 ft
E, A, I = constant
E = 29,000 ksi
A = 28.2 in.²
I = 833 in.⁴
(a)
y
1.447 in.
2′
0.0035434 rad
Displaced position
1.8828 in.
0.013559 rad
Initial position
3′
1.3533 in.
1.8454 in.
x
(b) Displaced Position of Member 2

Fig. 6.8



(c) End Displacements in the Global Coordinate System for Member 2



(d) End Forces in the Global Coordinate System for Member 2

Fig. 6.8 (*continued*)

and

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} 109.44 \text{ k} \\ -17.07 \text{ k} \\ -2{,}960.4 \text{ k-in.} \\ -145.44 \text{ k} \\ 65.07 \text{ k} \\ -6{,}896.7 \text{ k-in.} \end{bmatrix} \qquad \textbf{(2)}$$

Transformation Matrix: From Fig. 6.8(a), we can see that joint 2 is the beginning joint and joint 3 is the end joint for member 2. By applying Eqs. (3.62), we determine the member's direction cosines as

$$\cos\theta = \frac{X_3 - X_2}{L} = \frac{16 - 0}{20} = 0.8$$
$$\sin\theta = \frac{Y_3 - Y_2}{L} = \frac{12 - 24}{20} = -0.6$$

The transformation matrix for member 2 can now be evaluated, using Eq. (6.19).

$$\mathbf{T}_2 = \begin{bmatrix} 0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(3)}$$

Member Global End Displacements: By substituting the transpose of $\mathbf{T}_2$ from Eq. (3), and $\mathbf{u}_2$ from Eq. (1), into Eq. (6.25), we obtain

$$\mathbf{V}_2 = \mathbf{T}_2^T \mathbf{u}_2 = \begin{bmatrix} 2.3744 \text{ in.} \\ 0.02792 \text{ in.} \\ -0.0035434 \text{ rad} \\ 2.2883 \text{ in.} \\ -0.02460 \text{ in.} \\ -0.013559 \text{ rad} \end{bmatrix} \qquad \textbf{Ans}$$

These end displacements are depicted in Fig. 6.8(c).

Member Global End Forces: Similarly, by substituting the transpose of $\mathbf{T}_2$ from Eq. (3), and $\mathbf{Q}_2$ from Eq. (2), into Eq. (6.23), we determine the global end forces for member 2 to be

$$\mathbf{F}_2 = \mathbf{T}_2^T \mathbf{Q}_2 = \begin{bmatrix} 77.31 \text{ k} \\ -79.32 \text{ k} \\ \left\{ \begin{matrix} -2{,}960.4 \text{ k-in.} \\ (= -246.7 \text{ k-ft}) \end{matrix} \right\} \\ -77.31 \text{ k} \\ 139.32 \text{ k} \\ \left\{ \begin{matrix} -6{,}896.7 \text{ k-in.} \\ (= -574.73 \text{ k-ft}) \end{matrix} \right\} \end{bmatrix} \qquad \textbf{Ans}$$

The global member end forces are shown in Fig. 6.8(d).

Equilibrium Check: See Fig. 6.8(d).

$$+\rightarrow \sum F_X = 0 \qquad 77.31 - 77.31 = 0 \qquad \textbf{Checks}$$

$$+\uparrow \sum F_Y = 0 \qquad -79.32 - 3(20) + 139.32 = 0 \qquad \textbf{Checks}$$

$$+\circlearrowleft \sum M_{②} = 0 \qquad -246.7 - 3(20)\left(\frac{16}{2}\right) - 574.73 - 77.31(12)$$
$$+ 139.32(16) = -0.03 \text{ k-ft} \cong 0 \qquad \textbf{Checks}$$

Therefore, the member is in equilibrium. **Ans**

6.4 MEMBER STIFFNESS RELATIONS IN THE GLOBAL COORDINATE SYSTEM

The process of establishing the stiffness relationships for plane frame members in the global coordinate system is similar to that for the members of plane trusses (Section 3.6). We first substitute the local stiffness relations $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$ (Eq. (6.4)) into the force transformation relations $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (6.23)) to obtain

$$\mathbf{F} = \mathbf{T}^T\mathbf{Q} = \mathbf{T}^T\mathbf{ku} + \mathbf{T}^T\mathbf{Q}_f \tag{6.26}$$

Then, we substitute the displacement transformation relations $\mathbf{u} = \mathbf{Tv}$ (Eq. (6.20)) into Eq. (6.26) to determine the desired relationships between the member end forces $\mathbf{F}$ and end displacements $\mathbf{v}$, in the global coordinate system:

$$\mathbf{F} = \mathbf{T}^T\mathbf{kTv} + \mathbf{T}^T\mathbf{Q}_f \tag{6.27}$$

Equation (6.27) can be conveniently expressed as

$$\boxed{\mathbf{F} = \mathbf{Kv} + \mathbf{F}_f} \tag{6.28}$$

with

$$\boxed{\mathbf{K} = \mathbf{T}^T\mathbf{kT}} \tag{6.29}$$

$$\boxed{\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f} \tag{6.30}$$

The matrix $\mathbf{K}$ represents the member stiffness matrix in the global coordinate system; $\mathbf{F}_f$ is called the *member fixed-end force vector in the global coordinate system.*

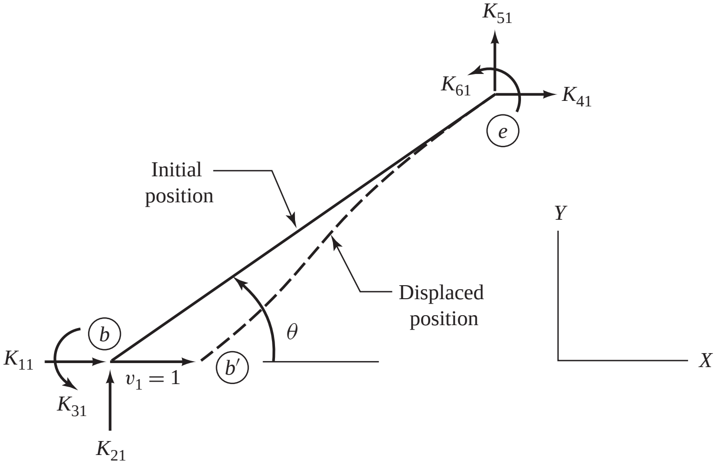
Member Global Stiffness Matrix K

The expression of the member global stiffness matrix $\mathbf{K}$ given in Eq. (6.29), as a product of the three matrices $\mathbf{T}^T$, $\mathbf{k}$, and $\mathbf{T}$, is sometimes referred to as the *matrix triple product* form of $\mathbf{K}$. The explicit form of $\mathbf{K}$, in terms of L, E, A, I, and θ of the member, can be determined by substituting the explicit forms of the member local stiffness matrix $\mathbf{k}$ from Eq. (6.6) and the member transformation matrix $\mathbf{T}$ from Eq. (6.19) into Eq. (6.29), and by multiplying the matrices $\mathbf{T}^T$, $\mathbf{k}$, and $\mathbf{T}$, in that order. The explicit form of the member global stiffness matrix $\mathbf{K}$ thus obtained is given in Eq. (6.31).

From a computer programming viewpoint, it is usually more convenient to evaluate $\mathbf{K}$ using the numerical values of $\mathbf{k}$ and $\mathbf{T}$ in the matrix triple product given in Eq. (6.29), rather than the explicit form of $\mathbf{K}$ given in Eq. (6.31). In Section 6.7, we will develop a computer subroutine to generate $\mathbf{K}$ by multiplying the numerical forms of $\mathbf{T}^T$, $\mathbf{k}$, and $\mathbf{T}$, in sequence. The explicit form of $\mathbf{K}$ (Eq. (6.31)), however, provides insight into the physical interpretation of the member global stiffness matrix, and proves convenient for evaluating $\mathbf{K}$ by hand calculations.

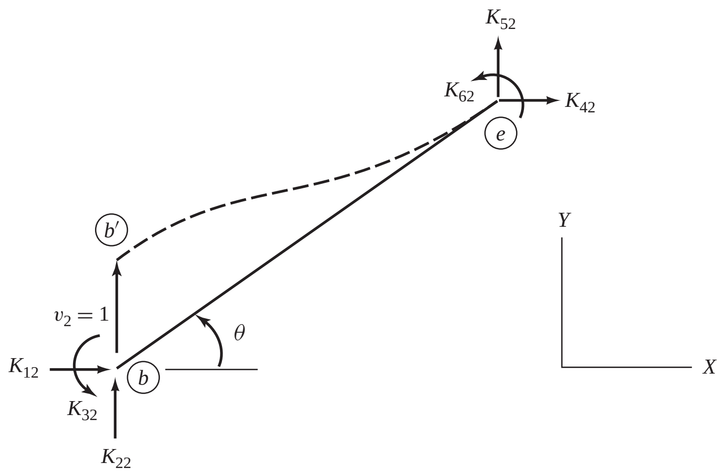
$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta & \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -6L\sin\theta & -\left(\frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta\right) & -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -6L\sin\theta \\ \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & \frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta & 6L\cos\theta & -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -\left(\frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta\right) & 6L\cos\theta \\ -6L\sin\theta & 6L\cos\theta & 4L^2 & 6L\sin\theta & -6L\cos\theta & 2L^2 \\ -\left(\frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta\right) & -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & 6L\sin\theta & \frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta & \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & 6L\sin\theta \\ -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -\left(\frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta\right) & -6L\cos\theta & \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & \frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta & -6L\cos\theta \\ -6L\sin\theta & 6L\cos\theta & 2L^2 & 6L\sin\theta & -6L\cos\theta & 4L^2 \end{bmatrix}. \tag{6.31}$$

The physical interpretation of the member global stiffness matrix $\mathbf{K}$ for plane frame members is similar to that of $\mathbf{K}$ for members of plane trusses. *A stiffness coefficient K_{ij} represents the force at the location and in the direction F_i required, along with other global end forces, to cause a unit value of displacement v_j, while all other global end displacements are 0, and the member is not subjected to any external loads between its ends.* In other words, as depicted in Figs. 6.9(a) through (f), the jth column of $\mathbf{K}$ ($j = 1$ through 6) represents the member end forces, in the global coordinate system, required to cause a unit value of the global end displacement v_j, while all other end displacements are 0, and the member is not subjected to any external loads.

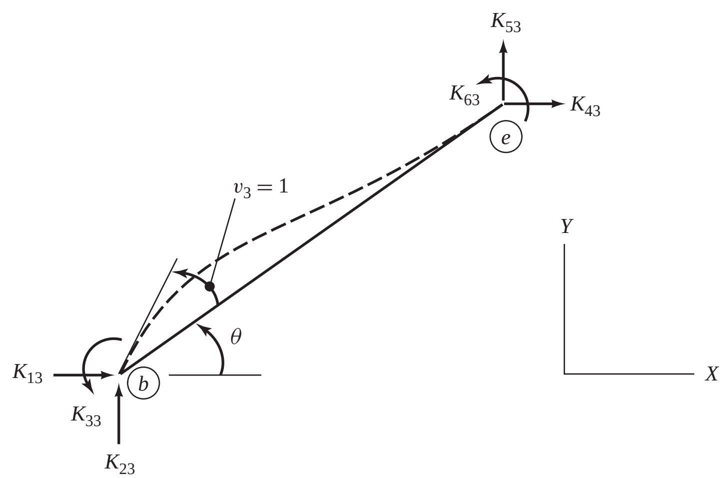


(a) First Column of $\mathbf{K}$ ($v_1 = 1$, $v_2 = v_3 = v_4 = v_5 = v_6 = 0$)

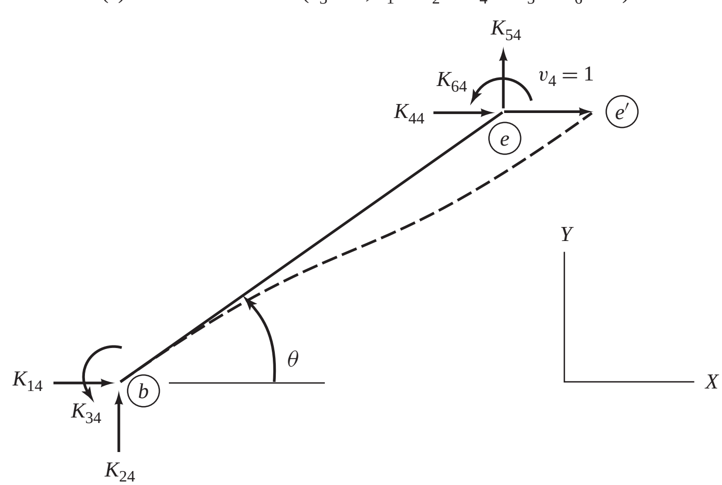
Fig. 6.9



(b) Second Column of **K** ($v_2 = 1$, $v_1 = v_3 = v_4 = v_5 = v_6 = 0$)

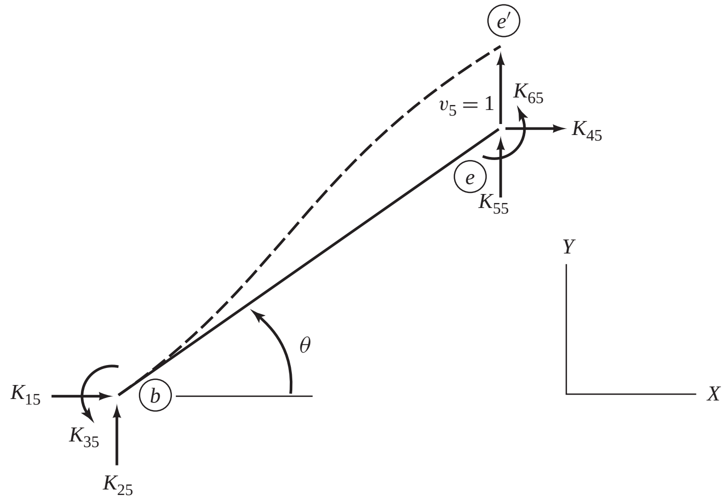


(c) Third Column of **K** ($v_3 = 1$, $v_1 = v_2 = v_4 = v_5 = v_6 = 0$)

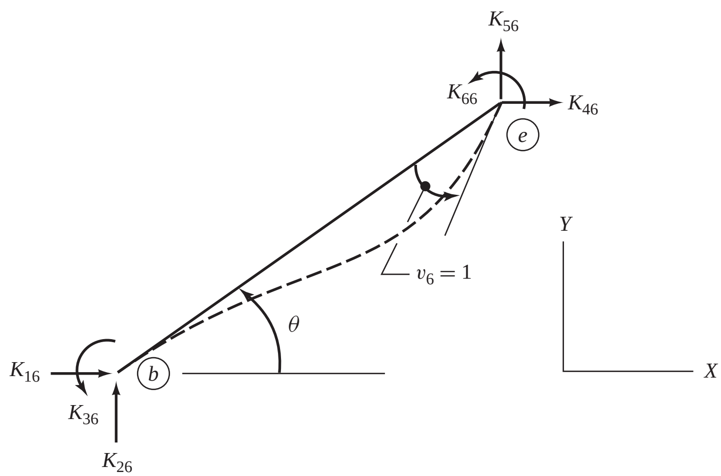


(d) Fourth Column of **K** ($v_4 = 1$, $v_1 = v_2 = v_3 = v_5 = v_6 = 0$)

Fig. 6.9 (*continued*)



(e) Fifth Column of **K** ($v_5 = 1$, $v_1 = v_2 = v_3 = v_4 = v_6 = 0$)



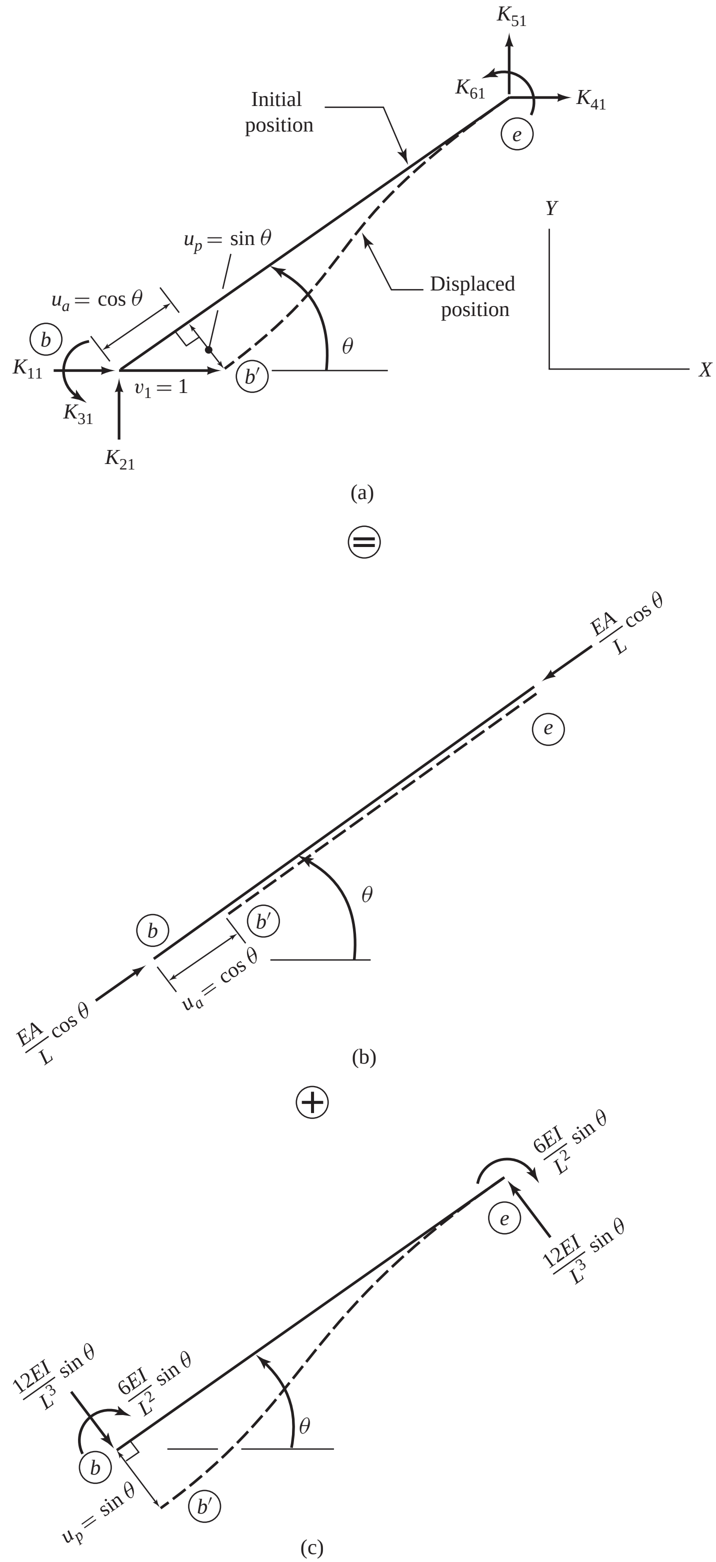
(f) Sixth Column of **K** ($v_6 = 1$, $v_1 = v_2 = v_3 = v_4 = v_5 = 0$)

Fig. 6.9 (*continued*)

We can use the foregoing interpretation of the member global stiffness matrix to check the explicit form of **K** given in Eq. (6.31). For example, to determine the first column of **K**, we subject the member to a unit end displacement $v_1 = 1$, while all other end displacements are held at 0. As shown in Fig. 6.10(a), the components of this global end displacement in the directions along, and perpendicular to, the member's longitudinal axis, respectively, are

$$u_a = v_1 \cos\theta = 1 \cos\theta = \cos\theta$$
$$u_p = v_1 \sin\theta = 1 \sin\theta = \sin\theta$$

The axial compressive force in the member caused by the axial deformation u_a is shown in Fig. 6.10(b), and the member end shears and moments due to the perpendicular displacement u_p are given in Fig. 6.10(c). Note that these



K_{51}
K_{61}
K_{41}
Initial position
e
Y
$u_p = \sin\theta$
Displaced position
$u_a = \cos\theta$
b
θ
K_{11}
$v_1 = 1$
b'
X
K_{31}
K_{21}
(a)
=
$\frac{EA}{L}\cos\theta$
e
θ
b
b'
$u_a = \cos\theta$
$\frac{EA}{L}\cos\theta$
(b)
+
$\frac{6EI}{L^2}\sin\theta$
e
$\frac{12EI}{L^3}\sin\theta$
$\frac{12EI}{L^3}\sin\theta$
$\frac{6EI}{L^2}\sin\theta$
θ
b
$u_p = \sin\theta$
b'
(c)

Fig. 6.10

member end shears and moments are obtained by multiplying the member end forces developed previously (Fig. 6.3(d)) by the negative of u_p (or by setting $u_2 = -u_p = -\sin\theta$ in Fig. 6.3(d)).

By comparing Figs. 6.10(a), (b), and (c), we realize that the global stiffness coefficients K_{11} and K_{21}, at end b of the member, must be equal to the algebraic sums in the global X and Y directions, respectively, of the member end axial force and shear at end b; that is,

$$K_{11} = \left(\frac{EA}{L}\cos\theta\right)\cos\theta + \left(\frac{12EI}{L^3}\sin\theta\right)\sin\theta$$

$$= \frac{EA}{L}\cos^2\theta + \frac{12EI}{L^3}\sin^2\theta \tag{6.32a}$$

and

$$K_{21} = \left(\frac{EA}{L}\cos\theta\right)\sin\theta - \left(\frac{12EI}{L^3}\sin\theta\right)\cos\theta$$

$$= \left(\frac{EA}{L} - \frac{12EI}{L^3}\right)\cos\theta\sin\theta \tag{6.32b}$$

Also, the global stiffness coefficient K_{31} in Fig. 6.10(a) must be equal to the member end moment in Fig. 6.10(c); that is,

$$K_{31} = -\frac{6EI}{L^2}\sin\theta \tag{6.32c}$$

Similarly, the global stiffness coefficients at end e of the member can be expressed as (see Figs. 6.10(a) through (c))

$$K_{41} = -\left(\frac{EA}{L}\cos\theta\right)\cos\theta - \left(\frac{12EI}{L^3}\sin\theta\right)\sin\theta$$

$$= -\frac{EA}{L}\cos^2\theta - \frac{12EI}{L^3}\sin^2\theta \tag{6.32d}$$

$$K_{51} = -\left(\frac{EA}{L}\cos\theta\right)\sin\theta + \left(\frac{12EI}{L^3}\sin\theta\right)\cos\theta$$

$$= -\left(\frac{EA}{L} - \frac{12EI}{L^3}\right)\cos\theta\sin\theta \tag{6.32e}$$

and

$$K_{61} = -\frac{6EI}{L^2}\sin\theta \tag{6.32f}$$

Note that the expressions for the member global stiffness coefficients, in Eqs. 6.32(a) through (f), are identical to those in the first column of the explicit form of **K** given in Eq. (6.31). The remaining columns of **K** can be verified in a similar manner.

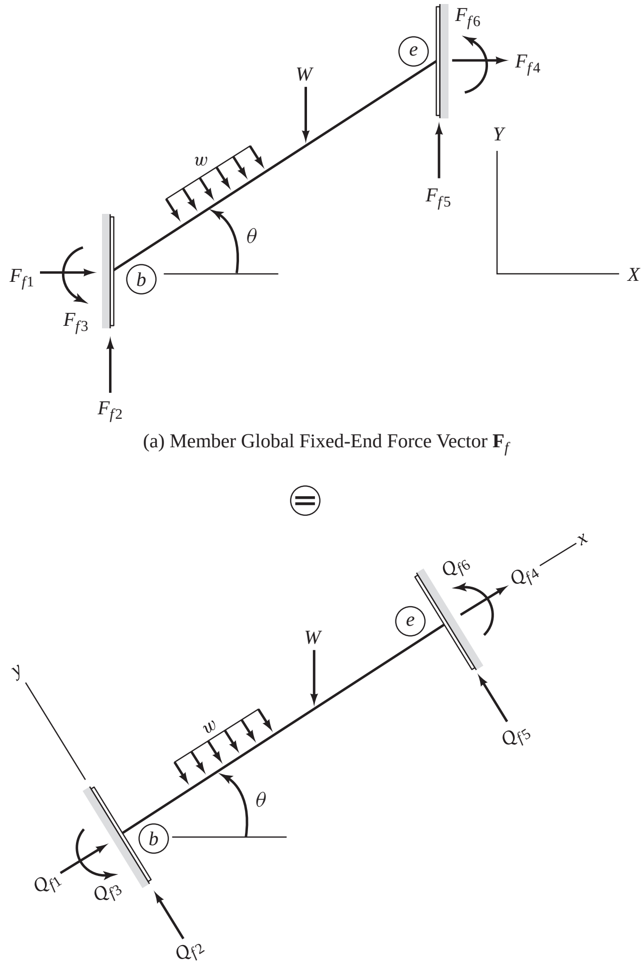
Member Global Fixed-End Force Vector $\mathbf{F}_f$

The explicit form of the member global fixed-end force vector $\mathbf{F}_f$ can be obtained by substituting Eqs. (6.19) and (6.15) into the relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$

(Eq. (6.30)). This yields

$$\mathbf{F}_f = \begin{bmatrix} FA_b \cos\theta - FS_b \sin\theta \\ FA_b \sin\theta + FS_b \cos\theta \\ FM_b \\ FA_e \cos\theta - FS_e \sin\theta \\ FA_e \sin\theta + FS_e \cos\theta \\ FM_e \end{bmatrix} \qquad \textbf{(6.33)}$$

The member global fixed-end forces $\mathbf{F}_f$, like the local fixed-end forces $\mathbf{Q}_f$, represent the forces that would develop at the member ends due to external loads, if both member ends were restrained against translations and rotations. However, the global fixed-end forces $\mathbf{F}_f$ are oriented in the global X and Y directions of the structure (Fig. 6.11(a)), whereas the local fixed-end forces $\mathbf{Q}_f$ are oriented in the local x and y directions of the member (Fig. 6.11(b)).



(a) Member Global Fixed-End Force Vector $\mathbf{F}_f$

(b) Member Local Fixed-End Force Vector $\mathbf{Q}_f$

Fig. 6.11

EXAMPLE 6.4

In Example 6.3, the global end displacement vector for member 2 of the frame of Fig. 6.8 was found to be

$$\mathbf{v}_2 = \begin{bmatrix} 2.3744 \text{ in.} \\ 0.02792 \text{ in.} \\ -0.0035434 \text{ rad} \\ 2.2883 \text{ in.} \\ -0.02460 \text{ in.} \\ -0.013559 \text{ rad} \end{bmatrix}$$

Calculate the end forces for this member in the global coordinate system using the member global stiffness relationship $\mathbf{F} = \mathbf{K}\mathbf{v} + \mathbf{F}_f$.

SOLUTION

Member Global Stiffness Matrix: It was shown in Example 6.3 that, for the member under consideration,

$$\cos\theta = 0.8 \qquad \text{and} \qquad \sin\theta = -0.6$$

By substituting these direction cosines, and the numerical values of $E = 29{,}000$ ksi, $A = 28.2$ in.^{2}, $I = 833$ in.^{4}, and $L = 240$ in., into Eq. (6.31), we evaluate the global stiffness matrix for member 2 as

$$\mathbf{K}_2 = \begin{bmatrix} 2{,}188.3 & -1{,}625.5 & 1{,}509.8 & -2{,}188.3 & 1{,}625.5 & 1{,}509.8 \\ -1{,}625.5 & 1{,}240.1 & 2{,}013.1 & 1{,}625.5 & -1{,}240.1 & 2{,}013.1 \\ 1{,}509.8 & 2{,}013.1 & 402{,}620 & -1{,}509.8 & -2{,}013.1 & 201{,}310 \\ -2{,}188.3 & 1{,}625.5 & -1{,}509.8 & 2{,}188.3 & -1{,}625.5 & -1{,}509.8 \\ 1{,}625.5 & -1{,}240.1 & -2{,}013.1 & -1{,}625.5 & 1{,}240.1 & -2{,}013.1 \\ 1{,}509.8 & 2{,}013.1 & 201{,}310 & -1{,}509.8 & -2{,}013.1 & 402{,}620 \end{bmatrix}$$

The matrix $\mathbf{K}_2$ can be obtained alternatively by substituting the numerical forms of $\mathbf{k}_2$ (Eq. (1) of Example 6.2) and $\mathbf{T}_2$ (Eq. (3) of Example 6.3) into the relationship $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$ (Eq. (6.29)), and by evaluating the matrix triple product. The reader is encouraged to use this alternative approach to verify the foregoing $\mathbf{K}_2$ matrix.

Member Global Fixed-end Force Vector: From Example 6.2: $FA_b = FA_e = -18$ k; $FS_b = FS_e = 24$ k; and $FM_b = -FM_e = 960$ k-in. By substituting these numerical values, and $\cos\theta = 0.8$ and $\sin\theta = -0.6$, into Eq. (6.33), we obtain

$$\mathbf{F}_{f2} = \begin{bmatrix} 0 \\ 30 \\ 960 \\ 0 \\ 30 \\ -960 \end{bmatrix}$$

Again, the reader is urged to verify this $\mathbf{F}_{f2}$ vector by substituting the numerical values of $\mathbf{Q}_{f2}$ (Eq. (3) of Example 6.2) and $\mathbf{T}_2$ (Eq. (3) of Example 6.3) into the relationship $\mathbf{F}_f = \mathbf{T}^T\,\mathbf{Q}_f$ (Eq. (6.30)), and by performing the matrix multiplication.

Member Global End Forces: The global end forces for member 2 can now be determined by applying Eq. (6.28):

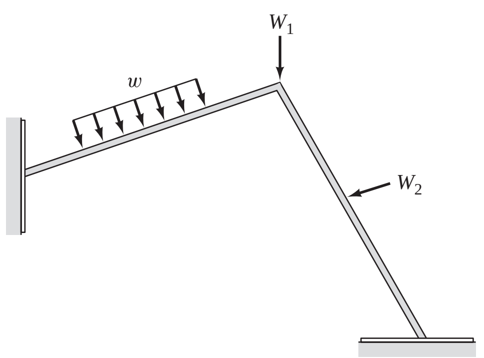
$$\mathbf{F}_2 = \mathbf{K}_2\mathbf{v}_2 + \mathbf{F}_{f2} = \begin{bmatrix} 77.22\text{ k} \\ -79.25\text{ k} \\ \left\{\begin{matrix} -2{,}960.5\text{ k-in.} \\ (= -246.7\text{ k-ft}) \end{matrix}\right\} \\ -77.22\text{ k} \\ 139.25\text{ k} \\ \left\{\begin{matrix} -6{,}896.7\text{ k-in.} \\ (= -574.73\text{ k-ft}) \end{matrix}\right\} \end{bmatrix} \qquad \textbf{Ans}$$

Note that this $\mathbf{F}_2$ vector is the same as the one obtained in Example 6.3 by transforming the member end forces from the local to the global coordinate system.

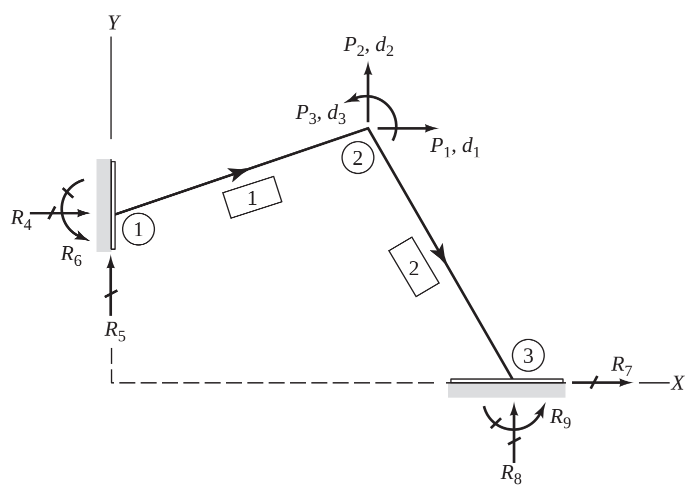
Equilibrium check: See Example 6.3.

6.5 STRUCTURE STIFFNESS RELATIONS

The process of establishing the structure stiffness relations for plane frames is essentially the same as that for beams (Section 5.5), except that the member global (instead of local) stiffness relations must now be used to assemble the structure stiffness matrices and the fixed-joint force vectors. Consider, for example, an arbitrary plane frame as shown in Fig. 6.12(a). As the analytical model of the frame in Fig. 6.12(b) indicates, the frame has three degrees of freedom, d_1, d_2, and d_3, with the corresponding joint loads designated P_1, P_2,



(a) Plane Frame



(b) Analytical Model

Fig. 6.12

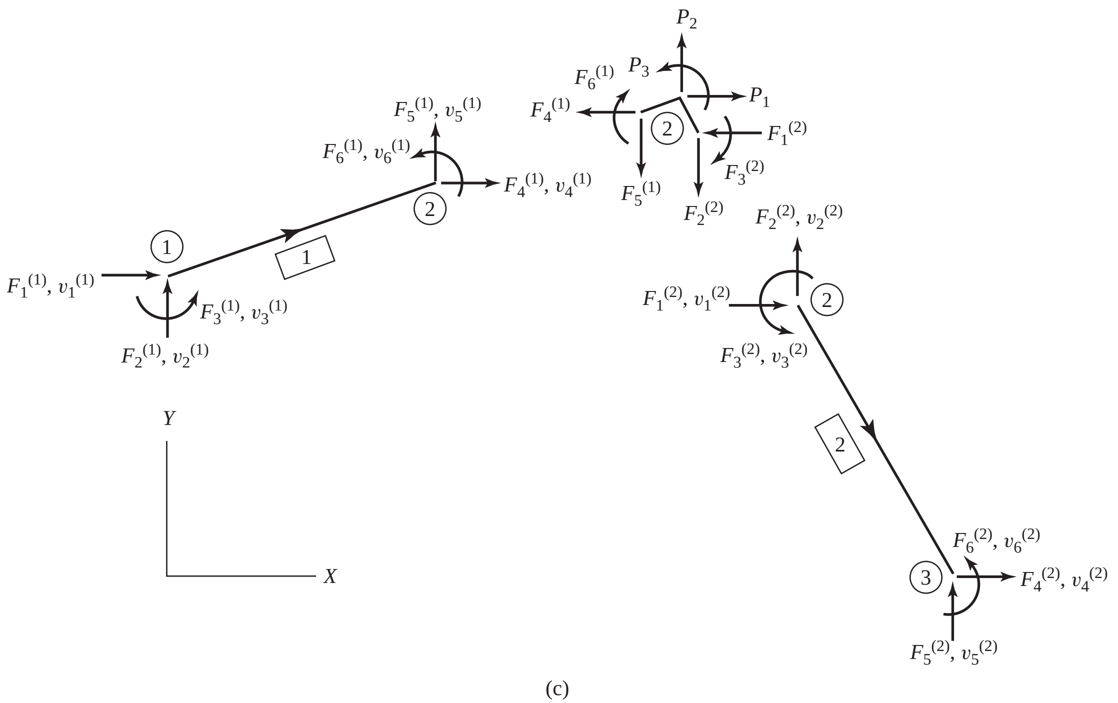


Fig. 6.12 (*continued*)

and P_3, respectively. Remember that our objective is to relate the known external joint (and member) loads to the as yet unknown joint displacements **d**.

To achieve our objective, we first relate the joint loads **P** to the member global end forces **F** by writing the joint equilibrium equations. By applying the three equations of equilibrium, $\sum F_X = 0$, $\sum F_Y = 0$, and $\sum M = 0$, to the free body of joint 2 drawn in Fig. 6.12(c), we obtain

$$P_1 = F_4^{(1)} + F_1^{(2)} \tag{6.34a}$$

$$P_2 = F_5^{(1)} + F_2^{(2)} \tag{6.34b}$$

$$P_3 = F_6^{(1)} + F_3^{(2)} \tag{6.34c}$$

in which the superscript (i) denotes the member number.

Next, we relate the joint displacements **d** to the member global end displacements **v** by applying the compatibility conditions that the member end displacements must be the same as the corresponding joint displacements. Thus, by comparing Figs. 6.12(b) and (c), we write the compatibility equations for members 1 and 2, respectively, as

$$v_1^{(1)} = v_2^{(1)} = v_3^{(1)} = 0 \qquad v_4^{(1)} = d_1 \qquad v_5^{(1)} = d_2 \qquad v_6^{(1)} = d_3 \tag{6.35}$$

$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = d_3 \qquad v_4^{(2)} = v_5^{(2)} = v_6^{(2)} = 0 \tag{6.36}$$

With the relationships between **P** and **F**, and **v** and **d**, now established, we express the member end forces **F** that appear in the equilibrium equations (Eqs. (6.34)) in terms of the member end displacements **v**, using the member global stiffness relations $\mathbf{F} = \mathbf{K}\mathbf{v} + \mathbf{F}_f$ (Eq. (6.28)). By writing this equation in expanded form for an arbitrary member i ($i = 1$ or 2), we obtain

$$\begin{bmatrix} F_1^{(i)} \\ F_2^{(i)} \\ F_3^{(i)} \\ F_4^{(i)} \\ F_5^{(i)} \\ F_6^{(i)} \end{bmatrix} = \begin{bmatrix} K_{11}^{(i)} & K_{12}^{(i)} & K_{13}^{(i)} & K_{14}^{(i)} & K_{15}^{(i)} & K_{16}^{(i)} \\ K_{21}^{(i)} & K_{22}^{(i)} & K_{23}^{(i)} & K_{24}^{(i)} & K_{25}^{(i)} & K_{26}^{(i)} \\ K_{31}^{(i)} & K_{32}^{(i)} & K_{33}^{(i)} & K_{34}^{(i)} & K_{35}^{(i)} & K_{36}^{(i)} \\ K_{41}^{(i)} & K_{42}^{(i)} & K_{43}^{(i)} & K_{44}^{(i)} & K_{45}^{(i)} & K_{46}^{(i)} \\ K_{51}^{(i)} & K_{52}^{(i)} & K_{53}^{(i)} & K_{54}^{(i)} & K_{55}^{(i)} & K_{56}^{(i)} \\ K_{61}^{(i)} & K_{62}^{(i)} & K_{63}^{(i)} & K_{64}^{(i)} & K_{65}^{(i)} & K_{66}^{(i)} \end{bmatrix} \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ v_3^{(i)} \\ v_4^{(i)} \\ v_5^{(i)} \\ v_6^{(i)} \end{bmatrix} + \begin{bmatrix} F_{f1}^{(i)} \\ F_{f2}^{(i)} \\ F_{f3}^{(i)} \\ F_{f4}^{(i)} \\ F_{f5}^{(i)} \\ F_{f6}^{(i)} \end{bmatrix} \tag{6.37}$$

From this, we determine the expressions for forces at end 2 of member 1 (i.e., $i = 1$) to be

$$\begin{aligned} F_4^{(1)} = K_{41}^{(1)} v_1^{(1)} + K_{42}^{(1)} v_2^{(1)} + K_{43}^{(1)} v_3^{(1)} + K_{44}^{(1)} v_4^{(1)} \\ + K_{45}^{(1)} v_5^{(1)} + K_{46}^{(1)} v_6^{(1)} + F_{f4}^{(1)} \end{aligned} \tag{6.38a}$$

$$\begin{aligned} F_5^{(1)} = K_{51}^{(1)} v_1^{(1)} + K_{52}^{(1)} v_2^{(1)} + K_{53}^{(1)} v_3^{(1)} + K_{54}^{(1)} v_4^{(1)} \\ + K_{55}^{(1)} v_5^{(1)} + K_{56}^{(1)} v_6^{(1)} + F_{f5}^{(1)} \end{aligned} \tag{6.38b}$$

$$\begin{aligned} F_6^{(1)} = K_{61}^{(1)} v_1^{(1)} + K_{62}^{(1)} v_2^{(1)} + K_{63}^{(1)} v_3^{(1)} + K_{64}^{(1)} v_4^{(1)} \\ + K_{65}^{(1)} v_5^{(1)} + K_{66}^{(1)} v_6^{(1)} + F_{f6}^{(1)} \end{aligned} \tag{6.38c}$$

Similarly, from Eq. (6.37), we determine the expressions for forces at end 2 of member 2 (i.e., $i = 2$) to be

$$\begin{aligned} F_1^{(2)} = K_{11}^{(2)} v_1^{(2)} + K_{12}^{(2)} v_2^{(2)} + K_{13}^{(2)} v_3^{(2)} + K_{14}^{(2)} v_4^{(2)} \\ + K_{15}^{(2)} v_5^{(2)} + K_{16}^{(2)} v_6^{(2)} + F_{f1}^{(2)} \end{aligned} \tag{6.39a}$$

$$\begin{aligned} F_2^{(2)} = K_{21}^{(2)} v_1^{(2)} + K_{22}^{(2)} v_2^{(2)} + K_{23}^{(2)} v_3^{(2)} + K_{24}^{(2)} v_4^{(2)} \\ + K_{25}^{(2)} v_5^{(2)} + K_{26}^{(2)} v_6^{(2)} + F_{f2}^{(2)} \end{aligned} \tag{6.39b}$$

$$\begin{aligned} F_3^{(2)} = K_{31}^{(2)} v_1^{(2)} + K_{32}^{(2)} v_2^{(2)} + K_{33}^{(2)} v_3^{(2)} + K_{34}^{(2)} v_4^{(2)} \\ + K_{35}^{(2)} v_5^{(2)} + K_{36}^{(2)} v_6^{(2)} + F_{f3}^{(2)} \end{aligned} \tag{6.39c}$$

By substituting the compatibility equations for members 1 and 2 (Eqs. (6.35) and (6.36)) into Eqs. (6.38) and (6.39), respectively, we obtain

$$F_4^{(1)} = K_{44}^{(1)} d_1 + K_{45}^{(1)} d_2 + K_{46}^{(1)} d_3 + F_{f4}^{(1)} \tag{6.40a}$$

$$F_5^{(1)} = K_{54}^{(1)} d_1 + K_{55}^{(1)} d_2 + K_{56}^{(1)} d_3 + F_{f5}^{(1)} \tag{6.40b}$$

$$F_6^{(1)} = K_{64}^{(1)} d_1 + K_{65}^{(1)} d_2 + K_{66}^{(1)} d_3 + F_{f6}^{(1)} \tag{6.40c}$$

$$F_1^{(2)} = K_{11}^{(2)} d_1 + K_{12}^{(2)} d_2 + K_{13}^{(2)} d_3 + F_{f1}^{(2)} \tag{6.40d}$$

$$F_2^{(2)} = K_{21}^{(2)} d_1 + K_{22}^{(2)} d_2 + K_{23}^{(2)} d_3 + F_{f2}^{(2)} \tag{6.40e}$$

$$F_3^{(2)} = K_{31}^{(2)} d_1 + K_{32}^{(2)} d_2 + K_{33}^{(2)} d_3 + F_{f3}^{(2)} \tag{6.40f}$$

Finally, by substituting Eqs. (6.40) into the joint equilibrium equations (Eqs. (6.34)), we obtain the desired structure stiffness relations for the plane frame:

$$\begin{aligned} P_1 = {} & \left(K_{44}^{(1)} + K_{11}^{(2)}\right) d_1 + \left(K_{45}^{(1)} + K_{12}^{(2)}\right) d_2 \\ & + \left(K_{46}^{(1)} + K_{13}^{(2)}\right) d_3 + \left(F_{f4}^{(1)} + F_{f1}^{(2)}\right) \end{aligned} \tag{6.41a}$$

$$\begin{aligned} P_2 = {} & \left(K_{54}^{(1)} + K_{21}^{(2)}\right) d_1 + \left(K_{55}^{(1)} + K_{22}^{(2)}\right) d_2 \\ & + \left(K_{56}^{(1)} + K_{23}^{(2)}\right) d_3 + \left(F_{f5}^{(1)} + F_{f2}^{(2)}\right) \end{aligned} \tag{6.41b}$$

$$\begin{aligned} P_3 = {} & \left(K_{64}^{(1)} + K_{31}^{(2)}\right) d_1 + \left(K_{65}^{(1)} + K_{32}^{(2)}\right) d_2 \\ & + \left(K_{66}^{(1)} + K_{33}^{(2)}\right) d_3 + \left(F_{f6}^{(1)} + F_{f3}^{(2)}\right) \end{aligned} \tag{6.41c}$$

The foregoing equations can be symbolically expressed as

$$\mathbf{P} = \mathbf{Sd} + \mathbf{P}_f$$

or

$$\boxed{\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}} \tag{6.42}$$

in which **S** represents the *NDOF* $\times$ *NDOF* structure stiffness matrix, and $\mathbf{P}_f$ is the *NDOF* $\times$ 1 structure fixed-joint force vector, for the plane frame with

$$\mathbf{S} = \begin{bmatrix} K_{44}^{(1)} + K_{11}^{(2)} & K_{45}^{(1)} + K_{12}^{(2)} & K_{46}^{(1)} + K_{13}^{(2)} \\ K_{54}^{(1)} + K_{21}^{(2)} & K_{55}^{(1)} + K_{22}^{(2)} & K_{56}^{(1)} + K_{23}^{(2)} \\ K_{64}^{(1)} + K_{31}^{(2)} & K_{65}^{(1)} + K_{32}^{(2)} & K_{66}^{(1)} + K_{33}^{(2)} \end{bmatrix} \tag{6.43}$$

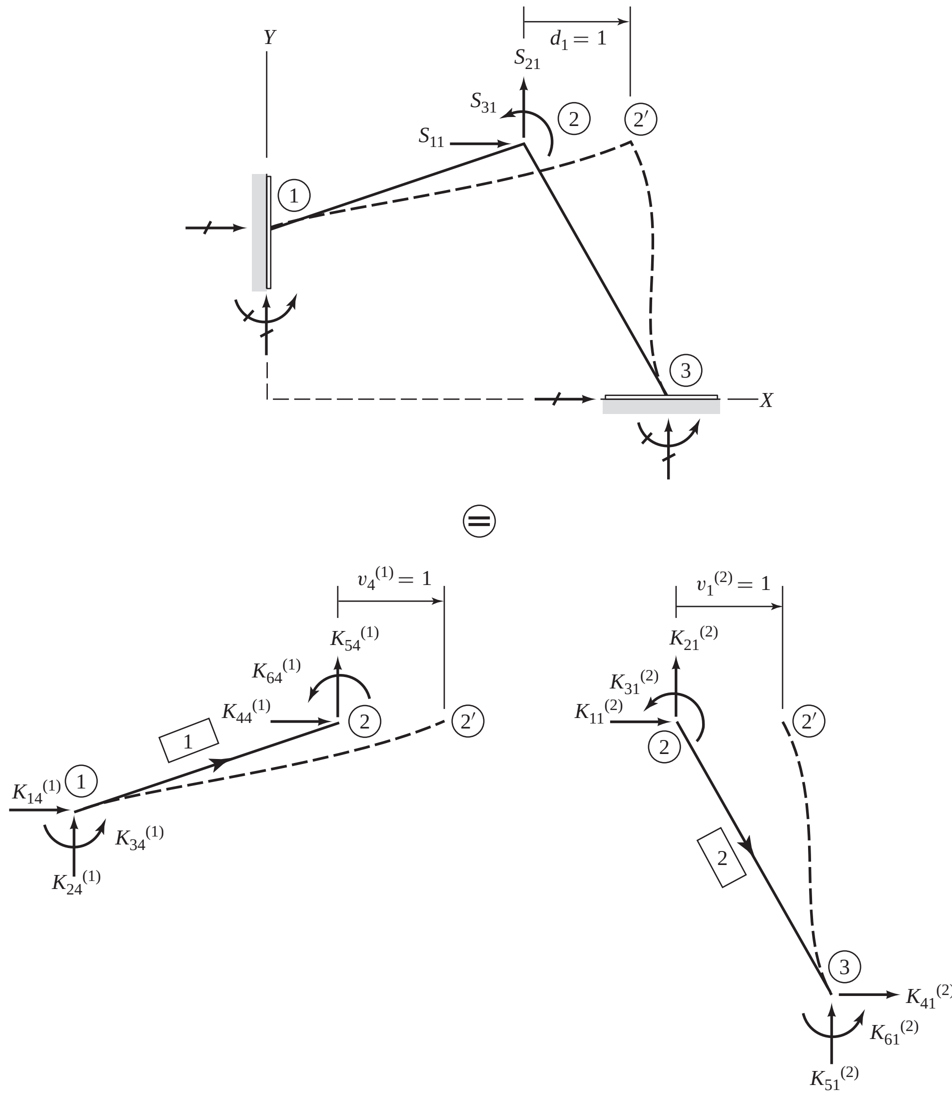
and

$$\mathbf{P}_f = \begin{bmatrix} F_{f4}^{(1)} + F_{f1}^{(2)} \\ F_{f5}^{(1)} + F_{f2}^{(2)} \\ F_{f6}^{(1)} + F_{f3}^{(2)} \end{bmatrix} \tag{6.44}$$

Structure Stiffness Matrix S

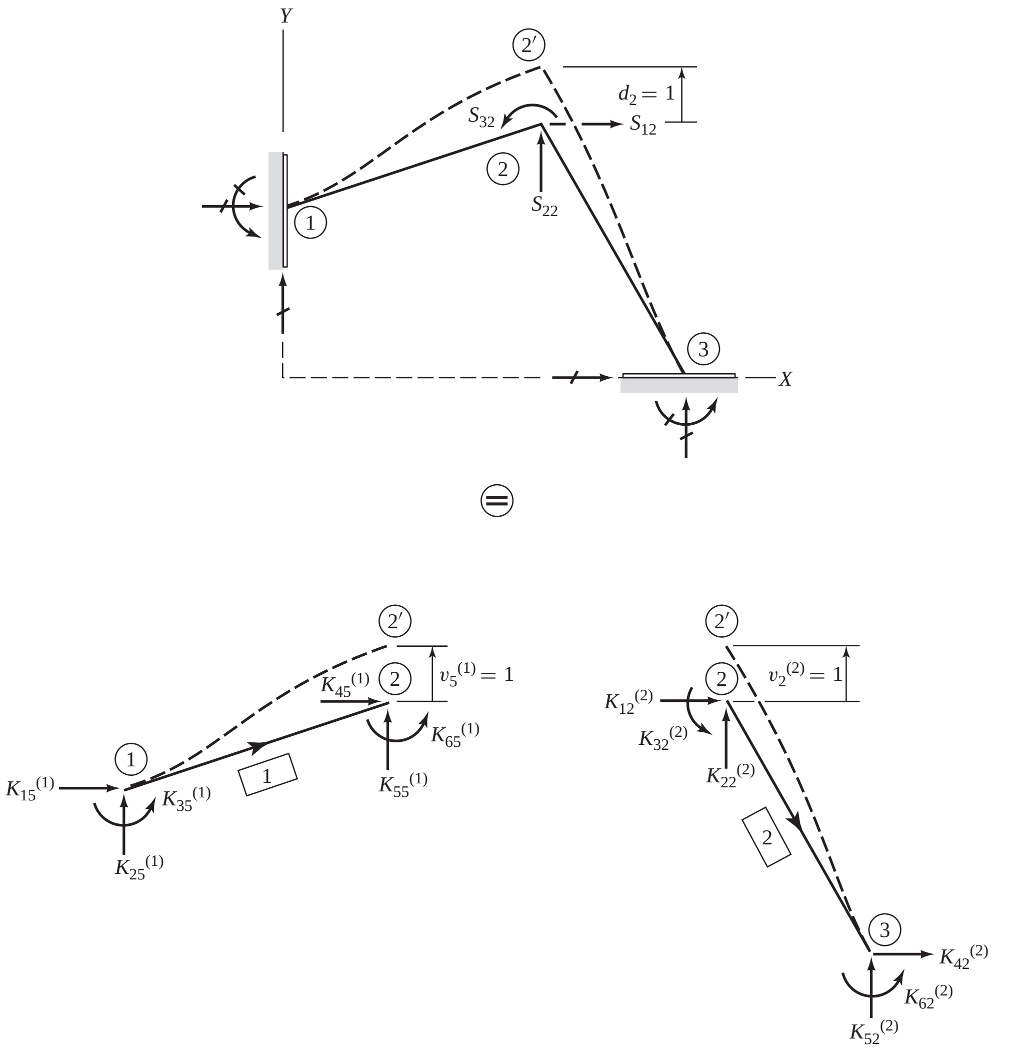
As discussed in Chapters 3 and 5, an element S_{ij} of the structure stiffness matrix **S** represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are 0, and the frame is subjected to no external loads. In other words, the jth column of **S** consists of joint forces required, at the locations and in the directions of all the degrees of freedom of the frame, to cause a unit value of the displacement d_j while all other joint displacements are 0.

We can use the foregoing interpretation to verify the **S** matrix given in Eq. (6.43) for the frame of Fig. 6.12. To obtain the first column of **S**, we subject the frame to a unit value of the joint displacement $d_1 = 1$ ($d_2 = d_3 = 0$), as shown in Fig. 6.13(a). As depicted there, this unit joint displacement induces



(a) First Column of **S** ($d_1 = 1$, $d_2 = d_3 = 0$)

Fig. 6.13



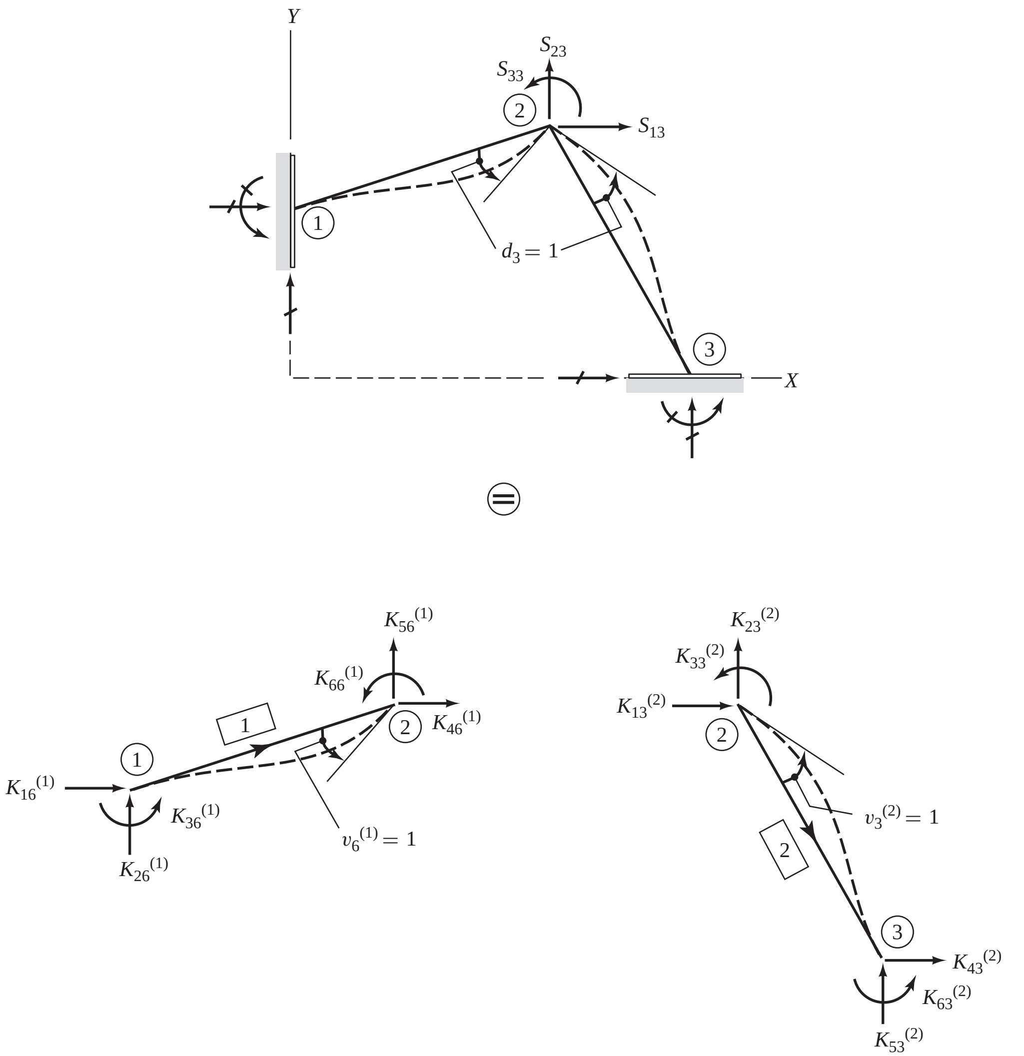
(b) Second Column of **S** ($d_2 = 1$, $d_1 = d_3 = 0$)

Fig. 6.13 (*continued*)

unit global end displacements $v_4^{(1)}$ at the end of member 1, and $v_1^{(2)}$ at the beginning of member 2. The member global stiffness coefficients, necessary to cause the foregoing end displacements, are also given in Fig. 6.13(a). From this figure, we can see that the structure stiffness coefficients (or joint forces) S_{11} and S_{21} at joint 2 must be equal to the algebraic sums of the forces in the X and Y directions, respectively, at the two member ends connected to the joint; that is,

$$S_{11} = K_{44}^{(1)} + K_{11}^{(2)} \tag{6.45a}$$

$$S_{21} = K_{54}^{(1)} + K_{21}^{(2)} \tag{6.45b}$$



(c) Third Column of **S** ($d_3 = 1$, $d_1 = d_2 = 0$)

Fig. 6.13 (*continued*)

Similarly, the structure stiffness coefficient (or joint moment) S_{31} at joint 2 must be equal to the algebraic sum of the moments at the two member ends connected to the joint; thus,

$$S_{31} = K_{64}^{(1)} + K_{31}^{(2)} \tag{6.45c}$$

Note that the expressions for S_{i1} ($i = 1$ to 3) given in Eqs. (6.45) are identical to those listed in the first column of **S** in Eq. (6.43).

The second and third columns of **S** can be verified in a similar manner using Figs. 6.13(b) and (c), respectively. It should be noted that the structure stiffness matrix **S** in Eq. (6.43) is symmetric, because of the symmetry of the member global stiffness matrices (i.e., $K_{ij} = K_{ji}$). The structure stiffness matrices of linear elastic structures must always be symmetric.

Structure Fixed-Joint Force Vector $\mathbf{P}_f$ and Equivalent Joint Loads

As discussed in Chapter 5, the structure fixed-joint forces represent the reactions that would develop at the locations and in the directions of the frame's degrees of freedom, due to member loads, if all the joints of the frame were fixed against translations and rotations. This definition enables us to directly express the structure fixed-joint forces in terms of the member global fixed-end forces (instead of deriving such expressions by combining the frame's equilibrium, compatibility, and member force-displacement relations, as was done in the earlier part of this section—see Eqs. (6.34) through (6.44)).

Let us verify the $\mathbf{P}_f$ vector, given in Eq. (6.44) for the frame of Fig. 6.12, using this direct approach. The frame is redrawn in Fig. 6.14(a) with its joint 2, which is actually free to translate in the X and Y directions and rotate, now restrained against these displacements by an imaginary restraint. When this hypothetical completely fixed frame is subjected to member loads only (note that the joint load W_1 shown in Fig. 6.12(a) is not drawn in Fig. 6.14(a)), the structure fixed-joint forces (or reactions) P_{f1}, P_{f2}, and P_{f3} develop at the imaginary restraint at joint 2. As shown in Fig. 6.14(a), the structure fixed-joint force at the location and in the direction of an ith degree of freedom is denoted by P_{fi}.

To relate the structure fixed-joint forces $\mathbf{P}_f$ to the member global fixed-end forces $\mathbf{F}_f$, we draw the free-body diagrams of the two members of the hypothetical fixed frame, as shown in Fig. 6.14(b). Note that, because all the joints of the frame are restrained, the member ends, which are rigidly connected to the joints, are also fixed against any displacements. Therefore, only the fixed-end forces due to member loads, $\mathbf{F}_f$, can develop at the ends of the members.

By comparing Figs. 6.14(a) and (b), we realize that the structure fixed-joint forces P_{f1} and P_{f2} at joint 2 must be equal to the algebraic sums of the fixed-end forces in the X and Y directions, respectively, at the two member ends connected to the joint; that is,

$$P_{f1} = F_{f4}^{(1)} + F_{f1}^{(2)} \tag{6.46a}$$

$$P_{f2} = F_{f5}^{(1)} + F_{f2}^{(2)} \tag{6.46b}$$

Similarly, the structure fixed-joint moment P_{f3} at joint 2 must be equal to the algebraic sum of the fixed-end moments at the two member ends connected to the joint. Thus,

$$P_{f3} = F_{f6}^{(1)} + F_{f3}^{(2)} \tag{6.46c}$$

Note that the expressions for P_{fi} ($i = 1$ to 3) given in Eqs. (6.46) are the same as those listed in the $\mathbf{P}_f$ vector in Eq. (6.44).

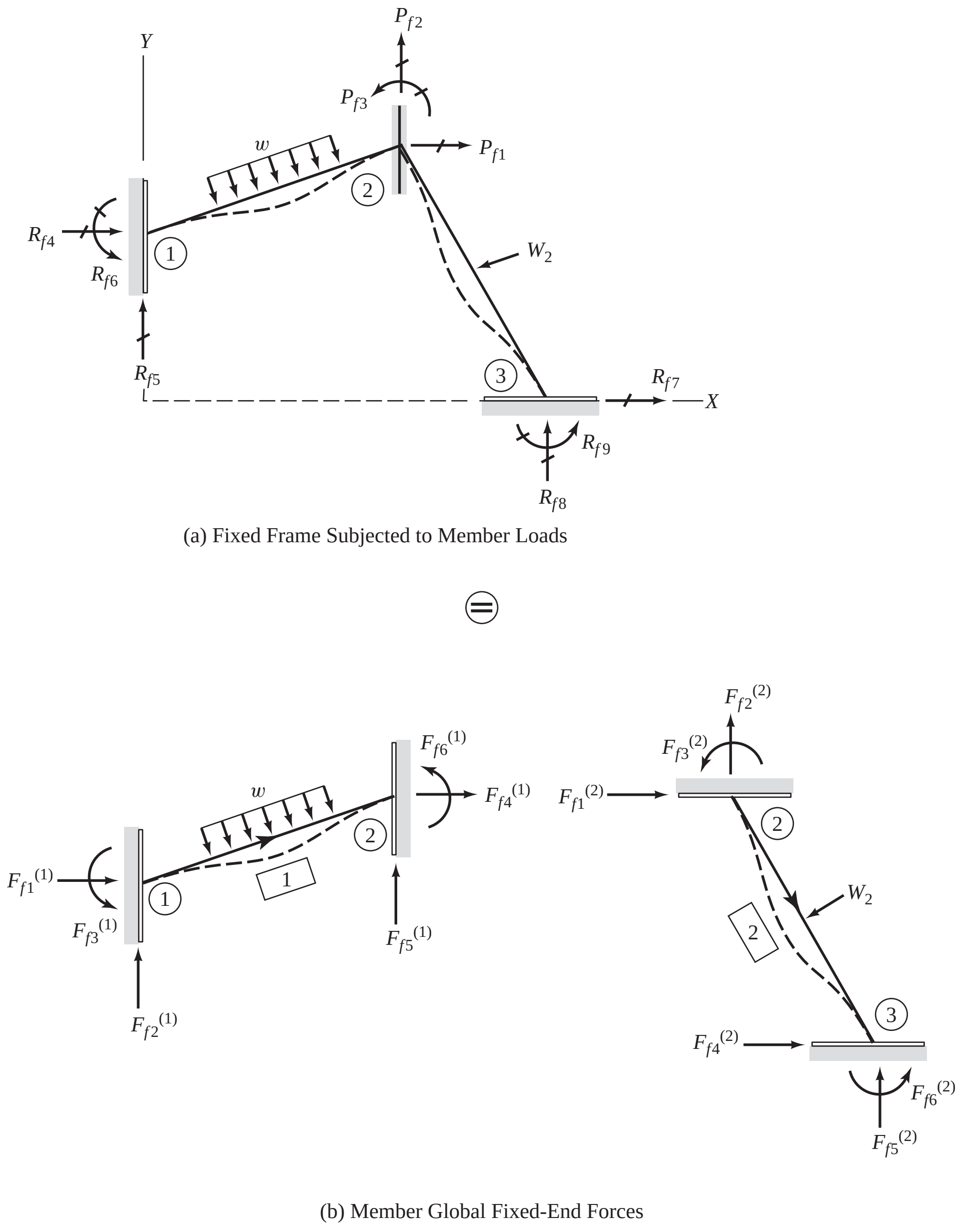


Fig. 6.14

It may be recalled from Section 5.6 that another interpretation of the structure fixed-joint forces due to member loads is that when they are applied to the structure with their directions reversed, the fixed-joint forces cause the same joint displacements as the actual member loads. The negatives of the structure fixed-joint forces are, therefore, referred to as the equivalent joint loads. We can show the validity of this interpretation by setting the joint loads equal to 0 (i.e., $\mathbf{P} = \mathbf{0}$) in Eq. (6.42), thereby reducing the structure stiffness relationship to

$$-\mathbf{P}_f = \mathbf{S}\mathbf{d} \tag{6.47}$$

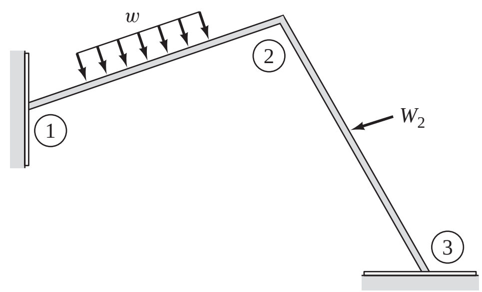
in which $\mathbf{d}$ now represents the joint displacements due to the negatives of the structure fixed-joint forces applied to the joints of the structure. However, since member loads are now the only external effects acting on the structure, the $\mathbf{d}$ vector in Eq. (6.47) must also represent the joint displacements due to member loads. Thus, we can conclude that the negatives of the structure fixed-joint forces must cause the same joint displacements as the actual member loads.

The validity of this interpretation can also be demonstrated using the principle of superposition. Figure 6.15(a) shows the two-member frame considered previously (Fig. 6.12), subjected to arbitrary member loads w and W_2. In Fig. 6.15(b), joint 2 of the frame is fixed by an imaginary restraint so that, when the fixed frame is subjected to member loads, the structure fixed-joint forces P_{f1}, P_{f2}, and P_{f3} develop at the imaginary restraint. Lastly, in Fig. 6.15(c), the actual frame is subjected to joint loads, which are equal in magnitude to the structure fixed-joint forces P_{f1}, P_{f2}, and P_{f3}, but reversed in direction.

By comparing Figs. 6.15(a) through (c), we realize that the actual loading applied to the actual frame in Fig. 6.15(a) equals the algebraic sum of the loadings in Figs. 6.15(b) and (c), because the reactions P_{f1}, P_{f2}, and P_{f3} in Fig. 6.15(b) cancel the corresponding applied loads in Fig. 6.15(c). Thus, in accordance with the superposition principle, any joint displacement of the actual frame due to the member loads (Fig. 6.15(a)) must be equal to the algebraic sum of the corresponding joint displacement of the fixed frame due to the member loads (Fig. 6.15(b)) and the corresponding joint displacement of the actual frame subjected to no member loads, but to the structure fixed-joint forces with their directions reversed. However, since the joint displacements of the fixed frame (Fig. 6.15(b)) are 0, the joint displacements of the frame due to the member loads (Fig. 6.15(a)) must be equal to the corresponding joint displacements of the frame due to the negatives of the fixed-joint forces (Fig. 6.15(c)). Thus, the negatives of the structure fixed-joint forces can be considered to be equivalent to member loads in terms of joint displacements. It should be noted that this equivalency is valid only for joint displacements, and it cannot be generalized to member end forces and support reactions.

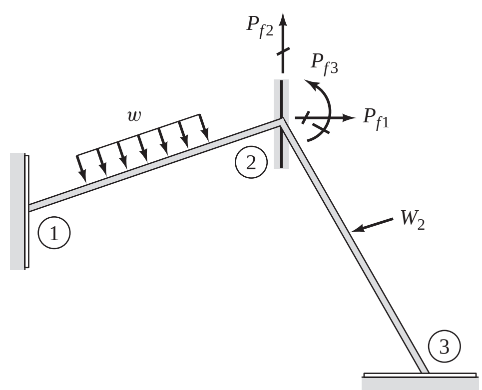
Assembly of S and $\mathbf{P}_f$, Using Member Code Numbers

In the preceding paragraphs of this section, we have demonstrated that the structure stiffness matrix $\mathbf{S}$ for plane frames can be formulated directly by algebraically adding the appropriate elements of the member global stiffness matrices $\mathbf{K}$ (see, for example, Eqs. (6.43) and (6.45), and Fig. 6.13). Furthermore, it has been shown that the structure fixed-joint force vector $\mathbf{P}_f$ for plane frames can also be established directly by algebraically adding the member global fixed-end forces $\mathbf{F}_f$ at the location, and in the direction, of each of the structure's degrees of freedom (see, for example, Eqs. (6.44) and (6.46), and Fig. 6.14).



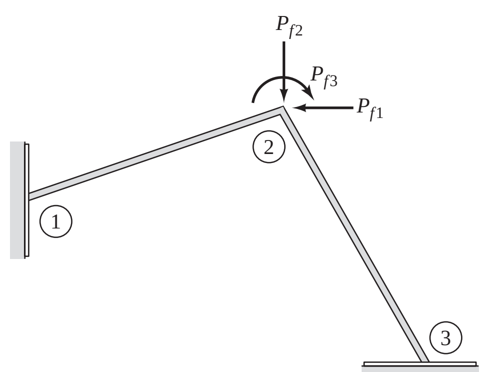
(a) Actual Frame Subjected to Member Loads

=



(b) Fixed Frame Subjected to Member Loads

+



(c) Actual Frame Subjected to Equivalent Joint Loads

Fig. 6.15

The foregoing process of directly generating **S** and $\mathbf{P}_f$ can be conveniently implemented by employing the member code number technique described in detail in Chapters 3 and 5. The application of this technique for plane frames remains essentially the same as that for the case of beams, except that the member global (instead of local) stiffness matrices **K** and the member global fixed-end force vectors $\mathbf{F}_f$ must now be used to form **S** and $\mathbf{P}_f$, respectively. It should also be realized that each member of the plane frame has six code numbers, arranged in the sequential order of the member's global end displacements **v**. The application of the member code number technique for plane frames is illustrated in the following example.

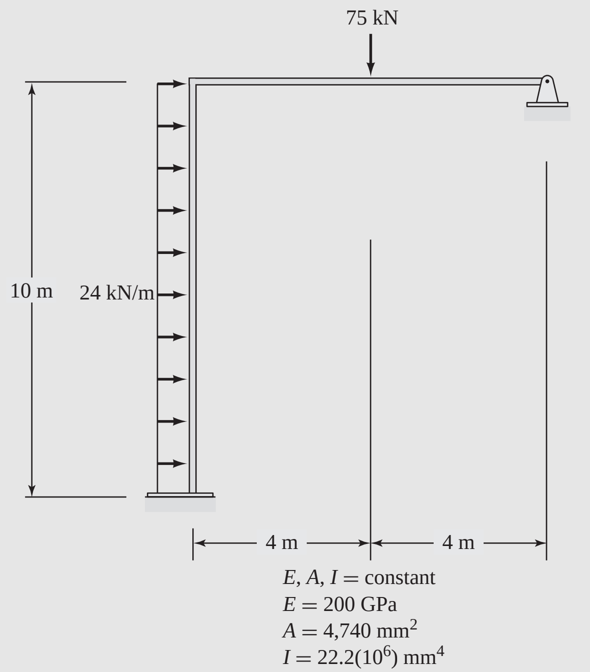
EXAMPLE 6.5

Determine the structure stiffness matrix, the fixed-joint force vector, and the equivalent joint loads for the frame shown in Fig. 6.16(a).

SOLUTION

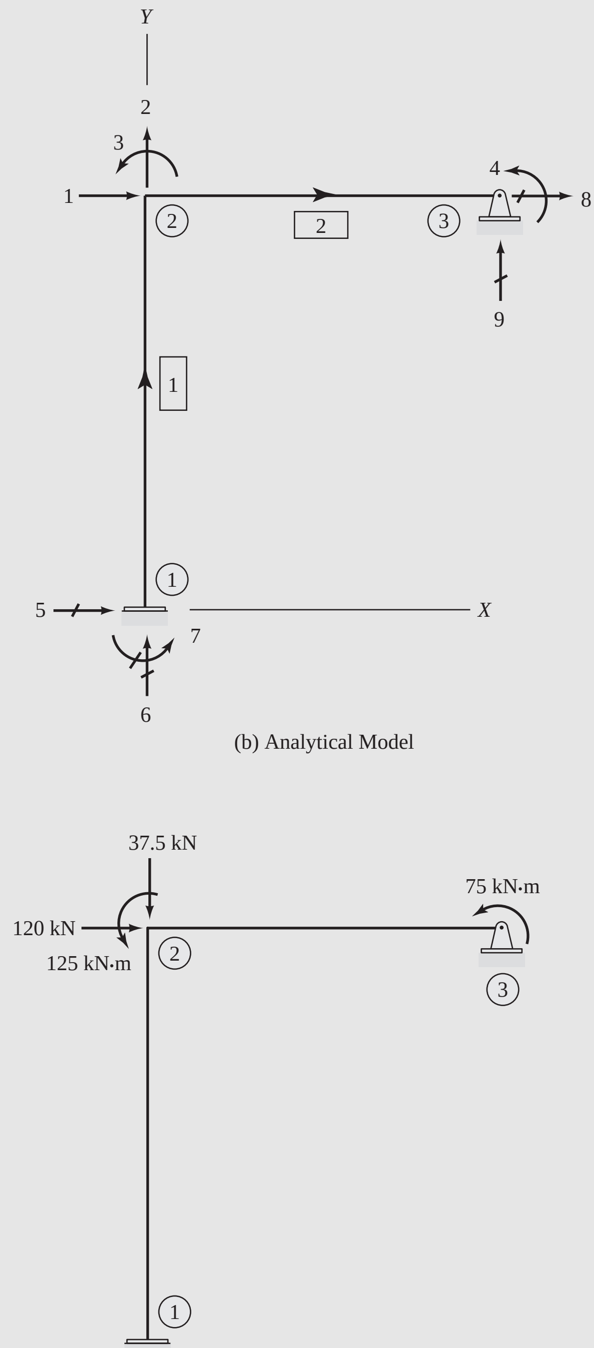
Analytical Model: See Fig. 6.16(b). The frame has four degrees of freedom and five restrained coordinates, as shown.

Structure Stiffness Matrix: The 4 × 4 structure stiffness matrix will be generated by evaluating each member's global stiffness matrix **K**, and storing its pertinent elements in **S** using the member code numbers.



(a) Frame

Fig. 6.16



(b) Analytical Model

(c) Equivalent Joint Loads

Fig. 6.16 (*continued*)

Member 1 As shown in Fig. 6.16(b), joint 1 is the beginning joint and joint 2 is the end joint for this member. Thus,

$$\cos\theta = \frac{X_2 - X_1}{L} = \frac{0-0}{10} = 0$$

$$\sin\theta = \frac{Y_2 - Y_1}{L} = \frac{10-0}{10} = 1$$

By substituting $E = 200(10)^6$ kN/m^2, $A = 0.00474$ m^2, $I = 0.0000222$ m^4, $L = 10$ m, and the foregoing values of the direction cosines into the expression for **K** given in Eq. (6.31), we obtain

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccccc} 5 & 6 & 7 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|ccc} 53.28 & 0 & -266.4 & -53.28 & 0 & -266.4 \\ 0 & 94{,}800 & 0 & 0 & -94{,}800 & 0 \\ -266.4 & 0 & 1{,}776 & 266.4 & 0 & 888 \\ \hline -53.28 & 0 & 266.4 & 53.28 & 0 & 266.4 \\ 0 & -94{,}800 & 0 & 0 & 94{,}800 & 0 \\ -266.4 & 0 & 888 & 266.4 & 0 & 1{,}776 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 1 \\ 2 \\ 3 \end{array} \qquad \textbf{(1)}$$

From Fig. 6.16(b), we observe that the code numbers for member 1 are 5, 6, 7, 1, 2, 3. These numbers are written on the right side and at the top of $\mathbf{K}_1$ (Eq. (1)), to indicate the rows and columns of **S** in which the elements of $\mathbf{K}_1$ are to be stored. Thus,

$$\mathbf{S} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 53.28 & 0 & 266.4 & 0 \\ 0 & 94{,}800 & 0 & 0 \\ 266.4 & 0 & 1{,}776 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \textbf{(2)}$$

Member 2 From Fig. 6.16(b), we can see that this member is horizontal, with its left-end joint 2 selected as the beginning joint, thereby orienting the positive directions of the member's local x and y axes in the positive directions of the global X and Y axes, respectively. Thus, no coordinate transformations are needed for this member (i.e., $\cos\theta = 1$, $\sin\theta = 0$, and $\mathbf{T} = \mathbf{I}$); and its stiffness relations, and fixed-end forces, are the same in the local and global coordinate systems.

By substituting the numerical values of E, A, and I, and $L = 8$ m into Eq. (6.6), we obtain

$$\mathbf{K}_2 = \mathbf{k}_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 8 & 9 & 4 \end{array} \\ \left[\begin{array}{ccc|cc|c} 118{,}500 & 0 & 0 & -118{,}500 & 0 & 0 \\ 0 & 104.06 & 416.25 & 0 & -104.06 & 416.25 \\ 0 & 416.25 & 2{,}220 & 0 & -416.25 & 1{,}110 \\ \hline -118{,}500 & 0 & 0 & 118{,}500 & 0 & 0 \\ 0 & -104.06 & -416.25 & 0 & 104.06 & -416.25 \\ \hline 0 & 416.25 & 1{,}110 & 0 & -416.25 & 2{,}220 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 8 \\ 9 \\ 4 \end{array}$$

The code numbers for this member—1, 2, 3, 8, 9, 4 (see Fig. 6.16(b))—are now used to add the pertinent elements of $\mathbf{K}_2$ in their proper positions in the structure stiffness matrix **S** given in Eq. (2), which now becomes

$$\begin{array}{c} \\ \mathbf{S} = \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 53.28+118{,}500 & 0 & 266.4 & 0 \\ 0 & 94{,}800+104.06 & 416.25 & 416.25 \\ 266.4 & 416.25 & 1{,}776+2{,}220 & 1{,}110 \\ 0 & 416.25 & 1{,}110 & 2{,}220 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Because the stiffnesses of both members of the frame have now been stored in **S**, the structure stiffness matrix for the given frame is

$$\begin{array}{c} \\ \mathbf{S} = \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 118{,}553 & 0 & 266.4 & 0 \\ 0 & 94{,}904 & 416.25 & 416.25 \\ 266.4 & 416.25 & 3{,}996 & 1{,}110 \\ 0 & 416.25 & 1{,}110 & 2{,}220 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \textbf{Ans}$$

Note that the structure stiffness matrix is symmetric.

Structure Fixed-Joint Force Vector: We will generate the 4×1 structure fixed-joint force vector by evaluating, for each member, the global fixed-end force vector $\mathbf{F}_f$, and storing its pertinent elements in $\mathbf{P}_f$ using the member code numbers.

Member 1 The 24 kN/m uniformly distributed load acting on this member is positive, because it acts in the negative direction of the member's local y axis. By substituting $w = 24$ kN/m, $L = 10$ m, and $l_1 = l_2 = 0$ into the fixed-end force expressions for loading type 3 listed inside the front cover, we evaluate

$$FS_b = FS_e = \frac{24(10)}{2} = 120 \text{ kN}$$

$$FM_b = -FM_e = \frac{24(10)^2}{12} = 200 \text{ kN} \cdot \text{m}$$

As the member is not subjected to any axial loads,

$$FA_b = FA_e = 0$$

By substituting the foregoing values of the member fixed-end forces, along with $\cos\theta = 0$ and $\sin\theta = 1$, into the explicit form of $\mathbf{F}_f$ given in Eq. (6.33), we obtain

$$\mathbf{F}_{f1} = \left[\begin{array}{r} -120 \\ 0 \\ 200 \\ \hdashline -120 \\ 0 \\ -200 \end{array}\right] \begin{array}{c} 5 \\ 6 \\ 7 \\ 1 \\ 2 \\ 3 \end{array} \qquad \textbf{(3)}$$

The code numbers of the member, 5, 6, 7, 1, 2, 3, are written on the right side of $\mathbf{F}_{f1}$ in Eq. (3) to indicate the rows of $\mathbf{P}_f$ in which the elements of $\mathbf{F}_{f1}$ are to be stored. Thus,

$$\mathbf{P}_f = \left[\begin{array}{r} -120 \\ 0 \\ -200 \\ 0 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \textbf{(4)}$$

Member 2 By substituting $P = 75$ kN, $L = 8$ m, and $l_1 = l_2 = 4$ m into the fixed-end force expressions for loading type 1, we determine the member fixed-end shears and moments to be

$$FS_b = FS_e = \frac{75}{2} = 37.5 \text{ kN}$$

$$FM_b = -FM_e = \frac{75(8)}{8} = 75 \text{ kN} \cdot \text{m}$$

As no axial loads are applied to this member,

$$FA_b = FA_e = 0$$

Thus,

$$\mathbf{F}_{f2} = \mathbf{Q}_{f2} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 37.5 \\ 75 \\ \hdashline 0 \\ 37.5 \\ \hdashline -75 \end{array}\right] \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 8 \\ 9 \\ 4 \end{array}$$

Using the member code numbers 1, 2, 3, 8, 9, 4, we add the pertinent elements of $\mathbf{F}_{f2}$ in their proper positions in $\mathbf{P}_f$ (as given in Eq. (4)), which now becomes

$$\mathbf{P}_f = \left[\begin{array}{c} -120 \\ 37.5 \\ -200 + 75 \\ -75 \end{array}\right] \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Because the fixed-end forces for both members of the frame have now been stored in $\mathbf{P}_f$, the structure fixed-joint force vector for the given frame is

$$\mathbf{P}_f = \left[\begin{array}{c} -120 \\ 37.5 \\ -125 \\ -75 \end{array}\right] \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}$$ **Ans**

Equivalent Joint Loads:

$$\mathbf{P}_e = -\mathbf{P}_f = \left[\begin{array}{c} 120 \\ -37.5 \\ 125 \\ 75 \end{array}\right] \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}$$ **Ans**

The equivalent joint loads are depicted in Fig. 6.16(c). These equivalent joint loads cause the same joint displacements of the frame as the actual member loads of Fig. 6.16(a).

6.6 PROCEDURE FOR ANALYSIS

Using the concepts discussed in the previous sections, we can now develop the following step-by-step procedure for the analysis of plane frames by the matrix stiffness method.

1. Prepare an analytical model of the structure, identifying its degrees of freedom and restrained coordinates (as discussed in Section 6.1). Recall that for horizontal members, the coordinate transformations can be avoided by selecting the left-end joint of the member as the beginning joint.

2. Evaluate the structure stiffness matrix $\mathbf{S}(NDOF \times NDOF)$ and fixed-joint force vector $\mathbf{P}_f$ $(NDOF \times 1)$. For each member of the structure, perform the following operations:

a. Calculate the length and direction cosines (i.e., $\cos\theta$ and $\sin\theta$) of the member (Eqs. (3.62)).

b. Compute the member stiffness matrix in the global coordinate system, $\mathbf{K}$, using its explicit form given in Eq. (6.31). The member global stiffness matrix alternatively can be obtained by first forming the member local stiffness matrix $\mathbf{k}$ (Eq. (6.6)) and the transformation matrix $\mathbf{T}$ (Eq. (6.19)), and then evaluating the matrix triple product, $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$ (Eq. (6.29)). The matrix $\mathbf{K}$ must be symmetric.

c. If the member is subjected to external loads, then evaluate the member fixed-end force vector in the global coordinate system, $\mathbf{F}_f$, using the expressions for fixed-end forces given inside the front cover, and the explicit form of $\mathbf{F}_f$ given in Eq. (6.33). The member global fixed-end force vector can also be obtained by first forming the member local fixed-end force vector $\mathbf{Q}_f$ (Eq. (6.15)), and then using the relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$ (Eq. (6.30)).

d. Identify the member code numbers and store the pertinent elements of $\mathbf{K}$ and $\mathbf{F}_f$ in their proper positions in the structure stiffness matrix $\mathbf{S}$ and the fixed-joint force vector $\mathbf{P}_f$, respectively.

The complete structure stiffness matrix $\mathbf{S}$, obtained by assembling the stiffness coefficients of all the members of the structure, must be symmetric.

3. If the structure is subjected to joint loads, then form the joint load vector $\mathbf{P}(NDOF \times 1)$.

4. Determine the joint displacements $\mathbf{d}$. Substitute $\mathbf{P}$, $\mathbf{P}_f$, and $\mathbf{S}$ into the structure stiffness relationship, $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$ (Eq. (6.42)), and solve the resulting system of simultaneous equations for the unknown joint displacements $\mathbf{d}$. To check the solution for correctness, substitute the numerical values of the joint displacements $\mathbf{d}$ back into the stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$. If the solution is correct, then the stiffness relationship should be satisfied. Note that joint translations are considered positive when in the positive directions of the global X and Y axes, and joint rotations are considered positive when counterclockwise.

5. Compute member end displacements and end forces, and support reactions. For each member of the structure, carryout the following steps:

a. Obtain member end displacements in the global coordinate system, $\mathbf{v}$, from the joint displacements, $\mathbf{d}$, using the member code numbers.

b. Form the member transformation matrix $\mathbf{T}$ (Eq. (6.19)), and determine the member end displacements in the local coordinate system, $\mathbf{u}$, using the transformation relationship $\mathbf{u} = \mathbf{Tv}$ (Eq. (6.20)).
c. Form the member local stiffness matrix $\mathbf{k}$ (Eq. (6.6)) and local fixed-end force vector $\mathbf{Q}_f$ (Eq. (6.15)); then calculate the member end forces in the local coordinate system, $\mathbf{Q}$, using the stiffness relationship $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$ (Eq. (6.4)).
d. Determine the member end forces in the global coordinate system, $\mathbf{F}$, using the transformation relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (6.23)).
e. If the member is attached to a support joint, then use the member code numbers to store the pertinent elements of $\mathbf{F}$ in their proper positions in the support reaction vector $\mathbf{R}$.

6. Check the calculation of member end forces and support reactions by applying the equilibrium equations $\left(\sum F_X = 0, \sum F_Y = 0, \text{and} \sum M = 0\right)$ to the free body of the entire structure. If the calculations have been carried out correctly, then the equilibrium equations should be satisfied.

Instead of following steps 5(c) and (d) of this procedure, the member end forces alternatively can be obtained by first calculating the global forces $\mathbf{F}$ using the global stiffness relationship $\mathbf{F} = \mathbf{Kv} + \mathbf{F}_f$ (Eq. (6.28)), and then evaluating the local forces $\mathbf{Q}$ from the transformation relationship $\mathbf{Q} = \mathbf{TF}$ (Eq. (6.18)). It should also be noted that it is usually not necessary to determine the global end forces for all the members of the structure, because such forces are not used for design purposes. However, $\mathbf{F}$ vectors for the members that are attached to supports are always evaluated, so that they can be used to form the support reaction vector $\mathbf{R}$.

EXAMPLE 6.6

Determine the joint displacements, member end forces, and support reactions for the two-member frame shown in Fig. 6.17(a) on the next page, using the matrix stiffness method.

SOLUTION

Analytical Model: See Fig. 6.17(b). The frame has three degrees of freedom—the translations in the X and Y directions, and the rotation, of joint 2—which are numbered 1, 2, and 3, respectively. The six restrained coordinates of the frame are identified by numbers 4 through 9, as shown in Fig. 6.17(b).

Structure Stiffness Matrix and Fixed-Joint Force Vector:

Member 1 As shown in Fig. 6.17(b), we have selected joint 1 as the beginning joint, and joint 2 as the end joint for this member. By applying Eqs. (3.62), we determine

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} = \sqrt{(10-0)^2 + (20-0)^2}$$
$$= 22.361 \text{ ft} = 268.33 \text{ in.} \quad \textbf{(1a)}$$

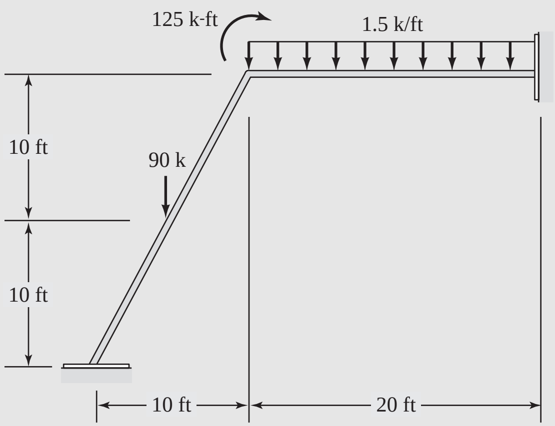
$$\cos\theta = \frac{X_2 - X_1}{L} = \frac{10-0}{22.361} = 0.44721 \quad \textbf{(1b)}$$

$$\sin\theta = \frac{Y_2 - Y_1}{L} = \frac{20-0}{22.361} = 0.89443 \quad \textbf{(1c)}$$

Using the units of kips and inches, we evaluate the member global stiffness matrix as (Eq. (6.31))

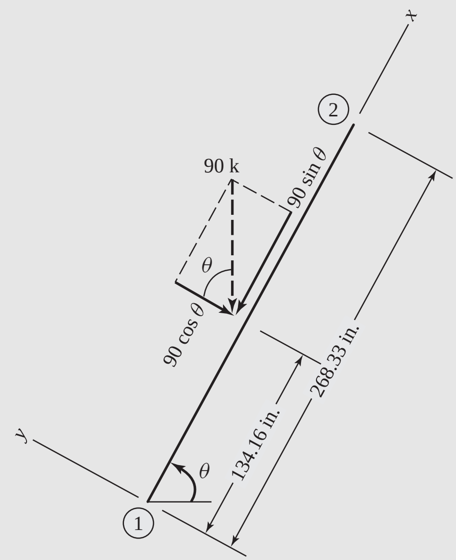
$$
\mathbf{K}_1 = \begin{array}{c} \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{ccc|ccc}
259.53 & 507.89 & -670.08 & -259.53 & -507.89 & -670.08 \\
507.89 & 1{,}021.4 & 335.04 & -507.89 & -1{,}021.4 & 335.04 \\
-670.08 & 335.04 & 134{,}015 & 670.08 & -335.04 & 67{,}008 \\
\hline
-259.53 & -507.89 & 670.08 & 259.53 & 507.89 & 670.08 \\
-507.89 & -1{,}021.4 & -335.04 & 507.89 & 1{,}021.4 & -335.04 \\
-670.08 & 335.04 & 67{,}008 & 670.08 & -335.04 & 134{,}015
\end{array}\right]
\end{array}
\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}
$$

Note that $\mathbf{K}_1$ is symmetric.

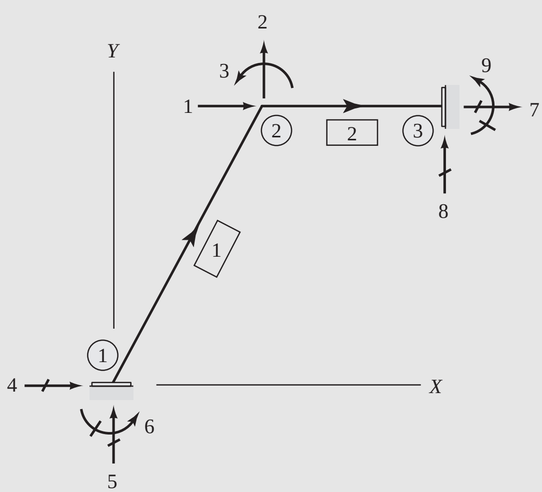


E, *A*, *I* = constant
$E = 29{,}000$ ksi
$A = 11.8$ in.^{2}
$I = 310$ in.^{4}

(a) Frame



(c) Loading on Member 1



(b) Analytical Model

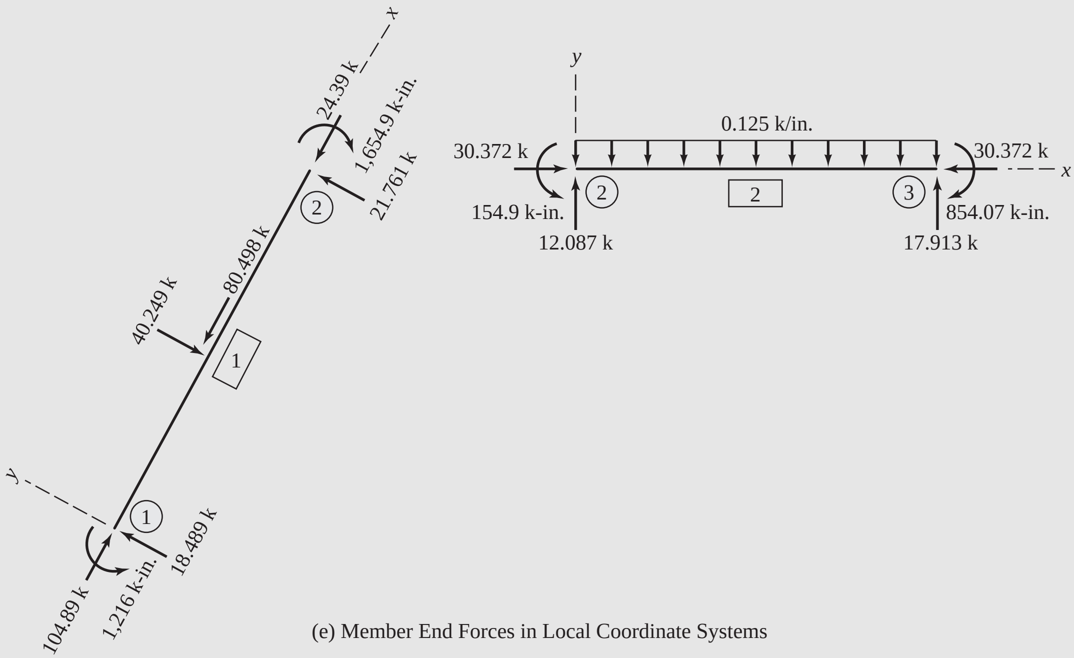
$$
\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\left[\begin{array}{ccc}
259.53 + 1{,}425.8 & 507.89 & 670.08 \\
507.89 & 1{,}021.4 + 7.8038 & -335.04 + 936.46 \\
670.08 & -335.04 + 936.46 & 134{,}015 + 149{,}833
\end{array}\right] \end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
$$

$$
= \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\left[\begin{array}{ccc}
1{,}685.3 & 507.89 & 670.08 \\
507.89 & 1{,}029.2 & 601.42 \\
670.08 & 601.42 & 283{,}848
\end{array}\right] \end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
$$

$$
\mathbf{P}_f = \left[\begin{array}{c} 0 \\ 45 + 15 \\ -1{,}350 + 600 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \left[\begin{array}{c} 0 \\ 60 \\ -750 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \end{array}
$$

(d) Structure Stiffness Matrix and Fixed-Joint Force Vector

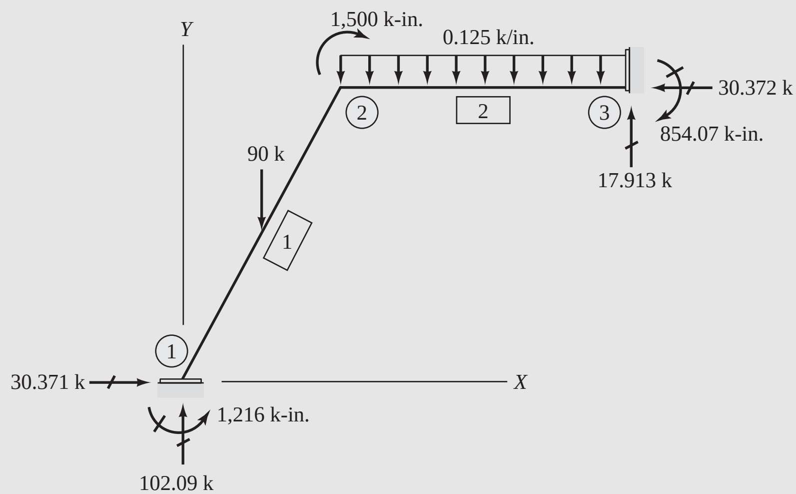
Fig. 6.17



(e) Member End Forces in Local Coordinate Systems

$$
\mathbf{R} = \begin{array}{c} \begin{bmatrix} 30.371 \text{ k} \\ 102.09 \text{ k} \\ 1,216 \text{ k-in.} \\ -30.372 \text{ k} \\ 17.913 \text{ k} \\ -854.07 \text{ k-in.} \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

(f) Support Reaction Vector



(g) Support Reactions

Fig. 6.17 (*continued*)

As the 90 k load applied to this member is inclined with respect to the member's local coordinate system, we evaluate the rectangular components of the load in the directions of the local x and y axes as (see Fig. 6.17(c))

$$W_x = 90 \sin\theta = 90(0.89443) = 80.498 \text{ k}$$

$$W_y = 90 \cos\theta = 90(0.44721) = 40.249 \text{ k}$$

Note that both W_x and W_y are considered positive because they act in the negative directions of the local x and y axes, respectively. The member's fixed-end axial forces can now be evaluated by substituting $W = W_x = 80.498$ k, $L = 268.33$ in., and $l_1 = l_2 = 134.16$ in. into the expressions for loading type 5 given inside the front cover. This yields

$$FA_b = FA_e = \frac{80.498}{2} = 40.249 \text{ k} \tag{2a}$$

Similarly, by substituting $W = W_y = 40.249$ k, and the numerical values of L, l_1, and l_2 into the equations for loading type 1, we obtain the fixed-end shears and moments as

$$FS_b = FS_e = \frac{40.249}{2} = 20.125 \text{ k} \tag{2b}$$

$$FM_b = -FM_e = \frac{40.249(268.33)}{8} = 1{,}350 \text{ k-in.} \tag{2c}$$

By substituting the numerical values of the member fixed-end forces and direction cosines into Eq. (6.33), we calculate the member global fixed-end force vector as

$$\mathbf{F}_{f1} = \begin{array}{c} \left[\begin{array}{r} 0 \\ 45 \\ 1{,}350 \\ \hdashline 0 \\ 45 \\ -1{,}350 \end{array}\right] \end{array} \begin{array}{l} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$

From Fig. 6.17(b), we observe that the code numbers for member 1 are 4, 5, 6, 1, 2, 3. Using these code numbers, we store the pertinent elements of $\mathbf{K}_1$ and $\mathbf{F}_{f1}$ in their proper positions in the 3×3 structure stiffness matrix $\mathbf{S}$ and the 3×1 structure fixed-joint force vector $\mathbf{P}_f$, respectively, as shown in Fig. 6.17(d).

Member 2 As this member is horizontal, with its left-end joint 2 selected as the beginning joint, no coordinate transformations are needed; that is, $\mathbf{T}_2 = \mathbf{I}$, $\mathbf{K}_2 = \mathbf{k}_2$, and $\mathbf{F}_{f2} = \mathbf{Q}_{f2}$. Thus, by substituting $L = 240$ in., $E = 29{,}000$ ksi, $A = 11.8$ in.^{2}, and $I = 310$ in.^{4} into Eq. (6.6), we obtain

$$\mathbf{K}_2 = \mathbf{k}_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{rrr:rrr} 1{,}425.8 & 0 & 0 & -1{,}425.8 & 0 & 0 \\ 0 & 7.8038 & 936.46 & 0 & -7.8038 & 936.46 \\ 0 & 936.46 & 149{,}833 & 0 & -936.46 & 74{,}917 \\ \hdashline -1{,}425.8 & 0 & 0 & 1{,}425.8 & 0 & 0 \\ 0 & -7.8038 & -936.46 & 0 & 7.8038 & -936.46 \\ 0 & 936.46 & 74{,}917 & 0 & -936.46 & 149{,}833 \end{array}\right] \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \tag{3}$$

Using the equations given for loading type 3 inside the front cover, we obtain the fixed-end forces due to the uniformly distributed load of magnitude 0.125 k/in. (= 1.5 k/ft):

$$FA_b = FA_e = 0$$

$$FS_b = FS_e = \frac{0.125(240)}{2} = 15 \text{ k}$$

$$FM_b = -FM_e = \frac{0.125(240)^2}{12} = 600 \text{ k-in.}$$

Thus (Eq. (6.15)),

$$\mathbf{F}_{f2} = \mathbf{Q}_{f2} = \begin{array}{c} \begin{bmatrix} 0 \\ 15 \\ 600 \\ \hdashline 0 \\ 15 \\ -600 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \qquad \textbf{(4)}$$

The relevant elements of $\mathbf{K}_2$ and $\mathbf{F}_{f2}$ are stored in $\mathbf{S}$ and $\mathbf{P}_f$, respectively, using the member code numbers 1, 2, 3, 7, 8, 9.

The completed structure stiffness matrix $\mathbf{S}$ and structure fixed-joint force vector $\mathbf{P}_f$ are given in Fig. 6.17(d). Note that $\mathbf{S}$ is symmetric.

Joint Load Vector: By comparing Figs. 6.17(a) and (b), we write the joint load vector, in kips and inches, as

$$\mathbf{P} = \begin{bmatrix} 0 \\ 0 \\ -1{,}500 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Joint Displacements: By substituting the numerical values of $\mathbf{P}$, $\mathbf{P}_f$, and $\mathbf{S}$ into Eq. (6.42), we write the stiffness relations for the entire frame as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

$$\begin{bmatrix} 0 \\ 0 \\ -1{,}500 \end{bmatrix} - \begin{bmatrix} 0 \\ 60 \\ -750 \end{bmatrix} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 \\ -60 \\ -750 \end{bmatrix} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Solving these equations, we determine the joint displacements to be

$$\mathbf{d} = \begin{bmatrix} 0.021302 \text{ in.} \\ -0.06732 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix} \qquad \textbf{Ans}$$

To check the foregoing solution, we substitute the numerical values of **d** back into the structure stiffness relationship, as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{Sd} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{bmatrix} \begin{bmatrix} 0.021302 \\ -0.06732 \\ -0.0025499 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -60 \\ -750 \end{bmatrix} \qquad \textbf{Checks}$$

Member End Displacements and End Forces:

Member 1 As in the case of plane trusses, the global end displacements **v** for a plane frame member can be obtained by applying the member's compatibility equations, using its code numbers. Thus, for member 1 of the frame under consideration, the global end displacement vector can be established as

$$\mathbf{v}_1 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \end{array}\!\!\begin{array}{l} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.021302 \text{ in.} \\ -0.06732 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix} \qquad \textbf{(5)}$$

As shown in Eq. (5), the code numbers for the member (4, 5, 6, 1, 2, 3) are first written on the right side of **v**. The fact that the code numbers corresponding to v_1, v_2, and v_3 are the restrained coordinate numbers 4, 5, and 6, respectively, indicates that $v_1 = v_2 = v_3 = 0$. Similarly, the code numbers 1, 2, and 3 corresponding to v_4, v_5, and v_6, respectively, indicate that $v_4 = d_1$, $v_5 = d_2$, and $v_6 = d_3$. Note that these compatibility equations can be verified easily by a visual inspection of the frame's line diagram given in Fig. 6.17(b).

To determine the member local end displacements, **u**, we first evaluate the transformation matrix **T** (Eq. (6.19)), using the direction cosines given in Eqs. (1):

$$\mathbf{T}_1 = \begin{bmatrix} 0.44721 & 0.89443 & 0 & 0 & 0 & 0 \\ -0.89443 & 0.44721 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.44721 & 0.89443 & 0 \\ 0 & 0 & 0 & -0.89443 & 0.44721 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The member local end forces can now be calculated using the relationship $\mathbf{u} = \mathbf{Tv}$ (Eq. (6.20)), as

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.050686 \text{ in.} \\ -0.04916 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix}$$

Before we can calculate the member's local end forces **Q**, we need to determine its local stiffness matrix **k** and fixed-end force vector $\mathbf{Q}_f$. Thus, using Eq. (6.6):

$$\mathbf{k}_1 = \begin{bmatrix} 1{,}275.3 & 0 & 0 & -1{,}275.3 & 0 & 0 \\ 0 & 5.584 & 749.17 & 0 & -5.584 & 749.17 \\ 0 & 749.17 & 134{,}015 & 0 & -749.17 & 67{,}008 \\ -1{,}275.3 & 0 & 0 & 1{,}275.3 & 0 & 0 \\ 0 & -5.584 & -749.17 & 0 & 5.584 & -749.17 \\ 0 & 749.17 & 67{,}008 & 0 & -749.17 & 134{,}015 \end{bmatrix}$$

and, by substituting Eqs. (2) into Eq. (6.15):

$$\mathbf{Q}_{f1} = \begin{bmatrix} 40.249 \\ 20.125 \\ 1{,}350 \\ 40.249 \\ 20.125 \\ -1{,}350 \end{bmatrix}$$

Now, using Eq. (6.4), we compute the member local end forces as

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 104.89 \text{ k} \\ 18.489 \text{ k} \\ 1{,}216 \text{ k-in.} \\ -24.39 \text{ k} \\ 21.761 \text{ k} \\ -1{,}654.9 \text{ k-in.} \end{bmatrix}$$ **Ans**

The local end forces for member 1 are depicted in Fig. 6.17(e), and we can check our calculations for these forces by considering the equilibrium of the free body of the member, as follows.

$+\nearrow \sum F_x = 0 \qquad 104.89 - 80.498 - 24.39 = 0.002 \cong 0$ **Checks**

$+\nwarrow \sum F_y = 0 \qquad 18.489 - 40.249 + 21.761 = 0.001 \cong 0$ **Checks**

$+\circlearrowleft \sum M_{\textcircled{2}} = 0 \qquad 1{,}216 - 18.489(268.33) + 40.249(134.16) - 1{,}654.9$

$= -0.25 \cong 0$ **Checks**

The member global end forces **F** can now be determined by applying Eq. (6.23), as

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{array}{c} \left[\begin{array}{r} 30.371 \\ 102.09 \\ 1{,}216 \\ \hdashline -30.371 \\ -12.083 \\ -1{,}654.9 \end{array}\right] \end{array} \begin{array}{l} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \qquad \textbf{(6)}$$

Next, to generate the support reaction vector **R**, we write the member code numbers (4, 5, 6, 1, 2, 3) on the right side of $\mathbf{F}_1$ as shown in Eq. (6), and store the pertinent elements of $\mathbf{F}_1$ in their proper positions in **R** by matching the code numbers on the side of $\mathbf{F}_1$ to the restrained coordinate numbers on the right side of **R** in Fig. 6.17(f).

Member 2 The global and local end displacements for this horizontal member are

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.021302 \text{ in.} \\ -0.06732 \text{ in.} \\ -0.0025499 \text{ rad} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By using $\mathbf{k}_2$ from Eq. (3) and $\mathbf{Q}_{f2}$ from Eq. (4), we compute the member local and global end forces to be

$$\mathbf{F}_2 = \mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 + \mathbf{Q}_{f2} = \begin{bmatrix} 30.372 \text{ k} \\ 12.087 \text{ k} \\ 154.9 \text{ k-in.} \\ \hdashline -30.372 \text{ k} \\ 17.913 \text{ k} \\ -854.07 \text{ k-in.} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}$$ **Ans**

These end forces for member 2 are depicted in Fig. 6.17(e). To check our calculations, we apply equilibrium equations to the free body of member 2, as follows.

$\overset{+}{\rightarrow} \sum F_x = 0$ $\quad 30.372 - 30.372 = 0$ **Checks**

$+\uparrow \sum F_y = 0$ $\quad 12.087 - 0.125(240) + 17.913 = 0$ **Checks**

$+\circlearrowleft \sum M_{②} = 0$ $\quad 154.9 - 0.125(240)(120) + 17.913(240)$
$\quad - 854.07 = -0.05 \cong 0$ **Checks**

Next, we store the pertinent elements of $\mathbf{F}_2$ in their proper positions in the reaction vector $\mathbf{R}$, using the member code numbers (1, 2, 3, 7, 8, 9), as shown in Fig. 6.17(f).

Support Reactions: The completed reaction vector $\mathbf{R}$ is given in Fig. 6.17(f), and the support reactions are depicted on a line diagram of the structure in Fig. 6.17(g). **Ans**

Equilibrium Check: Finally, we check our calculations by considering the equilibrium of the free body of the entire structure (Fig. 6.17(g)), as follows.

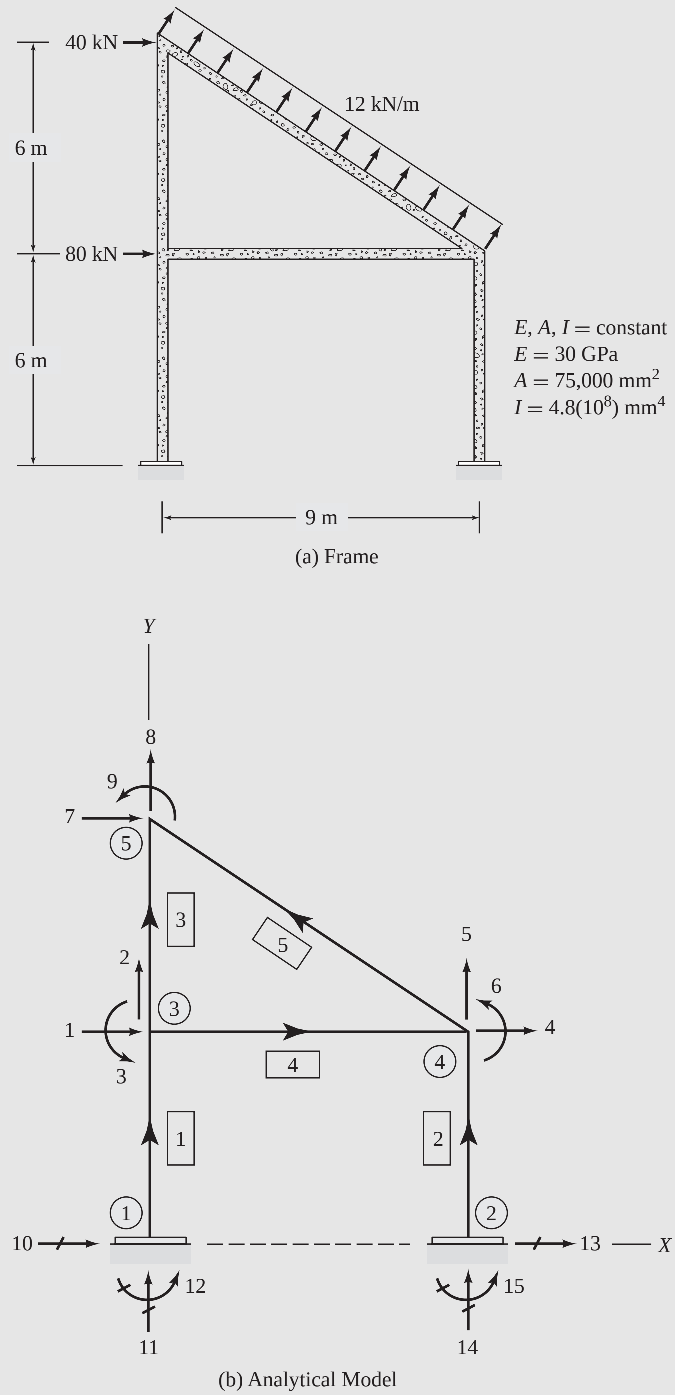
$\overset{+}{\rightarrow} \sum F_X = 0$ $\quad 30.371 - 30.372 = -0.001 \cong 0$ **Checks**

$+\uparrow \sum F_Y = 0$ $\quad 102.09 - 90 - 0.125(240) + 17.913 = 0.003 \cong 0$ **Checks**

$+\circlearrowleft \sum M_{①} = 0$ $\quad 1{,}216 - 90(60) - 1{,}500 - 0.125(240)(240) + 30.372(240)$
$\quad + 17.913(360) - 854.07 = -0.11 \cong 0$ **Checks**

EXAMPLE 6.7 Determine the joint displacements, member local end forces, and support reactions for the two-story frame, subjected to a wind loading, shown in Fig. 6.18(a).

SOLUTION *Analytical Model:* See Fig. 6.18(b). The frame has nine degrees of freedom, numbered 1 through 9; and six restrained coordinates, identified by the numbers 10 through 15.



40 kN
12 kN/m
6 m
80 kN
6 m
E, A, I = constant
E = 30 GPa
A = 75,000 mm²
I = 4.8(10⁸) mm⁴
9 m
(a) Frame
Y
8
9
7
5
3
5
5
2
6
1
3
4
3
4
4
1
2
1
2
10
13
X
12
15
11
14
(b) Analytical Model

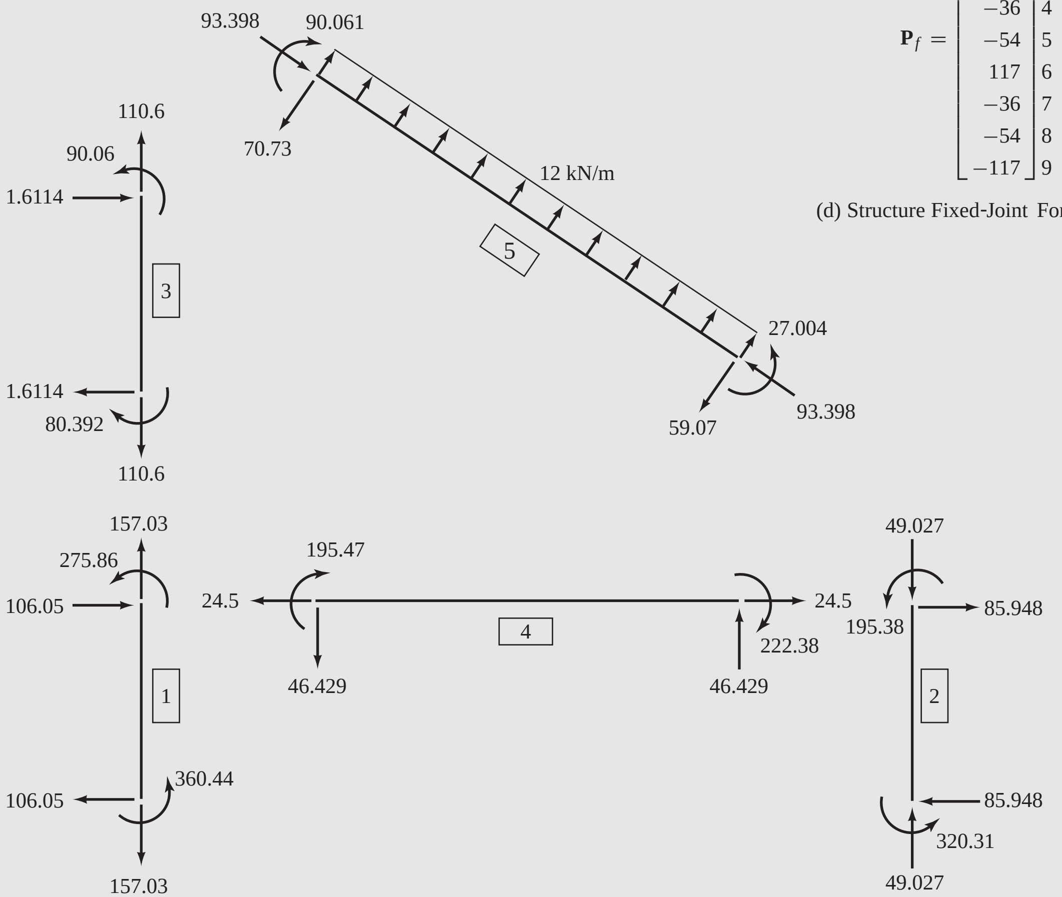
Fig. 6.18

$$
\mathbf{S} = \begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{rrrrrrrrr}
251{,}600 & 0 & 0 & -250{,}000 & 0 & 0 & -800 & 0 & -2{,}400 \\
0 & 750{,}237 & 1{,}066.7 & 0 & -237.04 & 1{,}066.7 & 0 & -375{,}000 & 0 \\
0 & 1{,}066.7 & 25{,}600 & 0 & -1{,}066.7 & 3{,}200 & 2{,}400 & 0 & 4{,}800 \\
-250{,}000 & 0 & 0 & 394{,}851 & -95{,}943 & 1{,}990.4 & -144{,}051 & 95{,}943 & -409.62 \\
0 & -237.04 & -1{,}066.7 & -95{,}943 & 439{,}335 & -1{,}681.1 & 95{,}943 & -64{,}098 & -614.44 \\
0 & 1{,}066.7 & 3{,}200 & 1{,}990.4 & -1{,}681.1 & 21{,}325 & 409.62 & 614.44 & 2{,}662.6 \\
-800 & 0 & 2{,}400 & -144{,}051 & 95{,}943 & 409.62 & 144{,}851 & -95{,}943 & 2{,}809.6 \\
0 & -375{,}000 & 0 & 95{,}943 & -64{,}098 & 614.44 & -95{,}943 & 439{,}098 & 614.44 \\
-2{,}400 & 0 & 4{,}800 & -409.62 & -614.44 & 2{,}662.6 & 2{,}809.6 & 614.44 & 14{,}925
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

(c) Structure Stiffness Matrix

$$
\mathbf{P}_f = \left[\begin{array}{r} 0 \\ 0 \\ 0 \\ -36 \\ -54 \\ 117 \\ -36 \\ -54 \\ -117 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

(d) Structure Fixed-Joint Force Vector

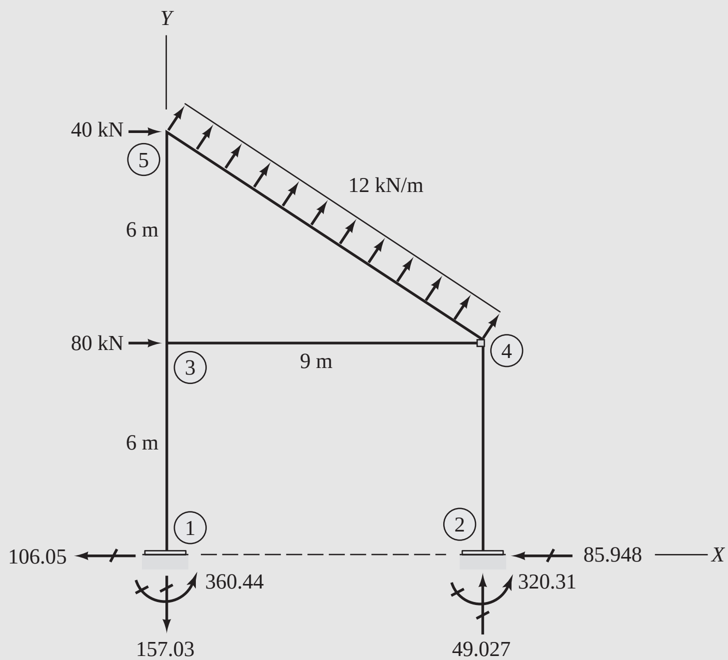


(e) Member Local End Forces

Fig. 6.18 (*continued*)

$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{l} -106.05 \text{ kN} \\ -157.03 \text{ kN} \\ \;\;360.44 \text{ kN}\cdot\text{m} \\ -85.948 \text{ kN} \\ \;\;49.027 \text{ kN} \\ \;\;320.31 \text{ kN}\cdot\text{m} \end{array}\right] \end{array} \begin{array}{l} 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{array}$$

(f) Support Reaction Vector



(g) Support Reactions

Fig. 6.18 (*continued*)

Structure Stiffness Matrix and Fixed-Joint Force Vector:

Members 1, 2, and 3 $E = 30(10^6)\,\text{kN/m}^2, A = 0.075\,\text{m}^2, I = 480(10^{-6})\,\text{m}^4, L = 6\,\text{m}$, $\cos\theta = 0$, and $\sin\theta = 1$. The member global stiffness matrix, in units of kN and meters, is given by the following (see Eq. (6.31)).

$$\begin{array}{l} \text{Member 3} \rightarrow \\ \text{Member 2} \rightarrow \\ \text{Member 1} \rightarrow \end{array} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \\ 13 & 14 & 15 & 4 & 5 & 6 \\ 10 & 11 & 12 & 1 & 2 & 3 \end{array}$$

$$\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{K}_3 = \left[\begin{array}{cccccc} 800 & 0 & -2{,}400 & -800 & 0 & -2{,}400 \\ 0 & 375{,}000 & 0 & 0 & -375{,}000 & 0 \\ -2{,}400 & 0 & 9{,}600 & 2{,}400 & 0 & 4{,}800 \\ -800 & 0 & 2{,}400 & 800 & 0 & 2{,}400 \\ 0 & -375{,}000 & 0 & 0 & 375{,}000 & 0 \\ -2{,}400 & 0 & 4{,}800 & 2{,}400 & 0 & 9{,}600 \end{array}\right] \begin{array}{ccc} 10 & 13 & 1 \\ 11 & 14 & 2 \\ 12 & 15 & 3 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}$$

As these members are not subjected to any loads, their fixed-end forces are 0. Thus,

$$\mathbf{F}_{f1} = \mathbf{F}_{f2} = \mathbf{F}_{f3} = \mathbf{0}$$

Using the code numbers for member 1 (10, 11, 12, 1, 2, 3), member 2 (13, 14, 15, 4, 5, 6), and member 3 (1, 2, 3, 7, 8, 9), the relevant elements of $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ are stored in their proper positions in the 9×9 structure stiffness matrix $\mathbf{S}$ (Fig. 6.18(c)).

Member 4 Substituting $L = 9$ m and the foregoing values of E, A, and I into Eq. (6.6), we obtain

$$\mathbf{K}_4 = \mathbf{k}_4 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 250{,}000 & 0 & 0 & -250{,}000 & 0 & 0 \\ 0 & 237.04 & 1{,}066.7 & 0 & -237.04 & 1{,}066.7 \\ 0 & 1{,}066.7 & 6{,}400 & 0 & -1{,}066.7 & 3{,}200 \\ -250{,}000 & 0 & 0 & 250{,}000 & 0 & 0 \\ 0 & -237.04 & -1{,}066.7 & 0 & 237.04 & -1{,}066.7 \\ 0 & 1{,}066.7 & 3{,}200 & 0 & -1{,}066.7 & 6{,}400 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \quad \textbf{(1)}$$

As no loads are applied to this member,

$$\mathbf{F}_{f4} = \mathbf{Q}_{f4} = \mathbf{0}$$

The pertinent elements of $\mathbf{K}_4$ are stored in $\mathbf{S}$ using the member code numbers 1, 2, 3, 4, 5, 6.

Member 5

$$L = \sqrt{(X_5 - X_4)^2 + (Y_5 - Y_4)^2} = \sqrt{(0-9)^2 + (12-6)^2} = 10.817 \text{ m} \quad \textbf{(2a)}$$

$$\cos\theta = \frac{X_5 - X_4}{L} = \frac{0-9}{10.817} = -0.83205 \quad \textbf{(2b)}$$

$$\sin\theta = \frac{Y_5 - Y_4}{L} = \frac{12-6}{10.817} = 0.5547 \quad \textbf{(2c)}$$

$$\mathbf{K}_5 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccc} 144{,}051 & -95{,}943 & -409.62 & -144{,}051 & 95{,}943 & -409.62 \\ -95{,}943 & 64{,}098 & -614.44 & 95{,}943 & -64{,}098 & -614.44 \\ -409.62 & -614.44 & 5{,}325.1 & 409.62 & 614.44 & 2{,}662.6 \\ -144{,}051 & 95{,}943 & 409.62 & 144{,}051 & -95{,}943 & 409.62 \\ 95{,}943 & -64{,}098 & 614.44 & -95{,}943 & 64{,}098 & 614.44 \\ -409.62 & -614.44 & 2{,}662.6 & 409.62 & 614.44 & 5{,}325.1 \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

From Figs. 6.18(a) and (b), we observe that the 12 kN/m uniformly distributed load applied to member 5 acts in the negative direction of the member's local y axis; therefore, it is considered positive for the purpose of calculating fixed-end forces. Thus,

$$FA_b = FA_e = 0 \quad \textbf{(3a)}$$

$$FS_b = FS_e = \frac{wL}{2} = \frac{12(10.817)}{2} = 64.9 \text{ kN} \quad \textbf{(3b)}$$

$$FM_b = -FM_e = \frac{wL^2}{12} = \frac{12(10.817)^2}{12} = 117 \text{ kN} \cdot \text{m} \quad \textbf{(3c)}$$

Using Eq. (6.33), we determine the global fixed-end force vector for the member to be

$$\mathbf{F}_{f5} = \begin{array}{c}\left[\begin{array}{r} -36 \\ -54 \\ 117 \\ -36 \\ -54 \\ -117 \end{array}\right]\end{array}\begin{array}{l} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Using the code numbers 4, 5, 6, 7, 8, 9, we store the pertinent elements of $\mathbf{K}_5$ and $\mathbf{F}_{f5}$ in their proper positions in the **S** matrix and the $\mathbf{P}_f$ vector, respectively.

The complete structure stiffness matrix **S** and structure fixed-joint force vector $\mathbf{P}_f$ are shown in Figs. 6.18(c) and (d), respectively.

Joint Load Vector:

$$\mathbf{P} = \left[\begin{array}{c} 80 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 40 \\ 0 \\ 0 \end{array}\right]\begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad \textbf{(4)}$$

Joint Displacements: By substituting the numerical values of **S** (Fig. 6.18(c)), $\mathbf{P}_f$ (Fig. 6.18(d)), and **P** (Eq. (4)) into the structural stiffness relationship $\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$ (Eq. (6.42)), and solving the resulting system of simultaneous equations, we obtain the following joint displacements:

$$\mathbf{d} = \left[\begin{array}{l} 0.185422 \text{ m} \\ 0.000418736 \text{ m} \\ -0.0176197 \text{ rad} \\ 0.18552 \text{ m} \\ -0.000130738 \text{ m} \\ -0.0260283 \text{ rad} \\ 0.186622 \text{ m} \\ 0.000713665 \text{ m} \\ 0.0178911 \text{ rad} \end{array}\right]\begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad \textbf{Ans}$$

To check this solution, we evaluate the matrix product **Sd**, using the foregoing values of the joint displacements **d**, and substitute the results into the structure stiffness relationship, as

$$\mathbf{P} - \mathbf{P}_f = \mathbf{Sd}$$

$$\left[\begin{array}{c} 80 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 40 \\ 0 \\ 0 \end{array}\right] - \left[\begin{array}{r} 0 \\ 0 \\ 0 \\ -36 \\ -54 \\ 117 \\ -36 \\ -54 \\ -117 \end{array}\right] \cong \left[\begin{array}{r} 79.939 \\ -0.001466 \\ 0.0013239 \\ 36.051 \\ 54.006 \\ -116.992 \\ 76.007 \\ 53.994 \\ 116.994 \end{array}\right] \qquad \textbf{Checks}$$

Member End Displacements and End Forces:

Member 1

$$\mathbf{v}_1 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \end{array} \begin{array}{c} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.185422 \\ 0.000418736 \\ -0.0176197 \end{bmatrix}$$

$\cos\theta = 0, \sin\theta = 1$

$$\mathbf{T}_1 = \mathbf{T}_2 = \mathbf{T}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.000418736 \\ -0.185422 \\ -0.0176197 \end{bmatrix}$$

$$\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \begin{bmatrix} 375{,}000 & 0 & 0 & -375{,}000 & 0 & 0 \\ 0 & 800 & 2{,}400 & 0 & -800 & 2{,}400 \\ 0 & 2{,}400 & 9{,}600 & 0 & -2{,}400 & 4{,}800 \\ -375{,}000 & 0 & 0 & 375{,}000 & 0 & 0 \\ 0 & -800 & -2{,}400 & 0 & 800 & -2{,}400 \\ 0 & 2{,}400 & 4{,}800 & 0 & -2{,}400 & 9{,}600 \end{bmatrix} \tag{6}$$

$$\mathbf{Q}_{f1} = \mathbf{0}$$

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 = \begin{bmatrix} -157.03 \text{ kN} \\ 106.05 \text{ kN} \\ 360.44 \text{ kN}\cdot\text{m} \\ 157.03 \text{ kN} \\ -106.05 \text{ kN} \\ 275.86 \text{ kN}\cdot\text{m} \end{bmatrix}$$ **Ans**

$$\mathbf{F}_1 = \mathbf{T}_1^T\mathbf{Q}_1 = \begin{bmatrix} -106.05 \\ -157.03 \\ 360.44 \\ 106.05 \\ 157.03 \\ 275.86 \end{bmatrix} \begin{array}{c} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array} \tag{7}$$

Member 2

$$\mathbf{v}_2 = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.18552 \\ -0.000130738 \\ -0.0260283 \end{bmatrix} \end{array} \begin{array}{c} 13 \\ 14 \\ 15 \\ 4 \\ 5 \\ 6 \end{array}$$

Using $\mathbf{T}_2$ from Eq. (5), we obtain

$$\mathbf{u}_2 = \mathbf{T}_2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.000130738 \\ -0.18552 \\ -0.0260283 \end{bmatrix}$$

Using $\mathbf{k}_2$ from Eq. (6), and realizing that $\mathbf{Q}_{f2} = \mathbf{0}$, we determine that

$$\mathbf{Q}_2 = \mathbf{k}_2\mathbf{u}_2 = \begin{bmatrix} 49.027 \text{ kN} \\ 85.948 \text{ kN} \\ 320.31 \text{ kN} \cdot \text{m} \\ -49.027 \text{ kN} \\ -85.948 \text{ kN} \\ 195.38 \text{ kN} \cdot \text{m} \end{bmatrix} \qquad \textbf{Ans}$$

$$\mathbf{F}_2 = \mathbf{T}_2^T\mathbf{Q}_2 = \begin{bmatrix} -85.948 \\ 49.027 \\ 320.31 \\ 85.948 \\ -49.027 \\ 195.38 \end{bmatrix} \begin{array}{c} 13 \\ 14 \\ 15 \\ 4 \\ 5 \\ 6 \end{array} \qquad \textbf{(8)}$$

Member 3

$$\mathbf{v}_3 = \begin{bmatrix} 0.185422 \\ 0.000418736 \\ -0.0176197 \\ 0.186622 \\ 0.000713665 \\ 0.0178911 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}$$

Using $\mathbf{T}_3$ from Eq. (5), we compute

$$\mathbf{u}_3 = \mathbf{T}_3\mathbf{v}_3 = \begin{bmatrix} 0.000418736 \\ -0.185422 \\ -0.0176197 \\ 0.000713665 \\ -0.186622 \\ 0.0178911 \end{bmatrix}$$

Using $\mathbf{k}_3$ from Eq. (6), and realizing that $\mathbf{Q}_{f3} = \mathbf{0}$, we obtain

$$\mathbf{Q}_3 = \mathbf{k}_3\mathbf{u}_3 = \begin{bmatrix} -110.6 \text{ kN} \\ 1.6114 \text{ kN} \\ -80.392 \text{ kN}\cdot\text{m} \\ 110.6 \text{ kN} \\ -1.6114 \text{ kN} \\ 90.06 \text{ kN}\cdot\text{m} \end{bmatrix} \qquad \textbf{Ans}$$

Note that it is not necessary to compute the member global end force vector $\mathbf{F}_3$, because this member is not attached to any supports (and, therefore, none of the elements of $\mathbf{F}_3$ will appear in the support reaction vector $\mathbf{R}$).

Member 4 $\quad \mathbf{T}_4 = \mathbf{I}$

$$\mathbf{u}_4 = \mathbf{v}_4 = \begin{array}{c} \begin{bmatrix} 0.185422 \\ 0.000418736 \\ -0.0176197 \\ 0.18552 \\ -0.000130738 \\ -0.0260283 \end{bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Using $\mathbf{k}_4$ from Eq. (1), and $\mathbf{Q}_{f4} = \mathbf{0}$, we obtain

$$\mathbf{Q}_4 = \mathbf{k}_4\mathbf{u}_4 = \begin{bmatrix} -24.5 \text{ kN} \\ -46.429 \text{ kN} \\ -195.47 \text{ kN}\cdot\text{m} \\ 24.5 \text{ kN} \\ 46.429 \text{ kN} \\ -222.38 \text{ kN}\cdot\text{m} \end{bmatrix} \qquad \textbf{Ans}$$

Member 5

$$\mathbf{v}_5 = \begin{bmatrix} 0.18552 \\ -0.000130738 \\ -0.0260283 \\ 0.186622 \\ 0.000713665 \\ 0.0178911 \end{bmatrix} \begin{array}{l} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

$\cos\theta = -0.83205, \sin\theta = 0.5547 \quad$ (Eqs. (2))

$$\mathbf{T}_5 = \begin{bmatrix} -0.83205 & 0.5547 & 0 & 0 & 0 & 0 \\ -0.5547 & -0.83205 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.83205 & 0.5547 & 0 \\ 0 & 0 & 0 & -0.5547 & -0.83205 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u}_5 = \mathbf{T}_5\mathbf{v}_5 = \begin{bmatrix} -0.154434 \\ -0.102799 \\ -0.0260283 \\ -0.154883 \\ -0.104113 \\ 0.0178911 \end{bmatrix}$$

$$\mathbf{k}_5 = \begin{bmatrix} 208{,}013 & 0 & 0 & -208{,}013 & 0 & 0 \\ 0 & 136.54 & 738.46 & 0 & -136.54 & 738.46 \\ 0 & 738.46 & 5{,}325.1 & 0 & -738.46 & 2{,}662.6 \\ -208{,}013 & 0 & 0 & 208{,}013 & 0 & 0 \\ 0 & -136.54 & -738.46 & 0 & 136.54 & -738.46 \\ 0 & 738.46 & 2{,}662.6 & 0 & -738.46 & 5{,}325.1 \end{bmatrix}$$

From Eqs. (3), we obtain

$$\mathbf{Q}_{f5} = \begin{bmatrix} 0 \\ 64.9 \\ 117 \\ 0 \\ 64.9 \\ -117 \end{bmatrix}$$

$$\mathbf{Q}_5 = \mathbf{k}_5\mathbf{u}_5 + \mathbf{Q}_{f5} = \begin{bmatrix} 93.398 \text{ kN} \\ 59.07 \text{ kN} \\ 27.004 \text{ kN}\cdot\text{m} \\ -93.398 \text{ kN} \\ 70.73 \text{ kN} \\ -90.061 \text{ kN}\cdot\text{m} \end{bmatrix}$$ **Ans**

The member local end forces are shown in Fig. 6.18(e).

Support Reactions: The reaction vector **R**, as assembled from the appropriate elements of the member global end force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ (Eqs. (7) and (8), respectively), is given in Fig. 6.18(f). Also, Fig. 6.18(g) depicts these support reactions on a line diagram of the frame. **Ans**

Equilibrium Check: Considering the equilibrium of the entire frame, we write (Fig. 6.18(g))

$$+\rightarrow \sum F_x = 0 \quad 40 + 80 + (12\sqrt{117})\frac{6}{\sqrt{117}} - 106.05 - 85.948 = 0.002 \cong 0$$

Checks

$$+\uparrow \sum F_Y = 0 \quad (12\sqrt{117})\frac{9}{\sqrt{117}} - 157.03 + 49.027 = -0.003 \cong 0$$ **Checks**

$$+\circlearrowleft \sum M_{①} = 0 \quad 360.44 - 80(6) - 40(12) - (12\sqrt{117})\left(\frac{6}{\sqrt{117}}\right)9$$

$$+ (12\sqrt{117})\left(\frac{9}{\sqrt{117}}\right)4.5 + 320.31 + 49.027(9)$$

$$= -0.007 \cong 0$$ **Checks**

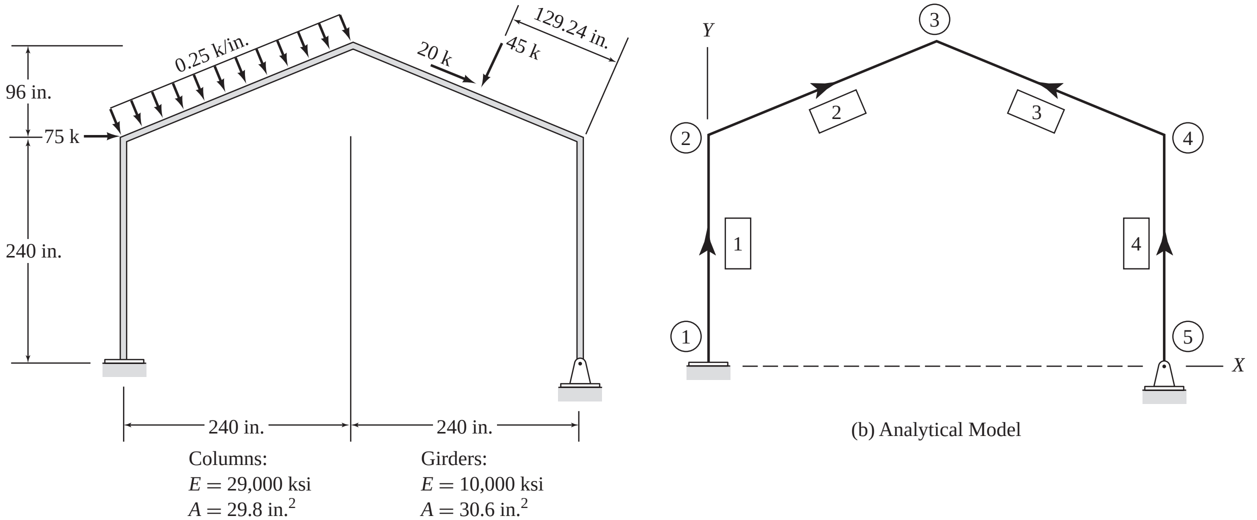
6.7 COMPUTER PROGRAM

The overall organization and format of the computer program for the analysis of plane frames remains the same as the plane truss and beam analysis programs developed previously. All the parts, and many subroutines, of the new program can be replicated from the previous programs with no, or relatively minor, modifications. In this section, we discuss the development of this program for the analysis of plane frames, with emphasis on the programming aspects not considered in previous chapters.

Input Module

Joint Data The part of the computer program for reading and storing the joint data for plane frames (i.e., the number of joints, NJ, and X and Y coordinates of each joint) remains the same as Part I of the plane truss analysis program (see flowchart in Fig. 4.3(a)). As discussed in Section 4.1, the program stores the joint coordinates in a $NJ \times 2$ joint coordinate matrix **COORD** in computer memory. As an example, let us consider the gable frame of Fig. 6.19(a), with its analytical model shown in Fig. 6.19(b). Since the frame has five joints, its **COORD** matrix has five rows, with the X and Y coordinates of a joint i stored in the first and second columns, respectively, of the ith row, as shown in Fig. 6.19(c). An example of the input data file for the gable frame is given in Fig. 6.20 on page 320.

Support Data The computer code written for Part II of the plane truss analysis program (see flowchart in Fig. 4.3(b)) can be used to input the support data for plane frames, provided that the number of structure coordinates per joint is set equal to 3 (i.e., $NCJT = 3$) in the program. A three-digit code is now used to specify the restraints at a support joint, with the first two digits representing the translational restraint conditions in the global X and Y directions, respectively, and the third digit representing the rotational restraint condition at the joint. As in the case of plane trusses and beams, each digit of the restraint



Columns:
$E = 29{,}000$ ksi
$A = 29.8$ in.^{2}
$I = 2{,}420$ in.^{4}

Girders:
$E = 10{,}000$ ksi
$A = 30.6$ in.^{2}
$I = 3{,}100$ in.^{4}

(a) Gable Frame

(b) Analytical Model

Fig. 6.19

$$\begin{array}{rl} & \begin{matrix} \text{X Coordinate} & \text{Y Coordinate} \end{matrix} \\ \mathbf{COORD} = & \begin{bmatrix} 0 & 0 \\ 0 & 240 \\ 240 & 336 \\ 480 & 240 \\ 480 & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{Joint 1} \\ \leftarrow \text{Joint 2} \\ \leftarrow \text{Joint 3} \\ \leftarrow \text{Joint 4} \\ \leftarrow \text{Joint 5} \end{matrix} \end{array}$$

$NJ \times 2$

(c) Joint Coordinate Matrix

$$\begin{array}{rl} & \begin{matrix} \text{Joint Number} & \text{Restraint in } X \text{ Direction (0 = free, 1 = restrained)} & \text{Restraint in } Y \text{ Direction (0 = free, 1 = restrained)} & \text{Rotational Restraint (0 = free, 1 = restrained)} \end{matrix} \\ \mathbf{MSUP} = & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$NS \times (NCJT + 1)$

(d) Support Data Matrix

$$\mathbf{EM} = \begin{bmatrix} 29000 \\ 10000 \end{bmatrix} \begin{matrix} \leftarrow \text{Material No. 1} \\ \leftarrow \text{Material No. 2} \end{matrix}$$

$NMP \times 1$

(e) Elastic Modulus Vector

$$\begin{array}{rl} & \begin{matrix} \text{Area} & \text{Moment of Inertia} \end{matrix} \\ \mathbf{CP} = & \begin{bmatrix} 29.8 & 2420 \\ 30.6 & 3100 \end{bmatrix} \begin{matrix} \leftarrow \text{Cross-Section Type No. 1} \\ \leftarrow \text{Cross-Section Type No. 2} \end{matrix} \end{array}$$

$NCP \times 2$

(f) Cross-Sectional Property Matrix

$$\begin{array}{rl} & \begin{matrix} \text{Beginning Joint} & \text{End Joint} & \text{Material No.} & \text{Cross-Section Type No.} \end{matrix} \\ \mathbf{MPRP} = & \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 4 & 3 & 2 & 2 \\ 5 & 4 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Member 1} \\ \leftarrow \text{Member 2} \\ \leftarrow \text{Member 3} \\ \leftarrow \text{Member 4} \end{matrix} \end{array}$$

$NM \times 4$

(g) Member Data Matrix

$$\begin{array}{rl} & \text{Joint Number} \\ \mathbf{JP} = & [2] \end{array} \qquad \begin{array}{rl} & \begin{matrix} \text{Force in } X \text{ Direction} & \text{Force in } Y \text{ Direction} & \text{Moment} \end{matrix} \\ \mathbf{PJ} = & [75 \quad 0 \quad 0] \end{array}$$

$NJL \times 1$ $\qquad$ $NJL \times NCJT$

(h) Joint Load Data Matrices

$$\begin{array}{rl} & \begin{matrix} \text{Member Number} & \text{Load Type Number} \end{matrix} \\ \mathbf{MP} = & \begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 3 & 5 \end{bmatrix} \end{array} \qquad \begin{array}{rl} \mathbf{PM} = & \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ -45 & 0 & 129.24 & 0 \\ 20 & 0 & 129.24 & 0 \end{bmatrix} \\ & \begin{matrix} W, M, w \text{ or } w_1 & \begin{cases} w_2 \text{ (if Load Type = 4)} \\ 0 \text{ (otherwise)} \end{cases} & l_1 & \begin{cases} l_2 \text{ (if Load Type = 3, 4 or 6)} \\ 0 \text{ (otherwise)} \end{cases} \end{matrix} \end{array}$$

$NML \times 2$ $\qquad$ $NML \times 4$

(i) Member Load Data Matrices

Fig. 6.19 (*continued*)

```
5
0, 0
0, 240            <------ Joint data
240, 336
480, 240
480, 0
2
1, 1, 1, 1        <------ Support data
5, 1, 1, 0
2
29000             <------ Material property data
10000
2
29.8, 2420        <------ Cross-sectional property data
30.6, 3100
4
1, 2, 1, 1
2, 3, 2, 2        <------ Member data
4, 3, 2, 2
5, 4, 1, 1
1                 <------ Joint load data
2, 75, 0, 0
3
2, 3, 0.25, 0, 0  <------ Member load data
3, 1, −45, 129.24
3, 5, 20, 129.24
```

Fig. 6.20 *An Example of an Input Data File*

code can be either 0 (indicating no restraint) or 1 (indicating restraint). The restraint codes for some common types of supports for plane frames are given in Fig. 6.21. The program stores the support data in a $NS \times 4$ **MSUP** matrix, as shown in Fig. 6.19(d) for the gable frame, and an example of how this data may appear in an input file is given in Fig. 6.20.

Material Property Data This part of the program remains the same as Part III of the plane truss analysis program (see flowchart in Fig. 4.3(c)). The program stores the moduli of elasticity in a $NMP \times 1$ **EM** vector, as shown in Fig. 6.19(e) for the example gable frame; Fig. 6.20 illustrates how this data may appear in an input data file.

Cross-sectional Property Data As two cross-sectional properties (namely, area and moment of inertia) are needed in the analysis of plane frames, the code written previously for Part IV of the plane truss program should be modified to increase the number of columns of the *cross-sectional property matrix* **CP** from one to two, as indicated by the flowchart in Fig. 6.22(a). As before, the number of rows of **CP** equals the number of cross-section types (NCP), with the area and moment of inertia of the cross-section i now stored in the first and second columns, respectively, of the ith row of the **CP** matrix of order $NCP \times 2$. For example, the **CP** matrix for the gable frame of Fig. 6.19(a) is shown in Fig. 6.19(f), and Fig. 6.20 shows how this data may appear in an input data file.

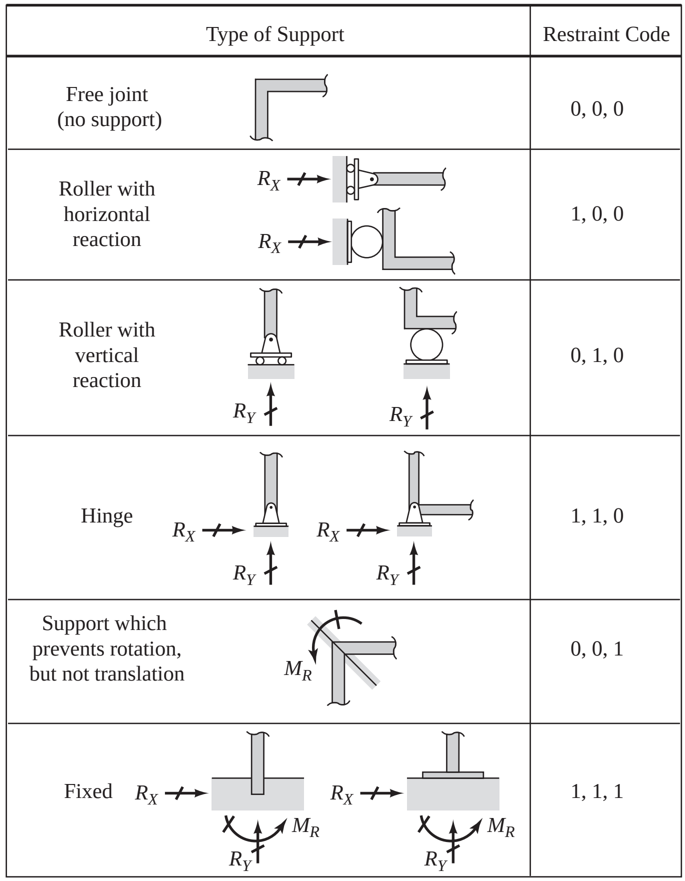
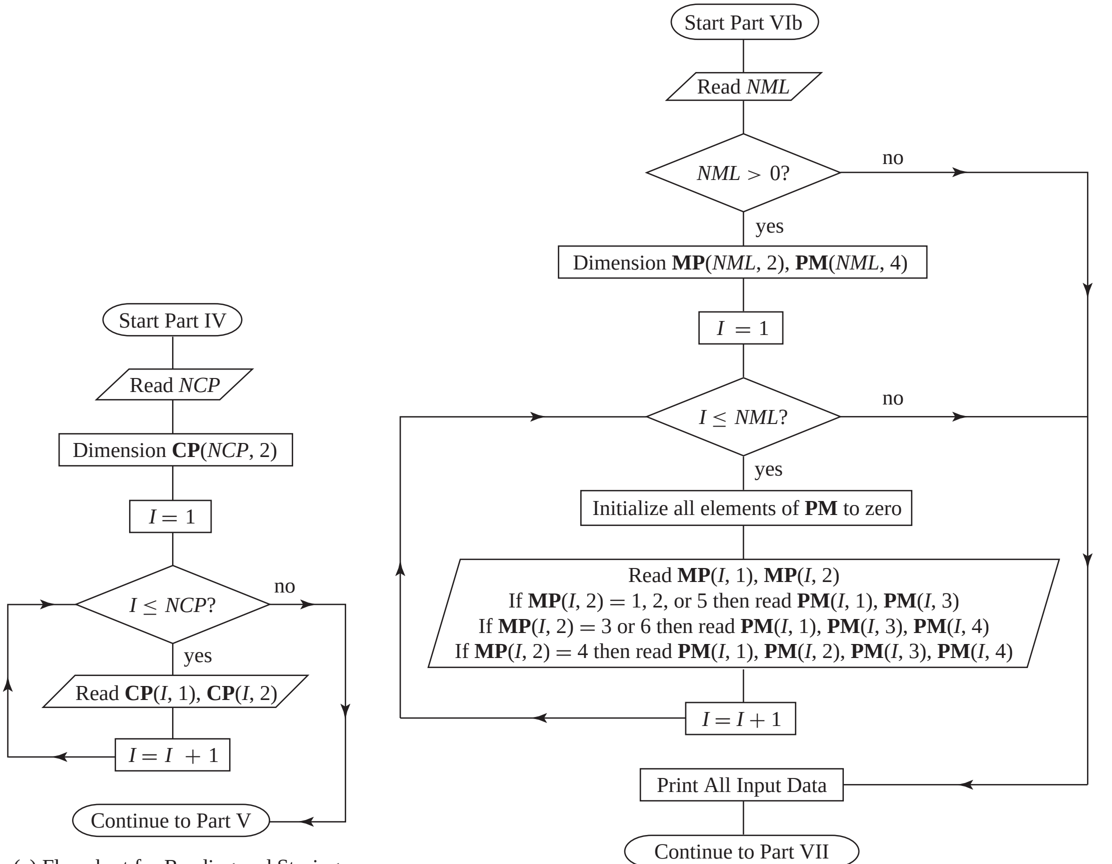


Fig. 6.21 *Restraint Codes for Plane Frames*

Member Data This part of the computer program remains the same as Part V of the plane truss analysis program (see flowchart in Fig. 4.3(e)). As discussed in Section 4.1, the program stores the member data in an integer matrix **MPRP** of order $NM \times 4$. The **MPRP** matrix for the example gable frame is shown in Fig. 6.19(g).

Joint Load Data The code written for Part VIa of the beam analysis program (see flowchart in Fig. 5.20(b)) can be used for inputting joint load data for plane frames, provided that $NCJT$ is set equal to 3. The program stores the numbers of the loaded joints in an integer vector **JP** of order $NJL \times 1$, with the corresponding loads in the X and Y directions and the couple being stored in the first, second, and third columns, respectively, of a real matrix **PJ** of order $NJL \times 3$. The joint load matrices for the example gable frame are shown in Fig. 6.19(h); Fig. 6.20 illustrates how this data may appear in an input data file.

Member Load Data As members of plane frames may be subjected to both axial and perpendicular loads, the code written for Part VIb of the beam



(a) Flowchart for Reading and Storing Cross-Sectional Property Data

(b) Flowchart for Reading and Storing Member Load Data

Fig. 6.22

analysis program should be modified to include inputting of the member axial loads. The flowchart shown in Fig. 6.22(b) can be used for programming the input of the four perpendicular, and two axial, member load types (i.e., load types 1 through 6) given inside the front cover. The format for reading and storing the member load data for plane frames remains the same as that for beams, as discussed in Section 5.8. The member load matrices, **MP** and **PM**, for the example gable frame are shown in Fig. 6.19(i); Fig. 6.20 shows this member load data in an input file that can be read by the program.

An example of a computer printout of the input data for the gable frame of Fig. 6.19 is given in Fig. 6.23.

Analysis Module

Assignment of Structure Coordinate Numbers The parts of the program for determining the number of degrees of freedom (*NDOF*) and forming the

```
*************************************
*        Computer Software          *
*               for                 *
*  MATRIX ANALYSIS OF STRUCTURES    *
*          Second Edition           *
*                by                 *
*          Aslam Kassimali          *
*************************************
```

General Structural Data

Project Title: Figure 6-19
Structure Type: Plane Frame
Number of Joints: 5
Number of Members: 4
Number of Material Property Sets (E): 2
Number of Cross-Sectional Property Sets: 2

Joint Coordinates

Joint No.	X Coordinate	Y Coordinate
1	0.0000E+00	0.0000E+00
2	0.0000E+00	2.4000E+02
3	2.4000E+02	3.3600E+02
4	4.8000E+02	2.4000E+02
5	4.8000E+02	0.0000E+00

Supports

Joint No.	X Restraint	Y Restraint	Rotational Restraint
1	Yes	Yes	Yes
5	Yes	Yes	No

Material Properties

Material No.	Modulus of Elasticity (E)	Co-efficient of Thermal Expansion
1	2.9000E+04	0.0000E+00
2	1.0000E+04	0.0000E+00

Cross-Sectional Properties

Property No.	Area (A)	Moment of Inertia (I)
1	2.9800E+01	2.4200E+03
2	3.0600E+01	3.1000E+03

Fig. 6.23 *A Sample Printout of Input Data*

Member Data

Member No.	Beginning Joint	End Joint	Material No.	Cross-Sectional Property No.
1	1	2	1	1
2	2	3	2	2
3	4	3	2	2
4	5	4	1	1

Joint Loads

Joint No.	X Force	Y Force	Moment
2	7.5000E+01	0.0000E+00	0.0000E+00

Member Loads

Member No.	Load Type	Load Magnitude (W or M) or Intensity (w or w1)	Load Intensity w2	Distance l1	Distance l2
2	Uniform	2.500E-1	---	0.00E+0	0.00E+0
3	Axial–C	2.000E+1	---	1.29E+2	----
3	Conc.	-4.500E+1	---	1.29E+2	----

************** End of Input Data **************

Fig. 6.23 (*continued*)

structure coordinate number vector (**NSC**), for plane frames, remain the same as Parts VII and VIII of the plane truss analysis program (see flowcharts in Figs. 4.8(a) and (b)), provided that *NCJT* is set equal to 3 in these programs.

Generation of the Structure Stiffness Matrix and Equivalent Joint Load Vector A flowchart for writing this part of the plane frame analysis program is given in Fig. 6.24. Comparing this flowchart with that for Part IX of the beam analysis program in Fig. 5.24, we can see that the two programs are similar; the present program, however, transforms the stiffness matrix and fixed-end force vector of each member from its local to the global coordinate system before storing their elements in the structure stiffness matrix and equivalent joint load vector, respectively (see Fig. 6.24). Recall from Chapter 5 that such coordinate transformations are not necessary for beams, because the local and global coordinate systems of such structures are oriented in the same direction. From the flowchart in Fig. 6.24, we can see that for each member of the plane frame, the program first reads the member's material and cross-sectional

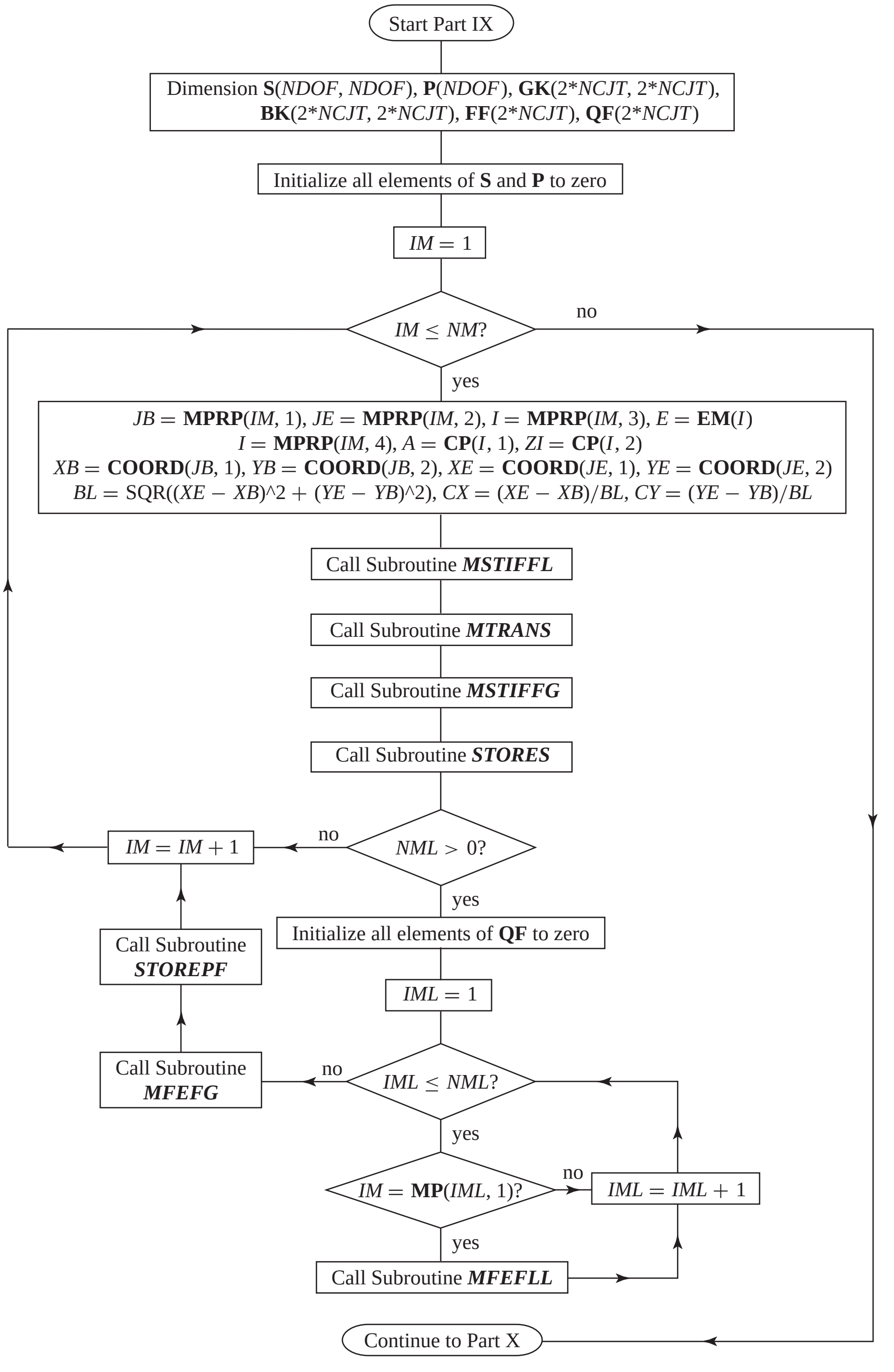


Fig. 6.24 *Flowchart for Generating Structure Stiffness Matrix and Equivalent Joint Load Vector*

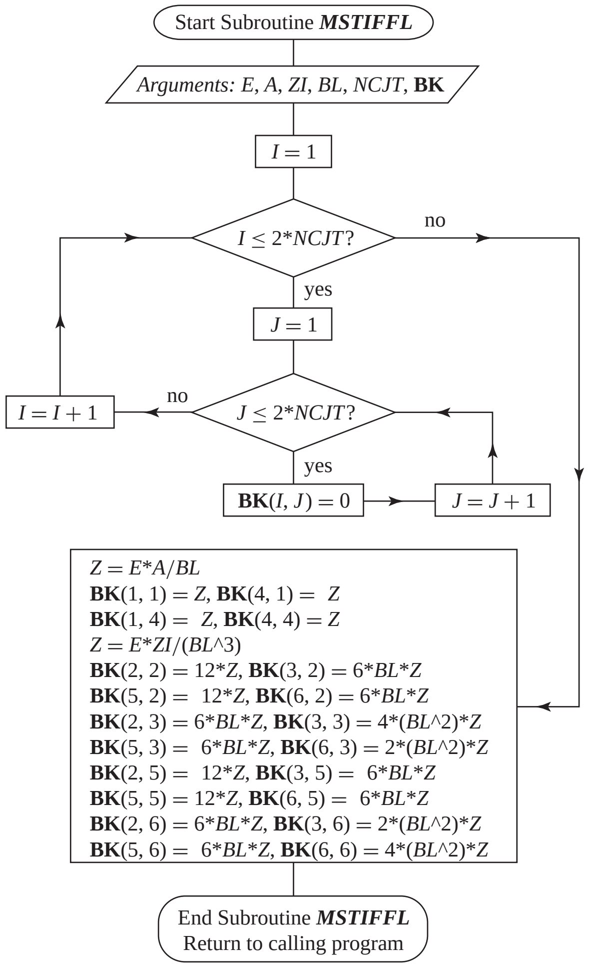


Fig. 6.25 *Flowchart of Subroutine **MSTIFFL** for Determining Member Local Stiffness Matrix for Plane Frames*

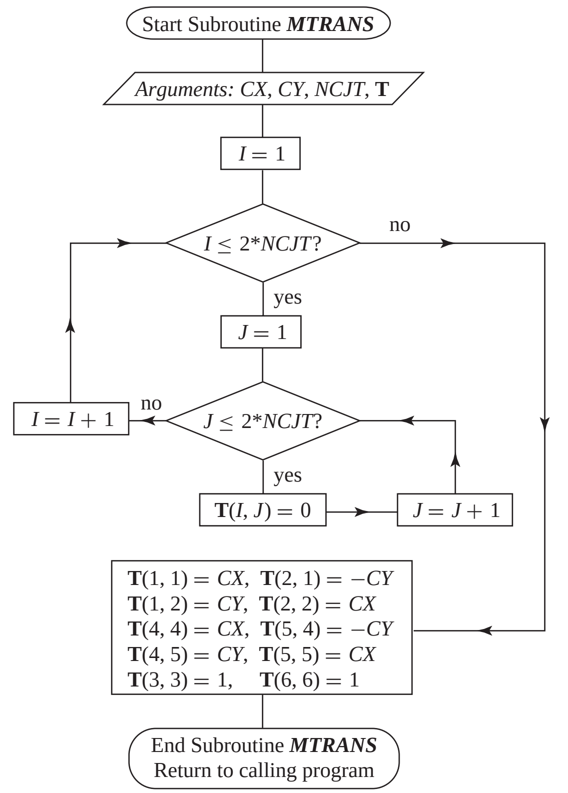


Fig. 6.26 *Flowchart of Subroutine **MTRANS** for Determining Member Transformation Matrix for Plane Frames*

properties, and calculates its length and direction cosines. Next, the program calls the subroutines ***MSTIFFL*** and ***MTRANS*** to form the member local stiffness matrix **BK** and transformation matrix **T**, respectively. As the flowcharts given in Figs. 6.25 and 6.26 indicate, these subroutines calculate the matrices **BK** and **T** in accordance with Eqs. (6.6) and (6.19), respectively. The program then calls the subroutine ***MSTIFFG*** to obtain the member global stiffness matrix **GK**. A flowchart of this subroutine, which evaluates the member global stiffness matrix using the matrix triple product $\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$ (Eq. (6.29)), is given in Fig. 6.27. As this flowchart indicates, the subroutine ***MSTIFFG*** uses two nested *Do Loops* to calculate the member global stiffness matrix **GK** (= **K**). In the first loop, the member local stiffness matrix **BK** (= **k**) is post-multiplied by its transformation matrix **T** to obtain an intermediate matrix **TS**;

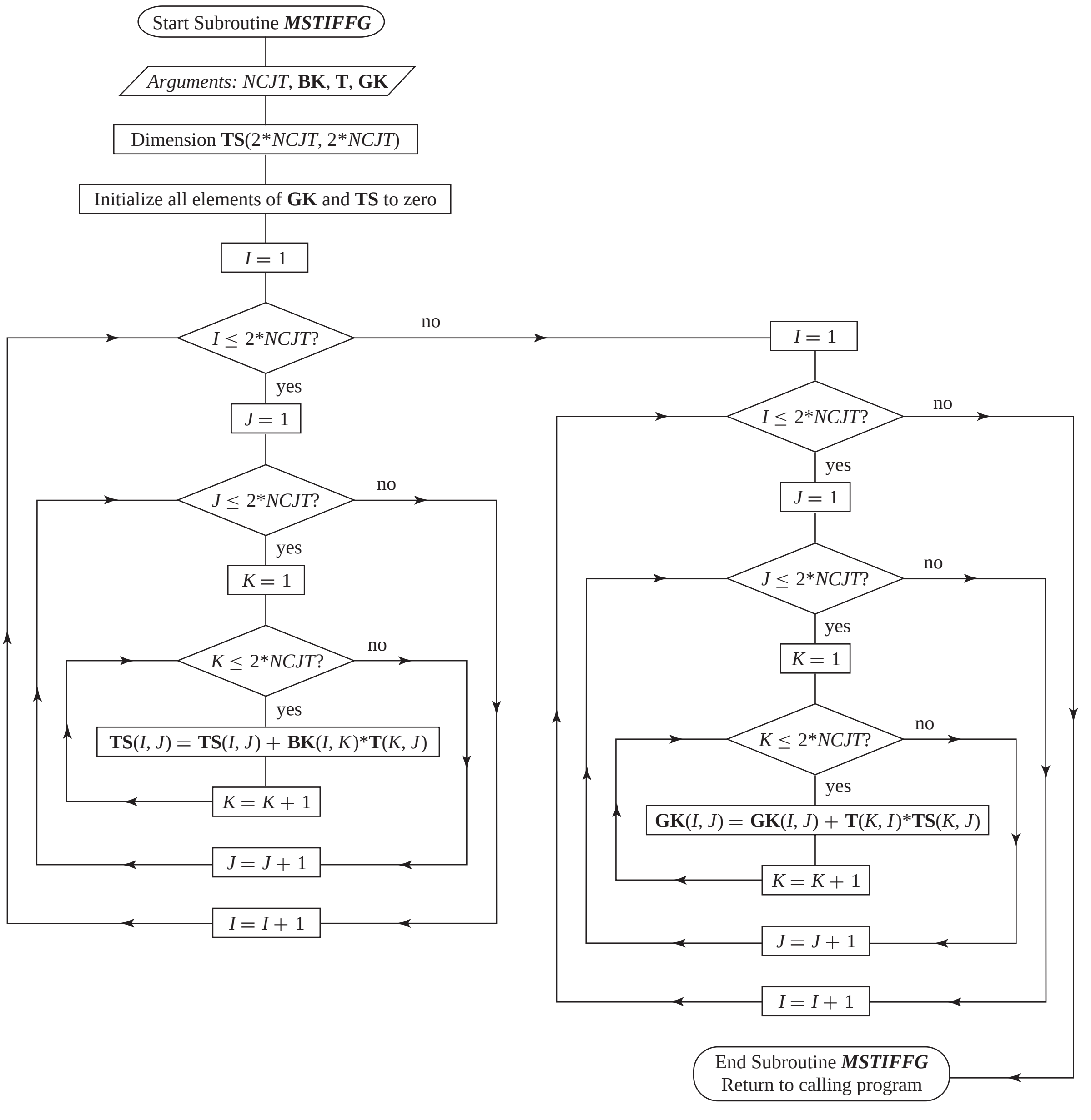


Fig. 6.27 *Flowchart of Subroutine* ***MSTIFFG*** *for Determining Member Global Stiffness Matrix*

in the second, the matrix **TS** is premultiplied by the transpose of the transformation matrix (i.e., $\mathbf{T}^T$) to obtain the desired member global stiffness matrix **GK** (= **K**). Returning our attention to Fig. 6.24, we can see that the program then calls the subroutine ***STORES*** to store the pertinent elements of **GK** in the structure stiffness matrix **S**. This subroutine remains the same as the ***STORES*** subroutine of the plane truss analysis program (see flowchart in Fig. 4.11). After ***STORES*** has been executed, the program (Fig. 6.24) forms the member

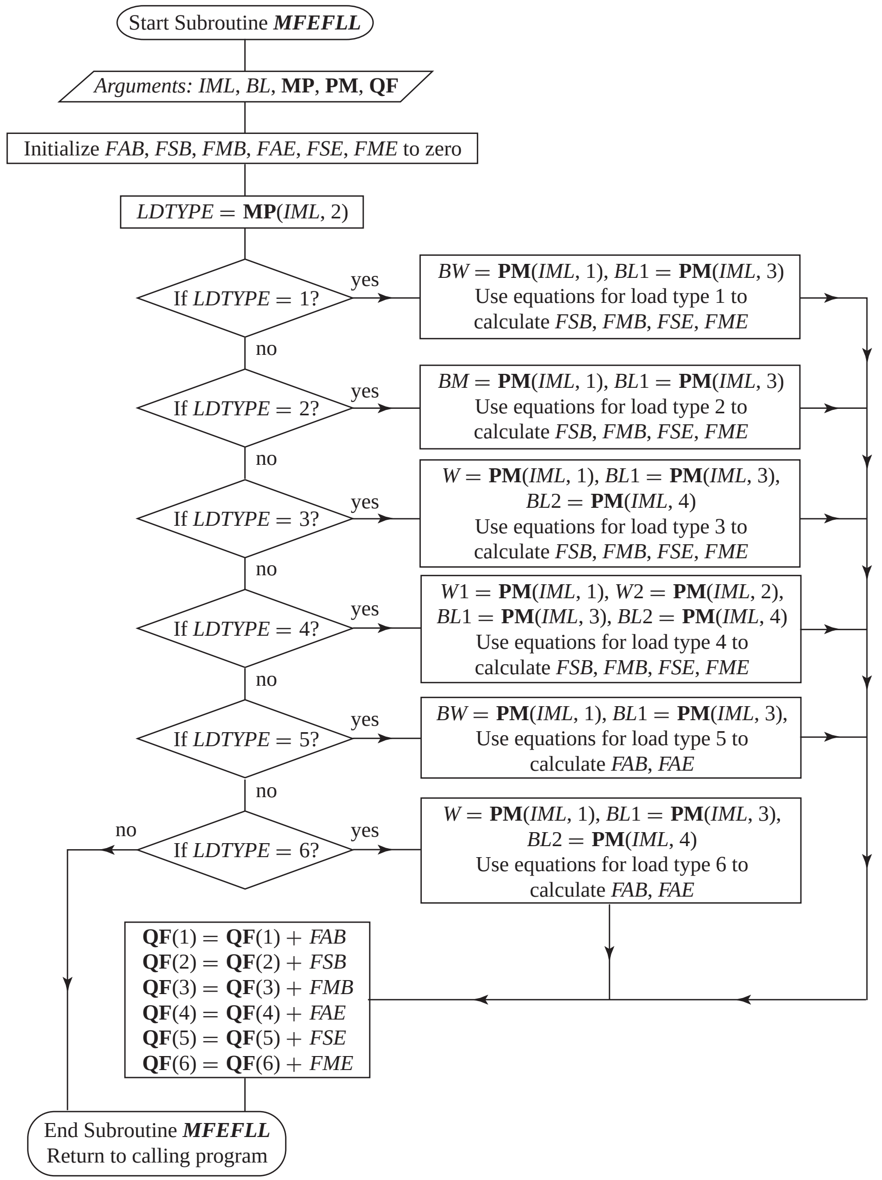


Fig. 6.28 *Flowchart of Subroutine **MFEFLL** for Determining Member Local Fixed-End Force Vector for Plane Frames*

local fixed-end force vector **QF** ($= \mathbf{Q}_f$) by calling the subroutine ***MFEFLL*** (Fig. 6.28), which calculates the values of the member fixed-end forces, for load types 1 through 6, using the equations given inside the front cover. Next, the program calls the subroutine ***MFEFG*** (Fig. 6.29), which evaluates the member global fixed-end force vector **FF** ($= \mathbf{F}_f$), using the relationship $\mathbf{F}_f = \mathbf{T}^T\mathbf{Q}_f$ (Eq. (6.30)). Finally, the program calls the subroutine ***STOREPF*** (Fig. 6.30) to store the negative values of the pertinent elements of **FF** in their

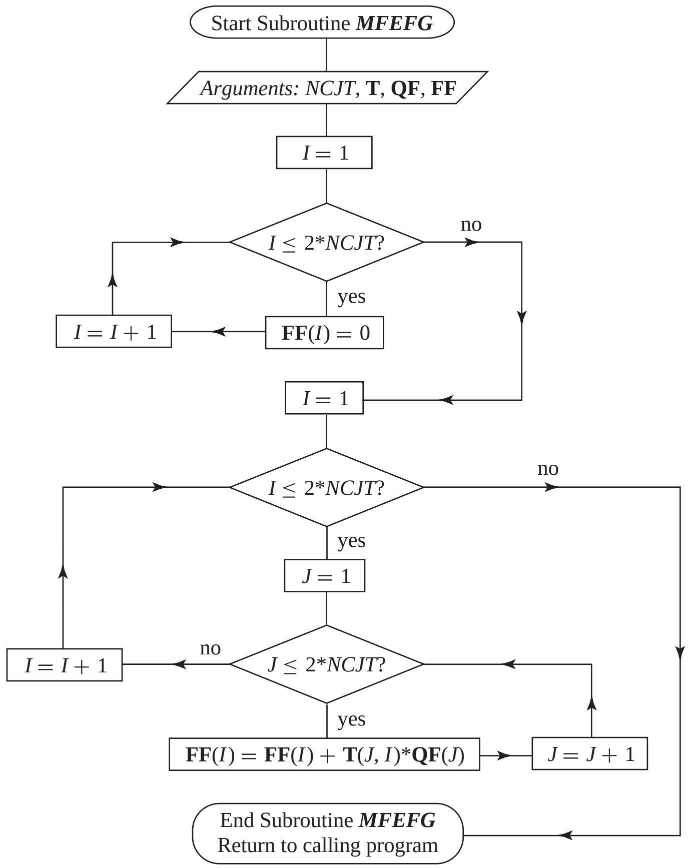


Fig. 6.29 *Flowchart of Subroutine* ***MFEFG*** *for Determining Member Global Fixed-End Force Vector*

proper positions in the structure load vector **P**. When all the operations shown in Fig. 6.24 have been performed for each member of the frame, the structure stiffness matrix **S** is complete, and the structure load vector **P** equals the negative of the structure fixed-joint force vector (i.e., $\mathbf{P} = -\mathbf{P}_f = \mathbf{P}_e$).

Storage of Joint Loads into the Structure Load Vector This is the same as Part X of the beam analysis program (see flowchart in Fig. 5.29).

Solution for Joint Displacements This part of the program remains the same as Part XI of the plane truss analysis program (see flowchart in Fig. 4.13). Recall that upon completion of this part, the vector **P** contains the values of the joint displacements **d**.

Calculation of Member Forces and Support Reactions A flowchart for writing this last part of the program is presented in Fig. 6.31. Note that this part of

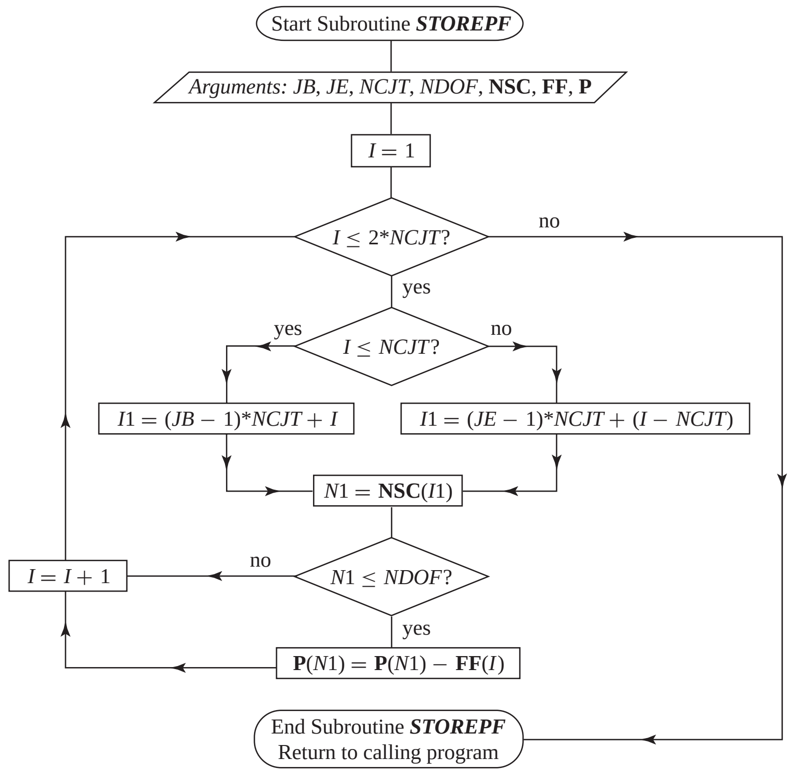


Fig. 6.30 *Flowchart of Subroutine **STOREPF** for Storing Member Global Fixed-End Force Vector in Structure Load Vector*

the plane frame analysis program is essentially a combination of the corresponding parts (XII) of the plane truss and beam analysis programs developed previously. From the flowchart in Fig. 6.31, we can see that, for each member of the frame, the program first reads the member's material and cross-sectional properties, and calculates its length and direction cosines. Next, the program calls the subroutine ***MDISPG*** to form the member global end displacement vector **V** (= **v**). This subroutine is the same as the ***MDISPG*** subroutine of the plane truss analysis program (see flowchart in Fig. 4.15). The program then calls the subroutine ***MTRANS*** (Fig. 6.26) to form the member transformation matrix **T**, and the subroutine ***MDISPL***, which calculates the member local end displacement vector **U** (= **u**), using the relationship $\mathbf{u} = \mathbf{T}\mathbf{v}$ (Eq. (6.20)). The subroutine ***MDISPL*** remains the same as the corresponding subroutine of the plane truss program (see flowchart in Fig. 4.17). Next, the subroutine ***MSTIFFL*** (Fig. 6.25) is called by the program to form the member local stiffness matrix **BK** (= **k**); if the member under consideration is subjected to loads, then its local fixed-end force vector **QF** (= $\mathbf{Q}_f$) is generated using the subroutine ***MFEFLL*** (Fig. 6.28). The program then calls the subroutines ***MFORCEL*** and ***MFORCEG***, respectively, to calculate the member's local and global end

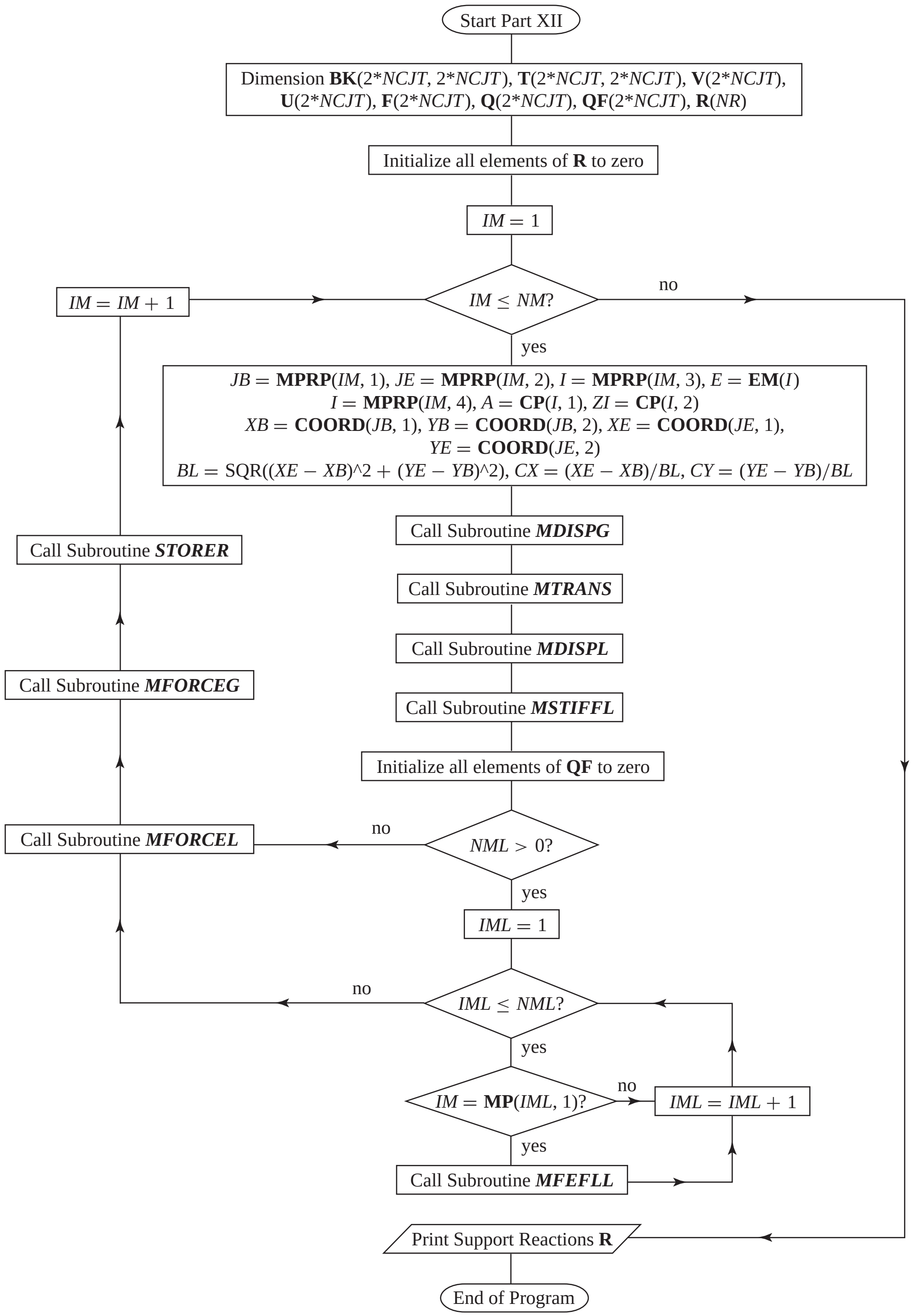


Fig. 6.31 *Flowchart for Determination of Member Forces and Support Reactions for Plane Frames*

force vectors **Q** and **F**. The subroutine ***MFORCEL***, which evaluates **Q** using the relationship $\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$ (Eq. (6.4)), is the same as the corresponding subroutine of the beam analysis program (see flowchart in Fig. 5.32); the subroutine ***MFORCEG***, which computes **F** using the relationship $\mathbf{F} = \mathbf{T}^T\mathbf{Q}$ (Eq. (6.23)), remains the same as the corresponding subroutine of the plane truss program (see flowchart in Fig. 4.20). Finally, the program stores the pertinent elements of **F** in the support reaction vector **R** by calling the subroutine ***STORER***, which remains the same as the corresponding subroutine of the plane truss program (see flowchart in Fig. 4.21). The computational process depicted in Fig. 6.31 can be somewhat expedited by calling the subroutines

```
****************************************************
*                Results of Analysis               *
****************************************************

                 Joint Displacements

Joint No.  X Translation  Y Translation   Rotation (Rad)
    1       0.0000E+00     0.0000E+00      0.0000E+00
    2       3.4472E+00    -9.1684E-03     -1.9513E-02
    3       3.9520E+00    -1.3152E+00      7.0646E-03
    4       4.4247E+00    -2.1160E-02     -9.2709E-03
    5       0.0000E+00     0.0000E+00     -2.3019E-02

          Member End Forces in Local Coordinates

Member  Joint    Axial Force   Shear Force     Moment
  1       1      3.3014E+01    6.7356E+01    1.3789E+04
          2     -3.3014E+01   -6.7356E+01    2.3767E+03

  2       2      1.9358E+01    2.7814E+01   -2.3767E+03
          3     -1.9358E+01    3.6808E+01    1.2142E+03

  3       4      5.9404E+01   -5.8303E+01   -8.0403E+03
          3     -3.9404E+01    1.3303E+01   -1.2142E+03

  4       5      7.6195E+01    3.3501E+01    1.5378E-03
          4     -7.6195E+01   -3.3501E+01    8.0403E+03

                  Support Reactions

 Joint No.    X Force        Y Force         Moment
     1      -6.7356E+01    3.3014E+01     1.3789E+04
     5      -3.3501E+01    7.6195E+01     0.0000E+00

***************** End of Analysis *****************
```

Fig. 6.32 *A Sample Printout of Analysis Results*

MFORCEG and ***STORER*** for only those members of the frame that are attached to supports. To check whether or not a member is attached to a support, its beginning and end joint numbers (i.e., *JB* and *JE*) can be compared with the support joint numbers stored in the first column of the support data matrix **MSUP**.

A sample printout, showing the results of analysis for the example gable frame of Fig. 6.19, is presented in Fig. 6.32, and the entire computer program for the analysis of plane frames is summarized in Table 6.1. As indicated in this table, the computer program consists of a main program (which is divided into twelve parts) and twelve subroutines. Of these, eight parts of the main program and six subroutines can be replicated from the previously developed plane truss and beam analysis programs without any modifications. Finally, it should be realized that the computer program, developed herein for the analysis of plane frames, can also be used to analyze beams, although it is not as efficient for beam analysis as the program developed specifically for that purpose in Chapter 5.

Table 6.1 *Computer Program for Analysis of Plane Frames*

Main program part	Description
I	Reads and stores joint data (Fig. 4.3(a))
II	Reads and stores support data (Fig. 4.3(b))
III	Reads and stores material properties (Fig. 4.3(c))
IV	Reads and stores cross-sectional properties (Fig. 6.22(a))
V	Reads and stores member data (Fig. 4.3(e))
VIa	Reads and stores joint loads (Fig. 5.20(b))
VIb	Reads and stores member loads (Fig. 6.22(b))
VII	Determines the number of degrees of freedom *NDOF* of the structure (Fig. 4.8(a))
VIII	Forms the structure coordinate number vector **NSC** (Fig. 4.8(b))
IX	Generates the structure stiffness matrix **S** and the structure load vector $\mathbf{P} = \mathbf{P}_e = -\mathbf{P}_f$ due to member loads (Fig. 6.24) Subroutines called: ***MSTIFFL***, ***MTRANS***, ***MSTIFFG***, ***STORES***, ***MFEFLL***, ***MFEFG***, and ***STOREPF***
X	Stores joint loads in the structure load vector **P** (Fig. 5.29)
XI	Calculates the structure's joint displacements by solving the stiffness relationship, $\mathbf{Sd} = \mathbf{P}$, using Gauss–Jordan elimination. The vector **P** now contains joint displacements (Fig. 4.13).
XII	Determines the member end force vectors **Q** and **F**, and the support reaction vector **R** (Fig. 6.31) Subroutines called: ***MDISPG***, ***MTRANS***, ***MDISPL***, ***MSTIFFL***, ***MFEFLL***, ***MFORCEL***, ***MFORCEG***, and ***STORER***

(*continued*)

Table 6.1 (*continued*)

Subroutine	Description
MDISPG	Forms the member global displacement vector **V** from the joint displacement vector **P** (Fig. 4.15)
MDISPL	Evaluates the member local displacement vector $\mathbf{U} = \mathbf{TV}$ (Fig. 4.17)
MFEFG	Determines the member global fixed-end force vector $\mathbf{FF} = \mathbf{T}^T\mathbf{QF}$ (Fig. 6.29)
MFEFLL	Calculates the member local fixed-end force vector **QF** (Fig. 6.28)
MFORCEG	Evaluates the member global force vector $\mathbf{F} = \mathbf{T}^T\,\mathbf{Q}$ (Fig. 4.20)
MFORCEL	Calculates the member local force vector $\mathbf{Q} = \mathbf{BK}\,\mathbf{U} + \mathbf{QF}$ (Fig. 5.32)
MSTIFFG	Determines the member global stiffness matrix $\mathbf{GK} = \mathbf{T}^T\,\mathbf{BK}\,\mathbf{T}$ (Fig. 6.27)
MSTIFFL	Forms the member local stiffness matrix **BK** (Fig. 6.25)
MTRANS	Forms the member transformation matrix **T** (Fig. 6.26)
STOREPF	Stores the negative values of the pertinent elements of the member global fixed-end force vector **FF** in the structure load vector **P** (Fig. 6.30)
STORER	Stores the pertinent elements of the member global force vector **F** in the reaction vector **R** (Fig. 4.21)
STORES	Stores the pertinent elements of the member global stiffness matrix **GK** in the structure stiffness matrix **S** (Fig. 4.11)

SUMMARY

In this chapter, we have developed the matrix stiffness method for the analysis of rigidly connected plane frames subjected to external loads. A block diagram summarizing the various steps of the analysis is shown in Fig. 6.33.

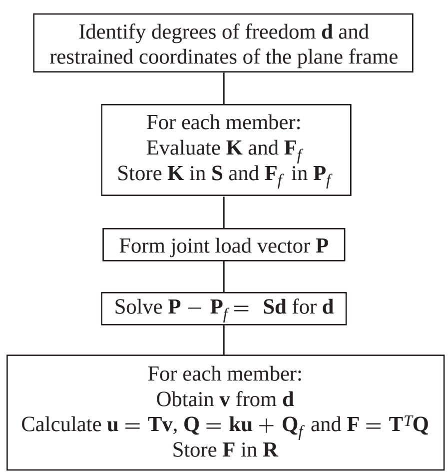
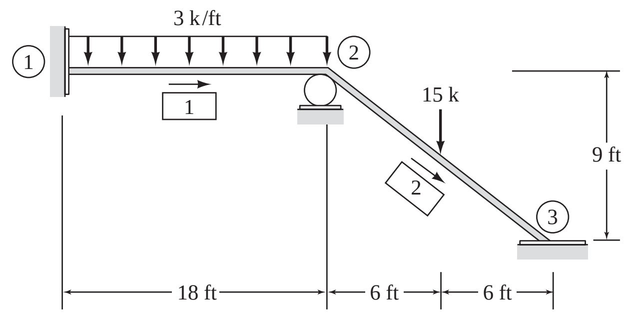


Fig. 6.33

PROBLEMS

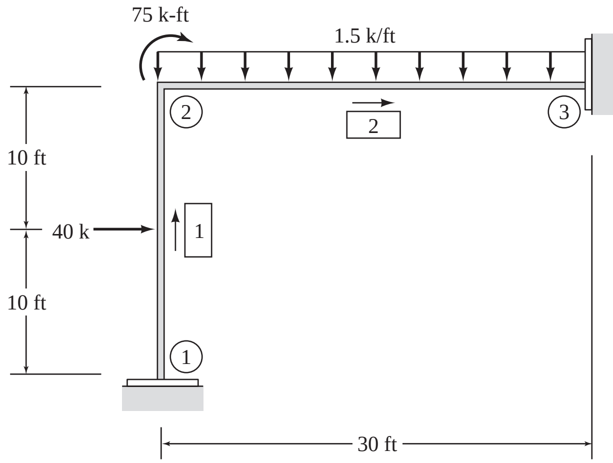
Section 6.1

6.1 through 6.6 Identify by numbers the degrees of freedom and restrained coordinates of the frames shown in Figs. P6.1 through P6.6. Also, form the joint load vector **P** for these frames.



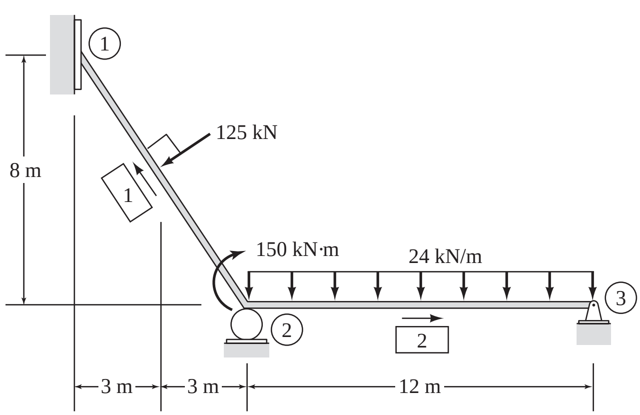
E, A, I = constant
E = 10,000 ksi
A = 8.84 in.^{2}
I = 170 in.^{4}

Fig. P6.3, P6.9, P6.18, P6.26, P6.34, P6.44



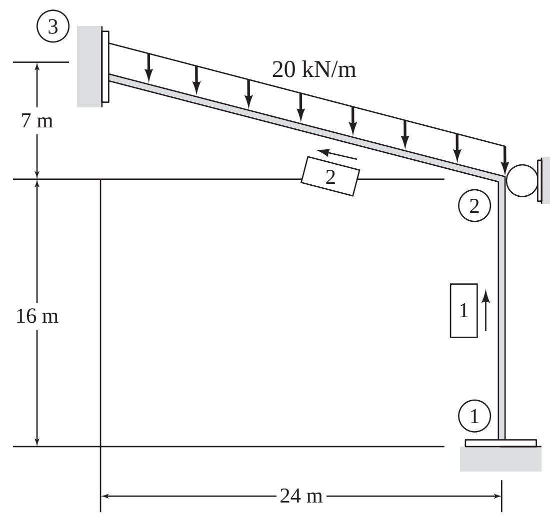
E, A, I = constant
E = 29,000 ksi
A = 10.3 in.^{2}
I = 510 in.^{4}

Fig. P6.1, P6.7, P6.16, P6.24, P6.32, P6.42



E, A, I = constant
E = 200 GPa
A = 13,000 mm^2
I = 762(10^6) mm^4

Fig. P6.2, P6.8, P6.17, P6.25, P6.33, P6.43



E, A, I = constant
E = 200 GPa
A = 16,000 mm^2
I = 1,186(10^6) mm^4

Fig. P6.4, P6.10, P6.19, P6.27, P6.35, P6.45

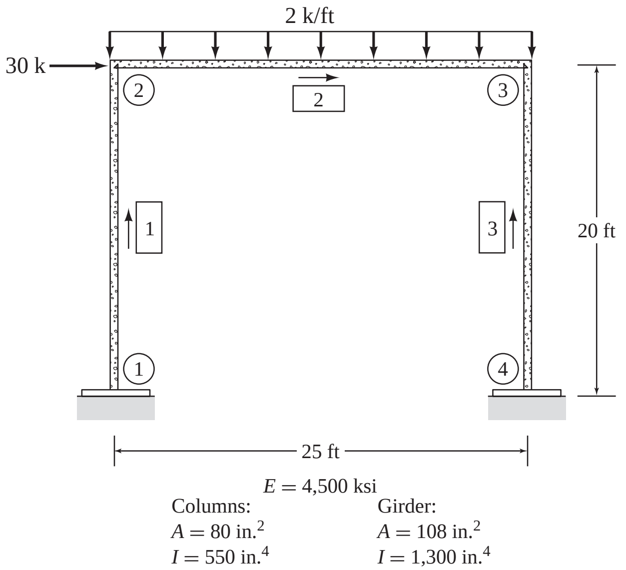


Fig. P6.5, P6.11, P6.20, P6.28, P6.36, P6.48

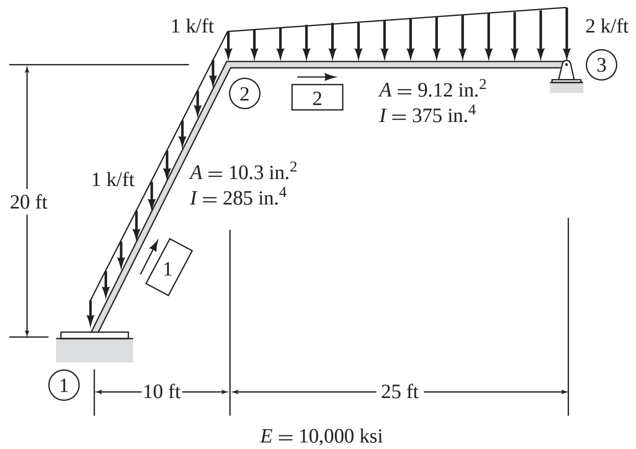


Fig. P6.6, P6.12, P6.21, P6.29, P6.37, P6.47

Section 6.2

6.7 through 6.12 Determine the local stiffness matrix **k**, and the fixed-end force vector $\mathbf{Q}_f$, for each member of the frames shown in Figs. P6.7 through P6.12. Use the fixed-end force equations given inside the front cover.

6.13 Using the integration approach, derive the equations of fixed-end forces due to the concentrated axial member load shown in Fig. P6.13. Check the results using the fixed-end force expressions given inside the front cover.

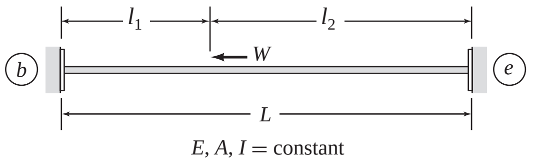


Fig. P6.13

6.14 Assume that the local end displacements for the members of the frame shown in Fig. P6.14 are

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5.2507 \text{ mm} \\ -12.251 \text{ mm} \\ -0.12416 \text{ rad} \end{bmatrix}; \qquad \mathbf{u}_2 = \begin{bmatrix} 9.2888 \text{ mm} \\ -9.5586 \text{ mm} \\ -0.12416 \text{ rad} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Calculate the member local end force vectors. Are the members in equilibrium under these forces?

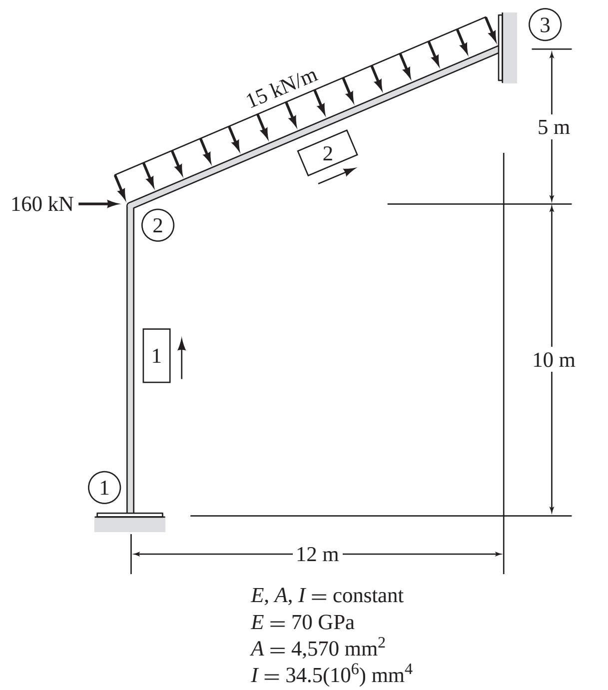


Fig. P6.14, P6.22, P6.30

6.15 Assume that the local end displacements for the members of the frame shown in Fig. P6.15 are

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.05147 \text{ in.} \\ 2.0939 \text{ in.} \\ 0.0079542 \text{ rad} \end{bmatrix}; \quad \mathbf{u}_2 = \begin{bmatrix} 2.0184 \text{ in.} \\ 0.14398 \text{ in.} \\ 0.0028882 \text{ rad} \\ 1.9526 \text{ in.} \\ -0.75782 \text{ in.} \\ 0.0079542 \text{ rad} \end{bmatrix};$$

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.04847 \text{ in.} \\ 2.023 \text{ in.} \\ 0.0028882 \text{ rad} \end{bmatrix}$$

Calculate the member local end force vectors. Are the members in equilibrium under these forces?

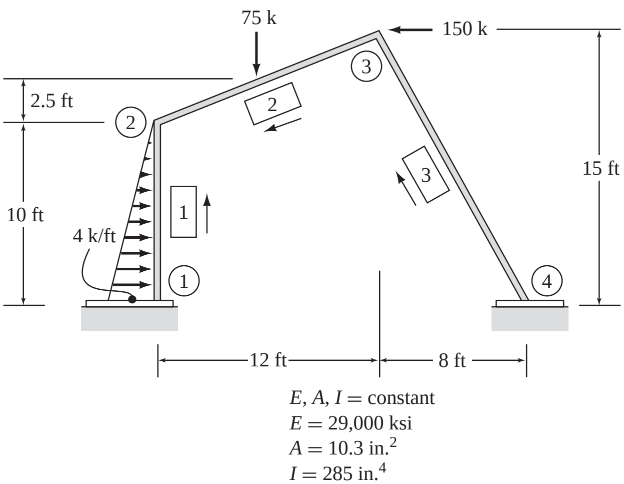


Fig. P6.15, P6.23, P6.31

Section 6.3

6.16 through 6.21 Determine the transformation matrix **T** for each member of the frames shown in Figs. P6.16 through P6.21.

6.22 Using the local end displacements given in Problem 6.14 for the members of the frame of Fig. P6.22, calculate the global end displacement vector and the global end force vector for each member of the frame. Are the members in equilibrium under the global end forces?

6.23 Using the local end displacements given in Problem 6.15 for the members of the frame of Fig. P6.23, calculate the global end displacement vector and the global end force vector for each member of the frame. Are the members in equilibrium under the global end forces?

Section 6.4

6.24 through 6.29 Determine the global stiffness matrix **K**, and fixed-end force vector $\mathbf{F}_f$, for each member of the frames shown in Figs. P6.24 through P6.29.

6.30 Calculate the member global end force vectors required in Problem 6.22 using the member global stiffness relationship $\mathbf{F} = \mathbf{K}\mathbf{v} + \mathbf{F}_f$.

6.31 Calculate the member global end force vectors required in Problem 6.23 using the member global stiffness relationship $\mathbf{F} = \mathbf{K}\mathbf{v} + \mathbf{F}_f$.

Section 6.5

6.32 through 6.37 Determine the structure stiffness matrix, the fixed-joint force vector, and the equivalent joint loads for the frames shown in Figs. P6.32 through P6.37.

6.38 Assume that the joint displacements for the frame of Fig. P6.38 are

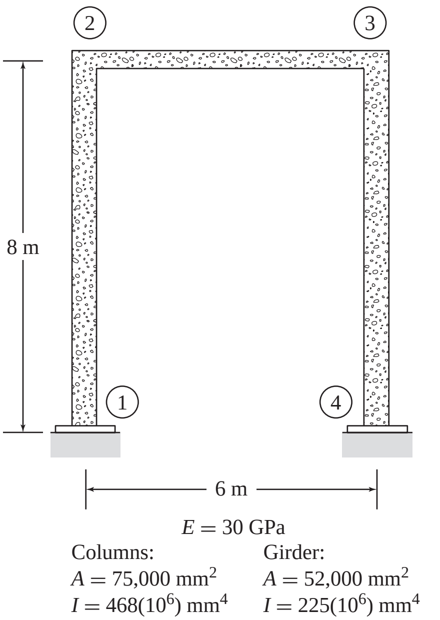


Fig. P6.38

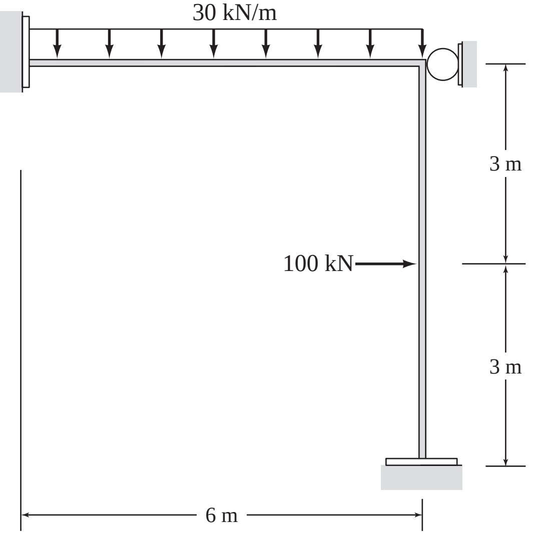
$$\mathbf{d} = \begin{bmatrix} 0.1965 \text{ m} \\ 0.00016452 \text{ m} \\ -0.017932 \text{ rad} \\ 0.19637 \text{ m} \\ -0.00016452 \text{ m} \\ -0.023307 \text{ rad} \end{bmatrix}$$

Calculate the joint loads causing these displacements. (No loads are applied to the members of the frame.)

6.39 Assume that the joint displacements for the frame of Fig. P6.39 are

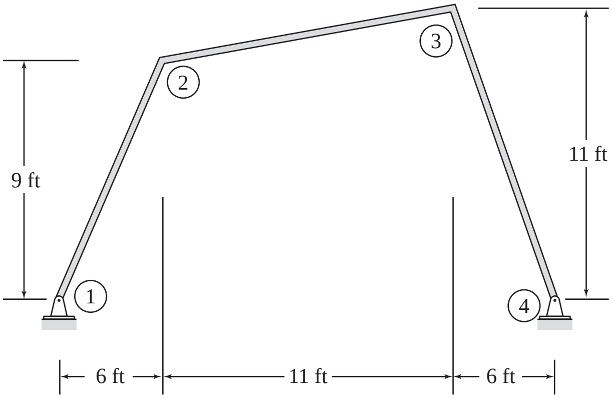
$$\mathbf{d} = \begin{bmatrix} -0.059209 \text{ rad} \\ 4.1192 \text{ in.} \\ -2.7371 \text{ in.} \\ 0.0041099 \text{ rad} \\ 3.3212 \text{ in.} \\ 1.7637 \text{ in.} \\ 0.02198 \text{ rad} \\ -0.048502 \text{ rad} \end{bmatrix}$$

Calculate the joint loads causing these displacements. (No loads are applied to the members of the frame.)



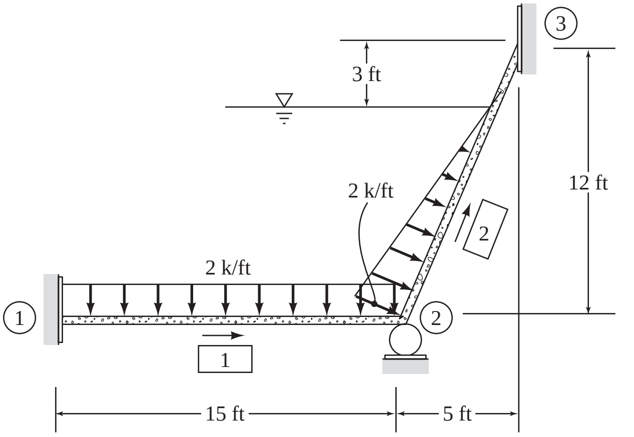
$E, A, I =$ constant
$E = 200$ GPa
$A = 11{,}800$ mm^2
$I = 554(10^6)$ mm^4

Fig. P6.40



$E, A, I =$ constant
$E = 10{,}000$ ksi
$A = 14.1$ in.^{2}
$I = 184$ in.^{4}

Fig. P6.39

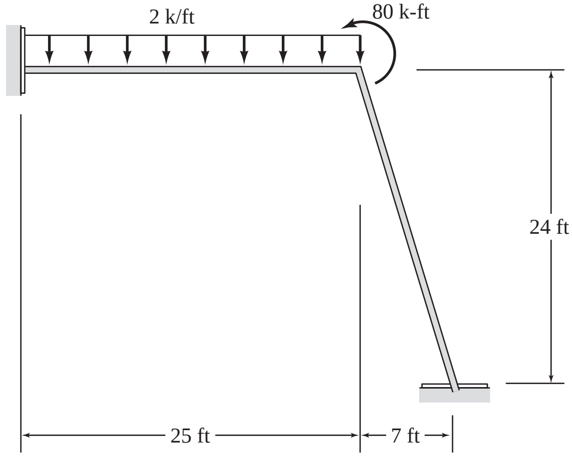


$E, A, I =$ constant
$E = 4{,}000$ ksi
$A = 96$ in.^{2}
$I = 512$ in.^{4}

Fig. P6.41

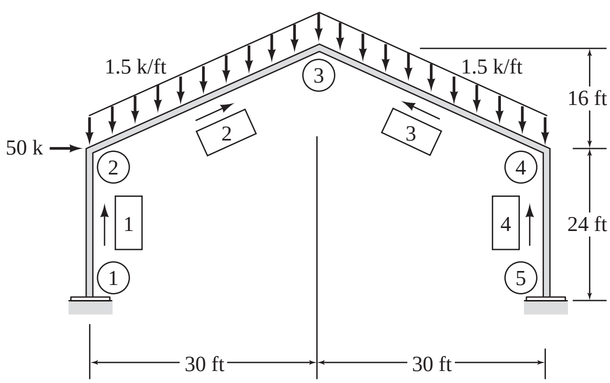
Section 6.6

6.40 through 6.50 Determine the joint displacements, member local end forces, and support reactions for the frames shown in Figs. P6.40 through P6.50, using the matrix stiffness method. Check the hand-calculated results by using the computer program which can be downloaded from the publisher's website for this book, or by using any other general purpose structural analysis program available.



E, A, I = constant
$E = 29{,}000$ ksi
$A = 14.7$ in.^{2}
$I = 800$ in.^{4}

Fig. P6.46

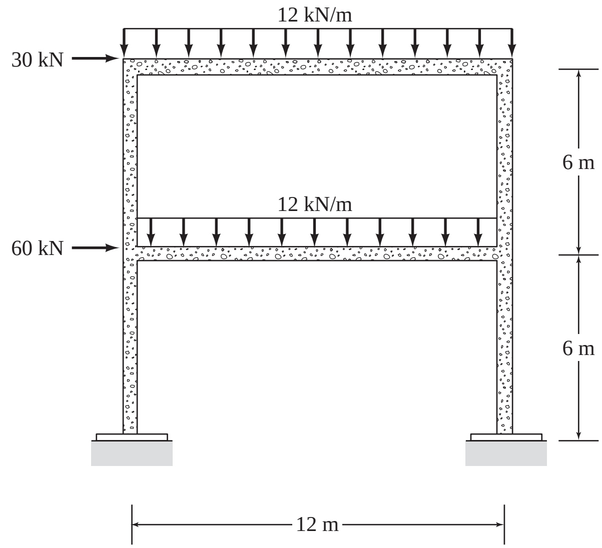


$E = 29{,}000$ ksi

Columns:
$A = 20$ in.^{2}
$I = 723$ in.^{4}

Girders:
$A = 20.1$ in.^{2}
$I = 1{,}830$ in.^{4}

Fig. P6.49



$E = 30$ GPa

Columns:
$A = 93{,}000$ mm^2
$I = 720(10^6)$ mm^4

Girders:
$A = 140{,}000$ mm^2
$I = 2{,}430(10^6)$ mm^4

Fig. P6.50

Section 6.7

6.51 Develop a general computer program for the analysis of rigidly connected plane frames by the matrix stiffness method. Use the program to analyze the frames of Problems 6.40 through 6.50, and compare the computer-generated results to those obtained by hand calculations.