Tema 1. Ejercicios propuestos

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Ejercicio 1.

$$
\begin{aligned}
&X \sim N_p(\mu_X, \Sigma_X) \qquad (\Sigma_X > 0)\\
&Y = BX + b \quad \text{con } B \text{ matriz (cte.) } p \times p \text{ no singular}\\
&\qquad\qquad\qquad\quad b \text{ vector (cte.) } p \times 1
\end{aligned}
$$

Entonces, $Y \sim N_p(\mu_Y, \Sigma_Y) = N_p(B\mu_X + b, B\Sigma_X B')$

Cambio de variables lineal: $\qquad$ (Abrevio $\Sigma_X = \Sigma$ y $\mu_X = \mu$)

$$
\begin{aligned}
f_Y(y) &= f_X(B^{-1}(y-b)) \cdot \operatorname{abs}(|B|^{-1})\\
&= \frac{1}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}\big(B^{-1}(y-b)-\mu\big)'\Sigma^{-1}\big(B^{-1}(y-b)-\mu\big)\Big\} \cdot \operatorname{abs}(|B|^{-1})\\
&= \frac{\operatorname{abs}(|B|^{-1})}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}\big((y-b)'(B^{-1})'-\mu'\big)\Sigma^{-1}\big(B^{-1}(y-b)-\mu\big)\Big\}\\
&= \frac{\operatorname{abs}(|B|^{-1})}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}\big(y'-b'-\mu'B'\big)(B^{-1})'\Sigma^{-1}B^{-1}\big(y-b-B\mu\big)\Big\}\\
&= \frac{\operatorname{abs}(|B|^{-1})}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}\big(y'-(b'+(B\mu)')\big)\big(B\Sigma B'\big)^{-1}\big(y-(b+B\mu)\big)\Big\}\\
&= \frac{\operatorname{abs}(|B|^{-1})}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}\big(y-(b+B\mu)\big)'\big(B\Sigma B'\big)^{-1}\big(y-(b+B\mu)\big)\Big\}\\
&= \frac{1}{(2\pi)^{p/2}|\Sigma_Y|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}\big(y-\mu_Y\big)'\big(\Sigma_Y\big)^{-1}\big(y-\mu_Y\big)\Big\}
\end{aligned}
$$

Para el factor de la izquierda en el último paso:

$$|\Sigma_Y| = |B\Sigma B'| = |B||\Sigma||B'| = |B|^2|\Sigma| \Rightarrow |\Sigma_Y|^{\frac{1}{2}} = \operatorname{abs}(|B|)|\Sigma|^{\frac{1}{2}}$$

$$\frac{1}{|\Sigma_Y|^{\frac{1}{2}}} = \frac{1}{\operatorname{abs}(|B|)}\frac{1}{|\Sigma|^{\frac{1}{2}}} = \frac{\operatorname{abs}(|B|^{-1})}{|\Sigma|^{\frac{1}{2}}}$$

Ejercicio 2.

$$X \sim N_p(\mu_X, \Sigma_X) \qquad (\Sigma_X > 0)$$
$$Y = BX + b \quad \text{con } B \text{ matriz (cte.) } p \times q \text{ de rango } q \leq p$$
$$b \text{ vector (cte.) } q \times 1$$

Entonces, $Y \sim N_q(\mu_Y, \Sigma_Y) = N_q(B\mu_X + b, B\Sigma_X B')$

(Abrevio $\Sigma_X = \Sigma$ y $\mu_X = \mu$)

Sabemos que $\phi_X(t) = \exp\left\{it'\mu - \frac{1}{2}t'\Sigma t\right\}$.

Tenemos

$$\begin{aligned}
\phi_Y(t) = e^{it'b}\phi_X(B't) &= e^{it'b}\exp\left\{i(B't)'\mu - \frac{1}{2}(B't)'\Sigma B't\right\} \\
&= \exp\left\{it'b + it'B\mu - \frac{1}{2}t'B\Sigma B't\right\} \\
&= \exp\left\{it'(b + B\mu) - \frac{1}{2}t'B\Sigma B't\right\} \\
&= \exp\left\{it'\mu_Y - \frac{1}{2}t'\Sigma_Y t\right\}
\end{aligned}$$