Ejercicios propuestos: Parte III

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Ejercicio 1

Función masa de probabilidad:

X/Y	0	2
2	1/16	1/16
4	1/8	1/4
6	0	1/2

Se pide calcular la función generatriz de momentos y, a partir de ella, la matriz de covarianzas.

f.g.m.:

$$M(t_1,t_2)=E[e^{t_1x+t_2y}]=\frac{1}{16}e^{2t_1}+\frac{1}{16}e^{2t_1+2t_2}+\frac{1}{8}e^{4t_1}+\frac{1}{4}e^{4t_1+2t_2}+\frac{1}{2}e^{6t_1+2t_2}=\frac{1}{16}(e^{2t_1}+e^{2t_1+2t_2}+2e^{4t_1}+4e^{4t_1+2t_2}+8e^{6t_1+2t_2})$$

Queremos calcular la **matriz de covarianzas**

$$\begin{pmatrix}\mathrm{Var}(X) & \mathrm{Cov}(X,Y)\\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y)\end{pmatrix}=\begin{pmatrix}\mu_{20} & \mu_{11}\\ \mu_{11} & \mu_{02}\end{pmatrix}$$

Para hallar los momentos centrados, podemos hallar primero los no centrados. Usaremos la función generatriz de momentos para hallarlos.

$$m_{10}=\frac{\partial M}{\partial t_1}(t_1,t_2)\Big|_{t_1=t_2=0}=\frac{1}{16}(2e^{2t_1}+2e^{2t_1+2t_2}+8e^{4t_1}+16e^{4t_1+2t_2}+48e^{6t_1+2t_2})\Big|_{t_1=t_2=0}=\frac{2+2+8+16+48}{16}=\frac{19}{4}$$

$$m_{01}=\frac{\partial M}{\partial t_2}(t_1,t_2)\Big|_{t_1=t_2=0}=\frac{1}{16}(2e^{2t_1+2t_2}+8e^{4t_1+2t_2}+16e^{6t_1+2t_2})\Big|_{t_1=t_2=0}=\frac{2+8+16}{16}=\frac{13}{8}$$

$$m_{20}=\frac{\partial^2 M}{\partial t_1^2}(t_1,t_2)\Big|_{t_1=t_2=0}=\frac{1}{16}(4e^{2t_1}+4e^{2t_1+2t_2}+32e^{4t_1}+64e^{4t_1+2t_2}+288e^{6t_1+2t_2})\Big|_{t_1=t_2=0}=\frac{4+4+32+64+288}{16}=\frac{49}{2}$$

$$m_{02}=\frac{\partial^2 M}{\partial t_2^2}(t_1,t_2)\Big|_{t_1=t_2=0}=\frac{1}{16}(4e^{2t_1+2t_2}+16e^{4t_1+2t_2}+32e^{6t_1+2t_2})\Big|_{t_1=t_2=0}=\frac{4+16+32}{16}=\frac{13}{4}$$

$$m_{11}=\frac{\partial^2 M}{\partial t_1 t_2}(t_1,t_2)\Big|_{t_1=t_2=0}=\frac{1}{16}(4e^{2t_1+2t_2}+32e^{4t_1+2t_2}+96e^{6t_1+2t_2})\Big|_{t_1=t_2=0}=\frac{4+32+96}{16}=\frac{33}{4}$$

Ya podemos hallar los momentos centrados:

$$\mu_{20}=m_{20}-m_{10}^2=\frac{49}{2}-\left(\frac{19}{4}\right)^2=\frac{31}{16}$$

$$\mu_{11}=m_{11}-m_{10}m_{01}=\frac{33}{4}-\frac{19}{4}\frac{13}{8}=\frac{17}{32}$$

$$\mu_{02}=m_{02}-m_{01}^2=\frac{13}{4}-\left(\frac{13}{8}\right)^2=\frac{39}{64}$$

Por tanto la matriz de covarianzas es

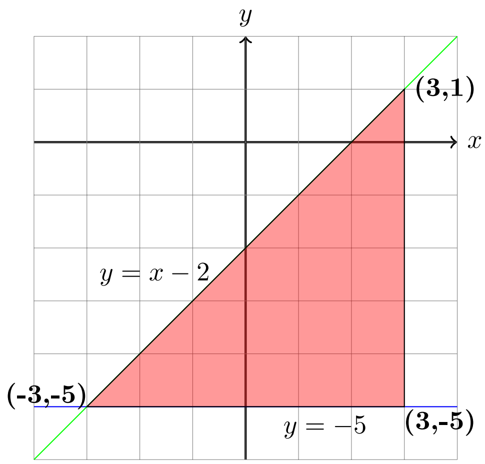
$$\begin{pmatrix}\frac{31}{16} & \frac{17}{32}\\ \frac{17}{32} & \frac{39}{64}\end{pmatrix}$$

Ejercicio 2

Función densidad:

$$f(x,y)=\frac{1}{18} \qquad -3<x<3,\ -5<y<-2+x$$

Se pide calcular la recta y la curva de regresión de Y sobre X, así como el coeficiente de determinación lineal y la razón de correlación.



Recta de regresión:

$$Y=aX+b$$

donde

$$a=\gamma_{Y/X}=\frac{\mathrm{Cov}(X,Y)}{Var(X)}, \qquad b=\mathrm{E}(Y)-a\,\mathrm{E}(X)$$

Covarianza:

$$\mathrm{Cov}(X,Y)=\mathrm{E}(XY)-\mathrm{E}(X)\,\mathrm{E}(Y)$$

$$\begin{aligned}
\mathrm{E}(XY)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf(x,y)dxdy=\int_{-3}^{3}\int_{-5}^{x-2}\frac{1}{18}xydydx=\frac{1}{18}\int_{-3}^{3}x\int_{-5}^{x-2}ydydx\\
&=\frac{1}{18}\int_{-3}^{3}x\left[\frac{y^2}{2}\right]_{-5}^{x-2}dx=\frac{1}{18}\int_{-3}^{3}x\left(\frac{(x-2)^2}{2}-\frac{(-5)^2}{2}\right)dx=\frac{1}{36}\int_{-3}^{3}x(x^2-4x+4-25)dx\\
&=\frac{1}{36}\int_{-3}^{3}x^3-4x^2-21xdx=\frac{1}{36}\left[\frac{x^4}{4}-\frac{4x^3}{3}-\frac{21x^2}{2}\right]_{-3}^{3}=\frac{1}{36}\left[\frac{4x^3}{3}\right]_{3}^{-3}=\frac{1}{36}\left(\frac{-108}{3}-\frac{108}{3}\right)\\
e&=\frac{-3}{3}-\frac{3}{3}=-2
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(X)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xf(x,y)dxdy=\int_{-3}^{3}\int_{-5}^{x-2}\frac{1}{18}xdydx=\frac{1}{18}\int_{-3}^{3}x\int_{-5}^{x-2}dydx\\
&=\frac{1}{18}\int_{-3}^{3}x(x-2-(-5))dx=\frac{1}{18}\int_{-3}^{3}x^2+3xdx=\frac{1}{18}\left[\frac{x^3}{3}+\frac{3x^2}{2}\right]_{-3}^{3}=\\
&=\frac{1}{18}\left(\frac{27}{3}-\frac{-27}{3}\right)=1
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}(Y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y f(x,y)dxdy = \int_{-3}^{3}\int_{-5}^{x-2}\frac{1}{18}ydydx = \frac{1}{18}\int_{-3}^{3}\int_{-5}^{x-2} ydydx \\
&= \frac{1}{18}\int_{-3}^{3}\left[\frac{y^2}{2}\right]_{-5}^{x-2}dx = \frac{1}{36}\int_{-3}^{3}(x-2)^2-(-5)^2dx = \frac{1}{36}\int_{-3}^{3}x^2-4x-21dx \\
&= \frac{1}{36}\left[\frac{x^3}{3}-2x^2-21x\right]_{-3}^{3} = \frac{1}{36}\left(\frac{27}{3}-63-\left(\frac{-27}{3}+63\right)\right) = -3
\end{aligned}
$$

$$
\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = -2 - 1\dot{(}-3) = 1
$$

Varianza de X:

$$
\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2
$$

$$
\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,y)dxdy = \int_{-3}^{3}\int_{-5}^{x-2}\frac{1}{18}x^2dydx = \frac{1}{18}\int_{-3}^{3}x^2\int_{-5}^{x-2}dydx \\
&= \frac{1}{18}\int_{-3}^{3}x^2(x-2-(-5))dx = \frac{1}{18}\int_{-3}^{3}x^3+3x^2dx = \frac{1}{18}\left[\frac{x^4}{4}+x^3\right]_{-3}^{3} = \\
&= \frac{1}{18}\Big(27-(-27)\Big) = 3
\end{aligned}
$$

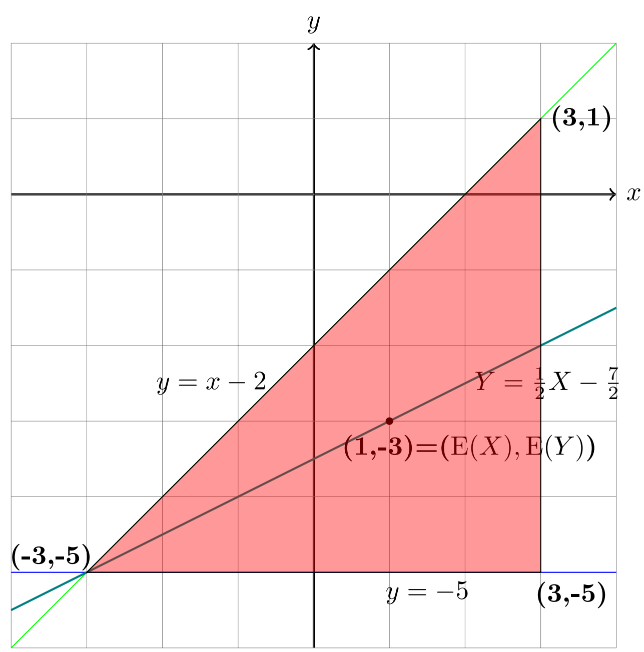
$$
\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 3 - 1^2 = 2
$$

Recta de regresión:

$$
a = \gamma_{Y/X} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} = \frac{1}{2}
$$

$$
b = \mathrm{E}(Y) - a\,\mathrm{E}(X) = -3 - 1\frac{1}{2} = -\frac{7}{2}
$$

$$
Y = \frac{1}{2}X - \frac{7}{2}
$$



Curva de regresión:

$$Y = \varphi(X) = E(Y/X)$$

Esperanza condicionada:

$$\mathrm{E}(Y/X = x) = \int_{-\infty}^{+\infty} y f(y/X = x) dy$$

Necesito la condicionada de Y a X.

Marginal:

$$f_1(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_{-5}^{x-2} \frac{1}{18}dy = \frac{1}{18}(x - 2 - (-5)) = \frac{1}{18}(x+3) \qquad x \in]-3,3[$$

Condicionada:

$x \in]-3,3[$

$$f(y/X = x) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{1}{18}}{\frac{1}{18}(x+3)} = \frac{1}{x+3} \qquad y \in]-5, x-2[$$

Curva de regresión:

$$\begin{aligned}\mathrm{E}(Y/X = x) &= \int_{-\infty}^{+\infty} y f(y/X = x)dy = \int_{-5}^{x-2} y\frac{1}{x+3}dy = \frac{1}{x+3}\left[\frac{y^2}{2}\right]_{-5}^{x-2} \\ &= \frac{1}{2x+6}\Big((x-2)^2 - (-5)^2\Big) = \frac{x^2 - 4x - 21}{2x+6} = \frac{(x+3)(x-7)}{2(x+3)} = \frac{1}{2}x - \frac{7}{2}\end{aligned}$$

$$Y = \varphi(X) = E(Y/X) = \frac{1}{2}X - \frac{7}{2}$$

En este caso coincide con la recta de regresión.

Razón de correlación:

$$\eta^2_{Y/X} = \frac{\mathrm{Var}(\mathrm{E}(Y/X))}{\mathrm{Var}(Y)} = 1 - \frac{\mathrm{E}(\mathrm{Var}(Y/X))}{\mathrm{Var}(Y)}$$

Varianza de la esperanza condicionada:

$$\mathrm{Var}(\mathrm{E}(Y/X)) = \mathrm{E}(\mathrm{E}(Y/X)^2) - \mathrm{E}(\mathrm{E}(Y/X))^2$$

$$\begin{aligned}\mathrm{E}(\mathrm{E}(Y/X = x)^2) &= \int_{-\infty}^{+\infty} E(Y/X = x)^2 f_1(x)dx = \int_{-3}^{3} \frac{1}{4}(x^2 - 14x + 49)\frac{1}{18}(x+3)dx \\ &= \frac{1}{72}\int_{-3}^{3} x^3 - 11x^2 + 7x + 147dx = \frac{1}{72}\left[\frac{x^4}{4} - 11\frac{x^3}{3} + 7\frac{x^2}{2} + 147x\right]_{-3}^{3} \\ &= \frac{1}{72}\left(-11\frac{27}{3} + 11\frac{-27}{3} + 147\cdot 6\right) = \frac{19}{2}\end{aligned}$$

$$\mathrm{E}(\mathrm{E}(Y/X)) = \mathrm{E}(Y) = -3$$

$$\mathrm{Var}(\mathrm{E}(Y/X)) = \mathrm{E}(\mathrm{E}(Y/X)^2) - \mathrm{E}(\mathrm{E}(Y/X))^2 = \frac{19}{2} - (-3)^2 = \frac{1}{2}$$

Varianza de Y:

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2$$

$$\begin{aligned}
\mathrm{E}(Y^2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2 f(x,y)dxdy = \int_{-3}^{3}\int_{-5}^{x-2} \frac{1}{18}y^2 dydx = \frac{1}{18}\int_{-3}^{3}\int_{-5}^{x-2} y^2 dydx \\
&= \frac{1}{18}\int_{-3}^{3}\left[\frac{y^3}{3}\right]_{-5}^{x-2} dx = \frac{1}{54}\int_{-3}^{3}(x-2)^3 - (-5)^3 dx = \frac{1}{54}\int_{-3}^{3} x^3 - 6x^2 + 12x + 117dx \\
&= \frac{1}{54}\int_{-3}^{3} -6x^2 + 117dx = \frac{1}{27}\Big[-2x^3 + 117x\Big]_{0}^{3} = \frac{1}{27}(-2\cdot 27 + 117\cdot 3) = 11
\end{aligned}$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 = 11 - (-3)^2 = 2$$

Razón de correlación:

$$\eta^2_{Y/X} = \frac{\mathrm{Var}(\mathrm{E}(Y/X))}{\mathrm{Var}(Y)} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

Coeficiente de determinación lineal:

$$\rho^2_{X,Y} = \frac{\mathrm{Cov}(X,Y)^2}{\mathrm{Var}(X)\,\mathrm{Var}(Y)} = \frac{1^2}{2\cdot 2} = \frac{1}{4}$$

La curva de regresión coincide con la recta de regresión, luego es lógico que la razón de correlación coincida con el coeficiente de determinación lineal.