Problemas de Análisis Matemático Avanzado

Tema 1: Funciones Armónicas

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Ejercicio 1:

a) Ecuaciones de Cauchy-Riemann en polares para $u^*(r,\theta) = u(re^{i\theta})$ y $v^*(r,\theta) = v(re^{i\theta})$.

b) Ecuacion de Laplace en coordenadas polares para $u^*(r,\theta) = u(re^{i\theta})$.

Solución:

Tenemos $u^*(r,\theta) = u(r\cos\theta, r\sin\theta)$. Por la regla de la cadena

$$\frac{\partial u^*}{\partial r}(r,\theta) = \frac{\partial u}{\partial r}(r\cos\theta, r\sin\theta) = \frac{\partial u}{\partial x}\frac{\partial(r\cos\theta)}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial(r\sin\theta)}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$

$$\frac{\partial u^*}{\partial \theta}(r,\theta) = \frac{\partial u}{\partial \theta}(r\cos\theta, r\sin\theta) = \frac{\partial u}{\partial x}\frac{\partial(r\cos\theta)}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial(r\sin\theta)}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta.$$

Luego,

$$\frac{\partial u^*}{\partial r}(r,\theta) = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta, \quad \frac{\partial u^*}{\partial \theta}(r,\theta) = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta. \tag{1}$$

Análogamente,

$$\frac{\partial v^*}{\partial r}(r,\theta) = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta, \quad \frac{\partial v^*}{\partial \theta}(r,\theta) = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}r\cos\theta. \tag{2}$$

Recíprocamente, recordando las relaciones entre (x,y) y (r,θ) tenemos

$$\frac{\partial u}{\partial x} = \frac{\partial u^*}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u^*}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u^*}{\partial r}\cos\theta - \frac{\partial u^*}{\partial \theta}\frac{\sin\theta}{r},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u^*}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u^*}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u^*}{\partial r}\sin\theta + \frac{\partial u^*}{\partial \theta}\frac{\cos\theta}{r}.$$

Por tanto,

$$\frac{\partial u}{\partial x} = \frac{\partial u^*}{\partial r}\cos\theta - \frac{\partial u^*}{\partial \theta}\frac{\sin\theta}{r}, \quad \frac{\partial u}{\partial y} = \frac{\partial u^*}{\partial r}\sin\theta + \frac{\partial u^*}{\partial \theta}\frac{\cos\theta}{r}. \tag{3}$$

a) Sustituyendo las ecuaciones de Cauchy-Riemann en (1) y (2) tenemos

$$\frac{\partial v^*}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}r\cos\theta = r\left(\frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta\right) = r\frac{\partial u^*}{\partial r},$$

$$\frac{\partial u^*}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta = -r\left(\frac{\partial v}{\partial y}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right) = -r\frac{\partial v^*}{\partial r}.$$

Las propiedades del cambio de variable a coordenadas polares nos dicen que estas ecuaciones son equivalentes a las ecuaciones clásicas de Cauchy-Riemann. También se pueden sustituir las ecuaciones del enunciado en (3) y su análoga para v.

b) Sacamos a partir de (3) las derivadas parciales del laplaciano.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u^*}{\partial r}\cos\theta - \frac{\partial u^*}{\partial\theta}\frac{\sin\theta}{r}\right)\\
&= \cos^2\theta\frac{\partial^2 u^*}{\partial r^2} - \frac{\partial^2 u^*}{\partial\theta\partial r}\cos\theta\frac{\sin\theta}{r} + \frac{\partial u^*}{\partial r}\left(\frac{1}{r} - \frac{\cos^2\theta}{r}\right)\\
&- \frac{\sin\theta}{r}\frac{\partial^2 u^*}{\partial r\partial\theta}\cos\theta + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u^*}{\partial\theta^2} + \frac{\partial u^*}{\partial\theta}\frac{2\cos\theta\sin\theta}{r^2}\\
&= \frac{\partial^2 u^*}{\partial r^2}\cos^2\theta - \frac{\partial^2 u^*}{\partial\theta\partial r}\frac{2\cos\theta\sin\theta}{r} + \frac{\partial^2 u^*}{\partial\theta^2}\frac{\sin^2\theta}{r^2} + \frac{\partial u^*}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial u^*}{\partial\theta}\frac{2\cos\theta\sin\theta}{r^2}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial^2 y} &= \frac{\partial}{\partial y}\left(\frac{\partial u^*}{\partial r}\sin\theta + \frac{\partial u^*}{\partial\theta}\frac{\cos\theta}{r}\right)\\
&= \sin^2\theta\frac{\partial^2 u^*}{\partial r^2} + \sin\theta\frac{\partial^2 u^*}{\partial\theta\partial r}\frac{\cos\theta}{r} + \frac{\partial u^*}{\partial r}\frac{\cos^2\theta}{r}\\
&+ \frac{\cos\theta}{r}\frac{\partial^2 u^*}{\partial r\partial\theta}\sin\theta + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u^*}{\partial\theta^2} - \frac{\partial u^*}{\partial\theta}\frac{2\cos\theta\sin\theta}{r^2}\\
&= \frac{\partial^2 u^*}{\partial r^2}\sin^2\theta + \frac{\partial^2 u^*}{\partial\theta\partial r}\frac{2\cos\theta\sin\theta}{r} + \frac{\partial^2 u^*}{\partial\theta^2}\frac{\cos^2\theta}{r^2} + \frac{\partial u^*}{\partial r}\frac{\cos^2\theta}{r} - \frac{\partial u^*}{\partial\theta}\frac{2\cos\theta\sin\theta}{r^2}.
\end{aligned}$$

La ecuación de Laplace queda así tras agrupar, cancelar y simplificar algunos términos

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} = \frac{\partial^2 u^*}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u^*}{\partial\theta^2} + \frac{1}{r}\frac{\partial u^*}{\partial r}.$$

Ejercicio 2: Probar que $u(re^{i\theta}) = \theta\log r$ es armónica en $\mathbb{C}\backslash\mathbb{R}^-$. Encontrar conjugada armónica v de u. ¿Qué función es $f(z) = u(z) + iv(z)$.

Solución: Calculamos las derivadas:

$$\frac{\partial u}{\partial r} = \frac{\theta}{r}, \quad \frac{\partial u}{\partial\theta} = \log r, \quad \frac{\partial^2 u}{\partial r^2} = -\frac{\theta}{r^2}, \quad \frac{\partial^2 u}{\partial\theta^2} = 0.$$

Sustituimos en la ecuación de Laplace:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = -\frac{\theta}{r^2} + \frac{1}{r^2}\cdot 0 + \frac{1}{r}\frac{\theta}{r} = 0.$$

Esto es válido en todo punto de $\mathbb{C}\backslash\mathbb{R}^-$, luego u es armónica.

Sea v la conjugada armónica de u, que existe porque el dominio es simplemente conexo. Imponemos condiciones para hallar v:

$$\begin{aligned}
\frac{\partial v}{\partial\theta} &= r\frac{\partial u}{\partial r} = r\frac{\theta}{r} = \theta,\\
\frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial\theta} = -\frac{1}{r}\log r.
\end{aligned}$$

La primera ecuación nos dice que v es de la forma $v(r,\theta) = \frac{\theta^2}{2} + \phi(r)$, y la segunda que $v(r,\theta) = -\frac{\log^2 r}{2} + \psi(\theta)$. Por tanto podemos tomar $v(r,\theta) = -\frac{\log^2 r}{2} + \frac{\theta^2}{2}$.

Tomando $f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}) = \theta\log r + i\left(-\frac{\log^2 r}{2} + \frac{\theta^2}{2}\right)$ para todo $re^{i\theta} \in \mathbb{C}\backslash\mathbb{R}^-$, tenemos $f \in \mathcal{H}(\mathbb{C}\backslash\mathbb{R}^-)$.