

Ejercicios propuestos: Parte III

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Ejercicio 1

Función masa de probabilidad:

X/Y	0	2
2	$1/16$	$1/16$
4	$1/8$	$1/4$
6	0	$1/2$

Se pide calcular la función generatriz de momentos y, a partir de ella, la matriz de covarianzas.

f.g.m.:

$$M(t_1, t_2) = E[e^{t_1 x + t_2 y}] = \frac{1}{16}e^{2t_1} + \frac{1}{16}e^{2t_1+2t_2} + \frac{1}{8}e^{4t_1} + \frac{1}{4}e^{4t_1+2t_2} + \frac{1}{2}e^{6t_1+2t_2} = \frac{1}{16}(e^{2t_1} + e^{2t_1+2t_2} + 2e^{4t_1} + 4e^{4t_1+2t_2} + 8e^{6t_1+2t_2})$$

Queremos calcular la **matriz de covarianzas**

$$\begin{pmatrix} \text{Var}(X) & \text{Cov}(X, Y) \\ \text{Cov}(X, Y) & \text{Var}(Y) \end{pmatrix} = \begin{pmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{pmatrix}$$

Para hallar los momentos centrados, podemos hallar primero los no centrados. Usaremos la función generatriz de momentos para hallarlos.

$$m_{10} = \frac{\partial M}{\partial t_1}(t_1, t_2) \Big|_{t_1=t_2=0} = \frac{1}{16}(2e^{2t_1} + 2e^{2t_1+2t_2} + 8e^{4t_1} + 16e^{4t_1+2t_2} + 48e^{6t_1+2t_2}) \Big|_{t_1=t_2=0} = \frac{2+2+8+16+48}{16} = \frac{19}{4}$$

$$m_{01} = \frac{\partial M}{\partial t_2}(t_1, t_2) \Big|_{t_1=t_2=0} = \frac{1}{16}(2e^{2t_1+2t_2} + 8e^{4t_1+2t_2} + 16e^{6t_1+2t_2}) \Big|_{t_1=t_2=0} = \frac{2+8+16}{16} = \frac{13}{8}$$

$$m_{20} = \frac{\partial^2 M}{\partial t_1^2}(t_1, t_2) \Big|_{t_1=t_2=0} = \frac{1}{16}(4e^{2t_1} + 4e^{2t_1+2t_2} + 32e^{4t_1} + 64e^{4t_1+2t_2} + 288e^{6t_1+2t_2}) \Big|_{t_1=t_2=0} = \frac{4+4+32+64+288}{16} = \frac{49}{2}$$

$$m_{02} = \frac{\partial^2 M}{\partial t_2^2}(t_1, t_2) \Big|_{t_1=t_2=0} = \frac{1}{16}(4e^{2t_1+2t_2} + 16e^{4t_1+2t_2} + 32e^{6t_1+2t_2}) \Big|_{t_1=t_2=0} = \frac{4+16+32}{16} = \frac{13}{4}$$

$$m_{11} = \frac{\partial^2 M}{\partial t_1 \partial t_2}(t_1, t_2) \Big|_{t_1=t_2=0} = \frac{1}{16}(4e^{2t_1+2t_2} + 32e^{4t_1+2t_2} + 96e^{6t_1+2t_2}) \Big|_{t_1=t_2=0} = \frac{4+32+96}{16} = \frac{33}{4}$$

Ya podemos hallar los momentos centrados:

$$\begin{aligned} \mu_{20} &= m_{20} - m_{10}^2 = \frac{49}{2} - \left(\frac{19}{4}\right)^2 = \frac{31}{16} \\ \mu_{11} &= m_{11} - m_{10}m_{01} = \frac{33}{4} - \frac{19}{4} \cdot \frac{13}{8} = \frac{17}{32} \\ \mu_{02} &= m_{02} - m_{01}^2 = \frac{13}{4} - \left(\frac{13}{8}\right)^2 = \frac{39}{64} \end{aligned}$$

Por tanto la matriz de covarianzas es

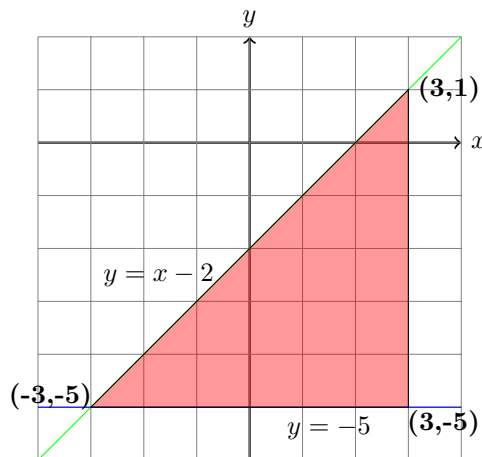
$$\begin{pmatrix} \frac{31}{16} & \frac{17}{32} \\ \frac{17}{32} & \frac{39}{64} \end{pmatrix}$$

Ejercicio 2

Función densidad:

$$f(x, y) = \frac{1}{18} \quad -3 < x < 3, \quad -5 < y < -2 + x$$

Se pide calcular la recta y la curva de regresión de Y sobre X , así como el coeficiente de determinación lineal y la razón de correlación.



Recta de regresión:

$$Y = aX + b$$

donde

$$a = \gamma_{Y/X} = \frac{\text{Cov}(X, Y)}{\text{Var}(X)}, \quad b = E(Y) - aE(X)$$

Covarianza:

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$\begin{aligned} E(XY) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x, y)dx dy = \int_{-3}^3 \int_{-5}^{x-2} \frac{1}{18} xy dy dx = \frac{1}{18} \int_{-3}^3 x \int_{-5}^{x-2} y dy dx \\ &= \frac{1}{18} \int_{-3}^3 x \left[\frac{y^2}{2} \right]_{-5}^{x-2} dx = \frac{1}{18} \int_{-3}^3 x \left(\frac{(x-2)^2}{2} - \frac{(-5)^2}{2} \right) dx = \frac{1}{36} \int_{-3}^3 x(x^2 - 4x + 4 - 25) dx \\ &= \frac{1}{36} \int_{-3}^3 x^3 - 4x^2 - 21x dx = \frac{1}{36} \left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{21x^2}{2} \right]_{-3}^3 = \frac{1}{36} \left[\frac{4x^3}{3} \right]_3^{-3} = \frac{1}{36} \left(\frac{-108}{3} - \frac{108}{3} \right) \\ e &= \frac{-3}{3} - \frac{3}{3} = -2 \end{aligned}$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xf(x, y)dx dy = \int_{-3}^3 \int_{-5}^{x-2} \frac{1}{18} x dy dx = \frac{1}{18} \int_{-3}^3 x \int_{-5}^{x-2} dy dx \\ &= \frac{1}{18} \int_{-3}^3 x(x - 2 - (-5)) dx = \frac{1}{18} \int_{-3}^3 x^2 + 3x dx = \frac{1}{18} \left[\frac{x^3}{3} + \frac{3x^2}{2} \right]_{-3}^3 = \\ &= \frac{1}{18} \left(\frac{27}{3} - \frac{-27}{3} \right) = 1 \end{aligned}$$

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y f(x, y) dx dy = \int_{-3}^3 \int_{-5}^{x-2} \frac{1}{18} y dy dx = \frac{1}{18} \int_{-3}^3 \int_{-5}^{x-2} y dy dx \\
&= \frac{1}{18} \int_{-3}^3 \left[\frac{y^2}{2} \right]_{-5}^{x-2} dx = \frac{1}{36} \int_{-3}^3 (x-2)^2 - (-5)^2 dx = \frac{1}{36} \int_{-3}^3 x^2 - 4x - 21 dx \\
&= \frac{1}{36} \left[\frac{x^3}{3} - 2x^2 - 21x \right]_{-3}^3 = \frac{1}{36} \left(\frac{27}{3} - 63 - \left(\frac{-27}{3} + 63 \right) \right) = -3
\end{aligned}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = -2 - 1(-3) = 1$$

Varianza de X:

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^2 f(x, y) dx dy = \int_{-3}^3 \int_{-5}^{x-2} \frac{1}{18} x^2 dy dx = \frac{1}{18} \int_{-3}^3 x^2 \int_{-5}^{x-2} dy dx \\
&= \frac{1}{18} \int_{-3}^3 x^2 (x-2 - (-5)) dx = \frac{1}{18} \int_{-3}^3 x^3 + 3x^2 dx = \frac{1}{18} \left[\frac{x^4}{4} + x^3 \right]_{-3}^3 = \\
&= \frac{1}{18} (27 - (-27)) = 3
\end{aligned}$$

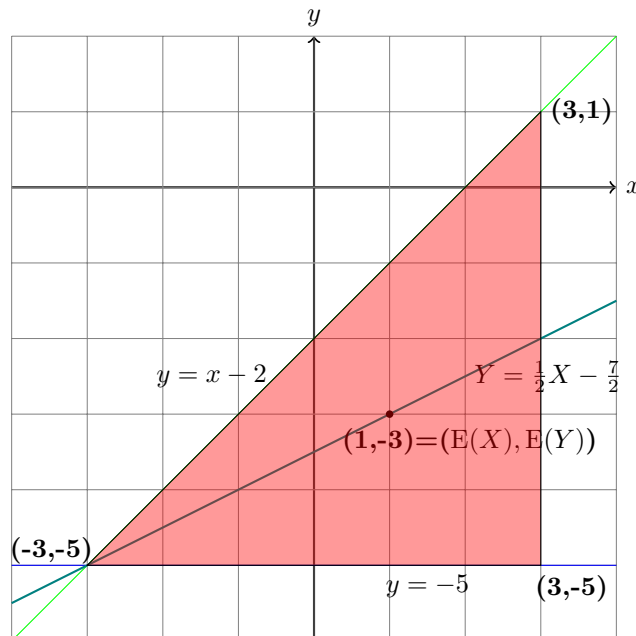
$$\text{Var}(X) = E(X^2) - E(X)^2 = 3 - 1^2 = 2$$

Recta de regresión:

$$a = \gamma_{Y/X} = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} = \frac{1}{2}$$

$$b = E(Y) - aE(X) = -3 - 1 \frac{1}{2} = -\frac{7}{2}$$

$$Y = \frac{1}{2}X - \frac{7}{2}$$



Coeficiente de determinación lineal:

Curva de regresión:

$$Y = \varphi(X) = E(Y/X)$$

Esperanza condicionada:

$$E(Y/X = x) = \int_{-\infty}^{+\infty} y f(y/X = x) dy$$

Necesito la condicionada de Y a X.

Marginal:

$$f_1(x) = \int_{-\infty}^{+\infty} f(x, y) dy = \int_{-5}^{x-2} \frac{1}{18} dy = \frac{1}{18}(x - 2 - (-5)) = \frac{1}{18}(x + 3) \quad x \in]-3, 3[$$

Condicionada:

$x \in]-3, 3[$

$$f(y/X = x) = \frac{f(x, y)}{f_1(x)} = \frac{\frac{1}{18}}{\frac{1}{18}(x + 3)} = \frac{1}{x + 3} \quad y \in]-5, x - 2[$$

Curva de regresión:

$$\begin{aligned} E(Y/X = x) &= \int_{-\infty}^{+\infty} y f(y/X = x) dy = \int_{-5}^{x-2} y \frac{1}{x + 3} dy = \frac{1}{x + 3} \left[\frac{y^2}{2} \right]_{-5}^{x-2} \\ &= \frac{1}{2x + 6} \left((x - 2)^2 - (-5)^2 \right) = \frac{x^2 - 4x - 21}{2x + 6} = \frac{(x + 3)(x - 7)}{2(x + 3)} = \frac{1}{2}x - \frac{7}{2} \end{aligned}$$

$$Y = \varphi(X) = E(Y/X) = \frac{1}{2}X - \frac{7}{2}$$

En este caso coincide con la recta de regresión.

Razón de correlación:

$$\eta_{Y/X}^2 = \frac{\text{Var}(E(Y/X))}{\text{Var}(Y)} = 1 - \frac{E(\text{Var}(Y/X))}{\text{Var}(Y)}$$

Varianza de la esperanza condicionada:

$$\text{Var}(E(Y/X)) = E(E(Y/X)^2) - E(E(Y/X))^2$$

$$\begin{aligned} E(E(Y/X)^2) &= \int_{-\infty}^{+\infty} E(Y/X = x)^2 f_1(x) dx = \int_{-3}^3 \frac{1}{4}(x^2 - 14x + 49) \frac{1}{18}(x + 3) dx \\ &= \frac{1}{72} \int_{-3}^3 x^3 - 11x^2 + 7x + 147 dx = \frac{1}{72} \left[\frac{x^4}{4} - 11 \frac{x^3}{3} + 7 \frac{x^2}{2} + 147x \right]_{-3}^3 \\ &= \frac{1}{72} \left(-11 \frac{27}{3} + 11 \frac{-27}{3} + 147 \cdot 6 \right) = \frac{19}{2} \end{aligned}$$

$$E(E(Y/X)) = E(Y) = -3$$

$$\text{Var}(E(Y/X)) = E(E(Y/X)^2) - E(E(Y/X))^2 = \frac{19}{2} - (-3)^2 = \frac{1}{2}$$

Varianza de Y:

$$\text{Var}(Y) = E(Y^2) - E(Y)^2$$

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y^2 f(x, y) dx dy = \int_{-3}^3 \int_{-5}^{x-2} \frac{1}{18} y^2 dy dx = \frac{1}{18} \int_{-3}^3 \int_{-5}^{x-2} y^2 dy dx \\ &= \frac{1}{18} \int_{-3}^3 \left[\frac{y^3}{3} \right]_{-5}^{x-2} dx = \frac{1}{54} \int_{-3}^3 (x-2)^3 - (-5)^3 dx = \frac{1}{54} \int_{-3}^3 x^3 - 6x^2 + 12x + 117 dx \\ &= \frac{1}{54} \int_{-3}^3 -6x^2 + 117 dx = \frac{1}{27} \left[-2x^3 + 117x \right]_0^3 = \frac{1}{27} (-2 \cdot 27 + 117 \cdot 3) = 11 \end{aligned}$$

$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = 11 - (-3)^2 = 2$$

Razón de correlación:

$$\eta_{Y/X}^2 = \frac{\text{Var}(E(Y/X))}{\text{Var}(Y)} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$