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MATH 6397 HW #4

1. A man has two sons, and he would like them to cooperate. Therefore in his will, it states that the two sons must each specify a sum of money, s_1 and s_2, that they are willing to accept for inheritance when he dies. The value of his estate is 1 000 000 USD.

Solution: Given:

If $s_1 + s_2 \leq 1\,000\,000$, then $g_1 = s_1$, $g_2 = s_2$, $g_3 = 1\,000\,000 - s_1 - s_2$

If $s_1 + s_2 > 1\,000\,000$, then $g_1 = g_2 = 0$, $g_3 = 1\,000\,000$

where s, g represents solicited and gained respectively. 1, 2, 3 represents also, son 1, son 2, and charity respectively.

I introduce 7 strategies: care free (cf), miser (ms), less (ls), equal (eq), more (me), greedy (gy) and charity (cy).

Assume cf, ms, ls, eq, me, gy, and cy is such that they mark 7 fixed points among 1,000,001 possible choices for each son based on the inheritance money (I), we obtain the table below:

Son 1 (s_1)

0	care free (cf)	g_1, g_2	g_1, g_2	g_1, g_2	g_1, g_2	g_1, g_2	g_1, g_2	* g_1, g_2
0.2 I	miser (ms)	g_1, g_2	g_1, g_2	g_1, g_2	g_1, g_2	g_1, g_2	* g_1, g_2	$0, g_2$
0.4 I	less (ls)	$g_1, 0$	g_1, g_2	g_1, g_2	g_1, g_2	* g_1, g_2	0, 0	$0, g_2$
0.5 I	equal (eq)	$g_1, 0$	g_1, g_2	g_1, g_2	* g_1, g_2	0, 0	0, 0	$0, g_2$
0.6 I	more (me)	$g_1, 0$	g_1, g_2	* g_1, g_2	0, 0	0, 0	0, 0	$0, g_2$
0.8 I	greedy (gy)	$g_1, 0$	* g_1, g_2	0, 0	0, 0	0, 0	0, 0	$0, g_2$
I	charity (cy)	* g_1, g_2	$0, g_2$	$0, g_2$	$0, g_2$	$0, g_2$	$0, g_2$	g_1, g_2
		0 cf	.2 ms	.4 ls	.5 eq	.6 me	.8 gy	1.0 cy

son 2 (s_2)

pure strategies Nash Equilibria include:

* – care free v charity

* – miser v greedy

* – less v more

* – equal v equal

⇒ Assuming they only care about their share : care free & charity are no longer strategies, then the pure strategy NE include:

- "miser" v "greedy"
- "less" v "more"
- "equal" v "equal"

- If s_1 decides to go from miser to less, s_1, s_2 gets zero

- similarly less → equal, more → greedy is not an improvement
  → miser → equal

- similarly equal → less, equal → greedy, s_1, s_2 gets zero.
  → miser → more

2.

		C	D
player 1	C	R, R	S, T
	D	T, S	P, P

$T > R > P > S$

a) Consider C, C until D and D ever after

* GRIM is the grudger
* ALLD

- games played for m rounds

Compute the pay off matrix of GRIM and ALLD after m rounds.

	ALLD	GRIM
ALLD	mp, mp	$mp, (m-1)p$
GRIM	$(m-1)p, mp$	mr, mr

p1 ALLD : mp

p2 ALLD : mp

p1 ALLD : mp

p2 GRIM : $(m-1)p$

p1 GRIM : mr

p2 GRIM : mr

(3)

	C	D
C	R, R	S, T
D	T, S	P, P

$T > R > P > S$

a)

$$\begin{array}{c|cc} & \text{GRIM} & \text{ALLD} \\ \hline \text{GRIM} & \multicolumn{2}{c}{\multirow{2}{*}{}} \\ \end{array}$$

$$\begin{array}{ccc} & \text{GRIM} & \text{ALLD} \\ \text{GRIM} & mR & S + (m-1)P \\ \text{ALLD} & T + (m-1)P & mP \end{array}$$

b) ALLD does not dominate GRIM if for m games

$$T + (m-1)P > mR$$

$$T + mP - P > mR$$

$$T - P > mR - mP$$

$$T - P > m(R - P)$$

$$m > \frac{T - P}{R - P} \qquad R > P$$

c) Given m rounds

player 2 cooperates on last round $\Rightarrow$ GRIM*

$$\text{payoff}_{\text{GRIM}} = S + (m-1)R$$

$$\text{payoff GRIM}^* = T + (m-1)R$$

Since $T > S$, GRIM* dominates GRIM

d) Considering from reverse:

	m	$m-1$	...	2	1
GRIM*	D	D	...	D	C
ALLD	D	D	...	D	D

The first to defect is likely to dominate the game.

game. These would include GRIM**, GRIM***, ..., GRIM$^{*(m-1)}$

the strategy that most ensures this is ALLD aka GRIM$^{(*)m}$.

ALLD does not dominate GRIM but it will dominate GRIM*.

e) we thus eventually end up with <u>ALLD</u>.

(3) say $\delta = 1$

time period	p (game finish)
1	δ
2	δ^2
3	δ^3
4	$\vdots$
$\vdots$	
m	δ^m

Expected number of rounds E_m is finite iff

$$m\delta < 1$$

From the table shown: assuming the game exceeds at least one round

the $\langle m \rangle = 1 + E_m$

where $E_{m'}$ is the expected number of rounds after first game.

Since δ is given as the continuation probability, the probability of m repetitions follow a geometric distribution with success probability s;

$$\langle m \rangle = 1 + \sum_{i=1}^{m} \delta^{i-1}$$

$$\langle m \rangle = 1 + \frac{\delta}{1-\delta}$$

b)

	C	D
C	R, R	S, T
D	T, S	P, P

$t > r > p > s$

	GRIM	ALLD
GRIM	GG	GA
ALLD	AG	AA

where

$$GG = R + R\delta + R\delta^2 + \cdots + R\delta^{m-1}$$

$$GA = S + p\delta + p\delta^2 + \cdots + p\delta^{m-1}$$

$$AG = T + p\delta + p\delta^2 + \cdots + p\delta^{m-1}$$

$$AA = p + p\delta + p\delta^2 + \cdots + p\delta^{m-1}$$

c) GRIM dominates ALLD if:

$$R + R\delta + R\delta^2 + \cdots + R\delta^{m-1} > T + p\delta + p\delta^2 + \cdots + p\delta^{m-1}$$

$$R + \frac{R\delta}{1-\delta} > T + \frac{p\delta}{1-\delta}$$

$$R - T > \frac{p\delta}{1-\delta} - \frac{R\delta}{1-\delta}$$

$$(R-T)\,1-\delta > p\delta - R\delta$$

$$R - T - \cancel{R\delta} + T\delta > p\delta - \cancel{R\delta}$$

$$T\delta - p\delta > T - R$$

$$\delta > \frac{T-R}{T-p} \qquad T > p$$

(4)

Consider an infinitely repeated game in which players make execution errors each round with probability ε.

Sticking with PD described in 2, we will consider

i) TFT with $p_T = \{1-\varepsilon, 1-\varepsilon, 1-\varepsilon, 1-\varepsilon\}$

ii) GRIM with $p_G = \{1-\varepsilon, \varepsilon, \varepsilon, \varepsilon\}$

iii) ALLC with $p_C = \{1-\varepsilon, 1-\varepsilon, 1-\varepsilon, 1-\varepsilon\}$

a) Write down the 6 transition matrices for the 6 pairwise matchups between these three strategies:

We thus have:

i. TFT v TFT

$$p_T = \{1-\varepsilon, \varepsilon, 1-\varepsilon, \varepsilon\}$$
$$q_T = \{1-\varepsilon, \varepsilon, 1-\varepsilon, \varepsilon\}$$

We have for the transition matrix the general form:

$$m = \begin{bmatrix} p_{cc}q_{cc} & p_{cc}(1-q_{cc}) & (1-p_{cc})q_{cc} & (1-p_{cc})(1-q_{cc}) \\ p_{cd}q_{dc} & p_{cd}(1-q_{dc}) & (1-p_{cd})q_{dc} & (1-p_{cd})(1-q_{dc}) \\ p_{dc}q_{cd} & p_{dc}(1-q_{cd}) & (1-p_{dc})q_{cd} & (1-p_{dc})(1-q_{cd}) \\ p_{dd}q_{dd} & p_{dd}(1-q_{dd}) & (1-p_{dd})q_{dd} & (1-p_{dd})(1-q_{dd}) \end{bmatrix}$$

I a) For TFT v TFT (TT)

$$m_{TT} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & (1-\varepsilon)\varepsilon \\ \varepsilon^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{bmatrix}$$

I b) Using Eigen analyses,

$$V = \begin{bmatrix} 1, & -1, & 0, & 1 \\ 1, & 0, & -1, & -1 \\ 1, & 0, & 1, & -1 \\ 1, & 1, & 0, & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 \\ 1-2\varepsilon \\ 2\varepsilon-1 \\ 4\varepsilon^2-4\varepsilon+1 \end{bmatrix}$$

vector   values

Normalised eigen vector and stationary distribution:

$$|V|_{TT} = \begin{bmatrix} 1/2 & -1 & 0 & 1 \\ 1/2 & 0 & -1 & -1 \\ 1/2 & 0 & 1 & -1 \\ 1/2 & 1 & 0 & 1 \end{bmatrix}$$

$$\pi_{TT} = [\, 1/2 \quad 1/2 \quad 1/2 \quad 1/2 \,]$$

II a) TFT v GRIM

$$M_{TG} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ \varepsilon^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 & \varepsilon^2 & (1-\varepsilon)\varepsilon \\ \varepsilon^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{bmatrix}$$

Using Eigen analyses:

$$V = \begin{bmatrix} 0 & \frac{\varepsilon}{2\varepsilon^2 - 2\varepsilon + 1} & -1 & \frac{\varepsilon^2}{\varepsilon^2 + \varepsilon - 1} \\ -1 & \frac{-2\varepsilon(\varepsilon - 1)}{2\varepsilon^2 - 2\varepsilon + 1} & 0 & \frac{-(\varepsilon^2 - 1)}{\varepsilon^2 + \varepsilon - 1} \\ 0 & \frac{\varepsilon}{2\varepsilon^2 - 2\varepsilon + 1} & 0 & \frac{-(\varepsilon^2 + \varepsilon)}{\varepsilon^2 + \varepsilon - 1} \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 1 \\ 1 - 2\varepsilon \\ 2\varepsilon^2 - \varepsilon \end{bmatrix}$$

II b)

$$\pi_{TG} = \frac{1}{\sqrt{14\varepsilon^2 - 4\varepsilon - 16\varepsilon^3 + 8\varepsilon^4 + 1}} \begin{bmatrix} \varepsilon \\ -2\varepsilon(\varepsilon - 1) \\ \varepsilon \\ 2\varepsilon^2 - 2\varepsilon + 1 \end{bmatrix}$$

III a) TFT v ALLC

$$M_{TA} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ (1-\varepsilon)\varepsilon & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ (1-\varepsilon)\varepsilon & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \end{bmatrix}$$

V = Eigen vector

$$\begin{bmatrix} -1 & 0 & \frac{2\varepsilon^2 - 2\varepsilon + 1}{2\varepsilon^2} \\ 0 & -1 & \frac{\cancel{\varepsilon}\,2\varepsilon^2 - 2\varepsilon + 1}{2\varepsilon(\varepsilon - 1)} \\ 1 & 0 & -(\varepsilon - 1)/\varepsilon \\ 0 & 1 & 1 \end{bmatrix}$$

$$D_{\text{Eigen values}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vector V_{TA} is 4×3 while D is 4×1.

I am unable to determine π_{TA}

IVa) GRIM v GRIM

$$m_{GG} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ \varepsilon^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ \varepsilon^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ \varepsilon^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{bmatrix}$$

IVb) Eigen vector

$$v = \begin{bmatrix} 0 & 0 & \varepsilon / 4\varepsilon^2 - 5\varepsilon + 2 & -1 \\ -1 & -1 & 2\varepsilon - 2\varepsilon^2 / 4\varepsilon^2 - 5\varepsilon + 2 & 0 \\ 1 & 0 & 2\varepsilon - 2\varepsilon^2 / 4\varepsilon^2 - 5\varepsilon + 2 & 1 \\ 0 & 1 & 1 & \end{bmatrix}$$

$$\text{Eigen value } D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1-2\varepsilon \end{bmatrix}$$

$$\therefore \pi_{GG} = \frac{1}{\sqrt{25\varepsilon^2 - 10\varepsilon - 28\varepsilon^3 + 12\varepsilon^4 + 2}} \begin{bmatrix} \frac{\varepsilon\sqrt{2}}{2} \\ -\varepsilon\sqrt{2}(\varepsilon - 1) \\ -\varepsilon\sqrt{2}(\varepsilon - 1 \\ \sqrt{2}(4\varepsilon^2 - 5\varepsilon + 2) \end{bmatrix}$$

Va) GRIM v ALLC

$$P_r = \{1-\varepsilon, \varepsilon, \varepsilon, \varepsilon\}$$

$$q_c = \{1-\varepsilon,\ 1-\varepsilon,\ 1-\varepsilon,\ 1-\varepsilon\}$$

$$M_{qc} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ (1-\varepsilon)\varepsilon & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ (1-\varepsilon)\varepsilon & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ (1-\varepsilon)\varepsilon & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \end{bmatrix}$$

Eigen vector, $V_{qc} = \begin{bmatrix} 0 & 0 & 1/2\varepsilon & (1-\varepsilon)/\varepsilon \\ -1 & -1 & \frac{1}{2(1-\varepsilon)} & -1 \\ 1 & 0 & (1-\varepsilon)\varepsilon & 1-\varepsilon/\varepsilon \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Eigen value, $D_{qc} = [\ 0 \quad 0 \quad 1 \quad 2\varepsilon^2 - 3\varepsilon + 1\ ]$

V b) $$\pi_{qc} = \frac{1}{\sqrt{2\varepsilon^2 - 2\varepsilon + 1\ (4\varepsilon^2 - 8\varepsilon + 5}} \begin{pmatrix} |\varepsilon - 1| \\ \varepsilon|1-\varepsilon| \\ 2|1-\varepsilon| \\ 2\varepsilon|\varepsilon - 1 \end{pmatrix}$$

V(a) ALLC v ALLC

$$P_c = q_c = \{1-\varepsilon,\ 1-\varepsilon,\ 1-\varepsilon,\ 1-\varepsilon\}$$

$$M_{cc} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 \end{bmatrix}$$

VIb) Eigen vector:

$$v_{cc} = \begin{bmatrix} -1 & -1 & -1 & \frac{(1-\varepsilon)^2}{2\varepsilon^2-2\varepsilon+1} \\ 1 & 0 & 0 & \frac{(1-\varepsilon)\varepsilon}{2\varepsilon^2-2\varepsilon+1} \\ 0 & 1 & 0 & \frac{(1-\varepsilon)\varepsilon}{2\varepsilon^2-2\varepsilon+1} \\ 0 & 0 & 1 & \frac{\varepsilon^2}{2\varepsilon^2-2\varepsilon+1} \end{bmatrix}$$

Eigen values:

$$p_{cc} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Stationary distribution:

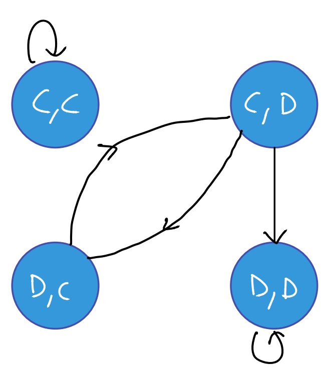
$$\pi_{cc} = \frac{1}{2\varepsilon^2-2\varepsilon+1} \begin{bmatrix} (1-\varepsilon)^2 \\ (1-\varepsilon)\varepsilon \\ (1-\varepsilon)\varepsilon \\ \varepsilon^2 \end{bmatrix}$$

(4Bii)

Explain what happens as $\varepsilon \to 0$ for π_i

$i \in \{TT, TG, TC, GG, GC, CC\}$

I) $$m_{TT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The stationary distribution:

$$\pi_1 = (0 \quad .7071 \quad .7071 \quad 0)$$
$$\pi_2 = (1 \quad 0 \quad 0 \quad 0)$$
$$\pi_3 = (0 \quad 0 \quad 0 \quad 1)$$

The first is a case of a quasi-stationary distribution.

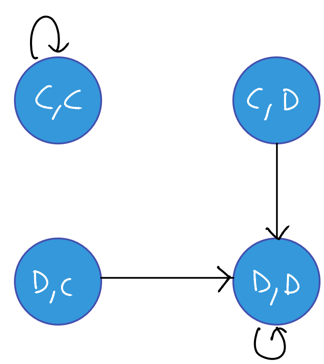
This is illustrated in the diagram.

II) TFT v GRIM

$$m_{TG}\big|_{\varepsilon \to 0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stationary distribution:

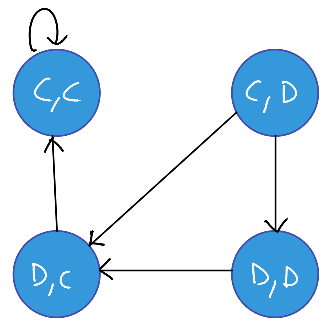
$$\pi_1 = (1 \quad 0 \quad 0 \quad 0)$$
$$\pi_2 = (0 \quad 0 \quad 0 \quad 1)$$



Without errors, TFT v GRIM ends up in either states CC or DD depending on starting points.

III) TFT v ALLC

$$m_{TC}\big|_{\varepsilon \to 0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



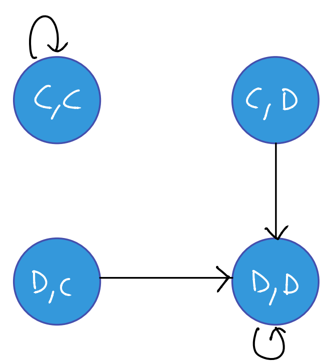
Stationary distribution:

$$\pi_{TC} = (1 \quad 0 \quad 0 \quad 0)$$

Without errors, TFT vs ALLC ends up in ALLC.

IV) GRIM v GRIM

$$m_{GG}\big|_{\varepsilon \to 0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Stationary distribution:

$$\pi_1 = (1 \quad 0 \quad 0 \quad 0)$$

$$\pi_2 = (0 \quad 0 \quad 0 \quad 1)$$

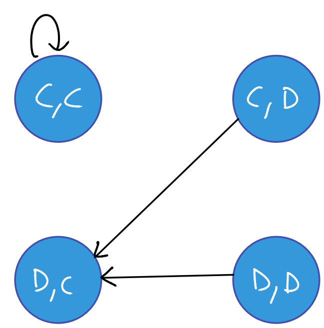
Without errors, GRIM v GRIM ends up in

either states CC or DD depending on starting points.

This is very similar to GRIM v TFT in the case without errors.

V) GRIM v ALLC

$$m_{TC}\Big|_{\varepsilon \to 0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



Stationary distribution:

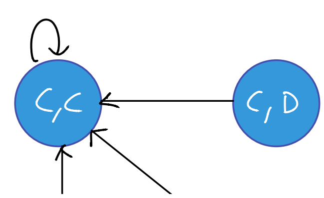
$$\pi_1 = (1 \quad 0 \quad 0 \quad 0)$$

$$\pi_2 = (0 \quad 0 \quad 1 \quad 0)$$

– Without errors, GRIM v ALLC ends up in either states CC or DC depending on starting points.

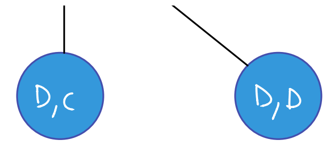
VI) ALLC v ALLC

$$m_{CC}\Big|_{\varepsilon \to 0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



Stationary distribution:

$$\pi_{CC} = (1 \quad 0 \quad 0 \quad 0)$$



- Without errors, ALLC v ALLC ends up in in state CC.

4(c)

Payoff $S = RV^*_{cc} + SV^*_{cd} + TV^*_{dc} + PV^*_{dd}$

I) $S_{TT} = R + S\frac{\sqrt{2}}{2} + T\frac{\sqrt{2}}{2} + P$

II) $S_{TG} = R + P$

III) $S_{TC} = R$

IV) $S_{GG} = R + P$

V) $S_{GC} = R + T$

VI) $S_{CC} = R$

	C	G	T
C	R, R	R+T, R+S	R, R
G	R+T, R+S	R+P, R+P	R+P, R+P
T	R, R	R+P, R+P	$R+\frac{\sqrt{2}}{2}(S+T)+P$, $R+\frac{\sqrt{2}}{2}(S+T)+P$

d) Assume $T > R > P > S > 0$

Then the Nash equilibria NE is TFT vs TFT

i) TFT v ALLC can improve by changing to any other strategy, same goes for ALLC vs ALLC

ii) GRIM strategy in GRIM v TFT can be improved by switching to TFT.

iii) one player in GRIM v GRIM can improve payoff by switching to ALLC

iv) ALLC in GRIM vs ALLC can improve payoff by switching to GRIM.

Summarily:

There is only one NE : TFT v TFT