BENDING LAB

RULES AND EXPECTATIONS:

1. It is expected that you should complete any pre-lab exercises *before* attending the lab session. These shall be turned in at the beginning of the lab period.

2. It is expected that you will discuss the lab assignment with classmates and teachers in order to improve your understanding of the material.

3. Everything written down that is turned in for credit should be from your own work based on your understanding of the material.

4. Any input from others should be cited in your assignment.

5. A perfect lab report should (a) be legible and well organized, (b) demonstrate a clear, reproducible, and extensible thought process, and (c) be correct. Your work will be evaluated on how nearly it meets this ideal.

1 Introduction

1.1 Objectives

1. To review bending theory and observe bending of uniform samples under various boundary conditions.

2. To test the validity of Euler-Bernoulli bending theory from the results of bending tests.

1.2 Theory and Background

Euler-Bernoulli theory of beam bending is applicable to the bending of long slender beams that are initially straight. The material comprising the beam is assumed to be linear elastic. The presentation here will be restricted to beams that are homogenous and uniform.
Elastic bending of beams is described by the following system of differential equations:

$$\frac{dV}{dx} = -q(x) \tag{1}$$
$$\frac{dM}{dx} = V(x) \tag{2}$$
$$M(x) = EI\kappa(x) \tag{3}$$
$$\frac{d\theta}{dx} = \kappa(x) \tag{4}$$
$$\frac{dv}{dx} = \theta(x) \tag{5}$$

In the above equations:

- $q(x)$ [force/distance] is the (downward) applied load intensity;
- $V(x)$ [force] is the internal shear force on the cross section at location x;
- $M(x)$ [force $\times$ distance] is the internal bending moment on the cross section at location x;
- E [force/area] is Young's (or elastic) modulus of material comprising the beam;
- I [distance^4] is the area moment of inertia of the cross section about its axis of bending (for circular cross section, $I = \pi d^4/64$, where $d =$ diameter of the circular cross section;)
- $\kappa(x)$ [distance^{-1}] is the local curvature of the neutral axis of the beam;
- $\theta(x)$ [radians] is the rotation of the cross section (at location x) due to the beam's deflection;
- $v(x)$ [distance] is the (upward) transverse deflection of the neutral axis of the beam.

In deriving these equations, it was assumed that $\theta \ll 1$ and that the material responds in its linear range. More precisely, we neglected quantities of $O(\theta^2)$ relative to 1. Therefore, we should expect the predictions of these equations to be accurate to within a factor of approximately (θ^2_{max}), where θ_{max} is the maximum rotation of all the beam's cross sections.
Equations (3) - (5) may be combined to relate the deflection $v(x)$ to the internal bending moment $M(x)$:

$$EI\frac{d^2v}{dx^2} = M(x) \tag{6}$$

1.2.1 Three-point bending deflection

"Three-point bending" refers to a loading condition in which a simply supported beam is acted upon by a concentrated load. In this case, let us suppose the beam is simply supported at its ends, and loaded at some point along its span, as shown in figure 1.

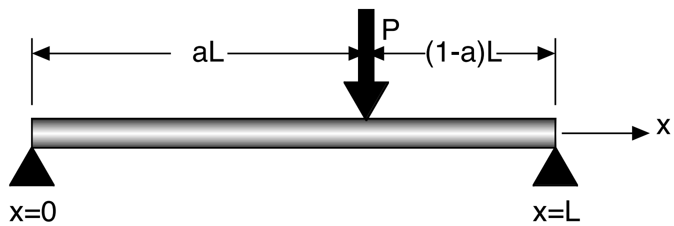

Figure 1: Schematic of beam in generic three point bending configuration.

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The bending moment distribution in this beam is given by:

$$M(x) = (L-a)Px/L - \frac{1}{2}P\left[|x-a| + (x-a)\right] \tag{7}$$

Substituting equation (7) into equation (6) and integrating twice with respect to x gives the deflection as:

$$v(x) = \frac{P}{EI}\left[\frac{(L-a)}{6L}(x^2 - 2aL + a^2)x - \frac{1}{12}\left[|x-a|^3 + (x-a)^3\right]\right] \tag{8}$$

In obtaining (8), we used the two boundary conditions:

$$v(0) = 0; \qquad v(L) = 0. \tag{9}$$

For a beam loaded at the center, $a = L/2$. The downward deflection at the same point, $\delta = -v(L/2)$, is given by (8) which simplifies in this case to:

$$\delta = \frac{PL^3}{48\,EI} \tag{10}$$

1.2.2 Cantilever bending deflection

A "cantilever" is a beam that has one end that has a fixed displacement and slope, while the motion at the other end is unconstrained. A cantilever loaded at its end is shown in figure 2.

Figure 2: Schematic of cantilever beam with load at end.

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The bending moment distribution in this beam is given by:

$$M(x) = P(x - L) \tag{11}$$

Substituting equation (11) into equation (6) and integrating twice with respect to x gives the deflection as:

$$v(x) = \frac{P}{EI}\left[\frac{1}{6}(x-L)^3 - \frac{1}{2}L^2x + \frac{1}{6}L^3\right] \tag{12}$$

In obtaining (12), we used the two boundary conditions:

$$v(0) = 0; \qquad \frac{dv}{dx}(0) = 0. \tag{13}$$

The downward deflection at the endpoint of the cantilever, $\delta = -v(L)$, is given by (12) which simplifies in this case to:

$$\delta = \frac{PL^3}{3\,EI} \tag{14}$$

1.3 Pre-lab questions

1. Look up the yield strengths and Young's moduli of 2011-T3 Aluminum, 360 Brass, and 304 Stainless Steel.

2. Consider an 18 in long simply supported beam, loaded at its center, with circular cross section of $1/8$ in diameter. Compute the maximum bending stress in the beam in terms of the magnitude of the load P.

3. Determine the load P that will cause yielding in the above beam, for the different materials considered in the first problem.

4. Determine the mass, in grams, whose weight on Earth gives the loads P computed above.

5. Assuming that the beam material behaves linearly up to the point of yielding, predict the maximum deflection for the load that causes yielding for each case.

   Note: While handling your beam specimen in the lab, please try to avoid deflections larger than this.

6. In your own words, briefly describe how to use the Vernier depth gauge. Sketch and label a picture of the device yourself to aid your description.

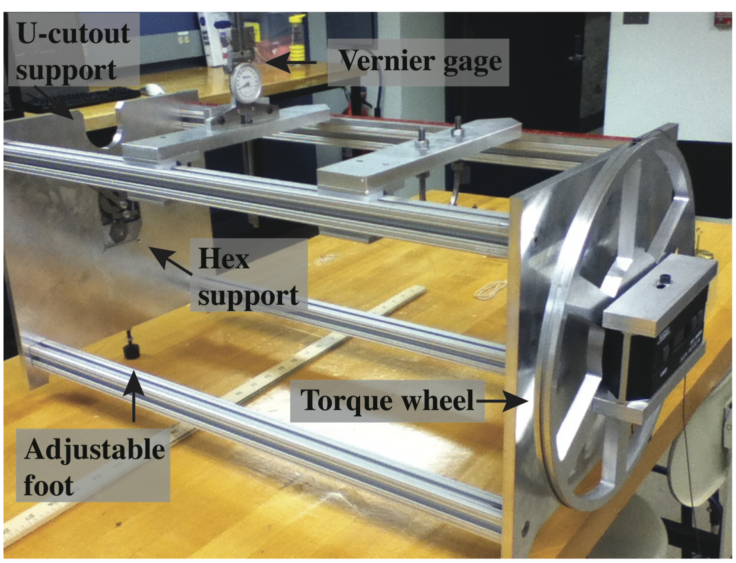
   This is an exercise intended to get you to think about the operation of the Vernier depth gauge so that you'll be able quickly and competently to use it during your lab period. If your time isn't used efficiently, you might have insufficient time to complete the necessary measurements. Therefore, neither text nor images may be copy and pasted from the lab handout in response to this question.

2 Lab procedure

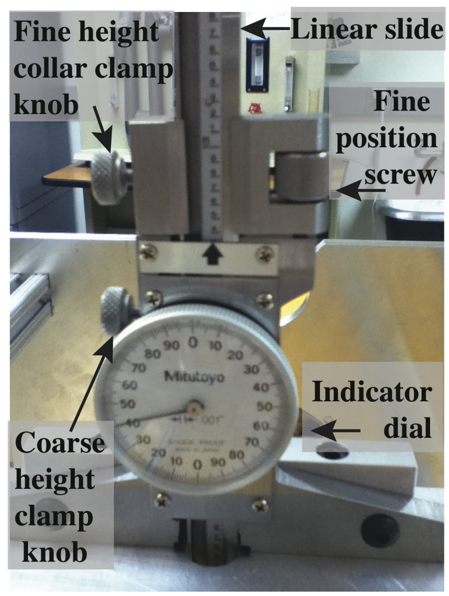
The experiment will be conducted using the Megazord [1] apparatus, shown in figures 5 and 4. The apparatus supports 3 different experimental configurations; you will use 2 of these arrangements in this lab.
The first configuration allows for a sample to be fixed by a wall support, thus creating a cantilever beam. A series of weights will be successively added to the free end of the beam, and the tip deflection will be measured using a manual micrometer. We will repeat for three different beam lengths, resulting in a total of three cantilever load cases.
The second configuration allows testing the sample in 3-point bending by suspending the sample between two simple supports. A series of weights will be suspended from the middle, and the resulting deflection of the loading point will be measured. We will repeat for three different beam lengths, resulting in a total of three load cases for three-point bending. Finally, we will measure the deflected shape of the longest 3-point bending sample when loaded at its largest load.

2.1 Vernier depth gage

All measurements of deflection will be accomplished using a Vernier depth gage, shown in Fig. 3. This type of gage is used for measuring either an absolute or relative change in height, based on the position of the tip of the gage slide. It is important to understand how to properly adjust and read the gage, so take a few minutes to familiarize yourself with its operation.



(a)



(b)

Figure 3: (a) Apparatus overview. (b) Vernier depth gage.

The gage consists of an adjustable vertical, linear slide marked with $0.1''$ resolution (note that $1'' = 1\,\text{in}$) and an adjustable round dial marked with $0.001''$ positions. The height of the linear slide may be adjusted using a coarse or fine position adjustment. The coarse height position can be adjusted by loosening the coarse height clamp knob and moving the linear slide by hand. Tightening this knob will apply a friction clamp to prevent the slide from moving. The slide position can be adjusted with finer resolution by using the fine height adjustment clamp and screw, which are attached to the gage support via a collar. When the fine height collar clamp knob is tightened, the screw may be used to move the slide up or down in small increments (note that the coarse height knob must not be tightened against the slide). The travel distance of this collar is limited to 0.02", so its position relative to the slide may be readjusted by loosening the knob and turning the screw appropriately. When repositioning the fine adjustment collar to allow for more travel, make sure to prevent the slide from moving by temporarily tightening the coarse position clamp.

2.2 Balancing the apparatus

The apparatus is supported by three feet, two under the torsion wheel wall, and a central adjustable foot under the opposite wall that features the U-shaped cutout. The apparatus may be leveled along this main axis by checking the height of the adjustable foot with a level and adjusting its height appropriately.

2.3 Cantilever bending procedure

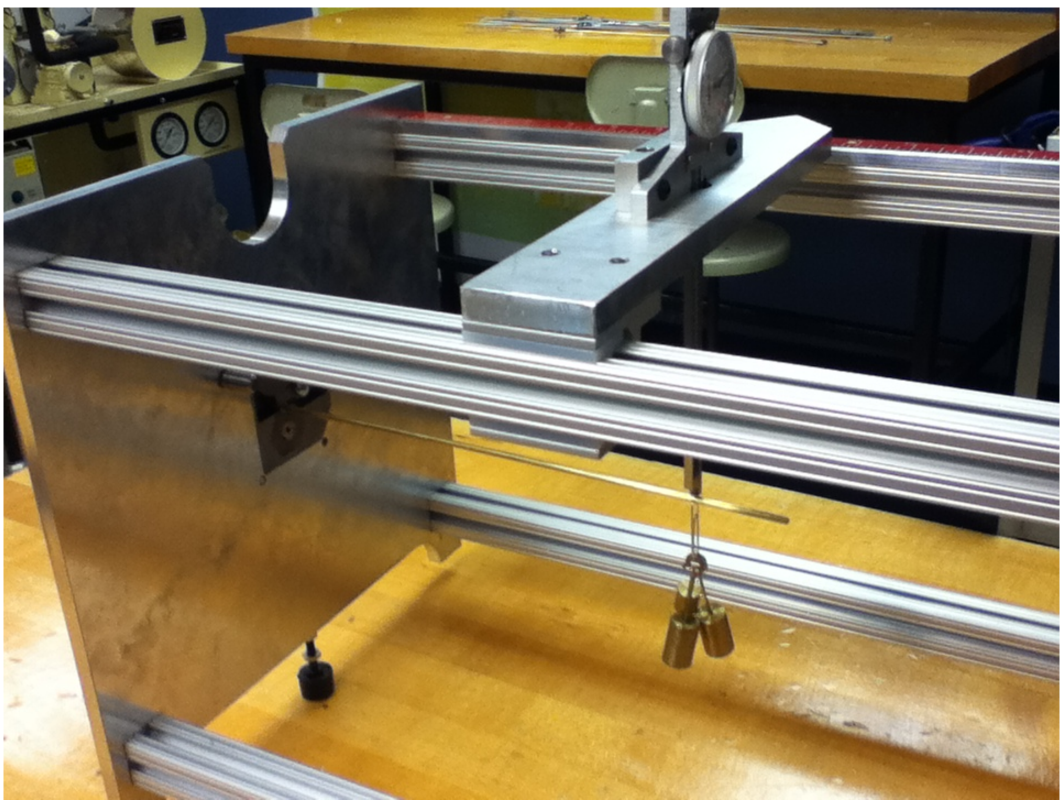


Figure 4: Cantilever bending configuration.

Nota Bene: Due to the size of the apparatus and range of loads involved, the samples are necessarily thin. Accordingly, they are very sensitive to vibration and touch, so be careful when you load the samples and make contact with the gage slide. Similarly, especially outside of the testing apparatus when they are not otherwise supported, handle the samples with care so they are not permanently bent.

Steps:

1. Select one $30''$ long, (nominally) $1/8''$ diameter cylindrical test sample. Make a note of the specimen identification marking and the material type. Before inserting the sample into the apparatus, note the

following three features of the rod: At one end is a hex collar; about $2/3$ distance from the end collar is a longer, double sided hex collar that we will call the central collar; near the far end of the rod a small hole has been drilled through its diameter. This part of the lab will test the sample section between the central collar and the hole. Carefully measure and record the distance between the end of the central collar and the hole. This represents the length L of the cantilever.

2. Insert the sample into the chuck opening through the back of the wall support so that the midpoint of the central collar is flush with the back wall, and so that the beam extends approximately 15 in into the apparatus. Tighten the chuck to fix the sample in place. The fixed-wall end of the cantilever is called "the root", and the free end of the cantilever is called the "free" or "tip end".

3. Insert the paperclip through the hole at the free end of the cantilever. This clip will be used to attach the weights to the sample.

4. Position the gage directly over the paperclip; c.f. figure 4.

5. Zero the gage dial: Turn the indicator dial to align the arrow with the zero marking on the dial. Complete the zeroing process by first loosening the coarse position knob, then moving the slide downward until it just comes in contact with the top of the sample. Tighten the thumbscrew and record the new position.

6. Apply a 20 g mass to the paperclip.

7. Measure the deflection: Using the fine adjustment screw, adjust the gauge slide until it again comes into contact with the sample. Record the new gauge position.

8. Increase mass by 20 g increments up to 80 g. Measure and record the new deflection for each load increment.

2.4 Three-point bending procedure

1. Use the same specimen as with the cantilever experiments.

2. Position the U-bolt bar 12 inches from the wall support.

3. Position the sample on the U-cutout support (on the wall) and the U-bolt so it is centered about the supports and sits in the groove notched into each support.

4. Position the gage mount bar so that it is 6 inches from the wall support.

5. Hang the load wire hanger on the sample so that the loops straddle the center of the sample (at the 6 inch mark on the rectangular support).

6. Zero the gage: First loosen the coarse height knob, then move the slide downward until it comes in contact with the top of the specimen but does not deform it. Re-tighten the knob. Zero the indicator dial so the arrow lines up with the zero position. Record the initial position of the slide. Adjust the load wire hanger positions as necessary; they should allow clearance for the gage slide to touch the sample but should allow the mass to be centered at 6 inches.

7. Apply a 50 g mass to the load wire. If the sample vibrates from the added load, wait for the vibrations to dampen out and the sample to become stationary before proceeding.

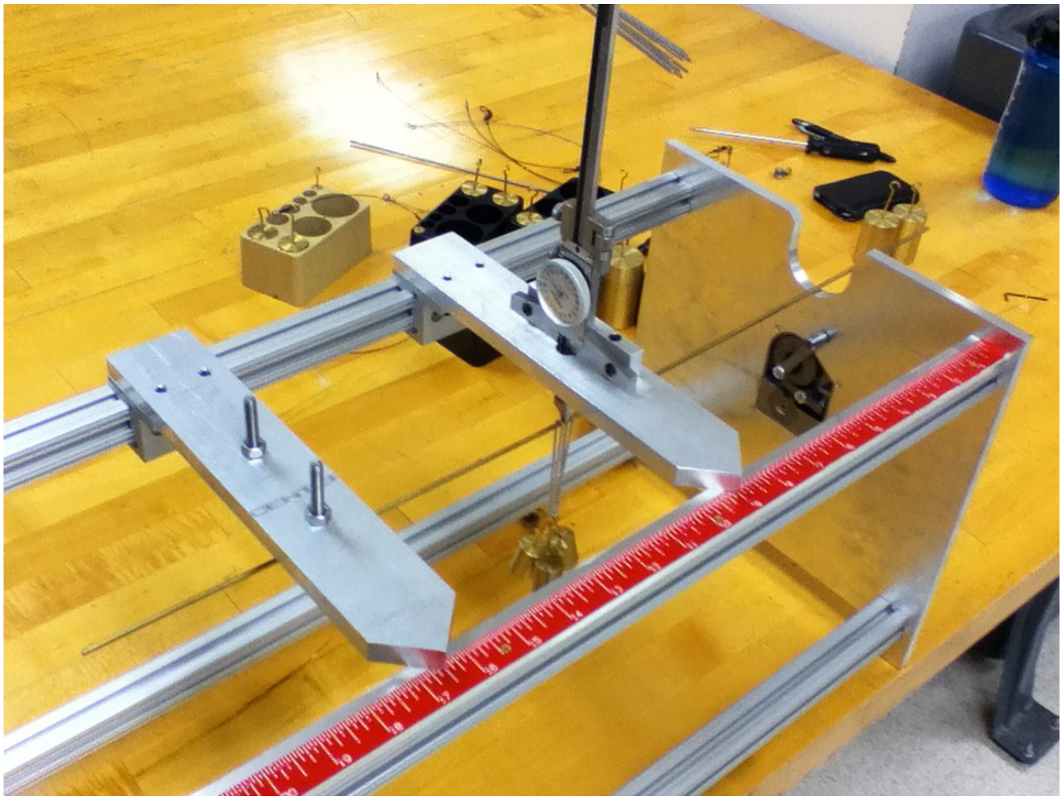


Figure 5: 3-point bending test.

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8. Adjust the gage until it comes into contact with the deflected specimen. **Be sure to track the number of rotations required to make contact,** as this number will provide the distance the sample has deflected. Make sure you understand the correspondence of one increment on the slide mark with a (full? partial?) rotation of the dial. Record the new gage position.

9. Increase mass by 50 g increments up to 200 g. Adjust and record the new gage position once the slide has been adjusted to reach the deflected sample for each mass increment. Once the 200 g position has been measured, retract the gage and unload the sample.

10. Move U-bolt to 15 in and the gage mount bar to 7.5 in, and repeat steps 3 - 9.

11. Move U-bolt to 18 in and the gage mount bar to 9 in. For the 18 in sample, you will measure the midpoint deflection as before. At the 100 g and 200 g loads, however, you will measure the deflection as a function of length along the sample as well.

12. Start by repeating steps 3 - 9. Once the deflection has been measured at the 100 g load, adjust the fine position screw to move the gauge slide back to its initial position (do not re-zero the gauge). When the slide is safely above the sample's undeflected position, move the gauge bar to the 1" position, near the cutout support. Adjust the gauge slide in order to measure the deflection at this position, and proceed with measurements of deflection along the sample at 1" increments, until the midpoint is reached again. These values are later called $\delta_j(100g)$.

13. Once back at the midpoint, continue loading the sample with 50 g increments up to 200 g. Measure the midpoint deflection at each new applied load. Once the midpoint deflection has been measured at 200 g,

repeat the transverse deflection measurements by retracting the gauge slide, moving the gauge bar back to the cutout support, and measuring the deflection at 1" increments between 1" and the midpoint. These values are later called $\delta_j(200g)$.

3 Analysis

3.1 Validation measures

We will validate Euler-Bernoulli beam theory by testing the predictions of equations (10), (14), and (8), as follows.
Equations (10) and (14) can be recast as:

$$P = k_{bending}\,\delta \tag{15}$$

$$k_{bending} = \frac{3EI}{L^3} \quad \text{cantilever} \tag{16}$$

$$k_{bending} = \frac{48EI}{L^3} \quad \text{3-point} \tag{17}$$

Here, $k_{bending}$ is to be interpreted as a spring constant due to beam bending. It clearly depends on the material properties, (e.g. through E), the geometry (e.g. through L and I), and the boundary conditions (e.g. 3 vs. 48 for the different boundary conditions.)
We will first test the prediction of equation (15): That load is a linear function of deflection. We will then test the predictions of equations (16) and (17), that $k_{bending}$ depends on L^3, and that EI is a "beam property", independent of the boundary conditions. Finally, we will test the prediction of equation (8) by simply plotting the predictions of (8) and comparing directly to measurements.
For all requested plots, use some analysis software package (e.g. Matlab). When plotting "A vs. B", the convention is to plot A on the y-axis and B on the x-axis; i.e. we plot y vs. x.

3.2 Initial display of results

1. On a single full-page plot, plot the total force (y-axis) vs midpoint deflection (x-axis) (or tip deflection for cantilever). Plot the points only and do not draw a connecting line. Include the data from all tests, but use separate symbols for each test. Are the data consistent for each material, set of boundary conditions, and all lengths?

   Our hope is that the theory will collapse all these data points onto a single curve or point. If so, then we will have shown that we use one simple material test to predict the response of any beam with any boundary conditions (covered within the assumptions of the theory.)

3.3 Testing linearity of response

We will first test the prediction of equation (15): That load is a linear function of deflection.

2. Fit a straight line to each of the data sets just plotted. The slope is called the "bending stiffness" of the beam, denoted here by $k_{bending}$.

   In Matlab, you may let: `>> P = [P_1; P_2; ...; P_N];` and `delta = [delta_1; delta_2; ...; delta_N];`. Here, N is the number of loads applied to a single beam. Then use:

```
>> bls = regress(P, [ones(10,1)delta]); P0 = bls(1); k = bls(2);
```

This will give the best fit:

$$P = P_0 + k\delta \tag{18}$$

Evaluate $k_{bending}$ for each beam tested. Add the straight lines predicted by equation (18) to your earlier plots.

3. Equation (15) predicts that for zero load P, there should be zero deflection δ. That is, P_0 should be zero in equation (18). The extent it is not zero is most likely indicative of an offset error in your measurement of δ. Compute $\delta_{offset} = -P_0/k$, the deflection at zero force predicted by equation (18). How large is δ_{offset} compared to your measured deflections, δ_j?

3.4 Testing dependence of bending stiffness on length and boundary conditions

We will now test the prediction of equations (16) and (17): that $k_{bending}$ depends on L^3, and that the same bending stiffness EI measured with one set of boundary conditions, applies to another set.

4. Define $\tilde{k}_{bending} = \frac{EI}{L^3}$, a "reduced" bending stiffness. Then equations (16) and (17) show that:

$$\tilde{k}_{bending} = \frac{1}{3} k_{bending} = \frac{EI}{L^3} \qquad \text{cantilever} \tag{19}$$

$$\tilde{k}_{bending} = \frac{1}{48} k_{bending} = \frac{EI}{L^3} \qquad \text{3-point} \tag{20}$$

Rearranging (19) and (20) shows:

$$\sqrt[3]{\frac{1}{\tilde{k}_{bending}}} = \sqrt[3]{\frac{1}{EI}}\, L \tag{21}$$

Compute $\tilde{k}_{bending}$ from the $k_{bending}$ values found in Analysis task 2. Compute $\tilde{k}_{bending} * L^3$ for all beams tested. Equation (21) predicts that these will all have the same value. Compute the percent differences between them. This analysis encompasses results for different lengths and different boundary conditions, hence implying that both effects are accounted for in the Euler-Bernoulli theory.

3.5 Testing predictions of the deflection curve for three-point bending

Up to this point, we've tested predictions of beam deflection at the load point. Here we test the predictions of equation (8) of the overall shape of the bent beam.

Let $\tilde{v}(x)$ represent the predicted shape of the bent beam, scaled by its maximum. Hence, from equation (8) (with $a = L/2$) we may write:

$$\tilde{v}(x) = \frac{v(x)}{v_{max}} = \frac{1}{L^3}\left[(3L^2 - 4x^2)x + 4\left(\left|x - L/2\right|^3 + (x - L/2)^3\right)\right] \tag{22}$$

We will compare the predictions of equation (22) to the measured deflection shape.

5. *Compute the deflection due to the load.* For each spanwise location measured x_j, compute the difference $\delta_j = \delta_j(200g) - \delta_j(100g)$; i.e. subtract the position of the beam when lightly loaded from the position when more heavily loaded with $200g$. (Taking this difference accounts for any initial predeformation (i.e. lack of straightness) of the rod.) In the calculations below, this is referred to as the Matlab array `delta`. It is a new array from that used in the previous sections.

6. *Compute the maximum deflection of the beam.* Equation (22) is scaled by v_{max} to remove material property or force dependence. Here we would like to do the same with the data. The maximum deflection of the beam is approximately the δ_j measured at the load point. We might use this value, but if it is in error, it can influence the rest of our comparison. Therefore we will compute an average δ_{max} for a single beam (i.e. a single length) by using all the data points for that beam.

   We will show how to do that in Matlab. First compute an array with entries x_j at each measurement point:
   `>> x = [1:9];`
   Then compute an array with $v(x)$ using equation (22):
   `>> v = (1/L^3)* ...`
   Note that the maximum value of $\tilde{v}(x)$ is one. What we want to find is the value of δ_{max} that minimizes $\sum_j(\delta_{max}v_j - \delta_j)^2$; i.e. we want to find δ_{max} that minimizes the sum squared error. This is a problem in single variable calculus. The solution is the least squares estimate of δ_{max}, which is given by:
   `>> delta_max = sum(v'*delta)/sum(v'*v);`

7. *Estimate error bars for the deflection δ_j.* The gauge width is about $g = 5/32$ in. This means that the point of contact of the gauge with the sample isn't necessarily at the location x_j, but might be to the right or left the width of the gauge. Because the beam is sloped, this effect can introduce error into our measurement of the beam deflection. We'll estimate this and use it as an indicator of the expected precision of our deflection measurements.

   Equation (8) predicts that the maximum slope of the beam is $\theta_{max} = 3\delta_{max}/L$. Assume that our fitting error is $\pm\Delta\delta = \pm\theta_{max} \times g$. Compute $\Delta\delta$. How large is $\Delta\delta$ compared to the precision of the micrometer ($1/1000$ in)?

8. *Visually compare measurements and predictions.* On the same figure, plot δ (with error bars $\pm\Delta\delta$) versus x, and $\delta_{max}\tilde{v}(x)$ versus x. (See Matlab help "errorbar".)

```
delta_delta = theta_max*g;                      % magnitude of the error.
error = ones(9,1)*delta_delta;                  % note there are 9 data points.
figure(3), errorbar(x,delta, error, error, 'o') % plot the data.
hold('on')
figure(3), plot(x, delta_max*v, 'red')          % add the theoretical curve.
hold('off')
```

We may consider the prediction of the deflected shape to be validated if the predicted curve $\tilde{v}(x)$ goes through the error bars of the measurements.

3.6 Discussion

Discuss the results (in 1-5 sentences) of each of the three validations:

- Linearity of force-deflection curve.
- Dependence of bending stiffness on beam length.
- Shape of deflection curve predictions.

References

[1] Megazord based on the original design from *Robust Revisionaries Senior Design Project*, M Taddonio, P DiTullio, M Brooks, M Sylvia, C Lee, ENG ME414, Mechanical Engineering, Boston University, 2012.