1 Cramer-Rao bound

Let us consider a single antenna transmitter and receiver that communicates through a channel that is composed by a single tap. The transmitted signal is defined as $s(t)$ and the received signal is $r(t)$. Assuming the receiver samples the received signal $r(t)$, the system model is written as

$$r(nT) = \underbrace{\beta s(nT-\tau)\mathrm{e}^{-2\pi \jmath f_D T n}}_{\mu(f_D,\tau,n)} + z(nT), \tag{1}$$

where β is the propagation loss, f_D is the doppler deviation, τ is the delay, and T is the sampling period. The pdf of the received signal is give by

$$g(f_D,\tau) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{\frac{|\mathbf{r}-\boldsymbol{\mu}(f_D,\tau)|^2}{2\sigma^2}} \tag{2}$$

Calculate the first derivative with respect the doppler.

$$\frac{\partial \boldsymbol{\mu}(f_D,\tau)}{\partial f_D} = -2\pi\jmath T\mathbf{s}(\tau)\odot\mathbf{e} \tag{3}$$

$$\frac{\partial \boldsymbol{\mu}^H(f_D,\tau)}{\partial f_D} = 2\pi\jmath T\mathbf{s}^H(\tau)\odot\mathbf{e}^* \tag{4}$$

$$\frac{\partial \boldsymbol{\mu}(f_D,\tau)}{\partial \tau} = -\mathbf{s}'(\tau)\odot\tilde{\mathbf{e}} \tag{5}$$

$$\frac{\partial \boldsymbol{\mu}^H(f_D,\tau)}{\partial \tau} = -\left(\mathbf{s}'(\tau)\odot\tilde{\mathbf{e}}\right)^H \tag{6}$$

where $\mathbf{e}[n] = n\mathrm{e}^{-2\pi\jmath f_D T n}$ and $\tilde{\mathbf{e}}[n] = \mathrm{e}^{-2\pi\jmath f_D T n}$

$$\mathbf{I}(f_D,\tau) = \begin{bmatrix} 2\mathrm{Re}\{\frac{\partial \boldsymbol{\mu}^H(f_D,\tau)}{\partial f_D}\frac{\partial \boldsymbol{\mu}(f_D,\tau)}{\partial f_D}\} & 2\mathrm{Re}\{\frac{\partial \boldsymbol{\mu}^H(f_D,\tau)}{\partial f_D}\frac{\partial \boldsymbol{\mu}(f_D,\tau)}{\partial \tau}\} \\ 2\mathrm{Re}\{\frac{\partial \boldsymbol{\mu}^H(f_D,\tau)}{\partial \tau}\frac{\partial \boldsymbol{\mu}(f_D,\tau)}{\partial f_D}\} & 2\mathrm{Re}\{\frac{\partial \boldsymbol{\mu}^H(f_D,\tau)}{\partial \tau}\frac{\partial \boldsymbol{\mu}(f_D,\tau)}{\partial \tau}\} \end{bmatrix}$$

$$\mathbf{I}(f_D,\tau) = \begin{bmatrix} 2\mathrm{Re}\{4\pi^2T^2|\beta|^2\sum_{n=0}^{N-1} n^2|\mathbf{s}_n(\tau)|^2\} & 2\mathrm{Re}\{2\pi\jmath T\sum_{n=0}^{N-1} n\mathbf{s}_n^*(\tau)\mathbf{s}_n'(\tau)\} \\ 2\mathrm{Re}\{2\pi\jmath T\sum_{n=0}^{N-1} n\mathbf{s}_n(\tau)\left(\mathbf{s}_n'(\tau)\right)^*\} & 2\mathrm{Re}\{4\pi^2T^2|\beta|^2\sum_{n=0}^{N-1}|\mathbf{s}_n'(\tau)|^2\} \end{bmatrix}$$

The inverse of Fisher matrix is

$$\mathbf{I}(f_D,\tau)^{-1} = \frac{1}{\det(\mathbf{I}(f_D,\tau))} \begin{bmatrix} 2\mathrm{Re}\{4\pi^2T^2|\beta|^2 \sum_{n=0}^{N-1} |\mathbf{s}'_n(\tau)|^2\} & -2\mathrm{Re}\{2\pi \jmath T \sum_{n=0}^{N-1} n\mathbf{s}^*_n(\tau)\mathbf{s}'_n \\ -2\mathrm{Re}\{2\pi\jmath T \sum_{n=0}^{N-1} n\mathbf{s}_n(\tau)\left(\mathbf{s}'_n(\tau)\right)^*\} & 2\mathrm{Re}\{4\pi^2T^2|\beta|^2 \sum_{n=0}^{N-1} n^2|\mathbf{s}_n(\end{bmatrix}$$

$$\mathbf{I}(f_D,\tau)^{-1} = \frac{1}{\det(\mathbf{I}(f_D,\tau))} \begin{bmatrix} 2\mathrm{Re}\{4\pi^2T^2|\beta|^2 \sum_{n=0}^{N-1} |\mathbf{s}'_n(\tau)|^2\} & -2\mathrm{Re}\{2\pi T \sum \\ -2\mathrm{Re}\{2\pi T \sum_{n=0}^{N-1} n|\mathbf{s}_n(\tau)||\mathbf{s}'_n(\tau)|\exp^{\jmath(\phi_s-\phi_{s'}+\pi/2)}\}\} & 2\mathrm{Re}\{ \end{bmatrix}$$

$$\mathbf{I}(f_D,\tau)^{-1} = \frac{1}{\det(\mathbf{I}(f_D,\tau))} \begin{bmatrix} 8\pi^2T^2|\beta|^2 \sum_{n=0}^{N-1} |\mathbf{s}'_n(\tau)|^2 & 4\pi T \sum_{n=0}^{N-1} n|\mathbf{s}^*_n \\ 4\pi T \sum_{n=0}^{N-1} n|\mathbf{s}_n(\tau)||\mathbf{s}'_n(\tau)|\sin\left(\phi_{s,n}(\tau)-\phi_{s',n}(\tau)\right) & 8\pi^2T \end{bmatrix}$$

where

$$\begin{aligned}\det(\mathbf{I}(f_D,\tau)) = 64\pi^4T^4|\beta|^4 \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} nm|\mathbf{s}'_n(\tau)|^2|\mathbf{s}_m(\tau)|^2 - \ldots \\ 16\pi^2T^2 \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} nm|\mathbf{s}_n(\tau)||\mathbf{s}'_n(\tau)||\mathbf{s}_m(\tau)||\mathbf{s}'_m(\tau)| \\ \times \sin\left(\phi_{s,n}(\tau)+\phi_{s',n}(\tau)\right)\sin\left(\phi_{s,m}(\tau)-\phi_{s',m}(\tau)\right) \qquad (7)\end{aligned}$$