Banach Tarski's Paradox

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Abstract

Investigation into the Banach–Tarski paradox which is a theorem that states: Given a solid mathematical sphere in 3D space, there exists a decomposition of the ball into a finite number of disjoint subsets, which are put through rotations , and then put back together to make two copies of the original sphere.

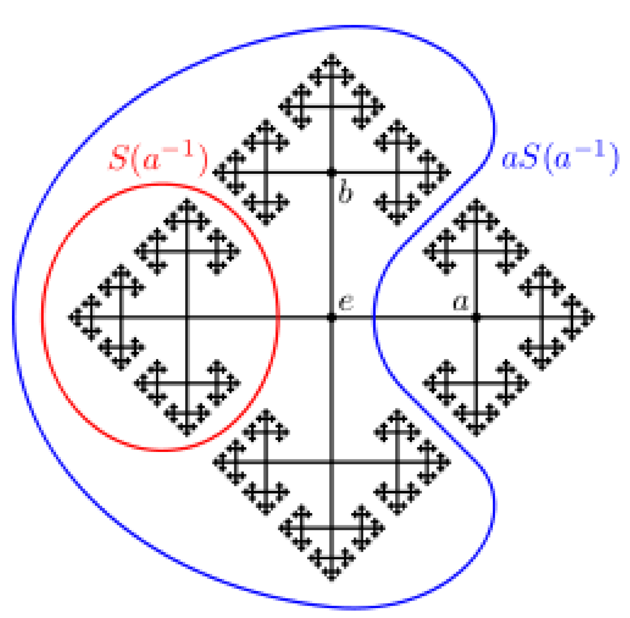
Introduction

This seems impossible to create a duplicate sphere from the original because one would expect the volume of the sphere to be preserved even through simple translations, but that is why it is called a paradox. In order to understand the Banach-Tarski paradox, we need to understand non-Lebesgue sets and free groups.

In proving the Banach-Tarski paradox, non-Lebesgue measurable sets are used. Lebesgue measurable sets are a way of assigning measures of volume, so the conservation of volume doesn't apply here. Non-Lebesgue measurable sets exist because of the Axiom of Choice. The axiom of choice states that given a collection of nonempty bins it is possible to make a selection of exactly one object from each bin even if the objects are non-distinguishable.

A free group is a group that any two words on a set of generators are different unless their equality follows from the group axioms. We will consider the two rotations in space up, ϕ and right, ψ. This leaves us with ϕ, ϕ^{-1}, ψ, ψ^{-1}. We will first create a decomposition of the free group on these two generators which will allows for the duplication. Then we will then show that there exists a free group of rotations in 3-D space. This will allow us to duplicate almost every point in the ball, and is the main idea of the theorem. Finally, we will apply everything to the sphere to prove the Banach-Tarski Paradox.

Figure 1: Free Group Image



To better understand Banach-Tarski Paradox one needs to look into Hilbert's Paradox of the Grand Hotel. It demonstrates the concept of infinity . The Grand Hotel is a full hotel with countably infinite many rooms. This hotel can occupy an additional guest by moving each guest to the previous room number plus one. Therefore the paradox proves that infinity plus one is still infinity.

Methods

Decomposition of a Free Group

Decomposition of the free group on two generators:

We will create a free group on two generators, denoted by F, with the generators ϕ and ψ. The group operation will be concatenation. We will have all expressions that can be made from and ϕ^{-1} as well as ψ^{-1} , and the empty expression e.

S(ϕ) will represent the set of all expressions in F that begin with ϕ, and has been simplified. This means there cannot be anything in S (ϕ) like $\phi\phi^{-1}\psi$, because $\phi\phi^{-1}\psi = \psi$ and ψ is not in S(ϕ). With this, we will have five disjoint sets whose union is F:

$F = \{e\} \cup S(\phi) \cup S(\phi^{-1}) \cup S(\psi) \cup S(\psi^{-1})$

Now, if we take two of these subsets and "shift" them based on their inverse, we will end up with two copies of F. If shift S(ϕ) by ϕ^{-1} we will obtain the sets of S(ϕ), S(ψ), and $S(\psi^{-1})$. A similar result will occur if we shift S(ψ) by ψ^{-1}. By doing these two shifts we will obtain two copies of F:

$F = \phi^{-1} S(\phi) \cup S(\phi^{-1})$

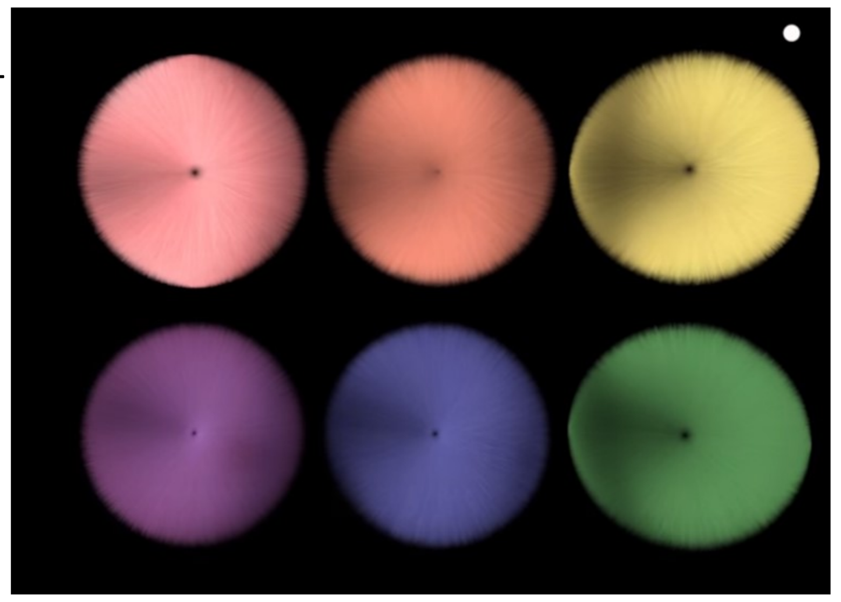
$F = \psi^{-1} S(\psi) \cup S(\psi^{-1})$

Do notice that {e} is included in $\phi^{-1}S(\phi)$ and $\psi^{-1} S(\psi)$, so we have a {e} left over. We will need to be careful about this when we deconstruct our sphere.

Deconstructing the Sphere

First we will pick a point on the sphere to start with and make that our starting point. From that point we will perform sequences of rotations starting (UP) (Down) (Right) and (Left). Now, there may be points we could have missed. That's okay, we will just pick a new starting point that hasn't been assigned a sequence yet, and then perform the sequence of rotations gaining more points that we have missed. This is done until all points on the sphere have been named. Now there is an exception of poles. Every point we rotate is rotated on either the x-axis or the z-axis. There is a sequence that leads to the same point. For example an up and down rotation it leads back to a point . We point these points in a set together. Once all the points have been given a name, we can put them into six different sets, all the rotations that start in a up, down, left, right and the points that represent the poles starting points and center point. The union of all these sets represents the sphere.

Figure 2: The Six Decomposed Spheres



Duplicating the Sphere

We take the set of all up rotations of the sphere and rotate it down, we will be left with all the starting points, all the left and right rotations and up rotations. With that we can add the center point the set of all pole points and the set of down rotations. We have created an exact copy of the original sphere with three sets left untouched, the right rotation, the left rotations and starting points. We will now make a second copy of the sphere with these leftover set.

We won't rotate all the set of all right rotations to the left because we then have a new set of all starting points. We will take the set of all right rotations and rotate every point that isn't just a string of right rotations to the left. From this we will have the set of all right rotations, up rotations and down rotations. Then we will add the leftover set of starting point and left rotations together. This sphere still doesn't have the center point and all the points of the poles. We take a line through the center of a sphere that doesn't intersect any of the pole holes where each pole hole lies on its own circle. Because it is a circle, it has infinitely many points minus the one missing point. We can use the concept of Hilbert's Paradox to fill in the missing point, and thus filling in all the pole holes. As for the center point we can apply the same concept as there is a circle that contains the origin and can fill in the center point as well. This leaves us with two identical copies of the original sphere.

Figure 3: Infinite Chocolate Bar

Free Group of Rotations

We pick a point $\theta = \arccos\left(\frac{1}{3}\right)$ on our 3-D sphere with our two generators, up, ϕ and right, ψ. We will choose the rotation angle to be because it is an irrational multiple of pi. This prevent any repetition when we perform rotations. We let ϕ be the rotation of the x-axis and ψ be the rotation of the z-axis. These rotations can be represented as matrices,

$$\theta = \arccos\left(\frac{1}{3}\right) = \sqrt{1 - \frac{1}{3^2}} = \frac{2\sqrt{2}}{3} \; and \;\; cos\left(arccos\left(\frac{1}{3}\right)\right) = \frac{1}{3}\cdot$$ since

$$A = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 1 \end{pmatrix} \quad B = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

We will denote the group from ϕ and ψ, G.

Then, we get the lemma, if $\rho : R^3 \rightarrow R^3$ is an $\frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})$ expression in G with length n in reduced form, then ρ(0, 1, 0) is of the following form, where a, b, and c are integers. We will use the lemma to prove that G is a free group because there is no nontrivial identity in G. By proof of contradiction, suppose there is a nontrivial identity in G. If this is the case, p of the point would have to equal itself because it is the identity. Based on the above lemma, p(0,1,0) is in the form so we have a = c and b = 3^n where n > 0. Therefore a ≡ b ≡ c ≡ 0 mod 3. It can be shown through induction listing all possible results of applying ϕ, ψ modulus 3 that this is not possible thus proving the theorem.

Discussions

From our study of Banach Tarski Paradox we learned a lot about modern algebra. We learned about infinite sets, set theory, group theory specifically free group. We learned about the Axiom of Choice more specifically Non-Lebesgue measurable sets and how this concept separates mathematical concepts and physics.

References

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