Remark 1. Notation: $K(-)$ is the connective K-theory spectrum, and $\mathbb{K}(-)$ is the non-connective K-theory spectrum.

Theorem 1. *There's a LES of K-groups*

$$\cdots \longrightarrow K_{n+1}(\mathbb{Q}) \longrightarrow \bigoplus_p K_n(\mathbb{F}_p) \longrightarrow K_n(\mathbb{Z}) \longrightarrow K_n(\mathbb{Q}) \longrightarrow \dots$$

Thanks to hard work about K theory of fields, gives lots of information about $K_*(\mathbb{Z})$.

This theorem is (non-trivially) a special case of:

Theorem 2. *(Quillen) Given $B \subset A$ a Serre subcategory of a small abelian category, then we have a cofiber sequence of spectra*

$$K(B) \to K(A) \to K(A/B).$$

Note that in particular, $K_0(A) \twoheadrightarrow K_0(A/B) \to 0$.

Remark 2. This is a premier computational tool. However, not all K-groups are equivalent to K-groups of some abelian category. Notably, K_* of a singular scheme, K_* of a ring spectrum, etc.

Remark 3. If $A \to B \to C$ is an exact sequence in Cat_∞^{perf}, it's not true in general that

$$K_0(B) \to K_0(C) \to 0$$

is exact. This is the first obstruction to a localization sequence.

Question 1. Is there an easy counterexample? I think possibly a singular cubic will work.

Fix: Negative K-theory.

Theorem 3. *If $A \to B \to C$ is an exact sequence in Cat_∞^{ex}, then*

$$\mathbb{K}(A) \to \mathbb{K}(B) \to \mathbb{K}(C)$$

is a cofiber sequence.

Construction 1. Idea: Want to find a C' s.t. $K(C') \cong *$ and a map $C \to C'$. Expect exact sequence

$$K_0(C') \longrightarrow K_0(C'/C) \longrightarrow K_{-1}(C) \longrightarrow K_{-1}(C')$$

but since $K_*(C') = 0$, just define $K_{-1}(C) = K_0(C'/C)$. Should be functorial: $C' = \mathcal{F}(C)$. Let $\Sigma C = cofib(C \to F(C))$, then $K_{-n}(C) = K_0(\Sigma^{(n)} C)$.

Definition 4. Say C is *flasque* if there are exact functors $F_1, F_2 : C \to C$ and equivalence

$$id \oplus F_1 \cong F_2,$$

and $(F_1)_* = (F_2)_* : K_*(C) \to K_*(C)$.

Then $id + (F_1)_* : K_*(C) \to K_*(C) \implies K_*(C) = 0$.

Example 5.

1. $Ind_\kappa(C)$, $\kappa > \omega$.
2. $F = (x \mapsto \oplus_{\mathbb{N}} x)$. Then $id \oplus F \cong F$, the Eilenberg-swindle, so $Ind_\kappa(\mathcal{C})$ is κ-acyclic.
3. A ring R is flasque if there's an R-bimodule M which is f.g. projective as a right R-mod, and there's a bimodule isomorphism $R \oplus M \cong M$. Then $Mod_R, Proj_R$ are flasque.
4. S any ring, $C(R) \subseteq End_S(S^\infty)$ row-finite column-finite infinite matrices (the cone ring). This is flasque.

5. The suspension ring $\Sigma S = C(S)/M(S)$, where $M(S)$ denotes the ring of finite matrices. Then $K_{-n}(S) = K_0(\Sigma^{(n)} S)$.

Remark 6. This construction works just fine for connective ring spectrum.

Construction 2. Model for K-theory of connective rings:
Can define

$$\begin{tikzcd} GL_n(R) \arrow[r] \arrow[d] & M_n(R) \arrow[d] \\ GL_n(\pi_0 R) \arrow[r] & M_n(\pi_0 R) \end{tikzcd}$$

Fact: $K_0(R) \cong K_0(\pi_0 R)$. Here the K-theory of a ring spectrum is the K-theory of the category of compact projective R-mods. Define $K_*(R) := \pi_*[K_0(\pi_0 R) \times BGL(R)^+]$.

Have a map $K_*(R) \to K_*(\pi_0(R))$. Compare the "ring suspensions":
have $\pi_0(\Sigma_{ring} R) = \Sigma_{ring} \pi_0 R$, so

$$\begin{tikzcd} K_{-1}(R) \arrow[r] \arrow[d] & K_0(\Sigma_{ring} R) \arrow[d] \\ K_{-1}(\pi_0 R) \arrow[r] & K_0(\pi_0 \Sigma_{ring} R) \end{tikzcd}$$

Upshot: $K_{-n}(R) = K_{-n}(\pi_0 R)$ for R connective.

Definition 7. $C \in Cat_\infty^{ex}$. Define $\mathcal{F}_k(C) = Ind_\kappa(C)$, $\Sigma_\kappa C = \mathcal{F}_\kappa(C)/C$, $K_{-n}(C) = K_0(\Sigma_\kappa^{(n)} C)$, $\mathbb{K}(C) = \operatorname{colim}_n \Omega^n K(\Sigma_\kappa^{(n)} C)$. This definition doesn't make any multiplicative properties apparent.

What is known:

1. K_{-n}(noetherian regular ring/scheme) $= 0$
2. K_{-1}(henselian ring) $= 0$ (hard-Drinfeld)
3. Weibel's conjecture: X: noetherian scheme of dimension $= d$. Then $K_{-n}(X)$ are zero if $n > d$. This is a theorem now ($d = 1$, Bass), ($d = 2$, Weibel), (X variety over field of char 0, Haesemeyer-Cortinas-Schlicting-Weibel), (X/F char$(F) > 0$ assuming res of singularites, Geisser-Hesselholt/Krishna), (whole thing, Kerz-Strunk-Tamme Nov '16).
4. Schlicting's conjecture: $K_{-n}(A) = 0$ if A (small) abelian. Still open in general. True if A is noetherian. Also true for $n = 1$ for any small abelian category.
5. If $E \in Cat_\infty^{ex}$ has a bounded t-structure, $K_{-1}(E) = 0$. If ... with E^{heart} noetherian, $K_{-n}(E) = 0$ for all $n \geq 1$.

Idea of $K_{-1}(A) = 0$.

$$\begin{tikzcd} D^b(A) \arrow[r] \arrow[d] & D^-(A) \arrow[d] \\ D^+ A \arrow[r] & D(A) \end{tikzcd}$$

induces pushout square on K-theory. $D^+ A$ and $D^- A$ are idempotent complete and K-theory acyclic. $K_*(A) = K_*(D^b(A))$, $K_0(D(A)) = K_{-1}(A)$.