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Contents

1 Spectral Categories

1.1 Summary

1.1.1 Important definitions

Term	Page Number	Loose Definition
Spectral category	8	Category enriched over the category of symmetric spectra.
DK equivalence	9	Functor which induces stable equivalence of mapping spectra + equivalence of "homotopy" categories.
Module over a spectral category	10	A spectral functor $A^{op} \to SymmSpt$ for a spectral category A.
Triangulated closure	10	Yoneda embed $A \hookrightarrow \widehat{A}^{cf}$ and take finite cell objects (pushouts of coproducts).
Thick closure	11	Same as above but take retracts of fintie cell objects.
Triangulated equivalence	11	Functor which induces DK equivalence of triangulated closures.
Morita equivalence of Spectral Categories	11	Functor which induces DK equivalence of thick closures.

1.1.2 Notation

- Cat_S is the category of small spectral categories and spectral functors.
- Cat_T is the category of small simplicial categories and simplicial functors.

1.1.3 Key results

1. There's a Quillen adjunction

$$Cat_S \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} Cat_T$$

where $\Omega^\infty(F(A,B))$ is the zeroth space functor or equivalently the simplicial set $[n] \mapsto Hom_{Spt}(\mathbb{S} \otimes \Delta^n, F(A,B)) \cong Hom_{Spc}(\Delta^n, \Omega^\infty F(A,B)) \cong \Omega^\infty F(A,B)_n$. This is the main theorem of Tabuada's paper "Homotopy Theory of Spectral Categories" (see references). We can modify Cat_S up to weak equivalence to make this a simplicial Quillen adjunction.

Remark 1. It might also be helpful to recall that, given a monoidal functor from a monoidal category M to a monoidal category N, any category enriched over M can be reinterpreted as a category enriched over N. Furthermore, we recover the underlying category of an enriched category by considering the functor $M(I,-) : M \to Set$ where I is the unit of the monoidal structure.

2. There's a model category $\widehat{\mathcal{A}}$ of spectral $\mathcal{A}$-modules (these are defined as functors, so we put the projective model structure on the functor category). The referenced paper here is Stable model categories are categories of modules by Schwede and Shipley.

1.2 Clarifying Remarks

- The definition of A-module is a generalization of the ordinary one. Recall that a ring is equivalently a preadditive category (an *Ab*-enriched category) with one object, and a module is a functor from this ring to abelian groups. Along the same vein, a ring spectrum is a spectral category with one object, and a module over this ring spectrum is a functor from this category to spectra.
- A category of enriched functors is itself enriched. Recall that $Nat(F,G) = \int_C Hom(F-, G-)$. Via the theory of enriched ends we can define a mapping spectrum to be $\int_C A(F-, G-)$, where A denotes the mapping spectrum of two objects in our enriched category.
- Given a functor $F : \mathcal{A} \to \mathcal{B}$, we get an obvious restriction functor $F^* : modB := \widehat{B} \to \widehat{A}$ given by precomposition with F (recall the definition of a module as a functor!). It has a left adjoint functor $F_!$ given by an enriched coend. It sends an A-module N to the coequalizer of

$$\bigvee_{o,p\in A} N(p) \wedge A(o,p) \wedge B(-, F(o)) \rightrightarrows \bigvee_{o\in A} N(o) \wedge B(-, F(o)) .$$

Recall that representable objects are "free on one generator". Indeed, the usual definition of a free object is that maps out of it are determined by maps out of a basis. We know, by the Yoneda lemma, that maps out of representable objects are determined by maps out of the identity morphism.

- The paper omits the definition of a pretriangulated spectral category. It's in the Mandell-Blumberg paper in the references, definition 5.4.

1.3 Background Material

While the theory of infinity categories allows us to work "coordinate-free" in some sense, we first need some examples of meaningful infinity categories and functors between them. One way to get a slew of presentable ∞-categories is to take the underlying infinity category of a left proper, simplicial, combinatorial model category. We'll define the simplicial nerve after introducing combinatorial model categories.

1.3.1 Combinatorial Model Categories

Definition 2. An infinite cardinal κ is a *regular cardinal* if it satisfies the following property, which we think of as requiring the collection of sets of smaller cardinality to be "closed under union":

- no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ.

Example 3. $\aleph_0$, or the cardinality of $\mathbb{N}$, is a regular cardinal. This is because any set of cardinality less than $\aleph_0$ is finite, and no infinite set is a finite union of finite sets.

Example 4. Any successor cardinal is regular. For $\aleph_1$, this follows from the fact that a countable union of countable sets is countable (we need choice here).

Definition 5. Let κ be an infinite regular cardinal. Then a *κ-filtered category* is one such that any diagram $F : D \to C$, where D has fewer than κ morphisms admits an extension $\widetilde{F} : D^+ \to C$ (i.e. F has a cocone). Here D^+ is the category obtained by freely adjoining a terminal object to D.

Example 6. Let $\kappa = \omega$ (or $\aleph_0$ in the notation we've been using). Then we're requiring every FINITE diagram in C to have a cocone. This is equivalent to the usual definition of a filtered category: a nonempty category s.t. each pair of objects has a join, and for any parallel morphisms $f, g : c_1 \to c_2$ in C there exists a morphism $h : c_2 \to c_3$ such that $hf = hg$. In other words, we can build a cocone for any finite diagram from these cocones.

Example 7. A preorder (there exists a unique morphism between any two objects) is ω-filtered precisely when it is *directed*, i.e. any two objects have a join.

Definition 8. Let κ be a regular cardinal. Then an object X such that $C(X, -)$ commutes with κ-filtered colimits is called *κ-compact*.

Definition 9. An object X of a category is *small* if it is κ-compact for some regular cardinal κ.

Definition 10. A category C is *locally presentable* if

1. C is a locally small category
2. C has all small colimits
3. there exists a small set $S \hookrightarrow Obj(C)$ of λ-small objects that generates C under λ-filtered colimits.
4. every obect in C is a small object.

Definition 11. Let C be a category and $I \subset Mor(C)$. Let $cell(I)$ be the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in I.

Definition 12. A model category C is *cofibrantly generated* if there are small sets of morphisms $I, J \subset Mor(C)$ such that

- $cof(I)$, the set of retracts of elements in $cell(I)$, is precisely the collection of cofibrations of C.
- $cof(J)$ is precisely the collection of acyclic cofibrations in C; and
- I and J permit the small object argument.

Definition 13. (Smith) A model category C is *combinatorial* if it is

- locally presentable as a category, and
- cofibrantly generated as a model category.

Example 14. The category *sSet* with the standand model structure on simplicial sets is a combinatorial model category.

Example 15. The category $sSet$ with the Joyal model structure (so that the quasi-categories are the fibrant objects) is combinatorial.

Question 1. Why should we care that a model category is combinatorial?

We'll define a left-proper model category, then give a fundamental result of Dugger's:

Definition 16. A model category is *left proper* if weak equivalence is preserved by pushout along cofibrations.

Example 17. A model category in which all objects are cofibrant is left proper. This includes the standard model structure on simplicial sets, as well the injective model structure on simplicial presheaves. This follows from the Reedy lemma, which allows us to calculate homotopy pushouts by considering diagrams s.t. the objects are cofibrant and one of the maps is a cofibration (the point is that replacing this diagram with a cofibrant diagram in the projective model structure on diagrams is an acyclic cofibration of diagrams, not just a weak equivalence).

Theorem 1. *(Dugger)*
Every combinatorial model category is Quillen equivalent to a left proper simplicial combinatorial model category.

Now, we have the following extremely useful result on Bousfield localizations:

Theorem 2. *If C is a left proper, simplicial, combinatorial model category, and $S \subset Mor(C)$ is a small set of morphisms, then the left Bousfield localization $L_S C$ does exist as a combinatorial model category. Moreover, the fibrant objects of $L_S C$ are precisely the S-local objects, and L_S is left proper and simplicial.*

In the context of infinity categories, we have some crucial results of Lurie. For these to make sense, however, we need to introduce the simplicial nerve construction. This is the generalization of the ordinary nerve construction to simplicially enriched categories.

Definition 18. We'll define a cosimplicial simplcially enriched category S. The objects of $S[n]$ are $\{0, 1, \dots, n\}$; the hom objects $S[n]_{i,j} \in sSet$ for $i, j \in \{0, 1, \dots, n\}$ are the nerves

$$S[n](i,j) = N(P_{i,j})$$

of the poset $P_{i,j}$ which is the poset of subsets of $[i, j]$ that contain both i and j with partial order given by inclusion.

Definition 19. The simplicial nerve of a simplicial category is the simplicial set characterized by

$$Hom_{sSet}(\Delta[n], N(C)) = Hom_{SSetCat}(S[n], C).$$

Theorem 3. *(HTT A.3.7.6) Let C be an ∞-category. The following conditions are equivalent:*

1. *The ∞-category C is presentable.*

2. *There exists a combinatorial simplicial model category A and an equivalence $C \cong N(A^\circ)$.*

Here A° is the underlying category of bifibrant objects.

Remark 20. (HTT A.3.7.7)
Let A and B be combinatorial simplicial model categories. Then the underlying ∞-categories $N(A^\circ)$ and $N(B^\circ)$ are equivalent iff A and B can be joined by a chain of simplicial Quillen equivalences.

Theorem 4. *(HTT 5.2.4.6) Let A and B be simplicial model categories, and let*

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

be a simplicial Quillen adjunction. This descends to an adjunction on the underlying ∞-categories.

2 Stable ∞-categories

2.1 Summary

2.1.1 Important definitions

Term	Page Number	Loose Definition
Stable ∞ category	14	∞-cat with finite (co)lims s.t. hopushouts coincide with hopullbacks.
Idempotent complete	14	Image under Yoneda embedding $C \to Fun(C^{op}, Gpd_\infty)$ is closed under retracts.
Morita equivalence of stab ∞-cats	14	Small stable ∞-cats A, B are ME if Idem(A) equiv to Idem(B).
Spectrum Object in Pted ∞-cat	15	A functor $F : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ s.t. certain diagrams are cartesian.
Spectral Yoneda embedding for stab ∞-cats	15	$C \approx Sp(C_*) \to Sp(Fun(C^{op}, Gpd_\infty)_*) \approx Fun(C^{op}, Sp(Gpd_\infty))$
Stably representable functor	16	Functor $C \to Spt$ equivalent to $Map(-, A)$ for a spectrum object A, where Map denotes mapping spectrum.
Accessible ∞-category	17	∞-cat equivalent to Ind_κ of a small ∞-cat.
Presentable ∞-category	17	∞ cat generated under sufficiently large filtered colimits by some small ∞-cat.
Localization of ∞-categories	18	∞-cat $C[S^{-1}]$ equipped with a map $C \to C[S^{-1}]$ which is universal for inverting elements of S.
Bousfield localization of presentable ∞-cat	19	Colimit preserving functor whose right adjoint is fully faithful.

2.1.2 Notation

- Cat_∞ is the ∞-category of small ∞ categories.
- Cat_∞^{ex} is the ∞-category of small stable ∞-categories and exact functors.
- Cat_∞^{perf} is the ∞-category of small idempotent complete stable ∞-categories (and exact functors).

2.1.3 Key results

1. Given a pretriangulated spectral category $\mathcal{C}$, the ∞-category $N((Mod(\mathcal{C}))^{cf})$ is stable.
2. If C is a pretriangulated spectral category, then $\pi_0 C$ admits a triangulated category structure.
3. If C is a stable ∞-category, then hoC admits a triangulated category structure.
4. The inclusion $Cat_\infty^{perf} \to Cat_\infty^{ex}$ has a left adjoint called idempotent completion.
5. If C is a ∞-category with finite limits, there is a stable ∞-category $Stab(C)$ with a limit preserving functor $\Omega^\infty : Stab(C) \to C$. In particular, this is accomplished by taking $Stab(C) = Sp(C)$, the category of spectrum objects of C. If C is already stable, then Ω^∞ is an equivalence, with inverse $\Sigma^\infty : C \to Sp(C)$.

6. If C is a stable ∞-category, there is a *spectral Yoneda embedding*

$$Y : C \simeq Sp(C_*) \to Sp(Fun(C^{op}, N(T)^{cf})_*) \simeq Fun(C^{op}, S_\infty)$$

with an adjoint *mapping spectrum functor*

$$Map : C^{op} \times C \to S_\infty$$

7. If C is a stable ∞-category, a $C^{op} \to S_\infty$ is stably representable if and only if it is represented by the suspension spectrum $\Sigma^\infty z$ of some $z \in C$, and this object is unique upto equivalence.

2.2 Clarifying Remarks

1. Idempotent completion is unique upto equivalence by HTT Proposition 5.1.4.9. Existence is obtained by taking the full subcategory of $Pre(C)$ spanned by objects that are retracts of objects in the image of C under the Yoneda embedding.

2. For this paper we assume stable ∞-categories to be pointed, i.e. they have zero objects. In the description of spectral Yoneda embedding above, $*$ indicates the subcategory of pointed objects.

2.3 Background Material

2.3.1 Pretriangulated Spectral Categories

The material here is from Section 5 of Mandell-Blumberg (in the higher K-theory paper, they say Section 4 but that's either a typo or just not updated).

Definition 21. A spectral category C is *pretriangulated* if

1. There exists an object 0 of C such that $C(-, 0)$ is homotopically trivial. This means that it is weakly equivalent to the constant functor $*$ at the one point symmetric spectrum.

2. Whenever a C-module M has the property that ΣM is weakly equivalent to a representable C-module $C(-, c)$, then M is weakly equivalent to some representable C-module $C(-, d)$.

3. Whenever the C-modules M and N are weakly equivalent to representables $C(-, a)$ and $C(-, b)$ respectively, the homotopy cofiber of any map of C-modules $M \to N$ is weakly equivalent to a representable C-module.

Remark 22. The first condition says that the homotopy category $\pi_0 C$ has a zero object. The second condition gives a desuspension functor on $\pi_0 C$ and the third condition gives a suspension functor on $\pi_0 C$.

Definition 23. A spectral functor between spectral categories $F : C \to D$ is a *DK-embedding* if for all objects a, b in C, the induced map of spectra $C(a, b) \to D(Fa, Fb)$ is a weak equivalence.

Remark 24. In Blumberg-Mandell, a DK equivalence is a DK embedding satisfying one of the following equivalent conditions

1. For all object d of D, there is an object c of C such that $D(-, d)$ and $D(-, Fc)$ are naturally isomorphic as D-modules.

2. The induced functor on the "graded homotopy categorie" $\pi_* C \to \pi_* D$ is an equivalence of categories.

In Blumberg-Gepner-Tabuada, this second condition is relaxed to $\pi_0 C \to \pi_0 D$ being an equivalence of categories. This is because a spectral functor between pretriangulated spectral categories is a DK equivalence if and only if it is a DK embedding and the induced functor on π_0 is an equivalence of categories.

Theorem 5. *(Blumberg-Mandell, Theorem 5.5) Any small spectral category C DK-embeds into a small pretriangulated spectral category.*

Remark 25. We should think of this as taking the closure of C under cofibration sequences and desuspensions in C-modules, via the Yoneda embedding. Note that this can be made functorial and it gives the "minimal pretriangulated closure" of C.

Definition 26. A *four-term Puppe sequence* in $\pi_0 C$ is a sequence of the form

$$a \to b \to c \to \Sigma a$$

if there exist a map of C-modules $f : M \to N$ such that the sequence

$$M \to N \to Cf \to \Sigma M$$

is isomorphic to the above sequence in the derived category of C-modules via the Yoneda embedding, and further the equvalence $\Sigma M \simeq C(-, \Sigma a) \simeq \Sigma C(-, a)$ is the suspension of the isomorphism $M \simeq C(-, a)$.

Theorem 6. *(Blumberg-Mandell, Theorem 5.6) Given a pretriangulated spectral category C, its homotopy category $\pi_0 C$ is triangulated with distinguished triangles the above four-term Puppe sequences.*

Proof. Proof of Theorem 5.6 is just observing that $\pi_0 C$ embeds as a full subcategory of the homotopy category of C-modules (with projective model structure) and checking that it is closed under suspensions, desuspensions and distinguished triangles. □

2.3.2 ∞-Categories

Definition 27. The **Joyal model structure** on simplicial sets is defined as follows :

- Cofibrations are levelwise monomorphisms.
- Weak equivalences are *(weak) categorical equivalence*. These are maps $f : A \to B$ such that for any ∞-category X, the map $X^B \to X^A$ induces an isomorphism on the fundamental category (the homotopy category) hoC.
- Fibrations are determined by the above.

Remark 28. All objects are cofibrant, and the fibrant objects are precisely ∞-categories.

Theorem 7. *(HA Theorem 1.1.2.15) Let C be a stable ∞-category. Then hoC has a triangulated category structure with distinguished triangles coming from cofiber sequences.*

Remark 29. There is a correspondance between pretriangulated spectral categories and stable ∞-categories. For example, if C is a pretriangulated spectral category, then $N((ModC)^{cf})$ is a stable ∞-category.

2.3.3 Idempotent completion

HTT 4.4.5 gives a good overview on the definition of idempotent completion for ∞-categories, while comparing it with the classical notion.

Remark 30. The notion of retracts between classical and ∞-categorical settings are a bit different. An ordinary category X is said to be *idempotent complete* if every idempotent map $X \to X$ comes from some retract Y of X. In such a situation Y can be determined uniquely as an equalizer (or a coequalizer). Hence, if C has finite limits or finite colimits, then C is idempotent complete.

This is not the case for ∞-categories. Consider the category $C_*(R)$ consisting of bounded chain complex of finite rank free R-modules and consider $N(C_*(R))$, which is actuallt a stable ∞-category. Hence it admits finite limits and colimits, but it is idempotent complete if and only if every finitely generated projective R-module is stably free.

The problem is that an idempotent in an ∞-category shouldn't be just a morphism e with $e \circ e \simeq e$ in hoC. It should specify homotopies on how to relate multiple compositions $e \circ e \circ ... \circ e \simeq e$. To achieve this, in HTT 4.4.5, simplicial sets called $Idem^+, Idem$ and Ret are introduced. Now idempotents, weak retractions, strong retractions in C are respectively defined to be maps of simplicial sets from $Idem, Ret, Idem^+$ to C. C is *idempotent complete* if every idempotent $F : Idem \to C$ has a colimit. This can be shown to be equivalent to the definition in the paper using results in HTT 5.1.4 and 5.1.5.

3 Symmetric Monoidal Structures and Dualizable Objects

3.1 Summary

3.1.1 Important definitions

Term	pg	Loose Definition
Tensor Product for $\mathrm{Cat}_\infty^{\mathrm{Perf}}$	20	If $\mathcal{A}$ and $\mathcal{B}$ are small stable idempotent-complete ∞-categories, we define $\mathcal{A}\widehat{\otimes}\mathcal{B} = (\mathrm{Ind}(\mathcal{A}) \otimes \mathrm{Ind}(\mathcal{B}))^\omega$.
Right-Compact Object	21	An object of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}\widehat{\otimes}\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$ is right-compact if putting in some object $a \in \mathcal{A}$ in the left argument always yields a compact object of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$.
Proper Stable ∞-Category	22	Mapping spectra are compact.
Smooth Stable ∞-Category	22	Perfect as a bimodule over itself. If $\mathcal{A}$ is idempotent-complete, then it is smooth if and only if it is a representable $\mathcal{A}^{\mathrm{op}}\widehat{\otimes}\mathcal{A}$-module. ("Coherent"?)
Dualiz-able	22	An object of a symmetric monoidal ∞-category is dualizable if it is dualizable in the homotopy category.

3.1.2 Notation

- The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Perf}}$ admits a symmetric monoidal structure. We write the tensor product as $\widehat{\otimes}$ to distinguish it from the usual tensor product of presentable stable ∞-categories.

3.1.3 Key results

1. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Perf}}$ admits the structure of a closed symmetric monoidal ∞-category with tensor product given by $\widehat{\otimes}$. The unit is the ∞-category $\mathcal{S}_\infty^\omega$ of compact spectra. Internal Hom is given by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$.

2. For any small stable ∞-category $\mathcal{A}$, the stable Yoneda embedding $\mathcal{A} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{op}}, \mathcal{S}_\infty)$ induces an equivalence $\mathrm{Ind}(A) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{op}}, \mathcal{S}_\infty)$. In particular, one can model the idempotent-completion of $\mathcal{A}$ by the Yoneda embedding $\mathcal{A} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{op}}, \mathcal{S}_\infty)^\omega$.

3. Let $\mathcal{A}$ and $\mathcal{B}$ be small stable idempotent-complete ∞-categories. The functor category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$ can be identified as a full subcategory of the ∞-category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}\widehat{\otimes}\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$ of $\mathcal{A}^{\mathrm{op}}\widehat{\otimes}\mathcal{B}$-modules. In fact, it is the full subcategory spanned by right-compact $\mathcal{A}^{\mathrm{op}}\widehat{\otimes}\mathcal{B}$-modules. (Note, I believe there is a small typo in the second-to-last paragraph of page 21, where it says "…certain subcategory of $\mathrm{Fun}^{\mathrm{L}}(\mathcal{A}\widehat{\otimes}\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$"; superscript should be ex, not L.)

4. An object $\mathcal{A}$ of $\mathrm{Cat}_\infty^{\mathrm{Perf}}$ is dualizable (with respect to the symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{Perf}}$) if and only if $\mathcal{A}$ is smooth and proper. Moreover, if $\mathcal{A}$ is dualizable, its dual is given by $\mathcal{A}^{\mathrm{op}}$.

3.2 Clarifying Remarks

1. This chapter assumes the monoidal structure on the ∞-category of presentable, stable ∞ categories, then defines some other monoidal products. The definition of the monoidal structure $\otimes$ on $\mathcal{P}r^L_{St}$ is involved. In fact, it must be involved; the unit of this monoidal structure is the category Sp of spectra. Hence, defining this monoidal structure gives us, in particular, a smash product of spectra.

2. Key result (2) above is one step in showing that the definition of Morita equivalence via Idempotent completion matches the definition in terms of module categories.

3. There's a slight difference in notation: BGT denote a symmetric monoidal infinity category by $\mathcal{C}^\infty$, whereas below we define a symmetric monoidal infinity category asa coCartesian fibration $\mathcal{C}^\infty \to \mathscr{F}\mathrm{in}_*$.

4. The terminology "smooth" and "proper" here comes from dg-categories (recall that this paper is a translation of Tabuada's work into the language of ∞-categories). In particular, a smooth proper dg category is one which closely resembles the category of perfect complexes on a smooth proper scheme, and these dg categories can be characterized as the dualizable objects in some category of dg categories.

3.3 Background Material

3.3.1 Symmetric Monoidal ∞-Categories

The following can be found in chapter 2 of HA. In particular, the introduction to this chapter is a very clear introduction to the main idea.

In ordinary category theory, a symmetric monoidal structure on a category $\mathcal{C}$ is usually described by a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an identity object $1 \in \mathcal{C}$ and natural isomorphisms describing associativity, commutativity, and unitality. In this setting, one demands that these natural isomorphisms satisfy coherence conditions. If one tries to do something analogous in the setting of ∞-categories, one will quickly find that higher and higher coherence conditions must be imposed, to the point where this is prohibitively complicated. We will thus need to try something.

Another way to describe a symmetric monoidal structure on an ordinary category $\mathcal{C}$ is by a Grothendieck opfibration. Let $\mathscr{F}\mathrm{in}_*$ denote the category consisting of objects $\langle n\rangle = \{1, \dots, n\} \sqcup \{*\}$ ($n \geq 0$) and functions between these sets that preserve $*$. By abuse, we will identify this category with the category of pointed finite sets. Some useful morphisms in $\mathscr{F}\mathrm{in}_*$ are the functions $\rho_i : \langle n\rangle \to \langle 1\rangle$ given by

$$\rho_i(j) = \begin{cases} 1 & \text{if } j = i \\ * & \text{otherwise.} \end{cases}$$

Given a symmetric monoidal category $\mathcal{C}$, we can form a category $\mathcal{C}^\otimes$ in which

- objects are finite (possibly empty) sequences of objects of $\mathcal{C}$, denoted by $[C_1, \dots, C_n]$
- A morphism $f : [C_1, \dots, C_n] \to [C'_1, \dots, C'_m]$ consists of a subset $S \subseteq \{1, \dots, n\}$, a map $\alpha : S \to \{1, \dots, m\}$, and a collection of morphisms $\{f_j : \bigotimes_{\alpha(i)=j} C_i \to C'_j\}_{1\leq j\leq m}$, and
- composition is defined in the only sensible way. See the introduction of chapter 2 of HA.

We get a forgetful functor $\mathcal{C}^\otimes \to \mathscr{F}\mathrm{in}_*$. In fact, this functor is a Grothendieck opfibration. We can identify $\mathcal{C}$ with the fibre $\mathcal{C}^\otimes_{\langle 1\rangle}$ over $\langle 1\rangle \in \mathscr{F}\mathrm{in}_*$. It has the special feature that the functors $\mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}$ induced by $\rho_i : \langle n\rangle \to \langle 1\rangle$ assemble into an equivalence $\mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^n$. (This is often referred to as a "Segal condition".) The main result is that a symmetric monoidal structure on $\mathcal{C}$ can be recovered from such a Grothendieck opfibration. Moreover, this construction generalizes readily to the ∞-category setting, as demonstrated in the following definition.

Remark 31. This Segal condition gives us functors $\mathcal{C}^2 \to \mathcal{C}^\otimes_{\langle 2\rangle} \to \mathcal{C}$, which is how we can recover the symmetric monoidal product from the category defined above. So just to clarify, defining the category $\mathcal{C}^\otimes$ will require us to specify what the objects $C_1 \otimes \cdots \otimes C_j$ are. The point is that defining the bifunctor $\otimes$ by specifying it in the form above lets us give a "minimal presentation" of the coherence conditions.

Definition 32. A *symmetric monoidal ∞-category* is a coCartesian fibration of simplicial sets $\mathcal{C}^\otimes \to \mathscr{F}\mathrm{in}_*$ such that, for each $n \geq 0$, the maps $\{\rho_i : \langle n\rangle \to \langle 1\rangle\}_{1\leq i\leq n}$ induce functors $\mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^\otimes_{\langle 1\rangle}$ that assemble into an equivalence $\mathcal{C}^\otimes_{\langle n\rangle} \to (\mathcal{C}^\otimes_{\langle 1\rangle})^n$.

Here, we make the usual abuse of notation of writing $\mathscr{F}\mathrm{in}_*$ instead of $N(\mathscr{F}\mathrm{in}_*)$. It should be noted that, given a symmetric monoidal ∞-category $\mathcal{C}^\otimes \to \mathscr{F}\mathrm{in}_*$, one typically thinks of $\mathcal{C}^\otimes_{\langle 1\rangle}$ as having been given a symmetric monoidal structure.

4 Morita Theory

4.1 Summary

4.1.1 Important definitions

Term	pg	Loose Definition
Stable simplicial category	25	Simplicial category s.t. the associated ∞-category is stable.
Stable spectral category	25	Spectral category whose associated simplicial category is stable.
Ψ_{tri}	24	Functor $N((Cat_S)^c)[W^{-1}] \to N((Set_\Delta)^c)[W^{-1}] \cong Cat_\infty$. Induced by the functor $Cat_S \to Set_\Delta$ given by $A \mapsto \widehat{A}_{tri} \mapsto N(\Omega^\infty(A)^{fib})$.
Ψ_{perf}	24	Same as above, but replace $\widehat{A}_{tri}$ with $\widehat{A}_{perf}$.
Triangu-lated equiva-lence v2	30	A map in the ∞-cat of small spectral categories s.t. $\Psi_{tri} f$ is an equivalence of stab ∞-cats.
Morita equiva-lence v2	30	$\Psi_{perf} f$ is an equivalence.
rep(B, A)	31	$\Upsilon(Fun^{ex}(B, A))$, the small pretriangulated spectral category associated to the small stable ∞-cat of exact functors from B to A.

4.1.2 Notation

- Υ is the right adjoint to Ψ_{tri}, and via inclusion to Ψ_{perf}.

4.1.3 Key Results

1. Ψ_{tri} lands in Cat_∞^{ex} =small stable ∞-cats.
2. Ψ_{perf} lands in Cat_∞^{perf} =idempotent complete small stable ∞-cats.
3. The ∞-category of stable ∞-cats is an accessible localization of the ∞-cat of spectral categories obtained by inverting the triangulated equivalences. In other words, the functor Ψ_{tri} has a fully faithful and accessible right adjoint Υ.
4. The ∞-category of stable idempotent complete ∞-cats is an accessible localization of the ∞-cat of spectral categories obtained by inverting the Morita equivalences.
5. The ∞-cats Cat_∞^{ex} and Cat_∞^{perf} are compactly generated, complete, and cocomplete.
6. Let I be a small category. Given a diagram $\mathcal{D}$ of small stable ∞-categories indexed by $N(I)$, there exists and I-diagram of pretriangulated spectral categories $\widetilde{\mathcal{D}}$ lifting $\mathcal{D}$.

4.2 Clarifying Remarks

1. This whole section is basically setting up technical machinery to allow us to lift stable ∞-categories to spectral categories and make arguments with these more rigid objects. This is the content of (3) and (4) above.

2. Given a small stable idempotent complete ∞-category A, we have that the conunit of the adjunction $\Psi_{perf}\Upsilon \to Id$ is a natural equivalence. Thus $A \cong \Psi_{perf}\Upsilon(A) \cong Idem \circ \Psi_{tri}\Upsilon(A)$, and recall that idempotent completion of a small stable ∞-cat can be modeled as $A \mapsto Fun^{ex}(A^{op}, S_\infty)^\omega$. It is in this sense that small stable idempotent complete ∞-categories are ∞-categories of modules.

4.3 Background Material

4.3.1 Accessible Localizations

Lemma 33. *If a right adjoint is full and faithful, the counit is an isomorphism.*

Proof. By definition of an adjunction $R \xrightarrow{\eta L} RLR \xrightarrow{R\epsilon} R$ is the identity, so that $R\epsilon$ is an isomorphism. Thus ϵ is an isomorphism since R is fully faithful. □

Remark 34. The analogous result holds in the ∞-categorical setting.

Definition 35. An $(\infty,1)$-functor $F : C \to D$ is accessible if C is an accessible $(\infty,1)$-category and there is a regular cardinal κ s.t. F preserves κ-small filtered colimits.

Remark 36. If an $(\infty,1)$-functor between accessible $(\infty,1)$-categories has a left or right adjoint $(\infty,1)$-functor, then it is itself accessible.

Definition 37. An $(\infty,1)$-functor $L : C \to C_0$ is called a (reflective) localization of the $(\infty,1)$-category C if it has a right adjoint $(\infty,1)$-functor $i : C_0 \hookrightarrow C$ that is full and faithful.

Definition 38. A localization is accessible if the localization functor is an accessible functor.

5 Exact Sequences

5.1 Summary

5.1.1 Important definitions

Term	pg	Loose Definition
Verdier quotient B/A	32	The cofiber of a fully faithful functor $f : A \to B$ in Pr^L_{St}
Exact sequence of presentable stable ∞-categories $A \to B \to C$	34	The composite is trivial, $A \to B$ is fully faithful, $B/A \to C$ is an equivalence.
Exact sequence in $Cat^{ex(\kappa)}_\infty$	35	If applying $Ind_\kappa(\_)$ gives an exact sequence in the above sense.
Split exact sequence $A \to B \to C$ in $Cat^{ex(\kappa)}_\infty$	37	It is exact (in the above sense) and there exist a right adjoint $B \to A$ with unit of adjunction being a natural iso and a right adjoint $C \to B$ with counit of adjunction being a natural iso
Exact sequence of spectral categories	38	If applying $N(\Omega^\infty Mod(\_)^{cf})$ gives an exact sequence of presentable stable ∞-categories.
$Split(Cat^{ex}_\infty)$ and $Split(Cat^{perf}_\infty)$	39	Subcategory of $Fun(\Delta^2, Cat^{ex}_\infty)$ consisting of split exact sequences. Similarly for the other case.
Strict exact sequence of small stable ∞-categories	40	An exact sequence of the form $A \to B \to B/A$ such that $A \to B$ is the inclusion of a full subcategory and every object of B that is a summand of an object of A is also in A.

5.1.2 Notation

- $Cat_\infty^{ex(\kappa)}$ is the ∞-category of κ-cocomplete stable ∞-categories and κ-small colimit preserving functors.

5.1.3 Key Results

1. If $A \to B$ is a fully faithful functor between presentable stable ∞-categories, the Verdier quotient B/A is the Bousfield localization B at the class of morphisms whose cones lie in the essential image of A.

2. Let $A \to B$ be a fully faithful functor between presentable stable ∞-categories. Then $Ho(A)/Ho(B) \to Ho(A/B)$ is an equivalence.

3. A functor between stable ∞-categories is fully faithful (*resp.* an equivalence) if and only if the induced functor on the homotopy categories is fully faithful (*resp.* an equivalence). No need to assume presentableness here.

4. A sequence in $Cat_\infty^{ex(\kappa)}$ is exact if and only if the induced sequence on homotopy categories is exact in the classical sense.

5. This proposition will be used later for the construction of nonconnective K-theory : let $A \to B \to C$ be an exact sequence of small stable ∞-categories. Then for any infinite regular cardinal κ, $Ind(A)^\kappa \to Ind(B)^\kappa \to Ind(C)^\kappa$ is an exact sequence of idempotent complete small stable ∞-categories.

6. $Split(Cat_\infty^{perf})$ is accessible.

7. Any split exact sequence is equivalent to a strict exact sequence.

5.2 Clarifying Remarks

1. In this section, a κ-continuous functor is defined to be one that preserves κ-filtered colimits.

2. A fully faithful functor of stable ∞-categories descend to a fully faithful triangulated functor on their homotopy categories.

5.3 Background Material

5.3.1 Classical notion of exactness

Remark 39. All categories and functors we consider here are triangulated.

Definition 40. A sequence of functors $A \to B \to C$ is exact if the composite is zero, $A \to B$ is fully faithful, and the induced functor on the Verdier quotient $B/A \to C$ is cofinal.

Definition 41. If A is a triangulated subcategory of B, then the Verdier quotient B/A is the universal triangulated category with a functor $B \to B/A$ such that every object of A is isomorphic to 0 under the functor.

Definition 42. A functor $C' \to C$ is cofinal if it becomes an equivalence after idempotent completion. Equivalently, if every object of C is a summand of an object in the image.

6 Additivity

6.1 Summary

6.1.1 Important definitions

Term	pg	Loose Definition
Additive invariant $E : Cat_\infty^{ex} \to D$	41	A functor where D is a stable presentable ∞-category, such that it inverts Morita equivalence (in the sense of Section 2), preserves filtered colimits, and is additive : every split exact sequence $A \to B \to C$ induces an equivalence $E(A) \vee E(C) \simeq E(B)$

6.1.2 Notation

- $Fun_{add}(Cat_\infty^{ex}, D)$ is the ∞-category of additive invariants with values in D.
- $Pre((Cat_\infty^{perf})^\omega)_*$ is the ∞-category $Fun(((Cat_\infty^{perf})^\omega)^{op}, \mathcal{T}_{\infty,*})$, presheaves of pointed spaces.
- $\phi : Cat_\infty^{perf} \to Pre((Cat_\infty^{perf})^\omega)_*$ is defined by first applying the Yoneda embedding and then restricting the domain to $(Cat_\infty^{perf})^\omega$
- $\mathcal{M}_{add}^{un}$ is the localization of $Pre((Cat_\infty^{perf})^\omega)_*$ with respect to the maps $\phi(B)/\phi(A) \to \phi(C)$ where $A \to B \to C$ is an element of a fixed set of representatives of split exact sequences in $(Cat_\infty^{perf})^\omega$. Let γ denote the localization functor.
- $\mathcal{U}_{add}^{un}$ is the composite :

$$Cat_\infty^{ex} \overset{Idem(\_)}{\to} Cat_\infty^{perf} \overset{\phi}{\to} Pre((Cat_\infty^{perf})^\omega)_* \overset{\gamma}{\to} \mathcal{M}_{add}^{un}$$

- $\mathcal{M}_{add}$ is the stabilization of $\mathcal{M}_{add}^{un}$. Recall this is obtained by taking the category of spectrum objects in $\mathcal{M}_{add}^{un}$.
- $\mathcal{U}_{add}$ is the composite

$$Cat_\infty^{ex} \overset{\mathcal{U}_{add}^{un}}{\to} \mathcal{M}_{add}^{un} \overset{\Sigma^\infty}{\to} \mathcal{M}_{add}$$

6.1.3 Key Results

1. "Unstable universal additive invariant" : The functor $\mathcal{U}_{add}^{un}$ inverts Morita equivalences, preserves filtered colimits and sends split exact sequences to cofiber sequences. Moreover, $\mathcal{U}_{add}^{un}$ is universal with respect to these properties. For any presentable pointed ∞-category D, there is an equivalence of ∞-categories

$$Fun^L(\mathcal{M}_{add}^{un}, D) \simeq Fun_{add}^{un}(Cat_\infty^{ex}, D)$$

   where the RHS denotes the full subcategory of $Fun(Cat_\infty^{ex}, D)$ of functors with the above listed properties.

2. "Stable universal additive invariant" : $\mathcal{U}_{add}$ is the unviersal additive invariant. Given any presentable stable ∞-category D, there is an equivalence of ∞-categories

$$Fun^L(\mathcal{M}_{add}, D) \simeq Fun_{add}(Cat_\infty^{ex}, D)$$

6.2 Clarifying Remarks

1. Examples of additive invariants : analogues of algebraic K-theory and topological Hochschild cohomology. They will be constructed in later sections.
2. There is an alternate description of the $\mathcal{M}_{add}$ by stablizing spaces first :

$$\begin{aligned} Stab(Pre((Cat_\infty^{perf})^\omega)_*) &= Stab(Fun(((Cat_\infty^{perf})^\omega)^{op}, \mathcal{T}_{\infty *,})) \\ &\simeq Fun(((Cat_\infty^{perf})^\omega)^{op}, Stab(\mathcal{T}_{\infty,*})) \\ &\simeq Fun(((Cat_\infty^{perf})^\omega)^{op}, S_\infty) \end{aligned}$$

   So there is a natural functor $\psi : Cat_\infty^{perf} \to Stab(Pre((Cat_\infty^{perf})^\omega)_*)$. Then $\mathcal{M}_{add}$ can be described as the localization of $Stab(Pre((Cat_\infty^{perf})^\omega)_*)$ at $\psi(A)/\psi(B) \to \psi(C)$ for all split exact sequence representatives we considered earlier.

6.3 Background Material

It may be helpful to look at Chapter 1 Section 4 of HA for details about stabilization and spectrum objects.

7 Connective K-Theory

7.1 Summary

7.1.1 Important definitions

Term	pg	Loose Definition
Gap($[n],\mathcal{C}$) (Waldhausen construction)	44	Full subcat of $Fun(N(Ar[n]),\mathcal{C})$ spanned by F s.t. $F(i,i)=0$ and if $i<j<k$, then $F(j,k)=p.o.(F(j,j)\leftarrow F(i,j)\rightarrow F(i,k))$.

7.1.2 Notation

- Ar$[n]$ is the category of arrows in $[n]$.
- $S^\infty_\bullet\mathcal{C}$ is the simplicial ∞-cat defined by $S^\infty_n\mathcal{C}=Gap([n],\mathcal{C})$.
- $\Omega|(S^\infty_\bullet\mathcal{C})_{iso}|$ is the ∞-categorical version of Waldhausen's K-theory space. Here iso means core.
- $(S^\infty_\bullet)^n$ is the nth iteration of the $S_\bullet$ construction. Makes sense because the output is a pted ∞-cat with fintie colimts.
- $|((S^\infty_\bullet)^n(\mathcal{C}))_{iso}|$ denotes the levels of the Waldhausen K-theory spectrum.
- Given a small stable ∞-cat $\mathcal{A}$, $K^\omega_\mathcal{A}$ is the object
$$\mathcal{B}\mapsto|(S^\infty_\bullet(Fun^{ex}(\mathcal{B},Idem(\mathcal{A}))))_{iso}|$$
in $Pre((Cat^{perf}_\infty)^\omega)_*$ and $K_\mathcal{A}$ is the object
$$\mathcal{B}\mapsto K(Fun^{ex}(\mathcal{B},Idem(\mathcal{A})))$$
in $Pre_{S_\infty}((Cat^{perf}_\infty)^\omega)$.

7.1.3 Key Results

1. Let $\mathcal{C}$ be an ∞-category with finite colimits. Then for each n, the forgetful functor
$$Gap([n],\mathcal{C})\rightarrow Fun(\Delta^{1,2,\dots,n},\mathcal{C})$$
is an equivalence of ∞-categories (and $\Delta^{1,2,\dots,b}\approx N([n-1])$).
2. Let $\mathcal{A}$ be a combinatorial simplicial model category and $C\subset\mathcal{A}$ a full subcategory. Then for each n, the induced map
$$N(C^{Ar[n]})^{cf})\rightarrow Fun(N(Ar[n]),N(C^{cf}))$$
is a categorical equivalence of simplicial sets.
3. Let $\mathcal{A}$ be a simplicial model category and $C\subset\mathcal{A}$ a small full subcategory of the cofibrants which admits all homotopy pushouts and is a Waldhausen category via the model structure on $\mathcal{A}$. Then there is an equivalence of spectra
$$K(C)\cong K(N((C)^{cf}))$$
which is natural is weakly exact functors.
4. A corollary of the above is: Let $\mathcal{C}$ be a small pretriangulated spectral category and let $\mathcal{M}_C$ denote the category of perfect $\mathcal{C}$-modules with its Waldhausen structure induced by the model structure on $\mathcal{C}$-modules. Then there is an isomorphism in the stable category
$$K(\mathcal{M}_C)\cong K(\psi_{perf}\mathcal{C}).$$
This is key because every category in Cat^{perf}_∞ is equivalent to $\psi_{perf}\mathcal{C}$ for some spectral category $\mathcal{C}$ by section 4 results.

5. The algebraic K-theory functor
$$K : Cat_\infty^{perf} \to S_\infty$$
is an additive invariant.

6. Let $\mathcal{A}$ be a small stable ∞-category and $\mathcal{B}$ be a compact idempotent-complete small stable ∞-category. Then there is a natural equivalence of spectra
$$Map(\mathcal{U}_{add}(\mathcal{B}), \mathcal{U}_{add}(\mathcal{A})) \cong K(Fun^{ex}(\mathcal{B}, Idem(\mathcal{A}))).$$
When $\mathcal{B}$ is the small stable ∞-cat S_∞^ω of compact spectra, there is a natural equivalence of spectra
$$Map(\mathcal{U}_{add}(S^\omega), \mathcal{U}_{add}(\mathcal{A})) \cong K(Idem(\mathcal{A})).$$
In particular, we have isomorphisms of abelian groups
$$Hom(\mathcal{U}_{add}(S_\infty^\omega), \Sigma^{-n}\mathcal{U}_{add}(\mathcal{A})) \cong K_n(Idem(\mathcal{A}))$$
in the triangulated category $Ho(\mathcal{M}_{add})$.

7. Let $\mathcal{A}$ be a small stable ∞-category. Then, we have a natural equivalence $\Sigma(\mathcal{U}_{add}^{un}(\mathcal{A})) \cong K_\mathcal{A}^\omega$ in $\mathcal{M}_{add}^{un}$ and a natural equivalence $\Sigma\mathcal{U}_{add}(\mathcal{A}) \cong \Sigma K_\mathcal{A}$ in $\mathcal{M}_{add}$.

8. Let $\mathcal{A}$ be a small stable ∞-category. Then the presheaves $K_\mathcal{A}^\omega$ and $K_\mathcal{A}$ are local, i.e. given any split exact sequence $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ in $\mathcal{E}$, the induced maps of spectra
$$map(\phi(\mathcal{D}), K_\mathcal{A}^\omega) \xrightarrow{\sim} Map(\phi(\mathcal{C})/\phi(\mathcal{A}), K_\mathcal{A}^\omega)$$
$$map(\psi(\mathcal{D}), K_\mathcal{A}) \xrightarrow{\sim} Map(\psi(\mathcal{C})/\psi(\mathcal{A}), K_\mathcal{A}^\omega)$$
are equivalences.

7.2 Clarifying Remarks

- The structure maps in the K-theory spectrum $S^1 \wedge (C_{iso}) \to |(S_\bullet^\infty \mathcal{C})_{iso}|$ are given by the following facts:
  1. $Gap([1], \mathcal{C})$ is equivalent to $\mathcal{C}$
  2. $Gap([0], \mathcal{C})$ is equivalent to pt
  3. The realization of a simplicial space K_n whose 0-simplices are a point contains a copy of $S^1 \wedge K_1$ (just think about the formula, we product with an interval, then crush the two ends).
- The corepresentability of K-theory says that, as a functor $Cat_\infty^{perf} \to S_\infty$, K is equivalent to the composite functor $Map(U_{add}(S_\infty^\omega), -) \circ U_{add}$. In particular, under the equivalence $(U_{add})^* : Fun^L(\mathcal{M}_{add}, S_\infty) \xrightarrow{\sim} Fun_{add}(Cat_\infty^{perf}, S_\infty)$, we can think of K as the colimit preserving functor $Map(U_{add}(S_\infty^\omega), -) : \mathcal{M}_{add} \to S_\infty$. Note that I'm restring to idempotent complete categories because I don't want to add the idempotent completion functor to the composite above.
- To summarize the results of this section: K theory is an additive invariant valued in spectra. Thus, it corresponds to a colimit preserving functor $\mathcal{M}_{add} \to S_\infty$. The functor corresponding to K is the representable functor $Map(U_{add}(S_\infty^\omega), -)$.

7.3 Background Material

7.3.1 Arrow Categories

See Charles' Notes section 7.6 for a description of the twisted arrow category, and note the remark just before 7.7: we have to consider spans such that all squares in the span are pullback squares if we want a quasi-category.

To get the arrow category from the twisted arrow category, we just flip all the arrows pointing left. Once we've done this, it's pretty clear pictorially what the nerve will be. We'll do this process in an example below:

The category $[2]^{tw}$ can be visualized as:

$$\begin{tikzcd}
& & 02 \arrow[dl] \arrow[dr] & & \\
& 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22
\end{tikzcd}.$$

Now we flip the arrows pointing left to obtain $Ar([2])$:

$$\begin{tikzcd}
& & 02 \arrow[dr] & & \\
& 01 \arrow[ur] \arrow[dr] & & 12 \arrow[dr] & \\
00 \arrow[ur] & & 11 \arrow[ur] & & 22
\end{tikzcd}.$$

Filling in the composites, we get

$$\begin{tikzcd}
& & 02 \arrow[dr] & & \\
& 01 \arrow[ur] \arrow[dr] \arrow[rr] & & 12 \arrow[dr] & \\
00 \arrow[ur] \arrow[rr] & & 11 \arrow[ur] \arrow[rr] & & 22
\end{tikzcd}.$$

making it more or less clear what the non-degenerate simplices of $N(Ar[2])$ are.

Now a functor out of this category can be written as:

$$\begin{tikzcd}
& & X_{02} \arrow[dr] & & \\
& X_{01} \arrow[ur] \arrow[dr] & & X_{12} \arrow[dr] & \\
X_{00} \arrow[ur] & & X_{11} \arrow[ur] & & X_{22}
\end{tikzcd}.$$

When we define $Gap([2], \mathcal{C})$, we have the additional requirements that the middle square is cocartesian and that X_{00}, X_{11}, X_{22} are the zero object. In particular, up to a contractible space of choices, the only data we need to specify is $X_{01} \to X_{02}$. Mapping such a functor to the functor obtained by restricting to the subcategory $01 \to 02$ yields a map $Gap([2], \mathcal{C}) \to Fun(\Delta^{1,2}, \mathcal{C})$ which is intuitively an equivalence.

7.3.2 K-theory via group completion

Remark 43. Motivation for some of this material comes from the following: given an honest monoid M, $\pi_1(BM)$ is the group completion of M. This is a consequence of the presentation of $\pi_1(BM)$ in terms of a maximal tree in M (which is in turn a consequence of Van Kampen). See IV.3.4 in the K-book and the corollaries in the K-book.

This is from Segal. Given a Γ-space A (see the background material of section 3 above), we can associate to it a spectrum with component spaces $A(1), BA(1), B^2A(1), \ldots$, where 1 is the set $\{1\}$, and BA is the Γ space s.t. for any finite set S, $BA(S)$ is the realization of the Γ-space $T \mapsto A(S \times T)$. The key here is that $B(-)$ of a Γ-space is again a Γ space, so that we can iterate the construction.

The important example to keep in mind is the Γ-space A which is $A(1) = \coprod BGL_n$, $A(2) = \coprod_{n\geq 0}(EG_m \times EG_n \times EG_{m+n})/(G_m \times G_n), \ldots$.

One important remark in the paper is that an H-space X has a homotopy inverse if it's grouplike and it has a numerable covering by sets which are contractible in X. The realization of a simplicial space has such a numerable covering if the space of 0-simplices is contractible.

Now $BA(1)$ is the realization of A, and by definition of a Γ-space, A_0 is contractible. Thus the product on $B^kA(1)$ has a homotopy inverse for any $k \geq 1$, and hence (by prop 1.4 in the paper), $B^kA(1) \to \Omega B^{k+1}A(1)$ is a homotopy equivalence. Thus the spectrum we formed above is ALMOST an Ω-spectrum, but it isn't necessarily true that $A(1) \to \Omega BA(1)$ is an equivalence. The spectrification of this prespectrum will be given by $\Omega BA(1), BA(1), B^2A(1), \ldots$, so we'd like to know what the precise relationship between $\Omega BA(1)$ and $A(1)$ is. Indeed, this is asking for the comparison between the monoid $[X, A(1)]$ and the zeroth cohomology group $[X, \Omega BA(1)]$. Indeed, going back to our above example, we see that $[X, A(1)]$ should be the monoid of vector bundles, so we'd hope that $[X, \Omega BA(1)]$ would be the group completion of this monoid. The upshot is that in nice cases (the ones we care about) this is true.

In particular, given a Γ-spaces A, one can naturally associate another Γ-space A' with a map $A \to A'$ with the following properties:

- $\pi_0(A')$ is the abelian group associated to the monoid $\pi_0(A)$; and
- $BA \to BA'$ is a weak equivalence of spectra.

7.3.3 Chunks

Definition 44. Say that a model category $\mathcal{S}$ is *excellent* if it is equipped with a symmetric monoidal structue and satisfies the following conditions:

1. The model category $\mathcal{S}$ is combinatorial.
2. Every monomorphism in $\mathcal{S}$ is a cofibration, and the collection of cofibrations is stable under products.
3. THe collection of weak equivalences in $\mathcal{S}$ is stable under filtered colimits.
4. The monoidal structure on $\mathcal{S}$ is compatible with the model structure. In other words, the tensor product functor $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a left Quillen bifunctor.
5. The monoidal model category $\mathcal{S}$ satisfies the invertibility hypothesis.

Remark 45. The invertibility hypothesis essentially says that inverting a morphism f in an $\mathcal{S}$-enriched category C does not change the homotopy type of C when f is already invertible up to homotopy.

Example 46. The canonical example is the category of simplicial sets when endowed with the Kan model structure and cartesian product.

Definition 47. Let $\mathcal{S}$ be an excellent model category, and let A be combinatorial $\mathcal{S}$-enriched model category. A *chunk* of $\mathcal{A}$ is a full subcategory $\mathcal{U} \subset \mathcal{A}$ with the following properties:

1. Let A be an object of $\mathcal{U}$ and let $\{\phi_i : A \to B_i\}_{i\in I}$ be a finite collection of morphisms in $\mathcal{U}$. Then there exists a factorization
$$A \xrightarrow{p} \overline{A} \xrightarrow{q} \prod_{i\in I} B_i$$
of the product map $\prod_{i\in I} \phi_i$, where p is a trivial cofibration, q is a fibration, and $\overline{A} \in \mathcal{U}$. Moreover, this factorization can be chosen to depend functorially on the collection $\{\phi_i\}$ via an $\mathcal{S}$-enriched functor.

2. Let A be an object of $\mathcal{U}$ and let $\{\phi_i : B_i \to A\}_{i\in I}$ be a finite collection of morphisms in $\mathcal{U}$. Then there exists a factorization
$$\coprod_{i\in I} B_i \xrightarrow{p} \overline{A} \xrightarrow{q} A$$
of the coproduct map $\coprod_{i\in I} \phi_i$, where p is a cofibration, q is a trivial fibration, and $\overline{A} \in \mathcal{U}$. Moreover, this factorization can be chosen to depend functorially on the collection $\{\phi_i\}$ via an $\mathcal{S}$-enriched functor.

Definition 48. Let $\mathcal{S}$ be an excellent model category, $\mathcal{A}$ a combinatorial $\mathcal{S}$-enriched model category, and C an $\mathcal{S}$-enriched category. We will say that a full subcategory $\mathcal{U} \subset \mathcal{A}$ is a C-chunk of $\mathcal{A}$ if it is a chunk of $\mathcal{A}$ and the subcategory $\mathcal{U}^C$ is a chunk of $\mathcal{A}^C$. Here we regards $\mathcal{A}^C$ as endowed with the projective model structure.

The main theorem about chunks is the following ridigification theorem:

Theorem 8. *(HTT 4.2.4.4) Let S be a small simplicial set, $\mathcal{C}$ a small simplicial category, and $u : \mathfrak{C}[S] \to \mathcal{C}$ an equivalence. Suppose that $\mathbf{A}$ is a combinatorial simplicial model category and let $\mathcal{U}$ be a $\mathcal{C}$-chunk of $\mathbf{A}$. Then the induced map*
$$N((\mathcal{U}^{\mathcal{C}})^\circ) \to Fun(S, N(\mathcal{U}^\circ))$$
is a categorical equivalence of simplicial sets.

This is a rigidification in the sense that the category on the left is the nerve of a category of honest (simplicial) functors.

8 Localization

8.1 Summary

8.1.1 Important definitions

Term	pg	Loose Definition
Localizing invariant $E : Cat^{ex}_\infty \to D$	41	A functor where D is a stable presentable ∞-category, such that it inverts Morita equivalences , preserves filtered colimits, and satisfies localization: every exact sequence $A \to B \to C$ becomes a cofiber sequence $E(A) \to E(B) \to E(C)$

8.1.2 Notation

- κ is an infinite regular cardinal larger than ω.
- $(Cat^{ex}_\infty)^\kappa$ will denote the category of κ-compact small stable ∞-categories.

8.1.3 Key Results

- There's a category M_{loc} and a functor $U_{loc} : Cat^{ex}_\infty \xrightarrow{M}_{loc}$ such that, given any stable presentable ∞-category D, there's an equivalence of ∞-categories
$$(U_{loc})^* : Fun^L(M_{loc}, D) \xrightarrow{\sim} Fun_{loc}(Cat^{ex}_\infty, D).$$

8.2 Clarifying Remarks

See background material

8.3 Background Material

8.3.1 Localization Theorems

Definition 49. Define $K(X \text{ on } Z)$ to be the K-theory space of the Waldhausen category $Ch_{perf,Z}(X)$ of perfect complexes on X which are exact on U.

Theorem 9. *(Thomason-Trobaugh localization)*

Let X be a quasi-compact, quasi-separated scheme, and let U be a quasi-compact open in X with complement Z. Then $K(X \text{ on } Z) \to K(X) \xrightarrow{j^*} K(U)$ is ALMOST a homotopy fibration, and there is a long exact sequence

$$K_{n+1}(U) \xrightarrow{\partial} K_n(X \text{ on } Z) \to K_n(X) \to K_n(U) \to \dots$$

ending in

$$K_0(X \text{ on } Z) \to K_0(X) \to K_0(U).$$

Because the map $K_0(X) \to K_0(U)$ is not surjective in general, this can't be an honest homotopy fibration.

Here the K-theory of a scheme X is the K-theory of $Ch_{perf}(X)$. Because X assumed to be quasi-compact, the inclusion $Ch^b_{perf}(X) \subset Ch_{perf}(X)$ induces an equivalence on derived categories, so that the K-theory of X is $K(Ch^b_{perf}(X))$.

Non-connective K-theory exists to rectify the "almost" in the theorem above.

Next we'll discuss Neeman's generalization of this theorem. First there's a notion of K-theory for triangulated categories such that, if A is an abelian category and $D^b(A)$ is its bounded derived category, then $K(D^b(A))$ agrees with $K(A)$ (a consequence of the "theorem of the heart").

Here's the (somewhat lengthy) theorem of Neeman, note that in the original source, Neeman uses the French word épaisse (which means thick) for thick subcategories:

Theorem 10. *(Neeman) Suppose S is any triangulated category closed with respect to arbitrary coproducts. Suppose that the subcategory S^c of $(\omega-)$ compact objects is small, and that S is the smallest localizing subcategory containing S^c. Suppose furthermore that there is a set R of objects in S^c, and $\mathcal{R}$ is the smallest localizing category containing R. Let T be the quotient category $S/\mathcal{R}$. Then the map $\mathcal{R} \to S$ carries $\mathcal{R}^c$ to S^c, the map $S \to T$ carries S^c to T^c, the natural functor $S^c/\mathcal{R}^c \to T^c$ is fully faithful, and T^c is the thick closure of the image.*

What in the blazes does this have to do with the TT localization theorem? Let $S = D(X), T = D(U), \mathcal{R} = D(X \text{ on } Z)$ where D here means the derived category of the abelian category of quasi-coherent sheaves. Passing to the compact objects will yield the perfect complexes, which is what we want to take the K-theory of. By the theorem of the heart and Quillen's localization theorem, there's a homotopy fibration sequence $K(\mathcal{R}^c) \to K(S^c) \to K(S^c/\mathcal{R}^c)$. However, the theorem above states that, although $S^c/\mathcal{R}^c \neq T^c$ in general, they're very closely related.

To make more precise how closely related the two are, we need one more lemma:

Lemma 50. *Suppose T is a triangulated category, and S is a full triangulated subcategory whose thick closure is all of T. Then a delooping of the map $K(S) \to K(T)$ is a covering space, and is a homotopy equivalence if and only if $S \cong T$, i.e. the inclusion $S \hookrightarrow T$ is an equivalence of categories.*

Now in our case, we can apply this theorem to the inclusion $S^c/\mathcal{R}^c$ as above to see that there's an isomorphism $\pi_n \Sigma K(S^c/\mathcal{R}^c) \cong \pi_n \Sigma K(T^c)$ for all $n \geq 2$, which by stability yields an isomorphism $\pi_n K(S^c/\mathcal{R}^c) \cong \pi_n K(T^c)$ for all $n \geq 1$. This recovers the TT-localization theorem as stated above.

8.3.2 Negative K-theory (Notes from Jeremiah's Talk)

Remark 51. Notation: $K(-)$ is the connective K-theory spectrum, and $\mathbb{K}(-)$ is the non-connective K-theory spectrum.

Theorem 11. *There's a LES of K-groups*

$$\cdots \longrightarrow K_{n+1}(\mathbb{Q}) \longrightarrow \bigoplus_p K_n(\mathbb{F}_p) \longrightarrow K_n(\mathbb{Z}) \longrightarrow K_n(\mathbb{Q}) \longrightarrow \dots$$

Thanks to hard work about K theory of fields, gives lots of information about $K_*(\mathbb{Z})$.

This theorem is (non-trivially) a special case of:

Theorem 12. *(Quillen) Given $B \subset A$ a Serre subcategory of a small abelian category, then we have a cofiber sequence of spectra*

$$K(B) \to K(A) \to K(A/B).$$

Note that in particular, $K_0(A) \twoheadrightarrow K_0(A/B) \to 0$.

Remark 52. This is a premier computational tool. However, not all K-groups are equivalent to K-groups of some abelian category. Notably, K_* of a singular scheme, K_* of a ring spectrum, etc.

Remark 53. If $A \to B \to C$ is an exact sequence in Cat_∞^{perf}, it's not true in general that

$$K_0(B) \to K_0(C) \to 0$$

is exact. This is the first obstruction to a localization sequence.

Question 2. Is there an easy counterexample? I think possibly a singular cubic will work.

Fix: Negative K-theory.

Theorem 13. *If $A \to B \to C$ is an exact sequence in Cat_∞^{ex}, then*

$$\mathbb{K}(A) \to \mathbb{K}(B) \to \mathbb{K}(C)$$

is a cofiber sequence.

Construction 1. Idea: Want to find a C' s.t. $K(C') \cong *$ and a map $C \to C'$. Expect exact sequence

$$K_0(C') \longrightarrow K_0(C'/C) \longrightarrow K_{-1}(C) \longrightarrow K_{-1}(C')$$

but since $K_*(C') = 0$, just define $K_{-1}(C) = K_0(C'/C)$. Should be functorial: $C' = \mathcal{F}(C)$. Let $\Sigma C = cofib(C \to F(C))$, then $K_{-n}(C) = K_0(\Sigma^{(n)} C)$.

Definition 54. Say C is *flasque* if there are exact functors $F_1, F_2 : C \to C$ and equivalence

$$id \oplus F_1 \cong F_2,$$

and $(F_1)_* = (F_2)_* : K_*(C) \to K_*(C)$.

Then $id + (F_1)_* : K_*(C) \to K_*(C) \implies K_*(C) = 0$.

Example 55.

1. $Ind_\kappa(C)$, $\kappa > \omega$.
2. $F = (x \mapsto \oplus_{\mathbb{N}} x)$. Then $id \oplus F \cong F$, the Eilenberg-swindle, so $Ind_\kappa(\mathcal{C})$ is κ-acyclic.
3. A ring R is flasque if there's an R-bimodule M which is f.g. projective as a right R-mod, and there's a bimodule isomorphism $R \oplus M \cong M$. Then $Mod_R, Proj_R$ are flasque.
4. S any ring, $C(R) \subseteq End_S(S^\infty)$ row-finite column-finite infinite matrices (the cone ring). This is flasque.
5. The suspension ring $\Sigma S = C(S)/M(S)$, where $M(S)$ denotes the ring of finite matrices. Then $K_{-n}(S) = K_0(\Sigma^{(n)} S)$.

Remark 56. This construction works just fine for connective ring spectrum.

Construction 2. Model for K-theory of connective rings:

Can define

$$\begin{tikzcd} GL_n(R) \arrow[r] \arrow[d] & M_n(R) \arrow[d] \\ GL_n(\pi_0 R) \arrow[r] & M_n(\pi_0 R) \end{tikzcd}$$

Fact: $K_0(R) \cong K_0(\pi_0 R)$. Here the K-theory of a ring spectrum is the K-theory of the category of compact projective R-mods. Define $K_*(R) := \pi_*[K_0(\pi_0 R) \times BGL(R)^+]$.

Have a map $K_*(R) \to K_*(\pi_0(R))$. Compare the "ring suspensions":

have $\pi_0(\Sigma_{ring} R) = \Sigma_{ring}\pi_0 R$, so

$$\begin{tikzcd} K_{-1}(R) \arrow[r] \arrow[d] & K_0(\Sigma_{ring} R) \arrow[d] \\ K_{-1}(\pi_0 R) \arrow[r] & K_0(\pi_0 \Sigma_{ring} R) \end{tikzcd}$$

Upshot: $K_{-n}(R) = K_{-n}(\pi_0 R)$ for R connective.

Definition 57. $C \in Cat_\infty^{ex}$. Define $\mathcal{F}_k(C) = Ind_\kappa(C)$, $\Sigma_\kappa C = \mathcal{F}_\kappa(C)/C$, $K_{-n}(C) = K_0(\Sigma_\kappa^{(n)} C)$, $\mathbb{K}(C) = \operatorname{colim}_n \Omega^n K(\Sigma_\kappa^{(n)} C)$. This definition doesn't make any multiplicative properties apparent.

What is known:

1. K_{-n}(noetherian regular ring/scheme) $= 0$
2. K_{-1}(henselian ring) $= 0$ (hard-Drinfeld)
3. Weibel's conjecture: X: noetherian scheme of dimension $= d$. Then $K_{-n}(X)$ are zero if $n > d$. This is a theorem now ($d = 1$, Bass), ($d = 2$, Weibel), (X variety over field of char 0, Haesemeyer-Cortinas-Schlicting-Weibel), (X/F char$(F) > 0$ assuming res of singularites, Geisser-Hesselholt/Krishna), (whole thing, Kerz-Strunk-Tamme Nov '16).
4. Schlicting's conjecture: $K_{-n}(A) = 0$ if A (small) abelian. Still open in general. True if A is noetherian. Also true for $n = 1$ for any small abelian category.
5. If $E \in Cat_\infty^{ex}$ has a bounded t-structure, $K_{-1}(E) = 0$. If ... with E^{heart} noetherian, $K_{-n}(E) = 0$ for all $n \geq 1$.

Idea of $K_{-1}(A) = 0$.

$$\begin{tikzcd} D^b(A) \arrow[r] \arrow[d] & D^-(A) \arrow[d] \\ D^+A \arrow[r] & D(A) \end{tikzcd}$$

induces pushout square on K-theory. D^+A and D^-A are idempotent complete and K-theory acyclic. $K_*(A) = K_*(D^b(A))$, $K_0(D(A)) = K_{-1}(A)$.