STUFF ABOUT QUASICATEGORIES

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Contents

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(Note: this is a draft, which can change daily. It includes some unorganized material which might later be incorporated into the narrative, or removed entirely.)

1. Introduction to ∞-categories

I'll give a brief discussion to motivate the notion of ∞-categories.

1.1. **Groupoids.** Modern mathematics is based on sets. The most basic way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring R, we can consider the set $X(R)$ of tuples $(x, y, z) \in R^3$ which satisfy the equation $x^5 + y^5 = z^5$. We can express such sets as limits; for instance, $X(R)$ is the pullback of the diagram of sets

$$R \times R \xrightarrow{(x,y)\mapsto x^5+y^5} R \xleftarrow{z^5 \leftarrow\!\shortmid z} R.$$

Another way to construct new sets is by taking "quotients"; e.g., as sets of equivalence classes of an equivalence relations. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that; but some have "higher" structure standing behind them, which is often encoded in the form of a *groupoid*[1]. Here are some examples.

- Given a topological space X, we can define an equivalence relation on the set of points, so $x \sim x'$ if and only if there is a continuous path connecting them. The set of equivalence classes is the set $\pi_0 X$ of path components. Standing behind this equivalence relation is the *fundamental groupoid* $\Pi_1 X$, whose objects are points, and whose morphisms are homotopy classes of paths.
- Given any category C, there is an equivalence relation on the collection of objects, so that $X \sim Y$ if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the *core* of C, which is the groupoid with the same objects as C, and only isomorphisms between them.

Note that many interesting problems are about describing isomorphism classes; e.g., classifying finite groups of a given order, or principal G-bundles on a space. In practice, one learns that in trying to classify objects up to isomorphism, you have to have a good handle on the isomorphisms between such objects. So you need to know about the groupoid, even if it is not the primary object of interest.

You can think of a problem like "describe the groupoid $\mathrm{Bun}_G(M)$ of principal G-bundles on a space M" as a more sophisticated version of "find the set $X(R)$ of solutions to $x^5 + y^5 = z^5$ in the ring R". (In fact, the theory of "moduli stacks" exactly develops the analogy between these two.) To do this, you can imagine having a "groupoid-based mathematics", generalizing the usual set-based one. Here are some observations about this.

- We regard two sets as "the same" if they are *isomorphic*, i.e., if there is a bijection $f: X \to X'$ between them. Any such bijection has a unique inverse bijection $f^{-1}: X' \to X$.

  On the other hand, we regard two groupoids as "the same" if they are merely *equivalent*, i.e., if there is a functor $f: G \to G'$ which admits an inverse *up to natural isomorphism.* It is not the case that such an inverse up to natural isomorphism is unique.

  Although any equivalence of groupoids admits some kind of inverse, the failure to be unique leads to complications. For example, for every nice space M there is an equivalence of groupoids
$$f_M: \mathrm{Fun}(\Pi_1 M, G) \to \mathrm{Bun}_G(M).$$
  The functor f_M is natural in M, but there is no way to construct inverse functors f_M^{-1} which are also natural in M.

[1] I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].

- We can consider "solutions to equations" in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids; thus, we need to consider "weak" or "homotopy" limits.

  For example, suppose M is a space which is a union of two open sets U and V. The *weak pullback* of
  $$\mathrm{Bun}_G(U) \to \mathrm{Bun}_G(U \cap V) \leftarrow \mathrm{Bun}_G(V)$$
  is a groupoid, whose objects are triples (P, Q, α), where $P \to U$ and $Q \to U$ are G-bundles, and $\alpha\colon P|_{U\cap V} \xrightarrow{\sim} Q|_{U\cap V}$ is an isomorphism of G-bundles over $U \cap V$; the morphisms $(P, Q, \alpha) \to (P', Q', \alpha')$ are pairs $(f\colon P \to P', g\colon Q \to Q')$ are pairs of bundle maps which are compatible over $U \cap V$ with the isomorphisms α, α'. Compare this with the *strict pullback*, which consists of (P, Q) such that $P|_{U\cap V} = Q|_{U\cap V}$ as bundles; in particular, $P|_{U\cap V}$ and $Q|_{U\cap V}$ must be the *identical sets*.

  A basic result about bundles is that $\mathrm{Bun}_G(M)$ is equivalent to this weak pullback. The strict limit may fail to be equivalent to this; in fact, it is impossible to describe the strict pullback without knowing precisely what definition of G-bundle we are using, whereas the identification of weak pullback is insensitive to the precise definition of G-bundle.

These kinds of issues persist when dealing with higher groupoids and categories.

1.2. **Higher groupoids.** There is a category Gpd of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here; there are *natural transformations* between functors $f, f'\colon G \to G'$ of groupoids. That is, $\mathrm{Fun}(G, G')$ forms a category. We can consider the collection consisting of (0) groupoids, (1) *equivalences* between groupoids, and (2) natural isomorphisms between equivalences; this is an example of a *2-groupoid*[2] There is no reason to stop at 2-groupoids: there are n-groupoids, the totality of which are an example of an $(n+1)$-groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider ∞-groupoids.

It is difficult to construct an "algebraic" definition of n-groupoid. The approach which works best is to use homotopy theory. Every groupoid G has a classifying space BG. This is defined explicitly as a quotient space
$$G \mapsto BG := \Big(\coprod_{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n} \Delta^n \Big) / (\sim),$$
where we put in a topological n-simplex for each n-fold sequence of composable arrows in G, modulo certain identifications. It turns out (i) the fundamental groupoid of BG is equivalent to G, and (ii) the higher homotopy groups π_k of BG are trivial, for $k \geq 2$. We thus say that the space BG is a *1-type*. Furthermore, (iii) there is a bijection between *equivalence classes of groupoids up to equivalence* and *CW-complexes which are 1-types, up to homotopy equivalence*. (More is true, but I'll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should *define* n-groupoids as n-types (CW complexes with trivial homotopy groups in dimensions $> n$), with equivalences being homotopy equivalences. Removing the restriction on homotopy groups leads to modelling ∞-groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classiyfing space construction factors through a "combinatorial" construction, called the nerve. That is, we have
$$(G \in \mathrm{Gpd}) \mapsto (NG \in s\mathrm{Set}) \mapsto (|NG| = BG \in \mathrm{Top}),$$
where NG is the *nerve* of the groupoid, and is an example of a *simplicial set*; $|X|$ denotes the *geometric realization* of a simplicial set X. In fact, the nerve is a particular kind of simplicial

[2]More precisely, a "quasistrict 2-groupoid".

set called a *Kan complex*. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that ∞-groupoids are precisely the Kan complexes.

1.3. **∞-categories.** There are a number of approaches to defining ∞-categories. Here are two which build on top of the identification of ∞-groupoids with Kan complexes.

- A category C consists of a set $\operatorname{ob} C$ of objects, and for each pair of objects a set $\hom_C(x, y)$ of maps from x to y. If we replace the set $\hom_C(x, y)$ with a Kan complex (or more generally a simplicial set) $\operatorname{map}_C(x, y)$, we obtain a category *enriched* over Kan complexes (or simplicial sets). This leads to one model for ∞-categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category C, we have a simplicial set NC. In general, NC is not a Kan complex; however, it does land in a special class of simplicial sets, which are called quasicategories. This leads to another model for ∞-categories: *quasicategories*.

I'm going to focus on the second model.

1.4. **Prerequisites.** I assume only familiarity with basic concepts of category theory, such as those discussed in the first few chapters of [Rie16]. It is helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher's textbook).

1.5. **Historical remarks.** Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of ∞-categories. This development began with the work of Joyal, starting in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls "∞-categories"), recasting and extending Joyal's work significantly.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a "categorical equivalence" between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I follow here is essentially that of Joyal.

Basic notions

2. Simplicial sets

Quasicategories will be defined as a particular kind of *simplicial set*.

2.1. **The simplicial indexing category Δ.** We write Δ for the category whose

- **objects** are the non-empty totally ordered sets sets $[n] := \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and
- **morphisms** $f\colon [n] \to [m]$ are order preserving functions.

I'll use notation such as $f = \langle f_0 \cdots f_n \rangle\colon [n] \to [m]$ with $f_0 \leq \cdots \leq f_n$ to represent the function $k \mapsto f_k$. Morphisms in Δ are often called **simplicial operators**.

Because $[n]$ is an ordered set, we can also think of it as a category: the objects are the elements of $[n]$, and there is a unique morphism $i \to j$ if and only if $i \leq j$. Thus, morphisms in the category Δ are precisely the functors between the categories $[n]$. We can, and will, also think of $[n]$ as the category "freely generated" by the picture

$$0 \to 1 \to \cdots \to (n-1) \to n.$$

Arbitrary morphisms in $[n]$ can be expressed uniquely as iterated composites of arrows in the picture; identity morphisms are thought of as "0-fold composites".

2.2. *Remark.* There are distinguished simplicial operators called **face** and **degeneracy** operators:

$$d^i := \langle 0 \dots \widehat{i} \dots n\rangle\colon \quad [n-1] \to [n], \quad 0 \le i \le n,$$
$$s^i := \langle 0 \dots i, i, \dots n\rangle\colon [n+1] \to [n], \quad 0 \le i \le n.$$

All maps in Δ are composites of these operators, and in fact Δ can be described as the category *generated* by the above symbols, subject to a set of *relations* called the "simplicial identities". You can look those up.

2.3. **Simplicial sets.** A **simplicial set** is a functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, i.e., a contravariant functor (or "presheaf") from Δ to sets. It is typical to write X_n for $X([n])$, and call it the set of n**-simplices** in X.

Given a simplex $a \in X_n$, and a simplicial operator $f\colon [m] \to [n]$, I will write $af \in X_m$ as shorthand for $X(f)(a)$. That is, we'll think of simplicial operators as acting on simplices from the right. Then we have the identity $(af)f' = a(ff')$, since the functor is *contra*variant. If I want to put the simplicial operator on the left, I'll write $f^*(a) = af$.

If I give the simplicial operator explicitly, as $f = \langle f_0 \cdots f_m\rangle\colon [m] \to [n]$, then I might also indicate the action of f using subscripts:

$$af = a\langle f_0 \cdots f_n\rangle = a_{f_0 \cdots f_m}.$$

In particular, applying simplicial operators of the form $\langle i\rangle\colon [0] \to [n]$ gives the **vertices** $a_0, \dots, a_n \in X_0$ of an n-simplex $a \in X_n$, while applying simplicial operators of the form $\langle ij\rangle\colon [1] \to [n]$ for $i \le j$ gives the **edges** $a_{ij} \in X_1$ of a.

A simplicial set is a functor; therefore a map of simplicial sets is a natural transformation. Explicitly, a map $f\colon X \to Y$ between simplicial sets is a collection of functions $f\colon X_n \to Y_n$, $n \ge 0$, which commute with simplicial operators: $(fx)\delta = f(x\delta)$ for all $\delta\colon [m] \to [n]$ and $x \in X_n$.

I'll write $s\mathrm{Set}$ for the category of simplicial sets[3].

2.4. **Standard n-simplex.** The **standard n-simplex** Δ^n is the simplicial set defined by

$$\Delta^n := \mathrm{Hom}_{\Delta}(-, [n]).$$

That is, the standard n-simplex is exactly the functor *represented* by the object $[n]$. The *Yoneda lemma* tells us that there is a bijection

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^n, X) \to X_n,$$
$$g \mapsto g\iota_n,$$

where $\iota_n = \langle 01 \cdots n\rangle = \mathrm{id}_{[n]} \in (\Delta^n)_n$.

We will often use this bijection implicitly. Thus, we will often represent a simplex $x \in X_n$ by its representing map, which we denote with the same symbol: $x\colon \Delta^n \to X$.

2.5. *Exercise.* The representing map of $x \in X_n$ sends $\langle f_0 \cdots f_k\rangle \in (\Delta^n)_k$ to $x\langle f_0 \cdots f_k\rangle \in X_k$.

Maps between standard simplices correspond exactly to simplicial operators, so $f\colon [m] \to [n]$ gives a map $\Delta^f\colon \Delta^m \to \Delta^n$ by $g \mapsto gf$. We can regard this as describing a simplex $f \in (\Delta^n)_m$, so I'll usually abuse notation and call this map $f\colon \Delta^m \to \Delta^n$.

The standard 0-simplex Δ^0 is the terminal object in $s\mathrm{Set}$. Sometimes I write $*$ instead of Δ^0 for this object.

2.6. *Exercise.* The **empty simplicial set** $\varnothing$ is the functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ sending each $[n]$ to the empty set. Show that a simplicial set X is the empty simplicial set if and only if $\mathrm{Hom}(\Delta^0, X) = \varnothing$.

[3]Lurie [Lur09] uses Set_Δ to denote the category of simplicial sets.

2.7. **Discrete simplicial sets.** Simplicial set X is **discrete** if every simplicial operator induces a bijection $f^*\colon X_n \to X_m$.

2.8. *Exercise.* Show that X is discrete if and only if the tautological map $\coprod_{x\in\mathrm{Hom}(\Delta^0,X)} \Delta^0 \to X$ is an isomorphism.

2.9. *Exercise.* Show that if Y is a discrete simplicial set, the evident restriction map

$$\mathrm{Hom}_{s\mathrm{Set}}(X,Y) \to \mathrm{Hom}_{\mathrm{Set}}(X_0,Y_0)$$

is a bijection.

Given a set S, we obtain an associated discrete simplicial set defined by $[n] \mapsto S$ for all S, and sending all simplicial operators to the identity map.

2.10. **Some notation and pictures.** I'm going to extend this kind of language to arbitrary totally ordered sets. The point is that given *any* non-empty finite totally ordered set $S = \{s_0 < s_1 < \cdots < s_n\}$, there is a *unique* order preserving bijection $S \approx [n]$ for a unique $n \geq 0$. We write Δ^S for the simplicial set with $(\Delta^S)_k = \{\text{order preserving } [k] \to S\}$. Of course, there is a unique isomorphism $\Delta^S \approx \Delta^n$. We can also apply this idea when $S = \varnothing$, in which case $(\Delta^\varnothing)_k = \varnothing$.

This is especially convenient for subsets $S \subseteq [n]$ with induced ordering. In this case, Δ^S is naturally a *subcomplex* of Δ^n (i.e., a collection of subsets of the $(\Delta^n)_k$ closed under simplicial operators), while also being naturally isomorphic to $\Delta^{|S|-1}$.

For instance, any simplicial operator $f\colon [m] \to [n]$ factors through its image $S = f([m]) \subseteq [n]$, giving a factorization

$$[m] \xrightarrow{f_{\mathrm{surj}}} S \xrightarrow{f_{\mathrm{inj}}} [n]$$

of maps between ordered sets, and thus a factorization $\Delta^m \xrightarrow{\Delta^{f_{\mathrm{surj}}}} \Delta^S \xrightarrow{\Delta^{f_{\mathrm{inj}}}} \Delta^n$ of Δ^f.

2.11. *Exercise.* Show that $\Delta^{f_{\mathrm{inj}}}$ and $\Delta^{f_{\mathrm{surj}}}$ are respectively injective and surjective maps between simplicial sets. (This is not completely formal.)

When we draw a "picture" of Δ^n, we draw a geometric n-simplex, with vertices labelled by $0,\dots,n$: the faces of the geometric simplex correspond exactly to injective simplicial operators: these are called *non-degenerate* simplices. For each non-degenerate simplex f in Δ^n, there is an infinite collection of *degenerate* simplices with the same "image" as f. Here are some "pictures" of standard simplices, which show their non-degenerate simplices.

Δ^0:
$$\langle 0\rangle$$

Δ^1:
$$\langle 0\rangle \longrightarrow \langle 1\rangle$$

Δ^2:
$$\begin{tikzcd}
 & \langle 1\rangle \arrow[d] \\
\langle 0\rangle \arrow[ur] \arrow[dr] & \\
 & \langle 2\rangle
\end{tikzcd}$$

Δ^3:
$$\begin{tikzcd}
 & \langle 1\rangle \arrow[dd] \arrow[dr] & \\
\langle 0\rangle \arrow[ur] \arrow[dr] \arrow[rr] & & \langle 3\rangle \\
 & \langle 2\rangle \arrow[ur] &
\end{tikzcd}$$

We'll extend the notions of degenerate and non-degenerate simplices to arbitrary simplicial sets in §14.5.

3. Nerve

The nerve of a category is a simplicial set, which retains all the information of the original category. In fact, the nerve construction provides a full embedding of Cat, the category of categories, into sSet, which means that we are able to think of categories as just a special kind of simplicial set.

3.1. **Nerve of a category.** Our basic example of a simplicial set is the **nerve** of a category: given a small category C, let

$$(NC)_n := \mathrm{Hom}_{\mathrm{Cat}}([n], C),$$

the set of functors from $[n]$ to C. This is obviously a simplicial set, because functors $[m] \to [n]$ are the same thing as simplicial operators.

We observe the following.

- $(NC)_0$ is canonically identified with the set of objects of C.
- $(NC)_1$ is canonically identified with the set of morphisms of C.
- The operators $\langle 0\rangle^*, \langle 1\rangle^* \colon (NC)_1 \to (NC)_0$ assign a morphism to its source and target respectively.
- The operator $\langle 00\rangle^* \colon (NC)_0 \to (NC)_1$ assigns an object to its identity map.
- $(NC)_2$ is in bijective correspondence with pairs (f, g) of morphisms such that gf is defined, i.e., such that $g_0 = f_1$. This bijection is given by sending $a \in (NC)_2$ to (a_{01}, a_{12}).
- The operator $\langle 02\rangle^* \colon (NC)_2 \to (NC)_1$ assigns to such a pair (f, g) its composite gf.
- $(NC)_n$ is in bijective correspondence with the set of sequences $(f_1, \dots, f_n)$ of arrows in C such that the target of f_i is the source of f_{i+1}.

In particular, you can recover the category from its nerve, up to isomorphism.

3.2. *Example.* We have that $N[n] \approx \Delta^n$.

This leads to the question: given a simplicial set X, how can we detect that it is isomorphic to the nerve of some category?

3.3. **Spines.** The **spine** of the n-simplex Δ^n is the subobject $I^n \subseteq \Delta^n$, defined by

$$(I^n)_k = \{\, \langle f_0 \cdots f_k\rangle \in (\Delta^n)_k \mid f_k \le f_0 + 1 \,\}.$$

That is, a k-simplex in I^n is a simplicial operator $[k] \to [n]$ whose image is of the form either $\{j\}$ or $\{j, j+1\}$.

$$\begin{tikzcd} & \langle 1\rangle \arrow[dd] & \\ \langle 0\rangle \arrow[ur] & & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd} \quad \text{is the spine inside} \quad \begin{tikzcd} & \langle 1\rangle \arrow[dd] \arrow[dr] & \\ \langle 0\rangle \arrow[ur] \arrow[rr] \arrow[dr] & & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd}$$

3.4. **Proposition.** *For any simplicial set X, there is a bijection*

$$\mathrm{Hom}(I^n, X) \xrightarrow{\sim} \{\, (a_1, \dots, a_n) \in (X_1)^{\times n} \mid a_i\langle 1\rangle = a_{i+1}\langle 0\rangle \,\},$$

defined by sending $f \colon I^n \to X$ to $(f(\langle 01\rangle), f(\langle 12\rangle), \cdots, f(\langle n-1, n\rangle))$.

This amounts to describing I^n as a colimit of a diagram of standard simplices; specifically, as a collection of 1-simplices "glued" together at their ends. We will give the proof below.

3.5. **Colimits of sets and simplicial sets.** Given any functor $F \colon C \to \mathrm{Set}$ from a small category to sets, there is a "simple formula" for its colimit. First consider the coproduct (i.e., disjoint union) $\coprod_{c \in \mathrm{ob}\, C} F(c)$ of the values of the functor; I'll write (c, x) for a typical element of this coproduct, with $c \in \mathrm{ob}\, C$ and $x \in F(c)$. Consider the relation $\sim$ on this defined by

$$(c, x) \sim (c', x') \quad \text{if } \exists\ \alpha \colon c \to c' \text{ in } C \text{ such that } F(\alpha)(x) = x'.$$

Then $\mathrm{colim}_C F$ is isomorphic to $\left(\coprod_c F(c)\right)\big/ \approx$, where "$\approx$" is the equivalence relation which is *generated* by the relation "$\sim$". The relation "$\sim$" is often not itself an equivalence relation, so it can be very difficult to figure out what "$\approx$" actually is: the simple formula may not be so simple in practice.

3.6. *Exercise.* If C is a groupoid, then $\sim$ is an equivalence relation.

There are cases when things are tractable.

3.7. **Proposition.** *Let $\mathcal{A}$ be a collection of subsets of a set S, which is a partially ordered set under containment and thus can be regarded as a category. Suppose $\mathcal{A}$ has the following property: for all $s \in S$, there exists a $T_s \in \mathcal{A}$ such that for all $T \in \mathcal{A}$, $s \in T$ if and only if $T_s \subseteq T$. (That is, for each element of S there is a minimal subset from $\mathcal{A}$ which contains it.) Then the tautological map*

$$\operatorname{colim}_{T\in\mathcal{A}} T \to \bigcup_{T\in\mathcal{A}} T.$$

is a bijection.

Proof. Show that $(T,t) \approx (T',t')$ if and only if $t = t'$. □

Given a functor $F: C \to s\mathrm{Set}$ to simplicial sets, we obtain a simplicial set X by taking colimits of sets in each degree: $X_n = \operatorname{colim}_C F(c)_n$. Then $X = \operatorname{colim}_C F$ is the colimit of F.

3.8. **Subcomplexes.** Given a simplicial set X, a **subcomplex** is just a subfunctor of X; i.e., a collection of subsets $A_n \subseteq X_n$ which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion $A \to X$ is a morphism of simplicial sets. We typically write $A \subseteq X$ when A is a subcomplex of X.

The image of any map of simplicial sets is a subcomplex. Also, for every set of simplicies in a simplicial set, there is a smallest subcomplex which contains the set.

3.9. *Example.* For a vertex $x \in X_0$, we write $\{x\} \subseteq X$ for the smallest subcomplex which contains x. This subcomplex has exactly one n-simplex for each $n \geq 0$, namely $x\langle 0 \cdots 0\rangle$, and thus is isomorphic to Δ^0.

3.10. **Subcomplexes of Δ^n.** We have already noted that for each $S \subseteq [n]$ we get a subcomplex $\Delta^S \subseteq \Delta^n$. The following says that every subcomplex of Δ^n is a union of Δ^Ss.

3.11. **Lemma.** *Let $K \subseteq \Delta^n$ be a subcomplex. If $(f: [m] \to [n]) \in K_m$ with $f([m]) = S$, then $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$.*

Proof. It is immediate that $f \in (\Delta^S)_m$. Consider the unique factorization

$$[m] \xrightarrow{f_{\mathrm{surj}}} S \xrightarrow{f_{\mathrm{inj}}} [n]$$

into a surjection followed by an injection. Every surjection between order preserving maps admits an order preserving section, so there exists $s: S \to [m]$ such that $f_{\mathrm{surj}} s = \mathrm{id}_S$. Therefore $fs = f_{\mathrm{inj}} f_{\mathrm{surj}} s = f_{\mathrm{inj}}$, where $f_{\mathrm{inj}}: S \to [n]$ is the inclusion map. Given a simplex $(g: [k] \to S) \in (\Delta^S)_k$ we thus have $f_{\mathrm{inj}} g = f(sg)$, whence $f_{\mathrm{inj}} g \in K_k$ since K is a subcomplex, $f \in K_m$, and $sg: [k] \to [n]$ is a simplicial operator. We have proved that $\Delta^S \subseteq K$. □

We can sharpen this: every subcomplex of Δ^n is a *colimit* of subcomplexes Δ^S.

3.12. **Proposition.** *Let $K \subseteq \Delta^n$ be a subcomplex. Let $\mathcal{A}$ be the poset of all non-empty subsets $S \subseteq [n]$ such that the inclusion $f: S \to [n]$ is a simplex in K. Then the tautological map*

$$\operatorname{colim}_{S\in\mathcal{A}} \Delta^S \to K$$

is an isomorphism.

Proof. We must show that for each $m \geq 0$, the map $\operatorname{colim}_{S\in\mathcal{A}} (\Delta^S)_m \to K_m$ is a bijection. Each $(\Delta^S)_m = \{\, f: [m] \to [n] \mid f([m]) \subseteq S \,\}$ is a distinct subset of $K_m \subseteq (\Delta^n)_m$; i.e., $S \neq S'$ implies $(\Delta^S)_m \neq (\Delta^{S'})_m$. In view of (3.7), it suffices to show that for each $f \in K_m$ there is a minimal S in $\mathcal{A}$ such that $f \in (\Delta^S)_m$. This is immediate from (3.11), which says that $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$ where $S = f([m])$, and it is obvious that this S is minimal with this property. □

Proof of (3.4). Let $\mathcal{A}$ be the poset of all non-empty $S \subseteq [n]$ which are simplices of I^n; i.e., subsets of the form $\{j\}$ or $\{j, j+1\}$. Explicitly this has the form

$$\{0\} \to \{0,1\} \leftarrow \{1\} \to \{1,2\} \leftarrow \{2\} \to \cdots \leftarrow \{n-1\} \to \{n-1,n\} \leftarrow \{n\}.$$

By (3.12), $\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \to I^n$ is an isomorphism. Thus $\operatorname{Hom}(I^n, X) \approx \lim_{S \in \mathcal{A}} \operatorname{Hom}(\Delta^S, X)$, and an elementary argument gives the result. □

3.13. **A characterization of nerves.** We can now characterize simplicial sets which are isomorphic to nerves by means of an "extension" condition: they are simplicial sets such that every map $I^n \to X$ from a spine extends *uniquely* along $I^n \subseteq \Delta^n$ to a map from the standard n-simplex. That is, nerves are precisely the simplicial sets with "unique spine extensions".

3.14. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if the restriction map* $\operatorname{Hom}(\Delta^n, X) \to \operatorname{Hom}(I^n, X)$ *is a bijection for all $n \geq 0$.*

Proof. Left as an exercise, using (3.4). □

Given this, the following is a standard exercise.

3.15. **Proposition.** *The nerve functor $N \colon$* Cat $\to s$Set *is fully faithful. That is, every simplicial set map $g \colon NC \to ND$ between nerves is of the form $g = N(f)$ for a unique functor $f \colon C \to D$.*

Proof sketch. We need to show that $\operatorname{Hom}_{\mathrm{Cat}}(C, D) \to \operatorname{Hom}_{s\mathrm{Set}}(NC, ND)$ is a bijection for all categories C and D. *Injectivity* is clear, as a functor is determined by its effect on objects and morphisms, which is exactly the effect on 0- and 1-simplices of the nerves. For *surjectivity*, observe that for any map $f \colon NC \to ND$ of simplicial sets, we can define a candidate functor $F \colon C \to D$, defined on object and morphisms by the action of f on 0- and 1- simplices. That F has the correct action on identity maps follows from the fact that f commutes with the simplicial operator $\langle 00 \rangle \colon [1] \to [0]$. That F preserves composition uses (3.14) and that f commutes with the simplicial operator $\langle 02 \rangle \colon [1] \to [2]$.

Next, note that for simplicial maps $f, f' \colon NC \to ND$ which agree on 0- and 1-simplices, we must have $f = f'$ (again using (3.14)), and thus in particular we have $f = N(F)$. □

3.16. *Exercise.* Show that if C is a category and X is any simplicial set (not necessarily a nerve), then two maps $f, f' \colon X \to NC$ are equal if and only if $f_0 = f_0'$ and $f_1 = f_1'$, i.e., if f and f' coincide on vertices and edges.

4. Horns and inner horns

We now are going to give another (less obvious) characterization of nerves, in terms of "extending inner horns", rather than "extending spines". We will then weaken this new characterization to obtain the definition of a quasicategory.

4.1. **Definition of horns and inner horns.** We define a collection of subobjects of the standard simplices, called "horns". For each $n \geq 1$, these are subsimplicial sets $\Lambda^n_j \subset \Delta^n$ for each $0 \leq j \leq n$.

The jth **horn** Λ^n_j is the subcomplex of Δ^n defined by

$$(\Lambda^n_j)_k = \{\, f \colon [k] \to [n] \mid ([n] \smallsetminus \{j\}) \not\subseteq f([k]) \,\}.$$

Using the fact (3.11) that subcomplexes of Δ^n are always unions of Δ^Ss, we see that Λ^n_j is the union of "faces" $\Delta^{[n] \smallsetminus i}$ of Δ^n other than the jth face:

$$\Lambda^n_j = \bigcup_{i \neq j} \Delta^{[n] \smallsetminus i} \subset \Delta^n.$$

When $0 < j < n$ we say that $\Lambda^n_j \subset \Delta^n$ is an **inner horn**.

4.2. *Example.* The horns inside Δ^1 are just the vertices viewed as subobjects: $\Lambda^1_0 = \Delta^{\{0\}} \subset \Delta^1$ and $\Lambda^1_1 = \Delta^{\{1\}} \subset \Delta^1$. Neither is an inner horn.

4.3. *Example.* These are the three horns inside the 2-simplex.

$$\begin{array}{ccc}
\begin{tikzcd} & \langle 1\rangle & \\ \langle 0\rangle \arrow[ur, "\langle 01\rangle"] \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle \end{tikzcd}
&
\begin{tikzcd} & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\ \langle 0\rangle \arrow[ur, "\langle 01\rangle"] & & \langle 2\rangle \end{tikzcd}
&
\begin{tikzcd} & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\ \langle 0\rangle \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle \end{tikzcd}
\\
\Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2
\end{array}$$

Only Λ^2_1 is an inner horn.

4.4. *Exercise.* Visualize the four horns inside the 3-simplex. The horn Λ^3_j actually kind of looks like a horn: you blow into the vertex $\langle j\rangle$, and sound comes out of the opposite missing face $\Delta^{[3]\smallsetminus j}$.

4.5. *Exercise.* Show that Λ^n_j is the largest subobject of Δ^n which does not contain the simplex $\langle 0\cdots\widehat{j}\cdots n\rangle \in (\Delta^n)_{n-1}$, the "face opposite the vertex j".

We note that inner horns always contain spines: $I^n \subseteq \Lambda^n_j$ if $0 < j < n$. This is also true for non-inner horns if $n \geq 3$, but not for non-inner horns with $n = 1$ or $n = 2$.

4.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit "unique inner horn extensions"; this is different but analogous to the characterization in terms of unique spine extensions (3.14).

4.7. **Proposition.** *A simplicial set X is isomorphic to the nerve of a category, if and only if* $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_j, X)$ *is a bijection for all $n \geq 2$, $0 < j < n$.*

The proof will take up the rest of the section.

4.8. **Nerves have unique inner horn extensions.** First we show that nerves have unique inner horn extensions.

4.9. **Proposition.** *If C is a category, then for every* inner *horn $\Lambda^n_j \subset \Delta^n$ the evident restriction map*

$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda^n_j, NC)$$

is a bijection.

Proof. Since inner horns contain spines, we can consider restriction along $I^n \subseteq \Lambda^n_j \subseteq \Delta^n$. The composite

$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda^n_j, NC) \xrightarrow{r} \mathrm{Hom}(I^n, NC)$$

of restriction maps is a bijection (3.14), so r is a surjection. Thus, it suffices to show that r is injective. This is immediate when $n = 2$, since $\Lambda^2_1 = I^2$, so we can assume $n \geq 3$.

We will show that for any inner horn Λ^n_j with $n \geq 3$ there exists a finite chain

$$I^n = F_0 \subset F_1 \subset \cdots \subset F_d = \Lambda^n_j$$

of subcomplexes and subsets $S_i \subset [n]$ such that (i) $F_i = F_{i-1} \cup \Delta^{S_i}$ and (ii) $I^{S_i} \subseteq F_{i-1} \cap \Delta^{S_i}$. Here I^{S_i} denotes the spine of Δ^{S_i}. Given this, we see that

$$F_i \approx \mathrm{colim}\big(F_{i-1} \leftarrow F_{i-1} \cap \Delta^{S_i} \to \Delta^{S_i}\big);$$

by (3.12). We obtain a commutative diagram

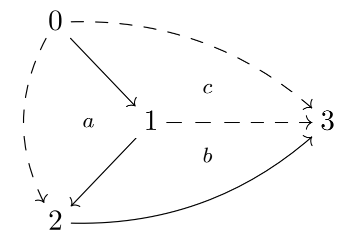
$$\begin{tikzcd}
\mathrm{Hom}(F_i, NC) \arrow[r, "b"] \arrow[d] & \mathrm{Hom}(F_{i-1}, NC) \arrow[d] & \\
\mathrm{Hom}(\Delta^{S_i}, NC) \arrow[r, "a"'] & \mathrm{Hom}(F_{i-1} \cap \Delta^{S_i}, NC) \arrow[r] & \mathrm{Hom}(I^{S_i}, NC)
\end{tikzcd}$$

in which the square is a pullback, and the horizontal composition on the bottom is a bijection. It immediately follows that a, and hence b, are injective. We can thus conclude that $\mathrm{Hom}(\Lambda^n_j, NC) \to \mathrm{Hom}(I^n, NC)$ is injective as desired.

When $n = 3$, we can "attach" simplices in order explicitly:

$$\Lambda^3_1 = I^3 \cup \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}} \cup \Delta^{\{0,1,3\}}, \qquad \Lambda^3_2 = I^3 \cup \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}} \cup \Delta^{\{0,2,3\}}.$$

Note that, for instance, in building Λ^3_1, we must add $\Delta^{\{0,1,3\}}$ after adding $\Delta^{\{1,2,3\}}$, so that the spine $I^{\{0,1,3\}}$ is already present.



When $n \geq 4$, we have that $(\Lambda^n_j)_1 = (\Delta^n)_1$ and $(\Lambda^n_j)_2 = (\Delta^n)_2$. The procedure to "build" Λ^n_j from I^n by adding subsimplices is: (1) first attach 2-simplices one at a time, in an allowable order; then (2) attach all higher dimensional subsimplices. In (2) the order doesn't matter since all 1-simplices (and hence all spines) are already present. $\square$

4.10. **Nerves are characterized by unique inner horn extension.** Let X be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_j, X)$ is a bijection for $0 < j < n$.

Considering the unique extensions along $\Lambda^2_1 \subset \Delta^2$, we see that this defines a "composition law" on the set X_1. Thati is, given $f, g \in X_1$ such that $f_1 = g_0$ in X_0,[4] consider

$$u\colon \Lambda^2_1 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \xrightarrow{(f,g)} X, \qquad \langle 01 \rangle \mapsto f \in X_1,\ \langle 12 \rangle \mapsto g \in X_1,$$

let $\widetilde{u}\colon \Delta^2 \to X$ be the *unique* extension of u along $\Lambda^2_1 \subset \Delta^2$, and define

$$g \circ f := \widetilde{u}_{02}.$$

The 2-simplex $\widetilde{u}$ is uniquely characterized by: $\widetilde{u}_{01} = f$, $\widetilde{u}_{12} = g$, $\widetilde{u}_{02} = g \circ f$.

This composition law is automatically unital. Given $x \in X_0$, write $1_x := x\langle 00 \rangle \in X_1$, so that $(1_x)_0 = x = (1_x)_1$. Then applying the composition law gives $1_x \circ f = f$ and $g \circ 1_x = g$. (Proof: consider $f\langle 011 \rangle, g\langle 001 \rangle \in X_2$.)

[4] Recall that $f_1 = f\langle 1 \rangle$ and $g_0 = g\langle 0 \rangle$, regarded as maps $\Delta^0 \to X$ and thus as elements of X_0, using the notation discussed in §2.3.

Now consider $\Lambda^3_1 \subset \Delta^3$. Recall (3.12) that Λ^3_1 is a union (and colimit) of $\Delta^S \subseteq \Delta^3$ such that $S \not\supseteq \{0,2,3\}$. A map $\Lambda^3_1 \to X$ can be pictured as

$$\begin{tikzcd}
0 \arrow[dr, "f"] \arrow[dd, bend right, "g\circ f"'] \arrow[drr, bend left, "(h\circ g)\circ f"] & & \\
& 1 \arrow[r, "h\circ g"] \arrow[dl, "g"] & 3 \\
2 \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

so that the planar 2-cells in the picture correspond to non-degenerate 2-simplices in Δ^3 which are contained in Λ^3_1, while the edges are labelled according to their images in X, using the composition law defined above. Let $v\colon \Delta^3 \to X$ be any extension of the above picture along $\Lambda^3_1 \subset \Delta^3$, and consider the restriction $w := v\langle 023\rangle\colon \Delta^2 \to X$ to the face $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$. Then $w_{01} = g \circ f$, $w_{12} = h$, and $w_{02} = (h \circ g) \circ f$, and thus the existence of w demonstrates that

$$h(g \circ f) = (h \circ g)f.$$

In other words, the *existence* of extensions along $\Lambda^3_1 \subset \Delta^3$ implies that the composition law we defined above is associative. The same argument also works with $\Lambda^3_2 \subset \Delta^3$.

Thus, given an X with unique inner horn extensions, we can construct a category C, so that objects of C are elements of X_0, morphisms of C are elements of X_1, and composition is given as above.

Next we construct a map $X \to NC$ of simplicial sets. There are obvious maps $\alpha_n\colon X_n \to (NC)_n$, corresponding to restriction along spines $I^n \subseteq \Delta^n$; i.e., $\alpha(x) = (x_{01}, \dots, x_{n-1,n})$. You can show that these maps are compatible with simplicial operators, so that they define a map $\alpha\colon X \to NC$ of simplicial sets. *Proof:* For any n-simplex $x \in X_n$, *all* of its edges are determined by edges on its spine via the composition law: $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \cdots \circ x_{i,i+1}$. for all $0 \le i \le j \le n$. Thus for $f\colon [m] \to [n]$ we have $\alpha(xf) = ((xf)_{01}, \dots, (xf)_{n-1,n}) = (x_{f_0 f_1}, \dots, x_{f_{n-1} f_n}) = (x_{01}, \dots, x_{n-1,n})_{f_0 \cdots f_n} = (\alpha x)f$.

Now we can prove that nerves are characterized by unique extension along inner horns.

Proof of (4.7). We have already shown (4.9) that nerves have unique extensions for inner horns. Consider a simplicial set X which has unique inner horn extension. By the discussion above, we obtain a category C and a map $\alpha\colon X \to NC$ of simplicial sets, which is clearly a bijection in degrees ≤ 2. We will show $\alpha_n\colon X_n \to (NC)_n$ is bijective by induction on n.

Fix $n \ge 3$, and consider the commutative square

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n, X) \arrow[r, "\sim"] \arrow[d, "\alpha_{\Delta^n}"'] & \mathrm{Hom}(\Lambda^n_1, X) \arrow[d, "\alpha_{\Lambda^n_1}"] \\
\mathrm{Hom}(\Delta^n, NC) \arrow[r, "\sim"'] & \mathrm{Hom}(\Lambda^n_1, NC)
\end{tikzcd}$$

The horizontal maps are induced by restriction, and are bijections (top by hypothesis, bottom by (4.9)). Because Λ^n_1 is a colimit of simplices of dimension $< n$ (3.12), the map $\alpha_{\Lambda^n_1}$ is a bijection by the induction hypothesis. Therefore so is α_{Δ^n}. $\square$

5. Quasicategories

We can now define the notion of a quasicategory, by removing the uniqueness part of the inner horn extension criterion for nerves.

5.1. **Identifying categories with their nerves.** From this point on, I will (at least informally) often not distinguish a category C with its nerve. In other words, when C is a category I'll sometimes write C where I mean NC, and so implicitly identify Cat with a full subcategory of sSet.

5.2. **Definition of quasicategory.** A **quasicategory** is a simplicial set C such that for every map $f\colon \Lambda^n_j \to C$ from an inner horn, there *exists* an extension of it to $g\colon \Delta^n \to C$. That is, C is a quasicategory if $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\Lambda^n_j, C)$ is surjective for all $0 < j < n$, $n \geq 2$.

Any category (more precisely, the nerve of any category) is a quasicategory.

We refer to elements of C_0 as the **objects** of C, and elements of C_1 as the **morphisms** of C. Every morphism $f \in C_1$ has a **source** and **target**, namely its vertices $f_0, f_1 \in C_0$. We write $f\colon f_0 \to f_1$, just as we would for morphisms in a category. Likewise, for every object $x \in C_0$, there is a distinguished morphism $1_x\colon x \to x$, called the **identity morphism**, defined by $1_x = x_{00}$.

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

5.3. **Products of quasicategories.** Simplicial sets are functors, so the product of simplicial sets X and Y is just the product of the functors. Thus, $(X \times Y)_n = X_n \times Y_n$.

5.4. **Proposition.** *The product of two quasicategories (as simplicial sets) is a quasicategory.*

Proof. Exercise. □

5.5. *Remark.* If C and D ordinary categories, then $N(C \times D) \approx NC \times ND$. Thus, the notion of product of quasicategories generalizes that of categories.

5.6. **Coproducts of quasicategories.** Similarly, the coproduct of simplicial sets X and Y is just the coproduct of functors, whence $(X \amalg Y)_n = X_n \amalg Y_n$. More generally, $\left(\coprod_s X_s\right)_n = \coprod_s (X_s)_n$ for an indexed collection $\{X_s\}$ of simplicial sets.

5.7. **Proposition.** *The coproduct of any indexed collection of quasicategories is a quasicategory.*

To prove this, we introduce the set of **components** of a simplicial set. Given a simplicial set X, we define the set of components by

$$\pi_0 X := \mathrm{colim}_{\Delta^{\mathrm{op}}} X.$$

Explicitly, this is the quotient of the set $\coprod_{n\geq 0} X_n$ of all simplices by the equivalence relation which is *generated* by the relation $\sim$, defined so that $a \sim af$ for any $a \in X_n$ and $f\colon [m] \to [n]$.

5.8. *Exercise.* Show that there is a canonical bijection

$$\mathrm{colim}\big[X_1 \rightrightarrows X_0\big] \xrightarrow{\sim} \pi_0 X,$$

where the left-hand side denotes the coequalizer of the two face maps $\langle 0\rangle^*, \langle 1\rangle^*\colon X_1 \to X_0$. That is, the components of X may be identified with the quotient of the set X_0 of vertices with respect to the equivalence relation generated by the relation $\sim$ defined by $e_0 \sim e_1$ any $e \in X_1$.

We say that a simplicial set is **connected** if it has exactly one component.

5.9. *Exercise.* Show that every standard simplex is connected, and that every horn is connected.

5.10. *Exercise.* Show that each equivalence class of simplices in a simplicial set X (i.e., corresponding to some element of $\pi_0 X$) forms a subcomplex of X, and that any simplicial set is isomorphic to the coproduct of its connected components.

Proof of (5.7). If $f\colon K \to X$ is a map from a connected simplicial set K, then f factors through the inclusion of some connected component of X. If $X = \coprod_s X_s$ is a coproduct, then any connected component of X must be contained in one of the X_s. The proof is now straightforward, using the fact that horns are connected (5.9). □

5.11. **Full subquasicategories.** Given a category C and a set of objects $S \subseteq \operatorname{ob} C$, the *full subcategory spanned by* S is the subcategory $C' \subseteq C$ with $\operatorname{ob} C' = S$ and with $\operatorname{mor} C' = \{ f \in \operatorname{mor} C \mid \operatorname{source}(f), \operatorname{target}(f) \in S \}$.

This has a straightforward generalization to quasicategories. Given a simplicial set C and a set $S \subseteq X_0$ of vertices, let

$$C'_n = \{ a \in C_n \mid a_j \in S \text{ for all } j = 0, \dots, n \}.$$

5.12. *Exercise.* Show that C' is a subcomplex of C, and that if C is a quasicategory then so is C'.

When C is a quasicategory, the subcomplex C' is called the **full subquasicategory spanned by** S.

5.13. **Opposite of a quasicategory.** Given a category C, the *opposite* category C^{op} has $\operatorname{ob} C^{\mathrm{op}} = \operatorname{ob} C$, and $\operatorname{Hom}_{C^{\mathrm{op}}}(x, y) = \operatorname{Hom}_C(y, x)$, and the sense of composition is reversed: $g \circ_{C^{\mathrm{op}}} f = f \circ_C g$.

The category Δ has a non-trivial involution $\mathrm{op}\colon \Delta \to \Delta$. This is the functor which sends $[n] \mapsto [n]$, and sends $\langle f_0, \dots, f_n \rangle\colon [n] \to [m]$ to $\langle m - f_n, \dots, m - f_0 \rangle$. You should think of this as the functor which "reverses the ordering" on the sets $[n]$.

The **opposite** of a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the composite functor $X^{\mathrm{op}} := X \circ \mathrm{op}$.

We have that $(\Delta^n)^{\mathrm{op}} = \Delta^n$, while $(\Lambda^n_j)^{\mathrm{op}} = \Lambda^n_{n-j}$. As a consequence, the opposite of a quasicategory is a quasicategory. We have that $(NC)^{\mathrm{op}} = N(C^{\mathrm{op}})$, so this generalizes the notion of opposite category. The functor $\mathrm{op}\colon \Delta \to \Delta$ satisfies $\mathrm{op} \circ \mathrm{op} = \mathrm{id}_\Delta$, so $(X^{\mathrm{op}})^{\mathrm{op}} = X$.

6. Functors and natural transformations

6.1. **Functors.** A **functor** between quasicategories is merely a map $f\colon C \to D$ between the simplicial sets.

We write QCat for the category of quasicategories and functors between them. Clearly $\mathrm{QCat} \subset s\mathrm{Set}$ is a full subcategory. Because the nerve functor is a full embedding of Cat into QCat, any functor between ordinary categories is also a functor between quasicategories.

6.2. **Natural transformations.** Given functors $F, G\colon C \to D$ between categories, a *natural transformation* $\phi\colon F \Rightarrow G$ is a choice, for each object c of C, of a map $\phi(c)\colon F(c) \to G(c)$ in D, such that for every morphism $\alpha\colon c \to c'$ in C the square

$$\begin{tikzcd} F(c) \arrow[r, "\phi(c)"] \arrow[d, "f(\alpha)"'] & G(c) \arrow[d, "g(\alpha)"] \\ F(c') \arrow[r, "\phi(c')"'] & G(c') \end{tikzcd}$$

commutes in D. There is a standard convenient reformulation of this: a natural transformation $\phi\colon F \Rightarrow G$ is the same thing as a functor

$$H\colon C \times [1] \to D,$$

so that $H|C \times \{0\} = F$, $H|C \times \{1\} = G$, and $H|\{c\} \times [1] = \alpha(c)$ for each $c \in \operatorname{ob} C$.

This admits a straightforward generalization to quasicategories. A **natural transformation** $f\colon f_0 \Rightarrow f_1$ of functors $f_0, f_1\colon C \to D$ between quasicategories is defined to be a map

$$f\colon C \times N[1] = C \times \Delta^1 \to D$$

of simplicial sets such that $f|C \times \{i\} = f_i$ for $i = 0, 1$. For ordinary categories this coincides with the classical notion.

7. Examples of quasicategories

There are many ways to produce quasicategories, as we will see. Unfortunately, "hands-on" constructions of quasicategories are relatively rare. Here I give a few reasonably explicit examples to play with.

7.1. **Large vs. small.** I have been implicitly assuming that certain categories are **small**; i.e., they have *sets* of objects and morphisms. For instance, for the nerve of a category C to be a simplicial set, we need $C_0 = \operatorname{ob} C$ to be a set.

However, in practice many categories of interest are only **locally small**; i.e., the collection of objects is not a set but is a "proper class", although for any pair of objects $\operatorname{Hom}_C(X,Y)$ is a set. For instance, the category Set of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These are called **large** categories.

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) "universes". A universe U is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe U, there is a larger universe U' such that $U \in U'$. Thus, if by "set" we mean "U-set", then the category Set is a "U'-category". This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable *strongly inaccessible cardinals.*

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the simplices form a set), while the nerve of a large category is a large simplicial set.

I'm not going to be pedantic about this. I'll usually assume categories like Set, Cat, sSet, etc., are categories whose objects are *small* sets, *small* categories, *small* simplicial sets, etc. However, I'll sometimes consider examples of such which are not small.

7.2. **Morita quasicategory.** Define a simplicial set C, so that C_n is a set whose elements are data $x := (A_i, M_{ij}, f_{ijk})$, where

- for each $i \in [n]$, A_i is an associative ring,
- for each $i < j$ in $[n]$, M_{ij} is an (A_i, A_j)-bimodule,
- for each $i < j < k$ in $[n]$, $f_{ijk}\colon M_{ij} \otimes_{A_j} M_{jk} \to M_{ik}$ is an isomorphism of (A_i, A_k)-bimodules, such that
- for each $i < j < k < \ell$, the diagram

$$\begin{tikzcd} M_{ij}\otimes M_{jk}\otimes M_{k\ell} \arrow[r, "\mathrm{id}\otimes f_{jk\ell}"] \arrow[d, "f_{ijk}\otimes\mathrm{id}"'] & M_{ij}\otimes M_{j\ell} \arrow[d, "f_{ij\ell}"] \\ M_{ik}\otimes M_{k\ell} \arrow[r, "f_{ik\ell}"'] & M_{i\ell} \end{tikzcd} \tag{7.3}$$

  commutes.

For an injective map $\delta\colon [m] \to [n]$, we define $x\delta := (A_{\delta(i)}, M_{\delta(i)\delta(j)}, f_{\delta(i)\delta(j)\delta(k)})$. If δ is not injective, we must set $M_{ij} := A_{\delta(i)}$ when $\delta(i) = \delta(j)$, and set f_{ijk} to the canonical isomorphism if $\delta(i) = \delta(j)$

or $\delta(i) = \delta(k)$.

$$
A_0 \qquad
\begin{tikzcd}
A_0 \arrow[d, "M_{01}"'] \\
A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[dd, "M_{01}"'] \arrow[dr, "M_{02}"] & \\
 \arrow[r, phantom, "f_{012}"] & A_2 \\
A_1 \arrow[ur, "M_{12}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[dd, bend right, "M_{01}"'] \arrow[dr, "M_{02}"] \arrow[drr, bend left, "M_{03}"] & \arrow[d, phantom, "f_{023}"] & \\
 \arrow[r, phantom, "f_{012}"] & A_2 \arrow[r, "M_{23}"] & A_3 \\
A_1 \arrow[ur, "M_{12}"'] \arrow[urr, bend right, "M_{13}"'] & \arrow[u, phantom, "f_{123}"] &
\end{tikzcd}
$$

The object C is a quasicategory. Fillers for $\Lambda_1^2 \subset \Delta^2$ always exist: a map $\Lambda_1^2 \to C$ is a choice of $(A_0, M_{01}, A_1, M_{12}, A_2)$, and an extension to Δ^2 can be given by setting M_{02} to be the tensor product, and f_{012} the identity map. However, there can be more than one choice: even keeping M_{02} the same, there is a choice of isomorphism f_{012}.

Fillers for $\Lambda_1^3 \subset \Delta^3$ and $\Lambda_2^3 \subset \Delta^3$ always exist, and are unique: finding a filler amounts to choosing isomorphisms $f_{023} = f_{ik\ell}$ (for Λ_1^3) or $f_{013} = f_{ij\ell}$ (for Λ_2^3) making (7.3) commute, and such choices are unique. Similarly, all fillers in higher dimensions $\Lambda_j^n \subset \Delta^n$ with $n \geq 4$ trivially exist and are unique.

7.4. **Quasicategory of categories.** Define a simplicial set $\mathcal{C}$ so that $\mathcal{C}_n$ is a set whose elements are data $x := (C_i, F_{ij}, \phi_{ijk})$ where

- for each $i \in [n]$, C_i is a (small) category,
- for each $i < j$ in $[n]$, $F_{ij} \colon C_i \to C_j$ is a functor,
- for each $i < j < k$ in $[n]$, $\phi_{ijk} \colon F_{jk}F_{ij} \Rightarrow F_{ik}$ is a natural *isomorphism* of functors $C_i \to C_k$, such that
- for each $i < j < k < \ell$, the diagram

$$
\begin{tikzcd}
F_{k\ell}F_{jk}F_{ij} \arrow[r, Rightarrow, "\phi_{jk\ell}\,\mathrm{id}_{F_{ij}}"] \arrow[d, Rightarrow, "\mathrm{id}_{F_{k\ell}}\,\phi_{ijk}"'] & F_{j\ell}F_{ij} \arrow[d, Rightarrow, "\phi_{ij\ell}"] \\
F_{k\ell}F_{ik} \arrow[r, Rightarrow, "f_{ik\ell}"'] & F_{i\ell}
\end{tikzcd}
$$

  commutes.

The action of simplicial operators is defined exactly as in the previous example, as is the proof that $\mathcal{C}$ is a quasicategory.

7.5. **Nerve of a crossed module.** A **crossed module** is data (G, H, ϕ, ρ), consisting of groups G and H, and homomorphisms $\phi \colon H \to G$ and $\rho \colon G \to \mathrm{Aut}H$, such that

$$\phi(\rho(g)(h)) = g\phi(h)g^{-1}, \qquad \rho(\phi(h))(h') = hh'h^{-1}, \qquad \text{for all } g \in G,\ h, h' \in H.$$

For instance: $G = H =$ the cyclic group of order 4, with $\phi(x) = x^2$ and ρ the non-trivial action. From this we can construct a quasicategory (in fact, a "quasigroupoid") much as in the last example: an n-simplex is data (g_{ij}, h_{ijk}) with $g_{ij} \in G$, $h_{ijk} \in H$, satisfying identities

$$g_{ij}g_{jk} = \phi(h_{ijk})g_{ik}, \qquad h_{ijk}h_{ik\ell} = \rho(g_{ij})(h_{jk\ell})h_{ij\ell}.$$

7.6. **Spans.** (See [Bar14, §§2–3], where this is called the *effective Burnside ∞-category.*) For each object $[n]$ of Δ, define $[n]^{\mathrm{tw}}$ to be the category with

- **objects** pairs (i, j) with $0 \leq i \leq j \leq n$, and
- a unique **morphism** $(i, j) \to (i', j')$ whenever $i' \leq i \leq j \leq j'$.

The construction $[n] \mapsto [n]^{\mathrm{tw}}$ defines a functor $\Delta \to \mathrm{Cat}$. (The category $[n]^{\mathrm{tw}}$ is called the *twisted arrow category* of $[n]$; you can define a twisted arrow category C^{tw} for any category C.)

Let C be a category which has pullbacks; for an explicit example, think of the category of finite sets. Let $R(C)$ be the simplicial set defined so that

$$\mathcal{R}(C)_n := \{\text{functors } ([n]^{\mathrm{tw}})^{\mathrm{op}} \to C\}.$$

Elements of $\mathcal{R}(C)_0$ are just objects of C. Elements of $\mathcal{R}(C)_1$, $\mathcal{R}(C)_2$, $\mathcal{R}(C)_3$ are respectively diagrams in C of shape

$$\begin{tikzcd}[column sep=tiny, row sep=small] & X_{01} \arrow[dl] \arrow[dr] & \\ X_{00} & & X_{11} \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny, row sep=small] & & X_{02} \arrow[dl] \arrow[dr] & & \\ & X_{01} \arrow[dl] \arrow[dr] & & X_{12} \arrow[dl] \arrow[dr] & \\ X_{00} & & X_{11} & & X_{22} \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny, row sep=small] & & & X_{03} \arrow[dl] \arrow[dr] & & & \\ & & X_{02} \arrow[dl] \arrow[dr] & & X_{13} \arrow[dl] \arrow[dr] & & \\ & X_{01} \arrow[dl] \arrow[dr] & & X_{12} \arrow[dl] \arrow[dr] & & X_{23} \arrow[dl] \arrow[dr] & \\ X_{00} & & X_{11} & & X_{22} & & X_{33} \end{tikzcd}$$

Let $\mathcal{A}(C)_n \subseteq \mathcal{R}(C)_n$ denote the subset whose n-simplices are functors $X\colon ([n]^{\mathrm{tw}})^{\mathrm{op}} \to C$ such that for every $i' \leq i \leq j \leq j'$ the square

$$\begin{tikzcd} X_{i'j'} \arrow[r] \arrow[d] & X_{ij'} \arrow[d] \\ X_{i'j} \arrow[r] & X_{ij} \end{tikzcd}$$

is a pullback in C. Then $\mathcal{A}(C)$ is a subcomplex, and in fact is a quasicategory. This is another example in which extensions along inner horns $\Lambda^n_j \subset \Delta^n$ exist for $n \geq 2$, and are unique for $n \geq 3$.

7.7. **Singular complex of a space.** The **topological n-simplex** is

$$\Delta^n_{\mathrm{top}} := \{\, (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1,\ x_i \geq 0 \,\},$$

the convex hull of the standard basis vectors. We get a functor $\Delta_{\mathrm{top}}\colon \Delta \to \mathrm{Top}$ by $\Delta_{\mathrm{top}}([n]) = \Delta^n_{\mathrm{top}}$. For a topological space T, we define $\operatorname{Sing} T$ to be the simplicial set $[n] \mapsto \operatorname{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, T)$.

Define **topological horns**

$$(\Lambda^n_j)_{\mathrm{top}} := \{\, x \in \Delta^n_{\mathrm{top}} \mid x_i = 0 \text{ for some } i \neq j \,\} \subset \Delta^n_{\mathrm{top}},$$

and observe that continuous maps $(\Lambda^n_j)_{\mathrm{top}} \to T$ correspond in a natural way with maps $\Lambda^n_j \to \operatorname{Sing} T$. (*Exercise.* This is a consequence of the fact that Λ^n_j is a colimit of all the $\Delta^S \subseteq \Lambda^n_j$, and that $(\Lambda^n_j)_{\mathrm{top}}$ is similarly a colimit of the corresponding Δ^S_{top}.) There exists a continuous retraction $\Delta^n_{\mathrm{top}} \to (\Lambda^n_j)_{\mathrm{top}}$, and thus we see that

$$\operatorname{Hom}(\Delta^n, \operatorname{Sing} T) \to \operatorname{Hom}(\Lambda^n_j, \operatorname{Sing} T)$$

is surjective for every horn (not just inner ones).

A simplicial set X which has extensions for all horns is called a **Kan complex**. Thus, $\operatorname{Sing} T$ is a Kan complex, and so in particular is a quasicategory (and as we will see, a quasigroupoid).

7.8. **Eilenberg-MacLane object.** Fix an abelian group A and an integer $d \geq 0$. We define a simplicial set $K = K(A, d)$, so that K_n is a set whose elements are data $a = (a_{i_0 \dots i_d})$ consisting of

- for each $0 \leq i_0 \leq \cdots \leq i_d \leq n$, an element $a_{i_0 \dots i_d} \in A$, such that
- $a_{i_0 \dots i_d} = 0$ if $i_{u-1} = i_u$ for any u, and
- for each $0 \leq j_0 \leq \cdots \leq j_{d+1} \leq n$ we have $\sum_u (-1)^u a_{j_0 \dots \widehat{j_u} \dots j_{d+1}} = 0$.

For a map $\delta\colon [m] \to [n]$ we define

$$(a\delta)_{i_0\dots i_d} = a_{\delta(i_0)\dots\delta(i_d)}.$$

The object $K(A,d)$ is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid). When $d=0$, it is just a "discrete" simplicial set, equal to A in each dimension. When $d=1$, it is isomorphic the nerve of A viewed as a category with one object.

7.9. *Exercise.* Show that $K(A,d)$ is a Kan complex, i.e., that $\mathrm{Hom}(\Delta^n, K(A,d)) \to \mathrm{Hom}(\Lambda^n_j, K(A,d))$ is surjective for all horns $\Lambda^n_j \subset \Delta^n$. In fact, this map is *bijective* unless $n=d$. (Hint: there are four distinct cases to check, namely $n<d$, $n=d$, $n=d+1$, and $n>d+1$.)

7.10. *Exercise.* Given a simplicial set X, a **normalized d-cocycle** with values in A is a function $f\colon X_d \to A$ such that

(1) $f(x_{0,\dots i,i,\dots d-1}) = 0$ for all $x \in X_{d-1}$ and $0 \le i \le d-1$, and
(2) $\sum(-1)^i f(x_{0,\dots,\widehat{i},\dots,d+1}) = 0$ for all $x \in X_{d+1}$ and $0 \le i \le d+1$.

Show that the set $Z^d_{\mathrm{norm}}(X;A)$ of normalized d-cocycles on X is in bijective correspondence with $\mathrm{Hom}_{s\mathrm{Set}}(X, K(A,d))$. (Hint: an element $a \in K_n$ is uniquely determined by the collection of elements $a\delta \in K_d = A$, as δ ranges over injective maps $[d] \to [n]$.)

7.11. *Remark.* This is an example of a *simplicial abelian group*: the map $+\colon K \times K \to K$ of simplicial sets defined by $(a+b)_{i_0\dots i_d} = a_{i_0\dots i_d} + b_{i_0\dots i_d}$ satisfies the axioms of an abelian group, reflecting the fact that $Z^d_{\mathrm{norm}}(X;A)$ is an abelian group.

8. Homotopy category of a quasicategory

Our next goal is to define the notion of an *isomorphism* in a quasicategory. This notion behaves much like that of *homotopy equivalence* in topology. Thus, we will define isomorphism by means of the *homotopy category* of a quasicategory.

8.1. **The fundamental category of a simplicial set.** A **fundamental category** for a simplicial set X consists of (i) a category hX, and (ii) a map $\alpha\colon X \to N(hX)$ of simplicial sets, such that for every category C, the map

$$\alpha^*\colon \mathrm{Hom}(N(hX), NC) \to \mathrm{Hom}(X, NC)$$

induced by restriction along α is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

8.2. **Proposition.** *Every simplicial set has a fundamental category.*

Proof sketch. Given X, we construct hX by generators and relations. First, consider the **free category** F, whose objects are the set X_0, and whose morphisms are finite "composable" sequences $[a_n,\dots,a_1]$ of edges of X_1. Thus, morphisms in F are "words", whose "letters" are 1-simplices a_i with $(a_{i+1})_0 = (a_i)_1$, and composition is concatenation of words; the element $[a_n,\dots,a_1]$ is then a morphism $(a_1)_0 \to (a_n)_1$. (Note: we suppose that there is an empty sequence $[]_x$ in F for each vertex $x \in X_0$; these correspond to identity maps in F.)

Then hX is defined to be the largest quotient category of F subject to the following relations:

- $[a] \sim []_x$ for each $x \in X_0$ where $a = x_{00} \in X_1$, and
- $[g,f] \sim [h]$ whenever there exists $a \in X_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$.

The map $\alpha\colon X \to N(hX)$ sends $x \in X_n$ to the equivalence class of $[x_{n-1,n},\dots,x_{0,1}]$. □

As a consquence: the fundamental category functor $h\colon s\mathrm{Set} \to \mathrm{Cat}$ is left adjoint to the nerve functor $N\colon \mathrm{Cat} \to s\mathrm{Set}$. We will give another construction of the fundamental category in (12.16).

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of. We won't use it very much. When C is a quasicategory, there is a much more concrete description of hC, which in this context is also called the **homotopy category** of C.

8.3. **The homotopy relation on morphisms.** Fix a quasicategory C. For $x, y \in C_0$, let $\hom_C(x, y) := \{ f \in C_1 \mid f_0 = x,\ f_1 = y \}$ denote the set of "morphisms" in C from x to y. We write 1_x for the degenerate element $x_{00} \in \hom_C(x, x)$.

Define relations $\sim_\ell$, $\sim_r$ on $\hom_C(x, y)$ (called **left homotopy** and **right homotopy**) by

- $f \sim_\ell g$ iff there exists $a \in C_2$ with $a_{01} = 1_x$, $a_{02} = f$, $a_{12} = g$,
- $f \sim_r g$ iff there exists $b \in C_2$ with $b_{12} = 1_y$, $b_{01} = f$, $b_{02} = g$.

$$f \sim_\ell g\colon \quad \begin{tikzcd} x \arrow[d, "1_x"'] \arrow[dr, "f"] \arrow[dr, phantom, near start, "a"'] & \\ x \arrow[r, "g"'] & y \end{tikzcd} \qquad\qquad f \sim_r g\colon \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g"'] \arrow[dr, phantom, near start, "b"] & y \arrow[d, "1_y"] \\ & y \end{tikzcd}$$

Note that $f \sim_\ell g$ in $\hom_C(x, y)$ coincides with $g \sim_r f$ on $\hom_{C^{\mathrm{op}}}(y, x)$.

8.4. **Proposition.** *The relations $\sim_\ell$ and $\sim_r$ are identical, and are an equivalence relation on $\hom_C(x, y)$.*

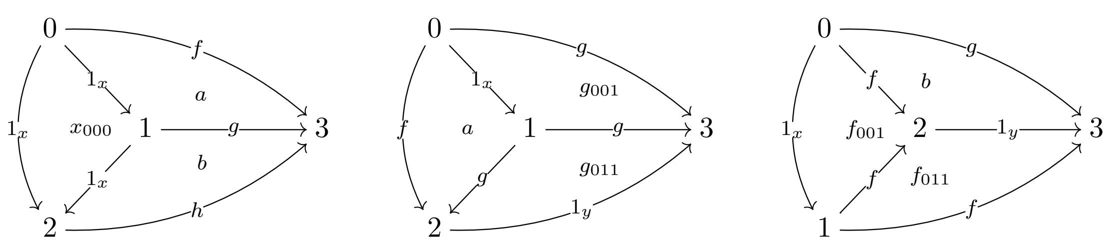
Proof. We will prove

(1) $f \sim_\ell f$,
(2) $f \sim_\ell g$ and $g \sim_\ell h$ imply $f \sim_\ell h$,
(3) $f \sim_\ell g$ implies $f \sim_r g$,
(4) $f \sim_r g$ implies $g \sim_\ell f$.

These show that $\sim_\ell$ is an equivalence relation, and also that $\sim_r$ and $\sim_\ell$ coincide. The idea is to use the inner-horn extension condition for C to produce the appropriate relations.

(1) $f \sim_\ell f$ is exhibited by $f_{001} \in C_2$.

(2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of Δ^3 to C constructed from the given data. The restriction of any extension to Δ^3 along the remaining face gives the conclusion.



□

8.5. **Composition of homotopy classes of morphisms.** We now define $f \approx g$ to mean $f \sim_\ell g$ (equivalently $f \sim_r g$). We can speak of **homotopy classes** $[f]$ of morphisms $f \in \hom_C(x, y)$. Next we observe that we can compose homotopy classes.

Given $f \in \hom_C(x, y)$, $g \in \hom_C(y, z)$, $h \in \hom_C(x, z)$, we say that h is a **composite** of (g, f) if there exists a 2-simplex $a \in C_2$ with $a\langle 01\rangle = f$, $a\langle 12\rangle = g$, $a\langle 02\rangle = h$; thus composition is a three-fold relation on $\hom(x, y) \times \hom(y, z) \times \hom(x, z)$. The composite relation is compatible with the homotopy relation, as shown by the following.

8.6. **Lemma.** *If $f \approx f'$, $g \approx g'$, h a composite of (g, f), and h' a composite of (g', f'), then $h \approx h'$.*

Proof. Since $\approx$ is an equivalence relation, it suffices prove the special cases (a) $f = f'$, and (b) $g = g'$. We prove case (b).

Let $a \in C_2$ exhibit $f \sim_\ell f'$, and let $b, b' \in C_2$ exhibit h as a composite of (g, f) and h' as a composite of (g, f') respectively. The inner horn $\Lambda^3_2 \to C$ defined by

$$
\begin{tikzcd}
0 \arrow[dr, "f"] \arrow[drr, bend left, "h"] \arrow[dd, bend right, "1_x"'] \arrow[dd, phantom, "a"] & & \\
& 2 \arrow[r, "g"] \arrow[ur, phantom, "b"] \arrow[dr, phantom, "b'"] & 3 \\
1 \arrow[ur, "f'"'] \arrow[urr, bend right, "h'"'] & &
\end{tikzcd}
$$

extends to $u\colon \Delta^3 \to C$, and $u|\Delta^{\{0,1,3\}}$ exhibits $h \sim_\ell h'$. □

Thus, any composite of (g, f) represents a well-defined homotopy class of morphisms in C, which I'll write as $[g] \circ [f]$.

I'll leave the proofs of the following as exercises; the proofs are much like what we have already seen.

8.7. **Lemma.** *Given $f\colon x \to y$, we have $[f] \circ [1_x] = [f] = [1_y] \circ [f]$.*

8.8. **Lemma.** *If $[g] \circ [f] = [u]$, $[h] \circ [g] = [v]$, then $[h] \circ [u] = [v] \circ [f]$.*

8.9. **The homotopy category of a quasicategory.** For any quasicategory, we define its **homotopy category** hC, so that $\mathrm{ob}(hC) := C_0$, $\hom_{hC}(x, y) := \hom_C(x, y)/\approx$, with composition defined by $[g] \circ [f]$. The above lemmas (8.7) and (8.8) exactly imply that hC is really a category.

We define a map $\pi\colon C \to N(hC)$ of simplicial sets as follows. On 0 simplices, π is the identity map $C_0 = N(hC)_0 = \mathrm{ob}\, hC$. On 1-simplices, the map is defined by the tautological quotient maps $\hom_C(x, y) \to \hom_C(x, y)/\approx$. The map π sends an n-simplex $a \in C_n$ to the unique n-simplex $\pi(a) \in N(hC)_n$ such that $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$. It is straightforward to check that this is well-defined: for instance, for $a \in C_2$, $\pi(a)$ is defined exactly because of the relation $[a_{12}] \circ [a_{01}] \approx [a_{02}]$.

Note that if C is an ordinary category, then $f \approx g$ if and only if $f = g$. Thus, $\pi\colon C \to N(hC)$ is an isomorphism of simplicial sets if and only if C is isomorphic to the nerve of a category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation "hC".

8.10. **Proposition.** *Let C be a quasicategory and D a small category, and let $\phi\colon C \to D$ be a map of simplicial sets. Then there exists a unique map $\psi\colon hC \to D$ such that $\psi\pi = \phi$.*

Proof. We first show existence, by constructing a suitable extension ψ, which is necessarily a functor between categories.

On objects, ψ sends $x \in \mathrm{ob}(hC) = C_0$ to $\phi(x) \in \mathrm{ob}(D) = D_0$. On morphisms, ψ sends $[f] \in \hom_{hC}(x, y)$ to $\phi(f) \in \hom_D(\phi(x), \phi(y)) \subseteq D_1$. Observe that the function on morphisms is well-defined since if $f \sim_\ell f'$, exhibited by some $a \in C_2$, then $\phi(a) \in D_2$ exhibits the identity $\phi(f) = \phi(f')\phi(1_x) = \phi(f')$. It is straightforward to show that ψ so defined is actually a functor, and that $\psi\pi = \phi$ as maps $C \to N(D)$.

The functor ψ defined above is the unique solution: the value of ψ on objects and morphisms is uniquely determined, and $\pi\colon C_k \to (hC)_k$ is bijective for $k = 0$ and surjective for $k = 1$. □

In particular, the homotopy category construction gives adjoint functors

$$h\colon \mathrm{QCat} \rightleftarrows \mathrm{Cat} : N.$$

8.11. *Remark.* If f, g, h are any edges in a quasicategory C which fit together to give $u\colon \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,2\}} \to C$, so that $u|\Delta^{\{01\}} = f$, $u|\Delta^{\{12\}} = g$, and $u|\Delta^{\{02\}} = h$, then there exists an extension of u to Δ^2 *if and only if* $\pi(h) = \pi(g)\pi(f)$ in hC. In particular, any edge in the $\approx$-equivalence class of h can be interpreted as a composite of g with f. To prove this, consider an extension of

$$\begin{tikzcd}
0 \arrow[dr, "h"] \arrow[drr, bend left, "h'"] \arrow[dd, bend right, "f"'] \arrow[dd, phantom, "a"] \arrow[drr, phantom, "b"'] & & \\
& 2 \arrow[r, "1_z"] & 3 \\
1 \arrow[ur, "g"] \arrow[urr, bend right, "g"'] \arrow[urr, phantom, "g_{011}"] & &
\end{tikzcd}$$

which exhibits $g \circ f \sim h'$ assuming $g \circ f \sim h$ and $h \sim_r h'$.

8.12. *Exercise.* Compute the homotopy categories of the various examples described in §7.

9. Saturated classes and inner-anodyne maps

Quasicategories are defined by an "extension property": they are the simplicial sets C such that any map $K \to C$ extends over L, whenever $K \subset L$ is an inner horn inclusion $\Lambda^n_j \subset \Delta^n$. The inner horn inclusions "generate" a larger class of inclusions $K \subseteq L$ which share this extension property for maps into quasicategories; this larger class includes the "saturation" of the inner horns.

For instance, we will observe that the spine inclusions $I^n \subset \Delta^n$ are in the saturation of the set of inner horn inclusions. This will imply that quasicategories admit "spine extensions", i.e., any $I^n \to C$ extends over $I^n \subset \Delta^n$ to a map $\Delta^n \to C$.

9.1. **Saturated classes.** Consider a category (such as sSet) which has all small colimits. A **saturated class**[5] is a class $\mathcal{A}$ of morphisms in the category, which

(1) contains all isomorphisms,
(2) is closed under cobase change,
(3) is closed under composition,
(4) is closed under transfinite composition,
(5) is closed under coproducts, and
(6) is closed under retracts.

Given a class of maps S, its **saturation** $\overline{S}$ is the smallest saturated class containing S.

We explain some of the elements of this definition.

- "Closed under cobase change" means that if f' is the pushout of f along some map g, then $f \in \mathcal{A}$ implies $f' \in \mathcal{A}$.
- We say that $\mathcal{A}$ is "closed under countable composition" if for every countable composable sequence
$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots ,$$
if each $f_k \in \mathcal{A}$, then the induced map $X_0 \to \operatorname{colim}_k X_k$ is in $\mathcal{A}$.
  The notion "closed under transfinite composition" is the generalization of this, where $\mathbb{N}$ is replaced by an arbitrary ordinal (i.e., a well-ordered set). Given an ordinal λ and a functor $X\colon \lambda \to s\mathrm{Set}$, if for every $i \in \lambda$ with $i \neq 0$ the map
$$\operatorname{colim}_{j<i} X(j) \to X(i)$$
is in $\mathcal{A}$, then the induced map $X(0) \to \operatorname{colim}_{j\in\lambda} X(j)$ is in $\mathcal{A}$.
- "Closed under coproducts" means that if $\{f_i\}$ is a set of maps in $\mathcal{A}$, then $\coprod_i f_i$ is in $\mathcal{A}$.

[5] These are called *weakly saturated* in [Lur09].

- We say that f is a **retract** of g if there exists a commutative diagram of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

  This is really a special case of the notion of a retract of an object in the functor category $\mathrm{Fun}([1], s\mathrm{Set})$.

9.2. *Remark.* This list of properties is not minimal. (3) is actually a special case of (4), and (5) can be deduced using (2) and (4).

9.3. *Example.* Consider the category of sets. Let $S = \{\varnothing \to \{1\}\}$ and $T = \{\{0,1\} \to \{1\}\}$. Then $\overline{S}$ = the class of injective maps, and $\overline{T}$ = the class of surjective maps.

9.4. *Example.* The class of monomorphisms of simplicial sets is a saturated class.

9.5. **Inner anodyne morphisms.** Let

$$\mathrm{InnHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k < n,\ n \geq 2 \,\}$$

denote the set of inner horn inclusions. The saturation $\overline{\mathrm{InnHorn}}$ is called the class of **inner anodyne**[6] morphisms. Note that inner anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets are themselves a saturated class.

9.6. **Proposition.** *If C is a quasicategory and $A \subseteq B$ is an inner anodyne inclusion, then any $f\colon A \to C$ admits an extension to $g\colon B \to C$ so that $g|A = f$.*

Proof. It suffices to show that the collection $\mathcal{A}$ of monomorphisms $i\colon A \to B$ such that every map from A to a quasicategory extends along i is saturated. Since $\mathrm{InnHorn} \subseteq \mathcal{A}$ we must have $\overline{\mathrm{InnHorn}} \subseteq \mathcal{A}$. This is a relatively straightforward exercise. It is highly recommended that you do this if you haven't seen it before. □

9.7. *Exercise* (Easy but important). Show that every inner anodyne map induces a bijection on vertices.

9.8. **Generalized horns and spines.** It is useful to be able to prove that certain explicit maps are inner anodyne.

Let $S \subseteq [n]$. A **generalized horn** the subcomplex $\Lambda^n_S \subset \Delta^n$ defined by

$$\Lambda^n_S := \bigcup_{i \in S} \Delta^{[n] \smallsetminus i},$$

i.e., the union of codimension one faces indexed by elements of S. In particular, $\Lambda^n_{[n] \smallsetminus \{j\}}$ is the usual horn Λ^n_j.

An **inner generalized horn** is $\Lambda^n_S \subseteq \Delta^n$ such that there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$.

9.9. *Example.* The union $\Lambda^n_{\{0,n\}} = \Delta^{\{0,1,\ldots,n-1\}} \cup \Delta^{\{1,2,\ldots n\}}$ of the "first and last" faces of Δ^n is an inner generalized horn when $n \geq 2$.

9.10. **Lemma.** *Inner generalized horn inclusions are inner anodyne.*

[6] Joyal calls these "mid-anodyne". According to the internet, "anodyne" derives from ancient Greek, meaning "without pain". The "anodyne" terminology was introduced in a related context by Gabriel and Zisman [GZ67], which we will see in §31.

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (56.1). The basic idea is simple: construct a sequence of subcomplexes

$$\Lambda^n_S = F_0 \subset F_1 \subset \cdots \subset F_d = \Delta^n$$

such that for each i, there is a subset $S_i \subseteq [n]$ such that (i) $F_i = F_{i-1} \cup \Delta^{S_i}$ and (ii) $F_{i-1} \cap \Delta^{S_i} = \Lambda^{S_i}_j \subset \Delta^{S_i}$, where this is the inclusion of an inner horn. This exhibits $\Lambda^n_S \to \Delta^n$ as a composite of maps, each of which is a cobase change of an inner horn inclusion, whence it is in $\overline{\mathrm{InnHorn}}$.

9.11. *Remark.* The idea is reminiscent of the proof of (4.9) (that nerves of categories have unique inner horn extensions). Note that in this case, however, it is crucial that when we "attach" a simplex to an F_{i-1}, the intersection must be *equal* to an inner horn.

Recall that every standard n-simplex contains a spine $I^n \subseteq \Delta^n$.

9.12. **Lemma.** *The spine inclusions $I^n \subset \Delta^n$ are inner anodyne for all n. Thus, for a quasicategory C, any $I^n \to C$ extends to $\Delta^n \to C$.*

This is proved in [Joy08a, Prop. 2.13]; we give the proof in the appendix appendix (56.2).

As a consquence, the tautological map $\pi\colon C \to hC$ from a quasicategory to its homotopy category is surjective in every degree.

10. Isomorphisms in a quasicategory

Let C be a quasicategory. We say that an edge $f \in C_1$ is an **isomorphism**[7] if its image in hC is an isomorphism.

Explicitly, $f\colon x \to y$ is an isomorphism if and only if there exists an edge $g\colon : y \to x$ such that $gf \approx 1_x$ and $fg \approx 1_y$; e.g., if there exist $g \in C_1$ and $a, b \in C_2$ such that

$$a_{01} = f = b_{12}, \qquad a_{12} = g = b_{01}, \qquad a_{02} = x_{00}, \qquad b_{02} = y_{00}.$$

$$\begin{tikzcd}
& x \arrow[rr, "1_x"] \arrow[dr, "f"] & \arrow[d, phantom, "a"] & x \\
y \arrow[ur, "g"] \arrow[rr, "1_y"'] & \arrow[u, phantom, "b"] & y \arrow[ur, "g"'] &
\end{tikzcd}$$

10.1. **Preinverses and postinverses.** Let C be a quasicategory. Given $f\colon x \to y \in C_1$, a **postinverse**[8] of f is a $g \in C_1$ such that $gf \approx 1_x$, and a **preinverse**[9] of f is an $e \in C_1$ such that $fe \approx 1_y$. An **inverse** is an $f' \in C_1$ which is both a postinverse and a preinverse. The following is trivial, but very handy.

10.2. **Proposition.** *Let $f \in C_1$. The following are equivalent.*

- *f is an isomorphism.*
- *f admits an inverse f'.*
- *f admits a postinverse g and a preinverse e.*
- *f admits a postinverse g and g admits a postinverse h.*
- *f admits a preinverse e and e admits a preinverse d.*

If these equivalent conditions hold, then $f \approx d \approx h$ and $f' \approx e \approx g$, and all of them are isomorphisms.

Proof. All of these are equivalent to the corresponding statements about morphisms in the homotopy category hC, where they are clearly equivalent. ☐

[7]Lurie uses the term "equivalence" for this. I prefer to go with "isomorphism" here, because they do in fact generalize the classical notion of isomorphism, and because so many other things also get to be called some kind of equivalence.

[8]or **left inverse**, or **retraction**,

[9]or **right inverse**, or **section**,

Note that inverses to a morphism in a quasicategory are generally far from unique, though necessarily they are unique up to homotopy.

10.3. **Quasigroupoids.** A **quasigroupoid** is a quasicategory C such that hC is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism.

10.4. *Exercise.* If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

10.5. **The core of a quasicategory.** Recall that for a category A, the **core** (or **maximal subgroupoid**) of A is the subcategory $A^{\mathrm{core}} \subseteq A$ consisting of all the objects, and all the *isomorphisms* between them.

For a quasicategory C, we define $C^{\mathrm{core}} \subseteq C$ to be the subsimplicial set consisting of simplices whose edges are all isomorphisms. That is, the diagram

$$\begin{tikzcd} C^{\mathrm{core}} \arrow[r, tail] \arrow[d] & C \arrow[d, "\pi"] \\ (hC)^{\mathrm{core}} \arrow[r, tail] & hC \end{tikzcd}$$

is a pullback. Observe that $N(A^{\mathrm{core}}) = (NA)^{\mathrm{core}}$ for a category A.

10.6. **Proposition.** *For a quasicategory C, C^{core} is a quasigroupoid, and every subcomplex of C which is a quasigroupoid is contained in C^{core}.*

Proof. First, note that C^{core} is a subcomplex by construction: if $a \in C_n$ is such that all edges are isomorphisms, then the same is true for af for any $f\colon [m] \to [n]$.

Next, we show that C^{core} is a quasicategory. In fact, we show that given $f\colon \Delta^n \to C$ such that $f(\Lambda^n_j) \subseteq C^{\mathrm{core}}$ for some $0 < j < n$, then $f(\Delta^n) \subseteq C^{\mathrm{core}}$, so that inner-horn-filling for C implies it for C^{core}. For $n = 2$ this is the fact that composites of isomorphisms are isomorphisms, while for $n \geq 3$ it is just the fact that an inner horn of Δ^n contains all its edges.

Thus, C^{core} is a quasicategory, and is easily seen to be a quasigroupoid, since an inverse of an isomorphism is also an isomorphism.

The final statement is clear: if $G \subseteq C$ is a subcomplex which is a quasigroupoid, then every edge in G has in inverse in G, and hence an inverse in C. □

10.7. **Kan complexes.** A **Kan complex** is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is, K is a Kan complex iff

$$\mathrm{Hom}(\Delta^n, K) \to \mathrm{Hom}(\Lambda^n_k, K)$$

is surjective for all $0 \leq k \leq n$, $n \geq 1$.

10.8. **Proposition.** *Every Kan complex is a quasigroupoid.*

Proof. It is immediate that a Kan complex K is a quasicategory. To show K is a quasigroupoid, note that the extension condition for $\Lambda^2_0 \subset \Delta^2$ implies that every morphism in hK admits a postinverse. Explicitly, if $f\colon x \to y$ is an edge in K, let $u\colon \Lambda^2_0 \to K$ with $u_{01} = f$ and $u_{02} = f_{00} = 1_x$, so there is an extension $v\colon \Delta^2 \to K$ and $g := v_{12}$ satisfies $gf \approx 1_x$. Use (10.4). □

This proposition has a converse.

A. **Deferred Proposition.** *Quasigroupoids are precisely the Kan complexes.*

This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will give the proof in (28.2).

Recall (§7.7) that we showed that the singular complex $\mathrm{Sing}\, T$ of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of $\mathrm{Sing}\, T$ as the **fundamental quasigroupoid** of the space T.

10.9. *Exercise.* Show that every simplicial set X has extensions for 1-dimensional horns; i.e., every $\Lambda^1_j \to X$ extends over $\Lambda^1_j \subset \Delta^1$. Thus, the stipulation that Kan complexes have extensions for 1-horns is not necessary.

10.10. *Exercise.* Show that if T is a space, then $h\,\mathrm{Sing}\,T$, the homotopy category of the singular complex of T, is precisely the usual fundamental groupoid of T.

10.11. **Quasigroupoids and components.** Recall (5.8) that the set of components of a simplicial set is given by

$$\pi_0 \approx \mathrm{colim}\big[X_1 \rightrightarrows X_0\big],$$

the coequalizer of the two face maps from edges to vertices. (If T is a topological space, then elements of $\pi_0\,\mathrm{Sing}\,T$ correspond exactly to path components of T.)

For *quasigroupoids*, π_0 recovers the set of isomorphism classes of objects.

10.12. **Proposition.** *If C is a quasicategory, then*

$$\pi_0\big(C^{\mathrm{core}}\big) \approx \textit{isomorphism classes of objects of } C.$$

10.13. *Exercise.* Show that for a quasicategory C, $\pi_0(C^{\mathrm{core}}) \approx \pi_0(h(C^{\mathrm{core}})) \approx \pi_0((hC)^{\mathrm{core}})$.

11. Function complexes and the functor quasicategory

11.1. **Function complexes.** Given simplicial sets X and Y, we may form the **function complex** $\mathrm{Map}(X, Y)$. This is a simplicial set with

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}(\Delta^n \times X, Y),$$

so that the simplicial operator δ^* is induced by $\mathrm{Hom}(\delta \times \mathrm{id}_X, Y) \colon \mathrm{Hom}(\Delta^n \times X, Y) \to \mathrm{Hom}(\Delta^m \times X, Y)$. The vertices of $\mathrm{Map}(X, Y)$ are precisely the maps $X \to Y$.

11.2. **Proposition.** *The function complex construction defines a functor*

$$\mathrm{Map} \colon s\mathrm{Set}^{\mathrm{op}} \times s\mathrm{Set} \to s\mathrm{Set}.$$

Proof. Left as an exercise. □

By construction, for each n, there is a bijective correspondence

$$\{\Delta^n \times X \to Y\} \longleftrightarrow \{\Delta^n \to \mathrm{Map}(X, Y)\}.$$

In fact, we can replace Δ^n with an arbitrary simplicial set.

11.3. **Proposition.** *For simplicial sets X, Y, Z, there is a bijection*

$$\mathrm{Hom}(X \times Y, Z) \longleftrightarrow \mathrm{Hom}(X, \mathrm{Map}(Y, Z))$$

natural in all three variables.

Proof. The bijection sends $f \colon X \times Y \to Z$ to $\widetilde{f} \colon X \to \mathrm{Map}(Y, Z)$ defined so that for $x \in X_n$, the simplex $\widetilde{f}(x) \in \mathrm{Map}(Y, Z)_n$ is represented by the composite

$$\Delta^n \times Y \xrightarrow{x \times \mathrm{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends $g \colon X \to \mathrm{Map}(Y, Z)$ to $\widetilde{g} \colon X \times Y \to Z$, defined so that for $(x, y) \in X_n \times Y_n$, the simplex $\widetilde{g}(x, y) \in Z_n$ is represented by

$$\Delta^n \xrightarrow{(\mathrm{id}, y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both $\widetilde{f}$ and $\widetilde{g}$ are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise. □

11.4. *Exercise.* Show, using the previous proposition, that there are natural isomorphisms

$$\mathrm{Map}(X \times Y, Z) \approx \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

of simplicial sets. This implies that the function complex construction makes $s\mathrm{Set}$ into a *cartesian closed category.* (Hint: show that both objects represent *isomorphic* functors $s\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$, and apply the Yoneda lemma.)

11.5. *Remark.* The construction of the function complex is not special to simplicial sets. The construction of $\mathrm{Map}(X, Y)$ (and proof of its properties) works the same way in any category of functors $C^{\mathrm{op}} \to \mathrm{Set}$, where C is a small category (such as Δ); in the general case, the role of the standard n-simplices is played by the representable functors $\mathrm{Hom}_C(-, c) \colon C^{\mathrm{op}} \to \mathrm{Set}$.

11.6. **Functor quasicategories.** Given ordinary categories C, D, we have a functor category $\mathrm{Fun}(C, D)$, with *objects:* the functors $C \to D$ and *morphisms:* the natural transformations between functors. A standard fact is that if A is another category, then there is a bijective correspondence between sets of functors

$$\{A \times C \to D\} \longleftrightarrow \{A \to \mathrm{Fun}(C, D)\}.$$

For instance, a functor $f \colon A \times C \to D$ corresponds to $\widetilde{f} \colon A \to \mathrm{Fun}(C, D)$ given on objects $a \in \mathrm{ob}\, A$ by $\widetilde{f}(a)(c) = g(a, c)$ and $\widetilde{f}(a)(\gamma) = f(1_a, \gamma)$, and on morphisms $\alpha \in \mathrm{mor}\, A$ by $\widetilde{f}(\alpha)(c) = f(\alpha, 1_c)$.

11.7. *Exercise.* Show that $N\,\mathrm{Fun}(C, D) \approx \mathrm{Map}(NC, ND)$. (Hint: use that $N[n] = \Delta^n$, and the fact that the nerve preserves finite products.)

Thus, we expect the generalization of functor category to quasicategories to be defined by the function complex. In fact, if C and D are quasicategories, then the vertices of $\mathrm{Map}(C, D)$ are precisely the functors $C \to D$, and the edges of $\mathrm{Map}(C, D)$ are precisely the natural transformations.

It turns out that a function complex between quasicategories is again a quasicategory. In fact, we have the following.

B. **Deferred Proposition.** *Let K be any simplicial set and C a quasicategory. Then* $\mathrm{Map}(K, C)$ *is a quasicategory.*

For this reason, we will sometimes write $\mathrm{Fun}(K, C)$ for $\mathrm{Map}(K, C)$ when C is a quasicategory.

To prove (B), we need a to take a detour to develop some more technology about saturated classes of maps and lifting properties. After this, we will complete the proof in §15.

Lifting properties

12. Lifting calculus

12.1. **The lifting relation.** Given morphisms f and g in a category, a **lifting problem** for (f, g) is any commutative square of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

A **lift** is a map s making the diagram commute, i.e., such that $sf = u$ and $gs = v$.

We write "$f \boxslash g$" if every lifting problem for (f, g) admits a lift. Equivalently, $f \boxslash g$ exactly if

$$\mathrm{Hom}(B, X) \xrightarrow{s \mapsto (sf, gs)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A, Y)} \mathrm{Hom}(B, Y)$$

is a surjection. One sometimes says f has the **left lifting property** relative to g, or that g has the **right lifting property** relative to f. Or we just say that f **lifts against** g**.**

We extend the notation to classes of maps, so $\mathcal{A} \boxslash \mathcal{B}$ means: $a \boxslash b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Given a class of morphisms $\mathcal{A}$, define the **right complement** $\mathcal{A}^{\boxslash}$ and **left complement** ${}^{\boxslash}\mathcal{A}$ by

$$\mathcal{A}^{\boxslash} = \{\, g \mid a \boxslash g \text{ for all } a \in \mathcal{A}\,\}, \qquad {}^{\boxslash}\mathcal{A} = \{\, f \mid f \boxslash a \text{ for all } a \in \mathcal{A}\,\}.$$

12.2. **Proposition.** *For any class $\mathcal{B}$, the left complement ${}^{\boxslash}\mathcal{B}$ is a saturated class.*

12.3. *Exercise* (Important). Prove (12.2).

The above proposition (12.2) has a dual statement: any right complement $\mathcal{B}^{\boxslash}$ is **cosaturated**, i.e., satisfies dual versions of the closure properties of a saturated class.

12.4. *Exercise* (Easy). Prove that if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^{\boxslash} \supseteq \mathcal{B}^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} \supseteq {}^{\boxslash}\mathcal{B}$. Use this to show $\mathcal{A}^{\boxslash} = ({}^{\boxslash}(\mathcal{A}^{\boxslash}))^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} = {}^{\boxslash}(({}^{\boxslash}\mathcal{A})^{\boxslash})$.

12.5. *Example.* Fix an abelian category $\mathcal{C}$. Let $\mathcal{A}$ be the class of morphisms of the form $0 \to P$ where P is projective, and let $\mathcal{B}$ be the class of epimorphisms. Then $\mathcal{A} \boxslash \mathcal{B}$; also, $\mathcal{B} = \mathcal{A}^{\boxslash}$ if $\mathcal{C}$ has enough projectives.

12.6. *Exercise.* In the previous example, identify the class ${}^{\boxslash}\mathcal{B}$.

12.7. **Inner fibrations.** A map p of simplicial sets is an **inner fibration**[10] if $\mathrm{InnHorn} \boxslash p$. The class of inner fibrations $\mathrm{InnFib} = \mathrm{InnHorn}^{\boxslash}$ is thus the right complement of the inner horns. Note that C is a quasicategory if and only if $C \to *$ is an inner fibration.

Because InnFib is a right complement, it is cosaturated. In particular, it is closed under composition. This implies that if $p\colon C \to D$ is an inner fibration *and* D is a quasicategory, then C is also a quasicategory.

12.8. *Exercise.* Show that if $f\colon C \to D$ is any functor from a quasicategory C to a *category* D, then f is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have *unique* extensions to simplices.)

12.9. **Factorizations.** It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation.

12.10. **Proposition** ("Small object argument")**.** *Let S be a* set *of morphisms in* $s\mathrm{Set}$*. Every map f between simplicial sets admits a factorization $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$.*

The proof of this proposition is by means of what is known as the "small object argument". I'll give the proof in the next section. For now we record a consequence.

12.11. **Corollary.** *For any set S of morphisms in* $s\mathrm{Set}$*, we have that $\overline{S} = {}^{\boxslash}(S^{\boxslash})$.*

Proof. That $\overline{S} \subseteq {}^{\boxslash}(S^{\boxslash})$ is immediate. Given f such that $f \boxslash S^{\boxslash}$, use the small object argument (12.10) to choose $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$. We have a commutative diagram of solid arrows

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{id}"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r, "\mathrm{id}"'] \arrow[d, "j"'] & \bullet \arrow[d, "f"'] \\
\bullet \arrow[r, dashed, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "p"] & \bullet
\end{tikzcd}$$

A map s exists making the diagram commute, because $f \boxslash p$. This exhibits f as a retract of j, whence $f \in \overline{S}$ since saturations are closed under retracts. $\square$

[10] Joyal calls these "mid-fibrations".

This is called the "retract trick": given $f = pj$, $f \boxslash p$ implies that f is a retract of j, while $j \boxslash f$ implies that f is a retract of p.

In particular, we have that $\overline{\text{InnHorn}} = {}^{\boxslash}\text{InnFib}$ and $\overline{\text{InnHorn}}^{\boxslash} = \text{InnFib}$, and thus any map can be factored into an inner anodyne map followed by an inner fibration.

12.12. **Weak factorization systems.** A **weak factorization system** in a category is a pair of classes of maps $(\mathcal{L}, \mathcal{R})$ such that

- every map f admits a factorization $f = r\ell$ with $r \in \mathcal{R}$ and $\ell \in \mathcal{L}$, and
- $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Thus, $(\overline{S}, S^{\boxslash})$ is a weak factorization in sSet for every set of maps S.

12.13. **Uniqueness of liftings.** The relation $f \boxslash g$ says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map $f\colon A \to B$, let $f^{\vee} := (f, f)\colon B \amalg_A B \to B$ be the evident fold map. It is straightforward to show that for a map $g\colon X \to Y$ we have that $\{f, f^{\vee}\} \boxslash g$ if and only if in every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

there exists a *unique* lift s.

12.14. *Example.* Consider the category of topological spaces. Let $\mathcal{A}$ be the class of morphisms of the form $A \times \{0\} \to A \times [0, 1]$, where A is an arbitrary space. Then $(\mathcal{A} \cup \mathcal{A}^{\vee})^{\boxslash}$ contains all covering maps (by the "Covering Homotopy Theorem").

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in which liftings of type $\mathcal{L} \boxslash \mathcal{R}$ are always unique is simply called a **factorization system**.

12.15. *Exercise.* Show that in a factorization system, the factorizations $f = r\ell$ are unique up to *unique isomorphism.*

12.16. *Example.* In simplicial sets, the projection map $C \to *$ is in the right complement to $S := \text{InnHorn} \cup \text{InnHorn}^{\vee}$ if and only if C is a nerve of a category. The small object argument using S, applied to a projection $X \to *$, produces a morphism $\pi\colon X \to Y$ in $\overline{S}$ with Y the nerve of a category. This Y is exactly the fundamental category hX of X described at the start of §8: given $f\colon X \to C$ with C a category, a unique extension of f over $X \to Y$ exists.

12.17. *Exercise.* Prove that if $f\colon X \to Y$ is any inner anodyne map, then the induced functor $h(f)\colon hX \to hY$ between fundamental categories is an isomorphism.

13. The small object argument

In this section we give the proof of (12.10). For the reader: it may be helpful to first work through the special case where $Y = \Delta^0$ (the terminal object in simplicial sets).

13.1. **A factorization construction.** Fix a set of maps $S = \{s_i\colon A_i \to B_i\}$ between simplicial sets. For each $f\colon X \to Y$, we will produce a factorization

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y, \qquad (Rf)(Lf) = f$$

as follows. Consider the set

$$[S,f] := \{\,(s_i,u,v)\mid s_i\in S,\ fu=vs_i\,\} = \left\{ \begin{tikzcd} A_i \arrow[r, "u"] \arrow[d, "s_i"'] & X \arrow[d, "f"] \\ B_i \arrow[r, "v"'] & Y \end{tikzcd} \right\}$$

of all commutative squares which have an arrow from S on the left-hand side, and f on the right-hand side. We define Ef, Lf, and Rf using the diagram

$$\begin{tikzcd} \coprod_{(s_i,u,v)} A_i \arrow[rr, "(u)"] \arrow[d, "\coprod s_i"'] && X \arrow[d, "Lf"'] \arrow[drr, "f"] && \\ \coprod_{(s_i,u,v)} B_i \arrow[rr] \arrow[rrrr, bend right, "(v)"'] && Ef \arrow[rr, "Rf"] && Y \end{tikzcd}$$

where the the coproducts are indexed by the set $[S,f]$, and the square is a pushout. Note that $Lf\in\overline{S}$ by construction, though we do not expect to have Rf in $S^{\boxslash}$.

We can iterate this:

$$\begin{tikzcd} X \arrow[d, equal] \arrow[dr, "Lf"] \arrow[drr, "L^2f"] \arrow[drrr, "L^3f"] \arrow[drrrrr, "L^\omega f"] &&&&& \\ X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[r, "LRf"] \arrow[dl, "Rf"] & E^2f \arrow[r, "LR^2f"] \arrow[dll, "R^2f"] & E^3f \arrow[r] \arrow[dlll, "R^3f"] & \cdots \arrow[r] & E^\omega f \arrow[dlllll, "R^\omega f"] \\ Y &&&&& \end{tikzcd}$$

here each triple $(E^\alpha f, L^\alpha f, R^\alpha f)$ is obtained by factoring the "R" map of the previous one, so that

$$E^{\alpha+1}f := E(R^\alpha f),\quad L^{\alpha+1}f := L(R^\alpha f)\circ(L^\alpha f),\quad R^{\alpha+1}f := R(R^\alpha f). \tag{13.2}$$

Taking direct limits gives a factorization $X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y$ of f, with $E^\omega f = \operatorname{colim}_{n\to\infty} E^n f$.

We can go even further, using the magic of transfinite induction, and define compatible factorizations $(E^\lambda f, L^\lambda f, R^\lambda f)$ for each ordinal[11] λ. For successor ordinals $\alpha+1$ use the prescription of (13.2), while for limit ordinals β take a direct limit $E^\beta f := \operatorname{colim}_{\alpha<\beta} E^\alpha f$ as in the construction of $E^\omega f$.

It is immediate that every $L^\alpha f\in\overline{S}$, because saturations are closed under transfinite composition. The maps $R^\alpha f$ are not generally contained in $S^{\boxslash}$, though they do satisfy a "partial" lifting property: whenever $\alpha<\beta$ there exists *by construction* a lift in any diagram of the form

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "s_i"'] & E^\alpha f \arrow[r] & E^{\alpha+1}f \arrow[r] & E^\beta f \arrow[d, "R^\beta f"] \\ B_i \arrow[rrr] \arrow[urr, dashed] &&& Y \end{tikzcd}$$

i.e., whenever $A_i\to E^\beta f$ can be factored through some $E^\alpha f\to E^\beta f$ with $\alpha<\beta$. The "small object argument" amounts to the following.

Claim. There exists an ordinal κ such that for every domain A_i of maps in S, every map $A_i\to E^\kappa f$ factors through some $E^\alpha f\to E^\kappa f$ with $\alpha<\kappa$.

Given this, we see that $R^\kappa f\in S^{\boxslash}$, and so we obtain the desired factorization. To prove the claim, we will choose κ to be a *regular cardinal* which is "bigger" than all the simplicial sets A_i.

[11]For a treatment of ordinals, see for instance the chapter on sets in [TS14].

13.3. **Regular cardinals.** The **cardinality** of a set X is the smallest ordinal λ such that there exists a bijection between X and λ; we write $|X|$ for this. Ordinals which can appear this way are called **cardinals**. For instance, the first infinite ordinal ω is the countable cardinal.

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol $\aleph_\alpha$ denotes the αth infinite cardinal, e.g., $\aleph_0 = \omega$.

Say that λ is a **regular cardinal**[12] if it is an infinite cardinal, and if for every set A of ordinals such that (i) $\alpha \in A$ implies $\alpha < \lambda$, and (ii) $|A| < \lambda$, then $\sup A < \lambda$. For instance, ω is a regular cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular[13]; however, there exist arbitrarily large regular cardinals[14].

Every ordinal α defines a category, which is the poset of ordinals strictly less than α. Colimits of functors $Y\colon \kappa \to \mathrm{Set}$ with κ a regular cardinal have the following property: the map

$$\operatorname{colim}_{\alpha<\kappa} \operatorname{Hom}(X, Y_\alpha) \to \operatorname{Hom}(X, \operatorname{colim}_{\alpha<\kappa} Y_\alpha) \tag{13.4}$$

is a bijection whenever $|X| < \kappa$. This generalizes the familiar case of $\kappa = \omega$: any map of a finite set into the colimit of a countable sequence factors through a finite stage.

13.5. *Exercise.* Prove that (13.4) is a bijection when $|X| < \kappa$.

13.6. **Small simplicial sets.** Given a regular cardinal κ, we say that a simplicial set is **κ-small** if it is isomorphic to the colimit of a functor $F\colon C \to s\mathrm{Set}$, such that (i) $|\operatorname{ob} C|, |\operatorname{mor} C| < \kappa$, and (ii) each $F(c)$ is isomorphic to a standard simplex Δ^n. Morally, we are saying that a simplicial set is κ-small if it can be "presented" with fewer than κ generators and fewer than κ relations.

Given a functor $Y\colon \kappa \to s\mathrm{Set}$ and a κ-small simplicial set X, we have a bijection as in (13.4). (This is sometimes phrased as: *κ-small simplicial sets are κ-compact.*) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal κ greater than $\sup\{|A_i|\}$.

13.7. *Example.* The standard simplices Δ^n, as well as any subcomplex such as the horns Λ^n_j, are ω-small: this is a consequence of (3.12). Thus, when we carry out the small object argument for $S = \mathrm{InnHorn}$, we can take $(E^\omega f, L^\omega f, R^\omega f)$ to be the desired factorization.

13.8. **Functoriality.** The construction $f \mapsto (X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y)$ is a functor $\mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([2], s\mathrm{Set})$, and it follows that so is $f \mapsto (X \xrightarrow{L^\alpha f} E^\alpha f \xrightarrow{R^\alpha f} Y)$ for any α. Because the choice of regular cardinal κ depends only on S, not on the map f, we see that the small object argument actually produces a *functorial factorization* of a map into a composite of an element of $\overline{S}$ with an element $S^{\boxslash}$. This will have use of this later.

14. Non-degenerate simplices and the skeletal filtration

We have noted that monomorphisms of simplicial sets form a saturated class. Here we identify a set of maps called Cell, so that the saturation of Cell is precisely the class of monomorphisms.

14.1. **Boundary of a standard simplex.** For each $n \geq 0$, we define

$$\partial\Delta^n := \bigcup_{k\in[n]} \Delta^{[n]\smallsetminus\{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the n-simplex. Equivalently,

$$(\partial\Delta^n)_k = \{\, f\colon [k] \to [n] \mid f([k]) \neq [n] \,\}.$$

Note that $\partial\Delta^0 = \varnothing$ and $\partial\Delta^1 = \Delta^{\{0\}} \amalg \Delta^{\{1\}}$.

[12]In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.

[13]For instance, $\aleph_\omega = \sup\{\, \aleph_k \mid k < \omega \,\}$.

[14]For instance, every successor cardinal $\aleph_{\alpha+1}$ is regular.

14.2. *Exercise.* Show that $\partial\Delta^n$ is the largest subcomplex of Δ^n which does not contain the "generator" $\langle 0 \dots n\rangle \in (\Delta^n)_n$. In other words, $\partial\Delta^n$ is the maximal *proper* subcomplex of Δ^n.

14.3. *Exercise.* Show that if C is a category, then the evident maps $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\partial\Delta^n, C)$ defined by restriction are isomorphisms when $n \geq 3$, but not necessarily when $n \leq 2$.

14.4. **Trivial fibrations and monomorphisms.** Let Cell be the set consisting of the inclusions $\partial\Delta^n \subset \Delta^n$ for $n \geq 0$. The resulting right complement is TFib := $\mathrm{Cell}^{\boxslash}$, the class of **trivial fibrations**.

Since the elments of Cell are monomorphisms, and the class of all monomorphisms is saturated, we see that all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. We are going to prove the converse, i.e., we will show that $\overline{\mathrm{Cell}}$ is precisely equal to the class of monomorphisms.

14.5. **Degenerate and non-degenerate simplices.** Recall $\Delta^{\mathrm{surj}}, \Delta^{\mathrm{inj}} \subset \Delta$, the subcategories consisting of all the objects and the *surjective* and *injective* order-preserving maps respectively.

A simplex $a \in X_n$ is said to be **degenerate** if there exists a *non-identity operator* $\sigma \in \Delta^{\mathrm{surj}}$ and simplex b in X such that $a = b\sigma$. Equivalently, a is degenerate if there exists a *non-injective* $f \in \Delta$ and b in X such that $a = bf$; this is because any $f \in \Delta$ factors as $f = f^{\mathrm{inj}} f^{\mathrm{surj}}$, a surjection followed by an injection, and if f is non-injective then f^{surj} is necessarily non-identity.

Likewise, a simplex $a \in X_n$ is **non-degenerate** if whenever $a = b\sigma$ for some $\sigma \in \Delta^{\mathrm{surj}}$ and simplex b in X, we must have that σ is an identity map. Equivalently, $a \in X_n$ is non-degenerate if whenever $a = bf$ for some $f \in \Delta$ and b in X, we have that $f \in \Delta^{\mathrm{inj}}$.

We write $X_n^{\mathrm{deg}}, X_n^{\mathrm{nd}} \subset X_n$ for the sets of degenerate and non-degenerate n-simplices in a simplicial set. Note that if $f \colon A \to X$ is a map of simplcial sets, then $f(A_n^{\mathrm{deg}}) \subseteq X_n^{\mathrm{deg}}$, while $f^{-1}(X_n^{\mathrm{nd}}) \subseteq A_n^{\mathrm{nd}}$.

14.6. **Proposition.** *If X is a simplicial set and $A \subseteq X$ is a subcomplex, then $A_n^{\mathrm{nd}} = X_n^{\mathrm{nd}} \cap A_n$.*

Proof. The inclusion $X_n^{\mathrm{nd}} \cap A_n \subseteq A_n^{\mathrm{nd}}$ is obvious. Conversely, let $a \in A_n^{\mathrm{nd}}$, and suppose there exists $b \in X_k$ and $\sigma \colon [n] \to [k] \in \Delta^{\mathrm{surj}}$ such that $b\sigma = a$. Observe that any surjection in Δ has a section (i.e., a preinverse), so there exists $\delta \colon [k] \to [n]$ such that $\sigma\delta = 1_{[k]}$. Thus $b = b\sigma\delta = a\delta \in A_k$. Thus b is a simplex of A, so the fact that $a \in A_n^{\mathrm{nd}}$ implies that σ is an identity map. Thus, we have proved that if a is nondegenerate in A, it must also be nondegenerate in X. $\square$

The observation is that degenerate simplicies in a simplicial set are precisely determined by knowledge of the non-degenerate simplices.

14.7. **Proposition** (Eilenberg-Zilber lemma)**.** *Let a be a simplex in X. Then there exists a* unique *pair (b, σ) consisting of a map σ in Δ^{surj} and a* non-degenerate *simplex b in X such that $a = b\sigma$.*

Proof. [GZ67, §II.3]. Given $\sigma \colon [n] \to [m]$, let $\Gamma(\sigma) = \{\, \delta \colon [m] \to [n] \mid \sigma\delta = \mathrm{id}_{[m]} \,\}$ denote the set of sections of σ. We note the following elementary observations:

- if $\sigma \in \Delta^{\mathrm{surj}}$ then $\Gamma(\sigma)$ is non-empty, and
- if $\sigma, \sigma' \in \Delta^{\mathrm{surj}}$ are such that $\Gamma(\sigma) = \Gamma(\sigma')$, then $\sigma = \sigma'$.

Let $a \in X_n$ be such that $a = b_i\sigma_i$ for $b_i \in X_{m_i}^{\mathrm{nd}}$, $\sigma_i \in \Delta^{\mathrm{surj}}([n], [m_i])$, for $i = 1, 2$. We want to show that $m_1 = m_2$, $b_1 = b_2$, and $\sigma_1 = \sigma_2$.

Pick any $\delta_1 \in \Gamma(\sigma_1)$ and $\delta_2 \in \Gamma(\sigma_2)$. Then we have

$$b_1 = b_1\sigma_1\delta_1 = a\delta_1 = b_2\sigma_2\delta_1, \qquad b_2 = b_2\sigma_2\delta_2 = a\delta_2 = b_1\sigma_1\delta_2.$$

Since b_1 and b_2 are non-degenerate, $\sigma_2\delta_1 \colon [m_1] \to [m_2]$ and $\sigma_1\delta_2 \colon [m_2] \to [m_1]$ must be injective. This implies $m_1 = m_2$, and since the only order-preserving injective map $[m] \to [m]$ is the identity map, we must have $\sigma_2\delta_1 = \mathrm{id} = \sigma_1\delta_2$, from which it follows that $b_1 = b_2$. Since δ_1 and δ_2 were arbitrary sections, we have shown that $\Gamma(\sigma_1) = \Gamma(\sigma_2)$, and therefore $\sigma_1 = \sigma_2$. $\square$

Here is an illustration of what can happen. Suppose $a \in X_2$ is a degeneracy of two simplices x and y.

- If $x, y \in X_0$, and $a = x_{000} = y_{000}$, then $x = a_0 = y$.
- If $x \in X_0$, $y \in X_1$, and $a = x_{000} = y_{001}$, then $y = a_{02} = x_{00}$, whence y is degenerate.
- If $x, y \in X_1$ and $a = x_{001} = y_{001}$, then $x = a_{01} = y$.
- If $x, y \in X_1$ and $a = x_{001} = y_{011}$, then $x = a_{12} = y_{11}$ and $y = a_{01} = x_{00}$, whence x and y are degenerate.

14.8. **Corollary.** *For any simplicial set X, the evident maps*

$$\coprod_{k \geq 0} X_k^{\mathrm{nd}} \times \mathrm{Hom}_{\Delta^{\mathrm{surj}}}([n], [k]) \to X_n$$

defined by $(x, \sigma) \mapsto x\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective morphisms $[n] \to [n']$.

Another way to say this: the restricted functor $X|(\Delta^{\mathrm{surj}})^{\mathrm{op}} \colon (\Delta^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Set}$ is *canonically isomorphic* to a coproduct of representable functors $\mathrm{Hom}_{\Delta^{\mathrm{surj}}}(-, [k])$ indexed by the nondegenerate simplicies of X. Or more simply: simplicial sets are *canonically free* with respect to degeneracy operators.

14.9. *Example.* Let $X = \Delta^k/\partial\Delta^k := \mathrm{colim}(\Delta^k \leftarrow \partial\Delta^k \to *)$. The simplicial set X has exactly two non-degenerate simplices: $* \in X_0$, and $\bar{\iota} \in X_k$ (the image of the generator $\iota \in (\Delta^k)_k$). The set X_n has exactly $1 + \binom{n}{k}$ elements: $1 = \left|\Delta^{\mathrm{surj}}([n], [0])\right|$ simplex associated to $*$, and $\binom{n}{k} = \left|\Delta^{\mathrm{surj}}([n], [k])\right|$ simplices associated to $\bar{\iota}$.

14.10. *Remark.* A simplicial set can be recovered up to isomorphism if you know (i) its sets of non-degenerate simplices, and (ii) the faces of the non-degenerate simplices. The above tells how to reconstruct the degenerate simplices; simplicial operators on degenerate simplices are computed using the fact that any simplicial operator factors into a surjection followed by an injection. *Warning.* As the above example shows, the faces of a non-degenerate simplex can be degenerate.

In the case that all faces of non-degenerate simplices in X are also non-degenerate, then we do get a functor $X^{\mathrm{nd}} \colon (\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$, and the full simplicial set X can be recovered from X^{nd}. For instance, this applies to the standard simplices Δ^n, as well as any subcomplexes of such. Functors $(\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$ are the combinatorial data behind the notion of a Δ-*complex*, as seen in Hatcher's textbook on algebraic topology [Hat02].

The following exercises give a different point of view of this principle.

14.11. *Exercise.* Fix an object $[n]$ in Δ, and consider the category $\Delta^{\mathrm{surj}}_{[n]/}$, which has

- **objects** the surjective morphisms $\sigma \colon [n] \to [k]$ in Δ, and
- **morphisms** commutative triangles in Δ of the form

$$\begin{tikzcd} & {[k]} \arrow[d, "\tau"] \\ {[n]} \arrow[ur, "\sigma"] \arrow[r, "\sigma'"'] & {[k']} \end{tikzcd}$$

Show that the category $\Delta^{\mathrm{surj}}_{[n]/}$ is *isomorphic* to the poset $\mathcal{P}(\underline{n})$ of subsets of the set $\underline{n} = \{1, \ldots, n\}$. In particular, $\Delta^{\mathrm{surj}}_{[n]/}$ is a lattice (i.e., has finite products and coproducts, called *meets* and *joins* in this context).

14.12. *Exercise.* Let X be a simplicial set. Given $n \geq 0$ and $\sigma \colon [n] \to [k]$ in Δ^{surj}, let $X_n^\sigma := \sigma^*(X_k)$, the image of the operator σ^* in X_n. Show that $X_n^{\sigma \vee \sigma'} = X_n^\sigma \cap X_n^{\sigma'}$, where $\sigma \vee \sigma'$ is join in the lattice $\Delta^{\mathrm{surj}}_{[n]/}$. Conclude that for each $x \in X_n$ there exists a maximal σ such that $x \in X_n^\sigma$.

14.13. **Skeleta.** Given a simplicial set X, the **k-skeleton** $\operatorname{Sk}_k X \subseteq X$ is the smallest subcomplex containing all simplicies of dimensions $\leq k$. For instance, $\operatorname{Sk}_{k-1} \Delta^k = \partial\Delta^k$. Using (14.8), we see that

$$(\operatorname{Sk}_k X)_n = \bigcup_{0\leq j\leq k} \{\, yf \mid y \in X_j,\ f\colon [n] \to [j] \in \Delta \,\} \approx \coprod_{0\leq j\leq k} X_j^{\mathrm{nd}} \times \operatorname{Hom}_{\Delta^{\mathrm{surj}}}([n],[j]).$$

Note that $X = \operatorname{colim}_{k\to\infty} \operatorname{Sk}_k X$. The complement of $\operatorname{Sk}_{k-1} X$ in $\operatorname{Sk}_k X$ consists precisely of the *nondegenerate* k-simplcies of X together with their degeneracies.

14.14. **Proposition.** *The evident square*

$$\begin{tikzcd} \coprod_{a\in X_k^{\mathrm{nd}}} \partial\Delta^k \arrow[r] \arrow[d, tail] & \operatorname{Sk}_{k-1} X \arrow[d, tail] \\ \coprod_{a\in X_k^{\mathrm{nd}}} \Delta^n \arrow[r] & \operatorname{Sk}_k X \end{tikzcd}$$

is a pushout of simplicial sets. More generally, for any subcomplex $A \subseteq X$, the evident square

$$\begin{tikzcd} \coprod_{a\in X_k^{\mathrm{nd}}\smallsetminus A_k^{\mathrm{nd}}} \partial\Delta^k \arrow[r] \arrow[d, tail] & A \cup \operatorname{Sk}_{k-1} X \arrow[d, tail] \\ \coprod_{a\in X_k^{\mathrm{nd}}\smallsetminus A_k^{\mathrm{nd}}} \Delta^k \arrow[r] & A \cup \operatorname{Sk}_k X \end{tikzcd}$$

is a pushout.

Proof. In each of the above squares, the complements of the vertical inclusions coincide precisely. □

14.15. **Corollary.** $\overline{\mathrm{Cell}}$ *is precisely the class of monomorphisms.*

Proof. We know all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. Any monomorphism is isomorphic to an inclusion $A \subseteq X$ of a subcomplex, so we only need show that such inclusions are contained in $\overline{\mathrm{Cell}}$. Since $X \approx \operatorname{colim}_k A \cup \operatorname{Sk}_k X$, (14.14) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of Cell. □

14.16. **Geometric realization.** Recall the the singular complex functor $\operatorname{Sing}\colon \mathrm{Top} \to s\mathrm{Set}$ (7.7). This functor has a left adjoint $|-| : s\mathrm{Set} \to \mathrm{Top}$, called **geometric realization**, constructed explicitly by

$$|X| := \operatorname{Cok}\left[\coprod_{f\colon [m]\to[n]} X_n \times \Delta^m_{\mathrm{top}} \rightrightarrows \coprod_{[p]} X_p \times \Delta^p_{\mathrm{top}}\right]; \tag{14.17}$$

that is, take a collection of topological simplices indexed by elements of X, and make identifactions according to the simplicial operators in X. (Here the symbol "Cok" represeents taking a "coequalizer", i.e., the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$.)

14.18. *Exercise.* Describe the two unlabelled maps in (14.17). Then show that $|-|$ is in fact left adjoint to Sing.

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that $|\Delta^n| \approx \Delta^n_{\mathrm{top}}$, and that $|\partial\Delta^n| \approx \partial\Delta^n_{\mathrm{top}}$. Applying this to the skeletal filtration, we discover that there are pushouts

$$\begin{tikzcd} \coprod_{a\in X_k^{\mathrm{nd}}} \partial\Delta^k_{\mathrm{top}} \arrow[r] \arrow[d, tail] & {|\mathrm{Sk}_{k-1} X|} \arrow[d, tail] \\ \coprod_{a\in X_k^{\mathrm{nd}}} \Delta^k_{\mathrm{top}} \arrow[r] & {|\mathrm{Sk}_k X|} \end{tikzcd}$$

of spaces, and that $|X| = \bigcup |\mathrm{Sk}_k X|$ with the direct limit topology. Thus, $|X|$ is given to us as a *CW-complex*; furthermore, every point in $|X|$ lies in the *interior* of a topological simplex associated to a unique non-degenerate simplex in X.

15. Box-product and box-power

We are going to prove several "enriched" versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we'll be able to prove that function complexes of quasicategories are themselves quasicategories.

15.1. **Definition of box-product and box-power.** Given maps $f\colon A \to B$, $g\colon K \to L$ and $h\colon X \to Y$ of simplicial sets, we define the **box-product**[15]

$$f\Box g\colon (A\times L)\amalg_{A\times K}(B\times K) \xrightarrow{(f\times L, B\times g)} B\times L$$

and **box-power**[16]

$$h^{\Box g}\colon \operatorname{Map}(L,X) \xrightarrow{(\operatorname{Map}(g,X),\operatorname{Map}(L,h))} \operatorname{Map}(K,X)\times_{\operatorname{Map}(K,Y)}\operatorname{Map}(L,Y).$$

15.2. *Remark.* Usually we only form $f\Box g$ when f and g are monomorphisms, in which case $f\Box g$ is also a monomorphism. In this case, the simplices $(b,\ell)\in B_n\times L_n$ which are *not* in the image of $f\Box g$ are exactly those such that $b\in B_n\smallsetminus A_n$ *and* $\ell\in L_n\smallsetminus K_n$.

15.3. *Remark* (Important!). On 0-simplices, the box-power $h^{\Box g}$ is just the "usual" map $\operatorname{Hom}(L,X)\to \operatorname{Hom}(K,X)\times_{\operatorname{Hom}(K,Y)}\operatorname{Hom}(L,Y)$ sending $s\mapsto (sg,hs)$. Thus, $h^{\Box g}$ is surjective on 0-simplcies if and only if $g \boxslash h$.

We think of the box-power as encoding "enriched" version of the lifting problem for (g,h). Thus, the target of $h^{\Box g}$ is an object which "parameterizes" commutative squares involving g and h, and often it itself will be a quasicategory or quasigroupoid. Similarly, the source of $h^{\Box g}$ "parameterizes" commutative squares together with lifts.

The product/mapping object adjunction gives rise to the following relationship between lifting problems.

15.4. **Proposition.** *We have that $(f\Box g)\boxslash h$ if and only if $f\boxslash (h^{\Box g})$.*

Proof. Compare the two lifting problems using the product/map adjunction.

$$\begin{tikzcd} (A\times L)\amalg_{A\times K}(B\times K) \arrow[r, "{(u,v)}"] \arrow[d, "f\Box g"'] & X \arrow[d, "h"] \\ B\times L \arrow[r, "w"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "\widetilde{u}"] \arrow[d, "f"'] & \operatorname{Map}(L,X) \arrow[d, "h^{\Box g}"] \\ B \arrow[r, "{(\widetilde{v},\widetilde{w})}"'] \arrow[ur, dashed, "\widetilde{s}"] & \operatorname{Map}(K,X)\times_{\operatorname{Map}(K,Y)}\operatorname{Map}(L,Y) \end{tikzcd}$$

[15] This is often called the **pushout-product**. Some also call it the **Leibniz** product, as its form is that of the Leibniz rule for boundary of a product space: $\partial(X\times Y) = (\partial X\times Y)\cup(X\times\partial Y)$ (which is itself reminiscent of the original Leibniz rule $D(fg) = (Df)g + f(Dg)$ of calculus).

[16] Sometimes called the **pullback-power** or **pullback-hom**. A common alternate notation is $g \pitchfork h$. This may also be called the **Leibniz-hom**, though I don't know what rule of calculus it is related to.

The data of (u,v,w) giving a commutative square as on the left corresponds bijectively to data $(\widetilde{u},\widetilde{v},\widetilde{w})$ giving a commutative square as on the right. Similarly, lifts s correspond bijectively to lifts $\widetilde{s}$. $\square$

It is important to note the special cases where one or more of $A=\varnothing$, $K=\varnothing$, or $Y=*$ hold. For instance, if $K=\varnothing$ and $Y=*$, the proposition implies

$$(A\times L \xrightarrow{f\times L} B\times L)\boxslash (X\to *) \quad\text{iff}\quad (A\xrightarrow{f} B)\boxslash (\mathrm{Map}(L,X)\to *).$$

This is the kind of case we are interested in for proving that $\mathrm{Map}(K,C)$ is a quasicategory whenever C is. The more general statement of the proposition is a kind of "relative" version of the thing we want; it is especially handy for carrying out inductive arguments.

15.5. *Exercise* (if you are comfortable with monoidal categories). Let $\mathcal{C}:=\mathrm{Fun}([1],s\mathrm{Set})$, the "arrow category" of simplicial sets. Show that $\square\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ defines a symmetric monoidal structure on $\mathcal{C}$, with unit object $(\varnothing\subset\Delta^0)$. Furthermore, show that this is a closed monoidal structure, with $-\square g$ left adjoint to $(-)^{\square g}\colon \mathcal{C}\to\mathcal{C}$.

15.6. **Inner anodyne maps and box-products.** The key fact we want to prove is the following.

15.7. **Proposition.** *We have that* $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}}\subseteq\overline{\mathrm{InnHorn}}$.

Proof. This is immediate from (15.8) and (15.9) below. $\square$

15.8. **Proposition.** *For any sets of maps S and T, we have* $\overline{S}\square\overline{T}\subseteq\overline{S\square T}$.

Proof. Let $\mathcal{F}=(S\square T)^{\boxslash}$. By the small object argument (12.11), $\overline{S\square T}={}^{\boxslash}\mathcal{F}$. First we show that $\overline{S}\square T\subseteq\overline{S\square T}$. Consider

$$\begin{aligned}\mathcal{A}&:=\{\,a\mid (a\square T)\boxslash\mathcal{F}\,\}\\ &\approx\{\,a\mid a\boxslash(\mathcal{F}^{\square T})\,\}\end{aligned}$$

by the previous proposition. Thus $\mathcal{A}$ is a left complement, and so is saturated. Since $S\subseteq\mathcal{A}$ then $\overline{S}\subseteq\mathcal{A}$, i.e., $\overline{S}\square T\subseteq{}^{\boxslash}\mathcal{F}=\overline{S\square T}$. Now consider

$$\mathcal{B}:=\{\,b\mid(\overline{S}\square b)\boxslash\mathcal{F}\,\}\approx\{\,b\mid b\boxslash(\mathcal{F}^{\square\overline{S}})\,\},$$

which is likewise saturated. We have just shown that $T\subseteq\mathcal{B}$, whence $\overline{T}\subseteq\mathcal{B}$, i.e., $\overline{S}\square\overline{T}\subseteq\overline{S\square T}$. $\square$

15.9. **Lemma.** *We have* $\mathrm{InnHorn}\square\mathrm{Cell}\subseteq\overline{\mathrm{InnHorn}}$.

Proof. This is a calculation, given in [Joy08a, App. H]. $\square$

Let's carry out the proof of (15.9) explicitly in the case of $(\Lambda^2_1\subset\Delta^2)\square(\partial\Delta^1\subset\Delta^1)$. This is an inclusion

$$(\Lambda^2_1\times\Delta^1)\cup_{\Lambda^2_1\times\partial\Delta^1}(\Delta^2\times\partial\Delta^1)\subset\Delta^2\times\Delta^1,$$

whose target is a "prism", and whose source is a "trough". To show this is in $\overline{\mathrm{InnHorn}}$, we'll give an explicit procedure for constructing the prism from the trough by succesively attaching simplices along inner horns.

Note that $\Delta^2\times\Delta^1=N([2]\times[1])$, so we are working inside the nerve of a poset, whose elements (objects) are "ij" with $i\in\{0,1,2\}$ and $j\in\{0,1\}$. Here is a picture of the trough, showing all the non-degenerate simplicies.

$$\begin{tikzcd}
01 \arrow[r] \arrow[rr, bend left] & 11 \arrow[r] & 21 \\
00 \arrow[u] \arrow[r] \arrow[ur] \arrow[rr, bend right] & 10 \arrow[u] \arrow[r] \arrow[ur] & 20 \arrow[u]
\end{tikzcd}$$

The complement of this in the prism consists of three non-degenerate 3-simplices, five non-degenerate 2-simplices (two of which form the "lid" of the trough, while the other three are in the interior of the prism), and one non-degenerate 1-simplex (separating the two 2-simplices which form the lid).

The following chart lists all non-degenerate simplices in the complement of the trough, along with their codimension one faces (in order). The "$\surd$" marks simplices which are contained in the trough.

$\langle 00,21\rangle$	$\langle 00,20,21\rangle$	$\langle 00,01,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,10,20,21\rangle$	$\langle 00,10,11,21\rangle$	$\langle 00,01,11,21\rangle$
$\surd\langle 21\rangle$	$\surd\langle 20,21\rangle$	$\surd\langle 01,21\rangle$	$\surd\langle 10,21\rangle$	$\surd\langle 11,21\rangle$	$\surd\langle 10,20,21\rangle$	$\surd\langle 10,11,21\rangle$	$\surd\langle 01,11,21\rangle$
$\surd\langle 10\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,20,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,11,21\rangle$
	$\surd\langle 00,20\rangle$	$\surd\langle 00,01\rangle$	$\surd\langle 00,10\rangle$	$\surd\langle 00,11\rangle$	$\langle 00,10,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,01,21\rangle$
					$\surd\langle 00,10,20\rangle$	$\surd\langle 00,10,11\rangle$	$\surd\langle 00,01,11\rangle$

Note that the simplices $\langle 00,21\rangle$, $\langle 00,10,21\rangle$, and $\langle 00,11,21\rangle$ of the complement appear multiple times as faces. We can attach simplices to the domain in the following order:

$$①\langle 00,20,21\rangle, \quad ②\langle 00,10,20,21\rangle, \quad ③\langle 00,10,11,21\rangle, \quad ④\langle 00,01,11,21\rangle.$$

In each case, the intersection of the simplex with (domain+previously attached simplices) is an inner horn. This directly exhibits the box product as an inner anodyne map.

16. Function complexes of quasicategories are quasicategories

16.1. **Enriched lifting properties.** We record the immediate consequences of $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (15.7).

16.2. **Proposition.**

(1) *If $i\colon A \to B$ is inner anodyne and $j\colon K \to L$ a monomorphism, then*
$$i\square j\colon (A\times L)\cup_{A\times K}(B\times K) \to B\times L$$
is inner anodyne.

(2) *If $j\colon K\to L$ is a monomorphism and $p\colon X\to Y$ is an inner fibration, then*
$$p^{\square j}\colon\ \mathrm{Map}(L,X)\to \mathrm{Map}(K,X)\times_{\mathrm{Map}(K,Y)}\mathrm{Map}(L,Y)$$
is an inner fibration.

(3) *If $i\colon A\to B$ is inner anodyne and $p\colon X\to Y$ is an inner fibration, then*
$$p^{\square i}\colon\ \mathrm{Map}(B,X)\to \mathrm{Map}(A,X)\times_{\mathrm{Map}(A,Y)}\mathrm{Map}(B,Y)$$
is a trivial fibration.

These can be summarized as
$$\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}, \qquad \mathrm{InnFib}^{\square\overline{\mathrm{Cell}}} \subseteq \mathrm{InnFib}, \qquad \mathrm{InnFib}^{\square\overline{\mathrm{InnHorn}}} \subseteq \mathrm{TFib}.$$

Statement (1) is just restating (15.7). Statements (2) and (3) can be thought of as "enriched" lifting properties. In particular, statement (3) implies $i \boxslash p$, since trivial fibrations are necessarily surjective on vertices.

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert "$\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$" without other explanation.

There are many useful special cases, obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If $i\colon A\to B$ is inner anodyne, so is $i\times \mathrm{id}_L\colon A\times L\to B\times L$.
- If $p\colon X\to Y$ is an inner fibration, then so is $\mathrm{Map}(L,p)\colon\ \mathrm{Map}(L,X)\to\mathrm{Map}(L,Y)$.
- If $j\colon K\to L$ is a monomorphism and C a quasicategory, then $\mathrm{Map}(j,C)\colon\ \mathrm{Map}(L,C)\to \mathrm{Map}(K,C)$ is an inner fibration.
- If $i\colon A\to B$ is inner anodyne and C a quasicategory, then $\mathrm{Map}(i,C)\colon\ \mathrm{Map}(B,C)\to \mathrm{Map}(A,C)$ is a trivial fibration.
- If C is a quasicategory, so is $\mathrm{Map}(L,C)$. Thus we have proved (B).

Let's spell out the proof of (B) in a little more detail. Because $\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$, we have (15.7) that

$$(\Lambda^n_j \subset \Delta^n)\square(\varnothing \subseteq K) = (\Lambda^n_j \times K \to \Delta^n \times K)$$

is inner anodyne for any K and $0 < j < n$. Thus, for any diagram

$$\begin{tikzcd} \Lambda^n_j \times K \arrow[r] \arrow[d] & C \\ \Delta^n \times K \arrow[ur, dashed] \end{tikzcd}$$

with C a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend $\Lambda^n_j \to \operatorname{Map}(K, C)$ along $\Lambda^n_j \subset \Delta^n$. That is, we have proved that $\operatorname{Map}(K, C)$ is a quasicategory.

16.3. *Remark.* Most saturated classes $\overline{S}$ that we will explictly discuss in these notes will have the property that $\overline{S}\square\overline{\text{Cell}} \subseteq \overline{S}$, and thus the analogues of the above remarks will hold for such classes.

16.4. *Exercise* (Important). Show that $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$. (Hint: (14.15).) State the analogue of (16.2) associated to this inclusion.

16.5. **Composition functors.** We can use the above theory to construct "composition functors". If C is an ordinary category, the operation of composing a sequence of n maps can be upgraded to a functor:

$$\operatorname{Fun}([1], C) \times \cdots \times \operatorname{Fun}([1], C) \to \operatorname{Fun}([1], C)$$

which on *objects* describes composition of a sequence of maps.

We can generalize this to quasicategories. We use the following observation: any trivial fibration admits a section, since $(\varnothing \to Y) \boxslash (p\colon X \to Y)$ if p is a trivial fibration (16.2).

Let C be a quasicategory. Then map $r\colon \operatorname{Fun}(\Delta^n, C) \to \operatorname{Fun}(I^n, C)$ induced by restriction along $I^n \subseteq \Delta^n$ is a trivial fibration by (16.2), since the spine inclusion is inner anodyne (9.12). Therefore r admits a section s, whence a diagram

$$\begin{tikzcd} \operatorname{Fun}(I^n, C) \arrow[r, bend left, "s"] & \operatorname{Fun}(\Delta^n, C) \arrow[l, "r"] \arrow[r, "r'"] & \operatorname{Fun}(\Delta^{\{0,n\}}, C) \end{tikzcd}$$

where r' is restriction along $\Delta^{\{0,n\}} \subset \Delta^n$. The composite $r's$ can be thought of as a kind of "n-fold composition" functor. It is not unique, since s isn't, but we'll see (**??**) this is not such a big deal: all functors constructed this way are "naturally isomorphic" to each other.

16.6. **A useful variant.** There is a variant of (15.7) which proves strictly more, and requires a bit less calculation.

16.7. **Proposition.** *We have that* $\overline{\{\Lambda^2_1 \subset \Delta^2\}\square\text{Cell}} = \overline{\text{InnHorn}}$.

Proof. See [Joy08a, §2.3.1], [Lur09, §2.3.2].

Here is the idea. First do the grungy calculation (part of the proof of (15.9), see (57.1)) to show that

$$\{\Lambda^2_1 \subset \Delta^2\}\square\text{Cell} \subseteq \overline{\text{InnHorn}}.$$

Next, consider $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$ defined by

$$s(y) = \begin{cases} (0, y) & \text{if } y < j, \\ (1, y) & \text{if } y = j, \\ (2, y) & \text{if } y > j, \end{cases} \qquad r(x, y) = \begin{cases} y & \text{if } x = 0 \text{ and } y < j, \\ y & \text{if } x = 2 \text{ and } y > j, \\ j & \text{otherwise.} \end{cases}$$

These explicitly exhibit $(\Lambda^n_j \subset \Delta^n)$ as a retract of $(\Lambda^2_1 \subset \Delta^2)\square(\Lambda^n_j \subset \Delta^n)$, so $\text{InnHorn} \subseteq \{\Lambda^2_1 \subset \Delta^2\}\square\overline{\text{Cell}}$. Taken together, these give the claim. $\square$

Given (16.7), we can recover (15.7) as a corollary. To see this, observe that if S is a set of maps and $\mathcal{A} = \overline{S\Box\mathrm{Cell}}$, then $\overline{\mathcal{A}\Box\mathrm{Cell}} \subseteq \mathcal{A}$; the proof is a straightforward consequence of the fact that $\mathrm{Cell}\Box\mathrm{Cell} \subseteq \overline{\mathrm{Cell}}$ (16.4). Apply this observation to $\mathcal{A} = \overline{\mathrm{InnHorn}}$.

Another consequence is a new characterization of quasicategories.

16.8. **Corollary.** *A simplicial set C is a quasicategory if and only if $f\colon \mathrm{Map}(\Delta^2, C) \to \mathrm{Map}(\Lambda^2_1, C)$ is a trivial fibration.*

Proof. Apply (16.7) together with the fact that $(\partial\Delta^k \subset \Delta^k) \boxslash f$ for all $k \geq 0$ iff $(\partial\Delta^k \subset \Delta^k)\Box(\Lambda^2_1 \subset \Delta^2) \boxslash (C \to *)$ for all $k \geq 0$. ◻

17. Natural isomorphisms

17.1. **Natural isomorphisms of functors.** Let C and D be quasicategories. Recall that a natural transformation between functors $f_0, f_1\colon C \to D$ is a map $a\colon C \times \Delta^1 \to D$ such that $a|C \times \Delta^{\{i\}} = f_i$, or equivalently a morphism $a\colon f_0 \to f_1$ in the quasicategory $\mathrm{Fun}(C, D)$.

Say that $a\colon f_0 \to f_1$ is a **natural isomorphism** if a is an isomorphism in the quasicategory of functors $\mathrm{Fun}(C, D)$. Thus, a is a natural isomorphism iff there exists a natural transformation $b\colon f_1 \to f_0$ such that $ba \sim 1_{f_0}$ and $ab \sim 1_{f_1}$.

This notion of natural isomorphism corresponds with the usual one for ordinary categories.

Observe that "there exists a natural isomorphism $f_0 \to f_1$" is an equivalence relation on functors $C \to D$, as this relation precisely coincides with "there exists an isomorphism $f_0 \to f_1$" in the category $h\,\mathrm{Fun}(C, D)$.

Furthermore, the "naturally isomorphic" relation is compatible with composition: if f, f' are naturally isomorphic and g, g' are naturally isomorphic, then so are gf and $g'f'$. This is read off from the fact that the composition map $\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, E)$ is a functor between quasicategories.

17.2. **Objectwise criterion for natural isomorphisms.** Recall that if A and B are categories, a natural transformation $\alpha\colon f_0 \to f_1$ between functors $f_0, f_1\colon A \to B$ is a natural isomorphism iff and only if α is "an isomorphism objectwise"; i.e., if for each object a of A the evident map $\alpha(a)\colon f_0(a) \to f_1(a)$ is an isomorphism in B. That natural isomorphisms have this property is immediate. The other direction follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its "values on objects". Thus, the inverse to α is uniquely determined by the inverses to the isomorphisms $\alpha(a)\colon f_0(a) \to f_1(a)$ for each $a \in \mathrm{ob}\,A$.

One of these directions is straightforward for quasicategories.

17.3. **Proposition.** *Let C and D be quasicategories. If $\alpha\colon C \times \Delta^1 \to D$ is a natural isomorphisms between functors $f_0, f_1\colon C \to D$, then for each object c of C the induced map $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism in D.*

Proof. The restriction map $\mathrm{Fun}(C, D) \to \mathrm{Fun}(\{c\}, D) = D$ is a functor between quasicategories, and so preserves isomorphisms. It sends α to $\alpha(c)$. ◻

The converse to this proposition is also true.

C. **Deferred Proposition.** *A natural transformation $\alpha\colon C \times \Delta^1 \to D$ of functors between quasicategories is a natural isomorphism if and only if each of the maps $\alpha(c)$ are isomorphisms in D.*

Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the $\alpha(c)$s to produce an inverse to α, which though it exists is not at all unique. We will prove this converse later.

18. Categorical equivalence

We are now in position to define the correct generalization of the notion of "equivalence" of categories. This will be called *categorical equivalence*.

We will first define a notion of categorical equivalence between quasicategories, by direct analogy to the notion of equivalences of categories. Then we will use this to give a definition of categorical equivalences which applies to arbitrary maps of simplicial sets. Finally, we will show that the two definitions agree for maps between quasicategories.

18.1. **Categorical equivalences between quasicategories.** A **categorical inverse** to a functor $f\colon C \to D$ between quasicategories is a functor $g\colon D \to C$ such that gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D. We provisionally say that a functor f between quasicategories is a **categorical equivalence** if it admits a categorical inverse.

18.2. *Remark.* Categorical equivalence between quasicategories is a kind of "homotopy equivalence", where homotopies are natural isomorphism between functors.

If C and D are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, and that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

18.3. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then $h(f)\colon hC \to hD$ is an equivalence of categories.*

Proof. Immediate, given that natural transformations $f \Rightarrow g\colon C \to D$ induce natural transformations $h(f) \Rightarrow h(g)\colon hC \to hD$. □

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to Δ^0, but whose homotopy categories are: e.g., $\operatorname{Sing} T$ for any non-contractible simply connected space T, or $K(A,d)$ for any non-trivial abelian group A and $d \geq 2$.

18.4. **General categorical equivalence.** We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map $f\colon X \to Y$ between arbitrary simplicial sets is a **categorical equivalence** if for every quasicategory C, the induced map $\operatorname{Fun}(f,C)\colon \operatorname{Fun}(Y,C) \to \operatorname{Fun}(X,C)$ of quasicategories admits a categorical inverse.

We claim that on maps between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

18.5. **Lemma.** *For a map $f\colon C \to D$ between quasicategories, the two notions of categorical equivalence described above coincide; i.e., f admits a categorical inverse iff it is a categorical equivalence in the general sense.*

To prove this, we will need the following observation. First note that for any simplicial sets X, Y, and E we have a natural map

$$\gamma_0\colon \operatorname{Hom}(X,Y) \to \operatorname{Hom}(\operatorname{Map}(Y,E), \operatorname{Map}(X,E))$$

of sets, and these fit together to give a functor $\operatorname{Map}(-,E)\colon s\mathrm{Set}^{\mathrm{op}} \to s\mathrm{Set}$. The observation we need is that this construction admits an "enrichment", to a map

$$\gamma\colon \operatorname{Map}(X,Y) \to \operatorname{Map}(\operatorname{Map}(Y,E), \operatorname{Map}(X,E)),$$

which coincides with γ_0 on vertices. The map γ is defined to be adjoint to the "composition" map $\operatorname{Map}(X,Y) \times \operatorname{Map}(Y,E) \to \operatorname{Map}(X,E)$. (*Exercise:* Describe explicitly what γ does to n-simplices.) We say that the functor $\operatorname{Map}(-,E)$ is an *enriched* functor, as it gives not merely a map between hom-sets (i.e., acts on 0-simplices in function complexes), but in fact gives a map between function complexes.

Proof. ($\Longrightarrow$) When C, D, and E are quasicategories so are the function complexes between them. In this case, the above map γ takes functors $C \to D$ to functors $\operatorname{Map}(D,E) \to \operatorname{Map}(C,E)$, natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Furthermore, the "enriched composition" is itself compatible with composition of functors. It follows a categorical inverse $g\colon D \to C$ to $f\colon C \to D$ gives rise to a categorical inverse $\operatorname{Map}(g,E)$ to the induced functor $\operatorname{Map}(f,E)\colon \operatorname{Map}(D,E) \to \operatorname{Map}(C,E)$.

($\Longleftarrow$) Conversely, suppose $f\colon C \to D$ is a categorical equivalence in the general sense, so that $f^* = \operatorname{Map}(f,E)$ admits a categorical inverse for every quasicategory E, which implies that each functor

$$h(f^*)\colon h\operatorname{Fun}(D,E) \to h\operatorname{Fun}(C,E)$$

is an equivalence of ordinary categories. In particular, it follows that f^* induces a bijection of sets

$$f^*\colon \pi_0(\operatorname{Fun}(D,E)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\operatorname{Fun}(C,E)^{\mathrm{core}});$$

recall that $\pi_0(\operatorname{Fun}(D,E)^{\mathrm{core}}) \approx \pi_0((h\operatorname{Fun}(D,E))^{\mathrm{core}})$ is precisely the set of *natural isomorphism classes* of functors $D \to E$.

Setting $E = C$, we use this to see that there must exist $g \in \operatorname{Fun}(D,C)_0$ together with a natural isomorphism $gf \to \mathrm{id}_C$ in $\operatorname{Fun}(C,C)_1$. Setting $E = D$, we note that since

$$f^*(\mathrm{id}_D) = \mathrm{id}_D f = f\,\mathrm{id}_C \approx fgf = f^*(fg),$$

we must have that $\mathrm{id}_D \approx fg$, i.e., there exists a natural isomorphism $\mathrm{id}_D \to fg$ in $\operatorname{Fun}(D,D)_1$. Thus, we have shown that g is a categorical inverse of f, as desired. $\square$

18.6. *Exercise.* In the above proof, we said that enriched composition is "compatible with composition". Give an explicit statement of what this means, and prove it.

18.7. *Remark.* The definition of categorical equivalence we are using here is *very* different to the definition adopted by Lurie [Lur09, §2.2.5]. It is also slightly different from the notion of "weak categorical equivalence" used by Joyal [Joy08a, 1.20]. As we will show soon (21.10), Joyal's weak categorical equivalence is equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] show's that Lurie's and Joyal's definitions are equivalent, and so they are both equivalent to the one we have used.

18.8. *Exercise.* Let $f\colon C \to D$ be a functor between quasicategories. Show that f is a categorical equivalence if and only if for all simplicial sets X, the induced functor $f_*\colon \operatorname{Map}(X,C) \to \operatorname{Map}(X,D)$ is a categorical equivalence.

19. Trivial fibrations and inner anodyne maps

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.

19.1. **Trivial fibrations.** Recall that a trivial fibration $p\colon X \to Y$ of simplicial sets is a map such that $(\partial\Delta^k \subset \Delta^k)\square p$ for all $k \geq 0$. That is, $\mathrm{TFib} = \mathrm{Cell}^{\boxslash}$, so p is a trivial fibration if and only if $\mathrm{Cell} \boxslash p$.

19.2. *Exercise.* Consider an indexed collection of trivial fibrations $p_i\colon X_i \to Y_i$. Show that $p := \coprod p_i\colon \coprod X_i \to \coprod Y_i$ is a trivial fibration.

19.3. **Trivial fibrations to the terminal object.**

19.4. **Proposition.** *Let $p\colon X \to *$ be a trivial fibration whose target is the terminal simplicial set. Then X is a Kan complex (and thus a quasigroupoid) and p is a categorical equivalence.*

Proof. The hypotheses together with $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ (16.4) imply that $\text{Hom}(B, X) \to \text{Hom}(A, X)$ is surjective for all inclusions $A \subseteq B$. Thus in particular X is a Kan complex, by taking $A \subset B$ to be a horn inclusion.

To show that p is a categorical equivalence, first note that X is non-empty, since $\text{Hom}(\Delta^0, X) \to \text{Hom}(\varnothing, X) = *$ is surjective. Choose any $s \in \text{Hom}(\Delta^0, X)$ (2.6). Clearly $ps = \text{id}_{\Delta^0}$. We will show that $sp\colon X \to X$ is naturally isomorphic to id_X. Consider

$$\begin{tikzcd} X \amalg X \arrow[r, equal] & X \times \partial\Delta^1 \arrow[r, "(\text{id}_X, sp)"] \arrow[d, tail] & X \arrow[d, "p"] \\ & X \times \Delta^1 \arrow[r] \arrow[ur, dotted, "h"] & * \end{tikzcd}$$

Since p is a trivial fibration, a lift h exists, which is a natural transformation $\text{id}_X \to sp$; note that h represents a morphism in $\text{Fun}(X, X)$. To show that h represents an isomorphism, it's enough to know that $\text{Fun}(X, X)$ is actually a quasigroupoid. Observe that (since p is a trivial fibration and $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$), the map

$$\text{Fun}(X, p)\colon \text{Fun}(X, X) \to \text{Fun}(X, *) = *$$

is also a trivial fibration, whence $\text{Fun}(X, X)$ is a Kan complex by the argument given above. $\square$

We will prove a converse to this later (32.9): if C is a quasicategory which is categorically equivalent to $*$, then $C \to *$ is a trivial fibration.

19.5. **Preisomorphisms.** We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories.

Let X be a simplicial set. Say that an edge $a \in X_1$ is a **preisomorphism** if it projects to an isomorphism under $\alpha\colon X \to hX$, the tautological map to the fundamental category. If X is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category.

19.6. **Proposition.** *An edge $a \in X_1$ is a preisomorphism if and only if for every map $g\colon X \to C$ to a quasicategory C, the image $g(a)$ is an isomorphism in C.*

Proof. Isomorphisms in C are exactly the edges which are sent to isomorphisms under $\gamma\colon C \to hC$. Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that hX is itself a category and hence a quasicategory. $\square$

As a consequence, any map $X \to Y$ of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map *from* a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.

19.7. *Example.* Consider the subcomplex $\Lambda^3_{\{0,3\}} = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$ of Δ^3. Define X to be the pushout of the diagram

$$\Lambda^3_{\{0,3\}} \xleftarrow{j} \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,3\}} \cup \Delta^{\{2,3\}} \xrightarrow{p} \Delta^{\{y<x\}},$$

where j is the evident inclusion, and p is the unique map to a 1-simplex given on vertices by $0, 2 \mapsto y$, $1, 3 \mapsto x$. The resulting complex X looks like

$$\begin{tikzcd} y \arrow[r, "y_{00}"] \arrow[d, "g"'] \arrow[dr, phantom, "a" near start] & y \arrow[d, "g"] \\ x \arrow[r, "x_{00}"'] \arrow[ur, "f"] \arrow[ur, phantom, "b" near end] & x \end{tikzcd}$$

with six non-degenerate simplices x, y, f, g, a, b. The simplicial set X is not a quasicategory; however, any map $\phi\colon X \to C$ to a quasicategory sends f and g to morphisms of C which are inverse to each other, and so these (and thus all) edges of X are preisomorphisms.

Say that vertices in a simplicial set X are **preisomorphic** if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map $g\colon X \to C$ to a quasicategory takes preisomorphic vertices to isomorphic objects.

We can apply this to function complexes. If two maps $f_0, f_1\colon X \to Y$ are preisomorphic (viewed as vertices in $\mathrm{Map}(X,Y)$), then for any quasicategory C, the induced functors $\mathrm{Map}(f_0, C), \mathrm{Map}(f_1, C)\colon \mathrm{Map}(Y,C) \to \mathrm{Map}(X,C)$ are naturally isomorphic. To see this, consider

$$\Delta^1 \xrightarrow{a} \mathrm{Map}(X,Y) \xrightarrow{b} \mathrm{Map}(\mathrm{Map}(Y,C), \mathrm{Map}(X,C))$$

where b is adjoint to the composition map $\mathrm{Map}(Y,C) \times \mathrm{Map}(X,Y) \to \mathrm{Map}(X,C)$. If a represents a preisomorphism $f_0 \to f_1$ in $\mathrm{Map}(X,Y)$, then ba represents an isomorphism $\mathrm{Map}(f_0, C) \to \mathrm{Map}(f_1, C)$, since the target of b is a quasicategory. As a consequence we get the following.

19.8. **Lemma.** *If $p\colon X \to Y$ and $q\colon Y \to X$ are maps of simplicial sets such that qp is preisomorphic to id_X and pq is preisomorphic to id_Y, then p and q are categorical equivalences.*

19.9. Trivial fibrations are always categorical equivalences.

19.10. **Proposition.** *Every trivial fibration between simplicial sets is a categorical equivalence.*

Here is some notation. Given maps $f\colon A \to Y$ and $g\colon B \to Y$, we write $\mathrm{Map}_{/Y}(f,g)$ or $\mathrm{Map}_{/Y}(A,B)$ for the simplicial set defined by the pullback square

$$\begin{array}{ccc} \mathrm{Map}_{/Y}(A,B) & \longrightarrow & \mathrm{Map}(A,B) \\ \downarrow & & \downarrow{\scriptstyle g_* = \mathrm{Map}(A,g)} \\ \{f\} & \longrightarrow & \mathrm{Map}(A,Y) \end{array}$$

That is, $\mathrm{Map}_{/Y}(A,B)$ can be thought of as a simplicial set which parameterizes sections of g over f. I'll call this the **relative function complex over** Y.

Proof. Fix a trivial fibration $p\colon X \to S$. We regard both X and S as objects *over* S, via p and id_s, and consider various relative function complexes over S.

Note that since p is a trivial fibration, so are $\mathrm{Map}(X,p)$ and $\mathrm{Map}(S,p)$ since $\overline{\mathrm{Cell}} \Box \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$, and therefore both

$$\mathrm{Map}_{/S}(S,X) \to \mathrm{Map}_{/S}(S,S) = * \qquad \text{and} \qquad \mathrm{Map}_{/S}(X,X) \to \mathrm{Map}_{/S}(X,S) = *$$

are trivial fibrations (since TFib is cosaturated and so closed under base change). It follows from (19.4) that both $\mathrm{Map}_{/S}(S,X)$ and $\mathrm{Map}_{/S}(S,Y)$ are quasigroupoids which are non-empty, and with the property that any pair of objects in each are isomorphic. These are contained in the simplicial sets $\mathrm{Map}(S,X)$ and $\mathrm{Map}(X,X)$ respectively, which however need not be quasicategories.

Pick any $s \in \mathrm{Map}_{/S}(S,X) \subseteq \mathrm{Map}(S,X)$, whence $ps = \mathrm{id}_S$, and pick any isomorphism $a\colon \mathrm{id}_X \to sp$ in $\mathrm{Map}_{/S}(X,X)$, which is hence a preisomorphism in $\mathrm{Map}(X,X)$.

Thus, we have exhibited maps p and s whose composites are preisomorphic to identity functors, and therefore they are categorical equivalences by (19.8). □

19.11. Inner anodyne maps are always categorical equivalences.

19.12. **Proposition.** *Every inner anodyne map between simplicial sets is a categorical equivalence.*

Proof. Let $j\colon X \to Y$ be a map in $\overline{\mathrm{InnHorn}}$, and let C be a quasicategory. The induced map $\mathrm{Map}(j,C)\colon \mathrm{Map}(Y,C) \to \mathrm{Map}(X,C)$ is a trivial fibration since $\overline{\mathrm{InnHorn}} \Box \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (16.2), and therefore a categorical equivalence. □

19.13. **Every simplicial set is categorically equivalent to a quasicategory.**

19.14. **Proposition.** *Fix a simplicial set X.*

(1) *There exists a quasicategory C and an inner anodyne map $f\colon X \to C$, which is therefore a categorical equivalence.*
(2) *For any two $f_i\colon X \to C_i$ as in (1), there exists a categorical equivalence $g\colon C_1 \to C_2$ such that $gf_1 = f_2$.*

Here is some more notation. Given maps $f\colon X \to A$ and $g\colon X \to B$, we write $\mathrm{Map}_{X/}(f,g)$ or $\mathrm{Map}_{X/}(A,B)$ for the simplicial set defined by the pullback square

$$\begin{tikzcd} \mathrm{Map}_{X/}(A,B) \arrow[r] \arrow[d] & \mathrm{Map}(A,B) \arrow[d, "f^*=\mathrm{Map}(f,B)"] \\ \{g\} \arrow[r] & \mathrm{Map}(X,B) \end{tikzcd}$$

This is the **relative function complex under** X.

Proof. (1) By the small object argument (12.10), we can factor $X \to *$ into $X \xrightarrow{j} C \xrightarrow{p} *$ where $j \in \overline{\mathrm{InnHorn}}$ and $p \in \mathrm{InnFib}$.

(2) For $i,j \in \{1,2\}$, we have a restriction map $f_i^*\colon \mathrm{Map}(C_i,C_j) \to \mathrm{Map}(X,C_j)$, which is necessarily a trivial fibration since $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$. Therefore the maps $\mathrm{Map}_{X/}(C_i,C_j) \to *$ are all trivial fibrations, i.e., each $\mathrm{Map}_{X/}(C_i,C_j)$ is a quasigroupoid with only one isomorphism class of objects (19.4). As in the proof of (19.10) we construct $g\colon C_1 \to C_2$ and $g'\colon C_2 \to C_1$ which are categorically inverse to each other; details are left to the reader. $\square$

Thus, we can always "replace" a simplicial set X by a categorically equivalent quasicategory C. Although such C is not unique, it is unique up to categorical equivalence.

We think of C as the quasicategory "freely generated" by the simplicial set X, an idea which is validated by the fact that $\mathrm{Fun}(j,D)\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(X,D)$ is a categorical equivalence for every quasicategory D.

20. Some examples of categorical equivalences

20.1. **Free monoid on one generator.** Let F denote the free monoid on one generator g. This is a category with one object x, and morphisms $\{\, g^n \mid n \geq 0\,\}$.

Associated to the generator g is a map

$$\gamma\colon S := \Delta^1/\partial\Delta^1 \to N(F).$$

We use "L/K" as a shorthand for "$L \amalg_K *$" whenever $K \subseteq L$. The object S is the "simplicial circle".

It is not hard to see that F is "freely generated" as a category by S, in the sense that $hS = F$. It turns out that $N(F)$ is also freely generated as a quasicategory by S.

20.2. **Proposition.** *The map $\gamma\colon S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is an explicit calculation. Let $a_k \in N(F)_k$ denote the k-simplex corresponding to the length k chain of maps $(g,g,\dots,g)$ in F where g is the generator, and let $Y_k \subseteq N(F)$ denote the subcomplex which is the image of $a_k\colon \Delta^k \to N(F)$. We have that $\bigcup Y_k = N(F)$, since a general d-simplex in $N(F)$ corresponds to a chain of maps $(g^{m_1},\dots,g^{m_d})$, which is evidently a face of a_m if $m \geq m_1 + \cdots + m_d$.

Clearly, $Y_1 \approx S$, while $Y_2 \approx Y_1 \cup_{\Lambda^2_1} \Delta^2$. In general, you can easily verify the following.

- A simplex x of Δ^k is such that $a_k(x)$ is in the subcomplex Y_{k-1} of Y_k if and only if x is in the subcomplex $\Lambda^k_{\{0,k\}} = \Delta^{\{0,\dots,k-1\}} \cup \Delta^{\{1,\dots,k\}}$.
- Every simplex y of Y_k not in Y_{k-1} is the image under a_k of a *unique* simplex in Δ^k.

In other words, the square

$$\begin{tikzcd} \Lambda^k_{\{0,k\}} \arrow[r] \arrow[d, tail] & Y_{k-1} \arrow[d, tail] \\ \Delta^k \arrow[r, "a_k"'] & Y_k \end{tikzcd}$$

is a pullback, and a_k induces in each degree n a bijection $(\Delta^k)_n \smallsetminus (\Lambda^k_{\{0,k\}})_n \xrightarrow{\sim} (Y_k)_n \smallsetminus (Y_{k-1})_n$. It follows (20.3) that the square is a pushout.

The inclusion $\Lambda^k_{\{0,k\}} \subset \Delta^k$ is a *generalized inner horn*, and we have noted this is inner anodyne when $k \geq 2$ (9.10). It follows that each $Y_{k-1} \to Y_k$ is inner anodyne, whence $S \to N(F)$ is inner anodyne. $\square$

In the proof, we used the following fact, which is worth recording.

20.3. **Lemma.** *If*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "i"'] & X \arrow[d, tail, "j"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback of simplicial sets such that (i) j is a monomorphism, and (ii) f induces in each degree n a bijection $Y'_n \smallsetminus i(X'_n) \xrightarrow{\sim} Y_n \smallsetminus i(X_n)$, then the square is a pushout square.

Proof. Verify the analogous statement for a pullback square of sets. $\square$

20.4. **Free categories.** We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let S be a **1-dimensional** simplicial set, i.e., one such that $S = \mathrm{Sk}_1 S$. These are effectively the same thing as *directed graphs* (allowed to have multiple parallel edges and loops): S_0 corresponds to the set of vertices, and S_1^{nd} corresponds to the set of edges.

Let $F := hS$. In this case, the morphisms of the fundamental category are precisely the words in the edges S_1^{nd} of the directed graph (including empty words for each vertex, corresponding to identity maps).

20.5. **Proposition.** *The evident map $\gamma\colon S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is virtually the same as the proof of (20.2). In this case, $Y_k \subseteq N(F)$ is the subcomplex generated by all $a\colon \Delta^k \to N(F)$ such that each spine-edge $a_{i-1,i}$ is in S_1^{nd}, and Y_k is obtained by attaching a generalized horn to Y_{k-1} for each such a. $\square$

As a consequence, it is "easy" to construct functors $F \to C$ from a free category to a quasicategory: start with a map $S \to C$, which amounts to specifying vertices and edges in C corresponding to S_0 and S_1^{nd}, and extend over $S \subseteq F$. The evident restriction map $\mathrm{Fun}(F, C) \to \mathrm{Map}(S, C)$ is a categorical equivalence, and in fact a trivial fibration.

20.6. *Exercise.* Describe the ordinary category $A := h\Lambda^3_0$ "freely generated" by Λ^3_0. Show that the tautological map $\Lambda^3_0 \to N(A)$ is inner anodyne.

20.7. **Free commutative monoids.** Let F be the free monoid on one generator again, with generator corresponding to simplicial circle $S = \Delta^1/\partial\Delta^1 \subset F$ Recall that $F^{\times n}$ is the free commutative monoid on n generators. Recall that the nerve functor preserves products, so $N(F^{\times n}) \approx N(F)^{\times n}$. We obtain a map

$$\delta = \gamma^{\times n} \colon S^{\times n} \to N(F^{\times n})$$

from the "simplicial n-torus".

20.8. **Proposition.** *The map $\delta \colon S^{\times n} \to N(F^{\times n})$ is a categorical equivalence, and in fact is inner anodyne.*

This is an easy consequnce of the following.

20.9. **Proposition.** *For any simplicial set K, if $j \colon A \to B$ is inner anodyne, so is $j \times K \colon A \times K \to B \times K$.*

Proof. The map $j\Box(\varnothing \subseteq K)$ is inner anodyne, since $\overline{\text{InnHorn}}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ by (15.7). $\square$

20.10. *Exercise.* Let $S \vee S \subseteq S^{\times 2}$ be the subcomplex obtained as the evident "one-point union" of the two "coordinate circles". Suppose given a map $\phi \colon S \vee S \to C$ to a quasicategory C, corresponding to a choice of object $x \in C_0$ together with two morphisms $f, g \colon x \to x$ in C_1. Show that there exists an extension of ϕ along $S \vee S \subset N(F^{\times 2})$ if and only if $fg \approx gf$ in hC.

20.11. *Remark.* The analogue of the above exercise for $n = 3$ isn't true. That is, given $S \vee S \vee S \to C$ corrposding to three morphisms $f, g, h \colon x \to x$ in C such that each pair "commutes" in hC, there need not exist an extension to $N(F^{\times 3})$. (For instance, take $C = \operatorname{Sing} T$, where $T \subseteq (S^1)^{\times 3}$ is the subspace of the 3-torus consisting of (x_1, x_2, x_3) such that at least one x_i is the basepoint of S^1.)

20.12. **Finite groups are not finite.** If A is any ordinary category, then $\operatorname{Sk}_2 N(A)$ "freely generates $N(A)$ as a category", in the sense that $h(\operatorname{Sk}_2 N(A)) \approx A$, or equivalently that $\operatorname{Map}(N(A), N(B)) \to \operatorname{Map}(\operatorname{Sk}_2 N(A), N(B))$ is an isomorphism for any category B. However, it is often the case that *no finite dimensional subcomplex* "freely generates $N(A)$ as a quasicategory". In fact, this is the case for every non-trivial finite group.

20.13. *Example.* Let G be a finite group, and let $C = N(G)$. The geometric realization $BG := |N(G)|$ is the *classifying space* of G. I want to show that if G is not the trivial group, then NG is not categorically equivalent to any *finite* dimensional simplicial set K (i.e., one with no non-degenerate simplices above a certain dimension). We need to use some homotopy theory, along with a fact to be proved later: if $f \colon X \to Y$ is any categorical equivalence of simplicial sets, then the induced map $|f| : |X| \to |Y|$ on geometric realizations is a homotopy equivalence of spaces.

First consider $G = \mathbb{Z}/n$ with $n > 1$. A standard calculation in topology says that $H^{2k}(|N(G)|, \mathbb{Z}) \approx \mathbb{Z}/n \not\approx 0$ for all $k > 0$. This implies that $|N(G)|$ cannot be homotopy equivalent to any finite dimensional complex.

Now consider a general non-trivial finite group G, and choose a non-trivial cyclic subgroup $H \leq G$. We know the fundamental group: $\pi_1 |K| \approx \pi_1 |N(G)| = G$. Covering space theory tells us we can construct a covering map $p \colon E \to |K|$ so that $\pi_1 E \to \pi_1 |K|$ is the inclusion $H \to G$. In fact, E is homotopy equivalent to the classifying space BH (because $\pi_k E \approx 0$ for $k \geq 2$). Since $|K|$ is a finite dimensional complex, so is E, whence $H^*(BH, \mathbb{Z}) \approx H^*(E, \mathbb{Z}) \approx 0$ for large $*$, contradicting the observation above.

Thus, non-trivial *finite* groups are never "freely generated as a quasicategory" by finite dimensional complexes.

20.14. *Remark.* Let T be a *finite* CW-complex, and G a finite group. A theorem of Haynes Miller (the "Sullivan conjecture") implies that every functor $N(G) \to \operatorname{Sing} T$ is naturally isomorphic to a constant functor (i.e., one which factors through Δ^0). Thus, the singular complex of a finite

CW-complex cannot "contain" any non-trivial finite group, even if its fundamental group does contain a non-trivial finite subgroup.

21. The homotopy category of quasicategories

21.1. **The homotopy category of** QCat**.** The homotopy category hQCat of quasicategories is defined as follows. The objects of hQCat are the quasicategories. Morphisms $C \to D$ in hQCat are natural isomorphism classes of functors. That is,

$$\mathrm{Hom}_{h\mathrm{QCat}}(C, D) := \text{isomorphism classes of objects in } h\,\mathrm{Fun}(C, D) = \pi_0\big(\mathrm{Fun}(C, D)^{\mathrm{core}}\big).$$

That this defines a category results from the fact that composition of functors passes to a functor $h\,\mathrm{Fun}(D, E) \times h\,\mathrm{Fun}(C, D) \to h\,\mathrm{Fun}(C, E)$, and thus is compatible with natural isomorphism.

Note that a map $f : C \to D$ of quasicategories is a categorical equivalence if and only if its image in hQCat is an isomorphism.

21.2. *Remark.* We can similarly define a category hCat, whose objects are categories and whose morphisms are *isomorphism classes* of functors. The nerve functor evidently induces a full embedding $h\mathrm{Cat} \to h\mathrm{QCat}$.

21.3. *Warning.* Although simililar in spirit, the definition of hQCat given above is not an example of the notion of the homotopy category of a quasicategory defined in §8: QCat is a (large) ordinary category, so is isomorphic to its own homotopy category in that sense. Here we are using the equivalence relation on morphisms(=functors) defined by natural isomorphism.

For future reference, we note that hQCat has finite products, which just amount to the usual products of simplicial sets.

21.4. **Proposition.** *The terminal simplicial set Δ^0 is a terminal object in hQCat. If C_1, C_2 are quasicategories, then the projection maps exhibit $C_1 \times C_2$ as a product in hQCat.*

Proof. This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each, and the fact that $\mathrm{Map}(X, C_1 \times C_2) = \mathrm{Map}(X, C_1) \times \mathrm{Map}(X, C_2)$. □

21.5. **The 2-out-of-3 property.** A class of morphisms $\mathcal{W}$ in a category is said to satisfy the **2-out-of-3 property** if whenever two maps in a triple of the form $(f, g, g \circ f)$ are in $\mathcal{W}$, so is the third.

21.6. *Example.* In any category, the class of isomorphisms satisfies 2-out-of-3, as does the class of identity maps.

21.7. *Example.* In Cat, the category of small categories, the class of equivalences satisfies 2-out-of-3.

21.8. **Proposition.** *The class* CatEq *of categorical equivalences in* sSet *satisfies 2-out-of-3.*

Proof. First, consider the special case of $h = gf$, where f, g, h are functors between *quasicategories.* In this case, we use the following fact:

- A map $f : C \to D$ between quasicategories is a categorical equivalence *if and only if* its image $[f]$ in hQCat is an isomorphism.

The proof of this fact amounts to showing that g is a categorical inverse of f exactly if $[g]$ is an inverse of $[f]$ in hQCat, which amounts to translating the definition of categorical inverse. Given this, 2-out-of-3 follows from the fact that the class of *isomorphisms* in hQCat satisfies 2-out-of-3.

For maps $h = gf$ between arbitrary simplicial sets, we reduce to the above case by considering $\mathrm{Fun}(h, B) = \mathrm{Fun}(f, B)\,\mathrm{Fun}(g, B)$ where B is an arbitrary quasicategory. □

21.9. **Weak categorical equivalence.** Joyal [Joy08a, 1.20] uses a variant of the notion of categorical equivalence, which turns out to be equivalent to what we are using. A map $f\colon X \to Y$ of simplicial sets is a **weak categorical equivalence**[17] if for every quasicategory C, the induced map $h\operatorname{Fun}(Y,C) \to h\operatorname{Fun}(X,C)$ is an equivalence of ordinary categories. Note that, like categorical equivalences, the class of weak categorical equivalences also satisfies 2-out-of-3.

21.10. **Proposition.** *A map is a categorical equivalence if and only if it is a weak categorical equivalence.*

Proof. ($\Longrightarrow$) Straightforward. ($\Longleftarrow$) In the case that f is a weak categorical equivalence between *quasicategories*, this is exactly what the second half of the proof of (18.5) really shows. For a general map f, use factorization to construct a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

so that u and v are inner anodyne (and so categorical equivalences), and X' and Y' are quasicategories. Applying $h\operatorname{Fun}(-,C)$ to the square with C a quasicategory, we see that the vertical maps become equivalences of categories, so if f is weak categorical equivalence so is f', which is then a categorical equivalence by what we have already proved, whence f is a categorical equivalence by 2-out-of-3. □

21.11. **The homotopy 2-category of** QCat**.** A **2-category** E is a category enriched over categories. That is, for each pair of objects $x, y \in \operatorname{ob} E$, there is a category $\operatorname{Hom}_E(x,y)$. The objects of $\operatorname{Hom}_E(x,y)$ are the **1-morphisms** $f\colon x \to y$ of E, and the morphisms of $\operatorname{Hom}_E(x,y)$ are the **2-morphisms** $\alpha\colon f \Rightarrow g$ of E. The **underlying category** of E consists of the objects and 1-morphisms only.

The standard example of a 2-category is Cat, the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category QCat of quasicategories to a **homotopy 2-category** $h_2\mathrm{QCat}$, so that

$$\operatorname{Hom}_{h_2\mathrm{QCat}}(C,D) := h\operatorname{Fun}(C,D).$$

That is,

- **objects** are quasicategories,
- **1-morphisms** are functors between quasicategories,
- **2-morphisms** are *isomorphism classes* of natural transformations of functors.

Note that QCat sits inside $h_2\mathrm{QCat}$ as its underlying category; thus, $h_2\mathrm{QCat}$ contains all the information of QCat. On the other hand $h\mathrm{QCat}$ is obtained from $h_2\mathrm{QCat}$ by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms.

Joins, slices, and Joyal's extension and lifting theorems

22. Joins

22.1. **Join of categories.** If A and B are ordinary categories, we can define a category $A \star B$ called the join. This has

$$\operatorname{ob}(A\star B) = \operatorname{ob} A \amalg \operatorname{ob} B, \qquad \operatorname{mor}(A\star B) = \operatorname{mor} A \amalg (\operatorname{ob} A \times \operatorname{ob} B) \amalg \operatorname{mor} B,$$

[17] This is not to be confused with "weak equivalence", which we will talk about later.

so that we put in a *unique* map from each object of A to each object of B. Explicitly,

$$\operatorname{Hom}_{A\star B}(x,y) := \begin{cases} \operatorname{Hom}_A(x,y) & \text{if } x,y\in \operatorname{ob} A,\\ \operatorname{Hom}_B(x,y) & \text{if } x,y\in \operatorname{ob} B,\\ \{*\} & \text{if } x\in \operatorname{ob} A,\ y\in \operatorname{ob} B,\\ \varnothing & \text{if } x\in \operatorname{ob} B,\ y\in \operatorname{ob} A,\end{cases}$$

with composition defined so that the evident inclusions $A \to A\star B \leftarrow B$ are functors. (Check that this really defines a category.)

22.2. *Example.* We have that $[p]\star[q] \approx [p+1+q]$.

An important special case are the **left cone** and **right cone** of a category, defined by $A^{\triangleleft} := [0]\star A$ and $A^{\triangleright} := A\star[0]$. For instance, $A^{\triangleright}$ is the category obtained by adjoining one additional object v to A, as well as a unique map $x\to v$ for each object x of $A^{\triangleright}$. In this case, v becomes a terminal object for $A^{\triangleright}$, and we can say that $A\mapsto A^{\triangleright}$ *freely adjoints a terminal object* to A. (Note that a terminal object of A will not be terminal in $A^{\triangleright}$ anymore; this is why we say "freely".)

Limits and colimits of functors can be characterized using cones: if $F\colon A\to B$ is a functor, a colimit of F is a functor $F'\colon A^{\triangleright}\to B$ which is initial among functors which extend F (more on this below).

22.3. *Exercise.* Show that functors $f\colon C\to A\star B$ are in bijective correspondence with triples of functors $(g\colon C\to[1],\ f_{\{0\}}\colon C^{\{0\}}\to A,\ f_{\{1\}}\colon C^{\{1\}}\to B)$, where $C^{\{j\}} := g^{-1}(\{j\})\subseteq C$.

22.4. **Ordered disjoint union.** As noted above (22.2), the join operation on categories effectively descends to Δ. We will call this the **ordered disjoint union**. It is a functor $\sqcup\colon \Delta\times\Delta\to\Delta$, defined so that $[p]\sqcup[q] := [p+1+q]$, to be thought of as the disjoint union of underlying sets, ordered so that the subsets $[p]$ and $[q]$ retain their ordering, and elements of $[p]$ come before elements of $[q]$.

It is handy to extend this to the category Δ_+, the subcategory of ordered sets obtained by adjoining $[-1] := \varnothing$ to Δ. The functor $\sqcup$ extends in an evident way to $\sqcup\colon \Delta_+\times\Delta_+\to\Delta_+$. This extended functor makes Δ_+ into a (nonsymmetric but strict) monoidal category, with unit object $[-1]$.

Note that for any map $f\colon [p]\to[q_1]\sqcup[q_2]$ in Δ_+, there is a *unique* decomposition $[p]=[p_1]\sqcup[p_2]$ such that $f=f_1\sqcup f_2$ for some (necessarily unique) $f_i\colon [p_i]\to[q_i]$ in Δ_+. Note that we need an object $[-1]$ to be able to say this, even if $p,q_1,q_2\geq 0$; if $f([p])\subseteq[q_1]$ then $p_2=-1$.

22.5. **Join of simplicial sets.** Let X and Y be simplicial sets. The **join** of X and Y is a simplicial set $X\star Y$ defined as follows.

The join of simplicial sets X and Y is defined by

$$(X\star Y)_n := \coprod_{[n]=[n_1]\sqcup[n_2]} X_{n_1}\times Y_{n_2},$$

where we set $X_{-1} = * = Y_{-1}$. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: $(x,y)f = (xf_1, yf_2)$ where $f=f_1\sqcup f_2$, $f_j\colon [m_j]\to[n_j]$.

In particular,

$$\begin{aligned}(X\star Y)_0 &= X_0 \amalg Y_0,\\ (X\star Y)_1 &= X_1 \amalg X_0\times Y_0 \amalg Y_1,\\ (X\star Y)_2 &= X_2 \amalg X_1\times Y_0 \amalg X_0\times Y_1 \amalg Y_2,\end{aligned}$$

and so on.

The functor $\star$ defines a monoidal structure on sSet, with unit object $\varnothing$. Note however that $\star$ is *not* symmetric monoidal, though it is true that $(Y \star X)^{\mathrm{op}} \approx X^{\mathrm{op}} \star Y^{\mathrm{op}}$. We have that

$$\Delta^p \star \Delta^q \approx \Delta^{p+1+q}.$$

An important example are the cones. The **left cone** and **right cone** of a simplicial set X are

$$X^{\triangleleft} := \Delta^0 \star X, \qquad X^{\triangleright} := X \star \Delta^0.$$

Note that outer horns are cones:

$$(\partial\Delta^n)^{\triangleleft} = \Delta^0 \star \partial\Delta^n \approx \Lambda^{n+1}_0, \qquad (\partial\Delta^n)^{\triangleright} = \partial\Delta^n \star \Delta^0 \approx \Lambda^{n+1}_{n+1}.$$

22.6. *Exercise.* Let $f\colon [m] \to [n]$ be any simplicial operator. Show that the induced map $f\colon \Delta^m \to \Delta^n$ is isomorphic to a join of maps $f_0 \star f_1 \star \cdots \star f_n$, of the form $f_j\colon \Delta^{m_j} \to \Delta^0$, where each $m_j \geq -1$ (with the convention that $\Delta^{-1} = \varnothing$).

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets: $N(A \star B) \approx N(A) \star N(B)$, and thus $N(A^{\triangleleft}) \approx (NA)^{\triangleleft}$ and $N(A^{\triangleright}) \approx (NA)^{\triangleright}$.

22.7. **The join of quasicategories is a quasicategory.** Here is a handy rule for constructing maps into a join. Note that every join admits a canonical map $\pi\colon X \star Y \to \Delta^0 \star \Delta^0 \approx \Delta^1$.

22.8. **Lemma** ([Joy08a, Prop. 3.5])**.** *Maps $f\colon K \to X \star Y$ are in bijective correspondence with the set of triples*

$$\big(g\colon K \to \Delta^1, \quad f_{\{0\}}\colon K^{\{0\}} \to X, \quad f_{\{1\}}\colon K^{\{1\}} \to Y\big),$$

where $K^{\{j\}} := g^{-1}(\{j\}) \subseteq K$.

Proof. This is a straightforward exercise; think about the special case where $K = \Delta^n$. In one direction, the correspondence sends f to $(\pi f, f|K^{\{0\}}, f|K^{\{1\}})$. $\square$

22.9. **Proposition.** *If C and D are quasicategories, so is $C \star D$.*

Proof. Use the previous lemma, together with the observation (not hard to prove) that for any map $g\colon \Lambda^n_j \to \Delta^1$ from an *inner* horn, the preimages $g^{-1}(\{0\})$ and $g^{-1}(\{1\})$ are either inner horns or simplices, or are empty. $\square$

23. Slices

23.1. **Slices of categories.** Given an ordinary category C, and an object $x \in \operatorname{ob} C$, we may form the **slice categories** $C_{x/}$ and $C_{/x}$, also called **undercategory** and **overcategory**.

For instance, the overcategory $C_{/x}$ is the category whose *objects* are maps $f\colon c \to x$ with target x, and whose *morphisms* $(f\colon c \to x) \to (f'\colon c' \to x)$ are maps $g\colon c \to c'$ such that $f'g = f$.

This can be reformulated in terms of joins. Let "T" denote the terminal category (isomorphic to $[0]$). Note that $\operatorname{ob} C_{/x}$ corresponds to the set of functors $f\colon [0] \star T \to C$ such that $f|T = x$, and $\operatorname{mor} C_{/x}$ corresponds to the set of functors $g\colon [1] \star T \to C$ such that $g|T = x$.

More generally, given a functor $p\colon A \to C$ of categories, we obtain slice categories $C_{p/}$ and $C_{/p}$ defined as follows. The category $C_{/p}$ has

- **objects:** functors $f\colon [0] \star A \to C$ such that $f|A = p$,
- **morphisms** $f \to f'$**:** functors $g\colon [1] \star A \to C$ such that $g|A = p$.

Likewise, the category $C_{p/}$ has

- **objects:** functors $f\colon A \star [0] \to C$ such that $f|A = p$,
- **morphisms** $f \to f'$**:** functors $g\colon A \star [1] \to C$ such that $g|A = p$.

23.2. *Exercise.* Show that $(C_{p/})^{\mathrm{op}} \approx (C^{\mathrm{op}})_{p^{\mathrm{op}}/}$ (isomorphism of categories).

23.3. *Exercise.* Fix a functor $p\colon A \to C$, and let B be a category. Construct bijections

$$\{\text{functors } f\colon B \to C_{/p}\} \leftrightarrow \{\text{functors } g\colon B \star A \to C \text{ s.t. } g|A = p\}$$

and

$$\{\text{functors } f\colon B \to C_{p/}\} \leftrightarrow \{\text{functors } g\colon A \star B \to C \text{ s.t. } g|A = p\}.$$

23.4. **Joins and colimits.** The join functor $\star\colon s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$ is in some ways analogous to the product functor $\times$, e.g., it is a monoidal functor.

Remember that $\times$ commutes with colimits in each input, and that the functors $X \times -$ and $- \times X$ admit right adjoints (in both cases, the right adjoint is $\mathrm{Map}(X, -)$). The join functor does not commute with colimits in each variable, but *almost* does so; the only obstruction is the value on the initial object

More precisely, the functors $X \star -$ and $- \star X\colon s\mathrm{Set} \to s\mathrm{Set}$ do not preserve the initial object, since $X \star \varnothing \approx X \approx \varnothing \star X$. However, (the identity map of) X is tautologically the initial object of $s\mathrm{Set}_{X/}$.

23.5. **Proposition.** *For every simplicial set X, the induced functors*

$$X \star -,\ - \star X\colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$$

preserve colimits.

Proof. This is immediate from the degreewise formula for the join:

$$(X \star Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg Y_n = X_n \amalg (\text{terms which are “linear” in } Y).$$

$\square$

23.6. *Exercise* (Trivial, but important). Show that the functors $X \star -$ and $- \star X\colon s\mathrm{Set} \to s\mathrm{Set}$ preserve *pushouts*.

23.7. **Slices of simplicial sets.** We have seen that the functors $S \star -\colon s\mathrm{Set} \to s\mathrm{Set}_{S/}$ and $- \star T\colon s\mathrm{Set} \to s\mathrm{Set}_{T/}$ preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called **slice** functors, denoted

$$(p\colon S \to X) \mapsto X_{p/}\colon s\mathrm{Set}_{S/} \to s\mathrm{Set}$$

and

$$(q\colon S \to X) \mapsto X_{/q}\colon s\mathrm{Set}_{S/} \to s\mathrm{Set}.$$

Explicitly, there are are bijective correspondences

$$\left\{ \begin{tikzcd} S \arrow[r, "p"] \arrow[d] & X \\ S \star K \arrow[ur, dashed] \end{tikzcd} \right\} \Longleftrightarrow \{K \dashrightarrow X_{p/}\}, \qquad \left\{ \begin{tikzcd} T \arrow[r, "q"] \arrow[d] & X \\ K \star T \arrow[ur, dashed] \end{tikzcd} \right\} \Longleftrightarrow \{K \dashrightarrow X_{/q}\}.$$

Taking $K = \Delta^n$ we obtain the formulas

$$(X_{p/})_n = \mathrm{Hom}_{s\mathrm{Set}_{S/}}(S \star \Delta^n, X), \qquad (X_{/q})_n = \mathrm{Hom}_{s\mathrm{Set}_{T/}}(\Delta^n \star T, X),$$

which we regard as the definition of slices.

In particular, we note the special cases associated to $x\colon \Delta^0 \to X$:

$$(X_{x/})_n = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(\Delta^0 \star \Delta^n, X) \approx \mathrm{Hom}_{s\mathrm{Set}_*}((\Delta^{n+1}, 0), (X, x)),$$

$$(X_{/x})_n = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(\Delta^n \star \Delta^0, X) \approx \mathrm{Hom}_{s\mathrm{Set}_*}((\Delta^{n+1}, n+1), (X, x)).$$

The notation (X, x) represents a pointed simplical set, the category of which is $s\mathrm{Set}_* := s\mathrm{Set}_{\Delta^0/}$. The associated adjunctions are

$$\mathrm{Hom}_{s\mathrm{Set}_*}(K^{\lhd}, (X, x)) \approx \mathrm{Hom}_{s\mathrm{Set}}(K, X_{x/}), \qquad \mathrm{Hom}_{s\mathrm{Set}_*}(K^{\rhd}, (X, x)) \approx \mathrm{Hom}_{s\mathrm{Set}}(K, X_{/x}).$$

The slice construction for simplicial sets agrees with that for categories.

23.8. **Proposition.** *The nerve preserves slices; i.e., if $p\colon A \to C$ is a functor, then $N(C_{p/}) \approx (NC)_{Np/}$ and $N(C_{/p}) \approx (NC)_{/Np}$.*

Proof. Straightforward. □

23.9. **Slice as a functor.** The function complex $\mathrm{Map}(-,-)$ is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables. A precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

$$\begin{tikzcd} S \arrow[r, "p"] & X \arrow[d, "f"] \\ T \arrow[u, "j"] \arrow[r, "fpj"'] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X_{/p} \arrow[r] \arrow[d] & Y_{/fp} \arrow[d] \\ X_{/pj} \arrow[r] & Y_{/fpj} \end{tikzcd} \qquad \begin{tikzcd} X_{p/} \arrow[r] \arrow[d] & Y_{fp/} \arrow[d] \\ X_{pj/} \arrow[r] & Y_{fpj/} \end{tikzcd}$$

There seems to be no decent notation for the maps in the right-hand squares. The whole business of joins and slices can get pretty confusing because of this. A precise formulation is that slice defines a functor $s\mathrm{Set}^{\mathrm{tw}} \to s\mathrm{Set}$ from the *twisted arrow category* of simplicial sets, whose objects are maps p of simplicial sets, and whose morphisms are $(j, f)\colon p \to fpj$.

Let's spell this out. Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider "restriction map" $X_{p/} \to Y_{fpj/}$. The composite of a map $u\colon K \to X_{p/}$ with this restriction map is described as follows. The map u corresponds to a dotted arrow in

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, tail] & S \arrow[r, "p"] \arrow[d, tail] & X \arrow[r, "f"] & Y \\ T \star K \arrow[r, "j \star K"'] & S \star K \arrow[ur, dashed, "\widetilde{u}"'] & & \end{tikzcd}$$

The composite $K \xrightarrow{u} X_{p/} \to Y_{fpj/}$ corresponds to $f\widetilde{u}(j \star K)$.

24. Initial and terminal objects

We now show that for a vertex x in a quasicategory C, the slice objects $C_{/x}$ and $C_{x/}$ are also quasicategories. Using this, we can make a definition of initial and terminal object in a quasicategory.

24.1. **Left and right fibrations.** We introduce the following notation. The **left horns** are the set of inclusions

$$\mathrm{LHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 \le k < n,\ n \ge 1 \,\} = \mathrm{InnHorn} \cup \{\, \Lambda^n_0 \subset \Delta^n \mid n \ge 1 \,\}$$

and the **right horns** are set of inclusions

$$\mathrm{RHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k \le n,\ n \ge 1 \,\} = \mathrm{InnHorn} \cup \{\, \Lambda^n_n \subset \Delta^n \mid n \ge 1 \,\}.$$

The associated saturations $\overline{\mathrm{LHorn}}$ and $\overline{\mathrm{RHorn}}$ are the **left anodyne** and **right anodyne** maps. The associated right complements

$$\mathrm{LFib} := \mathrm{LHorn}^{\boxslash}, \qquad \mathrm{RFib} := \mathrm{RHorn}^{\boxslash}$$

are the **left fibrations** and **right fibrations**. Note that $\mathrm{LFib} \cup \mathrm{RFib} \subseteq \mathrm{InnFib}$.

These classes correspond to each other under the opposite involution $(-)^{\mathrm{op}}\colon s\mathrm{Set} \to s\mathrm{Set}$; i.e., $\mathrm{LHorn}^{\mathrm{op}} = \mathrm{RHorn}$, $\mathrm{LFib}^{\mathrm{op}} = \mathrm{RFib}$.

24.2. **Proposition.** *Let C be a quasicategory and $x \in C_0$. The evident maps $C_{x/} \to C$ and $C_{/x} \to C$ which "forget x" are left fibration and right fibration respectively. In particular, $C_{x/}$ and $C_{/x}$ are also quasicategories.*

Proof. I claim that $\pi\colon C_{/x} \to C$ is a right fibration. Explicitly, this map sends the n-simplex $a\colon \Delta^n \to C_{/x}$, which corresponds to $\widetilde{a}\colon \Delta^n \star \Delta^0 \to C$ such that $\widetilde{a}|(\varnothing \star \Delta^0) = x$, to the n-simplex $\widetilde{a}|(\Delta^n \star \varnothing) \to C$. Using the join/slice adjunction, there is a bijective correspondence between lifting problems

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "f"] \arrow[d, tail] & C_{/x} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dashed] & C \end{tikzcd} \iff \begin{tikzcd} \varnothing \star \Delta^0 \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Lambda^n_j \star \Delta^0) \cup_{\Lambda^n_j \star \varnothing} (\Delta^n \star \varnothing) \arrow[r, "(\widetilde{f},g)"] \arrow[d, tail] & C \\ & \Delta^n \star \Delta^0 \arrow[ur, dashed] & \end{tikzcd}$$

The right hand diagram is isomorphic to

$$\begin{tikzcd} \{n+1\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_j \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

If C is a quasicategory, then an extension exists for $0 < j \leq n$.

Since right fibrations are inner fibrations, the composite $C_{/x} \to C \to *$ is an inner fibration, and thus $C_{/x}$ is a quasicategory.

The case of $C_{x/} \to C$ is similar, using the correspondence

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail] & C_{x/} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & C \end{tikzcd} \iff \begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_{j+1} \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

□

24.3. **Initial and terminal objects.** An **initial object** of a quasicategory C is an $x \in C_0$ such that every $f\colon \partial\Delta^n \to C$ (for all $n \geq 1$) such that $f|\{0\} = x$, there exists an extension $f'\colon \Delta^n \to C$.

A **terminal object** of C is an initial object of C^{op}. That is, a $y \in C_0$ such that every $f\colon \partial\Delta^n \to C$ with $f|\{n\} = y$ extends to Δ^n.

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \begin{tikzcd} \{n\} \arrow[r, tail] \arrow[rr, bend left, "y"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

If C is the nerve of a category, then these definitions turn out to coincide with the usual notion of initial and terminal objects in a category. For instance, consider the definition of initial object applied to $x \in C_0$.

- The condition for $n = 1$ says that for every object y in C there exists $f\colon x \to y$,
- The condition for $n = 2$ says that for every triple of maps $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$, we must have $h = gf$. In particular (taking $f = 1_x$), we see there is at most one map from x to any object.
- The conditions for $n \geq 3$ are trivially satisfied.

We can characterize the property of being initial or terminal in terms of the "forgetful" functor from the slice.

24.4. **Proposition.** *If C is a quasicategory, then $x \in C_0$ is initial if and only if $C_{x/} \to C$ is a trivial fibration, and terminal if and only if $C_{/x} \to C$ is a trivial fibration.*

Proof. Left as an exercise. □

24.5. *Remark.* This implies that if x is initial, then $C_{x/} \to C$ is a categorical equivalence. Later (**??**) we'll be able to show the converse: if $C_{x/} \to C$ is a categorical equivalence, then x is initial.

A crucial fact about initial and terminal objects in a category is that they are unique up to unique isomoprhism. There is an analogous fact for quasicategories.

24.6. **Proposition.** *Let C be a quasicategory. The full subquasicategory $C^{\mathrm{init}} \subseteq C$ spanned by the initial objects is either empty, or is such that $C^{\mathrm{init}} \to \Delta^0$ is a trivial fibration (whence C^{init} is a quasigroupoid equivalent to the terminal category).*

Proof. Immediate, since any $f \colon \partial\Delta^n \to C^{\mathrm{init}}$ with $n \geq 1$ can be extended to $f' \colon \Delta^n \to C$ by definition, and the image of f' must lie in the full subquasicategory C^{init} since all of its vertices do. □

There are some seemingly obvious facts about initial objects that we can't prove just yet.

D. **Deferred Proposition.**

(1) *Given an object x in a quasicategory C, the object i_x in the slice $C_{x/}$ corresponding to the edge $1_x \in C_1$ is an initial object of $C_{x/}$.*
(2) *Any object in a quasicategory isomorphic to an initial object is also an initial object.*

25. Joins and slices in lifting problems

Recall that for an object x in a quasicategory C, the slice objects $C_{x/}$ and $C_{/x}$ are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

25.1. **Proposition.** *Let $p \colon S \to C$ be a map of simplicial sets, and suppose C is a quasicategory. Then both $C_{p/}$ and $C_{/p}$ are quasicategories.*

The proof is just like that of (24.2): we will show below (25.12) that $C_{p/} \to C$ is a left fibration and $C_{/p} \to C$ is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

25.2. **Box-joins.** We define an analogue of the box-product for the the join. Given maps $i \colon A \to B$ and $j \colon K \to L$ of simplicial sets, the **box join** $i \circledast j$ is the map

$$i \circledast j \colon (A \star L) \amalg_{A \star K} (B \star K) \xrightarrow{(i \star L, B \star j)} B \star L.$$

Warning: unlike the box-product, the box-join is not symmetric: $i \circledast j \neq j \circledast i$.

25.3. *Example.* We have already observed examples of box joins in the proof of (24.2), namely

$$(\Lambda^n_j \subset \Delta^n) \circledast (\varnothing \subset \Delta^0) \approx (\Lambda^{n+1}_j \subset \Delta^{n+1}), \quad (\varnothing \subset \Delta^0) \circledast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{1+n}_{1+j} \subset \Delta^{1+n}).$$

A straightforward calculation shows that the box-join of a horn with a cell is always a horn:

$$(\Lambda^n_j \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) \approx (\Lambda^{n+1+k}_j \subset \Delta^{n+1+k}),$$
$$(\partial\Delta^k \subset \Delta^k) \circledast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{k+1+n}_{k+1+j} \subset \Delta^{k+1+n}).$$

Also, the box-join of a cell with a cell is always a cell:

$$(\partial\Delta^n \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) \approx (\partial\Delta^{n+1+k}, \Delta^{n+1+k}).$$

25.4. *Exercise.* Prove the isomorphisms asserted in (25.3).

25.5. *Remark.* The "box" construction is completely general: given any functor $F\colon s\mathrm{Set}\times s\mathrm{Set}\to s\mathrm{Set}$ of two variables, you get a corresponding "box-functor" $F^{\square}\colon \mathrm{Fun}([1],s\mathrm{Set})\times\mathrm{Fun}([1],s\mathrm{Set})\to\mathrm{Fun}([1],s\mathrm{Set})$.

25.6. **Box-slices.** Just as the box-product is associated to the box-power, so the box-join is associated to two kinds of **box-slices**. Given maps $K\xrightarrow{j}L\xrightarrow{q}X\xrightarrow{h}Y$, we define the map

$$h^{\boxast_q j}\colon X_{/q}\to X_{/qj}\times_{Y_{/hqj}}Y_{/hq},$$

where the components $X_{/q}\to X_{/qj}$ and $X_{/q}\to Y_{/hq}$ are the evident maps constructed by "restricting" along $p\colon K\to L$ in the first case, and "composing" with $h\colon X\to Y$ in the second case.

Similarly, given maps $A\xrightarrow{i}B\xrightarrow{p}X\xrightarrow{h}Y$ we have

$$h^{i\boxast_p}\colon X_{p/}\to X_{pi/}\times_{Y_{hpi/}}Y_{hp/}.$$

25.7. *Remark.* When $Y=*$, these box-slice maps are just the restriction maps $X_{/q}\to X_{/qj}$ and $X_{p/}\to X_{pi/}$.

The box-join and box-slice interact with lifting problems in much the same way that box-product and box-power do.

25.8. **Proposition.** *Given $i\colon A\to B$, $j\colon K\to L$, and $h\colon X\to Y$, the following are equivalent.*

(1) $(i\boxast j)\boxslash h$.
(2) $i\boxslash(h^{\boxast_q j})$ *for all* $q\colon L\to X$.
(3) $j\boxslash(h^{i\boxast_p})$ *for all* $p\colon B\to X$.

Proof. A straightforward exercise. The equivalence of (1) and (2) is

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_{/q} \arrow[d, "h^{\boxast_q j}"] \\ B \arrow[r] \arrow[ur, dashed] & X_{/qj}\times_{Y_{/hqj}}Y_{/hq} \end{tikzcd} \quad\Longleftrightarrow\quad \begin{tikzcd} \varnothing\star L \arrow[r] \arrow[rr, bend left, "q"] & (A\star L)\cup_{A\star K}(B\star K) \arrow[r] \arrow[d, "i\boxast j"'] & X \arrow[d, "h"] \\ & B\star L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

□

25.9. **Proposition.** *Let S and T be sets of maps in $s\mathrm{Set}$. Then $\overline{S}\boxast\overline{T}\subseteq\overline{S\boxast T}$.*

Proof. This is formal and nearly identical to the proof of the saturation result for box-products (15.8). □

25.10. **Proposition.** *We have $\overline{\mathrm{RHorn}}\boxast\overline{\mathrm{Cell}}\subseteq\overline{\mathrm{InnHorn}}$ and $\overline{\mathrm{Cell}}\boxast\overline{\mathrm{LHorn}}\subseteq\overline{\mathrm{InnHorn}}$.*

Proof. Use (25.3). □

25.11. **Corollary.** *Given $T\xrightarrow{j}S\xrightarrow{p}X\xrightarrow{f}Y$, consider*

$$\ell\colon X_{p/}\to X_{pj/}\times_{Y_{fpj/}}Y_{fp/},\qquad r\colon X_{/p}\to X_{/pj}\times_{Y_{/fpj}}Y_{/fp}.$$

We have the following.

(1) $j\in\overline{\mathrm{Cell}}$, $f\in\mathrm{InnFib}$ *implies* $\ell\in\mathrm{LFib}$, $r\in\mathrm{RFib}$.
(2) $j\in\overline{\mathrm{RHorn}}$, $f\in\mathrm{InnFib}$ *implies* $\ell\in\mathrm{TFib}$.
(3) $j\in\overline{\mathrm{LHorn}}$, $f\in\mathrm{InnFib}$ *implies* $r\in\mathrm{TFib}$.

We are mostly interested in special cases when $X=C$ is a quasicategory, and $Y=*$.

25.12. **Corollary.** *Given $A\xrightarrow{i}B\xrightarrow{p}C$ with C a quasicategory and i a monomorphism, the induced maps $C_{/p}\to C_{/pi}$ is a right fibration, and $C_{p/}\to C_{pi/}$ is a left fibration.*

25.13. **Corollary.** *Given $A\xrightarrow{i}B\xrightarrow{p}C$ with C a quasicategory, if i is right anodyne then $C_{p/}\to C_{pi/}$ is a trivial fibration, while if i is left anodyne then $C_{/p}\to C_{/pi}$ is a trivial fibration.*

25.14. **Composition functors for slices.** Here is a nice consequence of the above. Let C be a quasicategory and $f\colon x \to y$ a morphism in it. We obtain two functors

$$C_{/x} \xleftarrow{p} C_{/f} \xrightarrow{q} C_{/y}$$

associated to the inclusions $\{0\} \subset \Delta^1 \supset \{1\}$. The first inclusion $\{0\} \subset \Delta^1$ is a left-horn inclusion, and thus by (25.13) the restriction map p is a trivial fibration, and hence we can choose a section $s\colon C_{/x} \to C_{/f}$ of p. The resulting composite $qs\colon C_{/x} \to C_{/y}$ can be thought of as a functor realizing the operation which sends an object $(c \xrightarrow{g} x)$ of $C_{/x}$ to the object $(c \xrightarrow{fg} y)$ of $C_{/y}$ defined by composing f and g.

25.15. *Exercise.* Show that if C is a category, then p is an isomorphism, and that qs is precisely the functor $C_{/x} \to C_{/y}$ described above.

26. Limits and colimits in quasicategories

Now we can define the notion of a limit and colimit of a functor between quasicategories. Given a map $p\colon K \to C$ where C is a quasicategory, a **colimit** of p is defined to be an initial object of the slice quasicategory $C_{p/}$. Explicitly, a colimit of $p\colon K \to C$ is a map $\widetilde{p}\colon K \star \Delta^0 = K^{\triangleright} \to C$ extending p, such that a lift exists in every diagram of the form

$$\begin{tikzcd}
K \star \Delta^{\{0\}} \arrow[r, tail] \arrow[rr, bend left, "\widetilde{p}"] & K \star \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\
& K \star \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Sometimes it is better to call $\widetilde{p}$ a **colimit cone**, in which case the restriction $\widetilde{p}|\varnothing \star \Delta^0$ to the cone point is an object in C which can be called a "colimit of p".

Similarly, a **limit** of p is a terminal object of $C_{/p}$; explicitly, this is a map $\widetilde{p}\colon \Delta^0 \star K = K^{\triangleleft} \to C$ extending p such that a lift exists in every

$$\begin{tikzcd}
\Delta^{\{n\}} \star K \arrow[r, tail] \arrow[rr, bend left, "\widetilde{p}"] & \partial\Delta^n \star K \arrow[r] \arrow[d, tail] & C \\
& \Delta^n \star K \arrow[ur, dashed] &
\end{tikzcd}$$

Here is a handy characterization of limit and colimit cones, which we give first for initial and terminal objects.

26.1. **Proposition.** *An object x in a quasicategory is initial iff $C_{x/} \to C$ is a trivial fibration, and is final iff $C_{/x} \to C$ is a trivial fibration.*

Proof. This is just a straightforward translation of the definition of initial and terminal. □

26.2. **Proposition.** *Let C be a quasicategory. Let $\widetilde{p}\colon K^{\triangleright} \to C$ be a map, and write $p := \widetilde{p}|K$. Then $\widetilde{p}$ is a colimit diagram if and only if $C_{\widetilde{p}/} \to C_{p/}$ is a trivial fibration.*

Likewise, let $\widetilde{q}\colon K^{\triangleleft} \to C$ be a map, and write $q := \widetilde{q}|K$, then $\widetilde{q}$ is a limit diagram if and only if $C_{/\widetilde{q}} \to C_{/q}$ is a trivial fibration.

Proof. I'll just do the case of colimits.

We make an elementary observation: $(C_{p/})_{\widetilde{p}/} \approx C_{\widetilde{p}/}$. In this isomorphism, the symbol "$\widetilde{p}$" refers to both a morphism $\widetilde{p}\colon K^{\triangleright} \to C_{p/}$ and the corresponding object $\widetilde{p} \in (C_{p/})_0$. The point is that in either simplical set, a k-simplex corresponds to a map $K \star \Delta^0 \star \Delta^k \to C$ which restricts to $\widetilde{p}$ on $K \star \Delta^0 \star \varnothing$.

Using this, the statement reduces to the previous proposition. □

27. The Joyal extension and lifting theorems

We are now at the point where we can state and prove Joyal's theorems about extending or lifting maps along outer horns. This will allow us to prove many of the results whose proofs we have deferred up to now.

27.1. **Joyal extension theorem.** The following gives a condition for extending maps from *outer horns* into a quasicategory.

27.2. **Theorem** (Joyal extension)**.** [Joy02, Thm. 1.3] *Let C be a quasicategory, and fix a map $f\colon \Delta^1 \to C$. The following are equivalent.*

(1) *The edge represented by f is an isomorphism in C.*
(2) *Every $a\colon \Lambda^n_0 \to C$ with $n \geq 2$ such that $f = a|\Delta^{\{0,1\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*
(3) *Every $b\colon \Lambda^n_n \to C$ with $n \geq 2$ such that $f = b|\Delta^{\{n-1,n\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*

In particular, the implications (1) ⇒ (2) and (1) ⇒ (3) say that we can always extend $\Lambda^n_0 \to C$ to an n-simplex if the *leading edge* goes to an isomorphism in C, and extend $\Lambda^n_n \to C$ to an n-simplex if the *trailing edge* goes to an isomorphism in C.

The implications (2) ⇒ (1) and (3) ⇒ (1) are easy, and are left as an exercise.

27.3. *Exercise.* Suppose C is a quasicategory with edge $f \in C_1$, and suppose that every map $a\colon \Lambda^n_0 \to C$ with $n \in \{2,3\}$ and $f = a|\Delta^{\{0,1\}}$ admits an extension along $\Lambda^n_0 \subset \Delta^n$. Prove that f is an isomorphism.

The non-trivial implications of Joyal extension will lead to proofs of the deferred propositions (A), (C), and (D).

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are *conservative isofibrations.*

27.4. **Conservative functors.** A functor $p\colon C \to D$ between categories is **conservative** if whenever f is a morphism in C such that $p(f)$ is an isomorphism in D, then f is an isomorphism in C. The definition of a conservative functor between *quasicategories* is precisely the same.

27.5. **Proposition.** *All left fibrations and right fibrations between quasicategories are conservative.*

Proof. Consider a right fibration $p\colon C \to D$, and a morphism $f\colon x \to y$ in C such that $p(f)$ is an isomorphism. We first show that f admits a preinverse.

Let $a\colon \Lambda^2_2 \to C$ such that $a_{12} = f$ and $a_{02} = 1_y$. Let $b\colon \Delta^2 \to C$ be any 2-simplex exhibiting a preinverse of $p(f)$, i.e., such that $b_{12} = p(f)$ and $b_{02} = 1_{p(y)}$, so that b_{01} is a preinverse. Now have a diagram with a lift

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dashed] & D \end{tikzcd}$$

which exhibits a preinverse of f, which we will call g.

Because $p(f)$ was assumed invertible, its preinverse $p(g)$ is also invertible, and therefore by the above argument g admits a preinverse as well. We conclude that f is invertible. □

27.6. **Isofibrations.** We say that a functor $p\colon C \to D$ is an **isofibration**[18] if

(1) p is an inner fibration, and
(2) we have "isomorphism lifting" along p. That is, for any $c \in C_0$ and *isomorphism* $g\colon p(c) \to d'$, there exists a $c' \in C_0$ and isomorphism $f\colon c \to c'$ such that $p(f) = g$.

Recall that if C and D are nerves of ordinary categories, then any functor $C \to D$ is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories we can replace (2) with the dual condition

(2') for any $c \in C_0$ and *isomorphism* $g'\colon d' \to p(c)$, there exists a $c' \in C_0$ and isomorphism $f'\colon c' \to c$ such that $p(f) = g$.

To prove (2) from (2'), just apply condition (2') to the (unique) inverse of g.

The symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

27.7. **Proposition.** *An inner fibration $p\colon C \to D$ between quasicategories is an isofibration if and only if $h(p)\colon h(C) \to h(D)$ is an isofibration of ordinary categories.*

Proof. ($\Longrightarrow$) Straightforward. ($\Longleftarrow$) Suppose given an isomophism $g\colon p(c) \to d'$ in D. If $h(p)$ is an isofibration, there exists an isomorphism $f'\colon c \to c'$ in C such that $p(f') \sim_r g$. Now choose a lift in

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

where b exhibits $p(f') \sim_r g$ and $a(\langle 01\rangle) = f'$ and $a(\langle 12\rangle) = 1_{c'}$. The edge $f = s_{02}$ is a lift of g, and is an isomorphism since $f' \sim_r f$. $\square$

27.8. *Exercise.* (i) Let Group denote the category of groups, whose objects are pairs $G = (S, \mu)$ consisting of a set S and a function $\mu\colon S \times S \to S$ satisfying a well-known list of axioms. Show that the functor $U\colon \mathrm{Group} \to \mathrm{Set}$ which on objects sends $(S, \mu) \mapsto S$ is an isofibration between ordinary categories.

(ii) Consider the functor $U'\colon \mathrm{Group} \to \mathrm{Set}$ defined on objects by $G \mapsto \mathrm{Hom}(\mathbb{Z}, G)$. Explain why, although U' is naturally isomorphic to U, you don't know how to show whether U' is an isofibration without explicit reference to the axioms of your set theory. The moral is that the property of being an isofibration is not "natural isomorphism invariant".

27.9. **Left and right fibrations are isofibrations.**

27.10. **Proposition.** *All left fibrations and right fibrations between quasicategories are isofibrations.*

Proof. Suppose $p\colon C \to D$ is a right fibration (and hence an inner fibration) between quasicategories, and consider

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, dashed, "f"] & D \end{tikzcd}$$

where g represents an isomorphism. Because p is a right fibration, there exists a lift f. Because right fibrations are conservative, f represents an isomorphism. $\square$

[18] Joyal uses the term "quasifibration" in [Joy02]. Later in [Joy08a] this is called a "pseudofibration". Lurie uses this notion, but never names it. The term "isofibration" is used by Riehl and Verity [RV15].

Note that the above proof checked explicitly isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary *direct* proof that right-fibrations satisfy (2).

27.11. **Proof of the Joyal extension theorem.**

Proof of (27.2). We prove (1) $\Rightarrow$ (2). Suppose given $a\colon \Lambda^n_0 \to C$ such that $f = a|\Delta^{\{0,1\}}$ represents an isomorphism. Using the join/slice adjunction applied to

$$(\Lambda^n_0 \subset \Delta^n) \approx (\{0\} \subset \Delta^1) \circledast (\partial\Delta^{n-2} \subset \Delta^{n-2}),$$

there is a correspondence of lifting problems

$$\begin{tikzcd}
 & & & & \Delta^{\{0\}} \arrow[r, "h"] \arrow[d, tail] & C_{/(a|\Delta^{\{2,\dots,n\}})} \arrow[d, "q"] \\
\Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "a"] \arrow[d, tail] & C & \iff & \Delta^{\{0,1\}} \arrow[r, "g"] \arrow[ur, dashed] \arrow[dr, "f"'] & C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \arrow[d, "p"] \\
 & \Delta^n \arrow[ur, dashed] & & & & C
\end{tikzcd}$$

where g is adjoint to $a|(\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$, and h is adjoint to $a|(\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}})$. Because C is a quasicategory, both p and q are right fibrations (25.12), and therefore are conservative isofibrations (27.5), (27.10). Thus since f represents an isomorphism, so does g since p is conservative, and therefore a lift exists since q is an isofibration.

The proof of (2) $\Longrightarrow$ (1) is left as an exercise (27.3). The proof of (1) $\Longleftrightarrow$ (3) is similar. $\square$

27.12. **The Joyal lifting theorem.** There is a relative generalizaton.

27.13. **Theorem** (Joyal lifting)**.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $f \in C_1$ be an edge such that $p(f)$ is an isomorphism in D. The following are equivalent.*

(1) *The edge f is an isomorphism in C.*

(2) *For all $n \geq 2$, every diagram of the form*

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

admits a lift.

(3) *For all $n \geq 2$, every diagram of the form*

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

admits a lift.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are elementary, as in (27.3).

For (1) $\Rightarrow$ (2), the first step is to prove that

$$C_{/(a|\Delta^{\{2,\dots,n\}})} \xrightarrow{q} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p} C$$

are both right fibrations. For instance, the map q is the box-power of the inner fibration p by a monomorphism, so is a right fibration by (25.12). The map p is the composite

$$C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p'} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \xrightarrow{p''} C,$$

where p' is the base change of the right fibration $D_{/(pa|\Delta^2\{2,\dots,n\})} \to D_{/(pa|\partial\Delta^2\{2,\dots,n\})}$, and p'' is a right fibration (in both cases by (25.12)) Then the proof of (1) $\Longrightarrow$ (2) proceeds exactly as in (27.2). $\square$

28. Applications of the Joyal extension theorem

We can now prove all the statements whose proofs we have deferred until now, as well as some others.

28.1. **Quasigroupoids are Kan complexs.** First we prove (A), the identification of quasigroupoids with Kan complexes.

28.2. **Proposition.** *Every quasigroupoid is a Kan complex.*

Proof. In a quasigroupoid, the Joyal extension property applies to all maps from Λ^n_0 and Λ^n_n, since every edge is an isomorphism. $\square$

28.3. **Invariance of slice categories.** Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

28.4. *Exercise* (Reformulation of Joyal extension). If $f\colon x \to y$ is an edge in a quasicategory C, then the following are equivalent: (1) f is an isomorphism; (2) $C_{f/} \to C_{x/}$ is a trivial fibration; (3) $C_{/f} \to C_{/y}$ is a trivial fibration.

28.5. **Corollary.** *If $f\colon x \to y$ is an isomorphism in a quasicategory C, then $C_{x/}$ and $C_{y/}$ are categorically equivalent, and $C_{/x}$ and $C_{/y}$ are categorically equivalent.*

Proof. Consider $C_{x/} \xleftarrow{\pi} C_{f/} \xrightarrow{\rho} C_{y/}$. We have already observed (25.13) that $\rho \in \mathrm{TFib}$, since $\{1\} \subset \Delta^1$ is right anodyne. The reformulation of Joyal extension (28.4) implies that $\pi \in \mathrm{TFib}$ when f is an isomorphism. $\square$

28.6. **Initial objects.** Now we prove (D) about initial objects.

28.7. **Proposition.** *If x is an object in a quasicategory C, then the vertex $i_x \in (C_{x/})_0$ corresponding to $1_x \in C_1$ is an initial object of $C_{x/}$.*

Proof. The map $1_x\colon x \to x$ in C is obviously an isomorphism. We have a correspondence of lifting problems

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "i_x"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C_{x/} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd} \iff \begin{tikzcd} \Delta^0 \star \{0\} \arrow[r, tail] \arrow[rr, bend left, "1_x"] & \Delta^0 \star \partial\Delta^n \arrow[r] \arrow[d] & C \\ & \Delta^0 \star \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

and $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) \approx (\Lambda^{1+n}_0 \subseteq \Delta^{1+n})$, so a lift exists by the Joyal extension theorem. $\square$

28.8. **Proposition.** *Any object in a quasicategory isomorphic to an initial object is also initial.*

Proof. Let x be an initial object in C, and let c be an object isomorphic to x. It is easy to see that x is initial in the homotopy category hC, and therefore c is initial in hC also. This has a useful consequence: *any* map between x and c must be an isomorphism in C.

To show that c is initial in C, we want to extend $a\colon \partial\Delta^n \to C$ with $a_0 = c$ to Δ^n. Consider the left-hand diagram in

$$\begin{tikzcd} & \Delta^0 \arrow[r, "i_x"] \arrow[d, "x"'] & C_{x/} \arrow[d, "p"] \\ \partial\Delta^n \arrow[r, "a"] & C \arrow[r, equal] \arrow[ur, dashed, "s"] & C \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} (\Delta^0 \star \varnothing) \amalg (\varnothing \star \partial\Delta^n) \arrow[r, tail] \arrow[rrr, bend left, "{(x,a)}"] & \Delta^0 \star \partial\Delta^n \arrow[r, "\Delta^0\star a"] \arrow[d] & \Delta^0 \star C \arrow[r, "\widetilde{s}"] & C \\ & \Delta^0 \star \Delta^n \arrow[urr, dashed, "t"'] & & \end{tikzcd}$$

Because x is initial, p is a trivial fibration (26.2), and so a lift s exists. On vertices y of C, the map s determines a choice, of a morphism $s(y)\colon x \to y$; by what we said above, $s(c)\colon x \to c$ must be an isomorphism if c is isomorphic to x.

Now consider the right-hand diagram, in which $\widetilde{s}$ is adjoint to s. Write $b := \widetilde{s}(\Delta^0 \star a)$, a map from the horn $\Delta^0 \star \partial\Delta^n$ to C, so that so $b_0 = x$ and $b|(\varnothing \star \partial\Delta^n) = a$. The leading edge $b|\Delta^0 \star \{0\}$ is the edge $s(c)\colon x \to c$ in C, and we have observed any such is an isomorphism. Thus a lift t exists in the right-hand diagram by the Joyal lifting theorem since $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) = (\Lambda^{1+n}_0 \subset \Delta^{1+n})$. The face $t|(\varnothing \star \Delta^n)$ is the desired extension of a. □

28.9. **The objectwise criterion for natural isomorphisms.** Next we prove (C), the fact that natural isomorphisms are natural transformations which are isomorphisms *objectwise*. This is an immediate consequence of the the final statement of the following, applied to $C \times \partial\Delta^1 \subset C \times \Delta^1$.

28.10. **Proposition.** *Let $j\colon K \subseteq L$ be a monomorphism of simplicial sets such that $j\colon K_0 \xrightarrow{\sim} L_0$ is a bijection.*

1. *For every quasicategory C and commutative diagram*

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^2_0 \arrow[r, "a"] \arrow[d, tail] & \mathrm{Fun}(L,C) \arrow[d, "p=\mathrm{Fun}(j,C)"] \\ & \Delta^2 \arrow[r, "b"'] \arrow[ur, dashed, "s"] & \mathrm{Fun}(K,C) \end{tikzcd}$$

   such that pf represents an isomorphism in $\mathrm{Fun}(K,C)$*, a lift exists.*
2. *The functor $p = \mathrm{Fun}(j,C)$ is a conservative.*

Proof. Let $\mathcal{C}$ be the class of monomorphisms $j\colon K \to L$ for which statement (1) of the lemma holds.

We first prove that for $j \in \mathcal{C}$, statement (2) holds, i.e., that the map $p = \mathrm{Fun}(j,C)$ is conservative for any quasicategory C. To see this, consider $f\colon \Delta^1 \to \mathrm{Fun}(L,C)$ such that pf represents an isomorphism in $\mathrm{Fun}(K,C)$, and form the diagram (as in (1)) with $a_{01} = f$, a_{02} representing an identity map, and b exhibiting a postinverse of the edge represented by pf. Then existence of a lift s implies that f has a postinverse g in $\mathrm{Fun}(L,C)$. Since pg is a postinverse of the isomorphism pf it is also an isomorphism, so the same argument shows that g also has a postinverse. Thus f represents an isomorphism as desired.

To complete the proof, we show (i) that $\mathcal{C}$ is saturated, and (ii) that $\mathcal{C}$ contains $\partial\Delta^n \subset \Delta^n$ for $n \geq 1$. Given this, the result follows from the skeletal filtration (14.14).

The proof of (i) is a relatively straightforward exercise; note that for the composition and transfinite composition properties we need to use the fact just proved that $\mathrm{Fun}(j, C)$ is conservative when $j \in \mathcal{C}$.

By transforming the lifting problem to its adjoint, to prove claim (ii) it suffices to show that given

$$\begin{tikzcd}
\Delta^{\{0,1\}} \times \{0\} \arrow[r, tail] \arrow[rr, bend left, "f_0"] & (\Lambda^2_0 \times \Delta^n) \cup (\Delta^2 \times \partial\Delta^n) \arrow[r] \arrow[d, tail, "j"'] & C \\
& \Delta^2 \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

such that f_0 represents an isomorphism in C, a lift exists. This uses that f_0 is the composite

$$\Delta^{\{0,1\}} \xrightarrow{f} \mathrm{Fun}(\Delta^n, C) \xrightarrow{p} \mathrm{Fun}(\partial\Delta^n, C) \to \mathrm{Fun}(\{0\}, C)$$

induced by restriction along $\{0\} \subset \partial\Delta^n \subset \Delta^n$, which must represent an isomorphism in C if pf represents an isomorphism in $\mathrm{Fun}(\partial\Delta^n, C)$.

The proof is a straightfoward calculation, which is a special case of the following lemma (28.11), which can be thought of as a "box" version of Joyal's lifting theorem $\square$

28.11. **Proposition** (Box version of Joyal lifting)**.** *Suppose $p\colon C \to D$ is an inner fibration between quasicategories. Suppose $m, n \geq 1$. For any diagram*

$$\begin{tikzcd}
\Delta^{\{0,1\}} \times \{0\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\Lambda^m_0 \times \Delta^n) \cup_{\Lambda^m_0 \times \partial\Delta^n} (\Delta^m \times \partial\Delta^n) \arrow[r] \arrow[d, tail, "j"'] & C \arrow[d, "p"] \\
& \Delta^m \times \Delta^n \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

such that f represents an isomorphism in C, a lift exists.

Note: if we take instead $m \geq 2$ and $n = 0$, the above statement reduces to the Joyal lifting theorem.

Proof. Proved as [Joy08a, 5.8] (though there it is the $\Lambda^m_m \subset \Delta^m$ case that is proved). $\square$

The idea is to produce the map j by sucessively attaching either

- an inner horn $\Lambda^k_i \subset \Delta^k$ for $0 < i < k$, or
- a horn $\Lambda^k_0 \subset \Delta^k$ with $k \geq 2$ along a map from Λ^k_0 whose restriction along the leading edge $\Delta^{\{0,1\}}$ is $f\colon \Delta^1 \to X$.

The existence of the lifting follows, using the Joyal extension theorem.

Here is a picture for the case $m = 1$, $n = 1$. The source of $(\Lambda^1_0 \subset \Delta^1)\square(\partial\Delta^1 \subset \Delta^1)$ looks like

$$\begin{tikzcd}
01 \arrow[r] & 11 \\
00 \arrow[u] \arrow[r, "\sim"'] & 10
\end{tikzcd}$$

First attach the 2-simplex $\langle 00, 01, 11\rangle$, which intersects the sounce along the inner horn Λ^2_1. Then attach the 2-simplex $\langle 00, 10, 11\rangle$, which intersects what we have already built along the horn Λ^2_0, whose leading edge $\langle 00, 11\rangle$ is sent to an isomorphism in C.

Here are pictures for the case $m = 2$, $n = 1$. Here is the source of $(\Lambda^2_0 \subset \Delta^2)\Box(\partial\Delta^1 \subset \Delta^1)$.

$$\begin{tikzcd}
11 \arrow[rr, bend left] & 01 \arrow[l, hook'] \arrow[r, hook] & 21 \\
10 \arrow[u] \arrow[rr, bend right] & 00 \arrow[l, hook', "\sim"'] \arrow[r, hook] \arrow[u] \arrow[ul] \arrow[ur] & 20 \arrow[u]
\end{tikzcd}$$

The following chart lists all non-degenerate simplices in the complement, with "$\surd$" marking those in the source.

$\langle 10,21\rangle$	$\langle 10,20,21\rangle$	$\langle 10,11,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,10,20,21\rangle$	$\langle 00,10,11,21\rangle$	$\langle 00,01,11,21\rangle$
$\surd\langle 21\rangle$	$\surd\langle 20,21\rangle$	$\surd\langle 11,21\rangle$	$\langle 10,21\rangle$	$\surd\langle 11,21\rangle$	$\langle 10,20,21\rangle$	$\langle 10,11,21\rangle$	$\surd\langle 01,11,21\rangle$
$\surd\langle 10\rangle$	$\langle 10,21\rangle$	$\langle 10,21\rangle$	$\surd\langle 00,21\rangle$	$\langle 00,21\rangle$	$\surd\langle 00,20,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,11,21\rangle$
	$\surd\langle 10,20\rangle$	$\surd\langle 10,11\rangle$	$\surd\langle 00,10\rangle$	$\surd\langle 00,11\rangle$	$\langle 00,10,21\rangle$	$\langle 00,10,21\rangle$	$\surd\langle 00,01,21\rangle$
					$\surd\langle 00,10,20\rangle$	$\surd\langle 00,10,11\rangle$	$\surd\langle 00,01,11\rangle$

Note that the simplices $\langle 10,21\rangle$, $\langle 00,10,21\rangle$, and $\langle 00,11,21\rangle$ of the complement appear multiple times as faces. We attach simplices to the domain in the following order:

$$①\langle 10,11,21\rangle, \quad ②\langle 00,01,11,21\rangle, \quad ③\langle 00,10,11,21\rangle, \quad ④\langle 00,10,20,21\rangle.$$

Only the final step involves attaching along a non-inner horn; in that case, the attaching map sends the leading edge of Λ^3_0 to $\langle 00,10\rangle$.

The fundamental theorem

29. Mapping spaces of a quasicategory

For any pair of objects x, y in a category, there is a corresponding *hom-set* $\mathrm{Hom}_C(x,y)$. The analogue of the hom-set for a quasicategory will be a quasigroupoid called $\mathrm{map}_C(x,y)$.

Given a quasicategory C and objects $x, y \in C_0$, the **mapping space** from x to y is the simplicial set defined by the pullback square

$$\begin{tikzcd}
\mathrm{map}_C(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\
\{(x,y)\} \arrow[r] & C \times C
\end{tikzcd}$$

That is, $\mathrm{map}_C(x,y)$ is the fiber of the restriction map $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over the point (x,y).

If $C = N(A)$ is the nerve of a category, then $\mathrm{map}_C(x,y)$ is a discrete simplicial set (i.e., equal to its own 0-skeleton), and is isomorphic to $\mathrm{Hom}_C(x,y)$ (viewed as a discrete simplicial set).

29.1. **Mapping spaces are Kan complexes.** The terminology "space" is justified by the following

29.2. **Proposition.** *The simplicial sets* $\mathrm{map}_C(x,y)$ *are quasigroupoids (and hence Kan complexes).*

Proof. First note that $p\colon \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ is a inner fibration, since $\mathrm{InnHorn}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (15.7). Therefore the pullback $\mathrm{map}_C(x,y) \to \{(x,y)\}$ is an inner fibration, i.e., $\mathrm{map}_C(x,y)$ is a quasicategory.

To show that the quasicategory $\mathrm{map}_C(x,y)$ is a quasigroupoid, it suffices to show that every $\Lambda^2_0 \to \mathrm{map}_C(x,y)$ extends to a 2-simplex. Thus we must consider the lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, tail] \arrow[rrr, bend left, "f"] & \Lambda^2_0 \arrow[r] \arrow[d, tail] & \mathrm{map}_C(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "p"] \\
& \Delta^2 \arrow[r] \arrow[ur, dashed] & \{(x,y)\} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, C)
\end{tikzcd}$$

Since the right hand square is a pullback, it suffices to produce a lift $\Delta^2 \to \mathrm{Fun}(\Delta^1, C)$, which follows by (28.10)(2), since $\partial\Delta^1$ contains all the objects of Δ^1, and $p(f)$ is an identity map, and therefore certainly an isomorphism. □

29.3. *Remark.* Statement (2) of (28.10) implies that every edge in $\mathrm{map}_C(x, y)$ is an isomorphism in $\mathrm{Fun}(\Delta^1, C)$. However, we need to use statement (1) of (28.10) to conclude that such an edge in fact has an inverse in $\mathrm{map}_C(x, y)$.

More generally, the above proof shows that the fibers of $\mathrm{Map}(L, C) \to \mathrm{Map}(K, C)$ are quasigroupoids whenever C is a quasicategory and $K \subseteq L$ with $K_0 = L_0$.

29.4. **Mapping spaces and homotopy classes.** The set of morphisms $x \to y$ in a quasicategory is precisely the set of 0-simplices of $\mathrm{map}_C(x, y)$. It is straightforward to prove that for two such, $f \approx g$ (the relation we use to define the equivalence classes of hC) if and only if f and g are isomorphic objects in the Kan complex $\mathrm{map}_C(x, y)$. More precisely: it is straightforward to show that $f \approx g$ if and only if there exists a morphism $f \to g$ in $\mathrm{map}_C(x, y)$. We obtain the following.

29.5. **Proposition.** *If C is a quasicategory, then*

$$\mathrm{Hom}_{hC}(x, y) \approx \pi_0 \,\mathrm{map}_C(x, y)$$

for every pair of objects.

29.6. **Extended mapping spaces.** Given objects $x_0, \dots, x_n \in C_0$ in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

$$\begin{tikzcd} \mathrm{map}_C(x_0, \dots, x_n) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, C) \arrow[d] \\ \{(x_0, \dots, x_n)\} \arrow[r] & C^{\times(n+1)} \end{tikzcd}$$

29.7. **Lemma.** *The map*

$$g_n\colon \mathrm{map}_C(x_0, \dots, x_n) \to \mathrm{map}_C(x_{n-1}, x_n) \times \cdots \times \mathrm{map}_C(x_0, x_1)$$

induced by restriction along $I^n \subseteq \Delta^n$ is a trivial fibration. In particular, this map is a categorical equivalence between Kan complexes.

Proof. The map g_n is a base change of $p\colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(I^n, C)$. Since $I^n \subset \Delta^n$ is inner anodyne (9.12), and C is a quasicategory, the map p is a trivial fibration using $\overline{\mathrm{InnHorn}}\,\square\,\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (15.7). □

For any triple x_0, x_1, x_2 of objects we obtain a zig-zag of maps of Kan complexes

$$\mathrm{map}_C(x_1, x_2) \times \mathrm{map}_C(x_0, x_1) \xleftarrow[\sim]{g_2} \mathrm{map}_C(x_0, x_1, x_2) \to \mathrm{map}_C(x_0, x_2),$$

where the second map is induced by restriction along $\Delta^{\{0,2\}} \subset \Delta^2$, and the first map g_2 is a categorical equivalence. After choosing a categorical inverse to g_2, we obtain a "composition" map

$$\mathrm{comp}\colon \mathrm{map}_C(x_1, x_2) \times \mathrm{map}_C(x_0, x_1) \to \mathrm{map}_C(x_0, x_2). \tag{29.8}$$

This map is *not* uniquely determined, since it depends on a choice of categorical inverse to g_2. However, any two categorical inverses to g_2 are naturally isomorphic, and therefore comp is defined up to natural isomorphism. That is, it is a well-defined map in $h\mathrm{Kan}$, the homotopy category of Kan complexes (defined as a full subcategory of $h\mathrm{QCat}$).

29.9. **Proposition.** *The two maps obtained by composing the sides of the square*

$$\begin{tikzcd}
\mathrm{map}_C(x_2,x_3)\times\mathrm{map}_C(x_2,x_1)\times\mathrm{map}_C(x_0,x_1) \arrow[r, "\mathrm{id}\times\mathrm{comp}"] \arrow[d, "\mathrm{comp}\times\mathrm{id}"'] & \mathrm{map}_C(x_2,x_3)\times\mathrm{map}_C(x_0,x_2) \arrow[d, "\mathrm{comp}"] \\
\mathrm{map}_C(x_1,x_3)\times\mathrm{map}_C(x_0,x_1) \arrow[r, "\mathrm{comp}"'] & \mathrm{map}_C(x_0,x_3)
\end{tikzcd}$$

are naturally isomorphic. That is, the diagram commutes in $h\mathrm{Kan}$.

Proof. Here is a diagram of simplicial which actually commutes on the nose. I use "(x,y,z)" as shorthand for "$\mathrm{map}_C(x,y,z)$", etc.

$$\begin{tikzcd}
(x_2,x_3)\times(x_1,x_2)\times(x_0,x_1) & (x_2,x_3)\times(x_0,x_1,x_2) \arrow[l, "\sim"'] \arrow[r] & (x_2,x_3)\times(x_0,x_2) \\
(x_1,x_2,x_3)\times(x_0,x_1) \arrow[u, "\sim"'] \arrow[d] & (x_0,x_1,x_2,x_3) \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"'] \arrow[d] & (x_0,x_2,x_3) \arrow[u, "\sim"'] \arrow[d] \\
(x_1,x_3)\times(x_0,x_1) & (x_0,x_1,x_3) \arrow[l, "\sim"'] \arrow[r] & (x_0,x_3)
\end{tikzcd}$$

The maps labelled "$\xrightarrow{\sim}$" are categorical equivalences, and in fact are trivial fibrations. All the maps in the above diagram are obtained via restriction along inclusions in

$$\begin{tikzcd}
\Delta^{\{2,3\}}\cup\Delta^{\{1,2\}}\cup\Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \Delta^{\{2,3\}}\cup\Delta^{\{0,1,2\}} \arrow[d, "\sim"] & \Delta^{\{2,3\}}\cup\Delta^{\{0,2\}} \arrow[l] \arrow[d, "\sim"] \\
\Delta^{\{1,2,3\}}\cup\Delta^{\{0,1\}} \arrow[r, "\sim"] & \Delta^3 & \Delta^{\{0,2,3\}} \arrow[l] \\
\Delta^{\{1,3\}}\cup\Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[u] & \Delta^{\{0,1,3\}} \arrow[u] & \Delta^{\{0,3\}} \arrow[l] \arrow[u]
\end{tikzcd}$$

where the maps labelled "$\xrightarrow{\sim}$" are inner anodyne, and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define comp. After passing to $h\mathrm{Kan}$ the categorical equivalences become isomorphisms, and the result follows. ☐

29.10. **Segal categories.** Thus, a quasicategory does not quite give rise to a category "enriched over quasigroupoids". Although we can define a composition law, it is not uniquely determined, and is only associative "up to homotopy".

What we do get is a Segal category. A **Segal category** is a functor

$$M\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$$

such that

(1) the simplicial set $M([0])$ is discrete, i.e., $M([0]) = \mathrm{Sk}_0\, M([0])$, and
(2) for each $n \geq 1$ the "Segal map"

$$M([n]) \xrightarrow{(\langle n-1,n\rangle^*,\dots,\langle 0,1\rangle^*)} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$

is a weak equvialence of simplicial sets.

We will define "weak equivalence" of simplicial sets soon. For now, we note that a map between *Kan complexes* is a weak equivalence if and only if it is a categorical equivalence, and that if each $M([n])$ is a Kan complex, then so are the fiber products which appear in the above definition.

Given a quasicategory C, we obtain a functor $M_C\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$\begin{aligned} M_C([0]) &:= \mathrm{Sk}_0\, C, \\ M_C([n]) &:= \mathrm{Fun}(\Delta^n, C) \times_{\mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, C)} \mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, \mathrm{Sk}_0\, C) \\ &\approx \coprod_{x_0,\dots,x_n \in C_0} \mathrm{map}_C(x_0,\dots,x_n). \end{aligned}$$

This object encodes all the structure we used above. For instance, the zig-zag

$$M_C([1]) \times_{M_C([0]} M_C([1]) \xleftarrow{(\langle 12\rangle^*, \langle 01\rangle^*)} M_C([2]) \xrightarrow{\langle 02\rangle^*} M_C([1])$$

is a coproduct over all triples $x_0, x_1, x_2 \in C_0$ of the zig-zag (29.8) used to define "composition".

You also get a Segal category from any simplicially enriched category. If $\mathcal{C}$ is a (small) simplicially enriched category, with object set $\mathrm{ob}\,\mathcal{C}$, and function complexes $\mathcal{C}(x,x') \in s\mathrm{Set}$ for each x, x', define $M_{\mathcal{C}}\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$\begin{aligned} M_{\mathcal{C}}([0]) &:= \mathrm{ob}\,\mathcal{C}, \\ M_{\mathcal{C}}([n]) &:= \coprod_{x_0,\dots,x_n \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x_{n-1}, x_n) \times \cdots \times \mathcal{C}(x_0, x_1). \end{aligned}$$

We thus obtain functors

$$\mathrm{QCat} \to \mathrm{SeCat} \leftarrow s\mathrm{Cat}$$

relating quasicategories, Segal categories, and simplicially enriched categories. Simplicially enriched categories were proposed as a model for ∞-categories by Dwyer and Kan[19], while Segal categories were proposed as a model for ∞-categories by Hirschowitz and Simpson [HS01][20]. All of these models are known to be equivalent in a suitable sense; see [Ber10] for more about these models and their comparison.

29.11. **The enriched homotopy category of a quasicategory.** Given a quasicategory C we can produce a vestigial version of a category enriched over quasigroupoids, called the **enriched homotopy category** of C and denoted $\mathcal{H}C$.[21] This object will be a category *enriched over* hKan, the homotopy category of Kan complexes, whose underlying category is hC.

We now define $\mathcal{H}C$. The objects of $\mathcal{H}C$ are just the objects of C. For any two objects $x, y \in C_0$, we have the quasigroupoid

$$\mathcal{H}C(x,y) := \mathrm{map}_C(x,y)$$

which we will regard as an object of the homotopy category hKan of Kan complexes. Composition $\mathcal{H}C(x_1,x_2) \times \mathcal{H}C(x_0,x_1) \to \mathcal{H}C(x_0,x_2)$ is the composition map defined above. Composition is associative by the above proposition (29.9).

The *underlying* ordinary category of $\mathcal{H}C$ is just the ordinary homotopy category hC, since

$$\mathrm{Hom}_{h\mathrm{Kan}}(\Delta^0, \mathrm{map}_C(x,y)) \approx \pi_0\, \mathrm{map}_C(x,y) \approx \mathrm{Hom}_{hC}(x,y).$$

29.12. *Warning.* A quasicategory C *cannot* be recovered from its enriched homotopy category $\mathcal{H}C$, not even up to equivalence. In fact, there exist hKan-enriched categories which do not arise as $\mathcal{H}C$ for any quasicategory C. A proof is outside the scope of these notes; however, we note that counterexamples may be produced from associative H-spaces which are not loop spaces.

[19] They called them "homotopy theories" instead of "∞-categories.

[20] In fact, they generalize this to "Segal n-categories", which were the first effective model for (∞, n)-categories.

[21] Lurie usually calls this "hC", though he also uses that notation for the ordinary homotopy category of C. I prefer to have two separate notations.

29.13. *Exercise.* Let C be a quasicategory. Describe how to use the enriched homotopy category $\mathcal{H}C$ to define, for each morphism $f\colon x \to y$ in C and object c in C, maps

$$f^*\colon \operatorname{map}_C(y, c) \to \operatorname{map}_C(x, c), \qquad f_*\colon \operatorname{map}_C(c, x) \to \operatorname{map}_C(c, y)$$

in $h\mathrm{Kan}$ corresponding to pre- and post-composition with f. Show that these fit together to give a well-defined functor

$$\operatorname{map}(-, -)\colon hC^{\mathrm{op}} \times hC \to h\mathrm{Kan}.$$

30. The fundamental theorem of quasicategory theory

30.1. **The fundamental theorem of category theory.** A functor $f\colon A \to B$ between categories is **fully faithful** if for every pair $a, a' \in \operatorname{ob} A$ of objects, the resulting map $\operatorname{Hom}_A(a, a') \to \operatorname{Hom}_B(fa, fa')$ is a bijection. The functor is **essentially surjective** if for every $b \in \operatorname{ob} B$ there exists an object $a \in \operatorname{ob} A$ such that $f(a)$ and b are isomorphic.

It is a standard fact that a functor $f\colon A \to B$ is *an equivalence of categories if and only if it is fully faithful and essentially surjective.* One direction is easy. In the other direction, given f fully faithful and essentially surjective, to construct an inverse functor g one first makes a choice for each $b \in \operatorname{ob} B$ a pair (a, α) consisting of $a \in \operatorname{ob} A$ and an isomorphism $\alpha\colon f(a) \to b$. Then g is defined on objects by $g(b) = a$ and on maps by $\operatorname{Hom}_B(b, b') \approx \operatorname{Hom}_B(f(a), f(a')) \approx \operatorname{Hom}_A(a, a')$.

I will refer to this fact as the "Fundamental Theorem of Category Theory"[22]. This sounds a bit pretentious, but I think it can be justified; it is a simple but powerful technique for producing equivalences of categories. An even better case can be made for its generalization to quasicategories.

30.2. **Fully faithful and essentially surjective functors between quasicategories.** We say that a functor $f\colon C \to D$ between quasicategories is

- **fully faithful** if for every pair $c, c' \in C_0$, the resulting map $\operatorname{map}_C(c, c') \to \operatorname{map}_D(fc, fc')$ is a categorical equivalence, and
- **essentially surjective** if the induced functor $hf\colon hC \to hD$ is essentially surjective; i.e., if for every $d \in D_0$ there exists $c \in C_0$ and an isomorphism $fc \to d$ in D_1.

Another way to say this: $f\colon C \to D$ is fully faithful and essentially surjective iff the induced $h\mathrm{Kan}$-enriched functor $\mathcal{H}f\colon \mathcal{H}C \to \mathcal{H}D$ is an equivalence of *enriched* categories.

30.3. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then f is fully faithful and essentially surjective.*

Proof. We already know that $hf\colon hC \to hD$ is an equivalence of ordinary categories, which implies essential surjectivity. The proof that f is fully faithful is straightforward. The key step is to show that if $f_0, f_1\colon C \to D$ are naturally isomorphic functors, then f_0 is fully faithful iff f_1 is. To do this, observe that a natural isomorphism induces a map $\mathcal{H}(C \times N(\mathrm{Iso})) \to \mathcal{H}C$, and that there is an isomorphism of $h\mathrm{Kan}$-enriched categories $\mathcal{H}(C \times N(\mathrm{Iso})) \approx \mathcal{H}C \times \mathcal{H}(N\mathrm{Iso})$. □

The converse is true, but not as straightforward.

E. **Deferred Proposition** (Fundamental theorem of quasicategory theory)**.** *A map $f\colon C \to D$ between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

This is a non-trivial result. It gives a necessary and sufficient condition for $f\colon C \to D$ to admit a categorical inverse, but it does not spell out how to construct such an inverse.

[22] As a young mathematician, I imagined that every subject must have its own Fundamental Theorem, on the model of the Fundamental Theorems of Calculus, Arithmetic, Algebra, etc. I am sad that this is not generally the case.

30.4. **2-out-of-3 for fully faithful essentially surjective functors.** The following result will be useful in the proof of the fundamental theorem.

30.5. **Proposition.** *The class $\mathcal{C}$ of fully faithful and essentially surjective functors between quasicategories satisfies the 2-out-of-3 property.*

Proof. First note that if a functor $f\colon C \to D$ between quasicategories is fully faithful and essentially surjective, then the induced $hf\colon hC \to hD$ is an equivalence of ordinary categories. Conversely, if hf is an equivalence, then f is essentially surjective.

Thus, if $C \xrightarrow{f} D \xrightarrow{g} E$ are functors of quasicategories such that two of $\{f, g, gf\}$ are fully faithful and essentially surjective, then the third is also essentially surjective. Thus, we only need to deal with the fully faithful property, for which there are 3 cases: showing that f, g, or gf is fully faithful assuming the other two are categorical equivalences.

Given objects $x, x' \in C_0$, we have induced maps

$$\mathrm{map}_C(x, x') \xrightarrow{f} \mathrm{map}_D(fx, fx') \xrightarrow{g} \mathrm{map}_E(gfx, gfx').$$

Proofs of two of the three cases (for f and for gf) follow immediately using (21.8).

For the case of g, note that if f and gf are categorical equivalences, it follows by the same argument that $\mathrm{map}_D(y, y') \xrightarrow{g} \mathrm{map}_E(gy, gy')$ is a categorical equivalence for any y, y' in the image of $f\colon C_0 \to D_0$. This is good enough, which follows from the following lemma. □

30.6. **Lemma.** *Let $f\colon C \to D$ be a functor between quasicategories, and let $a\colon x \to y$ and $b\colon y' \to x'$ be isomorphisms in C. Then there is a commutative square*

$$\begin{tikzcd} \mathrm{map}_C(x, x') \arrow[r, "f"] \arrow[d, "\sim"'] & \mathrm{map}_D(fx, fx') \arrow[d, "\sim"] \\ \mathrm{map}_C(y, y') \arrow[r, "f"'] & \mathrm{map}_D(fy, fy') \end{tikzcd}$$

in $h\mathrm{Kan}$, where the vertical maps are categorical equivalences (i.e., isomorphisms in $h\mathrm{Kan}$) induced by composition with a and b.

Proof. Use (29.13). □

30.7. **Fully faithful and essentially surjective functors between quasigroupoids.** The special case for quasigroupoids is already interesting. Here, (E) specializes to the "Fundamental Theorem of Quasigroupoid Theory", which says that a map $f\colon X \to Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\mathrm{map}_X(x_0, x_1) \to \mathrm{map}_Y(fx_0, fx_1)$ on "path spaces", and
- induces a surjection $\pi_0 X \to \pi_0 Y$.

A mild variant says that a map $f\colon X \to Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\Omega_x X \to \Omega_{fx} Y$ on "loop spaces", and
- induces an isomorphism $\pi_0 X \to \pi_0 Y$.

Here $\Omega_x X := \mathrm{map}_X(x, x)$. Under the correspondence between quasigroupoids and classical homotopy theory, this turns out to be an analogue of the Whitehead theorem[23], which says that a map between CW-complexes is a homotopy equivalence iff it induces an isomorphism on all homotopy groups.

We will prove (E) after first considering the special case of Kan complexes.

[23] The "Fundamental Theorem of Classical Homotopy Theory"?

31. Anodyne maps and Kan fibrations

In the next few sections, we will develop some properties related to Kan complexes. As a byproduct, we'll obtain the proof of the specialization of (E) to Kan complexes.

31.1. **Weak equivalence.** Say that a map $f\colon X \to Y$ is a **weak equivalence** of simplicial sets if and only if $\mathrm{Map}(f, G)\colon \mathrm{Map}(Y, G) \to \mathrm{Map}(X, G)$ is a categorical equivalence for every *Kan complex* G.

Every categorical equivalence is a weak equivalence, but not conversely. For maps between Kan complexes, weak equivalences and categorical equivalences coincide, as is straightforward to show via the same proof as (18.5).

31.2. **Proposition.** *Weak equivalences of simplicial sets satisfy the 2-out-of-3 property.*

Proof. Proved just as for categorical equivalences (21.8). $\square$

31.3. **Simplicial homotopy equivalence.** A **simplicial homotopy inverse** to a map $f\colon X \to Y$ of simplicial sets is a map $g\colon Y \to X$ such that there exists a chain of edges in $\mathrm{Map}(X, X)$ connecting id_X with gf, and a chain of edges in $\mathrm{Map}(Y, Y)$ connecting id_Y with fg. Such an f is called a **simplicial homotopy equivalence**, and of course any homotopy inverse is also a simplicial homotopy equivalence.

31.4. **Proposition.** *Any simplicial homotopy equivalence is a weak equaivalence.*

Proof. First, if $f\colon X \to Y$ is a simplicial homotopy equivalence between Kan complexes, then it is clearly a categorical equivalence, because $\mathrm{Map}(X, X)$ and $\mathrm{Map}(Y, Y)$ are quasigroupoids.

In general, suppose K is a Kan complex and consider $f^*\colon \mathrm{Map}(Y, K) \to \mathrm{Map}(X, K)$. By the same reasoning as used in the proof of (18.5), we see that f^* is a simplicial homotopy equivalence between Kan complexes, so a categorical equivalence. $\square$

31.5. **Anodyne maps and Kan fibrations.** Let

$$\mathrm{Horn} = \{\, \Lambda^n_j \subset \Delta^n \mid n \geq 1,\ 0 \leq j \leq n \,\} = \mathrm{RHorn} \cup \mathrm{LHorn}.$$

A map is **anodyne** if it is in $\overline{\mathrm{Horn}}$, and is a **Kan fibration** if it is in $\mathrm{KFib} := \mathrm{Horn}^{\boxslash}$.

Since Horn is a set, the small object argument (12.10) applies to it: any map can be factored $f = pj$ with $j \in \overline{\mathrm{Horn}}$ and $p \in \mathrm{KFib}$.

31.6. **Proposition.** *We have that* $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$.

Proof. This amounts to the calculation $\mathrm{Horn} \square \mathrm{Cell} \subseteq \overline{\mathrm{Horn}}$. See [JT08, Theorem 3.2.2], or [GZ67]. $\square$

Thus, we have that

$$\mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$$

is a Kan fibration whenever $K \subseteq L$ and $X \to Y$ is a Kan fibration, and is a trivial fibration if $K \subseteq L$ is also anodyne.

As a special case, we learn that if X is a Kan complex and $K \subseteq L$, then $\mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X)$ is a Kan fibration. Here is another consequence.

31.7. **Proposition.** *Every anodyne map is a weak equivalence.*

Proof. If $f\colon A \to B$ is anodyne, then $\mathrm{Map}(f, G)$ is a trivial fibration for every Kan complex G, and hence a categorical equivalence (19.10). $\square$

31.8. *Exercise.* Show that the inclusion $\{j\} \subseteq \Delta^n$ of any vertex into any standard n-simplex is anodyne.

31.9. *Example.* From (31.8) it follows that any map $f\colon \Delta^m \to \Delta^n$ between standard simplicies is a weak equivalence, using the 2-out-of-3 property (31.2). Any such f which is not an isomorphism is an example of a weak equivalence between quasicategories which is not a categorical equivalence.

31.10. *Exercise.* Let $f\colon X \to Y$ be any map between Kan complexes. Show that f is a Kan fibration if and only if it is an isofibration. (Hint: Joyal lifting.)

31.11. *Exercise.* Give an example of an inner fibration between Kan complexes which is not a Kan fibration.

31.12. **The universal isomorphism.** Let Iso be the "walking isomorphism", i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Let $u\colon \Delta^1 \to N\mathrm{Iso}$ be the inclusion representing the unique map $0 \to 1$ in Iso.

31.13. **Proposition.** *The map $u\colon \Delta^1 \to N\mathrm{Iso}$ is anodyne, and hence a weak equivalence.*

Proof. The k-simplicies of $N(\mathrm{Iso})$ are in one-to-one correspondence with sequences $x_0x_1\cdots x_k$ with $x_i \in \{0,1\}$. There are exactly two *non-degenerate* k-simplices, corresponding to the alternating sequences $0101\ldots$ and $1010\ldots$.

Let $u_k\colon \Delta^k \to N\mathrm{Iso}$ be the non-degenerate simplex $0101\ldots$, and let $F_k \subset N\mathrm{Iso}$ be the smallest subcomplex containing u_k. Note that $N\mathrm{Iso} = \bigcup F_k$ and $F_1 = u(\Delta^1)$. The commutative square

$$\begin{tikzcd} \Lambda^k_0 \arrow[r] \arrow[d, tail] & F_{k-1} \arrow[d, tail] \\ \Delta^k \arrow[r, "u_k"'] & F_k \end{tikzcd}$$

is a pushout square for all $k \geq 1$. This is by (20.3), since (1) it is a pullback, and (2) any simplex in the complement of $F_{k-1} \subset F_k$ is the image of a unique simplex under u_k.

It follows that u is anodyne. $\square$

We obtain as a consquence the following criterion for an edge to be an isomorphism, which we will use later.

31.14. **Proposition.** *Let C be a quasicategory, and $f\colon \Delta^1 \to C$ representing morphism in C. Then there exists $f'\colon N(\mathrm{Iso}) \to C$ such that $f'u = f$ if and only if f represents an isomorphism.*

Proof. ($\Longrightarrow$) Clear: consider induced maps on homotopy categories. ($\Longleftarrow$) If f represents an isomorphism then it factors through $\Delta^1 \to C^{\mathrm{core}} \subseteq C$. Since the core is a quasigroupoid, and hence a Kan complex, an extension along the anodyne map u to a map $N\mathrm{Iso} \to C^{\mathrm{core}} \subseteq C$ exists. $\square$

31.15. *Remark.* Let $X \subset N\mathrm{Iso}$ be the subcomplex which is the union of the images of 2-simplices 010 and 101[24]. The inclusion $v\colon \Delta^1 \to X$ representing the edge 01 has the same property described in (31.14): $f\colon \Delta^1 \to C$ represents an isomorphism if and only if it extends along v. (The proof is easy.)

However, it turns out that $\Delta^1 \to X$ is *not* a weak equivalence, or what by 2-out-of-3 is the same thing, $X \to \Delta^0$ is not a weak equivalence. In particular, a map $X \to K$ to a Kan complex can fail to extend along $X \subset N\mathrm{Iso}$.

31.16. *Exercise.* Show that $X \to N\mathrm{Iso}$ is not anodyne, by constructing a map $X \to K(\mathbb{Z}, 2)$ which does not extend over $N\mathrm{Iso}$. (See (7.10).)

[24]This is isomorphic to the complex discussed in (19.7).

31.17. **Covering homotopy extension property.** This is the special case of $\overline{\text{Horn}}\square\overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ which we will use several times in the next few sections: for any inclusion $K \subseteq L$, the map

$$(K \times \Delta^1) \cup_{K\times\{j\}} (L \times \{j\}) \to L \times \Delta^1$$

with either $j = 0$ or $j = 1$ is anodyne. This amounts to saying that

$$\begin{tikzcd} (K \times \Delta^1) \cup (L \times \{j\}) \arrow[r] \arrow[d, tail] & X \arrow[d, "p"] \\ L \times \Delta^1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a lift whenever p is a Kan fibration. This is sometimes called the "covering homotopy extension property".

It may be helpful to think of this in the equivalent form, which asserts a lifting in

$$\begin{tikzcd} \{j\} \arrow[r] \arrow[d, tail] & \operatorname{Map}(L, X) \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \times \operatorname{Map}(L, Y) \end{tikzcd}$$

when $K \subseteq L$ and $p\colon X \to Y$ a Kan fibration. This gets used the following way: to demonstrate $(K \subseteq L) \boxslash p$ for a given Kan fibration p and inclusion $K \subseteq L$, we "deform" a lifting problem of this type along a "path" in the space $\operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$ of commutative squares to a lifting problem which we know has a lift.

31.18. **Fundamental theorem for Kan complexes: reduction to Kan fibrations.** We are going to show the following

31.19. **Theorem.** *A map $f\colon X \to Y$ between Kan complexes is a weak equivalence if and only if it is fully faithful and essentially surjective.*

We will prove this by reducing to Kan fibrations.

31.20. **Lemma.** *To prove* (31.19)*, it suffices to prove it for the case when f is a Kan fibration.*

Proof. ($\Longrightarrow$) Weak equivalences between Kan complexes are categorical equivalences, and we have already shown that these are fully faithful and essentially surjective (30.3).

($\Longleftarrow$) Given f between Kan complexes which is fully faithful and essentially surjective, use the small object argument (12.10) applied to Horn to factor it as

$$X \xrightarrow{j} V \xrightarrow{p} Y,$$

where j is anodyne and p is a Kan fibration. It follows that V is a Kan complex, j is a weak equivalence (31.7) between Kan complexes, and so is fully faithful and essentially surjective. Since the class of fully faithful and essentially surjective maps satisfy 2-out-of-3 (30.5), it follows that p also has this property. If we can use this to show p is a weak equivalence, it follows that f is a weak equivalence as desired. $\square$

We will prove the lemma in the next couple of sections, after analyzing Kan fibrations in more detail.

32. Kan fibrations between Kan complexes

In the next few sections, we are going to be considering various properties of Kan fibrations, with particular interest in Kan fibrations between Kan complexes.

In particualar, we are going to show that for a Kan fibration $p\colon X \to Y$ where X and Y are Kan complexes, all of the following are equivalent.

(1) p is a trivial fibration;
(2) p is a weak equivalence;
(3) p is a fiberwise deformation retraction;
(4) p has contractible fibers;
(5) p is fully faithful and essentially surjective.

The equivalence of (2) and (5) will complete the proof of the fundamental theorem for Kan complexs (31.19). In fact, (1)–(4) are equivalent without the hypothesis that the objects are Kan complexes, though we will not prove that in all cases.

32.1. **Fiberwise deformation retraction.** A map $p\colon X \to Y$ is said to be a **fiberwise deformation retraction** if there exists

- $s\colon Y \to X$ such that $ps = \mathrm{id}_Y$, and
- $k\colon X \times \Delta^1 \to X$ such that $k|X \times \{0\} = \mathrm{id}_X$, $k|X \times \{1\} = sp$, and $pk = p\pi$, where $\pi\colon X \times \Delta^1 \to X$ is projection; that is, the diagram

$$\begin{tikzcd} X \times \{0,1\} \arrow[rr, "(\mathrm{id}_X, sp)"] \arrow[d, tail] & & X \arrow[d, "p"] \\ X \times \Delta^1 \arrow[urr, "k"] \arrow[r, "\pi"'] & X \arrow[r, "p"'] & Y \end{tikzcd}$$

commutes.

Any fiberwise deformation retraction is a weak equivalence: s is a simplicial homotopy inverse to p (31.4).

32.2. *Exercise.* Show that the term "fiberwise" is justified: for each $y \in Y_0$, the projection $p^{-1}(y) \to \{y\}$ is a simplicial homotopy equivalence.

32.3. *Exercise.* Show that if $p\colon X \to Y$ is part of a deformation retraction as above, then any base change of p is also part of a deformation retraction.

Fiberwise deformation retractions of Kan fibrations are always trivial fibrations, as can be shown with the covering homotopy extension property.

32.4. **Lemma.** *Let $p\colon X \to Y$ be a Kan fibration between simplicial sets. Then p is part of a fiberwise deformation retraction if and only p is a trivial fibration.*

Proof. [JT08, Prop. 3.2.5]. ($\Longrightarrow$) We need to solve the lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "u"] & Y \end{tikzcd}$$

where i is a monomorphism. Using the data s and k of the fiberwise deformation retraction we obtain a commutative square

$$\begin{tikzcd}
(A\times\Delta^1)\cup(B\times\{1\}) \arrow[rr, "{(k(a\times\mathrm{id}_{\Delta^1}),sb)}"] \arrow[d, tail, "j"'] & & X \arrow[d, "p"] \\
B\times\Delta^1 \arrow[r, "\pi"'] \arrow[urr, dashed, "t"] & B \arrow[r, "b"'] & Y
\end{tikzcd}$$

Because p is a Kan fibration and j is anodyne by (31.6), a lift t exists. Then $u := t|B\times\{0\}$ is the desired lift.

($\Longleftarrow$) Left as an exercise. $\square$

32.5. *Exercise.* Show that any trivial fibration is a fiberwise deformation retraction.

32.6. **Trivial Kan fibrations between Kan complexes.** We know that trivial fibrations are always categorical equivalences (19.10). We now show that any Kan fibration between Kan complexes which is also a categorical equivalence is a trivial fibration.

32.7. **Proposition.** *A Kan fibration $p\colon X\to Y$ between Kan complexes is a trivial fibration if and only if it is a weak equivalence.*

Proof. [JT08, Prop. 3.2.6] ($\Longrightarrow$) We have already shown that trivial fibrations between quasicategories are always categorical equivalences, which implies they are weak equivalences if between Kan complexes.

($\Longleftarrow$) On the other hand, suppose p is a Kan fibration and a weak equivalence. Being a categorical equivalence between Kan complexes, p admits a categorical inverse: there exists $f\colon Y\to X$ and maps $u\colon X\times\Delta^1\to X$ and $v\colon Y\times\Delta^1\to Y$ which give natural isomorphisms $u\colon fp\to\mathrm{id}_X$ and $v\colon pf\to\mathrm{id}_Y$. We will "deform" this data to a fiberwise deformation retraction.

Step 1. Since $Y\times\{0\}\subset Y\times\Delta^1$ is anodyne by (31.6), a lift α exists in

$$\begin{tikzcd}
Y\times\{0\} \arrow[r, "f"] \arrow[d, tail] & X \arrow[d, "p"] \\
Y\times\Delta^1 \arrow[r, "v"'] \arrow[ur, dashed, "\alpha"] & Y
\end{tikzcd}$$

Let $s := \alpha|Y\times\{1\}$, so $ps = \mathrm{id}_Y$. The map α exhibits a natural isomorphism $\alpha\colon f\to s$ of functors $Y\to X$. Since fp is naturally isomorphic to id_X, we have $\mathrm{id}_X\approx fp\approx sp$, i.e., there exists a natural isomorphism $w\colon sp\to\mathrm{id}_X$.

Step 2. Consider the natural isomorphism $(sp)w\colon sp = spsp\to sp$ of functors $X\to X$. We have a commutative diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "a"] \arrow[d, tail] & \mathrm{Map}(X,X) \arrow[d, "{\mathrm{Map}(X,p)}"] \\
\Delta^2 \arrow[r, "b"'] \arrow[ur, dashed, "t"] & \mathrm{Map}(X,Y)
\end{tikzcd}$$

where $a_{01} = w$, $a_{02} = spw$, and b is the degeneracy $b = (pw)_{011}$; this commutes since $pspw = pw$. Since p is a Kan fibration, so is $\mathrm{Map}(X,p)$ by (31.6), and therefore a lift t exists. Let $k = t|\Delta^{\{1,2\}}\colon\Delta^1\to\mathrm{Map}(X,X)$. It is clear that this is a natural isomorphism $k\colon\mathrm{id}_Y\to sp$, and that this is "fiberwise", i.e., $pk = b_{12} = p\pi$ as maps $X\times\Delta^1\to Y$.

Thus, we have exhibited p as a fiberwise deformation retraction, so p is a trivial fibration by (32.4). $\square$

32.8. **Contractible Kan complexes.**

32.9. **Corollary.** *Let X be a simplicial set. The following are equivalent.*

(1) *X is a quasicategory which is categorically equivalent to Δ^0.*
(2) *$X \to \Delta^0$ is a trivial fibration.*
(3) *Every $\partial\Delta^n \to X$ extends over Δ^n.*

Such an X is necessarily a Kan complex.

Proof. We have (2) $\Leftrightarrow$ (3) by definition, and we know that (2) $\Rightarrow$ (1). Given (1), we have that X is a quasigroupoid, and hence a Kan complex (28.2), and (2) follows by the previous proposition (32.7). □

We say that an X satisfying these conditions is a **contractible Kan complex**.

32.10. **Monomorphisms which are weak equivalences.** We can now characterize the monomorphisms which are weak equivalences, in terms of maps into Kan complexes.

32.11. **Proposition.** *Let $j\colon A \to B$ be a monomorphism of simpicial sets. Then j is a weak equivalence if and only if* $\mathrm{Map}(j, X)\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ *is a trivial fibration for all Kan complexes X.*

Proof. Assume X is an arbitrary Kan complex. We know that $\mathrm{Map}(j, X)$ is always a Kan fibration between Kan complexes by (31.6). We have by definition that j is a weak equivalence iff all $\mathrm{Map}(j, X)$ are weak equivalences, which holds iff all $\mathrm{Map}(j, X)$ are trivial fibrations by (32.7). □

32.12. *Remark.* The class $\mathrm{WkEq}\cap\overline{\mathrm{Cell}}$ of monomorphisms which are weak equvialences is a saturated class. In fact, using (32.11) it is easy to show that it is the left complement of the class of maps of the form $p^{\square j}$, where $p\colon X \to \Delta^0$ is a projection from a Kan complex X, and $j\colon \partial\Delta^n \to \Delta^n$ is a cell inclusion. Furthermore, $\mathrm{WkEq}\cap\overline{\mathrm{Cell}}$ contains the saturated class $\overline{\mathrm{Horn}}$ of anodyne maps.

It turns out that $\overline{\mathrm{Horn}} = \mathrm{WkEq}\cap\overline{\mathrm{Cell}}$, i.e., the injective weak equivalences are *precisely the same* as the anodyne maps. This is a fairly non-trivial fact, and we will address it again later. **Maybe.** As a consequence, it will follow that (32.7) and (32.15) hold without the condition that the objects be Kan complexes.

32.13. **Enriched lifting for Kan fibrations between Kan complexes.**

32.14. **Proposition.** *If $j\colon A \to B$ is a monomorphism and a weak equivalence of simplicial sets, and $p\colon X \to Y$ is a Kan fibration between Kan complexes, then*

$$p^{\square i}\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a trivial fibration.

Proof. The map $p^{\square i}$ is a Kan fibration between Kan complexes, using $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ (31.6). Consider the diagram

$$\begin{tikzcd}
\mathrm{Map}(B,X) \arrow[r, "p^{\square i}"] & \mathrm{Map}(A,X)\times_{\mathrm{Map}(A,Y)}\mathrm{Map}(B,Y) \arrow[r, "q'"] \arrow[d] & \mathrm{Map}(A,X) \arrow[d] \\
& \mathrm{Map}(B,Y) \arrow[r, "q"'] & \mathrm{Map}(A,Y)
\end{tikzcd}$$

in which the square is a pullback. By (32.11) the maps q and $q'(p^{\square i})$ are trivial fibrations. The pullback q' of q is also a trivial fibration, and so $p^{\square i}$ is a weak equivalence by 2-out-of-3 (31.2), and therefore a trivial fibration since it is a Kan fibration between Kan complexes (32.7). □

We also obtain another characterization of Kan fibrations between Kan complexes.

32.15. **Corollary.** *A map $p\colon X \to Y$ between Kan complexes is a Kan fibration if and only if $j \boxslash p$ for all j which are monomorphisms and weak equivalences.*

Proof. ($\Longleftarrow$) Straightforward, since inner horn inclusions are monomorphisms and weak equivalences. ($\Longrightarrow$) Immediate from the previous proposition. $\square$

33. The fiberwise criterion for trivial fibrations

We give another criterion for Kan fibration to be a trivial fibration, in terms of its fibers.

33.1. **Fiberwise criterion for trivial fibrations.** The **fiber** $p^{-1}(y)$ of a map $p\colon X \to Y$ over a vertex $y \in Y_0$ is the pullback of p along $\{y\} \to Y$.

If $p\colon X \to Y$ is a trivial fibration, then since $\mathrm{TFib} = \mathrm{Horn}^{\boxslash}$ we see immediately that every projection $p^{-1}(y) \to *$ from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes.

33.2. **Proposition.** *Let $p\colon X \to Y$ be a Kan fibration. Then p is a trivial fibration if and only if every fiber of p is a contractible Kan complex.*

Proof. We have just observed ($\Longrightarrow$) . So suppose p is a Kan fibration whose fibers are contractible Kan complexes, and consider a lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "b"'] \arrow[ur, dashed, "t"] & Y
\end{tikzcd}$$

We will construct a lift t by using the covering homotopy extension property to "deform" this to a lifting problem involving a single fiber of p, which admits a solution by the hypothesis on the fibers of p.

Let $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ be the unique map given on vertices by $\gamma(i,j) = i$ if $j = 0$ and $\gamma(i,j) = n$ if $j = 1$. Thus $\gamma|\Delta^n \times \{0\} = \mathrm{id}$, while $\gamma|\Delta^n \times \{1\}$ factors through the vertex $\{n\} \subseteq \Delta^n$. Since $(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1)$ is anodyne by $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$, a lift exists in

$$\begin{tikzcd}
\partial\Delta^n \times \{0\} \arrow[rrr, "a"] \arrow[d, tail] & & & X \arrow[d, "p"] \\
\partial\Delta^n \times \Delta^1 \arrow[r, tail] \arrow[urrr, dashed, "c"] & \Delta^n \times \Delta^1 \arrow[r, "\gamma"'] & \Delta^n \arrow[r, "b"'] & Y
\end{tikzcd}$$

Since $b\gamma|\Delta^n \times \{1\}$ factors through a vertex $y \in Y_0$, we have a lift d in

$$\begin{tikzcd}
\partial\Delta^n \times \{1\} \arrow[r, "c|\partial\Delta^n\times\{1\}"] \arrow[d, tail] & p^{-1}(y) \arrow[r, tail, "j"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \{1\} \arrow[r] \arrow[ur, dashed, "d"] & \{y\} \arrow[r, "y"'] & Y
\end{tikzcd}$$

since by hypothesis $p^{-1}(y)$ is a contractible Kan complex. Putting this together we obtain a commutative square

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n\times\{1\}} (\Delta^n \times \{1\}) \arrow[r, "{(c,jd)}"] \arrow[d, tail] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

which admits a lift s since $(\partial\Delta^n \subset \Delta^n)\square(\{1\} \subset \Delta^1)$ is anodyne. The restriction $t := s|\Delta^n \times \{0\}$ provides a solution to the original lifting problem, since $t|\partial\Delta^n \times \{0\} = c|\partial\Delta^n \times \{0\} = a$ and $pt = (b\gamma)|\Delta^n \times \{0\} = b$. □

We often apply the above result in the following way.

33.3. **Corollary.** *Suppose*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback square such that (1) p is a Kan fibration and (2) $g_0\colon Y'_0 \to Y_0$ is surjective. Then p is a trivial fibration if and only if p' is a trivial fibration. Furthermore, if all objects are Kan complexes, then p is a weak equivalence if and only if p' is a weak equivalence.

Proof. The fibers of p all appear as fibers of p' by (2). Use the fiberwise criterion (33.2) and (32.7). □

33.4. **Criterion for fully faithful map between Kan complexes.** As an application of the fiberwise criterion, we have the following criterion for a Kan fibration between Kan complexes to be fully faithful.

33.5. **Proposition.** *A Kan fibration $p\colon X \to Y$ between Kan complexes is fully faithful iff the induced map*

$$q = p^{\square(\partial\Delta^1 \subset \Delta^1)}\colon \operatorname{Map}(\Delta^1, X) \to \operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial\Delta^1, Y)} \operatorname{Map}(\partial\Delta^1, X)$$

is a trivial fibration

Proof. Consider the pullback square

$$\begin{tikzcd} \coprod_{x,x'} \operatorname{map}_X(x,x') \arrow[r] \arrow[d] & \operatorname{Map}(\Delta^1, X) \arrow[d, "q"] \\ \coprod_{x,x'} \operatorname{map}_Y(fx,fx') \arrow[r] & \operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial\Delta^1, Y)} \operatorname{Map}(\partial\Delta^1, X) \end{tikzcd}$$

where the products are over all pairs of elements of X_0. The map along the bottom is a bijection on vertices. The map q is a Kan fibration by (31.6), and all the objects are Kan complexes, again using (31.6). The result follows using (33.3). □

34. Fundamental theorem for Kan complexes

In this section, we will prove quasigroupoid version of the fundamental theorem (31.19), i.e., that fully faithful and essentially surjective maps between quasigroupoids (=Kan complexes) are weak equivalences. We note that we have already reduced (31.20) to the case of Kan fibrations.

34.1. **Proposition.** *A Kan fibration $p\colon X \to Y$ between Kan complexes which is fully faithful and essentially surjective is a weak equivalence.*

Proof. Consider

$$X \xrightarrow{j=(ip,\mathrm{id}_X)} \operatorname{Map}(\Delta^1, Y) \times_Y X \xrightarrow{\pi_1} \operatorname{Map}(\Delta^1, Y) \xrightarrow{\langle 1\rangle^*} Y$$

where the fiber product is constructed from $\operatorname{Map}(\Delta^1, Y) \xrightarrow{\langle 0\rangle^*} Y \xleftarrow{p} X$, and $i\colon Y \to \operatorname{Map}(\Delta^1, Y)$ is adjoint to the projection $Y \times \Delta^1 \to \Delta^1$. Since $\langle 1\rangle^* i = \mathrm{id}_Y$ we see that $\langle 1\rangle^*\pi_1 j = p$.

A straightforward argument shows that the map $\langle 0\rangle^*\colon \operatorname{Map}(\Delta^1, Y) \to Y$ is the projection part of a fiberwise deformation retraction, with section i; the homotopy is the adjoint to restriction along a suitable map $\Delta^1 \times \Delta^1 \to \Delta^1$. The projection map $\pi_2\colon \operatorname{Map}(\Delta^1, Y) \times_Y X \to X$ is the base change of $\langle 0\rangle^*$ along p, and so is also a fiberwise deformation retraction (32.3), and so is a simplicial homotopy equivalence, and hence a weak equivalence.

Since $\pi_2 j = \mathrm{id}_X$, it follows that j is a weak equivalence by 2-out-of-3 (31.2). Again using 2-out-of-3, we see that to show p is a weak equivalence it suffices to show that $r_1 := \langle 1\rangle^* \pi_1$ is a weak equivalence.

Now consider the diagram

$$\begin{tikzcd}
\operatorname{Map}(\Delta^1, Y) \times_{Y \times Y} (X \times X) \arrow[r, "\pi_2'"'] \arrow[d, "\mathrm{id} \times (\mathrm{id} \times p)"'] \arrow[rr, bend left, "r_1'"] & X \times X \arrow[r, "\pi_2'"'] & X \arrow[d, "p"] \\
\operatorname{Map}(\Delta^1, Y) \times_{Y \times Y} (X \times Y) \arrow[r, "\pi_2"] \arrow[rr, bend right, "r_1"'] & X \times Y \arrow[r, "\pi_2"] & Y
\end{tikzcd}$$

in which the composite along the bottom is also r_1, and the rectangle is a pullback. We complete the proof using the fiberwise criterion (33.3) to show that r_1 (which is a Kan fibration) is a trivial fibration; i.e., we need that p is surjective on 0-simplices and that r_1' is a trivial fibration.

First, note that since p is an essentially surjective Kan fibration, p must actually be surjective on 0-simplices, by a straightforward argument using the fact that $(\{0\} \subset \Delta^1) \boxslash p$.

Both projections on the top row are Kan fibrations, since $X \to \Delta^0$ and $\operatorname{Map}(\Delta^1, Y) \to \operatorname{Map}(\partial\Delta^1, Y)$ are Kan fibrations. Thus r_1' is a Kan fibration between Kan complexes, so we only need to show that r_1' is a weak equivalence, since then (32.7) will give that r_1' is a trivial fibration.

Now consider

$$\operatorname{Map}(\Delta^1, X) \xrightarrow{q} \operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial\Delta^1, Y)} \operatorname{Map}(\partial\Delta^1, X) \xrightarrow{r_1'} X$$

where q is the evident box-power map. The composite $r_1' q$ is equal to the evident restriction map along $\{1\} \subset \Delta^1$. Since $\{1\} \subset \Delta^1$ is anodyne it follows that $r_1' q$ is a trivial fibration. Finally, since p is fully faithful, (33.5) gives that q is a trivial fibration, and hence it follows that r_1' is a weak equivalence by 2-out-of-3. The proof is complete. $\square$

35. Isofibrations

In this section, we return to isofibrations. The moral is that isofibrations between quasicategories play a role analogous to Kan fibrations between Kan complexes.

35.1. **Characterizations of isofibrations.** Recall that a functor $f\colon C \to D$ between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every

$$\begin{tikzcd}
\{j\} \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
\Delta^1 \arrow[r, "f"'] \arrow[ur, dashed, "g"] & D
\end{tikzcd}$$

with $j = 0$ such that f represents an isomorphism admits a lift g which is also represents isomorphism. Furthermore, it is equivalent to require (2') instead of (2), where (2') is the same statement with $j = 1$.

Also, note that $C \to *$ is an isofibration for any quasicategory C.

We have the following "lifting criterion" for isofibrations.

35.2. **Proposition.** *An map p between quasicategories is an isofibration iff (1) it is an inner fibration and (2) $(\{0\} \subset N(\mathrm{Iso})) \boxslash p$.*

Proof. ($\Longleftarrow$) Straightforward, using the fact (31.14) that every $f\colon \Delta^1 \to D$ representing an isomorphism factors through a map $N(\mathrm{Iso}) \to D$.

($\Longrightarrow$) Let p be an isofibration. To solve the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\ N(\mathrm{Iso}) \arrow[r] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

recall from the proof of (31.13) that $N(\mathrm{Iso}) = \bigcup F_k$ where F_k is obtained from F_{k-1} by gluing along $\Lambda^k_0 \subset \Delta^k$; we construct lifts $s_k\colon F_k \to C$ inductively. A lift $s_1\colon F_1 = \Delta^1 \to C$ exists by the definition of isofibration, and we may assume that s_1 is an isomorphism in C. Then the Joyal lifting theorem (27.13) provides lifts s_k for $k \geq 2$. $\square$

In other words, the isofibrations are precisely the maps *between quasicategories* which are contained in $\bigl(\mathrm{InnHorn} \cup \{\{0\} \subset N(\mathrm{Iso})\}\bigr)^{\boxslash}$. In particular, the pullback of an isofibration along a map from a quasicategory is also a quasicategory.

35.3. *Remark.* We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called "categorical fibration", and will be discussed later.

Here is another characterization of isofibrations in terms of cores. Remember that any functor $p\colon C \to D$ restricts to a functor $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ between Kan complexes.

35.4. **Proposition.** *A map $p\colon C \to D$ between quasicategories is a isofibration if and only if (1) it is an inner fibration, and (2) $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a Kan fibration.*

Proof. ($\Longrightarrow$) Let p be an isofibration. Then p^{core} is also an isofibration, by an elementary argument. (The point is that the relevant lifting problems for p^{core} clearly have lifts s with target C, since p is an isofibration; an easy argument shows that the image of such lifts s must actually land in C^{core}.)

Thus have reduced to showing that any isofibration between Kan complexes is a Kan fibration, which is a straightforward exercise using Joyal lifting (31.10).

($\Longleftarrow$) If p^{core} is a Kan fibration, then it is immediate that property (2) of an isofibration holds. $\square$

In particular, isofibrations between Kan complexes are precisely Kan fibrations. (This can be proved directly using Joyal lifting, as in (31.10).)

35.5. **Lifting properties for isofibrations.** We are now ready to prove the following proposition, which will be the key tool in what follows.

35.6. **Proposition.** *Let $p\colon C \to D$ be an isofibration between quasicategories, and $i\colon K \to L$ any monomorphism of simplicial sets. Then the induced box-power map*

$$p^{\square i}\colon\ \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$$

is an isofibration.

Proof. Fix a map $p\colon C \to D$ between categories. We first note that the class maps $i\colon K \to L$ such that $p^{\square i}$ is an isofibration is saturated. In fact, if $S := \mathrm{InnHorn} \cup \{\{0\} \subset N(\mathrm{Iso})\}$, then $p^{\square i}$ is an isofibration iff $S \boxslash (p^{\square i})$ iff $i \boxslash (p^{\square S})$ by (35.2) and (15.4). Therefore the class $\mathcal{C} := \{\, i \mid p^{\square i} \in \mathrm{IsoFib}\,\}$ is the left complement of $p^{\square S}$, and thus saturated.

Therefore, to show that $\mathcal{C}$ contains all monomorphisms, it suffices show that it contains $i = (\partial\Delta^n \subset \Delta^n)$ for $n \geq 0$.

Note that we will certainly have that $p^{\square i}$ is an inner fibration, using using $\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (15.7).

If $n = 0$, then $p^{\square i} = p$ so the claim is trivial[25].

Now assume $n \geq 1$. Since $p^{\square i}$ is an inner fibration between quasicategories, it suffices to solve the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, tail] & \mathrm{Fun}(\Delta^n, C) \arrow[d, "p^{\square i}"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dashed] & \mathrm{Fun}(\partial\Delta^n, C) \times_{\mathrm{Fun}(\partial\Delta^n, D)} \mathrm{Fun}(\Delta^n, D) \end{tikzcd} \iff \begin{tikzcd} (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) \arrow[r, "g"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

where f represents an isomorphism in the target.

The edge $f' := (g|\Delta^1 \times \Delta^{\{0\}})$ is the same as the composite

$$\Delta^1 \xrightarrow{f} \mathrm{Fun}(\partial\Delta^n, C) \times_{\mathrm{Fun}(\partial\Delta^n, D)} \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(\partial\Delta^n, C) \to \mathrm{Fun}(\{0\}, C).$$

Since f is an isomorphism in the fiber product, it follows that f' is an isomorphism in C. Therefore a lift exists by the box-version of Joyal lifting (28.11), since $n \geq 1$. $\square$

One consequence of the above is that if p is an isofibration and i is an inclusion, then

$$(p^{\square i})^{\mathrm{core}} \colon \mathrm{Fun}(L, C)^{\mathrm{core}} \to \big(\mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)\big)^{\mathrm{core}}$$

is a Kan fibration. It turns out we can replace the target (the core of a pullback) with a pullback of cores.

35.7. **Corollary.** *If $p \colon C \to D$ is any functor between quasicategories and $K \subseteq L$, then*

$$\big(\mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)\big)^{\mathrm{core}} = \mathrm{Fun}(K, C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L, D)^{\mathrm{core}}. \tag{35.8}$$

Proof. Both sides of (35.8) can be regarded as subobjects of $\mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$, which we note is a quasicategory since $r \colon \mathrm{Fun}(L, D) \to \mathrm{Fun}(K, D)$ is an inner fibration and $\mathrm{Fun}(K, C)$ is a quasicategory. The left-hand side is clearly contained in the right-hand side, since any functor between quasicategories takes isomorphisms to isomorphisms. By (35.6), $r \colon \mathrm{Fun}(L, D) \to \mathrm{Fun}(K, D)$ is actually a isofibration, and so r^{core} is a Kan fibration (35.4). Thus the right-hand side of (35.8) is a pullback of Kan complexes along a Kan fibration, and thus is a Kan complex, which is necesarily contained in the left-hand side of (35.8). $\square$

35.9. **Trivial fibrations between quasicategories.** Now we can prove a generalization of (32.7), which identified trivial fibrations between Kan complexes as the Kan fibrations which are weak equivalences. In the proof we will make essential use of this special case.

35.10. **Proposition.** *Let $p \colon C \to D$ be a map between quasicategories. Then p is a trivial fibration if and only if it is an isofibration and a categorical equivalence.*

Proof. [Joy08a, Theorem 5.15]. ($\Longrightarrow$) If p is a trivial fibration, it is an inner fibration and $(\{0\} \subset N(\mathrm{Iso})) \boxslash p$, so it is an isofibration (35.2). We have already shown that p is a categorical equivalence (19.10).

($\Longleftarrow$) Conversely, suppose p is isofibration and categorical equivalence. It suffices to show that for any inclusion $i \colon K \subset L$, the map

$$g := (p^{\square i})^{\mathrm{core}} \colon \mathrm{Fun}(L, C)^{\mathrm{core}} \to \mathrm{Fun}(K, C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L, D)^{\mathrm{core}}$$

is surjective on 0-simplicies; in fact, we will show that g is a trivial fibration. By (35.6), the map $p^{\square i} \colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$ is an isofibration; by (35.4), the restriction

[25] This step is the only place in the proof where we actually use the fact that p is an isofibration, and not merely an inner fibration! In fact, if p is merely an inner fibration, but $K_0 = L_0$, then $p^{\square i}$ is a isofibration.

$(p^{\square i})^{\mathrm{core}}$ of this map to cores is a Kan fibration; by (35.7), this restriction is precisely g. Thus g is a Kan fibration between Kan complexes. It will therefore suffice by (32.7) to show that g is also a categorical equivalence.

The maps

$$\mathrm{Fun}(L,C) \to \mathrm{Fun}(L,D), \qquad \mathrm{Fun}(K,C) \to \mathrm{Fun}(K,D)$$

are categorical equivalences since p is, and so induce categorical equivalences on cores. These maps are also isofibrations by (35.6), and therefore the restrictions to cores are Kan fibrations. Thus $\mathrm{Fun}(L,C)^{\mathrm{core}} \to \mathrm{Fun}(L,D)^{\mathrm{core}}$ and $\mathrm{Fun}(K,C)^{\mathrm{core}} \to \mathrm{Fun}(K,D)^{\mathrm{core}}$ are Kan fibrations and weak equivalences between Kan complexes, and thus are trivial fibrations (32.7). Thus in

$$\mathrm{Fun}(L,C)^{\mathrm{core}} \xrightarrow{g} \mathrm{Fun}(K,C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L,D)^{\mathrm{core}} \to \mathrm{Fun}(L,D)^{\mathrm{core}}$$

the second map is a trivial fibration being a pullback of $\mathrm{Fun}(K,p)^{\mathrm{core}}$, and the composite is a weak equivalence. It follows that g is a weak equivalence, and the result is proved. $\square$

This proof made essential use of our characterization of trivial fibrations between Kan complexes, which is why we had to prove that special case first.

35.11. **Proposition.** *A map $p\colon C \to D$ with D a quasicategory is an isofibration if and only if $j \boxslash p$ for every $j\colon K \to L$ which is both a monomorphism and a categorical equivalence.*

Proof. ($\Longleftarrow$) Immediate from the the characterization of isofibrations as maps between quasicategories in the right complement of $\mathrm{InnHorn} \cup \{\{0\} \subset N\mathrm{Iso}\}$ (35.2).

($\Longrightarrow$) Suppose p is an isofibration. It suffices to show that for $j \in \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$,

$$q\colon\ \mathrm{Fun}(K,C) \to \mathrm{Fun}(L,C) \times_{\mathrm{Fun}(L,D)} \mathrm{Fun}(K,D)$$

is a trivial fibration, whence it is surjective on 0-simplices and thus $j \boxslash p$. Since j is a monomorphisms, (35.6) says q is an isofibration between quasicategories, so by the criterion for an isofibration to be a trivial fibration (35.10) it suffices to show that q is a categorical equivalence. Since j is a categorical equivalence both $\mathrm{Fun}(j,C)$ and $\mathrm{Fun}(j,D)$ are categorical equivalences. The map $\mathrm{Fun}(j,D)$ is also an isofibration by (35.6), so it is a trivial fibration by (35.10). Therefore the basechange $\mathrm{Fun}(L,C) \times_{\mathrm{Fun}(L,D)} \mathrm{Fun}(K,D) \to \mathrm{Fun}(L,C)$ of $\mathrm{Fun}(j,D)$ is a trivial fibration, and the result follows using the 2-out-of-3 property of categorical equivalences (21.8). $\square$

35.12. **Monomorphisms which are categorical equivalences.** We can now prove a generalization of (32.11), which characterized the injective weak equivalences.

35.13. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets. Then j is a categorical equivalence if and only if* $\mathrm{Map}(j,C)\colon \mathrm{Map}(L,C) \to \mathrm{Map}(K,C)$ *is a trivial fibration for all quasicategories C.*

Proof. Straightforward using the fact that $\mathrm{Map}(j,C)$ is an isofibration for any inclusion (35.6), and that isofibrations which are categorical equivalences are trivial fibrations (35.10). $\square$

35.14. *Remark.* As a consequence of (35.13), we see that the class $\overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ of injective categorical equivalences is a saturated class: it is the left complement of the class of maps of the form $p^{\square i}$ where $p\colon C \to \Delta^0$ is a projection from a quasicategory, and i is a cell inclusion. Furthermore, this class contains the class of inner anodyne maps.

This class is *not the same* as $\overline{\mathrm{InnHorn}}$. For instance, every inner anodyne map is a bijection on vertices, but $\{0\} \to N\mathrm{Iso}$ which is not bijective on vertices is an injective categorical equivalence. Neither is it the same as the saturation of $\mathrm{InnHorn} \cup \{\{0\} \subset N\mathrm{Iso}\}$. This is a significant way in which the theory of quasicategories is not entirely parallel with the theory of Kan complexes.

36. Localization of quasicategories

36.1. **Quasigroupoidification.** Let C be a quasicategory, and X a simplicial set. Let

$$\mathrm{Fun}^{(X)}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subquasicategory spanned by objects $f\colon X \to C$ which have the property that $f(X) \subseteq C^{\mathrm{core}}$.

Note that $\mathrm{Fun}^{(X)}(X, C)$ is a quasicategory, but not necessarily a quasigroupoid: for instance, morphisms are functors $f\colon X \times \Delta^1 \to C$ such that $f(X \times \partial\Delta^1) \subseteq C^{\mathrm{core}}$, but need not satisfy $f(X \times \Delta^1) \subseteq C^{\mathrm{core}}$.

36.2. **Lemma.** *Consider* $\Delta^1 \subset N(\mathrm{Iso})$. *The restriction map* $\mathrm{Fun}(N(\mathrm{Iso}), C) \to \mathrm{Fun}(\Delta^1, C)$ *factors through a trivial fibration*

$$\mathrm{Fun}(N(\mathrm{Iso}), C) \to \mathrm{Fun}^{(\Delta^1)}(\Delta^1, C).$$

Proof. We've already proved this is surjective on zero-simplices (31.14). In that proof, we noted that $N(\mathrm{Iso}) = \bigcup F_k$ with $F_k = F_{k-1} \cup_{\Lambda^k_0} \Delta^k$ and $F_1 = \Delta^1$, and the result was immediate using the Joyal extension theorem.

To prove this lemma, it suffices to show that each $\mathrm{Fun}^{(\Delta^1)}(F_k, C) \to \mathrm{Fun}^{(\Delta^1)}(F_{k-1}, C)$ is a trivial fibration, and note that $\mathrm{Fun}^{(\Delta^1)}(F_k, C) = \mathrm{Fun}(F_k, C)$ as long as $k \geq 3$. This amounts to an application of the "box version" of the Joyal extension theorem (28.11), i.e., solving the lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \times \{0\} \arrow[r, hook] \arrow[rr, bend left, "f"] & (\Lambda^k_0 \times \Delta^n) \cup_{\Lambda^k_0 \times \partial\Delta^n} (\Delta^k \times \partial\Delta^n) \arrow[r] \arrow[d, hook] & C \arrow[d] \\
& \Delta^k \times \Delta^n \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

when f represents an isomorphism in C. □

36.3. **Proposition.** *Let* $i\colon X \to X'$ *be any anodyne map to a Kan complex. Then the restriction map* $\mathrm{Fun}(i, C)\colon \mathrm{Fun}(X', C) \to \mathrm{Fun}(X, C)$ *factors through a trivial fibration*

$$q\colon \mathrm{Fun}(X', C) \to \mathrm{Fun}^{(X)}(X, C).$$

Proof. Every edge in X maps to an isomorphism in X'. Therefore $\mathrm{Fun}(i, C)$ must factor through a map q into the subcomplex $\mathrm{Fun}^{(X)}(X, C) \subseteq \mathrm{Fun}(X, C)$. We also know that $\mathrm{Fun}(i, C)$ is an isofibration by (35.6), and therefore the map q is also an isofibration. Therefore by (35.10) it is enough to show that q is a categorical equivalence.

Given any two anodyne maps $i\colon X \to X'$ and $i'\colon X \to X''$ to Kan complexes, there exists a categorical equivalence $f\colon X' \to X''$ such that $fi = i'$ (because $\mathrm{Fun}(X', X'') \to \mathrm{Fun}(X, X'')$ is a trivial fibration by (31.6)). Therefore in the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(X', C) \arrow[rr, "\mathrm{Fun}(f,C)"] \arrow[dr, "q"'] & & \mathrm{Fun}(X'', C) \arrow[dl, "q'"] \\
& \mathrm{Fun}^{(X)}(X, C) &
\end{tikzcd}$$

the map $\mathrm{Fun}(f, C)$ is a categorical equivalence, so q is a categorical equivalence iff q' is. In other words, we only need to prove the result for a single choice of i for each X.

Now consider an arbitrary simplicial set X, and form

$$X \xrightarrow{f} Y := X \cup_{\coprod \Delta^1} \coprod N(\mathrm{Iso}) \xrightarrow{g} X',$$

where the coproducts are taken over all maps $\Delta^1 \to X$, and g is some inner anodyne map to a quasicategory. Since f is anodyne, so is $i = gf$.

I claim that the quasicategory X' is actually a Kan complex, i.e., a quasigroupoid, i.e., a quasicategory such that $h(X')$ is a groupoid. To see this, note that every edge in Y factors through a map $\mathrm{Iso} \to Y$ by construction; from this, we see that every edge in the simplicial set Y must be a preisomorphism, whence hY is a groupoid. Since g is inner anodyne, $hY \to hX'$ is an equivalence, so hX' is a groupoid as desired.

Now $\mathrm{Fun}(i, C)$ is the composite of

$$\mathrm{Fun}(X', C) \xrightarrow{\mathrm{Fun}(g,C)} \mathrm{Fun}(Y, C) \xrightarrow{p} \mathrm{Fun}^{(X)}(X, C) \subseteq \mathrm{Fun}(X, C).$$

Since g is inner anodyne, $\mathrm{Fun}(g, C)$ is a trivial fibration by (35.6). The map p is a pullback of maps $\mathrm{Fun}(N(\mathrm{Iso}), C) \to \mathrm{Fun}^{(\Delta^1)}(\Delta^1, C)$, which are also trivial fibrations by (36.2). The result is proved. □

We write $X \to X_{\mathrm{Kan}}$ for any choice of anodyne map to a Kan complex, and call it a **quasi-groupoidification** of X. The key observation is that for any quasicategory C we get a categorical equivalence

$$\mathrm{Fun}(X_{\mathrm{Kan}}, C) \approx \mathrm{Fun}^{(X)}(X, C).$$

We can apply this construction when X is a quasicategory, or even when X is the nerve of an ordinary category, and obtain interesting new Kan complexes.

36.4. *Example.* It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact the nerve of some poset [Tho80]. Thus, for every Kan complex K, there exists an ordinary category A and a weak equivalence $NA \to K$, and hence a categorical equivalence $(NA)_{\mathrm{Kan}} \to K$ between Kan complexes.

We note that there is also a classical groupoidification construction, which given an ordinary category A produces a groupoid A_{Gpd}. We have that $h(NA_{\mathrm{Kan}}) \approx N(A_{\mathrm{Gpd}})$, but in general $(NA)_{\mathrm{Kan}}$ is not weakly equivalent to $N(A_{\mathrm{Gpd}})$.

36.5. *Exercise.* Let A be the poset of *proper and non-empty* subsets of $\{0, 1, 2, 3\}$. Show that A_{Gpd} is equivalent to the one-object category, but that NA_{Kan} is not equivalent to the one-object category. (In the second case, the idea is that the geometric realization of NA is a 2-sphere. Explicitly, you can prove non-equivalence by showing $\pi_0 \mathrm{Fun}(NA, K(\mathbb{Z}, 2)) \approx \mathbb{Z}$, using the Eilenberg-MacLane object of §7.8.)

36.6. **Localization of quasicategories.** There is a more general construction, which applies to a simplicial set X equipped with a subcomplex $W \subseteq X$. Let

$$\mathrm{Fun}^{(W)}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subcategory spanned by objects $f \colon X \to C$ such that $f(W) \subseteq C^{\mathrm{core}}$. Note that this really only depends on the 1-simplices in W.

36.7. **Proposition.** *Given an inclusion $W \subseteq X$, choose an anodyne map $W \to W_{\mathrm{Kan}}$ to a Kan complex, and then choose an inner anodyne map $g \colon Y := X \cup_W W_{\mathrm{Kan}} \to X'$ to a quasicategory. Then for any quasicategory C, the restriction map $\mathrm{Fun}(X', C) \to \mathrm{Fun}(X, C)$ factors through a trivial fibration*

$$\mathrm{Fun}(X', C) \to \mathrm{Fun}^{(W)}(X, C).$$

Proof. We have

$$\begin{tikzcd} \mathrm{Fun}(X',C) \arrow[r, "\mathrm{Fun}(g,C)"] & \mathrm{Fun}(Y,C) \arrow[r] \arrow[d] & \mathrm{Fun}^{(W)}(X,C) \arrow[r, tail] \arrow[d] & \mathrm{Fun}(X,C) \arrow[d] \\ & \mathrm{Fun}(W_{\mathrm{Kan}},C) \arrow[r, "p"'] & \mathrm{Fun}^{(W)}(W,C) \arrow[r, tail] & \mathrm{Fun}(W,C) \end{tikzcd}$$

in which both squares are pullbacks. The map $\mathrm{Fun}(g,C)$ is a trivial fibration since g is inner anodyne, while p is a trivial fibration as we have shown (36.3). □

We sometimes write $X \to X_{(W)}$ for any map $X \to X'$ obtained as in the proposition. Note that any $X \to X_{(X)}$ is an example of $X \to X_{\mathrm{Kan}}$. The observation is that for any quasicategory C, we have a categorical equivalence

$$\mathrm{Fun}^{(W)}(X,C) \approx \mathrm{Fun}(X_{(W)},C).$$

36.8. **Quasicategories from relative categories.** A **relative category** is a pair $W \subseteq C$ consisting of an *ordinary category* C and a subcategory W containing all the objects of C. The above construction gives, for any relative category, a map

$$C \to C_{(W)},$$

unique up to categorical equivalence.

37. The path fibration

To prove the fundamental theorem of quasicategories for a general map between quasicategories, we will reduce to the special case of isofibrations. We do this by means of the "path fibration" construction, which provides a factorization of a map into a categorical fibration followed by an isofibration.

37.1. **The path fibration for quasicategories.** Let $f\colon C \to D$ be a functor between quasicategories. We define a factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ by means of the following diagram.

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[dr, equal] & P(f) \arrow[r] \arrow[d, "s_0"'] \arrow[rr, bend left, "p"] & \mathrm{Fun}^{(\Delta^1)}(\Delta^1,D) \arrow[r, "r_1"'] \arrow[d, "r_0"] & D \\ & C \arrow[r, "f"'] & D & \end{tikzcd}$$

Here the the square is a pullback square. The map j is the unique one so that $s_0 j = \mathrm{id}_C$, and tj is induced by $C \xrightarrow{f} D \xrightarrow{\widetilde{\pi}} \mathrm{Fun}(\Delta^1,D)$ where $\widetilde{\pi}$ is adjoint to the projection $D \times \Delta^1 \to D$. The maps $r_i\colon \mathrm{Fun}^{(\Delta^1)}(\Delta^1,D) \subseteq \mathrm{Fun}(\Delta^1,D) \to D$ are induced by restriction along $\{i\} \subset \Delta^1$.

In particular, the objects of $P(f)$ are pairs (c,α) consisting of an object $c \in C_0$ and an isomorphism $\alpha\colon f(c) \to d$ in D. The map j sends an object c to $(c, 1_{f(c)})$, while p sends (c,α) to d.

The factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ is called the **path fibration** of f, because of the following.

37.2. **Lemma.** *The map j is a categorical equivalence and p is an isofibration.*

Proof. The composite map

$$\mathrm{Fun}(N(\mathrm{Iso}),D) \xrightarrow{q} \mathrm{Fun}^{(\Delta^1)}(\Delta^1,D) \xrightarrow{r_i} D$$

induced by restiction along $\{i\} \subset N(\mathrm{Iso})$ is a categorical equivalence. By earlier, q is a trivial fibration, and therefore the r_i are categorical equivalences.

The the restriction map $\mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\partial\Delta^1, D) = D \times D$ is an isofibration. We claim that $r\colon \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \to D \times D$ is also an isofibration. It easily seen that r is an inner fibration, since $\mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$ is full. To show that r is also isofibration is then straightforward (e.g., if $\alpha\colon \alpha_0 \to \alpha_1$ is a natural isomorphism of functors $\Delta^1 \to D$, and $\alpha_0(\Delta^1) \subseteq D^{\mathrm{core}}$, then $\alpha_1(\Delta^1) \subseteq D^{\mathrm{core}}$).

Since the projections $D \times D \to D$ are isofibrations, it follows that the r_i are isofibrations and hence trivial fibrations.

Therefore s, being a pullback of r_0, is a trivial fibration. Therefore $P(f)$ is a quasicategory and j is a categorical equivalence.

In the commutative diagram

$$\begin{tikzcd} & P(f) \arrow[r] \arrow[dl, "p"'] \arrow[d, "s"] & \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \arrow[d, "r=(r_0,r_1)"] \\ D & C \times D \arrow[l, "\pi"] \arrow[r, "f\times \mathrm{id}_D"'] & D \times D \end{tikzcd}$$

the square is a pullback. We have shown that r is an isofibration, and hence so is its pullback s. Therefore $p = \pi s$ is an isofibration, as desired. $\square$

37.3. **The path fibration for Kan complexes.** If $f\colon C \to D$ is a functor between quasicategories, and D is a Kan complex, then $\mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) = \mathrm{Fun}(\Delta^1, D)$, and the diagram defining the path fibration takes the form

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[dr, equal] & P(f) \arrow[r] \arrow[d, "s_0"] \arrow[rr, bend left, "p"] & \mathrm{Fun}(\Delta^1, D) \arrow[r, "r_1"'] \arrow[d, "r_0"] & D \\ & C \arrow[r, "f"'] & D & \end{tikzcd}$$

37.4. **Proposition.** *If $f\colon C \to D$ is a functor between Kan complexes, then j is a weak equivalence and p is a Kan fibration.*

Proof. Immediate. $\square$

38. Proof of the fundamental theorem

We are ready to finish the proof of (E), The Fundamental Theorem of Quasicategories.

38.1. **Proposition.** *If $f\colon C \to D$ is a fully faithful and essentially surjective functor between quasicategories, then f is a categorical equivalence.*

We will prove this below. First note that the quasigroupoid version of this result (which we proved earlier) gives us the following.

38.2. **Lemma.** *If $f\colon C \to D$ is a fully faithful and essentially surjective functor between quasicategories, then $f\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a categorical equivalence (in fact, a weak equivalence).*

Proof. Note that if f is fully faithful and essentially surjective, so is f^{core}. Apply (31.19). $\square$

Note that the path fibration construction gives us a factorization of f into $C \xrightarrow{j} P(f) \xrightarrow{p} D$, where j is a categorical equivalence and p is an isofibration. Because both categorical equivalences and (fully faithful + essentially surjective) are classes which satisfy 2-out-of-3 (21.8), (30.5), proving the fundamental theorem reduces to the case that f is itself an isofibration.

To prove the isofibration case of (38.1), we will deduce it from the following.

38.3. **Proposition.** *If $p\colon C \to D$ is an isofibration which is fully fathiful and essentially surjective, then $q = (p^{\square i})^{\mathrm{core}}\colon \operatorname{Fun}(L,C)^{\mathrm{core}} \to \operatorname{Fun}(K,C)^{\mathrm{core}} \times_{\operatorname{Fun}(K,D)^{\mathrm{core}}} \operatorname{Fun}(L,D)^{\mathrm{core}}$ is a trivial fibration for every monomorphism $i\colon K \to L$.*

Proof that (38.3) *implies* (38.1). As noted above, it is enough to consider isofibrations $p\colon C \to D$ which are fully faithful and essentially surjective. By (38.3), for any monomorphism $i\colon K \to L$ the map $q = (p^{\square i})^{\mathrm{core}}$ is a trivial fibration, and therefore surjective on vertices. The core of a quasicategory has all its objects, and thus the box power map $p^{\square i}\colon \operatorname{Fun}(L,C) \to \operatorname{Fun}(K,C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L,D)$ is surjective on vertices. Thuse, we have $i \boxslash p$ for every monomorphism i, whence p is a trivial fibration, and thus a categorical equivalence. $\square$

38.4. **Proof of** (38.3)**.** We start with the following lemma, which says that isofibrations between quasicategories which are trivial fibrations on cores are characterized by a lifting property.

38.5. **Lemma.** *There exists a set of maps S such that for any isofibration $q\colon C \to D$ between quasicategories, we have $S \boxslash q$ iff $q^{\mathrm{core}} \in \mathrm{TFib}$.*

Proof. Given an inclusion $K \subseteq L$ of simplicial sets, we can use two applications of the small object argument (12.10) to construct

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K_{\mathrm{Kan}} \arrow[d] \arrow[dr] & \\ L \arrow[r] & L' \arrow[r] & L_{\mathrm{Kan}} \end{tikzcd}$$

in which the square is pushout, the horizontal maps are anodyne, and the objects K_{Kan} and L_{Kan} are Kan complexes.

We will show that $(K \subseteq L) \boxslash q^{\mathrm{core}}$ iff $(K_{\mathrm{Kan}} \subseteq L_{\mathrm{Kan}}) \boxslash q$. The lemma will follow immediately by taking $S = \{\, (\partial\Delta^n)_{\mathrm{Kan}} \subset (\Delta^n)_{\mathrm{Kan}} \mid n \geq 0 \,\}$.

($\Longrightarrow$) Suppose $(K \subseteq L) \boxslash q^{\mathrm{core}}$. Since K_{Kan} and L_{Kan} are Kan complexes, any maps from them to quasicategories must factor through cores. Thus it suffices to find a lift in the right-hand square of

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K_{\mathrm{Kan}} \arrow[r, equal] \arrow[d] & K_{\mathrm{Kan}} \arrow[r] \arrow[d] & C^{\mathrm{core}} \arrow[d, "q^{\mathrm{core}}"] \\ L \arrow[r] \arrow[urrr, dashed, "s"] & L' \arrow[r] \arrow[urr, dashed, "s'"] & L_{\mathrm{Kan}} \arrow[r] \arrow[ur, dashed, "s''"] & D^{\mathrm{core}} \end{tikzcd}$$

By hypothesis, a lift s exists, and therefore a lift s' since the left-hand square is a pushout. Because $L' \to L_{\mathrm{Kan}}$ is anodyne and q^{core} is a Kan fibration (35.4), an extension to a lift s'' exists.

($\Longleftarrow$) Suppose $(K_{\mathrm{Kan}} \subseteq L_{\mathrm{Kan}}) \boxslash q$. Consider a lifting problem

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d] & C^{\mathrm{core}} \arrow[d, "q^{\mathrm{core}}"] \\ L \arrow[r] \arrow[ur, dashed] & D^{\mathrm{core}} \end{tikzcd}$$

Because C^{core} is a Kan complex and $K \to K_{\mathrm{Kan}}$ is anodyne, the map a factors through some $a'\colon K_{\mathrm{Kan}} \to C^{\mathrm{core}}$, and there is a unique compatible map $b'\colon L' \to D^{\mathrm{core}}$ from the pushout along $K \subset L$. Again, b' factors through $b''\colon L_{\mathrm{Kan}} \to D^{\mathrm{core}}$. Thus we have extended the original square to a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K_{\mathrm{Kan}} \arrow[r, equal] \arrow[d] & K_{\mathrm{Kan}} \arrow[r, "a'"] \arrow[d] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d] & C \arrow[d, "q"] \\ L \arrow[r] & L' \arrow[r] & L_{\mathrm{Kan}} \arrow[r, "b''"'] \arrow[ur, dashed, "t'"] \arrow[urr, dashed, "t"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

A lift t exists by hypothesis, and since L_{Kan} is a Kan complex it factors through a unique lift t' (using that $C^{\mathrm{core}} \to C$ and $D^{\mathrm{core}} \to D$ are monomorphisms). The composite $L \to L_{\mathrm{Kan}} \to C^{\mathrm{core}}$ is the desired lift. $\square$

Fix an isofibration $p\colon C \to D$ between quasicategories which is fully faithful and essentially surjective. Consider the class

$$\mathcal{C}_p := \{\, i \mid i \text{ is a monomorphism and } (p^{\square i})^{\mathrm{core}} \in \mathrm{TFib}\,\}.$$

The statement of (38.3) amounts to showing that $\mathcal{C}_p$ contains every monomorphism.

38.6. **Lemma.** *The class $\mathcal{C}_p$ is saturated.*

Proof. First note that for any monomorphism i, the map $p^{\square i}$ is an isofibration since p is (35.6). Using the set of maps S provided by the previous lemma (38.5), for a monomorphism i we have that $(p^{\square i})^{\mathrm{core}} \subseteq \mathrm{TFib}$ iff $S \boxslash (p^{\square i})$ iff $i \boxslash (p^{\square S})$. Thus $\mathcal{C}_p$ is the intersection of ${}^{\boxslash}(p^{\square S})$ with $\overline{\mathrm{Cell}}$, and so is saturated. $\square$

38.7. **Lemma.** *Let $p\colon C \to D$ be an isofibration between quasicategories. If $K \subseteq L$, and if $(\varnothing \subset K)$ and $(\varnothing \subset L)$ are elements of $\mathcal{C}_p$, then $(K \subseteq L) \in \mathcal{C}_p$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(L,C)^{\mathrm{core}} \arrow[r, "(p^{\square i})^{\mathrm{core}}"'] \arrow[rr, bend left, "\mathrm{Map}(L,p)^{\mathrm{core}}"] & \bigl(\mathrm{Map}(K,C) \times_{\mathrm{Map}(K,D)} \mathrm{Map}(L,D)\bigr)^{\mathrm{core}} \arrow[r, "q"'] \arrow[d] & \mathrm{Map}(L,D)^{\mathrm{core}} \arrow[d] \\
& \mathrm{Map}(K,C)^{\mathrm{core}} \arrow[r, "\mathrm{Map}(K,p)^{\mathrm{core}}"'] & \mathrm{Map}(K,D)^{\mathrm{core}}
\end{tikzcd}$$

where the square is a pullback (35.7). By hypothesis, both $\mathrm{Map}(K,p)^{\mathrm{core}}$ and $\mathrm{Map}(L,p)^{\mathrm{core}}$ are trivial fibrations, whence q is also a trivial fibration. By 2-out-of-3, we have that $(p^{\square i})^{\mathrm{core}}$ is a categorical equivalence. Since p is an isofibration, we have that $(p^{\square i})^{\mathrm{core}}$ is a Kan fibration (35.6), (35.7) between Kan complexes, and therefore is a trivial fibration (32.7) as desired. $\square$

Next we observe that $(\varnothing \subseteq \Delta^n) \in \mathcal{C}_p$ if p is fully faithful and essentially surjective.

38.8. **Proposition.** *If $p\colon C \to D$ is an isofibration which is fully faithful and essentially surjective, then $\mathrm{Fun}(\Delta^n, C)^{\mathrm{core}} \to \mathrm{Fun}(\Delta^n, D)^{\mathrm{core}}$ is a trivial fibration for all $n \geq 0$.*

Proof. In the case $n = 0$, this means showing that $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a trivial fibration, which we have already observed (38.2).

For $n \geq 1$, consider the diagram

$$\begin{tikzcd}
\coprod_{c_0,\dots,c_n} \mathrm{map}_C(c_0,\dots,c_n) \arrow[r, tail] \arrow[d, "\coprod q_{c_0,\dots,c_n}"'] & \mathrm{Fun}(\Delta^n, C)^{\mathrm{core}} \arrow[d, "q"] \arrow[dr, "\mathrm{Fun}(\Delta^n,p)^{\mathrm{core}}"] & \\
\coprod_{c_0,\dots,c_n} \mathrm{map}_D(pc_0,\dots,pc_n) \arrow[r, tail, "j"'] \arrow[d] & (C^{\times n+1})^{\mathrm{core}} \times_{(D^{\times n+1})^{\mathrm{core}}} \mathrm{Fun}(\Delta^n, D)^{\mathrm{core}} \arrow[r, "r"'] \arrow[d] & \mathrm{Fun}(\Delta^n, D)^{\mathrm{core}} \arrow[d] \\
C_0^{\times n+1} \arrow[r, tail] & (C^{\times n+1})^{\mathrm{core}} \arrow[r, "(p^{\times n+1})^{\mathrm{core}}"] & (D^{\times n+1})^{\mathrm{core}}
\end{tikzcd}$$

in which all the squares are pullbacks. The map r is a base change of $(p^{\times n+1})^{\mathrm{core}}\colon (C^{\times n+1})^{\mathrm{core}} \to (D^{\times n+1})^{\mathrm{core}}$, which is isomorphic to the $(n+1)$-fold product of $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$, which we have just noted is a trivial fibration. Thus, r is a trivial fibration, so it will suffice to show that q is a trivial fibration, for which we will use the fiberwise criterion (33.3).

We know that $p^{\square(\mathrm{Sk}_0 \Delta^n \subset \Delta^n)}$ is an isofibration (35.6), and thus $q = (p^{\square(\mathrm{Sk}_0 \Delta^n \subset \Delta^n)})^{\mathrm{core}}$ is a Kan fibration (35.4). Because p is an essentially surjective isofibration, p^{core} is surjective on vertices, and thus j is surjective on 0-simplices. Thus by the fiberwise criterion (33.3) we need to show that $\coprod q_{c_0,\dots,c_n}$ is a trivial fibration. Since coproducts of trivial fibrations are trivial fibrations (19.2) we thus reduce to showing that each $q_{c_0,\dots,c_n}$ (which is a Kan fibration between Kan complexes being a pullback of q) is a weak equivalence, and hence a trivial fibration (32.7). This is immediate from the fact that p is fully faithful and induces maps compatible with the weak equivalences $\mathrm{map}_C(c_0,\dots,c_n) \to \mathrm{map}_C(c_0,c_1) \times \cdots \times \mathrm{map}_C(c_{n-1},c_n)$ and $\mathrm{map}_D(pc_0,\dots,pc_n) \to \mathrm{map}_D(pc_0,pc_1) \times \cdots \mathrm{map}_D(pc_{n-1},pc_n)$. □

Proof of (38.3). Fix $p\colon C \to D$ a fully faithful and essentially surjective isofibration. We want to show that $\overline{\mathrm{Cell}} \subseteq \mathcal{C}_p = \{\, i \mid i \in \overline{\mathrm{Cell}},\ (p^{\square i})^{\mathrm{core}} \in \mathrm{TFib}\,\}$. Since $\mathcal{C}_p$ is saturated (38.6), it suffices to show that $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for all $n \geq 0$. We will do this by induction on n.

For the case $n = 0$, this is immediate from (38.8). For $n \geq 1$, suppose we have $(\partial\Delta^k \subset \Delta^k) \in \mathcal{C}_p$ for $k < n$. Since $\partial\Delta^n$ is itself an $(n-1)$-skeleton, skeletal filtration (14.14) gives that $(\varnothing \subset \partial\Delta^n) \in \mathcal{C}_p$. Then use (38.7) and (38.8) to conclude that $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ as desired. □

Model categories

39. Categorical fibrations

A map $p\colon X \to Y$ of simplicial sets is a **categorical fibration** if and only if $j \boxslash p$ for all j which are monomorphisms and categorical equivalences. I'll write CatFib for the class of categorical fibrations.

Categorical fibrations generalize isofibrations. In fact, a map $p\colon C \to D$ with D a quasicategory is a categorical fibration if and only if it is an isofibration, as we proved in (35.11).

39.1. **Proposition.** *A map $p\colon X \to Y$ of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.*

Proof. ($\Longrightarrow$) Clear. ($\Longleftarrow$) If p is a categorical fibration and a categorical equivalence, factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ with j a monomorphism and q a trivial fibration. Then the usual argument shows that p is a retract of q, using the fact that $j \boxslash p$ since j is a categorical equivalence by 2-of-3. □

39.2. **Proposition.** *If $p\colon X \to Y$ is a categorical fibration and $j\colon K \to L$ is a monomorphism, then*

$$q\colon\ \mathrm{Map}(L,X) \to \mathrm{Map}(K,X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L,Y)$$

is a categorical fibration. Furthermore, if either j or p is also a categorical equivalence, then so is q.

Proof. For the first, let $i\colon A \to B$ be a monomorphism which is a categorical equivalence. We have $i \boxslash q$ iff $(i \square j) \boxslash p$. By definition of categorical fibration, it suffices to show that $i\square j$ is a categorical equivalence, i.e., to show $\mathrm{Map}(i\square j, C)$ is a categorical equivalence for every quasicategory C. In fact, $\mathrm{Map}(i, C)$ is an isofibration and a categorical equivalence, hence a trivial fibration, and therefore $j \square \mathrm{Map}(i, C)$.

If p is also a categorical equivalence, then it is a trivial fibration, and the result follows.

If j is also a categorical equivalence, then for any monomorphism i, we have $i \boxslash q$ iff $(i\square j) \boxslash p$ iff $j \boxslash (p^{\square i})$. But $p^{\square i}$ is a categorical fibration by what we have just proved, so the result holds. □

39.3. **Categorical fibrations and the small object argument.** Clearly, $\mathrm{CatFib} = \big(\overline{\mathrm{Cell}} \cap \mathrm{CatEq}\big)^{\boxslash}$ is a right complement to a class of maps. We would like to know that CatFib is the right complement to a *set* of maps; then we could use the small object argument to factor any map into an injective categorical equivalence followed by a categorical fibration.

Unfortunately, it's apparently not known how to write down an explicit set of maps S so that $S^{\boxslash} = \mathrm{CatFib}$. What is known is that such a set *exists*. In particular, we let S be the set of all injective categorical equivalences $K \to L$ such that the number of simplices in K and L is bounded by some sufficiently large cardinal.

We will define a **detection functor** $F\colon \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], \mathrm{Set})$ on categories of morphisms. This will have the following properties:

- For each map $f\colon X \to Y$, the map $F(f)$ is a monomorphism of sets.
- A map $f\colon X \to Y$ is a categorical equivalence if and only if $F(f)$ is a bijection.
- The functor F commutes with κ-filtered colimits for some regular cardinal κ.
- The functor F takes κ-small simplicial sets to κ-small sets.

We define F as the composite of several intermediate steps.

Step 1: Recall that the small object argument gives a *functorial* way to factor a map f as $f = pi$, with $i \in \overline{S}$ and $p \in S^{\boxslash}$. "Funtorial factorization" means that we get a section of the functor $\mathrm{Fun}([2], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$ defining composition.

We can apply this using $S = \mathrm{InnHorn}$. Thus, given any simplicial set, we *functorially* obtain an inner anodyne map $X \to X_{\mathrm{QCat}}$ to a quasicategory X_{QCat}. As a result, we have a functor $f \mapsto f_{\mathrm{QCat}}\colon \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$, with the property that f is a categorical equivalence if and only if f_{QCat} is, and both source and target of f_{QCat} are quasicategories.

Step 2: Form the path fibration $Q(f)\colon P(f_{\mathrm{QCat}}) \to Y_{\mathrm{QCat}}$ of f_{QCat}. The map $Q(f)$ is thus an isofibration between quasicategories, and is a trivial fibration if and only if f is a categorical equivalence.

Step 3: Write $Q(f)\colon X' \to Y'$. Define $E(f)$ to be the map of sets

$$E(f)\colon \coprod_n \mathrm{Hom}(\Delta^n, X') \to \coprod_n \mathrm{Hom}(\partial\Delta^n, X') \times_{\mathrm{Hom}(\partial\Delta^n, Y')} \mathrm{Hom}(\Delta^n, Y').$$

Thus, f is a categorical equivalence if and only if $E(f)$ is *surjective.*

Step 4: Write $E(f)\colon E_0(f) \to E_1(f)$, and define $F(f)$ by

$$F(f)\colon \operatorname{colim}\big[E_0(f) \times_{E_1(f)} E_0(f) \rightrightarrows E_0(f)\big] \to E_1(f).$$

In other words, $F(f)$ is the map from the image of $E(f)$ to $E_1(f)$. Thus, $F(f)$ is always a monomorphism, and f is a categorical equivalence if and only if $F(f)$ is a bijection.

There exists a regular cardinal κ such that F commutes with κ-filtered colimits, and takes κ-small simplicial sets to κ-small sets. (In fact, we can take $\kappa = \omega^+$, the sucessor to the countable cardinal).

Using the detection functor, we can prove the following key lemma.

39.4. **Lemma.** *Let $f\colon X \subseteq Y$ be an inclusion which is a categorical equivalence. Every κ-small subcomplex $A \subseteq Y$ is contained in a κ-small subcomplex $B \subseteq Y$ with the property that $B \cap X \subseteq B$ is a categorical equivalence.*

Proof. For a subcomplex $A \subseteq Y$ let f_A denote the inclusion $A \cap X \subseteq A$. The collection of all κ-small subcomplexes of Y is κ-filtered. Thus

$$\operatorname{colim}_{\kappa\text{-small } A \subseteq Y} F(f_A) = F(f),$$

which we have assumed is an isomorphism. Thus for any κ-small $A \subseteq$ there must exist a κ-small $A' \supset A$ such that a lift exists in

$$\begin{tikzcd}
F_0(f_A) \arrow[r] \arrow[d, tail] & F_0(f_{A'}) \arrow[d, tail] \\
F_1(f_A) \arrow[r] \arrow[ur, dashed] & F_1(f_{A'})
\end{tikzcd}$$

This is because $F_1(f_A)$ is a κ-small set, so any lift $F_1(f_A) \to F_0(F)$ factors through some stage of the κ-filtered colimit.

We use transfinite induction to obtain a sequence $\{A_i\}$ indexed by $i < \kappa$, where at limit ordinals we take a colimit. Set $B := \operatorname{colim} A_i$. Because κ is regular $|B| < \kappa$, and we have that $F(f_B)$ is an isomorphism by construction. □

Consider the collection of monomorphisms $i\colon A \to B$ such that i is a categorical equivalence and $|B| < \kappa$. Choose a set S of such spanning all isomorphism classes of such maps; this is a set because of the cardinality bound. Clearly $S \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$.

39.5. **Proposition.** *We have* $\overline{S} = \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$.

Proof. [Joy08a, D.2.16]. Given an injective categorical equivalence $X \subseteq Y$, we consider the following poset $\mathcal{P}$. The objects of $\mathcal{P}$ are subobjects $P \subseteq Y$ such that $X \subseteq P$ so that the inclusion $X \to P$ is contained in $\overline{S}$. The morphisms of $\mathcal{P}$ are inclusions $P \to Q$ of subobjects of Y which are contained in $\overline{S}$. Because $\overline{S}$ is saturated, the hypotheses of Zorn's lemma apply to give a maximal element M of $\mathcal{P}$. Since $X \subseteq Y$ is assumed to be a categorical equivalence, 2-out-of-3 gives that $M \subseteq Y$ is a categorical equivalence.

If $M = Y$ we are done, so suppose $M \neq Y$. Then there exists a κ-small $A \subseteq Y$ not contained in M, which by the above lemma can be chosen so that $A \cap M \subseteq A$ is a categorical equivalence, and thus an element of S. The pushout $M \subseteq A \cup M$ of this map is thus in $\overline{S}$ contradicting the maximality of M. □

In particular, we learn that every map can be factored into an injective categorical equivalence followed by a categorical fibration.

40. The Joyal model structure on simplicial sets

40.1. **Model categories.** A **model category** (in the sense of Quillen) is a category $\mathcal{M}$ with three classes of maps: W, Cof, Fib, conventionally called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying the following axioms.

- $\mathcal{M}$ has all small limits and colimits.
- W satisfies the 2-out-of-3 property.
- $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems (12.12).

Conventionally, an object X is **cofibrant** if the map from the initial object is a cofibration, and **fibrant** if the map to the terminal object is a fibration.

40.2. *Remark.* The third axiom implies that Cof, $\mathrm{Cof} \cap \mathrm{W}$, Fib, and $\mathrm{Fib} \cap \mathrm{W}$ are closed under retracts.

40.3. *Exercise.* Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. *Hint:* construct a factorization of f which is itself a retract of a factorization of g[26].

40.4. *Exercise* (Slice model categories). Let $\mathcal{M}$ be a model category, and let X be an object of $\mathcal{M}$. Show that the slice categories $\mathcal{M}_{X/}$ and $\mathcal{M}_{/X}$ admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under $\mathcal{M}_{/X} \to \mathcal{M}$ or $\mathcal{M}_{X/} \to \mathcal{M}$ are weak equivalences, cofibrations, and fibrations in $\mathcal{M}$.

[26] In many formulations of model categories, weak equivalences being closed under retracts is taken as an axiom. The formulation we use is described in Riehl, "A concise definition of a model category", which gives a solution to this exercise.

40.5. **The Joyal model category.**

40.6. **Theorem** (Joyal). *The category of simplicial sets admits a model structure, in which*

- W = *categorical equivalences (*CatEq*),*
- Cof = *monomorphims (*$\overline{\text{Cell}}$*),*
- Fib = *categorical fibrations (*CatFib*).*

Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.

Proof. Categorical equivalences satisfy 2-out-of-3 by (21.8). We have that

- Cof = $\overline{\text{Cell}}$ by definition,
- $\text{Fib} \cap \text{W} = \text{TFib} = \text{Cell}^{\boxslash}$ by (39.1),
- $\text{Cof} \cap \text{W} = \overline{S}$ for some set S (39.5),
- $\text{Fib} = \text{CatFib} = (\text{Cof} \cap \text{W})^{\boxslash} = S^{\boxslash}$ by definition,

so both $(\text{Cof} \cap \text{W}, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap \text{W})$ are weak factorization systems via the small object argument (12.10). Thus, we get a model category.

We have shown (35.11) that the categorical fibrations $p\colon C \to D$ with D a quasicategory are precisely the isofibrations. Applied when $D = *$, this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations. □

40.7. *Remark.* It is a standard fact that a model category structure is uniquely determined by its cofibrations and fibrant objects. Thus, the Joyal model structure is the unique model structure on simplicial sets with Cof = monomorphisms and with fibrant objects the quasicategories.

40.8. **Cartesian model categories.** Recall that the category of simplicial sets is *cartesian closed*. A **cartesian model category** is a model category which is cartesian closed, with the following properties. Suppose $i\colon A \to B$ and $j\colon K \to L$ are cofibrations and $p\colon X \to Y$ is a fibration. Then

- $$i \Box j\colon (A \times L) \cup_{A \times K} (B \times K) \to B \times L$$
  is a cofibration, and is in addition a weak equivalence if either i or j is also a weak equivalence, and
- $$p^{\Box j}\colon \operatorname{Map}(L, X) \to \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$$
  is a fibration, and is in addition a weak equivalence if either j or p is also a weak equivalence.

In fact, we only need to specify *one* of the above two properties, as they imply each other.

40.9. **Proposition.** *The Joyal model structure is cartesian.*

Proof. This is just (39.2). □

41. The Kan-Quillen model structure on simplicial sets

A map $p\colon X \to Y$ is a **groupoidal fibration** if and only if $j \boxslash p$ for all j which are monomorphisms and weak equivalences. I write GpdFib for the class of categorical fibrations.

41.1. **The Kan-Quillen model structure.**

41.2. **Theorem** (Cisinski). *The category of simplicial sets admits a model structure, in which*

- W = *weak equivalences,*
- Cof = *monomorphims,*
- Fib = *groupoidal fibrations.*

Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.

Proof. This goes very much the same way as the Joyal model structure, and I won't spell it out in detail. First build a detecting functor F so that a map f is a weak equivalence iff $F(f)$ is a bijection; this is just as in the categorical equivalence case, except that we make use of functorial replacement $X \mapsto X_{\mathrm{Kan}}$ by Kan complexes, rather than by quasicategories. Using this, we can show that $\overline{\mathrm{Cell}} \cap \mathrm{WkEq} = \overline{S}$ and $\mathrm{GpdFib} = S^{\boxslash}$ for some set S, giving the factorization system $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$.

We know that trivial fibrations are weak equivalences and are certainly groupoidal fibrations. The converse is proved just as in the categorical fibration case (39.1). This gives the other factorization system $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$.

We have already proved that Kan fibrations between Kan complexes have the lifting property of groupoidal fibrations (32.15), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case. $\square$

41.3. **Proposition.** *The Quillen model structure is cartesian.*

Proof. We must show that $p^{\square j}$ is a groupoidal fibration if j is a monomorphsm and p a groupoidal fibration, and also that it is a weak equivalence if either j or p is. This is proved by an argument nearly identical to the proof of (39.2). $\square$

41.4. **Kan fibrations are groupoidal fibrations.** The above model structure was actually first produced by Quillen. In Quillen's formulation, the fibrations were taken to be the *Kan fibrations.* In fact, this is the same model structure, by

41.5. **Proposition** (Quillen)**.** $\mathrm{KFib} = \mathrm{GpdFib}$.

The non-trivial part is to show that $\mathrm{KFib} \subseteq \mathrm{GpdFib}$; note that we already know that a Kan fibration between Kan complexes is a groupoidal fibration by (32.15). This is usually done via an argument (due to Quillen) based on the theory of *minimal fibrations.* See for instance Quillen's original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that KFib is the cosaturation of the class of *Kan fibrations between Kan complexes.* In fact one can show that every Kan fibration is a *base change* of a Kan fibration between Kan complexes, see [KLV12].

42. Model categories and homotopy colimits

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of weak equivalences.

42.1. **Reedy lemma.**

42.2. **Proposition** (Reedy lemma)**.** *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories which takes trivial cofibrations to weak equivalences. Then F takes weak equivalences between cofibrant objects to weak equivalences.*

Proof. Let $f\colon X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. Form the commutative diagram

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dr, "b"] \arrow[drr, "\mathrm{id}_Y"] & & \\
X \arrow[r] \arrow[rr, bend right, "a"'] \arrow[rrr, bend right=40, "f"'] & X \amalg Y \arrow[r] & C \arrow[r, "p"] & Y
\end{tikzcd}$$

where the square is a pushout, and we have chosen a factorization $X \amalg Y \to C \to Y$ into a cofibration followed by a weak equivalence (e.g., a trivial fibration). Because X and Y are cofibrant, the maps

$X \to X \amalg Y \leftarrow Y$ are cofibrations. Using this and the 2-of-3 property, we see that a and b are trivial cofibrations. Applying F gives

$$\begin{tikzcd} & F(Y) \arrow[d, "F(b)"'] \arrow[dr, "\mathrm{id}"] & \\ F(X) \arrow[r, "F(a)"'] & F(C) \arrow[r, "F(p)"'] & F(Y) \end{tikzcd}$$

in which $F(b)$ and $F(a)$ are weak equivalences by hypothesis, whence $F(b)$ is a weak equivalence by 2-of-3, and therefore $F(f) = F(p)F(b)$ is a weak equivalence, as desired. ☐

The opposite of a model categry is also a model category, by switching the roles of fibrations and cofibrations. Thus, there is a dual formulation of the Reedy lemma.

42.3. **Proposition** (Reedy lemma, dual form). *Let $G\colon \mathcal{N} \to \mathcal{M}$ be a functor between model categories which takes trivial fibrations to weak equivalences. Then G takes weak equivalences between fibrant objects to weak equivalences.*

42.4. **Quillen pairs.** Given an adjoint pair of functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$ between model categories, we see from the properties of weak factorization systems that

- F preserves cofibrations if and only if G preserves trivial fibrations, and
- F preserves trivial cofibrations if and only if G preserves fibrations.

If both of these are true, we say that (F, G) is a **Quillen pair**.

Note that if (F, G) is a Quillen pair, then the Reedy lemma (42.2) applies to F, and the dual form of the Reedy lemma (42.3) applies to G.

42.5. **Good colimits.** We can apply the above to certain examples of colimit functors.

42.6. *Exercise.* Let S be a small discrete category (i.e., all maps are identities). Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(S, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence, cofibration, or fibration iff each $\alpha_s\colon X_s \to X'_s$ is one in $\mathcal{M}$.

Thes show that $\mathrm{colim}\colon \mathrm{Fun}(S, \mathcal{M}) \leftrightarrows \mathcal{M} : \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition.

42.7. **Proposition.** *Given a collection $f_s\colon X_s \to X'_s$ of weak equivalences between cofibrant objects in $\mathcal{M}$, the induced map $\amalg f_s\colon \coprod X_s \to \coprod X'_s$ is a weak equivalence.*

42.8. *Exercise.* Let V be the category

$$0 \xleftarrow{10} 1 \xrightarrow{12} 2.$$

Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(V, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if $\alpha(i)\colon X(i) \to X'(i)$ is a weak equivalence for $i = 0, 1, 2$ (i.e., an **objectwise weak equivalence**), and is
- a cofibration if $\alpha(0)$ and $\alpha(1)$ are cofibrations, and the evident map $X(2) \cup_{X(1)} X'(1) \to X'(2)$ is a cofibration.

Then show that $\mathrm{colim}\colon \mathrm{Fun}(V, \mathcal{M}) \leftrightarrows \mathcal{M} : \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition. (Hint: determine what the fibrations in $\mathrm{Fun}(V, \mathcal{M})$ must be according to the lifting property.)

42.9. **Proposition.** *Given a natural transformation $\alpha\colon X \to X'$ of functors $V \to \mathcal{M}$, i.e., a diagram*

$$\begin{tikzcd} X(0) \arrow[d, "\sim"'] & X(1) \arrow[l] \arrow[r, "X(12)"] \arrow[d, "\sim"'] & X(2) \arrow[d, "\sim"'] \\ X'(0) & X'(1) \arrow[l] \arrow[r, "X'(12)"] & X'(2) \end{tikzcd}$$

in which the vertical maps are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(12)$ and $X'(12)$ are cofibrations, the induced map $\operatorname{colim}_V X \to \operatorname{colim}_V X'$ *is a weak equivalence.*

42.10. *Exercise.* Let ω be the category

$$0 \to 1 \to 2 \to \cdots$$

with objects indexed by natural numbers. Show that if $\mathcal{M}$ is a model category, then $\operatorname{Fun}(\omega, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if each $\alpha(i)$ is a weak equivalence,
- a cofibration if each map $X'(i) \cup_{X(i)} X(i+1) \to X'(i+1)$ is a cofibration.

Then show that $\operatorname{colim}\colon \operatorname{Fun}(\omega, \mathcal{M}) \leftrightarrows \mathcal{M} : \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition.

42.11. **Proposition.** *Give a natural transformation $\alpha\colon X \to X'$ of functors $\omega \to \mathcal{M}$ such that all maps $\alpha(i)\colon X(i) \to X'(i)$ are weak equivalences, all objects $X(i)$ and $X(i')$ are cofibrant, and the maps $X(i) \to X(i+1)$ and $X'(i) \to X'(i+1)$ are cofibrations, the induced map* $\operatorname{colim}_\omega X \to \operatorname{colim}_\omega X'$ *is a weak equivalence.*

In the Joyal and Quillen model structures, all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (42.7), (42.9), (42.11) a **good colimit**. Thus, we see that good colimits are homotopy invariant. These "good colimits" are examples of what are called *homotopy colimits.*

Since he opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that **good limits** are homotopy invariant.

42.12. *Exercise.* State and prove the dual versions of all the results in this section.

43. A simplicial set is weakly equivalent to its opposite

Recall the opposite functor $\mathrm{op}\colon X \to X$. We will now prove the following, which is a kind of a generalization of the fact that every groupoid is isomorphic to its own opposite.

43.1. **Proposition.** *Every simplicial set X is weakly equivalent to its opposite, in the sense that there exists a zig-zag of weak equivalences connecting X and X^{op}.*

43.2. *Remark.* Here is one possible proof (in some sense, the most natural proof). Note that there is a *homeomorphism* of geometric realizations $|X| \approx |X^{\mathrm{op}}|$. Then use the fact that geometric realization induces an equivalence $h(s\mathrm{Set}, \mathrm{WkEq}) \approx h(\mathrm{Top}, \mathrm{WkEq})$. Of course, we haven't actually proved this fact about homotopy categories yet.

I'll give some different proofs which are internal to simplicial sets. I do this in part because it's interesting to see how this is done, but also because it allows me to set up some technology which will be useful later.

43.3. **General "singular" and "realization" functors.** Consider a functor $C\colon \Delta \to \mathcal{A}$ into some category $\mathcal{A}$, i.e., a **cosimplicial object** in $\mathcal{A}$. Often we write $C^n \in \mathcal{A}$ instead of $C([n])$ for the values of this functor, and so write $C^\bullet$ for the whole functor. In most of our examples, we will actually have $\mathcal{A} = s\mathrm{Set}$, so $C^\bullet$ will be a cosimplicial simplicial set.

Given such a $C^\bullet$, **realization** and **singular** functors are an adjoint pair

$$\mathrm{Re} = \mathrm{Re}_{C^\bullet} : s\mathrm{Set} \rightleftarrows \mathcal{A} : \mathrm{Si}_{C^\bullet} = \mathrm{Si}$$

associated to $C^\bullet$. The singular functor Si is always defined, and is defined by

$$(\mathrm{Si}\, X)_n := \operatorname{Hom}_{s\mathrm{Set}}(C([n]), X),$$

with simplicial operators induced by the fact that C is a functor on Δ. The left adjoint Re is defined if $\mathcal{A}$ is cocomplete[27].

43.4. *Exercise.* Show that if $\mathcal{A}$ is cocomplete, and $X \in s\mathrm{Set}$, then

$$\operatorname{Re} X \approx \operatorname{Cok}\left[\coprod_{f\colon [m]\to[n]} \coprod_{x\in X_n} C^m \rightrightarrows \coprod_{[p]} \coprod_{x\in X_p} C^p\right].$$

(Part of the exercise is to identity the two maps.)

43.5. *Example.* For the cosimplicial space $\Delta^\bullet_{\mathrm{top}}\colon \Delta \to \mathrm{Top}$ taking $[n]$ to the topological n-simplex, $\mathrm{Re}_{\Delta^\bullet_{\mathrm{top}}}$ and $\mathrm{Si}_{\Delta^\bullet_{\mathrm{top}}}$ are just the usual geometric realization and singular complex functors.

43.6. *Example.* For the tautological functor $\Delta^\bullet\colon \Delta \to s\mathrm{Set}$ sending $[n] \mapsto \Delta^n$, the functors $\mathrm{Re}_{\Delta^\bullet}$ and $\mathrm{Si}_{\Delta^\bullet}$ are isomorphic to the identity functor on simplicial sets.

43.7. *Example.* For the functor $(\Delta^\bullet)^{\mathrm{op}} = \Delta^\bullet \circ \mathrm{op}\colon \Delta \to s\mathrm{Set}$, the functors $\mathrm{Re}_{\Delta^\bullet\circ\mathrm{op}}$ and $\mathrm{Si}_{\Delta^\bullet\circ\mathrm{op}}$ are both isomorphic to the *opposite* functor $X \mapsto X^{\mathrm{op}}$.

43.8. **The cosimplicial object $\mathrm{Iso}^\bullet$.** Consider $\mathrm{Iso}^\bullet\colon \Delta \to s\mathrm{Set}$, where Iso^n is the (nerve of the) groupoid with objects $\{0, 1, \dots, n\}$ and unique isomophisms between every object. This has corresponding realization and singular functors

$$\mathrm{Re} = \mathrm{Re}_{\mathrm{Iso}^\bullet}\colon s\mathrm{Set} \to s\mathrm{Set} : \mathrm{Si}_{\mathrm{Iso}^\bullet} = \mathrm{Si}\,.$$

There is an evident natural transformation $\alpha\colon \Delta^\bullet \to \mathrm{Iso}^\bullet$, by the unique functors $[k] \to \mathrm{Iso}^k$ sending object j to j. This gives an adjoint-related pair of natural transformations

$$\eta_X\colon X \to \mathrm{Re}_{\mathrm{Iso}^\bullet} X, \qquad \epsilon_X\colon \mathrm{Si}_{\mathrm{Iso}^\bullet} X \to X.$$

If we precompose with op, we obtain a cosimplicial object $\mathrm{Iso}^\bullet \circ \mathrm{op} = (\mathrm{Iso}^\bullet)^{\mathrm{op}}$. One sees that there are natural isomorphisms

$$\mathrm{Re}_{\mathrm{Iso}^\bullet\circ\mathrm{op}} \approx (\mathrm{Re}_{\mathrm{Iso}^\bullet})^{\mathrm{op}}, \qquad \mathrm{Si}_{\mathrm{Iso}^\bullet\circ\mathrm{op}} \approx (\mathrm{Si}_{\mathrm{Iso}^\bullet})^{\mathrm{op}}, \tag{43.9}$$

and that the evident natural transformation $\alpha\circ\mathrm{op}\colon \Delta^\bullet\circ\mathrm{op} \to \mathrm{Iso}^\bullet\circ\mathrm{op}$ gives rise to an adjoint-related pair of natural transformations

$$\eta'_X\colon X^{\mathrm{op}} \to \mathrm{Re}_{\mathrm{Iso}^\bullet\circ\mathrm{op}} X, \qquad \epsilon'_X\colon \mathrm{Si}_{\mathrm{Iso}^\bullet\circ\mathrm{op}} X \to X^{\mathrm{op}}.$$

Furthermore, under the isomorphisms of (43.9) the maps η'_X and ϵ'_X are identified with $\eta_{X^{\mathrm{op}}}$ and $\epsilon_{X^{\mathrm{op}}}$.

43.10. *Exercise.* Prove the statements of the previous paragraph.

Although $\Delta^\bullet$ and $\Delta^\bullet \circ \mathrm{op}$ are not isomorphic as functors $\Delta \to s\mathrm{Set}$, it is the case that $\mathrm{Iso}^\bullet \approx \mathrm{Iso}^\bullet \circ \mathrm{op}$, using the isomorphism of categories $\mathrm{Iso}^n \to \mathrm{Iso}^n$ given on objects by $x \mapsto n - x$. Putting all this together, we obtain natural transformations

$$X \xrightarrow{\eta_X} \mathrm{Re}_{\mathrm{Iso}^\bullet} X \approx \mathrm{Re}_{\mathrm{Iso}^\bullet} X^{\mathrm{op}} \xleftarrow{\eta_{X^{\mathrm{op}}}} X^{\mathrm{op}}, \qquad X \xleftarrow{\epsilon_X} \mathrm{Si}_{\mathrm{Iso}^\bullet} X \approx \mathrm{Si}_{\mathrm{Iso}^\bullet} X^{\mathrm{op}} \xrightarrow{\epsilon_{X^{\mathrm{op}}}} X^{\mathrm{op}}.$$

We'll show that that η_X, and hence $\eta_{X^{\mathrm{op}}}$, are always weak equivalences, and that ϵ_X, and hence $\epsilon_{X^{\mathrm{op}}}$, are weak equivalences whenever X is a Kan complex. In the following, $\mathrm{Re} = \mathrm{Re}_{\mathrm{Iso}^\bullet}$ and $\mathrm{Si} = \mathrm{Si}_{\mathrm{Iso}^\bullet}$.

43.11. **Lemma.** *For each monomorphism $K \to L$, the induced map $(\operatorname{Re} K) \amalg_K L \to \operatorname{Re} L$ is a monomorphism. In particular,*

- Re *preserves monomorphisms and* Si *preserves trivial fibrations, and*
- $\eta_L\colon L \to \operatorname{Re} L$ *is always a monomorphism.*

[27]This means that $\mathcal{A}$ has colimits for all functors $F\colon B \to \mathcal{A}$ from small categories B.

Proof. Formally, it is enough to check the case of $\partial\Delta^n \subset \Delta^n$. To see this, check that the lifting problems

$$\begin{tikzcd}
(\operatorname{Re} K) \cup_K L \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Re} L \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\iff
\begin{tikzcd}
K \arrow[r] \arrow[d, hook] & \operatorname{Si} X \arrow[d] \\
L \arrow[r] \arrow[ur, dashed] & (\operatorname{Si} Y) \times_Y X
\end{tikzcd}$$

are equivalent. This means we need to show that all monomorphisms are contained in the saturated class ${}^{\boxslash}\mathcal{C}$, where $\mathcal{C}$ is the class of all the maps $(\operatorname{Si} p, \epsilon_X)\colon \operatorname{Si} X \to (\operatorname{Si} Y) \times_Y X$ such that $p \in \mathrm{TFib}$, which means we only need to show that Cell is contained in it.

This is a calculation: $\operatorname{Re}(\partial\Delta^n) \to \operatorname{Re}(\Delta^n) = N(\mathrm{Iso}^n)$ is isomorphic to inclusion of the subcomplex $K \subseteq N(\mathrm{Iso}^n)$ whose k-simplices are sequences $x_0 \to \cdots \to x_k$ such that $\{x_0, \dots, x_k\} \neq \{0, \dots, n\}$. To show this, use the fact that Δ^n is a colimit of its $(n-1)$-dimensional faces along their intersections, and that Re preserves colimits. The image of the simplex $(0, 1, \dots, n)$ in $\operatorname{Re}(\Delta^n)$ intersects K exactly in its boundary, so $(\operatorname{Re} \partial\Delta^n) \cup_{\partial\Delta^n} \Delta^n \to \operatorname{Re} \Delta^n$ is a monomorphism as desired. □

43.12. *Remark.* Given any natural transformation $\lambda\colon F \to G$ of functors, and map $f\colon X \to Y$, we get induced maps

$$F(Y) \cup_{F(X)} G(X) \to G(Y), \qquad F(X) \to F(Y) \times_{G(Y)} G(X).$$

These can be thought of as a variant of the "box" construction we've considered elsewhere (25.5), but associated to the "evaluation pairing" $\mathrm{Fun}(s\mathrm{Set}, s\mathrm{Set}) \times s\mathrm{Set} \to s\mathrm{Set}$ rather than a functor $s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$.

43.13. **Skeletal induction.** The next step is to show that $K \to \operatorname{Re} K$ is a weak equivalence for every simplicial set K. To do this, we will use the following bit of machinery.

43.14. **Proposition** (Skeletal induction)**.** *Let $\mathcal{C}$ be a class of simplicial sets with the following properties.*

(1) *If $X \in \mathcal{C}$, then every object isomorphic to X is in $\mathcal{C}$.*
(2) *Every $\Delta^n \in \mathcal{C}$.*
(3) *The class $\mathcal{C}$ is closed under* good colimits. *That is:*
  (a) *any coproduct of objects of $\mathcal{C}$ is in $\mathcal{C}$;*
  (b) *any pushout of a diagram $X_0 \leftarrow X_1 \to X_2$ of objects in $\mathcal{C}$ along a monomorphism $X_1 \to X_2$ is in $\mathcal{C}$;*
  (c) *any colimit of a countable sequence $X_0 \to X_1 \to X_2 \to \cdots$ of objects in $\mathcal{C}$, such that each $X_k \to X_{k+1}$ is a monomorphism, is in $\mathcal{C}$.*

Then $\mathcal{C}$ is the class of all simplicial sets.

Proof. This is a straightforward consequence of the skeletal filtration (14.14). To show $X \in \mathcal{C}$, it suffices to show each $\operatorname{Sk}_n X \in \mathcal{C}$ by (3c). So we show that all n-skeleta are in $\mathcal{C}$ by induction on n, with the case $n = -1$ (the empty simplicial set), which is really a special case of (3a). Since $\operatorname{Sk}_{n-1} X \subseteq \operatorname{Sk}_n X$ is a pushout along a coproduct of maps $\partial\Delta^n = \operatorname{Sk}_{n-1} \Delta^n \to \Delta^n$, this follows using (2), (3a), (3b), and the inductive hypothesis. □

43.15. **Proposition.** *For every simplical set X, the map $X \to \operatorname{Re} X$ is a weak equivalence.*

Proof. Let $\mathcal{C}$ be the class of X such that $\eta\colon X \to \operatorname{Re} X$ is a weak equivalence. We verify the hypotheses of the above proposition. Property (1) is obvious.

To prove property (2) recall that $\eta_{\Delta^1}\colon \Delta^1 \to \mathrm{Iso}^1$ is anodyne (31.13). We can identify η_{Δ^n} as a retract of $(\eta_{\Delta^1})^{\times n}\colon (\Delta^1)^{\times n} \to (\mathrm{Iso}^1)^{\times n}$, which is necessarily anodyne since a product of an anodyne map with any identity map is anodyne (similar to (20.9)). The retractions $\Delta^n \xrightarrow{f} (\Delta^1)^{\times n} \xrightarrow{g} \Delta^n$

and $\mathrm{Iso}^n \xrightarrow{f} (\mathrm{Iso}^1)^{\times n} \xrightarrow{g} \mathrm{Iso}^n$ are maps which are given on vertices by

$$f(k) = (\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{n-k}), \qquad g(k_1,\dots,k_n) = \max\{\, j \mid k_j = 1\,\}.$$

Property (3) involves colimits, which in every case are *good colimits*. In each case, we need to show that a map $\operatorname{colim} X_i \to \operatorname{Re}\operatorname{colim} X_i$ is a weak equivalence when each $X_i \to \operatorname{Re} X_i$ is. The functor Re preserves colimits and monomorphisms (43.11), so in every case we are comparing good colimits, so the result follows from (42.7), (42.9), (42.11). □

We thus obtain the desired result.

43.16. **Corollary.** *Every simplicial set is weakly equivalent to its opposite X^{op}.*

Proof. Both maps in $X \xrightarrow{\eta_X} \operatorname{Re} X \approx \operatorname{Re} X^{\mathrm{op}} \xleftarrow{\eta_{X^{\mathrm{op}}}} X^{\mathrm{op}}$ are weak equivalences (43.15). □

43.17. **Proposition.** *For each monomorphism $K \to L$, the induced map $(\operatorname{Re} K) \amalg_K L \to \operatorname{Re} L$ is a monomorphism and a weak equivalence.*

Proof. Both squares

$$\begin{tikzcd} K \arrow[r, "\eta"] \arrow[d] & \operatorname{Re} K \arrow[d] \\ L \arrow[r, "\eta"'] & (\operatorname{Re} K) \amalg_K L \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Re} K \arrow[r, "\mathrm{id}"] \arrow[d] & \operatorname{Re} K \arrow[d] \\ \operatorname{Re} L \arrow[r, "\mathrm{id}"] & \operatorname{Re} L \end{tikzcd}$$

are good pushouts, using (43.11). The evident map from the left square to the right square is a weak equivalence at the upper left, upper right, and lower left corners (43.15), so the result follows from the invariance of good pushouts (42.9). □

43.18. **Corollary.** *If $p\colon X \to Y$ is a Kan fibration, then $\operatorname{Si} X \to \operatorname{Si} Y \times_Y X$ is a trivial fibration. In particular, if X is a Kan complex, then $\operatorname{Si} X \to X$ is a trivial fibration.*

In particular, for any Kan complex X, both maps in $X \xleftarrow{\epsilon_X} \operatorname{Si} X \approx \operatorname{Si} X^{\mathrm{op}} \xrightarrow{\epsilon_{X^{\mathrm{op}}}} X^{\mathrm{op}}$ are trivial fibrations.

Proof. Straightforward, using (43.17). □

43.19. *Remark.* The object $\operatorname{Re} X$ is not generally categorically equivalent to X.

It can be shown that if C is a quasicategory, then $\operatorname{Si} C$ is categorically equivalent to C^{core}.

44. Initial objects, revisited

Recall the definition of an initial object in a quasicategory. One characterization was: x is an initial object of C iff the left fibration $p\colon C_{x/} \to C$ is a trivial fibration.

When C is the nerve of an ordinary category, this reduces to the usual definition of initial object. In this case, there is an equivalent characterization: x is initial if and only if $\operatorname{Hom}_C(x,y)$ is a singleton set for all objects y of C.

We would like to generalize this to the case of quasicategories.

F. **Deferred Proposition.** *An object x of a quasicategory is initial if and only if $\operatorname{map}_C(x,c)$ is a contractible Kan complex for all objects c of C.*

This is not straightforward to prove, because one has to relate the mapping space to the join/slice construction used for talking about initial objects.

44.1. **Right and left mapping spaces.** Let x, y be objects of a quasicategory C. We define the **right mapping space** $\operatorname{map}^R_C(x, y)$ and **left mapping space** $\operatorname{map}^L_C(x, y)$ by pullback diagrams

$$\begin{tikzcd} \operatorname{map}^R_C(x,y) \arrow[r] \arrow[d] & C_{x/} \arrow[d, "\pi"'] \\ \Delta^0 \arrow[r, "y"'] & C \end{tikzcd} \qquad \begin{tikzcd} \operatorname{map}^L_C(x,y) \arrow[r] \arrow[d] & C_{/y} \arrow[d, "\pi"'] \\ \Delta^0 \arrow[r, "x"'] & C \end{tikzcd}$$

where the maps labelled π are the evident forgetful maps.

For instance, an n-simplex of $\operatorname{map}^R_C(x, y)$ is precisely a map $a\colon \Delta^{n+1} \to C$ such that $a|\Delta^{\{0,\dots,n\}}$ is the constant simplex at x, and $a|\Delta^{\{n+1\}} = y$. In particular, a 0-simplex in $\operatorname{map}^R_C(x, y)$ is a morphism $x \to y$ in C, while a 1-simplex in $\operatorname{map}^R_C(x, y)$ is a 2-simplex in C exhibiting the $\sim_r$ relation between two maps, which we used to define the homotopy category in §8.

We have shown (25.13) that when C is a quasicategory, the maps $C_{x/} \to C$ and $C_{/y} \to C$ are left fibrations and right fibrations respectively. Thus both $\operatorname{map}^R_C(x, y)$ and $\operatorname{map}^L_C(x, y)$ are Kan complexes (using the Joyal extension theorem.) Furthermore, by the above remarks we know that $\pi_0 \operatorname{map}^R_C(x, y) \approx \pi_0 \operatorname{map}^L_C(x, y) \approx \operatorname{Hom}_{hC}(x, y)$.

We will show below that both $\operatorname{map}^R_C(x, y)$ and $\operatorname{map}^L_C(x, y)$ are weakly equivalent to the standard mapping space $\operatorname{map}_C(x, y)$.

44.2. **Box products and right and left anodyne maps.** Recall that $\overline{\overline{\mathrm{InnHorn}}\Box\overline{\mathrm{Cell}}} \subseteq \overline{\mathrm{InnHorn}}$ (15.7) and $\overline{\overline{\mathrm{Horn}}\Box\overline{\mathrm{Cell}}} \subseteq \overline{\mathrm{Horn}}$. We have an analogous fact for left or right anodyne maps.

44.3. **Proposition.** *We have that* $\overline{\overline{\mathrm{LHorn}}\Box\overline{\mathrm{Cell}}} \subseteq \overline{\mathrm{LHorn}}$ *and* $\overline{\overline{\mathrm{RHorn}}\Box\overline{\mathrm{Cell}}} \subseteq \overline{\mathrm{RHorn}}$.

Proof. See .... □

44.4. **Fiberwise criterion for trivial fibrations, revisited.** Recall the fiberwise criterion for trivial fibrations (33.2): a Kan fibration p is a trivial fibration if and only if the fibers of p are contractible Kan complexes. In fact, this still holds if we only know p is a left or right fibration.

44.5. **Proposition.** *Suppose $p\colon X \to Y$ is a right fibration or left fibration of simplicial sets. Then p is a trivial fibration if and only if it has contractible fibers.*

Proof. [Lur09, 2.1.3.4]. Let's consider the case of $p\colon X \to Y$ a left fibration. The direction ($\Longrightarrow$) is immediate, so we only need to prove ($\Longleftarrow$) .

We attempt to carry out the argument of the proof of (33.2), and show that $(\partial\Delta^n \subset \Delta^n) \boxslash p$ for all $n \geq 0$. The case of $n = 0$ is immediate, since the fibers of p must be non-empty, since they are contractible, so we can assume $n \geq 1$.

Examining that proof of (33.2), we see that we used only the hypothesis that p is a Kan fibration in order to have that

$$(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1) \boxslash p, \qquad \big((\partial\Delta^n \subset \Delta^n)\Box(\{1\} \subset \Delta^1)\big) \boxslash p.$$

In the first case, the inclusion $(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1)$ is left anodyne by (44.3), so this remains true if p is a left fibration.

In the second case, we need to argue a little differently. In the proof of (33.2) this fact is used to produce a lifting (for $n \geq 1$) in a diagram of the form

$$\begin{tikzcd} (\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) \arrow[r, "{(c,jd)}"] \arrow[d, tail] & X \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

Pulling back along the factorization of the bottom map $b\gamma$, we obtain a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n\times\{1\}} (\Delta^n \times \{1\}) \arrow[r] \arrow[d, tail] & C \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "\gamma"'] \arrow[ur, dashed, "s'"'] & \Delta^n \arrow[r, "b"'] & Y
\end{tikzcd}$$

where the right-hand square is a pullback. Observe that (i) p' is a left fibration, and hence an inner fibration, between quasicategories, and that (ii) the map γ (as defined in the proof of (33.2)) sends the edge $\{n\} \times \Delta^{\{n-1,n\}}$ to the vertex n in Δ^n. Therefore we can apply the box-version of Joyal lifting (28.11) (or rather, the dual version of it using a right-horn instead of a left horn) to produce a lift s'. □

44.6. **Corollary.** *An object x of a quasicategory C is initial if and only if $\operatorname{map}^R_C(x,c)$ is contractible for all objects c of C, and is final if and only if $\operatorname{map}^L_C(c,x)$ is contractible for all objects c of C.*

Proof. The fibers of the left fibration $C_{x/} \to C$ are precisely the right mapping spaces $\operatorname{map}^R_C(x,c)$. By what we just proved (44.5) these fibers are all contractible if and only if $C_{x/} \to C$ is a trivial fibration, which we have noted (24.4) is equivalent to x being initial in C. □

Next, we need to show that the right and left mapping spaces are weakly equivalent to the standard mapping space.

45. The alternate join and slice

We want to compare the right and left mapping spaces, which are fibers of the projections $C_{x/} \to C$ and $C_{/x} \to C$, to the ordinary mapping spaces, which are fibers of $\operatorname{Fun}(\Delta^1, C) \to \operatorname{Fun}(\partial\Delta^1, C)$. We do this using constructions called the "alternate join" and "alternate slice" [Lur09, §4.2.1].

Given an object x in C, consider the map

$$q\colon \operatorname{Fun}(\Delta^1, C) \times_{\operatorname{Fun}(\{0\},C)} \{x\} \to \operatorname{Fun}(\{1\}, C) = C.$$

Note that the fiber of q over some object c of C is precisely $\operatorname{map}_C(x,c)$.

When C is actually the nerve of an ordinary category, the domain of q is isomorphic to the usual slice $C_{x/}$, and q is isomorphic to the usual map $p\colon C_{x/} \to C$.

For a general quasicategory, q is not at all isomorphic p. What is true is that there is a commutative diagram

$$\begin{tikzcd}
C_{x/} \arrow[r, "f"] \arrow[dr, "p"'] & \operatorname{Fun}(\Delta^1, C) \times_{\operatorname{Fun}(\{0\},C)} \{x\} \arrow[d, "q"] \\
& C
\end{tikzcd}$$

The map f sends a simplex $a\colon \Delta^k \to C_{/x}$, which corresponds to $\widetilde{a}\colon \Delta^{k+1} \to C$ such that $\widetilde{a}_0 = x$, to the map

$$\Delta^k \times \Delta^1 \xrightarrow{r} \Delta^{k+1} \xrightarrow{\widetilde{a}} C,$$

where r is the unique map given on vertices by $r(i,0) = 0$, $r(i,1) = i+1$.

The characterization of initial objects in terms of contractible mapping spaces (F) thus amounts to the claim that p is a trivial fibration if and only if q has contractible fibers. In fact, we'll prove that

- both p and q are left fibrations,
- f is a categorical equivalence.

Because p and q are left fibrations, they are trivial fibrations iff their fibers are contractible (44.5). Because f is a categorical equivalence p is a categorical equivalence if and only if q is. The result follows because p and q are in particular isofibrations (27.10), and an isofibration is a trivial fibration if and only if it is a categorical equivalence (35.10).

In other words, we can regard q as an alternate version of the slice construction. In fact, we will use the (unmemorable) notation $C^{x/}$ for the domain of q, and call it the "alternate slice".

45.1. **The alternate join.** Given simplicial sets X and Y, define the **alternate join** by the pushout diagram

$$\begin{tikzcd} (X \times \{0\} \times Y) \amalg (X \times \{1\} \times Y) \arrow[r, tail] \arrow[d] & X \times \Delta^1 \times Y \arrow[d] \\ (X \times \{0\} \times \Delta^0) \amalg (\Delta^0 \times \{1\} \times Y) \arrow[r, tail] & X \diamond Y \end{tikzcd}$$

The alternate join comes with a natural comparison map

$$X \diamond Y \to X \star Y,$$

which is the unique map compatible with the evident projections to Δ^1 which restricts to the identity maps of X and Y over the endpoints of Δ^1.

45.2. *Example.* We have

$$X \diamond \Delta^0 \approx (X \times \Delta^1)/(X \times \{1\}), \qquad \Delta^0 \diamond Y \approx (\Delta^1 \times Y)/(\{0\} \times Y).$$

These come with evident maps $X \diamond \Delta^0 \to X^{\triangleright}$ and $\Delta^0 \diamond Y \to Y^{\triangleleft}$.

Like the true join, $X \diamond \varnothing \approx X \approx \varnothing \diamond X$, and the functors $X \diamond -: s\mathrm{Set} \to s\mathrm{Set}_{X/}$ and $- \diamond Y: s\mathrm{Set} \to s\mathrm{Set}_{Y/}$ commute with colimits.

Unlike the true join, the alternate join is not monoidal. Also, the alternate join of two quasicategories is not usually a quasicategory.

45.3. **Proposition.** *The alternate join $\diamond$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \diamond Y \to X \diamond Y'$ and $Y \diamond Z \to Y' \diamond Z$.*

Proof. The $\diamond$ product is constructed using a "good" pushout, i.e., a pushout along a cofibration. The result follows because both products and good pushouts preserve categorical equivalences (42.9). □

45.4. **Equivalence of join and alternate join.** The key result of this section is the following.

45.5. **Proposition.** *The canonical map $X \diamond Y \to X \star Y$ is a categorical equivalence for all simplicial sets X and Y.*

Note that, in view of the previous proposition, this implies that the ordinary join $\star$ also preserves categorical equivalences in each variable.

The strategy for the proof is based on the following idea.

45.6. **Proposition.** *Let $\alpha: F \to F'$ be a natural transformation between functors $s\mathrm{Set} \to \mathcal{M}$, where $\mathcal{M}$ is some model category. If F and F'*

- *commute with colimits,*
- *take monomorphisms to cofibrations,*
- *take categorical equivalences to weak equivalences in $\mathcal{M}$, and*
- *if $\alpha(\Delta^1): F(\Delta^1) \to F'(\Delta^1)$ is a weak equivalence in $\mathcal{M}$,*

then $\alpha(X): F(X) \to F'(X)$ is a weak equivalence in $\mathcal{M}$ for all simplicial sets X.

Proof. [Lur09, 4.2.1.2] Consider the class of simplicial sets $\mathcal{C} := \{\, X \mid \alpha(X) \text{ is a weak equivalence}\,\}$. We use skeletal induction (43.14) to show that $\mathcal{C}$ contains all simplicial sets.

It is clear that $\mathcal{C}$ is closed under isomorphic objects, and in fact is closed under categorically equivalent objects: if $X \in \mathcal{C}$ and X is categorically equivalent to Y, then $Y \in \mathcal{C}$. Because F and F' preserve colimits and cofibrations, they take good colimit diagrams in $s\mathrm{Set}$ to good colimit diagrams in $\mathcal{M}$. Since good colimits are weak equivalence invariant (42.7), (42.9), (42.11), we see that $\mathcal{C}$ is closed under forming good colimits. It remains to show that $\Delta^n \in \mathcal{C}$ for all n.

By hypothesis $\Delta^1 \in \mathcal{C}$. Since Δ^0 is a retract of Δ^1, it follows that $\Delta^0 \in \mathcal{C}$. Consider the inclusions $I^n = \bigcup \Delta^{\{k-1,k\}} \subseteq \Delta^n$. Since $\mathcal{C}$ is closed under good colimits and contains the 0-simplex and 1-simplex, we have $I^n \in \mathcal{C}$. Because F and F' take categorical equivalences to weak equivalences, and $I^n \to \Delta^n$ is a categorical equivalence, it follows that $\Delta^n \in \mathcal{C}$. $\square$

We will apply this idea to functors $s\mathrm{Set} \to s\mathrm{Set}_{X/}$, where $s\mathrm{Set}$ is given the Joyal model structure, and the slice category $s\mathrm{Set}_{X/}$ inherits its model structure from the Joyal model structure on $s\mathrm{Set}$ (40.4).

Proof of (45.5). The functors $X\diamond(-), X\star(-), (-)\diamond X, (-)\star X \colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$ satisfy the first three properties required of the functors in the previous proposition (45.6), where the model structure on $s\mathrm{Set}_{X/}$ is inherited from the Joyal model structure. That is, they preserve colimits, they take monomorphisms to monomorphisms, and take categorical equivalences to categorical equivalences (45.3).

Thus, to show $X \diamond Y \to X \star Y$ is a categorical equivalence for a fixed X and arbitrary Y, it suffices by the previous proposition to show that $X \diamond \Delta^1 \to X \star \Delta^1$ is a categorical equivalence. The same argument lets us reduce to the case when $X = \Delta^1$, i.e., to showing that a single map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ is a categorical equivalence.

We will show $\overline{f}$ is a categorical equivalence by producing a map $\overline{g} \colon \Delta^1 \star \Delta^1 \to \Delta^1 \diamond \Delta^1$ such that $\overline{f}\overline{g} = \mathrm{id}_{\Delta^1 \star \Delta^1}$ and $\overline{g}\overline{f}$ is preisomorphic to the identity map of $\Delta^1 \diamond \Delta^1$, via (19.8) .

Since $\Delta^1 \diamond \Delta^1$ is a quotient of a cube, we start with maps involving the cube. Let

$$f \colon (\Delta^1)^{\times 3} \to \Delta^3 = \Delta^1 \star \Delta^1$$

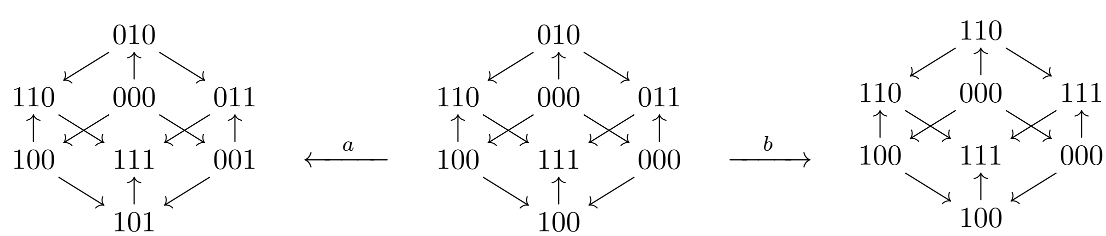
be the map which on vertices sends

$$(a_1 a_2 a_3) \mapsto \sup\{\, k \mid a_k = 1 \,\}.$$

On passage to quotients this gives the comparison map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$.

Let $g \colon \Delta^3 \to (\Delta^1)^{\times 3}$ be the map classifying the simplex $\langle (000), (100), (110), (111) \rangle$, and let $\overline{g} \colon \Delta^3 \to \Delta^1 \diamond \Delta^1$ be the composite with the quotient map. We have $fg = \mathrm{id}_{\Delta^3} = \overline{f}\overline{g}$.

Let $h \in \mathrm{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$ and $a, b \in \mathrm{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$ be as indicated in the following picture.



id $\qquad$ h $\qquad$ gf

These pass to simplices $\overline{h}$, $\overline{a}$, $\overline{b}$ in $\mathrm{Map}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$. The edges $\overline{a}$ and $\overline{b}$ are preisomorphisms, as one sees that for each vertex $v \in (\Delta^1 \diamond \Delta^1)$, the induced maps $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1) \xrightarrow{\overline{a} \text{ or } \overline{b}} \Delta^1 \diamond \Delta^1$

represent degenerate edges. Thus $\overline{f}\overline{g}$ and $\overline{g}\overline{f}$ are preisomorphic to identity maps, and hence $\overline{f}$ is a categorical equivalence as desired. □

45.7. **Alternate box-join.** Just as we defined the "box-join" $\boxast$, we can define the "alternate box-join" $\boxbslash$: given $f\colon A \to B$ and $g\colon K \to L$, we obtain

$$f \boxbslash g\colon (B \diamond K) \cup_{A \diamond K} (A \diamond L) \to B \diamond L.$$

45.8. **Proposition.** *We have that*

$$\overline{\mathrm{RHorn}} \boxbslash \overline{\mathrm{Cell}} \quad \cup \quad \overline{\mathrm{Cell}} \boxbslash \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}.$$

Proof. We'll show that $\overline{\mathrm{RHorn}} \boxbslash \overline{\mathrm{Cell}} \subseteq \mathrm{Cell} \cap \mathrm{CatEq}$. It is straightforward to show that the $\boxbslash$-product of two monomorphisms is a monomorphism. Thus, it suffices to show that for $f\colon A \to B$ right anodyne and any inclusion $g\colon K \to L$, the map $f \boxbslash g$ is a categorical equivalence. We know that $\overline{\mathrm{RHorn}} \boxast \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}} \subseteq \mathrm{CatEq}$ (25.10), so $f \boxast g$ is a categorical equivalence. Furthermore, in

$$\begin{tikzcd}
(B \diamond K) \cup_{A \diamond K} (A \diamond L) \arrow[r] \arrow[d] & B \diamond L \arrow[d] \\
(B \star K) \cup_{A \star K} (A \star L) \arrow[r] & B \star L
\end{tikzcd}$$

the vertical maps are categorical equivalences; this uses the result proved above (45.5), as well as the fact that since f and g are monomorphisms, the domains of $f \boxbslash g$ and $f \boxast g$ are constructed from good pushouts. □

Question: is $\overline{\mathrm{LHorn}} \boxbslash \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$?

45.9. **Alternate slice.** Given $p\colon S \to X$ and $q\colon T \to X$, we define the **alternate slices** $X^{p/}$ and $X^{/q}$ via the bijective correspondences

$$\left\{ \begin{tikzcd} S \arrow[d] \arrow[dr, "p"] & \\ S \diamond K \arrow[r, dashed] & X \end{tikzcd} \right\} \iff \{K \dashrightarrow X^{p/}\}, \qquad \left\{ \begin{tikzcd} T \arrow[d] \arrow[dr, "q"] & \\ K \diamond T \arrow[r, dashed] & X \end{tikzcd} \right\} \iff \{K \dashrightarrow X^{/q}\}.$$

just as we defined ordinary slices using joins. These constructions give right adjoints to the alternate join functors:

$$S \diamond (-)\colon s\mathrm{Set} \rightleftarrows s\mathrm{Set}_{S/} : (p \mapsto X^{p/}), \qquad (-) \diamond T\colon s\mathrm{Set} \rightleftarrows s\mathrm{Set}_{T/} : (q \mapsto X^{/q}).$$

45.10. **Proposition.** *Given $K \xrightarrow{j} L \xrightarrow{p} C$, if C is a quasicategory and j is a monomorphism, then $C^{p/} \to C^{pj/}$ is a left fibration, and $C^{/p} \to C^{/pj}$ is a right fibration.*

Proof. Follows from $\overline{\mathrm{LHorn}} \boxbslash \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ and $\overline{\mathrm{Cell}} \boxbslash \overline{\mathrm{RHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$. □

45.11. **Proposition.** *For any quasicategory C and map $p\colon S \to C$, the comparison map $C_{p/} \to C^{p/}$ is a categorical equivalence.*

Proof. Write $G(C,p) := C_{p/}$ and $G'(C,p) := C^{p/}$. The comparison map is thus a natural transformation $\beta\colon G \to G'$ of functors $s\mathrm{Set}_{S/} \to s\mathrm{Set}$. The left adjoints are $F(K) := S * K$ and $F'(K) := S \diamond K$; the adjoint to β is a natural transformation $\alpha\colon F' \to F$ of functors $s\mathrm{Set} \to s\mathrm{Set}_{S/}$.

Both F and F' take monomorphisms to monomorphisms. Also, both F and F' take categorical equivalences to categorical equivalences (45.3), (45.11). Thus (F, G) and (F', G') are Quillen pairs, where $s\mathrm{Set}$ is given the Joyal model structure, and the slice category $s\mathrm{Set}_{X/}$ inherits the Joyal model structure.

Thus we have a natural transformation between Quillen adjunctions with respect to Joyal model structures:

$$F \overset{\alpha}{\Leftarrow} F'\colon (s\mathrm{Set})^{\mathrm{Joyal}} \rightleftarrows (s\mathrm{Set}_{S/})^{\mathrm{Joyal}} : G \overset{\beta}{\Rightarrow} G'.$$

The claim is immediate from the following. □

45.12. **Proposition.** *Consider*

$$F \overset{\alpha}{\Leftarrow} F' \colon \mathcal{M} \rightleftarrows \mathcal{N} : G \overset{\beta}{\Rightarrow} G'.$$

a natural transformation between Quillen adjunctions (F,G) *and* (F',G')*. If* $\alpha(X)\colon F'(X) \to F(X)$ *is weak equivalence for every cofibrant* X *in* $\mathcal{M}$*, then* $\beta(Y)\colon G(Y) \to G'(Y)$ *is a weak equivalence for every fibrant* Y*.*

Proof. ... □

45.13. **Proposition.** *For any quasicategory* C *and object* $x \in C_0$*, the natural comparison maps* $\mathrm{map}^R_C(x,y) \to \mathrm{map}_C(x,y) \leftarrow \mathrm{map}^L_C(x,y)$ *are weak equivalences.*

Proof. In

$$\begin{tikzcd} C_{x/} \arrow[rr, "f"] \arrow[dr, "p"'] & & C^{x/} \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

the map f is a categorical equivalence (45.11) and p and q are left fibrations, and hence are categorical fibrations. It follows that the induced maps on fibers $\mathrm{map}^R_C(x,c) \to \mathrm{map}_C(x,c)$ are categorical equivalences and hence weak equivalences, since the pullbacks describing the pullbacks are good pullbacks (with respect to the Joyal model structure). □

45.14. **Slices as fibers.** The alternate slice $C^{f/}$ has another convenient characterization: it is the fiber over f of a map between functor categories.

45.15. **Proposition.** *For a map* $f\colon S \to X$ *of simplicial sets, the alternate slice* $X^{f/}$ *is isomorphic to the fiber of the restriction map*

$$\mathrm{Map}(S \diamond \Delta^0, X) \to \mathrm{Map}(S, X).$$

over f*.*

Proof. Let F be the fiber of the restriction map. There is an evident correspondence

$$K \dashrightarrow F \qquad \Longleftrightarrow \qquad \begin{tikzcd} S \times K \arrow[r, "\pi"] \arrow[d] & S \arrow[d, "f"] \\ (S \diamond \Delta^0) \times K \arrow[r, dashed] & C \end{tikzcd}$$

The claim follows by showing that the evident quotient map $S \times \Delta^1 \times K \to (S \diamond \Delta^0) \times K$ extends to an isomorphism

$$S \diamond K \xrightarrow{\sim} ((S \diamond \Delta^0) \times K) \cup_{S \times K} S$$

compatible with the standard inclusions of S. □

We can also cosnider the fiber of the inclusion $S \subset S \star \Delta^0$ into the standard cone. This gives yet another version of the slice.

45.16. **Corollary.** *Let* C *be a quasicategory, and let* $F(f) :=$ *the fiber of* $\mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$ *over* f*. Then there is a chain of categorical equivalences*

$$F(f) \to C^{f/} \leftarrow C_{f/}.$$

Furthermore, $F(f)$ *and* $C_{f/}$ *have the same set of* 0*-simplices, and both arrows above coincide on* 0*-simplices.*

Proof. The second equivalence is just (45.11). For the first equivalence, note that

$$\begin{tikzcd} \mathrm{Fun}(S \star \Delta^0, C) \arrow[r] \arrow[d] & \mathrm{Fun}(S \diamond \Delta^0, C) \arrow[d] \\ \mathrm{Fun}(S, C) \arrow[r, equal] & \mathrm{Fun}(S, C) \end{tikzcd}$$

the top horizontal map is a categorical equivalence using (45.5), while the vertical maps are both categorical fibrations. Therefore the induced map on fibers over f is a categorical equivalence, since the pullback squares in question are good.

The 0-simplices of $F(f)$ and $C_{f/}$ are exactly the set $\{S^{\triangleright} \to C\}$. Both inclusions $F(f)_0 \to (C^{f/})_0 \leftarrow (C_{f/})_0$ are induced by restriction along the standard comparison map $S \diamond \Delta^0 \to S \star \Delta^0$. $\square$

Cartesian fibrations

46. Quasicategories as a quasicategory

46.1. **Functors to sets and categories.** In ordinary category theory, the category Set of sets plays a distinguished role. Notably, for every category[28] C and every object $x \in \operatorname{ob} C$, we have the **representable** and **corepresentable** functors

$$\rho_x = \mathrm{Hom}_C(-, x) \colon C^{\mathrm{op}} \to \mathrm{Set}, \qquad \rho^x = \mathrm{Hom}_C(x, -) \colon C \to \mathrm{Set}.$$

These are found everywhere, and are the subject of the *Yoneda lemma*: natural transformations $\rho_x \to F$ are in natural bijective correspondence with elements of $F(x)$.

In many situation we have examples of functors $C \to \mathrm{Cat}$ to the category of categories.

46.2. *Example.* Let C be the category of associative rings. We have a functor

$$\mathrm{Mod} \colon C^{\mathrm{op}} \to \mathrm{Cat},$$

which sends R to Mod_R, the category of left R-modules, and sends a homomorphism $f \colon R \to S$ to the restriction functor $f^* \colon \mathrm{Mod}_S \to \mathrm{Mod}_R$, so f^*M has underlying abelian group M and R-module structure given by $r \cdot m := f(r)m$.

Sometimes we have something that looks like a functor $C \to \mathrm{Cat}$, but isn't quite: these are **pseudofunctors**.

46.3. *Example.* Let C be associative rings as in the last example. We have an operation

$$\mathrm{Mod}' \colon C \to \mathrm{Cat},$$

which sends R to Mod_R, but sends a homomorphism $f \colon R \to S$ to the "extension of scalars functor", given on objects by

$$f_* M := S \otimes_R M.$$

Given $R \xrightarrow{f} S \xrightarrow{g} T$, we only have a natural isomorphism $T \otimes_S (S \otimes_R M) \approx T \otimes_R M$, so $g_* f_* \neq (gf)_*$, and thus Mod' is not really a functor. If we incorporate the data of suitable natural isomorphisms $g_* f_* \approx (gf)_*$ and $\mathrm{id}_* \approx \mathrm{id}$ into the definition, we get the notion of a pseudofunctor. **Give reference here.**

[28] locally small category: the hom-sets of C must actually be objects of Set

46.4. **Functors to ∞-groupoids and ∞-categories?** Given a quasicategory C and an object x, we would like to consider the functor represented by x. Roughly, this associates to an object c in C the object $\mathrm{map}_C(c,x)$, which is a Kan complex. Thus, we might imagine that we get a functor

$$\mathrm{map}_C(-,x)\colon C^{\mathrm{op}} \to \mathrm{Kan}.$$

Of course, $\mathrm{map}_C(-,x)$ is not a functor: given a morphism $f\colon c \to c'$ in C, we only know how to produce a zig-zag $\mathrm{map}_C(c',x) \xleftarrow{\sim} \bullet \to \mathrm{map}_C(c,x)$.

46.5. **Three models for an ∞-category.** We have touched on several different models for an ∞-category:

(1) quasicategories,
(2) categories enriched over Kan complexes.

Although QCat and Kan are not "naturally" quasicategories, they can be regarded naturally as of type (2).

Thus, we have a simplicially enriched category $\mathrm{QCat}_{s,\mathrm{core}}$, whose underlying objects are quasicategories, and whose mapping spaces are

$$\mathrm{Fun}(C,D)^{\mathrm{core}}.$$

We can extract an actual quasicategory from this by using some kind of machine. For instance, Lurie defines

$$\mathrm{Cat}_\infty := \mathcal{N}(\mathrm{QCat}_{s,\mathrm{core}})$$

where $\mathcal{N}$ denotes the "simplicial nerve" construction.

If we are interested in functors from C^{op} to ∞-categories, we can model this by the simplicially enriched category

$$(s\mathrm{Set}^{\mathrm{Cart}}_{/C})_{s,\mathrm{core}}.$$

Here, $s\mathrm{Set}_{/C}$ is the slice category, which has a simplicial enrichment with function objects

$$\mathrm{Map}_{/C}(p,p') \subseteq \mathrm{Map}(E,E')$$

for $p\colon E \to C$ and $p'\colon E' \to C$. We can consider a full subcategory consisting of $p\colon E \to C$ which are "Cartesian fibrations" (to be defined below). Cartesian fibrations are a special kind of inner fibration, so $\mathrm{Map}_{/C}(p,p')$ is a quasicategory in this case. We obtain a category enriched over Kan complexes, with function objects

$$\mathrm{Map}_{/C}(p,p')^{\mathrm{core}}.$$

This is the candidate for $\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Cat}_\infty)$.

47. Coherent nerve

47.1. **The coherent nerve.** The coherent nerve $\mathcal{N}$ is a construction which turns a simplicially enriched category into a simplicial set, and in particular turns a Kan-enriched category into a quasicategory. It was invented by Cordier [Cor82]. The coherent nerve is constructed as right adjoint of a "realization/singular" pair

$$\mathfrak{C}\colon s\mathrm{Set} \rightleftarrows s\mathrm{Cat} : \mathcal{N}.$$

Given a finite totally ordered set S, define

$$\mathcal{P}(S) := \{\, A \subseteq S \mid \{\min,\max\} \subseteq A \subseteq S \,\}.$$

This is a poset, ordered by set containment; here $\min,\max$ denote the least and greatest elements of S (possibly the same). If S is empty, so is $P(S)$.

Let $\mathfrak{C}(\Delta^n)$ denote the simplicially enriched category defined as follows.

- The objects are elements of $[n] = \{0,\dots,n\}$.

- For objects $x, y \in [n]$, the function complex is
$$\mathrm{Map}_{\mathfrak{C}(\Delta^n)}(x,y) := N\mathcal{P}([x,y]), \qquad [x,y] := \{\, t \in [n] \mid x \le t \le y \,\},$$
which is set to be empty if $x > y$.
- Composition is induced by the union operation on subsets:
$$(T,S) \mapsto T \cup S \colon \mathcal{P}([y,z]) \times \mathcal{P}([x,y]) \to \mathcal{P}([x,z]).$$

Every $f\colon [m] \to [n]$ in Δ gives rise to an enriched functor $\mathfrak{C}(f)\colon \mathfrak{C}(\Delta^m) \to \mathfrak{C}(\Delta^n)$, which on objects operates as f does on elements of the ordered sets, and is induced on morphisms by
$$S \mapsto f(S)\colon \mathcal{P}([x,y]) \to \mathcal{P}([fx,fy])$$
We obtain (after identifying Δ with its image in $s\mathrm{Set}$) a functor $\mathfrak{C}\colon \Delta \to s\mathrm{Cat}$.

Given a simplicially enriched category C, its **coherent nerve** (or **simplicial nerve**) is the simplicial set $\mathcal{N}C$ defined by
$$(\mathcal{N}C)_n = \mathrm{Hom}_{s\mathrm{Cat}}(\mathfrak{C}(\Delta^n), C).$$
...

47.2. **Quasicategories from simplicial nerves.**

47.3. **Proposition.** *If $\mathcal{C}$ is a category enriched over Kan complexes, then $\mathcal{N}(\mathcal{C})$ is a quasicategory.*

Proof. □

48. Correspondences

A **correspondence** is defined to be an inner fibration $p\colon M \to \Delta^1$. A map of correspondences is a morphism in the slice category $s\mathrm{Set}_{/\Delta^1}$.

48.1. **Correspondences of ordinary categories.** If M is an ordinary category, then any functor $p\colon M \to \Delta^1$ is an inner fibration. Given such a functor, we can identify the following data:

- categories $C := p^{-1}(\{0\})$ and $D := p^{-1}(\{1\})$, the preimages of the vertices, and
- for each pair of objects $c \in \mathrm{ob}\, C$, $d \in \mathrm{ob}\, D$, a set
$$\mathcal{M}(c,d) := \mathrm{Hom}_M(c,d),$$
which
- fit together to give a functor
$$\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}.$$

Conversely, given the data of categories C and D, and a functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$, we can construct a category M with functor $p\colon M \to \Delta^1$ in the evident way, with
$$\mathrm{ob}\, M := \mathrm{ob}\, C \amalg \mathrm{ob}\, D, \qquad \mathrm{mor}\, M := \mathrm{mor}\, C \amalg \left(\coprod_{c,d} \mathcal{M}(c,d)\right) \amalg \mathrm{mor}\, D.$$

Under the above, maps $f\colon M \to M'$ between correspondences which are categories are sent to data consisting of: functors $u\colon C \to C'$ and $v\colon D \to D'$, and natural transformations
$$\mathcal{M} \to \mathcal{M}' \circ (u \times v) \quad \text{of functors} \quad C^{\mathrm{op}} \times D \to \mathrm{Set}.$$

48.2. *Example.* If C and D are categories, then the functor $C \star D \to \Delta^0 \star \Delta^0 \approx \Delta^1$ is an example of a correspondence. The corresponding functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ is the one with $\mathcal{M}(c,d) = \{*\}$ for all objects.

48.3. *Example.* Let $F\colon C\to D$ be a functor between categories. Then we get a functor $\mathcal{M}\colon C^{\mathrm{op}}\times D\to \mathrm{Set}$ defined by
$$\mathcal{M}(c,d):=\mathrm{Hom}_D(F(c),D),$$
and thus an associated correspondence $p\colon M\to\Delta^1$.

Similarly, let $G\colon D\to C$ be a functor between categories. Then we get a functor $\mathcal{M}'\colon C^{\mathrm{op}}\times D\to\mathrm{Set}$ defined by
$$\mathcal{M}'(c,d):=\mathrm{Hom}_C(c,G(d)),$$
and thus an associated correspondence $p'\colon M'\to\Delta^1$.

48.4. *Example.* Suppose $F\colon C\rightleftarrows D: G$ is an adjoint pair of functors. If we form $\mathcal{M}$ and $\mathcal{M}'$ as in the previous example, we see that the adjunction gives a natural isomorphism $\mathcal{M}\approx\mathcal{M}'$ of functors $C^{\mathrm{op}}\times D\to\mathrm{Set}$. The associated correspondences $M\to\Delta^1$ and $M'\to\Delta^1$ are isomorphic.

49. Cartesian and cocartesian morphisms

In the following, we fix an inner fibration $p\colon M\to S$. We will often assume that S (and thus M) is a quasicategory.

Consider an edge $f\colon x\to y$ in M. We say that the edge represented by $f\colon\Delta^1\to M$ is **p-cartesian** if a lift exists in every diagram of the form

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, tail] & M \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

for all $n\geq 2$.

There is a dual notion of a **p-cocartesian edge**, where Λ^n_n is replaced by Λ^n_0, and we use the leading edge of the simplex instead of the trailing edge.

We have already seen examples of this property.

- Let $p\colon C\to *$ where C is a quasicategory By the Joyal extension theorem (27.2), we have that an edge in C is p-cartesian if and only if it is p-cocartesian if and only if it is an isomorphism.
- Let $p\colon M\to S$ be an inner fibration between quasicategories, and suppose $f\in M_1$ is an edge such that $p(f)$ is an isomorphism in S. By the Joyal lifting theorem (27.13), f is p-cartesian if and only if it is p-cocartesian if and only if f is an isomorphism in M.
- If $p\colon M\to S$ is a right fibration, then every edge in M is p-cartesian. Likewise, if p is a left fibration, then every edge in M is p-cocartesian.

Thus, Joyal's theorem completely describe cartesian/cocartesian edges over an *isomorphism* in a quasicategory.

We have an equivalent formulation: f is p-cartesian if and only if
$$M_{/f}\to M_{/y}\times_{S_{/p(y)}} S_{/pf}$$
is a trivial fibration.

49.1. **Cartesian edges and correspondence.** Let $p\colon M\to\Delta^1$ be a correspondence, with M an ordinary category. We write $C:=p^{-1}(\{0\})$, $D:=p^{-1}(\{1\})$, and $\mathcal{M}\colon C^{\mathrm{op}}\times D\to\mathrm{Set}$ for the associated functor.

Suppose $f\colon c\to d$ is an edge such that $p(f)=\langle 01\rangle$.

49.2. **Lemma.** *The edge f is p-catesian if and only if, for each $u\colon x\to d$ with $p(u)=\langle 01\rangle$, there exists a unique $v\colon x\to c$ such that $fu=v$.*

In particular, if f is p-cartesian, then composition

$$f_*\colon \mathrm{Hom}_M(x,c) \to \mathrm{Hom}_M(x,d)$$

is a bijection for all $x \in \operatorname{ob} C$. Equivalently, the map

$$\mathrm{Hom}_C(x,c) \to \mathcal{M}(x,d), \qquad v \mapsto fv$$

is a bijection, so $\mathcal{M}(-,d)\colon C^{\mathrm{op}} \to \mathrm{Set}$ is represented by c.

49.3. **Box criterion for cartesian edges.**

49.4. **Proposition.** [Lur09, 2.4.1.8] *Let $p\colon M \to S$ be an inner fibration, and $f \in M_1$ an edge. Then f is p-cartesian if and only if a lift exists in every diagram of the form*

$$\begin{tikzcd}
\Delta^1 \times \{n\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{1\}\times\Delta^n) \cup_{\{1\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & M \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

for all $n \geq 1$.

Proof. The if part is just like the proof of the box version of Joyal lifting. $\square$

We reformulate this criterion. Consider the box power map

$$q := p^{\square(\{1\}\subset\Delta^1)}\colon \mathrm{Map}(\Delta^1, M) \to \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\},S)} \mathrm{Map}(\{1\}, M).$$

Then the above proposition says that f is p-cartesian iff a lift exists in every diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & \mathrm{Map}(\Delta^1, M) \arrow[d, "q"] \\
\Delta^n \arrow[r, "b"'] \arrow[ur, dashed] & \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\},S)} \mathrm{Map}(\{1\}, M)
\end{tikzcd}$$

such that $n \geq 1$ and $a(n) = f \in \mathrm{Map}(\Delta^1, M)_0$.

49.5. **Uniqueness of lifts to Cartesian edges.** Let $U \subseteq \mathrm{Map}(\Delta^1, M)$ be the full subsimplicial set spanned by the vertices which represent p-cartesian edges. Likewise, let $V \subseteq \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\},S)} \mathrm{Map}(\{1\}, M)$ denote the essential image of U under q, i.e., the full subsimplicial set spanned by the vertices $q(U_0)$. Obviously, the map q restricts to a map $q'\colon U \to V$.

Note in particular that V_0 is the subset of $\{\, (g,y) \in S_1 \times M_0 \mid g_1 = p(y) \,\}$ such that there *exists* a Cartesian edge $f \in M_1$ with $f_1 = y$ and $p(f) = g$, and the preimage of (g,y) under $q'\colon U \to V$ is the set of all choices of lifts. The following in particular asserts a kind of uniqueness for choices of lifts.

49.6. **Proposition.** *The map $q'\colon U \to V$ is a trivial fibration.*

Proof. Consider

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & U \arrow[r, tail] \arrow[d, "q|U"] & \mathrm{Map}(\Delta^1, M) \arrow[d, "q"] \\
\Delta^n \arrow[r] & V \arrow[r, tail] & \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\},S)} \mathrm{Map}(\{1\}, M)
\end{tikzcd}$$

If $n \geq 1$, then a lift $s\colon \Delta^n \to \mathrm{Map}(\Delta^1, M)$ exists by the previous proposition, since $a(n) \in U_0 \subseteq \mathrm{Map}(\Delta^1, M)_0$ represents a p-cartesian edge. Because $(\partial\Delta^n)_0 = (\Delta^n)_0$ when $n \geq 1$, we see that s maps into the full subcomplex U.

If $n = 0$, this amounts to $U_0 \to V_0$ being surjective, which holds by definition. $\square$

49.7. **Cartesian fibration.** A **cartesian fibration** is a map $p\colon M \to S$ which is an inner fibration, and is such that for all $(g, y) \in S_1 \times M_0$ with $g_1 = p(y)$, there exists a p-cartesian edge f with $p(f) = g$ and $f_1 = y$.

49.8. *Example.* Every left or right fibration is a cartesian fibration, since all edges are cartesian.

By the above, we see that an inner fibration $p\colon M \to S$ is a cartesian fibration if and only if $V = \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\},S)} \mathrm{Map}(\{1\}, M)$.

49.9. **Cartesian correspondences.** Given a map $p\colon M \to S$, for any simplex $a \in S_k$ write

$$M_a := \mathrm{Map}_{/S}(\Delta^k, M) = \mathrm{Map}_{/S}(a, p).$$

Note that if $a = bf$ for some simplicial operator $f\colon [k] \to [l]$, we obtain an induced restriction map

$$f^*\colon M_b \to M_a.$$

Given a correspondence $p\colon M \to \Delta^1$, we obtain

$$C = M_{\langle 0\rangle} \xleftarrow{\langle 0\rangle^*} M_{\langle 01\rangle} \xrightarrow{\langle 1\rangle^*} M_{\langle 1\rangle} = D.$$

Note that these are all quasicategories. The objects of $M_{\langle 01\rangle}$ are precisely the edges in M lying over $\langle 01\rangle$.

49.10. **Proposition.** *Let $p\colon M \to S$ be a cartesian fibration, and let $M^{\mathrm{cart}}_{\langle 01\rangle} \subseteq M_{\langle 01\rangle}$ be the full subcategory spanned by elements corresponding to cartesian edges. Then $M^{\mathrm{cart}}_{\langle 01\rangle} \to M_{\langle 1\rangle}$ is a trivial fibration.*

Proof. Every square in

$$\begin{tikzcd}
M^{\mathrm{cart}}_{\langle 01\rangle} \arrow[r] \arrow[d, hook] & U \arrow[d, hook, "i"] & \\
M_{\langle 01\rangle} \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1, M) \arrow[d, "q"] & \\
M_{\langle 1\rangle} \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1, \Delta^1) \times_{\mathrm{Map}(\{1\},\Delta^1)} \mathrm{Map}(\{1\}, M) \arrow[r] \arrow[d] & \mathrm{Map}(\{1\}, M) \arrow[d] \\
\{\mathrm{id}_{\Delta^1}\} \arrow[r] & \mathrm{Map}(\Delta^1, \Delta^1) \arrow[r] & \mathrm{Map}(\{1\}, \Delta^1)
\end{tikzcd}$$

is a pullback. The result follows because $qi = q'$ is a trivial fibration. $\square$

More generally, given an inner fibration $p\colon M \to S$ and a simplex $a \in S_k$, the objects of the quasicategory M_a correspond to k-simplices $b \in M_k$ such that $p(b) = a$. Let $M^{\mathrm{cart}}_a \subseteq M_a$ denote the full subcategory spanned by objects corresponding to $b \in M_k$ such that all edges of b are p-cartesian.

49.11. **Proposition.** *Let $p\colon M \to S$ be an inner fibration, and $f \in M_1$ an edge. Consider*

$$\begin{tikzcd}
\Delta^k \times \{n\} \arrow[r, hook] \arrow[rr, bend left, "f"] & (\Lambda^k_j \times \Delta^n) \cup_{\Lambda^k_j \times \partial\Delta^n} (\Delta^k \times \partial\Delta^n) \arrow[r] \arrow[d, hook] & M \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where $\Lambda^k_j \subset \Delta^k$ is a right horn inclusion, and f represents a p-cartesian edge. Then a lift exists whenever $n, k \geq 1$, and also when $k \geq 2$, $n = 0$.

Proof. This should also be like the box version of Joyal lifting. Note that if $k = 0$, we recover the definition of p-cartesian edge. □

This admits a reformulation: if f is p-cartesian, then if $0 < j \leq k$ and $a(n) = f$ is p-cartesian there is a lift in

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & \mathrm{Map}(\Delta^k, M) \arrow[d, "q"] \\ \Delta^n \arrow[r, "b"'] \arrow[ur, dashed] & \mathrm{Map}(\Delta^k, S) \times_{\mathrm{Map}(\Lambda^k_j, S)} \mathrm{Map}(\Lambda^k_j, M) \end{tikzcd}$$

when $k \geq 1$ and $n \geq 1$, or for all $n \geq 0$ if $k \geq 2$.

49.12. **Cartesian fibrations and right fibrations.**

49.13. **Proposition.** [Lur09, 2.4.2.4] *A map $p: M \to S$ is a right fibration iff it is a cartesian fibration whose fibers are Kan complexes.*

Proof. We have already seen that a right fibration is a cartesian fibration, and has Kan complexes as fibers.

Now suppose p is cartesian fibration with Kan complex fibers. Let $f: x \to y$ be an edge in M. Since p is cartesian, there exists a p-cartesian edgee $f': x' \to y$ over $p(f)$. Since p is cartesian fibration and f' a cartesian edge, there exists $a \in M_2$ with $a_{02} = f$ and $a_{12} = f'$ and $p(a) = (p(f))_{001}$. Thus $g := a_{01}$ is an edge in the fiber over $(p(f))_0$, so is an isomorphism in that fiber. □

49.14. **Mapping space criterion for cartesian edges.**

49.15. **Proposition.** [Lur09, 2.4.4.3] *Let $p: C \to D$ be an inner fibration between quasicategories, and $f: x \to y$ a morphism in C. The following are equivalent.*

1. *f is p-cartesian.*
2. *For every $c \in C_0$, the diagram*

$$\begin{tikzcd} \mathrm{map}_C(c, x) \arrow[r, "f_*"] \arrow[d] & \mathrm{map}_C(c, y) \arrow[d] \\ \mathrm{map}_D(p(c), p(x)) \arrow[r, "p(f)_*"'] & \mathrm{map}_D(p(c), p(y)) \end{tikzcd}$$

   is a homotopy pullback.

50. Limits and colimits as functors

Suppose J and C are categories. We say that C has all J-colimits if every functor $F: J \to C$ has a colimit in J. It is a standard observation that if F is such a functor, then we can assemble a functor

$$\mathrm{colim}_J: \mathrm{Fun}(J, C) \to C.$$

In fact, we can regard this functor as a composite of functors

$$\mathrm{Fun}(J, C) \xrightarrow{s} \mathrm{Fun}(J^{\triangleright}, C) \xrightarrow{\text{eval. at } v} C,$$

where s is some section of the restriction functor $\mathrm{Fun}(J^{\triangleright}, C) \to \mathrm{Fun}(J, C)$ which takes values in colimit cones.

Even when C does not have all J-colimits, we can assert the following. Consider the diagram

$$\begin{tikzcd} \mathrm{Fun}^{\text{colim cone}}(J^{\triangleright}, C) \arrow[r, tail] \arrow[d, "p"'] & \mathrm{Fun}(J^{\triangleright}, C) \arrow[d] \\ \mathrm{Fun}^{\exists\ \text{colim}}(J, C) \arrow[r, tail] & \mathrm{Fun}(J, C) \end{tikzcd}$$

in which the objects on the left are the evident full subcategories of the corresponding objects on the right, i.e., the ones consisting of colimit cones, and of functors which admit colimits. Then p is an equivalence of categories, and in fact is a trivial fibration. Therefore there is a contractible groupoid of sections of p, and any section s gives rise to a colimit functor

$$\mathrm{Fun}^{\exists\ \text{colim}}(J, C) \xrightarrow{s} \mathrm{Fun}^{\text{colim cone}}(J^{\triangleright}, C) \xrightarrow{\text{eval. at } v} C.$$

We want to prove the analogous statement for quasicategories. Thus, given a quasicategory C and a simplicial set S, let $\mathrm{Fun}^{\text{colim cone}}(S^{\triangleright}, C) \subseteq \mathrm{Fun}(S^{\triangleright}, C)$ denote the full subquasicategory spanned by $S^{\triangleright} \to C$ which are colimit cones, and let $\mathrm{Fun}^{\exists\ \text{colim}}(S, C) \subseteq \mathrm{Fun}(S, C)$ denote the full subquasicategory spanned by $S \to C$ for which a colimit exists.

50.1. **Proposition.** *The induced projection* $q\colon \mathrm{Fun}^{\textit{colim cone}}(S^{\triangleright}, C) \to \mathrm{Fun}^{\exists\ \textit{colim}}(S, C)$ *is a trivial fibration.*

We refer to this as the *functoriality of colimits*. We will prove it below.

The strategy is to show (1) that q is an isofibration, and (2) that q is fully faithful and essentially surjective. Then (38.1) applies to show that q is a categorical equivalence, and so a trivial fibration by (39.1).

Parts of this are already clear. For instance, q is certainly an inner fibration, since $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$ is one, and q is the restriction of p to full subquasicategories. Likewise, q is manifestly essentially surjective.

50.2. **Conical maps.** In what follows, C will be a quasicategory and S a simplicial set, and we write

$$V = V(S) := \mathrm{Fun}(S^{\triangleright}, C), \qquad U = U(S) := \mathrm{Fun}(S, C).$$

Let $p\colon V \to U$ be the evident restriction map.

Let's say that a morphism $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V is **conical** if its evalutation $\widehat{\alpha}(v)\colon \widehat{f}(v) \to \widehat{g}(v)$ at the cone point of $S^{\triangleright}$ is an isomorphism in C.

What follows are two propositions involving conical maps. We will prove them soon. The first says that any morphism in U can be lifted to a *conical* morphism in V with prescribed target.

50.3. **Proposition.** *Fix a quasicategory C and a simplicial set S. Suppose given*

- *a functor $\widehat{g}\colon S^{\triangleright} \to C$, and*
- *a natural transformation $\alpha\colon f \Rightarrow g$ of functors $S \to C$ such that $g = \widehat{g}|S$.*

Then there exists a conical morphism $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V such that $\widehat{\alpha}|S = \alpha$.

$$\begin{tikzcd} \{1\} \arrow[r, "\widehat{g}"] \arrow[d, tail] & \mathrm{Fun}(S^{\triangleright}, C) = V \arrow[d] \\ \Delta^1 \arrow[r, "\alpha"'] \arrow[ur, dashed, "\widehat{\alpha}", "conical"'] & \mathrm{Fun}(S, C) = U \end{tikzcd}$$

The second says that morphisms in V can be "transported" along conical maps.

50.4. **Proposition.** *Fix a quasicategory C, simplicial set S, and a map $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V, and let $\alpha\colon f \to g$ denote $\widehat{\alpha}|S$. For any object $\widehat{h}$ of V, consider the square*

$$\begin{tikzcd}
\mathrm{map}_V(\widehat{h},\widehat{f}) \arrow[r, "\widehat{\alpha}\circ"] \arrow[d] & \mathrm{map}_V(\widehat{h},\widehat{g}) \arrow[d] \\
\mathrm{map}_U(h,f) \arrow[r, "\alpha\circ"'] & \mathrm{map}_U(h,g)
\end{tikzcd}$$

where $h = \widehat{h}|S$, and the horizontal maps are induced by postcomposition with $\widehat{\alpha}$ and α respectively. If $\widehat{\alpha}$ is conical, then the above square is a homotopy pullback square.

We will explain and prove these two propositions soon. For the time being, you should convince yourself that if C is the nerve of an ordinary category, then both propositions are entirely straightforward to prove.

50.5. **Proof of functoriality of colimits, using properties of conical maps.** Recall that $\widehat{f}\colon S^{\triangleright} \to C$ extending $f\colon S \to C$ is a colimit cone if and only if it corresponds to an initial object of $C_{f/}$. Using the categorical equivalences

$$F(f) \to C^{f/} \leftarrow C_{f/}$$

where $F(f) \subseteq V$ is the fiber of $p\colon V \to U$ over f, we see that it is equivalent to say that $\widehat{f}$ is initial in $F(f)$.

The following gives a criterion for being a colimit cone in terms of the whole functor category $V = \mathrm{Fun}(S^{\triangleright}, C)$, rather than just in terms of the fiber over some f.

50.6. **Proposition.** *A functor $\widehat{f}\colon S^{\triangleright} \to C$ is a colimit cone if and only if*

$$p'\colon\ \mathrm{map}_V(\widehat{f},\widehat{g}) \to \mathrm{map}_U(f,g)$$

is a weak equivalence for every $\widehat{g}\colon S^{\triangleright} \to C$, $g = \widehat{g}|S = p(\widehat{g})$.

Proof. Since $p\colon V \to U$ is a categorical fibration, the induced maps p' on mapping spaces are Kan fibrations. Thus, p' is a weak equivalence if and only if its fibers are contractible.

($\Longleftarrow$) Suppose every p' is a weak equivalence. Then in particular p' is a weak equivalence for any $\widehat{g}\colon S^{\triangleright} \to C$ such that $\widehat{g}|S = f$. In this case, the fiber of p' over $1_f \in \mathrm{map}_U(f,f)$ is precisely the mapping space $\mathrm{map}_{F(f)}(\widehat{f},\widehat{g})$ in the fiber quasicategory $F(f) \subseteq \mathrm{Fun}(S^{\triangleright}, C)$, and this fiber is contractible. Therefore, $\widehat{f}$ is an initial object of $F(f)$, and therefore $\widehat{f}$ is initial in $C_{f/}$ by the above discussion. We have shown that $\widehat{f}$ is a colimit cone.

($\Longrightarrow$) Suppose $\widehat{f}$ is a colimit cone. Therefore for $\widehat{f'}$ such that $\widehat{f'}|S = f$ the fiber of $\mathrm{map}_V(\widehat{f},\widehat{f'}) \to \mathrm{map}_U(f,f)$ over 1_f is contractible. We need to show that the fiber of $p'\colon \mathrm{map}_V(\widehat{f},\widehat{g}) \to \mathrm{map}_U(f,g)$ over a general $\alpha \in \mathrm{map}_U(f,g)$ is contractible.

Given such an α, choose a conical map $\widehat{\alpha}\colon \widehat{f'} \to \widehat{g}$ with $\widehat{\alpha}|S = \alpha$ (50.3), and consider the resulting square

$$\begin{tikzcd}
\mathrm{map}_V(\widehat{f},\widehat{f'}) \arrow[r, "\widehat{\alpha}\circ"] \arrow[d, "p'"'] & \mathrm{map}_V(\widehat{f},\widehat{g}) \arrow[d, "p''"] \\
\mathrm{map}_U(f,f) \arrow[r, "\alpha\circ"'] & \mathrm{map}_U(f,g)
\end{tikzcd}$$

$$1_f \longmapsto \alpha$$

Since $\widehat{\alpha}$ is conical, the square is a homotopy pullback square (50.4). Therefore, the fiber of p'' over α is weakly equivalent to the fiber of p' over 1_f, which is contractible since $\widehat{f}$ is a colimit cone. $\square$

Proof of (50.1). First we show that $q\colon \mathrm{Fun}^{\text{colim cone}}(S^{\rhd}, C) \to \mathrm{Fun}^{\exists\,\text{colim}}(S, C)$ is an isofibration; we have already observed that it is an inner fibration. Given an isomorphism $\alpha\colon f \to g$ between objects in $\mathrm{Fun}^{\exists\,\text{colim}}(S, C) \subseteq U$ and a choice of colimit cone $\widehat{g}$ over g, chose a conical lift $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$. The arrow $\widehat{\alpha}\colon S^{\rhd} \times \Delta^1 \to C$ restricts to an isomorphism at each vertex of $S^{\rhd}$, and so is a natural isomorphism by the objectwise criterion for natural isomorphisms. Thus $\widehat{f}$ is also a colimit cone by (50.6), so $\widehat{\alpha}$ is an isomorphism in $\mathrm{Fun}^{\text{colim cone}}(S^{\rhd}, C)$.

We have already observed that q is essentially surjective (in fact, it is surjective on 0-simplices). That q is fully faithful is immediate from (50.6). $\square$

50.7. Proof of properties of conical maps.

Proof of (50.3). Recall the situation: we are given a natural transformation $\alpha\colon f \Rightarrow g$ of functors $S \to C$, and a lift $\widehat{g}\colon S^{\rhd} \to C$ of the target to the cone, and we want to find a *conical* lift of α:

$$\begin{tikzcd} \{1\} \arrow[r, "\widehat{g}"] \arrow[d, tail] & \mathrm{Fun}(S^{\rhd}, C) \arrow[d] \\ \Delta^1 \arrow[r, "\alpha"'] \arrow[ur, dashed, "\widehat{\alpha}", "\text{conical}"'] & \mathrm{Fun}(S, C) \end{tikzcd}$$

We make use of a natural map

$$\kappa\colon S^{\rhd} \times K \to (S \times K)^{\rhd}.$$

Note that this map sends $\{v\} \times K$ to the cone point $\{v\}$. Consider the composite

$$\lambda\colon (S \times \Delta^1) \cup_{S \times \{1\}} (S^{\rhd} \times \{1\}) \to S^{\rhd} \times \Delta^1 \xrightarrow{\kappa} (S \times \Delta^1)^{\rhd}$$

where the first map is the box-product $(S \subset S^{\rhd}) \square (\{1\} \subset \Delta^1)$. By inspection, we see that the composite map can be identified with the box-join

$$(S \times \{1\} \subseteq S \times \Delta^1) \circledast (\varnothing \subseteq \Delta^0).$$

Since $\mathrm{RHorn} \square \mathrm{Cell} \subseteq \overline{\mathrm{RHorn}}$ (44.3) we have that $(S \times \{1\} \subseteq S \times \Delta^1)$ is right anodyne. Likewise, since $\mathrm{RHorn} \circledast \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (25.10), we conclude that λ is inner anodyne. Therefore, an extension $\overline{\alpha}$ exists in

$$\begin{tikzcd} (S \times \Delta^1) \cup_{S \times \{1\}} (S^{\rhd} \times \{1\}) \arrow[r, "(\alpha, \widehat{g})"] \arrow[d, tail] & C \\ S^{\rhd} \times \Delta^1 \arrow[d, "\kappa"'] & \\ (S \times \Delta^1)^{\rhd} \arrow[uur, dashed, "\overline{\alpha}"'] & \end{tikzcd}$$

We set $\widetilde{\alpha} := \overline{\alpha} \circ \kappa$. It is clear that $\widetilde{\alpha}$ is conical: $\widehat{\alpha}(v)$ is the identity map of $\overline{\alpha}(v)$. $\square$

For the proof of (50.4), let's first note that, as stated, it actually doesn't make sense! This proposition asserts that for conical $\widehat{\alpha}$, the diagram

$$\begin{tikzcd} \mathrm{map}_V(\widehat{h}, \widehat{f}) \arrow[r, "\widehat{\alpha}\circ"] \arrow[d] & \mathrm{map}_V(\widehat{h}, \widehat{g}) \arrow[d] \\ \mathrm{map}_U(h, f) \arrow[r, "\alpha\circ"'] & \mathrm{map}_U(h, g) \end{tikzcd}$$

is a homotopy pullback. However, the horizontal maps ("postcomposition" with α and $\widehat{\alpha}$) are only defined as a homotopy class of maps in $h\mathrm{Kan}$. For instance, "$\alpha\circ$" is the homotopy class defined by the zig-zag around the left and top of the diagram

$$\begin{tikzcd}
\mathrm{map}_U(h,f,g)_\alpha \arrow[r] \arrow[d, "\sim"'] & \mathrm{map}_U(h,f,g) \arrow[r, "\mathrm{comp}"] \arrow[d, "\sim"] & \mathrm{map}_U(h,g) \\
\mathrm{map}_U(h,f)\times\{\alpha\} \arrow[r] & \mathrm{map}_U(h,f)\times \mathrm{map}(f,g) &
\end{tikzcd}$$

where the left-hand square is a pullback. The correct statement of (50.4) is that in

$$\begin{tikzcd}
\mathrm{map}_V(\widehat{h},\widehat{f}) \arrow[d] & \mathrm{map}_V(\widehat{h},\widehat{f},\widehat{g})_{\widehat{\alpha}} \arrow[l, "\sim"'] \arrow[r] \arrow[d] & \mathrm{map}_V(\widehat{h},\widehat{g}) \arrow[d] \\
\mathrm{map}_U(h,f) & \mathrm{map}_U(h,f,g)_\alpha \arrow[l, "\sim"] \arrow[r] & \mathrm{map}_U(h,g)
\end{tikzcd}$$

the right-hand square is a homotopy pullback.

We can refine this a little further. Fix a map $e\colon \Delta^{\{1,2\}}\to C$. For a simplicial set S, let $K\subseteq S^{\triangleright}\times\Delta^2$ be a subcomplex containing the edge $\{v\}\times\Delta^{\{1,2\}}$, and define $\mathrm{Map}(K,C)_e$ by the pullback square

$$\begin{tikzcd}
\mathrm{Map}(K,C)_e \arrow[r, tail] \arrow[d] & \mathrm{Map}(K,C) \arrow[d] \\
\{e\} \arrow[r, tail] & \mathrm{Map}(\{v\}\times\Delta^{\{1,2\}},C)
\end{tikzcd}$$

To prove our proposition, it suffices to show that for every isomorphism e in C, the map

$$\mathrm{Map}(S^{\triangleright}\times\Delta^2,C)_e\to \mathrm{Map}((S^{\triangleright}\times\Lambda^2_2)\cup_{S\times\Lambda^2_2}(S\times\Delta^2),C)_e$$

is a trivial fibration. Equivalently, we must produce a lift in each diagram of the form

$$\begin{tikzcd}
& (S^{\triangleright}\times\partial\Delta^m)\cup_{S\times\partial\Delta^m}(S\times\Delta^m) \arrow[r] \arrow[d, tail] & \mathrm{Map}(\Delta^2,C) \arrow[d] & \\
\{v\}\times\Delta^{\{1,2\}} \arrow[r, tail] \arrow[rrrd] & S^{\triangleright}\times\Delta^m \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(\Lambda^2_2,C) \arrow[r] & \mathrm{Map}(\Delta^{\{1,2\}},C) \\
& & & \{e\} \arrow[u, tail]
\end{tikzcd}$$

We reduce to producing a lift in

$$\begin{tikzcd}
\{n\}\times\Delta^{\{1,2\}} \arrow[r, tail] \arrow[rr, bend left, "e"] & (\Delta^n\times\Lambda^2_2)\cup_{\partial\Delta^n\times\Lambda^2_2}(\partial\Delta^n\times\Delta^2) \arrow[r] \arrow[d, tail] & C \\
& \Delta^n\times\Delta^2 \arrow[ur, dashed] &
\end{tikzcd}$$

where e is an isomorphism in C. This is precisely the box-version of Joyal extension.

51. More stuff

Recall that the join constructions $K\star -$ and $-\star K$ are colimit preserving functors $s\mathrm{Set}\to s\mathrm{Set}_{K/}$ to the category of simplicial sets *under* K. In particular, viewed as functors $s\mathrm{Set}\to s\mathrm{Set}$ to plain simplicial sets, they preserve pushouts, and transfinite compositions.

51.1. **Proposition.** *If $\mathcal{A}$ is a class of maps in $s\mathrm{Set}$, then $K\star\overline{\mathcal{A}}\subseteq\overline{K\star\mathcal{A}}$ and $\overline{\mathcal{A}}\star K\subseteq\overline{\mathcal{A}\star K}$.*

Proof. Check that $K \star -\colon s\mathrm{Set} \to s\mathrm{Set}$ preserves isomorphisms, transfinite composition, pushouts, and retracts. $\square$

51.2. *Remark.* Given $f\colon X \to Y$ and K, we have a factorization of $K \star f$ as

$$K \star X \to (K \star X) \amalg_{\varnothing \star X} (\varnothing \star Y) \xrightarrow{(\varnothing \subseteq K) \circledast f} K \star Y.$$

51.3. **Proposition.** *We have* $\Delta^0 \star \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$ *and* $\overline{\mathrm{Cell}} \star \Delta^0 \subseteq \overline{\mathrm{RHorn}}$.

51.4. **Proposition.** *Let C be a quasicategory and x an object of C. Then x is an initial object iff $\{x\} \to C$ is left anodyne, and x is a terminal object iff $\{x\} \to C$ is right anodyne.*

Proof. ($\Longrightarrow$) Let x be terminal, and consider $j\colon \{x\} \to C$. Since $j^{\triangleright}$ is right anodyne, it suffices to show that j is a retract of $j^{\triangleright}$. To do this, we construct a map r fitting into

$$\begin{tikzcd}
\{x\} \arrow[r, tail] \arrow[d, "j"'] \arrow[rr, bend left, "\mathrm{id}"] & \{x\}^{\triangleright} \arrow[r] \arrow[d, "j^{\triangleright}"'] & \{x\} \arrow[d, "j"'] \\
C \arrow[r, tail] \arrow[rr, bend right, "\mathrm{id}"'] & C^{\triangleright} \arrow[r, dashed, "r"] & C
\end{tikzcd}$$

This amounts to solving the lifting problem

$$\begin{tikzcd}
C \cup \{x\}^{\triangleright} \arrow[r, "(\mathrm{id}, 1_x)"] \arrow[d, tail] & C \\
C^{\triangleright} \arrow[ur, dashed]
\end{tikzcd}
\qquad \Longleftrightarrow \qquad
\begin{tikzcd}
\{x\} \arrow[r, "1_x"] \arrow[d, tail] & C_{x/} \arrow[d] \\
C \arrow[r, equal] \arrow[ur, dashed] & C
\end{tikzcd}$$

Since x is terminal, $C_{x/} \to C$ is a trivial fibration (26.2), so a lift exists.

($\Longleftarrow$) Suppose $j\colon \{x\} \to C$ is right anodyne. Since $C_{/x} \to C$ is a right fibration, a lift exists in

$$\begin{tikzcd}
\{x\} \arrow[r, "1_x"] \arrow[d, tail] & C_{x/} \arrow[d] \\
C \arrow[r, equal] \arrow[ur, dashed] & C
\end{tikzcd}$$

which is equivalent to x being terminal. $\square$

51.5. **Corollary.** *Let $p\colon D \to C$ be a right fibration between quasicategories, and let x be an object of C. Then the induced map*

$$\mathrm{Map}(C_{/x}, D) \to \mathrm{Map}(\{1_x\}, D) \times_{\mathrm{Map}(\{1_x\}, C)} \mathrm{Map}(C_{/x}, C)$$

is a trivial fibration. In particular, the map

$$\mathrm{Map}_C(C_{/x}, D) \to \mathrm{Map}_C(\{1_x\}, D)$$

induced by restriction over the projection map $(C_{/x} \to C) \in \mathrm{Map}(C_{/x}, C)$ is a trivial fibration between Kan complexes.

52. Introduction: the cover/functor correspondence

Consider the following classes of maps between simplicial sets, called respectively **covers**, **left covers** and **right covers**.

$$\mathrm{Cover} := (\mathrm{Horn} \amalg \mathrm{Horn}^{\vee})^{\boxslash},$$

$$\mathrm{LCover} := (\mathrm{LHorn} \amalg \mathrm{LHorn}^{\vee})^{\boxslash}, \qquad \mathrm{RCover} := (\mathrm{RHorn} \amalg \mathrm{RHorn}^{\vee})^{\boxslash}.$$

Thus, Cover is the subclass of Kan fibrations, which admit *unique* lifting for every lifting problem with respect to horns, and therefore with respect to all anodyne maps. Similar statements hold for LCover and RCover.

We obtain for each simplicial set X full subcategories $s\mathrm{Set}_{/X}^{\mathrm{Cover}}$, $s\mathrm{Set}_{/X}^{\mathrm{LCover}}$, $s\mathrm{Set}_{/X}^{\mathrm{RCover}}$, of the slice category $s\mathrm{Set}_{/X}$. We will see that each of these categories is equivalent to a certain category of functors from hX to sets.

52.1. **Simplicial covering maps.** Covers $p\colon E \to X$ are precisely the simplicial analogues of covering maps of spaces. To see this, will first note that covers are "locally trivial".

52.2. **Lemma.** *Cover* $= (S \amalg S^\vee)^\boxslash$*, where S is the set of all maps $\Delta^m \to \Delta^n$ between standard simplices. Thus, $p\colon E \to X$ is a cover if and only if for each simplex $a \in X_n$ and each simplicial operator $\delta\colon [m] \to [n]$, the function $x \mapsto x\delta\colon p^{-1}(a) \to p^{-1}(a\delta)$ is a bijection of sets.*

Proof. Given a class $\mathcal{B}$ of maps, let ${}^{\boxslash!}\mathcal{B}$ denote the class of maps f which have *unique* lifting with respect to maps in $\mathcal{B}$; i.e., such that $\{f, f^\vee\} \boxslash \mathcal{B}$. An easy argument shows that ${}^{\boxslash!}\mathcal{B}$ is saturated, and also has the property that if $f, g \circ f \in {}^{\boxslash!}\mathcal{B}$, then $g \in {}^{\boxslash!}\mathcal{B}$.

Because any inclusion $\Delta^0 \to \Delta^n$ of a vertex is anodyne and thus in ${}^{\boxslash!}$Cover, applying the observation of the previous paragraph to a composite $\Delta^0 \to \Delta^m \xrightarrow{\delta} \Delta^n$ shows that every such $\delta \in {}^{\boxslash!}$Cover. Hence Cover $\subseteq (S \amalg S^\vee)^\boxslash$.

Conversely, let $\mathcal{B} := (S \amalg S^\vee)^\boxslash$, so that $S \subseteq {}^{\boxslash!}\mathcal{B}$. Since ${}^{\boxslash!}\mathcal{B}$ is saturated, we can show that Horn $\subseteq {}^{\boxslash!}\mathcal{B}$, by building horn inclusions out of (injective) maps between simplices. Thus $(S \amalg S^\vee)^\boxslash \subseteq$ Cover. $\square$

52.3. **Corollary.** *A map $p\colon E \to X$ is a cover if and only if for each map $f\colon \Delta^n \to X$ there is a pullback square of the form*

$$\begin{tikzcd} \coprod \Delta^n \arrow[r] \arrow[d, "proj"'] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Left as an exercise, using the lemma. $\square$

Given a cover $p\colon E \to X$, we define a functor $F\colon hX \to \mathrm{Set}^{\mathrm{core}}$ as follows. For each object $x \in X_0$, we let $F(x) := p^{-1}(x)$. For each edge $f \in X_1$, we define a bijection $F(f)\colon F(f_0) \to F(f_1)$ as the composite

$$p^{-1}(f_0) \xleftarrow{\langle 0\rangle} p^{-1}(f) \xrightarrow{\langle 1\rangle} p^{-1}(f_1),$$

where both maps are bijections by the lemma. For each 2-simplex $a \in X_2$, we have a commutative diagram of bijections which shows that $F(a_{12})F(a_{01}) = F(a_{02})$. By the universal property of fundamental category, this extends uniquely to a functor $F\colon hX \to \mathrm{Set}^{\mathrm{core}}$.

Conversely, given a functor $F\colon hX \to \mathrm{Set}^{\mathrm{core}}$, we define a cover $p\colon E \to X$ as follows. Set

$$E_n := \coprod_{x \in X_n} F(x_0),$$

and for each simplicial operator $\delta\colon [m] \to [n]$, define $\delta^*\colon E_n \to E_m$ by

$$\delta^*(x, t) := \big(x\delta, F(x_{0,\delta(0)})(t)\big).$$

52.4. **Proposition.** *The above constructions give an adjoint equivalence of categories*

$$s\mathrm{Set}_{/X}^{\mathrm{Cover}} \approx \mathrm{Fun}(hX, \mathrm{Set}^{\mathrm{core}}).$$

Proof. Left as an exercise, including showing that $\mathrm{Un}(F)$ is in fact a cover. $\square$

Let $\Pi_1 X := (hX)_{\mathrm{Gpd}}$, the groupoid obtained by formally inverting all morphisms in hX. The above passes to an equivalence of categories $s\mathrm{Set}_{/X}^{\mathrm{Cover}} \approx \mathrm{Fun}(\Pi_1 X, \mathrm{Set})$. This is the simplicial set analogue of the classification of covering spaces.

52.5. **Left and right covers correspond to functors to sets.** I'll state the analogues of the above results for left and right covers. Proofs are left for the reader.

52.6. **Lemma.** *LCover* $= (S \amalg S^{\vee})^{\boxslash}$ *and* RCover $= (T \amalg T^{\vee})^{\boxslash}$ *, where* $S = \{\, \delta\colon \Delta^m \to \Delta^n \mid \delta(0) = 0 \,\}$ *and* $T = \{\, \delta\colon \Delta^m \to \Delta^n \mid \delta(m) = n \,\}$.

Given a left cover $p\colon E \to X$, we define a functor $F\colon hX \to \mathrm{Set}$ much as we did above, so that $F(x) := p^{-1}(x)$, while $F(f)$ is the map obtained as the composite

$$p^{-1}(f_0) \xleftarrow[\sim]{\langle 0 \rangle} p^{-1}(f) \xrightarrow{\langle 1 \rangle} p^{-1}(f_1),$$

where the first map is a bijection by the lemma. We write $\mathrm{St}(p) := F$ and call it the **straightening** of p.

Conversely, given a functor $F\colon hX \to \mathrm{Set}$, we define a left cover $p\colon E \to X$ by

$$E_n := \coprod_{x \in X_n} F(x_0), \qquad \delta^*(x,t) := \big(x\delta, F(x_{0,\delta(0)})(t)\big).$$

The constructions for right covers are similar, except that the corresponding functors are contravariant. Thus, given a right cover $p\colon E \to X$ we obtain a straightening $\mathrm{St}(p) = F\colon hX^{\mathrm{op}} \to \mathrm{Set}$ given by $F(x) = p^{-1}(x)$ and $F(f)$ the composite

$$p^{-1}(f_0) \xleftarrow{\langle 0 \rangle} p^{-1}(f) \xrightarrow[\sim]{\langle 1 \rangle} p^{-1}(f_1),$$

while for a functor $F\colon hX^{\mathrm{op}} \to \mathrm{Set}$, we obtain a right cover $p\colon E \to X$ by

$$E_n := \coprod_{x \in X_n} F(x_n), \qquad \delta^*(x,t) := \big(x\delta, F(x_{\delta(m),m})(t)\big).$$

52.7. **Proposition.** *The above constructions give adjoint equivalences of categories*

$$\mathrm{St}\colon s\mathrm{Set}_{/X}^{\mathrm{LCover}} \rightleftarrows \mathrm{Fun}(hX, \mathrm{Set}) : \mathrm{Un}, \qquad \mathrm{St}\colon s\mathrm{Set}_{/X}^{\mathrm{RCover}} \rightleftarrows \mathrm{Fun}(hX^{\mathrm{op}}, \mathrm{Set}) : \mathrm{Un}\,.$$

Note also that left covers of X correspond to right covers of X^{op}.

52.8. **Straightening and unstraightening over a category.** Suppose $X = hX = C$ is itself a category and $F\colon C \to \mathrm{Set}$ a functor, and consider its unstraightening $p\colon E \to C$. It is straightforward to show that E is the nerve of a category, namely the **category of elements** of F, which has

- **objects:** pairs (c, x), $c \in C_0$ and $x \in F(x)$, and
- **morphisms** $(c, x) \to (c', x')$**:** morphisms $\alpha\colon c \to c'$ in C such that $F(\alpha)(x) = x'$.

The left cover map p is then just the evident forgetful functor. An analogous statement holds for contravariant functors.

Given any object x in C, we have the slice projections $C_{x/} \to C$ and $C_{/x} \to C$.

52.9. **Proposition.** *The projections* $C_{x/} \to C$ *and* $C_{/x} \to C$ *are left and right covers respectively. Their straightenings are representable functors:* $\mathrm{St}(C_{x/} \to C) = \mathrm{Hom}_C(x, -)$ *and* $\mathrm{St}(C_{/x} \to C) = \mathrm{Hom}_C(-, x)$.

Proof. Exercise. $\square$

52.10. **Fiber product corresponds to tensor product.** Let C be a category. Given functors $F\colon C^{\mathrm{op}} \to \mathrm{Set}$ and $G\colon C \to \mathrm{Set}$, the **tensor product** is the set defined by

$$F \otimes_C G := \operatorname{colim}\Big[\coprod_{c_0 \to c_1 \in C_1} F(c_1) \times G(c_0) \rightrightarrows \coprod_{c \in C} F(c) \times G(c)\Big].$$

52.11. *Exercise.* Given $x \in C_0$, let $\rho_x := \operatorname{Hom}_C(-,x)\colon C^{\mathrm{op}} \to \mathrm{Set}$ and $\rho^x := \operatorname{Hom}_C(x,-)\colon C \to \mathrm{Set}$ denote the representable functors. Show that $\rho_x \otimes_C G \approx G(x)$ and $F \otimes_C \rho^x \approx F(x)$.

If $A \to C$ and $B \to C$ are the unstraightenings of F and G respectively, then

$$F \otimes_C G \approx \pi_0(A \times_C B).$$

52.12. **The universal left cover.** What happens if we unstraighten the identity functor of Set? We get the category of elements of Set, which is precisely $\mathrm{Set}_* := \mathrm{Set}_{*/}$, the category of based sets.

53. Straightening and unstraightening

Let $\mathfrak{D}_{\Delta^n}\colon \mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$ be the simplicialy enriched functor defined as follows.

- For each object $x \in \{0,\dots,n\}$, set
$$\mathfrak{D}_{\Delta^n}(x) := N\mathcal{P}_\ell(x),$$
the nerve of the poset
$$\mathcal{P}_\ell(x) := \big\{S \mid \{x\} \subseteq S \subseteq [x,n]\big\}$$
of subsets of the interval $[x,n] = \{x,\dots,n\}$ which contain the left endpoint.
- The structure of enriched functor is induced by the union operation on subsets:
$$(T,S) \mapsto T \cup S\colon \mathcal{P}(x,y) \times \mathcal{P}_\ell(y) \to \mathcal{P}_\ell(x).$$
- For each map $\delta\colon \Delta^m \to \Delta^n$, we define a natural transformation
$$\mathfrak{D}_\delta\colon \mathfrak{D}_{\Delta^m} \to \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}}$$
of simplicially enriched functors $\mathfrak{C}(\Delta^m)^{\mathrm{op}} \to s\mathrm{Set}$, which at each object x of $\mathfrak{C}(\Delta^m)^{\mathrm{op}}$ is a map $\mathfrak{D}_{\Delta^m}(x) \to \mathfrak{D}_{\Delta^n}(\delta x)$ induced by the map of posets
$$S \mapsto \delta(S)\colon \mathcal{P}_\ell(x) \to \mathcal{P}_\ell(\delta x).$$

53.1. *Remark.* The functor $\mathcal{D}_{\Delta^n}\colon \mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$ is isomorphic to the representable functor $\operatorname{Map}_{\mathfrak{C}((\Delta^n)^\triangleright)}(-,v)$, where v represents the cone point of $(\Delta^n)^\triangleright$. Likewise, the natural transformation $\mathfrak{D}_\delta\colon \mathfrak{D}_{\Delta^m} \to \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}}$ coincides with the transformation

$$\operatorname{Map}_{\mathfrak{C}((\Delta^m)^\triangleright)}(-,v) \xrightarrow{\mathfrak{C}(\delta^\triangleright)} \operatorname{Map}_{\mathfrak{C}((\Delta^n)^\triangleright)}(\delta(-),v)$$

induced by $\delta^\triangleright\colon (\Delta^m)^\triangleright \to (\Delta^n)^\triangleright$.

Fix a simplical set S, and consider a simplicially enriched functor $F\colon \mathfrak{C}(S)^{\mathrm{op}} \to s\mathrm{Set}$. We define a morphism

$$\mathrm{Un}_S(F)\colon X \to S$$

of simplicial sets, called the **unstraightening** of F over S, as follows.

- An n-simplex of $\mathrm{Un}_S F$ is a pair
$$f\colon \Delta^n \to S, \qquad t\colon \mathfrak{D}_{\Delta^n} \to F \circ \mathfrak{C}(f),$$
where f is a map of simplicial sets, and t is a map of simplicially enriched functors $\mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$.

- To a map $\delta\colon \Delta^m \to \Delta^n$ we have an induced map $(\mathrm{Un}_S F)_n \to (\mathrm{Un}_S F)_m$, which sends an n-simplex (f,t) to the pair

$$\Delta^m \xrightarrow{\delta} \Delta^n \xrightarrow{f} S, \qquad \mathfrak{D}_{\Delta^m} \xrightarrow{\mathfrak{D}_\delta} \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}} \xrightarrow{t\circ\mathfrak{C}(\delta)^{\mathrm{op}}} F \circ \mathfrak{C}(f) \circ \mathfrak{C}(\delta)^{\mathrm{op}}.$$

54. Pullback of right anodyne map along left fibration

The arguments here are based on the appendix to [Mos15].

54.1. **Lemma.** *Consider a pullback square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ \Lambda^n_k \arrow[r, tail] & \Delta^n \end{tikzcd}$$

If p is a left fibration and $k \geq 1$, then $A \to B$ is right anodyne.

54.2. **Contraction maps.** Let Δ_R denote the category whose

- objects are totally ordered sets $[n]^R := [n] \sqcup \{R\} = \{0 < 1 < \cdots < n < R\}$ for $n \geq -1$, and
- morphisms are order preserving functions which take the "right basepoint" R to R.

There is an evident "inclusion" functor $\Delta \to \Delta_R$ sending $[n] \mapsto [n]_R$.

A **right contraction** of a simplicial set X is a choice of extension of $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ to a functor $X\colon (\Delta_R)^{\mathrm{op}} \to \mathrm{Set}$. There is an evident dually defined category Δ_L and corresponding notion of **left contraction**.

A right contraction of a simplicial set X is completely determined by the inclusion $X_{-1} \to X_0$ together with its **contraction operators**, which are the maps $X_{n-1} \to X_n$ for $n \geq 0$ induced by the surjective map $Q\colon [n]_R \to [n-1]_R$ which sends $n \mapsto R$.

Let $\Delta_R^{\mathrm{surj}} \subset \Delta_R$ denote the subcategory consisting of all objects, and all *surjective* maps. Given a right-contracted simplicial set $X\colon (\Delta^R)^{\mathrm{op}} \to \mathrm{Set}$, say that $a \in X_n$ is **contracted** if there exists a *non-identity* $\sigma\colon [n]_R \to [k]_R$ in Δ_R^{surj} and $b \in X_k$ such that $a = b\sigma$. That is, a contracted simplex is one which is *either* degenerate or in the image of the contraction operators Q. Say $a \in X_n$ is **non-contracted** if $a = b\sigma$ for $\sigma \in \Delta_R^{\mathrm{surj}}$ we have $\sigma = \mathrm{id}$.

We have an anologue of the Eilenberg-Zilber lemma (14.7) for right contracted simplicial sets, which is proved in the same way.

54.3. **Proposition.** *For a in X, there exists a* unique *pair (b,σ) consisting of a map $\sigma \in \Delta_R^{\mathrm{surj}}$ and a* non-contracted *simplex $b \in X$ such that $a = b\sigma$.*

54.4. **Corollary.** *For any right contracted simplicial set, the evident maps*

$$\coprod_{k\geq -1} \coprod_{b \in X_k^{\mathrm{nc}}} \mathrm{Hom}_{\Delta_R^{\mathrm{surj}}}([n]_R, [k]_R) \to X([n]_R)$$

defined by $(b,\sigma) \mapsto b\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective morphisms $[n]_R \to [n']_R$.

There is an analogue of the skeletal filtration: let $F_n X \subseteq X$ denote the smallest subobject containing all simplices of dimensions $\leq n$. Then the simplices of $F_n X$ consist of those which are degeracies or contractions of noncontracted simplices of dimension $\leq n$.

Comparing this with the skeletal filtration, we discover the following.

54.5. **Lemma.** *For all $n \geq 0$, the contraction operators $Q\colon X_{n-1} \to X_n$ restrict to bijections $X_{n-1}^{\mathrm{nc}} \to (X_n^{\mathrm{nd}} \smallsetminus X_n^{\mathrm{nc}})$.*

54.6. **Proposition.** *Let X be a simplicial set equipped with a right contraction. Consider the evident inclusion $f\colon S \to X$, where S is the discrete simpicial set with underlying set X_{-1}. Then f is right anodyne; in fact, $f \in \overline{\{ \Lambda^n_n \subset \Delta^n \mid n \geq 1 \}}$.*

We note that for $x \in X_{n-1}$,

- $d_i Q(x) = Q d_i(x)$ if $i \in \{0, \dots, n-1\}$, i.e., the contraction operators commute with most face maps;
- we have $d_n Q(x) = x$;
- $s_i Q(x) = Q s_i(x)$ if $i \in \{0, \dots, n-1\}$, i.e., the contraction operators commute with most degeneracy maps;
- we have $s_n Q(x) = QQ(x)$.

In fact, a right contraction for a simplicial set X is equivalent to choosing data $(X_{-1}, d_0\colon X_0 \to X_{-1}, \{Q\colon X_{n-1} \to X_n\})$, where the Q satisfy the above identities.

There is an evident complementary notion of **left contraction**.

If X admits a right contraction, then it is "right contractible" to a discrete simplicial set.

54.7. **Proposition.** *Let X be a simplicial set equipped with a right contraction. Consider the evident inclusion $f\colon S \to X$, where S is the discrete simpicial set with underlying set X_{-1}. Then f is right anodyne; in fact, $f \in \overline{\{ \Lambda^n_n \subset \Delta^n \mid n \geq 1 \}}$.*

Proof. Consider the collection X^{nd} of non-degenerate simplices of X not contained in S. Partition X^{nd} into disjoint subsets $X^I \amalg X^{II}$, where $X^I = (X^{\mathrm{nd}} \smallsetminus S^{\mathrm{nd}}) \cap Q(X)$, the set of nondegenerate simplices which are in the image of the contraction operators.

The claim is that the contraction operators define bijections $\phi\colon X^{II}_{n-1} \to X^I_n$ for $n \geq 1$. To see this, we note the following.

- If $Qx = s_i y \in X_n$ with $i \in \{0, \dots, n-2\}$, then $x = d_n Q x = d_n s_i y = s_i d_n y$. Thus, x is degenerate.
- If $Qx = s_{n-1} y \in X_n$, then $x = d_n Q x = d_n s_{n-1} y = y$ and $Q d_{n-1} x = d_{n-1} Q x = d_{n-1} s_{n-1} y = y$. Thus, x is in the image of Q.
- Taken together, the last two statements imply that Q applied to an element of X^{II} is non-degenerate. Thus, ϕ is a well-defined map.
- If $x = s_i y \in X_{n-1}$ with $i \in \{0, \dots, n-2\}$, then $Q s_i y = s_i Q y$, i.e., Qx is degenerate whenever x is degenerate. Thus, ϕ is surjective.
- We have $d_n Q x = x$ for $x \in X_{n-1}$. Therefore, ϕ is injective.

Now we can filter X by subcomplexes E_n, where $E_{-1} = S$, while E_n is the smallest subcomplex containing $\mathrm{Sk}_{n-1} X$ and X^I_n. For each $Qx \in X^I_n$ we have $d_i Q x = Q d_i x \in E_{n-1}$ when $i \in \{0, \dots, n-1\}$, while $d_n Q x = x \notin E_{n-1}$. Thus each inclusion $E_{n-1} \subseteq E_n$ is obtained by attaching the collection X^I_n of n-simplices along $\Lambda^n_n \subset \Delta^n$. □

54.8. **Proposition.** *Let X be a simplicial set and $x \in X_0$. Then the inclusion $\{1_x\} \to X_{/x}$ is right anodyne, and the inclusion $\{1_x\} \to X_{x/}$ is left anodyne.*

Proof. The functor $\Delta \to s\mathrm{Set}_*$ to pointed simplicial sets defined by $[n] \mapsto (\Delta^n)^{\triangleright}$ manifestly extends to a functor $\Delta^R \to s\mathrm{Set}_*$. From this we obtain a canonical contraction on $X_{/x}$. □

54.9. **Proposition.** *Suppose X is a contractible Kan complex. Then for any choice of vertex $* \in X_0$, the inclusion $\{*\} \to X$ admits a right contraction.*

Proof. We define a right contraction by inductively constructing contraction operators. Set $X_{-1} := \{*\}$, and let $Q\colon X_{-1} \to X_0$ denote the evident inclusion.

Suppose we have already defined contraction operators $Q\colon X_{k-1} \to X_k$ for $k < n$, which satisfy all the identities which make sense. That is, we have a presheaf of sets on the subcategory of Δ^R

generated by Δ and by the contraction operators $[k]_R \to [k-1]_R$ for $k < n$. We will construct $Q\colon X_{n-1} \to X_n$ so that the following hold.

(1) If $x = s_i y$, then $Qx = s_i Qy$.
(2) If $x = Qy$, then $Qx = s_{n-1} Qy$.
(3) We have that $d_n Qx = x$ and $d_i Qx = Q d_i x$ for $i \in \{0, \dots, n-1\}$.

Doing so precisely gives an extension of the given presheaf to the subcategory of Δ^R generated by Δ and contraction operators $[k]_R \to [k-1]_R$ for $k \le n$.

The idea is to use the above as a prescription for $Q\colon X_{n-1} \to X_n$: that is,

(1) if $x = s_i y$, set $Qx := s_i Qy$;
(2) if $x = Qy$, set $Qx := s_{n-1} Qy$;
(3) otherwise, choose any Qx such that $d_n Qx = x$ and $d_i Qx = Q d_i x$ for $i \in \{0, \dots, n-1\}$.

We need to check that this makes sense: that cases (1) and (2) agree when they overlap, and that a choice in (3) is always possible. We consider the various cases, for $x \in X_{n-1}$.

- Suppose $x = s_i y = s_j z$ for some $i, j \in \{0, \dots, n-2\}$. Thus, both y and z are common degeneracies of some non-degenerate simplex, i.e., $y = \sigma u$ and $z = \tau u$ for some $u \in X_d$ with $d \le n-3$, while $\sigma, \tau\colon [n-2] \to [d]$ are surjective operators with $\sigma s_i = \tau s_j$. Then $yQs_i = u\sigma Q s_i = uQ\sigma s_i = uQ\tau s_j = u\tau Q s_j = zQs_j$.
- Suppose $x = s_i y = Qz$ for some $i \in \{0, \dots, n-2\}$. Then there exists

That a choice in (3) exists follows from the fact that the collection of $Qd_0 x, \dots, Qd_{n-1}x, x$ fit together to give a map $\partial\Delta^n \to X$. $\square$

Fix a vertex k of Δ^n. We define the **excess** of a simplex $\delta\colon \Delta^m \to \Delta^n$ (relative to k) as follows.

- If some vertex of Δ^n other than k is *not* in the image of δ, the excess of δ is -1.
- Otherwise, the excess of δ is

$$|\{\, j \in (\Delta^m)_0 \mid \delta(j) \neq k \,\}| - n.$$

Note that the simplices of excess $= -1$ form the subcomplex Λ^n_k.

Given a map $p\colon B \to \Delta^n$, define the excess of a simplex $\Delta^m \to B$ (relative to k and p) to be the excess of $px\colon \Delta^m \to \Delta^n$. Observe that the collection of simplices B with excess $\le d$ form a subcomplex $E_d \subseteq B$, and that E_{-1} is the pullback of B over Λ^n_k.

54.10. **Lemma.** *If $k \in \{1, \dots, n\}$ and p is a left fibration, then each map $E_{d-1} \to E_d$ is right anodyne.*

Here is a formalism for "simplicial operators"[29]. Given a finite totally ordered set I, we can define $\Delta^I := N(I)$. We have that Δ^I is isomorphic to a *unique* standard simplex Δ^n (or to the empty simplicial set if I is empty) by a *unique* isomorphism. The **I-simplices** X_I of a simplicial set X are the set of maps $\Delta^I \to X$; we have a canonical identification $X_I \approx X_n$ where $|I| = n+1$.

We define the following maps between totally ordered sets.

- $d^a\colon I \smallsetminus \{a\} \to I$ for $a \in I$ is the evident inclusion.
- $s^a_{b,c}\colon I \to I \cup_{\{b,c\}} \{a\}$ for consecutive elements $b < c$ in I is a projection map; the target is the totally ordered set obtained by identifying b and c and relabelling the new element as "a" (which is assumed not to be an element of $I \smallsetminus \{b, c\}$, though we are allowed to consider $s^a_{a,b}$ or $s^b_{a,b}$).
- $s^a_b\colon I \to I \cup_{\{b\}} \{a\}$ is an isomorphism, where the target is obtained by removing b and replacing it with a.

The resulting operators on a simplicial set are $d_a\colon X_i \to X_{I \smallsetminus \{a\}}$ (face operators), $s_a^{b,c}\colon X_{I \cup_{\{b,c\}} \{a\}} \to X_I$ (degeneracy operators), and $s_a^b\colon X_{I \cup_{\{b\}} \{a\}} \to X_I$. Then we have the following simplicial identities:

[29] Adapted from the appendix to [Mos15].

- $d_a d_b = d_b d_a$,
- $s_a^{b,c} s_d^{e,f} = s_d^{e,f} s_a^{b,c}$ if a,b,c,d,e,f are pairwise distinct,
- $s_c^{d,c} s_a^{b,c} = s_b^{b,d} s_a^{b,c}$,
- $d_a s_b^{c,d} = s_b^{c,d} d_a$ if a,b,c,d pairwise distinct,
- $d_b s_a^{b,c} = s_a^c$, $d_c s_a^{b,c} = s_a^b$.

In other words, these identies say that simplicial operators generally commute, except when we compose $f s_a^{b,c}$ where f is an operator involving b or c.

Fix a map $p\colon B \to \Delta^n$ and a vertex k of Δ^n. For each simplex $u\colon \Delta^d \to B$ such that pu factors through the face $\Delta^{[n]\smallsetminus\{k\}} \subset \Delta^n$ opposite k, we obtain a coaugmented simplicial set $B(u)\colon (\Delta^+)^{\mathrm{op}} \to \mathrm{Set}$. Here $B(u)_j$ is the the set of fillers in

$$\begin{tikzcd}
\Delta^d \approx \Delta^{d_L} \star \varnothing \star \Delta^{d_R} \arrow[r, "u"] \arrow[d, tail] & B \arrow[d, "p"] \\
\Delta^{d_L} \star \Delta^j \star \Delta^{d_R} \arrow[r] \arrow[ur, dashed] & \Delta^{k-1} \star \{k\} \star \Delta^{n-k-1} \approx \Delta^n
\end{tikzcd}$$

Note that any simplicial operator $\delta\colon \Delta^{d'} \to \Delta^d$ induces a map $B(\delta)\colon B(u) \to B(u\delta)$ of coaugmented simplicial sets.

54.11. **Lemma.** *Suppose there exist, for each simplex $u\colon \Delta^d \to B$ such that the image of pu is equal to $\Delta^{[n]\smallsetminus k}$, a right contraction Q of $B(u)$, and that these contraction maps are compatible with degeneracy operators, in the sense that for surjective $\sigma\colon \Delta^{d'} \to \Delta^d$ we have $Q \circ B(\sigma) = B(\sigma) \circ Q$. Then $A \to B$ is right anodyne.*

Proof. Observe that the set of simplices of B in the complement of A is precisely the disjoint union of the $B(u)$, where u ranges over simplices such that pu surjects to the face opposite k.

Let $S_n := B_n^{\mathrm{nd}} \smallsetminus A_n^{\mathrm{nd}}$, the set of non-degenerate n-simplices in the complement of A. Let $S'_n =$ the set of such which are in the image of the degeneracy operators, and let S''_n be its complement in S_n. We claim that the Qs induce a bijection

$$\phi\colon S''_{n-1} \to S'_n.$$

To see this, we observe the following for simplices in the complement of A.

- If $Q(x)$ is a degenerate simplex, then x is also degenerate. Therefore Q induces a function $\phi\colon S''_{n-1} \to S'_n$.
- If x is a degenerate simplex, then so is $Q(x)$. Therefore ϕ is surjective.
- We have that $d_z(Q(x)) = x$, where $z = \sup V_x$. Therefore ϕ is injective.

Furthermore, we have for $x \in S''_{n-1}$ that

- For $a \in V_{Q(x)}$ but $a \neq \sup V_{Q(x)}$, we have that $d_a(Q(x)) = Q(d_a(x))$. Therefore these faces $d_a(Q(x))$ are either degenerate or are in the image of Q.
- For $a \in [n] \smallsetminus V_{Q(x)}$, we have that $d_a(Q(x))$ has strictly smaller excess than x and $Q(x)$.
- $z = \sup V_{Q(x)}$ is not equal to 0 (since $k > 0$.).

We can now construct B from A by attaching right horns. Let $E_{r,n} \subseteq E_r$ denote the smallest subcomplex of E_r containing S'_d for $d \leq n$; let $E_{r,-1} = E_{r-1}$. Then each inclusion $E_{r,n-1} \to E_{r,n}$ is obtained by attaching right horns of dimension n, according to the elements of S'_n which have excess r. $\square$

54.12. **Lemma.** *Suppose $B \to \Delta^n$ is a left fibration. Then contraction maps as in the previous lemma exist.*

Proof. For non-degenerate simplices in the complement of A, we define Q inductively, as follows.

(1) If $x = s_a^{b,c} y$, set $Q_z x := s_a^{b,c} Q_z y$.

(2) If $x = Q_w y$, set $Q_z x := s_w^{w,z} Q_w y$.
(3) Otherwise, choose $Q_z x$ to be any n-simplex such that $d_z Q_z x = x$ and $d_a Q_z x = Q_z d_a x$ for $a \in V_x$.

□

55. Cartesian fibrations

Let $p\colon C \to D$ a functor between ordinary categories. A morphism $f\colon x' \to x$ in C is called **p-Cartesian** if for every object c of C the evident commutative square

$$\begin{tikzcd} \mathrm{Hom}_C(c, x') \arrow[r, "f\circ"] \arrow[d, "p"'] & \mathrm{Hom}_C(c, x) \arrow[d, "p"] \\ \mathrm{Hom}_D(p(c), p(x')) \arrow[r, "p(f)\circ"'] & \mathrm{Hom}_D(p(c), p(x)) \end{tikzcd}$$

is a pullback square of sets.

Given an object $x \in \mathrm{ob}\, C$ and a morphism $g\colon y' \to p(x)$ in D, a **Cartesian lift of g at x** is a p-Cartesian morphism $f\colon x' \to x$ such that $p(f) = g$.

We say that $p\colon C \to D$ is a **Cartesian fibration** of categories if every pair $(x \in \mathrm{ob}\, C, g\colon y' \to p(x) \in D)$ admits a Cartesian lift.

Here are some observations, whose verification we leave to the reader. Fix a functor $p\colon C \to D$.

- Every isomorphism in C is p-Cartesian.
- Every Cartesian lift of an isomorphism in D is itself an isomorphism.
- If $f\colon x' \to x$ is p-Cartesian, then for any $g\colon x'' \to x'$ in C, we have that g is p-Cartesian if and only if gf is p-Cartesian.
- Any two Cartesian lifts of g at x are "canonically isomorphic".
  Explicitly, fix $g\colon y' \to y$ in D and an object x in C such that $y = p(x)$. If $f_1\colon x'_1 \to x$ and $f_2\colon x'_2 \to x$ are any two Cartesian lifts of g, then there exists a unique map $u\colon x'_1 \to x'_2$ such that $p(u) = 1_{y'}$ and $f_2 u = f_1$; the map u is necessarily an isomorphism.
- The map p is a right fibration if and only if it is a Cartesian fibration and every morphism in C is p-Cartesian.

Now suppose that $p\colon C \to D$ is a Cartesian fibration. For an object y of D, we write $C_y := p^{-1}(y)$ for the fiber of C over y.

- The map p is an isofibration.
- For each morphism $g\colon y' \to y$ in D and object x in C with $p(x) = y$, fix a choice of Cartesian lift $\widetilde{g}_x$ of g at x. Using this data, we obtain functors
  $$g^!\colon C_y \to C'_y$$
  so that for morphism $\alpha\colon x_1 \to x_2$ in C_y, the map $g^!(\alpha)$ in $C_{y'}$ is the unique one fitting into
  $$\begin{tikzcd} x'_1 \arrow[r, "\widetilde{g}_{x_1}"] \arrow[d, "g^!(\alpha)"'] & x_1 \arrow[d, "\alpha"] \\ x'_2 \arrow[r, "\widetilde{g}_{x_2}"'] & x_2 \end{tikzcd}$$
  The functor $g^!$ depends on the choices of Cartesian lifts of g. Any two set of choices of lifts give rise to isomorphic functors.
- For each pair of morphisms $y'' \xrightarrow{h} y' \xrightarrow{g} y$, we obtain a natural isomorphism of functors
  $$\gamma\colon h^! \circ g^! \overset{\sim}{\Rightarrow} (hg)^!\colon C_y \to C_{y''}.$$

This natural transformation is given by the unique maps γ_x in $C_{y''}$ fitting into

$$\begin{tikzcd} x'' \arrow[r, "\widetilde{h}_{x'}"] \arrow[d, "\gamma_x"'] & x' \arrow[d, "\widetilde{g}_x"] \\ x''' \arrow[r, "\widetilde{(gh)}_x"'] & x \end{tikzcd}$$

Similarly, there is a natural isomorphism $\mathrm{id} \overset{\sim}{\Rightarrow} (1_y)_! \colon C_y \to C_y$. The data of the functors $g^!$ together with these natural isomorphisms define a **pseudofunctor** $D^{\mathrm{op}} \to \mathrm{Cat}$, which on objects sends $y \mapsto C_y$.

- We can produce an actual functor $F \colon D^{\mathrm{op}} \to \mathrm{Cat}$ with $F(y)$ equivalent to C_y as follows. Given functors $p' \colon C' \to D$ and $p \colon C \to D$, let $\mathrm{Fun}_D(C', C)$ denote the category of fiberwise functors and natural transformations; i.e., the fiber of $p\circ \colon \mathrm{Fun}(C', C) \to \mathrm{Fun}(C', D)$ over q. Let $\mathrm{Fun}^+_D(C', C) \subseteq \mathrm{Fun}_D(C', C)$ denote the full subcategory of functors $f \colon C' \to C$ which take p'-Cartesian morphisms to p-Cartesian morphims.

  We obtain a functor $F \colon D^{\mathrm{op}} \to \mathrm{Cat}$, given on objects by
  $$F(y) := \mathrm{Fun}^+_D(D_{/y}, C).$$
  One can show that restriction to $\{1_y\} \subseteq D_{/y}$ defines an equivalence of categories $F(y) \to C_y$.
- Given D, there is a 2-category $\mathcal{F}_D$, whose objects are Cartesian fibrations $p \colon C \to D$; for any two objects $p \colon C \to D$ and $p' \colon C' \to D$ we take $\mathrm{Fun}^+_D(C', C)$ as the category of morphisms from p' to p. One can show that $\mathcal{F}_D$ is 2-equivalent to the 2-category $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat})$.

Appendices

56. Appendix: Generalized horns

A **generalized horn**[30] is a subcomplex $\Lambda^n_S \subset \Delta^n$ of the standard n-simplex, where $S \subseteq [n]$ and
$$(\Lambda^n_S)_k := \{\, f \colon [k] \to [n] \mid S \not\subseteq f([k]) \,\}.$$
In other words, a generalized horn is a union of some codimension 1 faces of the n-simplex:
$$\Lambda^n_S = \bigcup_{s \in S} \Delta^{[n] \smallsetminus s}.$$
In particular,
$$\Lambda^n_{[n]} = \partial\Delta^n, \quad \Lambda^n_{[n]\smallsetminus j} = \Lambda^n_j, \quad \Lambda^n_{\{j\}} = \Delta^{[n]\smallsetminus j}, \quad \Lambda^n_{\varnothing} = \varnothing.$$
In general $S \subseteq T$ implies $\Lambda^n_S \subseteq \Lambda^n_T$.

56.1. **Proposition** (Joyal [Joy08a, Prop. 2.12])**.** *Let $S \subsetneq [n]$ be a* proper *subset.*

(1) $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{Horn}}$ *if $S \neq \varnothing$.*
(2) $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{LHorn}}$ *if $n \in S$.*
(3) $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{RHorn}}$ *if $0 \in S$.*
(4) $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{InnHorn}}$ *if S is not an interval; i.e., if there exist $a < b < c$ with $a, c \in S$ and $b \notin S$.*

[30] This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our Λ^n_S is Joyal's $\Lambda^{[n]\smallsetminus S}$.

Proof. Consider $S \subsetneq [n]$ and $t \in [n] \smallsetminus S$. Observe the diagram

$$\begin{tikzcd} \Delta^{[n]\smallsetminus t} \cap \Lambda^n_S \arrow[r] \arrow[d] & \Delta^{[n]\smallsetminus t} \arrow[d] & \\ \Lambda^n_S \arrow[r] & \Lambda^n_{S\cup t} \arrow[r] & \Delta^n \end{tikzcd}$$

in which the square is a pushout, and the top arrow is isomorphic to the generalized horn $\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t}$. Thus, $(\Lambda^n_S \subset \Delta^n)$ is contained in the saturation of the set contaning the two inclusions

$$\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t} \quad \text{and} \quad \Lambda^n_{S\cup t} \subset \Delta^n.$$

Each of the statements of the proposition follows by an evident induction on the size of $[n] \smallsetminus S$. I'll do case (4). If $S \subset [n]$ is not an interval, there exists some $s < u < s'$ with $s, s' \in S$ and $u \notin S$. If $[n] \smallsetminus S = \{u\}$ then we already have an inner horn. If not, then choose $t \in [n] \smallsetminus (S \cup \{u\})$, in which case $S \cup t$ is not an interval in $[n]$, and S is not an interval in $[n] \smallsetminus t$. □

56.2. **Proposition** (Joyal [Joy08a, Prop. 2.13])**.** *For all $n \geq 2$, we have that $(I^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. We can factor the inclusion spine inclusion $h_n = g_n f_n$ as

$$I^n \xrightarrow{f_n} \Delta^{\{1,\dots,n\}} \cup I^n \xrightarrow{g_n} \Delta^n.$$

We show by induction on n that $f_n, g_n \in \overline{\text{InnHorn}}$, noting that the case $n = 2$ is immediate.

To show that $f_n \in \overline{\text{InnHorn}}$, consider the pushout square

$$\begin{tikzcd} I^{\{1,\dots,n\}} \arrow[r] \arrow[d] & \Delta^{\{1,\dots,n\}} \arrow[d] \\ I^n \arrow[r, "f_n"'] & \Delta^{\{1,\dots,n\}} \cup I^n \end{tikzcd}$$

in which the top arrow is equal to $h_{n-1} = g_{n-1} f_{n-1}$, which is in $\overline{\text{InnHorn}}$ by induction.

To show that $g_n \in \overline{\text{InnHorn}}$, consider the diagram

$$\begin{tikzcd} \Delta^{\{1,\dots,n-1\}} \cup I^{\{0\dots,n-1\}} \arrow[r, "g_{n-1}"] \arrow[d] & \Delta^{\{0,\dots,n-1\}} \arrow[d] & \\ \Delta^{\{1,\dots,n\}} \cup I^n \arrow[r] & \Delta^{\{1,\dots,n\}} \cup \Delta^{\{0,\dots,n-1\}} \arrow[r] & \Delta^n \end{tikzcd}$$

in which the square is a pushout, the top horizontal arrow is equal to g_{n-1}, an element of $\overline{\text{InnHorn}}$ by induction, and the bottom right horizontal arrow is equal to $\Lambda^n_{\{0,n\}} \subset \Delta^n$, which is in $\overline{\text{InnHorn}}$ by (56.1)(4).

□

57. Appendix: Prisms

57.1. **Lemma.** *For all $n \geq 0$ we have that $(\Lambda^2_1 \subset \Delta^2) \square (\partial\Delta^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. [Lur09, 2.3.2.1]. For each $0 \leq a \leq b \leq n$, let τ_{ab} be the $(n+2)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\tau_{ab} = \langle (0,0), \dots, (0,a),\ (1,a), \dots, (1,b),\ (2,b), \dots, (2,n) \rangle.$$

The set $\{\tau_{ab}\}$ consists of all the non-degenerate $(n+2)$-simplices.

For each $0 \leq a \leq b < n$, let σ_{ab} be the $(n+1)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\sigma_{ab} = \langle (0,0), \dots, (0,a),\ (1,a), \dots, (1,b),\ (2,b+1), \dots, (2,n) \rangle.$$

Note that σ_{ab} is a face of τ_{ab} and of $\tau_{a,b+1}$.

We attach simplices to $K := (\Lambda^2_1 \times \Delta^n) \cup (\Delta^2 \times \partial\Delta^n)$ in the following order:

$$\sigma_{00},\ \sigma_{01}, \sigma_{11},\ \sigma_{02}, \sigma_{12}, \sigma_{22},\ \dots\ \sigma_{0,n-1}, \dots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00},\ \tau_{01}, \tau_{11},\ \tau_{02}, \tau_{12}, \tau_{22},\ \dots\ \tau_{0,n}, \dots, \tau_{n,n}.$$

The claim is that each attachment is along an inner horn inclusion. The proof is a lengthy verification, which is left to the reader.

We note that each σ_{ab} gets attached along the horn at the vertex $(1, a)$ in σ_{ab}, i.e., via a $\Lambda^{n+1}_{a+1} \subset \Delta^{n+1}$ horn inclusion, which is always inner since $a \leq b < n$. Likewise, each τ_{ab} gets attached along the horn at vertex $(1, a)$ in τ_{ab}, i.e., via a $\Lambda^{n+2}_{a+1} \subset \Delta^{n+2}$ horn inclusion, which is always inner since $a \leq b \leq n$.

□

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