I'm using the Arxiv version 5 Feb 2013

1 Spectral Categories

1.1 Summary

1.1.1 Important definitions

Definition	Page Number
Spectral category	8
DK equivalence	9
Module over a spectral category	10
Triangulated closure	10
Triangulated equivalence	11
Morita equivalence	11

1.1.2 Notation

- Cat_S is the category of small spectral categories and spectral functors.
- Cat_T is the category of small simplicial categories and simplicial functors.

1.1.3 Key results

1. There's a Quillen adjunction
$$Cat_S \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} Cat_T$$
where $\Omega^\infty(F(A,B))$ is the zeroth space functor or equivalently the simplicial set $[n] \mapsto Hom_{Spt}(\mathbb{S} \otimes \Delta^n, F(A,B)) \cong Hom_{Spc}(\Delta^n, \Omega^\infty F(A,B)) \cong \Omega^\infty F(A,B)_n$. This is the main theorem of Tabuada's paper "Homotopy Theory of Spectral Categories" (see references). We can modify Cat_S up to weak equivalence to make this a simplicial Quillen adjunction.

2. There's a model category $\widehat{\mathcal{A}}$ of spectral $\mathcal{A}$-modules (these are defined as functors, so we put the projective model structure on the functor category).

1.2 Background Material

While the theory of infinity categories allows us to work "coordinate-free" in some sense, we first need some examples of meaningful infinity categories and functors between them. One way to get a slew of presentable ∞-categories is to take the underlying infinity category of a left proper, simplicial, combinatorial model category. We'll define the simplicial nerve after introducing combinatorial model categories.

1.2.1 Combinatorial Model Categories

Definition 1. An infinite cardinal κ is a *regular cardinal* if it satisfies the following property, which we think of as requiring the collection of sets of smaller cardinality to be "closed under union":

- no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ.

Example 2. $\aleph_0$, or the cardinality of $\mathbb{N}$, is a regular cardinal. This is because any set of cardinality less than $\aleph_0$ is finite, and no infinite set is a finite union of finite sets.

Example 3. Any successor cardinal is regular. For $\aleph_1$, this follows from the fact that a countable union of countable sets is countable (we need choice here).

Definition 4. Let κ be an infinite regular cardinal. Then a *κ-filtered category* is one such that any diagram $F : D \to C$, where D has fewer than κ morphisms admits an extension $\widetilde{F} : D^+ \to C$ (i.e. F has a cocone). Here D^+ is the category obtained by freely adjoining a terminal object to D.

Example 5. Let $\kappa = \omega$ (or $\aleph_0$ in the notation we've been using). Then we're requiring every FINITE diagram in C to have a cocone. This is equivalent to the usual definition of a filtered category: a nonempty category s.t. each pair of objects has a join, and for any parallel morphisms $f, g : c_1 \to c_2$ in C there exists a morphism $h : c_2 \to c_3$ such that $hf = hg$. In other words, we can build a cocone for any finite diagram from these cocones.

Example 6. A preorder (there exists a unique morphism between any two objects) is ω-filtered precisely when it is *directed*, i.e. any two objects have a join.

Definition 7. Let κ be a regular cardinal. Then an object X such that $C(X, -)$ commutes with κ-filtered colimits is called *κ-compact*.

Definition 8. An object X of a category is *small* if it is κ-compact for some regular cardinal κ.

Definition 9. A category C is *locally presentable* if

1. C is a locally small category
2. C has all small colimits
3. there exists a small set $S \hookrightarrow Obj(C)$ of λ-small objects that generates C under λ-filtered colimits.
4. every obect in C is a small object.

Definition 10. Let C be a category and $I \subset Mor(C)$. Let *cell*(I) be the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in I.

Definition 11. A model category C is *cofibrantly generated* if there are small sets of morphisms $I, J \subset Mor(C)$ such that

- $cof(I)$, the set of retracts of elements in *cell*(I), is precisely the collection of cofibrations of C.
- $cof(J)$ is precisely the collection of acyclic cofibrations in C; and
- I and J permit the small object argument.

Definition 12. (Smith) A model category C is *combinatorial* if it is

- locally presentable as a category, and
- cofibrantly generated as a model category.

Example 13. The category $sSet$ with the standand model structure on simplicial sets is a combinatorial model category.

Example 14. The category $sSet$ with the Joyal model structure (so that the quasi-categories are the fibrant objects) is combinatorial.

Question 1. Why should we care that a model category is combinatorial?

We'll define a left-proper model category, then give a fundamental result of Dugger's:

Definition 15. A model category is *left proper* if weak equivalence is preserved by pushout along cofibrations.

Example 16. A model category in which all objects are cofibrant is left proper. This includes the standard model structure on simplicial sets, as well the injective model structure on simplicial presheaves. This follows from the Reedy lemma, which allows us to calculate homotopy pushouts by considering diagrams s.t. the objects are cofibrant and one of the maps is a cofibration (the point is that replacing this diagram with a cofibrant diagram in the projective model structure on diagrams is an acyclic cofibration of diagrams, not just a weak equivalence).

Theorem 1. *(Dugger)*

Every combinatorial model category is Quillen equivalent to a left proper simplicial combinatorial model category.

Now, we have the following extremely useful result on Bousfield localizations:

Theorem 2. *If C is a left proper, simplicial, combinatorial model category, and $S \subset Mor(C)$ is a small set of morphisms, then the left Bousfield localization $L_S C$ does exist as a combinatorial model category. Moreover, the fibrant objects of $L_S C$ are precisely the S-local objects, and L_S is left proper and simplicial.*

In the context of infinity categories, we have the following crucial results of Lurie:

Theorem 3. *(HTT A.3.7.6) Let C be an ∞-category. The following conditions are equivalent:*

1. *The ∞-category C is presentable.*
2. *There exists a combinatorial simplicial model category A and an equivalence $C \cong N(A^\circ)$.*

Remark 17. (HTT A.3.7.7)

Let A and B be combinatorial simplicial model categories. Then the underlying ∞-categories $N(A^\circ)$ and $N(B^\circ)$ are equivalent iff A and B can be joined by a chain of simplicial Quillen equivalences.

Theorem 4. *(HTT 5.2.4.6) Let A and B be simplicial model categories, and let*

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

be a simplicial Quillen adjunction. This descends to an adjunction on the underlying ∞-categories.