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1 Spectral Categories

1.1 Summary

1.1.1 Important definitions

Definition	Page Number
Spectral category	8
DK equivalence	9
Module over a spectral category	10
Triangulated closure	10
Triangulated equivalence	11
Morita equivalence	11

1.1.2 Notation

- Cat_S is the category of small spectral categories and spectral functors.
- Cat_T is the category of small simplicial categories and simplicial functors.

1.1.3 Key results

1. There's a Quillen adjunction

$$Cat_S \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} Cat_T$$

where $\Omega^\infty(F(A,B))$ is the zeroth space functor or equivalently the simplicial set $[n] \mapsto Hom_{Spt}(\mathbb{S} \otimes \Delta^n, F(A,B)) \cong Hom_{Spc}(\Delta^n, \Omega^\infty F(A,B)) \cong \Omega^\infty F(A,B)_n$. This is the main theorem of Tabuada's paper "Homotopy Theory of Spectral Categories" (see references). We can modify Cat_S up to weak equivalence to make this a simplicial Quillen adjunction.

Remark 1. It might also be helpful to recall that, given a monoidal functor from a monoidal category M to a monoidal category N, any category enriched over M can be reinterpreted as a category enriched over N. Furthermore, we recover the underlying category of an enriched category by considering the functor $M(I, -) : M \to Set$ where I is the unit of the monoidal structure.

2. There's a model category $\widehat{\mathcal{A}}$ of spectral $\mathcal{A}$-modules (these are defined as functors, so we put the projective model structure on the functor category). The referenced paper here is Stable model categories are categories of modules by Schwede and Shipley.

1.2 Clarifying Remarks

- The definition of A-module is a generalization of the ordinary one. Recall that a ring is equivalently a preadditive category (an Ab-enriched category) with one object, and a module is a functor from this ring to abelian groups. Along the same vein, a ring spectrum is a spectral category with one object, and a module over this ring spectrum is a functor from this category to spectra.
- A category of enriched functors is itself enriched. Recall that $Nat(F,G) = \int_C Hom(F-, G-)$. Via the theory of enriched ends we can define a mapping spectrum to be $\int_C A(F-, G-)$, where A denotes the mapping spectrum of two objects in our enriched category.
- Given a functor $F : \mathcal{A} \to \mathcal{B}$, we get an obvious restriction functor $F^* : modB := \widehat{B} \to \widehat{A}$ given by precomposition with F (recall the definition of a module as a functor!). It has a left adjoint functor $F_!$ given by an enriched coend. It sends an A-module N to the coequalizer of

$$\bigvee_{o,p \in A} N(p) \wedge A(o,p) \wedge B(-, F(o)) \rightrightarrows \bigvee_{o \in A} N(o) \wedge B(-, F(o)) .$$

Recall that representable objects are "free on one generator". Indeed, the usual definition of a free object is that maps out of it are determined by maps out of a basis. We know, by the Yoneda lemma, that maps out of representable objects are determined by maps out of the identity morphism.

- The paper omits the definition of a pretriangulated spectral category. It's in the Mandell-Blumberg paper in the references

1.3 Background Material

While the theory of infinity categories allows us to work "coordinate-free" in some sense, we first need some examples of meaningful infinity categories and functors between them. One way to get a slew of presentable ∞-categories is to take the underlying infinity category of a left proper, simplicial, combinatorial model category. We'll define the simplicial nerve after introducing combinatorial model categories.

1.3.1 Combinatorial Model Categories

Definition 2. An infinite cardinal κ is a *regular cardinal* if it satisfies the following property, which we think of as requiring the collection of sets of smaller cardinality to be "closed under union":

- no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ.

Example 3. $\aleph_0$, or the cardinality of $\mathbb{N}$, is a regular cardinal. This is because any set of cardinality less than $\aleph_0$ is finite, and no infinite set is a finite union of finite sets.

Example 4. Any successor cardinal is regular. For $\aleph_1$, this follows from the fact that a countable union of countable sets is countable (we need choice here).

Definition 5. Let κ be an infinite regular cardinal. Then a *κ-filtered category* is one such that any diagram $F : D \to C$, where D has fewer than κ morphisms admits an extension $\widetilde{F} : D^+ \to C$ (i.e. F has a cocone). Here D^+ is the category obtained by freely adjoining a terminal object to D.

Example 6. Let $\kappa = \omega$ (or $\aleph_0$ in the notation we've been using). Then we're requiring every FINITE diagram in C to have a cocone. This is equivalent to the usual definition of a filtered category: a nonempty category s.t. each pair of objects has a join, and for any parallel morphisms $f, g : c_1 \to c_2$ in C there exists a morphism $h : c_2 \to c_3$ such that $hf = hg$. In other words, we can build a cocone for any finite diagram from these cocones.

Example 7. A preorder (there exists a unique morphism between any two objects) is ω-filtered precisely when it is *directed*, i.e. any two objects have a join.

Definition 8. Let κ be a regular cardinal. Then an object X such that $C(X, -)$ commutes with κ-filtered colimits is called *κ-compact*.

Definition 9. An object X of a category is *small* if it is κ-compact for some regular cardinal κ.

Definition 10. A category C is *locally presentable* if

1. C is a locally small category
2. C has all small colimits
3. there exists a small set $S \hookrightarrow Obj(C)$ of λ-small objects that generates C under λ-filtered colimits.
4. every obect in C is a small object.

Definition 11. Let C be a category and $I \subset Mor(C)$. Let *cell*(I) be the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in I.

Definition 12. A model category C is *cofibrantly generated* if there are small sets of morphisms $I, J \subset Mor(C)$ such that

- $cof(I)$, the set of retracts of elements in *cell*(I), is precisely the collection of cofibrations of C.

- $cof(J)$ is precisely the collection of acyclic cofibrations in C; and
- I and J permit the small object argument.

Definition 13. (Smith) A model category C is *combinatorial* if it is

- locally presentable as a category, and
- cofibrantly generated as a model category.

Example 14. The category $sSet$ with the standand model structure on simplicial sets is a combinatorial model category.

Example 15. The category $sSet$ with the Joyal model structure (so that the quasi-categories are the fibrant objects) is combinatorial.

Question 1. Why should we care that a model category is combinatorial?

We'll define a left-proper model category, then give a fundamental result of Dugger's:

Definition 16. A model category is *left proper* if weak equivalence is preserved by pushout along cofibrations.

Example 17. A model category in which all objects are cofibrant is left proper. This includes the standard model structure on simplicial sets, as well the injective model structure on simplicial presheaves. This follows from the Reedy lemma, which allows us to calculate homotopy pushouts by considering diagrams s.t. the objects are cofibrant and one of the maps is a cofibration (the point is that replacing this diagram with a cofibrant diagram in the projective model structure on diagrams is an acyclic cofibration of diagrams, not just a weak equivalence).

Theorem 1. *(Dugger)*
Every combinatorial model category is Quillen equivalent to a left proper simplicial combinatorial model category.

Now, we have the following extremely useful result on Bousfield localizations:

Theorem 2. *If C is a left proper, simplicial, combinatorial model category, and $S \subset Mor(C)$ is a small set of morphisms, then the left Bousfield localization $L_S C$ does exist as a combinatorial model category. Moreover, the fibrant objects of $L_S C$ are precisely the S-local objects, and L_S is left proper and simplicial.*

In the context of infinity categories, we have some crucial results of Lurie. For these to make sense, however, we need to introduce the simplicial nerve construction. This is the generalization of the ordinary nerve construction to simplicially enriched categories.

Definition 18. We'll define a cosimplicial simplcially enriched category S. The objects of $S[n]$ are $\{0, 1, \ldots, n\}$; the hom objects $S[n]_{i,j} \in sSet$ for $i, j \in \{0, 1, \ldots, n\}$ are the nerves

$$S[n](i, j) = N(P_{i,j})$$

of the poset $P_{i,j}$ which is the poset of subsets of $[i, j]$ that contain both i and j with partial order given by inclusion.

Definition 19. The simplicial nerve of a simplicial category is the simplicial set characterized by

$$Hom_{sSet}(\Delta[n], N(C)) = Hom_{SSetCat}(S[n], C).$$

Theorem 3. *(HTT A.3.7.6) Let C be an ∞-category. The following conditions are equivalent:*

1. *The ∞-category C is presentable.*
2. *There exists a combinatorial simplicial model category A and an equivalence $C \cong N(A^\circ)$.*

Here A° is the underlying category of bifibrant objects.

Remark 20. (HTT A.3.7.7)

Let A and B be combinatorial simplicial model categories. Then the underlying ∞-categories $N(A^\circ)$ and $N(B^\circ)$ are equivalent iff A and B can be joined by a chain of simplicial Quillen equivalences.

Theorem 4. *(HTT 5.2.4.6) Let A and B be simplicial model categories, and let*

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

be a simplicial Quillen adjunction. This descends to an adjunction on the underlying ∞-categories.