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1 Spectral Categories

1.1 Summary

1.1.1 Important definitions

Term	Page Number	Loose Definition
Spectral category	8	Category enriched over the category of symmetric spectra.
DK equivalence	9	Functor which induces stable equivalence of mapping spectra + equivalence of "homotopy" categories.
Module over a spectral category	10	A spectral functor $A^{op} \to SymmSpt$ for a spectral category A.
Triangulated closure	10	Yoneda embed $A \hookrightarrow \widehat{A}^{cf}$ and take finite cell objects (pushouts of coproducts).
Thick closure	11	Same as above but take retracts of fintie cell objects.
Triangulated equivalence	11	Functor which induces DK equivalence of triangulated closures.
Morita equivalence of Spectral Categories	11	Functor which induces DK equivalence of thick closures.

1.1.2 Notation

- Cat_S is the category of small spectral categories and spectral functors.
- Cat_T is the category of small simplicial categories and simplicial functors.

1.1.3 Key results

1. There's a Quillen adjunction
$$Cat_S \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} Cat_T$$
where $\Omega^\infty(F(A,B))$ is the zeroth space functor or equivalently the simplicial set $[n] \mapsto Hom_{Spt}(\mathbb{S} \otimes \Delta^n, F(A,B)) \cong Hom_{Spc}(\Delta^n, \Omega^\infty F(A,B)) \cong \Omega^\infty F(A,B)_n$. This is the main theorem of Tabuada's paper "Homotopy Theory of Spectral Categories" (see references). We can modify Cat_S up to weak equivalence to make this a simplicial Quillen adjunction.

   Remark 1. It might also be helpful to recall that, given a monoidal functor from a monoidal category M to a monoidal category N, any category enriched over M can be reinterpreted as a category enriched over N. Furthermore, we recover the underlying category of an enriched category by considering the functor $M(I, -) : M \to Set$ where I is the unit of the monoidal structure.

2. There's a model category $\widehat{\mathcal{A}}$ of spectral $\mathcal{A}$-modules (these are defined as functors, so we put the projective model structure on the functor category). The referenced paper here is Stable model categories are categories of modules by Schwede and Shipley.

1.2 Clarifying Remarks

- The definition of A-module is a generalization of the ordinary one. Recall that a ring is equivalently a preadditive category (an Ab-enriched category) with one object, and a module is a functor from this ring to abelian groups. Along the same vein, a ring spectrum is a spectral category with one object, and a module over this ring spectrum is a functor from this category to spectra.

- A category of enriched functors is itself enriched. Recall that $Nat(F,G) = \int_C Hom(F-, G-)$. Via the theory of enriched ends we can define a mapping spectrum to be $\int_C A(F-, G-)$, where A denotes the mapping spectrum of two objects in our enriched category.

- Given a functor $F : \mathcal{A} \to \mathcal{B}$, we get an obvious restriction functor $F^* : modB := \widehat{B} \to \widehat{A}$ given by precomposition with F (recall the definition of a module as a functor!). It has a left adjoint functor $F_!$ given by an enriched coend. It sends an A-module N to the coequalizer of

$$\bigvee_{o,p\in A} N(p) \wedge A(o,p) \wedge B(-, F(o)) \rightrightarrows \bigvee_{o\in A} N(o) \wedge B(-, F(o)) \; .$$

  Recall that representable objects are "free on one generator". Indeed, the usual definition of a free object is that maps out of it are determined by maps out of a basis. We know, by the Yoneda lemma, that maps out of representable objects are determined by maps out of the identity morphism.

- The paper omits the definition of a pretriangulated spectral category. It's in the Mandell-Blumberg paper in the references, definition 5.4.

1.3 Background Material

While the theory of infinity categories allows us to work "coordinate-free" in some sense, we first need some examples of meaningful infinity categories and functors between them. One way to get a slew of presentable ∞-categories is to take the underlying infinity category of a left proper, simplicial, combinatorial model category. We'll define the simplicial nerve after introducing combinatorial model categories.

1.3.1 Combinatorial Model Categories

Definition 2. An infinite cardinal κ is a *regular cardinal* if it satisfies the following property, which we think of as requiring the collection of sets of smaller cardinality to be "closed under union":

- no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ.

Example 3. $\aleph_0$, or the cardinality of $\mathbb{N}$, is a regular cardinal. This is because any set of cardinality less than $\aleph_0$ is finite, and no infinite set is a finite union of finite sets.

Example 4. Any successor cardinal is regular. For $\aleph_1$, this follows from the fact that a countable union of countable sets is countable (we need choice here).

Definition 5. Let κ be an infinite regular cardinal. Then a *κ-filtered category* is one such that any diagram $F : D \to C$, where D has fewer than κ morphisms admits an extension $\widetilde{F} : D^+ \to C$ (i.e. F has a cocone). Here D^+ is the category obtained by freely adjoining a terminal object to D.

Example 6. Let $\kappa = \omega$ (or $\aleph_0$ in the notation we've been using). Then we're requiring every FINITE diagram in C to have a cocone. This is equivalent to the usual definition of a filtered category: a nonempty category s.t. each pair of objects has a join, and for any parallel morphisms $f, g : c_1 \to c_2$ in C there exists a morphism $h : c_2 \to c_3$ such that $hf = hg$. In other words, we can build a cocone for any finite diagram from these cocones.

Example 7. A preorder (there exists a unique morphism between any two objects) is ω-filtered precisely when it is *directed*, i.e. any two objects have a join.

Definition 8. Let κ be a regular cardinal. Then an object X such that $C(X, -)$ commutes with κ-filtered colimits is called *κ-compact*.

Definition 9. An object X of a category is *small* if it is κ-compact for some regular cardinal κ.

Definition 10. A category C is *locally presentable* if

1. C is a locally small category

2. C has all small colimits

3. there exists a small set $S \hookrightarrow Obj(C)$ of λ-small objects that generates C under λ-filtered colimits.

4. every obect in C is a small object.

Definition 11. Let C be a category and $I \subset Mor(C)$. Let *cell*(I) be the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in I.

Definition 12. A model category C is *cofibrantly generated* if there are small sets of morphisms $I, J \subset Mor(C)$ such that

- $cof(I)$, the set of retracts of elements in *cell*(I), is precisely the collection of cofibrations of C.
- $cof(J)$ is precisely the collection of acyclic cofibrations in C; and
- I and J permit the small object argument.

Definition 13. (Smith) A model category C is *combinatorial* if it is

- locally presentable as a category, and
- cofibrantly generated as a model category.

Example 14. The category $sSet$ with the standand model structure on simplicial sets is a combinatorial model category.

Example 15. The category $sSet$ with the Joyal model structure (so that the quasi-categories are the fibrant objects) is combinatorial.

Question 1. Why should we care that a model category is combinatorial?

We'll define a left-proper model category, then give a fundamental result of Dugger's:

Definition 16. A model category is *left proper* if weak equivalence is preserved by pushout along cofibrations.

Example 17. A model category in which all objects are cofibrant is left proper. This includes the standard model structure on simplicial sets, as well the injective model structure on simplicial presheaves. This follows from the Reedy lemma, which allows us to calculate homotopy pushouts by considering diagrams s.t. the objects are cofibrant and one of the maps is a cofibration (the point is that replacing this diagram with a cofibrant diagram in the projective model structure on diagrams is an acyclic cofibration of diagrams, not just a weak equivalence).

Theorem 1. *(Dugger)*
Every combinatorial model category is Quillen equivalent to a left proper simplicial combinatorial model category.

Now, we have the following extremely useful result on Bousfield localizations:

Theorem 2. *If C is a left proper, simplicial, combinatorial model category, and $S \subset Mor(C)$ is a small set of morphisms, then the left Bousfield localization $L_S C$ does exist as a combinatorial model category. Moreover, the fibrant objects of $L_S C$ are precisely the S-local objects, and L_S is left proper and simplicial.*

In the context of infinity categories, we have some crucial results of Lurie. For these to make sense, however, we need to introduce the simplicial nerve construction. This is the generalization of the ordinary nerve construction to simplicially enriched categories.

Definition 18. We'll define a cosimplicial simplcially enriched category S. The objects of $S[n]$ are $\{0, 1, \ldots, n\}$; the hom objects $S[n]_{i,j} \in sSet$ for $i, j \in \{0, 1, \ldots, n\}$ are the nerves

$$S[n](i, j) = N(P_{i,j})$$

of the poset $P_{i,j}$ which is the poset of subsets of $[i, j]$ that contain both i and j with partial order given by inclusion.

Definition 19. The simplicial nerve of a simplicial category is the simplicial set characterized by

$$Hom_{sSet}(\Delta[n], N(C)) = Hom_{SSetCat}(S[n], C).$$

Theorem 3. *(HTT A.3.7.6) Let C be an ∞-category. The following conditions are equivalent:*

1. *The ∞-category C is presentable.*
2. *There exists a combinatorial simplicial model category A and an equivalence $C \cong N(A^\circ)$.*

Here A° is the underlying category of bifibrant objects.

Remark 20. (HTT A.3.7.7)
Let A and B be combinatorial simplicial model categories. Then the underlying ∞-categories $N(A^\circ)$ and $N(B^\circ)$ are equivalent iff A and B can be joined by a chain of simplicial Quillen equivalences.

Theorem 4. *(HTT 5.2.4.6) Let A and B be simplicial model categories, and let*

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

be a simplicial Quillen adjunction. This descends to an adjunction on the underlying ∞-categories.

2 Stable ∞-categories

2.1 Summary

2.1.1 Important definitions

Term	Page Number	Loose Definition
Stable ∞ category	14	∞-cat with finite (co)lims s.t. hopushouts coincide with hopullbacks.
Idempotent complete	14	Image under Yoneda embedding $C \to Fun(C^{op}, Gpd_\infty)$ is closed under retracts.
Morita equivalence of stab ∞-cats	14	Small stable ∞-cats A, B are ME if Idem(A) equiv to Idem(B).
Spectrum Object in Pted ∞-cat	15	A functor $F : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ s.t. certain diagrams are cartesian.
Spectral Yoneda embedding for stab ∞-cats	15	$C \approx Sp(C_*) \to Sp(Fun(C^{op}, Gpd_\infty)_*) \approx Fun(C^{op}, Sp(Gpd_\infty))$
Stably representable functor	16	Functor $C \to Spt$ equivalent to $Map(-, A)$ for a spectrum object A, where Map denotes mapping spectrum.
Accessible ∞-category	17	∞-cat equivalent to Ind_κ of a small ∞-cat.
Presentable ∞-category	17	∞ cat generated under sufficiently large filtered colimits by some small ∞-cat.
Localization of ∞-categories	18	∞-cat $C[S^{-1}]$ equipped with a map $C \to C[S^{-1}]$ which is universal for inverting elements of S.
Bousfield localization of presentable ∞-cat	19	Colimit preserving functor whose right adjoint is fully faithful.

2.1.2 Notation

- Cat_∞ is the ∞-category of small ∞ categories.
- Cat_∞^{ex} is the ∞-category of small stable ∞-categories and exact functors.
- Cat_∞^{perf} is the ∞-category of small idempotent complete stable ∞-categories (and exact functors).

2.1.3 Key results

1. Given a pretriangulated spectral category $\mathcal{C}$, the ∞-category $N((Mod(\mathcal{C}))^{cf})$ is stable.
2. If C is a pretriangulated spectral category, then $\pi_0 C$ admits a triangulated category structure.
3. If C is a stable ∞-category, then hoC admits a triangulated category structure.
4. The inclusion $Cat_\infty^{perf} \to Cat_\infty^{ex}$ has a left adjoint called idempotent completion.
5. If C is a ∞-category with finite limits, there is a stable ∞-category $Stab(C)$ with a limit preserving functor $\Omega^\infty : Stab(C) \to C$. In particular, this is accomplished by taking $Stab(C) = Sp(C)$, the category of spectrum objects of C. If C is already stable, then Ω^∞ is an equivalence, with inverse $\Sigma^\infty : C \to Sp(C)$.
6. If C is a stable ∞-category, there is a *spectral Yoneda embedding*
$$Y : C \simeq Sp(C_*) \to Sp(Fun(C^{op}, N(T)^{cf})_*) \simeq Fun(C^{op}, S_\infty)$$
with an adjoint *mapping spectrum functor*
$$Map : C^{op} \times C \to S_\infty$$
7. If C is a stable ∞-category, a $C^{op} \to S_\infty$ is stably representable if and only if it is represented by the suspension spectrum $\Sigma^\infty z$ of some $z \in C$, and this object is unique upto equivalence.

2.2 Clarifying Remarks

1. Idempotent completion is unique upto equivalence by HTT Proposition 5.1.4.9. Existence is obtained by taking the full subcategory of $Pre(C)$ spanned by objects that are retracts of objects in the image of C under the Yoneda embedding.
2. For this paper we assume stable ∞-categories to be pointed, i.e. they have zero objects. In the description of spectral Yoneda embedding above, $*$ indicates the subcategory of pointed objects.

2.3 Background Material

2.3.1 Pretriangulated Spectral Categories

The material here is from Section 5 of Mandell-Blumberg (in the higher K-theory paper, they say Section 4 but that's either a typo or just not updated).

Definition 21. A spectral category C is *pretriangulated* if

1. There exists an object 0 of C such that $C(-, 0)$ is homotopically trivial. This means that it is weakly equivalent to the constant functor $*$ at the one point symmetric spectrum.
2. Whenever a C-module M has the property that ΣM is weakly equivalent to a representable C-module $C(-, c)$, then M is weakly equivalent to some representable C-module $C(-, d)$.
3. Whenever the C-modules M and N are weakly equivalent to representables $C(-, a)$ and $C(-, b)$ respectively, the homotopy cofiber of any map of C-modules $M \to N$ is weakly equivalent to a representable C-module.

Remark 22. The first condition says that the homotopy category $\pi_0 C$ has a zero object. The second condition gives a desuspension functor on $\pi_0 C$ and the third condition gives a suspension functor on $\pi_0 C$.

Definition 23. A spectral functor between spectral categories $F : C \to D$ is a *DK-embedding* if for all objects a, b in C, the induced map of spectra $C(a,b) \to D(Fa, Fb)$ is a weak equivalence.

Remark 24. In Blumberg-Mandell, a DK equivalence is a DK embedding satisfying one of the following equivalent conditions

1. For all object d of D, there is an object c of C such that $D(-,d)$ and $D(-,Fc)$ are naturally isomorphic as D-modules.

2. The induced functor on the "graded homotopy categorie" $\pi_* C \to \pi_* D$ is an equivalence of categories.

In Blumberg-Gepner-Tabuada, this second condition is relaxed to $\pi_0 C \to \pi_0 D$ being an equivalence of categories. This is because a spectral functor between pretriangulated spectral categories is a DK equivalence if and only if it is a DK embedding and the induced functor on π_0 is an equivalence of categories.

Theorem 5. *(Blumberg-Mandell, Theorem 5.5) Any small spectral category C DK-embeds into a small pretriangulated spectral category.*

Remark 25. We should think of this as taking the closure of C under cofibration sequences and desuspensions in C-modules, via the Yoneda embedding. Note that this can be made functorial and it gives the "minimal pretriangulated closure" of C.

Definition 26. A *four-term Puppe sequence* in $\pi_0 C$ is a sequence of the form

$$a \to b \to c \to \Sigma a$$

if there exist a map of C-modules $f : M \to N$ such that the sequence

$$M \to N \to Cf \to \Sigma M$$

is isomorphic to the above sequence in the derived category of C-modules via the Yoneda embedding, and further the equvalence $\Sigma M \simeq C(-,\Sigma a) \simeq \Sigma C(-,a)$ is the suspension of the isomorphism $M \simeq C(-,a)$.

Theorem 6. *(Blumberg-Mandell, Theorem 5.6) Given a pretriangulated spectral category C, its homotopy category $\pi_0 C$ is triangulated with distinguished triangles the above four-term Puppe sequences.*

Proof. Proof of Theorem 5.6 is just observing that $\pi_0 C$ embeds as a full subcategory of the homotopy category of C-modules (with projective model structure) and checking that it is closed under suspensions, desuspensions and distinguished triangles. □

2.3.2 ∞-Categories

Definition 27. The **Joyal model structure** on simplicial sets is defined as follows :

- Cofibrations are levelwise monomorphisms.
- Weak equivalences are *(weak) categorical equivalence*. These are maps $f : A \to B$ such that for any ∞-category X, the map $X^B \to X^A$ induces an isomorphism on the fundamental category (the homotopy category) hoC.
- Fibrations are determined by the above.

Remark 28. All objects are cofibrant, and the fibrant objects are precisely ∞-categories.

Theorem 7. *(HA Theorem 1.1.2.15) Let C be a stable ∞-category. Then hoC has a triangulated category structure with distinguished triangles coming from cofiber sequences.*

Remark 29. There is a correspondance between pretriangulated spectral categories and stable ∞-categories. For example, if C is a pretriangulated spectral category, then $N((ModC)^{cf})$ is a stable ∞-category.

2.3.3 Idempotent completion

HTT 4.4.5 gives a good overview on the definition of idempotent completion for ∞-categories, while comparing it with the classical notion.

Remark 30. The notion of retracts between classical and ∞-categorical settings are a bit different. An ordinary category X is said to be *idempotent complete* if every idempotent map $X \to X$ comes from some retract Y of X. In such a situation Y can be determined uniquely as an equalizer (or a coequalizer). Hence, if C has finite limits or finite colimits, then C is idempotent complete.

This is not the case for ∞-categories. Consider the category $C_*(R)$ consisting of bounded chain complex of finite rank free R-modules and consider $N(C_*(R))$, which is actuallt a stable ∞-category. Hence it admits finite limits and colimits, but it is idempotent complete if and only if every finitely generated projective R-module is stably free.

The problem is that an idempotent in an ∞-category shouldn't be just a morphism e with $e \circ e \simeq e$ in hoC. It should specify homotopies on how to relate multiple compositions $e \circ e \circ ... \circ e \simeq e$. To achieve this, in HTT 4.4.5, simplicial sets called $Idem^+, Idem$ and Ret are introduced. Now idempotents, weak retractions, strong retractions in C are respectively defined to be maps of simplicial sets from $Idem, Ret, Idem^+$ to C. C is *idempotent complete* if every idempotent $F : Idem \to C$ has a colimit. This can be shown to be equivalent to the definition in the paper using results in HTT 5.1.4 and 5.1.5.

3 Symmetric Monoidal Structures and Dualizable Objects

3.1 Summary

3.1.1 Important definitions

Term	pg	Loose Definition
Tensor Product for $\mathrm{Cat}_\infty^{\mathrm{Perf}}$	20	If $\mathcal{A}$ and $\mathcal{B}$ are small stable idempotent-complete ∞-categories, we define $\mathcal{A}\widehat{\otimes}\mathcal{B} = (\mathrm{Ind}(\mathcal{A}) \otimes \mathrm{Ind}(\mathcal{B}))^\omega$.
Right-Compact Object	21	An object of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}\widehat{\otimes}\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$ is right-compact if putting in some object $a \in \mathcal{A}$ in the left argument always yields a compact object of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$.
Proper Stable ∞-Category	22	Mapping spectra are compact.
Smooth Stable ∞-Category	22	Perfect as a bimodule over itself. If $\mathcal{A}$ is idempotent-complete, then it is smooth if and only if it is a representable $\mathcal{A}^{\mathrm{op}}\widehat{\otimes}\mathcal{A}$-module. ("Coherent"?)
Dualiz-able	22	An object of a symmetric monoidal ∞-category is dualizable if it is dualizable in the homotopy category.

3.1.2 Notation

- The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Perf}}$ admits a symmetric monoidal structure. We write the tensor product as $\widehat{\otimes}$ to distinguish it from the usual tensor product of presentable stable ∞-categories.

3.1.3 Key results

1. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Perf}}$ admits the structure of a closed symmetric monoidal ∞-category with tensor product given by $\widehat{\otimes}$. The unit is the ∞-category $\mathcal{S}_\infty^\omega$ of compact spectra. Internal Hom is given by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$.

2. For any small stable ∞-category $\mathcal{A}$, the stable Yoneda embedding $\mathcal{A} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{op}}, \mathcal{S}_\infty)$ induces an equivalence $\mathrm{Ind}(A) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{op}}, \mathcal{S}_\infty)$. In particular, one can model the idempotent-completion of $\mathcal{A}$ by the Yoneda embedding $\mathcal{A} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{op}}, \mathcal{S}_\infty)^\omega$.

3. Let $\mathcal{A}$ and $\mathcal{B}$ be small stable idempotent-complete ∞-categories. The functor category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$ can be identified as a full subcategory of the ∞-category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}\widehat{\otimes}\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$ of $\mathcal{A}^{\mathrm{op}}\widehat{\otimes}\mathcal{B}$-modules. In fact, it is the full subcategory spanned by right-compact $\mathcal{A}^{\mathrm{op}}\widehat{\otimes}\mathcal{B}$-modules. (Note, I believe there is a small typo in the second-to-last paragraph of page 21, where it says "…certain subcategory of $\mathrm{Fun}^{\mathrm{L}}(\mathcal{A}\widehat{\otimes}\mathcal{B}^{\mathrm{op}}, \mathcal{S}_\infty)$"; superscript should be ex, not L.)

4. An object $\mathcal{A}$ of $\mathrm{Cat}_\infty^{\mathrm{Perf}}$ is dualizable (with respect to the symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{Perf}}$) if and only if $\mathcal{A}$ is smooth and proper. Moreover, if $\mathcal{A}$ is dualizable, its dual is given by $\mathcal{A}^{\mathrm{op}}$.

3.2 Clarifying Remarks

1. This chapter assumes the monoidal structure on the ∞-category of presentable, stable ∞ categories, then defines some other monoidal products. The definition of the monoidal structure $\otimes$ on $\mathcal{P}r^L_{St}$ is involved. In fact, it must be involved; the unit of this monoidal structure is the category Sp of spectra. Hence, defining this monoidal structure gives us, in particular, a smash product of spectra.

2. Key result (2) above is one step in showing that the definition of Morita equivalence via Idempotent completion matches the definition in terms of module categories.

3. There's a slight difference in notation: BGT denote a symmetric monoidal infinity category by $\mathcal{C}^\infty$, whereas below we define a symmetric monoidal infinity category asa coCartesian fibration $\mathcal{C}^\infty \to \mathscr{F}\mathrm{in}_*$.

4. The terminology "smooth" and "proper" here comes from *dg*-categories (recall that this paper is a translation of Tabuada's work into the language of ∞-categories). In particular, a smooth proper *dg* category is one which closely resembles the category of perfect complexes on a smooth proper scheme, and these *dg* categories can be characterized as the dualizable objects in some category of *dg* categories.

3.3 Background Material

3.3.1 Symmetric Monoidal ∞-Categories

The following can be found in chapter 2 of HA. In particular, the introduction to this chapter is a very clear introduction to the main idea.

In ordinary category theory, a symmetric monoidal structure on a category $\mathcal{C}$ is usually described by a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an identity object $1 \in \mathcal{C}$ and natural isomorphisms describing associativity, commutativity, and unitality. In this setting, one demands that these natural isomorphisms satisfy coherence conditions. If one tries to do something analogous in the setting of ∞-categories, one will quickly find that higher and higher coherence conditions must be imposed, to the point where this is prohibitively complicated. We will thus need to try something.

Another way to describe a symmetric monoidal structure on an ordinary category $\mathcal{C}$ is by a Grothendieck opfibration. Let $\mathscr{F}\mathrm{in}_*$ denote the category consisting of objects $\langle n \rangle = \{1, \ldots, n\} \sqcup \{*\}$ $(n \geq 0)$ and functions between these sets that preserve $*$. By abuse, we will identify this category with the category of pointed finite sets. Some useful morphisms in $\mathscr{F}\mathrm{in}_*$ are the functions $\rho_i : \langle n \rangle \to \langle 1 \rangle$ given by

$$\rho_i(j) = \begin{cases} 1 & \text{if } j = i \\ * & \text{otherwise.} \end{cases}$$

Given a symmetric monoidal category $\mathcal{C}$, we can form a category $\mathcal{C}^\otimes$ in which

- objects are finite (possibly empty) sequences of objects of $\mathcal{C}$, denoted by $[C_1, \ldots, C_n]$
- A morphism $f : [C_1, \ldots, C_n] \to [C'_1, \ldots, C'_m]$ consists of a subset $S \subseteq \{1, \ldots, n\}$, a map $\alpha : S \to \{1, \ldots, m\}$, and a collection of morphisms $\{f_j : \bigotimes_{\alpha(i)=j} C_i \to C'_j\}_{1 \leq j \leq m}$, and
- composition is defined in the only sensible way. See the introduction of chapter 2 of HA.

We get a forgetful functor $\mathcal{C}^\otimes \to \mathscr{F}\mathrm{in}_*$. In fact, this functor is a Grothendieck opfibration. We can identify $\mathcal{C}$ with the fibre $\mathcal{C}^\otimes_{\langle 1 \rangle}$ over $\langle 1 \rangle \in \mathscr{F}\mathrm{in}_*$. It has the special feature that the functors $\mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}$ induced by $\rho_i : \langle n \rangle \to \langle 1 \rangle$ assemble into an equivalence $\mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}^n$. (This is often referred to as a "Segal condition".)

The main result is that a symmetric monoidal structure on $\mathcal{C}$ can be recovered from such a Grothendieck opfibration. Moreover, this construction generalizes readily to the ∞-category setting, as demonstrated in the following definition.

Remark 31. This Segal condition gives us functors $\mathcal{C}^2 \to \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}$, which is how we can recover the symmetric monoidal product from the category defined above. So just to clarify, defining the category $\mathcal{C}^{\otimes}$ will require us to specify what the objects $C_1 \otimes \cdots \otimes C_j$ are. The point is that defining the bifunctor $\otimes$ by specifying it in the form above lets us give a "minimal presentation" of the coherence conditions.

Definition 32. A *symmetric monoidal ∞-category* is a coCartesian fibration of simplicial sets $\mathcal{C}^{\otimes} \to \mathscr{F}\text{in}_*$ such that, for each $n \geq 0$, the maps $\{\rho_i : \langle n\rangle \to \langle 1\rangle\}_{1\leq i\leq n}$ induce functors $\mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}$ that assemble into an equivalence $\mathcal{C}^{\otimes}_{\langle n\rangle} \to (\mathcal{C}^{\otimes}_{\langle 1\rangle})^n$.

Here, we make the usual abuse of notation of writing $\mathscr{F}\text{in}_*$ instead of $N(\mathscr{F}\text{in}_*)$. It should be noted that, given a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathscr{F}\text{in}_*$, one typically thinks of $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ as having been given a symmetric monoidal structure.

4 Morita Theory

4.1 Summary

4.1.1 Important definitions

Term	pg	Loose Definition
Stable simplicial category	25	Simplicial category s.t. the associated ∞-category is stable.
Stable spectral category	25	Spectral category whose associated simplicial category is stable.
Ψ_{tri}	24	Functor $N((Cat_S)^c)[W^{-1}] \to N((Set_\Delta)^c)[W^{-1}] \cong Cat_\infty$. Induced by the functor $Cat_S \to Set_\Delta$ given by $A \mapsto \widehat{A}_{tri} \mapsto N(\Omega^\infty(A)^{fib})$.
Ψ_{perf}	24	Same as above, but replace $\widehat{A}_{tri}$ with $\widehat{A}_{perf}$.
Triangulated equivalence v2	30	A map in the ∞-cat of small spectral categories s.t. $\Psi_{tri}f$ is an equivalence of stab ∞-cats.
Morita equivalence v2	30	$\Psi_{perf}f$ is an equivalence.
rep(B, A)	31	$\Upsilon(Fun^{ex}(B, A))$, the small pretriangulated spectral category associated to the small stable ∞-cat of exact functors from B to A.

4.1.2 Notation

- Υ is the right adjoint to Ψ_{tri}, and via inclusion to Ψ_{perf}.

4.1.3 Key Results

1. Ψ_{tri} lands in Cat^{ex}_∞ =small stable ∞-cats.
2. Ψ_{perf} lands in Cat^{perf}_∞ =idempotent complete small stable ∞-cats.

3. The ∞-category of stable ∞-cats is an accessible localization of the ∞-cat of spectral categories obtained by inverting the triangulated equivalences. In other words, the functor Ψ_{tri} has a fully faithful and accessible right adjoint Υ.

4. The ∞-category of stable idempotent complete ∞-cats is an accessible localization of the ∞-cat of spectral categories obtained by inverting the Morita equivalences.

5. The ∞-cats Cat_{∞}^{ex} and Cat_{∞}^{perf} are compactly generated, complete, and cocomplete.

6. Let I be a small category. Given a diagram $\mathcal{D}$ of small stable ∞-categories indexed by $N(I)$, there exists and I-diagram of pretriangulated spectral categories $\widetilde{\mathcal{D}}$ lifting $\mathcal{D}$.

4.2 Clarifying Remarks

1. This whole section is basically setting up technical machinery to allow us to lift stable ∞-categories to spectral categories and make arguments with these more rigid objects. This is the content of (3) and (4) above.

2. Given a small stable idempotent complete ∞-category A, we have that the conunit of the adjunction $\Psi_{perf}\Upsilon \to Id$ is a natural equivalence. Thus $A \cong \Psi_{perf}\Upsilon(A) \cong Idem \circ \Psi_{tri}\Upsilon(A)$, and recall that idempotent completion of a small stable ∞-cat can be modeled as $A \mapsto Fun^{ex}(A^{op}, S_{\infty})^{\omega}$. It is in this sense that small stable idempotent complete ∞-categories are ∞-categories of modules.

4.3 Background Material

4.3.1 Accessible Localizations

Lemma 33. *If a right adjoint is full and faithful, the counit is an isomorphism.*

Proof. By definition of an adjunction $R \xrightarrow{\eta L} RLR \xrightarrow{R\epsilon} R$ is the identity, so that $R\epsilon$ is an isomorphism. Thus ϵ is an isomorphism since R is fully faithful. $\square$

Remark 34. The analogous result holds in the ∞-categorical setting.

Definition 35. An $(\infty, 1)$-functor $F : C \to D$ is accessible if C is an accessible $(\infty, 1)$-category and there is a regular cardinal κ s.t. F preserves κ-small filtered colimits.

Remark 36. If an $(\infty, 1)$-functor between accessible $(\infty, 1)$-categories has a left or right adjoint $(\infty, 1)$-functor, then it is itself accessible.

Definition 37. An $(\infty, 1)$-functor $L : C \to C_0$ is called a (reflective) localization of the $(\infty, 1)$-category C if it has a right adjoint $(\infty, 1)$-functor $i : C_0 \hookrightarrow C$ that is full and faithful.

Definition 38. A localization is accessible if the localization functor is an accessible functor.