HOMOTOPY THEORY OF SPECTRAL CATEGORIES

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Abstract. We construct a Quillen model structure on the category of spectral categories, where the weak equivalences are the symmetric spectra analogue of the notion of equivalence of categories.

Contents

1. Introduction

In the past fifteen years, the discovery of highly structured categories of spectra (S-modules [12], symmetric spectra [16], simplicial functors [22], orthogonal spectra [23], ...) has opened the way for an importation of more and more algebraic techniques into stable homotopy theory [1] [10] [11]. In this paper, we study a new ingredient in this 'brave new algebra': *Spectral categories*.

Spectral categories are categories enriched over the symmetric monoidal category of symmetric spectra. As linear categories can be understood as *rings with several objects*, spectral categories can be understood as *symmetric ring spectra with several objects*. They appear nowadays in several (related) subjects:

On one hand, they are considered as the 'topological' analogue of differential graded (=DG) categories [7] [18] [30]. The main idea is to replace the monoidal category $Ch(\mathbb{Z})$ of complexes of abelian groups by the monoidal category Sp^{Σ} of symmetric spectra, which one should imagine as 'complexes of abelian groups up to homotopy'. In this way, spectral categories provide a non-additive framework for non-commutative algebraic geometry in the sense of Bondal, Drinfeld, Kapranov, Kontsevich, Toën, Van den Bergh ... [2] [3] [7] [8] [20] [21] [31]. They can be seen as non-additive derived categories of quasi-coherent sheaves on a hypothetical non-commutative space.

On the other hand they appear naturally in stable homotopy theory by the work of Dugger, Schwede-Shipley, ... [6] [27]. For example, it is shown in [27, 3.3.3]

Key words and phrases. Symmetric spectra, Spectral category, Quillen model structure, Bousfield's localization Q-functor, Non-additive filtration.

that stable model categories with a set of compact generators can be characterized as modules over a spectral category. In this way several different subjects such as: equivariant homotopy theory, stable motivic theory of schemes, ... and all the classical algebraic situations [27, 3.4] fit in the context of spectral categories.

It turns out that in all the above different situations, spectral categories should be considered only up to the notion of *stable quasi-equivalence* (5.1): a mixture between stable equivalences of symmetric spectra and categorical equivalences, which is the correct notion of equivalence between spectral categories.

In this article, we construct a Quillen model structure [24] on the category Sp^Σ-Cat of spectral categories, with respect to the class of stable quasi-equivalences. Starting from simplicial categories [4], we construct in theorem 4.8 a 'levelwise' cofibrantly generated Quillen model structure on Sp^Σ-Cat. Then we adapt Schwede-Shipley's non-additive filtration argument (Appendix A) to our situation and prove our main theorem:

Theorem (5.10)**.** *The category Sp^Σ-Cat admits a right proper Quillen model structure whose weak equivalences are the stable quasi-equivalences and whose cofibrations are those of theorem 4.8.*

Using theorem 5.10 and the same general arguments of [31], we can describe the mapping space between two spectral categories $\mathcal{A}$ and $\mathcal{B}$ in terms of the nerve of a certain category of $\mathcal{A}$-$\mathcal{B}$-bimodules and prove that that the homotopy category $\mathsf{Ho}(\mathsf{Sp}^\Sigma$-Cat) possesses internal Hom's relative to the derived smash product of spectral categories.

Acknowledgments : It is a great pleasure to thank B. Toën for stating this problem to me and G. Granja for several discussions and references. I am also very grateful to S. Schwede for his interest and for kindly suggesting several improvements and references. I would like also to thank A. Joyal and A. E. Stanculescu for pointing out an error in a previous version and the anonymous referee for numerous suggestions for the improvement of this paper.

2. Preliminaries

Throughout this article the adjunctions are displayed vertically with the left, resp. right, adjoint on the left side, resp. right side.

2.1. **Definition.** *Let $(\mathcal{C}, -\otimes -, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, -\wedge -, \mathbb{I}_{\mathcal{D}})$ be two symmetric monoidal categories. A* strong monoidal functor *is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with an isomorphism $\eta : \mathbb{I}_{\mathcal{D}} \to F(\mathbb{I}_{\mathcal{C}})$ and natural isomorphisms*

$$\psi_{X,Y} : F(X) \wedge F(Y) \to F(X \otimes Y), \quad X, Y \in \mathcal{C}$$

which are coherently associative and unital (see diagrams 6.27 *and* 6.28 *in* [5]*). A* strong monoidal adjunction *between monoidal categories is an adjunction for which the left adjoint is strong monoidal.*

Let $s\mathbf{Set}$, resp. $s\mathbf{Set}_\bullet$, be the (symmetric monoidal) category of simplicial sets, resp. pointed simplicial sets. By a *simplicial category*, resp. *pointed simplicial category*, we mean a category enriched over $s\mathbf{Set}$, resp. over $s\mathbf{Set}_\bullet$. We denote by $s\mathbf{Set}$-Cat, resp. $s\mathbf{Set}_\bullet$-Cat, the category of small simplicial categories, resp.

pointed simplicial categories. Observe that the usual adjunction [17] (on the left)

$$\begin{tikzcd} s\mathbf{Set}_\bullet \arrow[d, shift left] & & s\mathbf{Set}_\bullet\text{-Cat} \arrow[d, shift left] \\ s\mathbf{Set} \arrow[u, shift left, "(-)_+"] & & s\mathbf{Set}\text{-Cat} \arrow[u, shift left, "(-)_+"] \end{tikzcd}$$

is strong monoidal and so it induces the adjunction on the right.

Let Sp^Σ be the (symmetric monoidal) category of symmetric spectra of pointed simplicial sets [16] [25]. We denote by $\wedge$ its smash product and by $\mathbb{S}$ its unit, i.e. the sphere symmetric spectrum [25, I-3]. Recall that the projective level model structure on Sp^Σ [25, III-1.9] and the projective stable model structure on Sp^Σ [25, III-2.2] are monoidal with respect to the smash product.

2.2. **Lemma.** *The projective level model structure on* Sp^Σ *satisfies the monoid axiom* [26, 3.3].

Proof. Let Z be a symmetric spectrum and $f : X \to Y$ a trivial cofibration in the projective level model structure. By proposition [25, III-1.11] the morphism

$$Z \wedge f : Z \wedge X \to Z \wedge Y$$

is a trivial cofibration in the injective level model structure [25, III-1.9]. Since trivial cofibrations are stable under co-base change and transfinite composition, we conclude that each map in the class

$$(\{\text{projective trivial cofibration}\} \wedge \mathsf{Sp}^\Sigma) - \text{cof}_{reg}$$

is in particular a level equivalence. This proves the lemma. $\surd$

2.3. **Definition.** *A* spectral category $\mathcal{A}$ *is a* Sp^Σ*-category* [5, 6.2.1].

Recall that this means that $\mathcal{A}$ consists in the following data:

- a class of objects $\text{obj}(\mathcal{A})$ (usually denoted by $\mathcal{A}$ itself);
- for each ordered pair of objects (x, y) of $\mathcal{A}$, a symmetric spectrum $\mathcal{A}(x, y)$;
- for each ordered triple of objects (x, y, z) of $\mathcal{A}$, a composition morphism in Sp^Σ
$$\mathcal{A}(x, y) \wedge \mathcal{A}(y, z) \to \mathcal{A}(x, z)\,,$$
satisfying the usual associativity condition;
- for any object x of $\mathcal{A}$, a morphism $\mathbb{S} \to \mathcal{A}(x, x)$ in Sp^Σ, satisfying the usual unit condition with respect to the above composition.

If $\text{obj}(\mathcal{A})$ is a set we say that $\mathcal{A}$ is a *small* spectral category.

2.4. **Definition.** *A* spectral functor $F : \mathcal{A} \to \mathcal{B}$ *is a* Sp^Σ*-functor* [5, 6.2.3].

Recall that this means that F consists in the following data:

- a map $\text{obj}(\mathcal{A}) \to \text{obj}(\mathcal{B})$ and
- for each ordered pair of objects (x, y) of $\mathcal{A}$, a morphism in Sp^Σ
$$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)$$
satisfying the usual unit and associativity conditions.

2.5. *Notation.* We denote by Sp^Σ-Cat the category of small spectral categories.

Observe that the classical adjunction [25, I-2.12] (on the left)

$$
\begin{array}{ccc}
\mathsf{Sp}^{\Sigma} & \qquad & \mathsf{Sp}^{\Sigma}\text{-Cat} \\
\Sigma^{\infty}\uparrow \downarrow (-)_0 & & \Sigma^{\infty}\uparrow \downarrow (-)_0 \\
s\mathbf{Set}_{\bullet} & & s\mathbf{Set}_{\bullet}\text{-Cat}
\end{array}
$$

is strong monoidal and so it induces the adjunction on the right.

3. Simplicial categories

In this chapter we give a detailed proof of a technical lemma concerning simplicial categories, which is due to A. E. Stanculescu.

3.1. *Remark.* Notice that we have a fully faithful functor:

$$
\begin{array}{ccc}
s\mathbf{Set}\text{-Cat} & \longrightarrow & \mathrm{Cat}^{\Delta^{op}} \\
\mathcal{A} & \mapsto & \mathcal{A}_*
\end{array}
$$

given by $\mathrm{obj}(\mathcal{A}_n) = \mathrm{obj}(\mathcal{A})$, $n \geq 0$ and $\mathcal{A}_n(x, x') = \mathcal{A}(x, x')_n$.

Recall from [4, 1.1], that the category $s\mathbf{Set}$-Cat carries a cofibrantly generated Quillen model structure whose weak equivalences are the Dwyer-Kan (=DK) equivalences, i.e. the simplicial functors $F : \mathcal{A} \to \mathcal{B}$ such that:

- for all objects $x, y \in \mathcal{A}$, the map

$$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)$$

  is a weak equivalence of simplicial sets and
- the induced functor

$$\pi_0(F) : \pi_0(\mathcal{A}) \longrightarrow \pi_0(\mathcal{B})$$

  is an equivalence of categories.

3.2. *Notation.* Let $\mathcal{A}$ be an (enriched) category and $x \in \mathrm{obj}(\mathcal{A})$. We denote by $x^*\mathcal{A}$ the full (enriched) subcategory of $\mathcal{A}$ whose set of objects is $\{x\}$.

3.3. **Lemma.** *(Stanculescu* [29, 4.7]*) Let $\mathcal{A}$ be a cofibrant simplicial category. Then for every $x \in obj(\mathcal{A})$, the simplicial category $x^*\mathcal{A}$ is also cofibrant (as a simplicial monoid).*

Proof. Let $\mathcal{O}$ be the set of objects of $\mathcal{A}$. Notice that if the simplicial category $\mathcal{A}$ is cofibrant then it is also cofibrant in $s\mathbf{Set}^{\mathcal{O}}$-Cat [9, 7.2]. Moreover a simplicial category with one object (for example $x^*\mathcal{A}$) is cofibrant if and only if it is cofibrant as a simplicial monoid, i.e. cofibrant in $s\mathbf{Set}^{\{x\}}$-Cat.

Now, by [9, 7.6] the cofibrant objects in $s\mathbf{Set}^{\mathcal{O}}$-Cat can be characterized as the retracts of the free simplicial categories. Recall from [9, 4.5] that a simplicial category $\mathcal{B}$ (i.e. a simplicial object $\mathcal{B}_*$ in Cat) is *free* if and only if:

(1) for every $n \geq 0$, the category $\mathcal{B}_n$ is free on a graph G_n of generators and
(2) all degenerancies of generators are generators.

Therefore it is enough to show the following: if $\mathcal{A}$ is a free simplicial category, then $x^*\mathcal{A}$ is also free (as a simplicial monoid). Since for every $n \geq 0$, the category $\mathcal{A}_n$ is free on a graph, lemma 3.4 implies that the simplicial category $x^*\mathcal{A}$ satisfies condition (1). Moreover, since the degenerancies in $\mathcal{A}_*$ induce the identity map on objects and send generators to generators, the simplicial category $x^*\mathcal{A}$ satisfies also condition (2). This proves the lemma. $\surd$

3.4. **Lemma.** *Let $\mathcal{C}$ be a category which is free on a graph G of generators. Then for every object $x \in obj(\mathcal{C})$, the category $x^*\mathcal{C}$ is also free on a graph $\widetilde{G}$ of generators.*

Proof. We start by defining the generators of $\widetilde{G}$. An element of $\widetilde{G}$ is a path in $\mathcal{C}$ from x to x such that:

(i) every arrow in the path belongs to G and
(ii) the path starts in x, finishes in x and *never* passes throught x in an intermediate step.

Let us now show that every morphism in $x^*\mathcal{C}$ can be written uniquely as a finite composition of elements in $\widetilde{G}$. Let f be a morphism in $x^*\mathcal{C}$. Since $x^*\mathcal{C}$ is a full subcategory of $\mathcal{C}$ and $\mathcal{C}$ is free on the graph G, the morphism f can be written uniquely as a finite composition

$$f = g_n \cdots g_i \cdots g_2\, g_1\,,$$

where g_i , $1 \leq i \leq n$ belongs to G. Now consider the partition

$$1 \leq m_1 < \cdots < m_j < \cdots < m_k = n\,,$$

where m_j is such that the target of the morphism g_{m_j} is the object x. If we denote by $M_1 = g_{m_1} \cdots g_1$ and by $M_j = g_{m_j} \cdots g_{m_{(j-1)}+1}$, $j \geq 2$ the morphisms in $\widetilde{G}$, we can factor f as

$$f = M_k \cdots M_j \cdots M_1\,.$$

Notice that our arguments shows us also that this factorization is unique and so the lemma is proven. $\surd$

4. Levelwise quasi-equivalences

In this chapter we construct a cofibrantly generated Quillen model structure on Sp^{Σ}-Cat whose weak equivalences are defined as follows.

4.1. **Definition.** *A spectral functor $F : \mathcal{A} \to \mathcal{B}$ is a* levelwise quasi-equivalence *if:*

L1) *for all objects $x, y \in \mathcal{A}$, the morphism of symmetric spectra*

$$F(x,y) : \mathcal{A}(x,y) \to \mathcal{B}(Fx, Fy)$$

is a level equivalence of symmetric spectra [25, III-1.9] *and*

L2) *the induced simplicial functor*

$$F_0 : \mathcal{A}_0 \to \mathcal{B}_0$$

*is a DK-equivalence in s***Set***-Cat.*

4.2. *Notation.* We denote by $\mathcal{W}_l$ the class of levelwise quasi-equivalences in Sp^{Σ}-Cat.

4.3. *Remark.* Notice that if condition $L1)$ is verified, condition $L2)$ is equivalent to:

L2') the induced functor

$$\pi_0(F_0) : \pi_0(\mathcal{A}_0) \longrightarrow \pi_0(\mathcal{B}_0)$$

is essentially surjective.

We now define our sets of (trivial) generating cofibrations in Sp^{Σ}-Cat.

4.4. **Definition.** *The set I of* generating cofibrations *consists in:*

- *the spectral functors obtained by applying the functor U (A.1) to the set of generating cofibrations of the projective level model structure on* Sp^Σ [25, III-1.9]. *More precisely, we consider the spectral functors*

$$C_{m,n} : U(F_m \partial\Delta[n]_+) \longrightarrow U(F_m\Delta[n]_+),\ m,n \geq 0,$$

where F_m denotes the level m free symmetric spectra functor [25, I-2.12].
- *the spectral functor*

$$C : \emptyset \longrightarrow \underline{\mathbb{S}}$$

from the empty spectral category $\emptyset$ (which is the initial object in Sp^Σ-Cat) to the spectral category $\underline{\mathbb{S}}$ with one object $*$ and endomorphism ring spectrum $\mathbb{S}$.

4.5. **Definition.** *The set J of* trivial generating cofibrations *consists in:*

- *the spectral functors obtained by applying the functor U (A.1) to the set of trivial generating cofibrations of the projective level model structure on Sp^Σ. More precisely, we consider the spectral functors*

$$A_{m,k,n} : U(F_m\Lambda[k,n]_+) \longrightarrow U(F_m\Delta[n]_+),\ m\geq 0,\ n\geq 1,\ 0\leq k\leq n\,.$$

- *the spectral functors obtained by applying the composed functor $\Sigma^\infty(-_+)$ to the set $(A2)$ of trivial generating cofibrations in $s\mathbf{Set}$-Cat* [4]. *More precisely, we consider the spectral functors*

$$A_{\mathcal{H}} : \underline{\mathbb{S}} \longrightarrow \Sigma^\infty(\mathcal{H}_+),$$

where $A_{\mathcal{H}}$ sends $*$ to the object x.

4.6. *Notation.* We denote by J', resp. J'', the subset of J consisting of the spectral functors $A_{m,k,n}$, resp. $A_{\mathcal{H}}$. In this way $J' \cup J'' = J$.

4.7. *Remark.* By definition [4] the simplicial categories $\mathcal{H}$ have weakly contractible function complexes and are cofibrant in $s\mathbf{Set}$-Cat. By lemma 3.3, we conclude that $x^*\mathcal{H}$ (i.e. the full simplicial subcategory of $\mathcal{H}$ whose set of objects is $\{x\}$) is a cofibrant simplicial category.

4.8. **Theorem.** *If we let $\mathcal{M}$ be the category Sp^Σ-Cat, W be the class $\mathcal{W}_l$, I be the set of spectral functors of definition 4.4 and J the set of spectral functors of definition 4.5, then the conditions of the recognition theorem* [15, 2.1.19] *are satisfied. Thus, the category Sp^Σ-Cat admits a cofibrantly generated Quillen model structure whose weak equivalences are the levelwise quasi-equivalences.*

Proof of Theorem 4.8. We start by observing that the category Sp^Σ-Cat is complete and cocomplete and that the class $\mathcal{W}_l$ satisfies the two out of three axiom and is stable under retracts. Since the domains of the (trivial) generating cofibrations in Sp^Σ are sequentially small, the same holds by [19] for the domains of spectral functors in the sets I and J. This implies that the first three conditions of the recognition theorem [15, 2.1.19] are verified.

We now prove that J-inj $\cap\, \mathcal{W}_l = I$-inj. For this we introduce the following auxiliary class of spectral functors:

4.9. **Definition.** *Let* **Surj** *be the class of spectral functors $F : \mathcal{A} \to \mathcal{B}$ such that:*

Sj1) *for all objects $x, y \in \mathcal{A}$, the morphism of symmetric spectra*

$$F(x,y) : \mathcal{A}(x,y) \to \mathcal{B}(Fx,Fy)$$

is a trivial fibration in the projective level model structure [25, III-1.9] *and*

Sj2) *the spectral functor F induces a surjective map on objects.*

4.10. **Lemma.** *I-inj* $=$ **Surj**.

Proof. Notice that a spectral functor satisfies condition $Sj1$) if and only if it has the right lifting property (=R.L.P.) with respect to the spectral functors $C_{m,n}$, $m, n \geq 0$. Clearly a spectral functor has the R.L.P. with respect to the spectral functor C if and only if it satisfies condition $Sj2$). $\surd$

4.11. **Lemma.** **Surj** $=$ *J-inj* $\cap \mathcal{W}_l$.

Proof. We prove first the inclusion $\subseteq$. Let $F : \mathcal{A} \to \mathcal{B}$ be a spectral functor which belongs to **Surj**. Conditions $Sj1$) and $Sj2$) clearly imply conditions $L1$) and $L2$) and so F belongs to $\mathcal{W}_l$. Notice also that a spectral functor which satisfies condition $Sj1$) has the R.L.P. with respect to the trivial generating cofibrations $A_{m,k,n}$. It is then enough to show that F has the R.L.P. with respect to the spectral functors $A_{\mathcal{H}}$. By adjunction, this is equivalent to demand that the simplicial functor $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ has the R.L.P. with respect to the set $(A2)$ of trivial generating cofibrations $\{x\} \to \mathcal{H}$ in s**Set**-Cat [4]. Since F satisfies conditions $Sj1$) and $Sj2$), proposition [4, 3.2] implies that F_0 is a trivial fibration in s**Set**-Cat and so the claim follows.

We now prove the inclusion $\supseteq$. Observe that a spectral functor satisfies condition $Sj1$) if and only if it satisfies condition $L1$) and it has moreover the R.L.P. with respect to the trivial generating cofibrations $A_{m,k,n}$. Now, let $F : \mathcal{A} \to \mathcal{B}$ be a spectral functor which belongs to J-inj $\cap \mathcal{W}_l$. It is then enough to show that it satisfies condition $Sj2$). Since F has the R.L.P. with respect to the trivial generating cofibrations

$$A_{\mathcal{H}} : \underline{\mathbb{S}} \longrightarrow \Sigma^{\infty}(\mathcal{H}_+)$$

the simplicial functor $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ has the R.L.P. with respect to the inclusions $\{x\} \to \mathcal{H}$. This implies that F_0 is a trivial fibration in s**Set**-Cat and so by proposition [4, 3.2], the simplicial functor F_0 induces a surjective map on objects. Since F_0 and F induce the same map on the set of objects, the spectral functor F satisfies condition $Sj2$). $\surd$

We now characterize the class J-inj.

4.12. **Lemma.** *A spectral functor $F : \mathcal{A} \to \mathcal{B}$ has the R.L.P. with respect to the set J of trivial generating cofibrations if and only if it satisfies:*

F1) *for all objects $x, y \in \mathcal{A}$, the morphism of symmetric spectra*

$$F(x, y) : \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$$

is a fibration in the projective level model structure [25, III-1.9] *and*

F2) *the induced simplicial functor*

$$F_0 : \mathcal{A}_0 \to \mathcal{B}_0$$

*is a fibration in the Quillen model structure on s**Set**-Cat.*

Proof. Observe that a spectral functor F satisfies condition $F1$) if and only if it has the R.L.P. with respect to the trivial generating cofibrations $A_{m,k,n}$. By adjunction F has the R.L.P. with respect to the spectral functors $A_{\mathcal{H}}$ if and only if the simplicial functor F_0 has the R.L.P. with respect to the inclusions $\{x\} \to \mathcal{H}$. In conclusion F has the R.L.P. with respect to the set J if and only if it satisfies conditions $F1$) and $F2$) altogether. $\surd$

4.13. **Lemma.** *J'-cell* $\subseteq \mathcal{W}_l$.

Proof. Since the class $\mathcal{W}_l$ is stable under transfinite compositions [14, 10.2.2] it is enough to prove the following: let $m \geq 0$, $n \geq 1$, $0 \leq k \leq n$ and $R : U(F_m\Lambda[k,n]_+) \to \mathcal{A}$ a spectral functor. Consider the following pushout:

$$\begin{tikzcd} U(F_m\Lambda[k,n]_+) \arrow[r, "R"] \arrow[d, "A_{m,k,n}"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, "P"] \\ U(F_m\Delta[n]_+) \arrow[r] & \mathcal{B}\,. \end{tikzcd}$$

We need to show that P belongs to $\mathcal{W}_l$. Since the symmetric spectra morphisms

$$F_m\Lambda[k,n]_+ \longrightarrow F_m\Delta[n]_+,\ m \geq 0,\, n \geq 1,\, 0 \leq k \leq n$$

are trivial cofibrations in the projective level model structure, lemma 2.2 and proposition A.2 imply that the spectral functor P satisfies condition $L1$). Since P induces the identity map on objects, condition $L2'$) is automatically satisfied and so P belongs to $\mathcal{W}_l$. $\surd$

4.14. **Proposition.** *J''-cell* $\subseteq \mathcal{W}_l$.

Proof. Since the class $\mathcal{W}_l$ is stable under transfinite compositions, it is enough to prove the following: let $\mathcal{A}$ be a small spectral category and $R : \underline{\mathbb{S}} \to \mathcal{A}$ a spectral functor. Consider the following pushout

$$\begin{tikzcd} \underline{\mathbb{S}} \arrow[r, "R"] \arrow[d, "A_{\mathcal{H}}"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, "P"] \\ \Sigma^{\infty}(\mathcal{H}_+) \arrow[r] & \mathcal{B}\,. \end{tikzcd}$$

We need to show that P belongs to $\mathcal{W}_l$. We start by showing condition $L1$). Factor the spectral functor $A_{\mathcal{H}}$ as

$$\underline{\mathbb{S}} \longrightarrow x^*\Sigma^{\infty}(\mathcal{H}_+) \hookrightarrow \Sigma^{\infty}(\mathcal{H}_+)\,,$$

where $x^*\Sigma^{\infty}(\mathcal{H}_+)$ is the full spectral subcategory of $\Sigma^{\infty}(\mathcal{H}_+)$ whose set of objects is $\{x\}$ (3.2). Consider the iterated pushout

$$\begin{tikzcd} \underline{\mathbb{S}} \arrow[r, "R"] \arrow[d, tail, "\sim"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, "P_0"'] \arrow[dd, bend left, "P"] \\ x^*\Sigma^{\infty}(\mathcal{H}_+) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathcal{A}} \arrow[d, "P_1"'] \\ \Sigma^{\infty}(\mathcal{H}_+) \arrow[r] & \mathcal{B}\,. \end{tikzcd}$$

In the lower pushout, since $x^*\Sigma^{\infty}(\mathcal{H}_+)$ is a full spectral subcategory of $\Sigma^{\infty}(\mathcal{H}_+)$, proposition [13, 5.2] implies that $\widetilde{\mathcal{A}}$ is a full spectral subcategory of $\mathcal{B}$ and so P_1 satisfies condition $L1$).

In the upper pushout, since $x^*\Sigma^{\infty}(\mathcal{H}_+) = \Sigma^{\infty}((x^*\mathcal{H})_+)$ and $x^*\mathcal{H}$ is a cofibrant simplicial category (4.7), the spectral functor $\underline{\mathbb{S}} \overset{\sim}{\rightarrowtail} x^*\Sigma^{\infty}(\mathcal{H}_+)$ is a trivial cofibration. Now, let $\mathcal{O}$ denote the set of objects of $\mathcal{A}$ (notice that if $\mathcal{A} = \emptyset$, then there is no spectral functor R) and $\mathcal{O}' := \mathcal{O}\backslash R(*)$. By lemma 2.2 and proposition [28, 6.3], the category $(\mathsf{Sp}^{\Sigma})^{\mathcal{O}}$-Cat of spectral categories with a fixed set of objects $\mathcal{O}$ carries a natural Quillen model structure. Notice that $\widetilde{\mathcal{A}}$ identifies with the following pushout in $(\mathsf{Sp}^{\Sigma})^{\mathcal{O}}$-Cat

$$\begin{tikzcd} \coprod_{\mathcal{O}'} \underline{\mathbb{S}} \amalg \underline{\mathbb{S}} \arrow[r, "R"] \arrow[d, tail, "\sim"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, tail, "\sim"', "P_0"] \\ \coprod_{\mathcal{O}'} \underline{\mathbb{S}} \amalg x^*\Sigma^{\infty}(\mathcal{H}_+) \arrow[r] & \widetilde{\mathcal{A}}\,. \end{tikzcd}$$

Since the left vertical arrow is a trivial cofibration so it is P_0. In particular P_0 satisfies condition L_1) and so we conclude that the composed spectral functor P satisfies also condition L_1).

We now show that P satisfies condition $L2'$). Let f be a 0-simplex in $\mathcal{H}(x, y)$. By construction [4] of the simplicial categories $\mathcal{H}$, f becomes invertible in $\pi_0(\mathcal{H})$. We consider it as a morphism in the spectral category $\Sigma^{\infty}(\mathcal{H}_+)$. Notice that the spectral category $\mathcal{B}$ is obtained from $\mathcal{A}$, by gluing $\Sigma^{\infty}(\mathcal{H}_+)$ to the object $R(*)$. Since f clearly becomes invertible in $\pi_0(\Sigma^{\infty}(\mathcal{H}_+))_0$, its image by the spectral functor $\Sigma^{\infty}(\mathcal{H}_+) \to \mathcal{B}$ becomes invertible in $\pi_0(\mathcal{B}_0)$. This implies that the functor

$$\pi_0(P_0) : \pi_0(\mathcal{A}_0) \longrightarrow \pi_0(\mathcal{B}_0)$$

is essentially surjective and so P satisfies condition $L2'$). In conclusion, P satisfies condition $L1$) and $L2'$) and so it belongs to $\mathcal{W}_l$. $\surd$

We have shown that J-cell $\subseteq \mathcal{W}_l$ (lemma 4.13 and proposition 4.14) and that I-inj $= J$-inj $\cap \mathcal{W}_l$ (lemmas 4.10 and 4.11). This implies that the last three conditions of the recognition theorem [15, 2.1.19] are satisfied. This finishes the proof of theorem 4.8.

Properties.

4.15. **Proposition.** *A spectral functor $F : \mathcal{A} \to \mathcal{B}$ is a fibration with respect to the model structure of theorem 4.8, if and only if it satisfies conditions $F1$) and $F2$) of lemma 4.12.*

Proof. This follows from lemma 4.12, since by the recognition theorem [15, 2.1.19], the set J is a set of generating trivial cofibrations. $\surd$

4.16. **Corollary.** *A spectral category $\mathcal{A}$ is fibrant with respect to the model structure of theorem 4.8, if and only if $\mathcal{A}(x, y)$ is a levelwise Kan simplicial set for all objects $x, y \in \mathcal{A}$.*

Notice that by proposition 4.15 we have a Quillen adjunction

$$\begin{tikzcd} \mathsf{Sp}^{\Sigma}\text{-Cat} \arrow[d, shift left, "(-)_0"] \\ s\mathbf{Set}\text{-Cat}\,. \arrow[u, shift left, "\Sigma^{\infty}(-_+)"] \end{tikzcd}$$

4.17. **Proposition.** *The Quillen model structure on Sp^{Σ}-Cat of theorem 4.8 is right proper.*

Proof. Consider the following pullback square in Sp^Σ-Cat

$$\begin{tikzcd} \mathcal{A}\underset{\mathcal{B}}{\times}\mathcal{C} \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{A} \arrow[r, "\sim", "R"'] & \mathcal{B}, \end{tikzcd}$$

with R a levelwise quasi-equivalence and F a fibration. We need to show that P is a levelwise quasi-equivalence. Notice that pullbacks in Sp^Σ-Cat are calculated on objects and on symmetric spectra morphisms. Since the projective level model structure on Sp^Σ is right proper [25, III-1.9] and F satisfies condition $F1$), the spectral functor P satisfies condition $L1$). Notice that the composed functor

$$\mathsf{Sp}^\Sigma\text{-Cat} \xrightarrow{(-)_0} s\mathbf{Set}_\bullet\text{-Cat} \longrightarrow s\mathbf{Set}\text{-Cat}$$

commutes with limits and that by proposition 4.15, F_0 is a fibration in $s\mathbf{Set}$-Cat. Since the model structure on $s\mathbf{Set}$-Cat is right proper [4, 3.5] and R_0 is a DK-equivalence, we conclude that the spectral functor P satisfies also condition $L2$). $\surd$

4.18. **Proposition.** *Let $\mathcal{A}$ be a cofibrant spectral category (in the Quillen model structure of theorem 4.8). Then for all objects $x, y \in \mathcal{A}$, the symmetric spectra $\mathcal{A}(x,y)$ is cofibrant in the projective level model structure on Sp^Σ [25, III-1.9].*

Proof. The Quillen model structure of theorem 4.8 is cofibrantly generated and so any cofibrant object in Sp^Σ-Cat is a retract of a I-cell complex [14, 11.2.2]. Since cofibrations are stable under transfinite composition it is enough to prove the proposition for pushouts along a generating cofibration. Let $\mathcal{A}$ a spectral category such that $\mathcal{A}(x,y)$ is cofibrant for all objects $x, y \in \mathcal{A}$:

- consider the following pushout

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "C"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d] \\ \underline{\mathbb{S}} \arrow[r] & \mathcal{B}. \end{tikzcd}$$

  Notice that $\mathcal{B}$ is obtained from $\mathcal{A}$, by simply introducing a new object. It is then clear that, for all objects $x, y \in \mathcal{B}$, the symmetric spectra $\mathcal{B}(x,y)$ is cofibrant.
- Now, consider the following pushout

$$\begin{tikzcd} U(F_m\partial\Delta[n]_+) \arrow[r] \arrow[d, "C_{m,n}"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, "P"] \\ U(F_m\Delta[n]_+) \arrow[r] & \mathcal{B}. \end{tikzcd}$$

  Notice that $\mathcal{A}$ and $\mathcal{B}$ have the same set of objects and P induces the identity map on the set of objects. Since $F_m\partial\Delta[n]_+ \to F_m\Delta[n]_+$ is a projective cofibration, proposition A.3 implies that the morphism of symmetric spectra

$$P(x,y) : \mathcal{A}(x,y) \longrightarrow \mathcal{B}(x,y)$$

is still a projective cofibration. Finally, since the I-cell complexes in Sp^Σ-Cat are built from $\emptyset$ (the initial object), the proposition is proven.

$\surd$

4.19. **Lemma.** *The functor*

$$U : \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma\text{-}Cat \qquad (A.1)$$

sends projective cofibrations to cofibrations.

Proof. The Quillen model structure of theorem 4.8 is cofibrantly generated and so any cofibration in Sp^Σ-Cat is a retract of a transfinite composition of pushouts along the generating cofibrations. Since the functor U preserves retractions, colimits and send the generating projective cofibrations to (generating) cofibrations, the lemma is proven. $\surd$

5. Stable quasi-equivalences

In this chapter we construct a 'localized' Quillen model structure on Sp^Σ-Cat. We denote by $[-,-]$ the set of morphisms in the stable homotopy category $\mathsf{Ho}(\mathsf{Sp}^\Sigma)$ of symmetric spectra. From a spectral category $\mathcal{A}$ one can form a genuine category $[\mathcal{A}]$ by keeping the same set of objects and defining the set of morphisms between x and y in $[\mathcal{A}]$ to be $[\mathbb{S}, \mathcal{A}(x,y)]$. We obtain in this way a functor

$$[-] : \mathsf{Sp}^\Sigma\text{-Cat} \longrightarrow \text{Cat}\,,$$

with values in the category of small categories.

5.1. **Definition.** *A spectral functor $F : \mathcal{A} \to \mathcal{B}$ is a* stable quasi-equivalence *if:*

S1) *for all objects $x, y \in \mathcal{A}$, the morphism of symmetric spectra*

$$F(x,y) : \mathcal{A}(x,y) \to \mathcal{B}(Fx, Fy)$$

is a stable equivalence [25, II-4.1] *and*

S2) *the induced functor*

$$[F] : [\mathcal{A}] \longrightarrow [\mathcal{B}]$$

is an equivalence of categories.

5.2. *Notation.* We denote by $\mathcal{W}_s$ the class of stable quasi-equivalences.

5.3. *Remark.* Notice that if condition $S1)$ is verified, condition $S2)$ is equivalent to:

S2') the induced functor

$$[F] : [\mathcal{A}] \longrightarrow [\mathcal{B}]$$

is essentially surjective.

Functor Q. In this subsection we construct a functor

$$Q : \mathsf{Sp}^\Sigma\text{-Cat} \longrightarrow \mathsf{Sp}^\Sigma\text{-Cat}$$

and a natural transformation $\eta : \text{Id} \to Q$, from the identity functor on Sp^Σ-Cat to the functor Q. We start with a few definitions (see the proof of proposition [25, II-4.21]). Let $m \geq 0$ and $\lambda_m : F_{m+1}S^1 \to F_m S^0$ the morphism of symmetric spectra which is adjoint to the wedge summand inclusion $S^1 \to (F_m S^0)_{m+1} = \Sigma^+_{m+1} \wedge S^1$ indexed by the identity element. The morphism λ_m factors through the mapping cylinder as $\lambda_m = r_m c_m$ where $c_m : F_{m+1}S^1 \to Z(\lambda_m)$ is the 'front' mapping cylinder inclusion and $r_m : Z(\lambda_m) \to F_m S^0$ is the projection (which is a homotopy equivalence). Notice that c_m is a trivial cofibration [16, 3.4.10] in the projective stable model structure. Define the set K as the set of all pushout product maps

$$\begin{array}{c} \Delta[n]_+ \wedge F_{m+1}S^1 \coprod_{\partial\Delta[n]_+\wedge F_{m+1}S^1} \partial\Delta[n]_+ \wedge Z(\lambda_m) \\ \Big\downarrow{\scriptstyle i_{n_+}\wedge c_m} \\ \Delta[n]_+ \wedge Z(\lambda_m) \end{array},$$

where $i_{n_+} : \partial\Delta[n] \to \Delta[n],\ n \geq 0$ is the inclusion map. Let FI_Λ be the set of all morphisms of symmetric spectra $F_m\Lambda[k,n]_+ \to F_m\Delta[n]_+$ [25, I-2.12] induced by the horn inclusions for $m \geq 0, n \geq 1, 0 \leq k \leq n$.

5.4. *Remark.* By adjointness, a symmetric spectrum X has the R.L.P. with respect to the set FI_Λ if and only if for all $n \geq 0$, X_n is a Kan simplicial set and it has the R.L.P. with respect to the set K if and only if the induced map of simplicial sets

$$\mathrm{Map}(c_m, X) : \mathrm{Map}(Z(\lambda_m), X) \longrightarrow \mathrm{Map}(F_{m+1}S^1, X) \simeq \Omega X_{m+1}$$

has the R.L.P. with respect to all inclusions i_n, $n \geq 0$, i.e. it is a trivial Kan fibration of simplicial sets. Since the mapping cylinder $Z(\lambda_m)$ is homotopy equivalent to F_mS^0, $\mathrm{Map}(Z(\lambda_m), X)$ is homotopy equivalent to $\mathrm{Map}(F_mS^0, X) \simeq X_m$.

So altogether, the R.L.P. with respect to the union set $K \cup FI_\Lambda$ implies that for $n \geq 0$, X_n is a Kan simplicial set and for $m \geq 0$, $\widetilde{\delta_m} : X_m \to \Omega X_{m+1}$ is a weak equivalence, i.e. X is an Ω-spectrum.

Notice that the converse is also true. Let X be an Ω-spectrum. For all $n \geq 0$, X_n is a Kan simplicial set, and so X has the R.L.P. with respect to the set FI_Λ. Moreover, since c_m is a cofibration, the map $\mathrm{Map}(c_m, X)$ is a Kan fibration [17, II-3.2]. Since for $m \geq 0$, $\widetilde{\delta_m} : X_m \to \Omega X_{m+1}$ is a weak equivalence, the map $\mathrm{Map}(c_m, X)$ is in fact a trivial Kan fibration.

Now consider the set $U(K \cup FI_\Lambda)$ of spectral functors obtained by applying the functor U (A.1) to the set $K \cup FI_\Lambda$. Since the domains of the elements of the set $K \cup FI_\Lambda$ are sequentially small in Sp^Σ, the same holds by [19] to the domains of the elements of $U(K \cup FI_\Lambda)$. Notice that $U(K \cup FI_\Lambda) = U(K) \cup J'$ (4.5).

5.5. **Definition.** *Let $\mathcal{A}$ be a small spectral category. The functor $Q : \mathsf{Sp}^\Sigma\text{-}Cat \to \mathsf{Sp}^\Sigma\text{-}Cat$ is obtained by applying the small object argument, using the set $U(K) \cup J'$ to factor the spectral functor*

$$\mathcal{A} \longrightarrow \bullet,$$

where $\bullet$ denotes the terminal object in Sp^Σ-Cat.

5.6. *Remark.* We obtain in this way a functor Q and a natural transformation $\eta : \mathrm{Id} \to Q$. Notice also that $Q(\mathcal{A})$ has the same objects as $\mathcal{A}$, and the R.L.P. with respect to the set $U(K) \cup J'$. By remark 5.4 and [19], we get the following property:

- Ω) for all objects $x, y \in Q(\mathcal{A})$, the symmetric spectrum $Q(\mathcal{A})(x, y)$ is an Ω-spectrum.

5.7. **Proposition.** *Let $\mathcal{A}$ be a small spectral category. The spectral functor*

$$\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow Q(\mathcal{A})$$

is a stable quasi-equivalence.

Proof. The elements of the set $K \cup FI_\Lambda$ are trivial cofibrations in the projective stable model structure. This model structure is monoidal and satisfies the monoid axiom [16, 5.4.1]. This implies, by proposition A.2, that the spectral functor $\eta_{\mathcal{A}}$ satisfies condition $S1$). Since the spectral functor $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow Q(\mathcal{A})$ induce the identity on sets of objects, condition $S2'$) is automatically verified. $\surd$

Main theorem.

5.8. **Definition.** *A spectral functor $F : \mathcal{A} \to \mathcal{B}$ is:*

- *a Q*-weak equivalence *if $Q(F)$ is a levelwise quasi-equivalence (4.1).*
- *a* cofibration *if it is a cofibration in the model structure of theorem 4.8.*
- *a Q*-fibration *if it has the R.L.P. with respect to all cofibrations which are Q-weak equivalences.*

5.9. **Lemma.** *A spectral functor $F : \mathcal{A} \to \mathcal{B}$ is a Q-weak equivalence if and only if it is a stable quasi-equivalence.*

Proof. We have at our disposal a commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\eta_{\mathcal{A}}"] \arrow[d, "F"'] & Q(\mathcal{A}) \arrow[d, "Q(F)"] \\ \mathcal{B} \arrow[r, "\eta_{\mathcal{B}}"'] & Q(\mathcal{B})\,, \end{tikzcd}$$

where by proposition 5.7, the spectral functors $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$ are stable quasi-equivalences. Since the class $\mathcal{W}_s$ satisfies the two out of three axiom, the spectral functor F is a stable quasi-equivalence if and only if $Q(F)$ is a stable quasi-equivalence. The spectral categories $Q(\mathcal{A})$ and $Q(\mathcal{B})$ satisfy condition Ω) and so by lemma [16, 4.2.6], $Q(F)$ satisfies condition $S1$) if and only if it satisfies condition $L1$).

Notice that, since $Q(\mathcal{A})$ (and $Q(\mathcal{B})$) satisfy condition Ω), the set $[\mathbb{S}, Q(\mathcal{A})(x,y)]$ can be canonically identified with $\pi_0(Q(\mathcal{A})(x,y))_0$ and so the categories $[Q(\mathcal{A})]$ and $\pi_0(Q(\mathcal{A}))$ are naturally identified. This allows us to conclude that $Q(F)$ satisfies condition $S2'$) if and only if it satisfies condition $L2'$) and so the lemma is proven. $\surd$

5.10. **Theorem.** *The category Sp^Σ-Cat admits a right proper Quillen model structure whose weak equivalences are the stable quasi-equivalences (5.1) and the cofibrations those of theorem 4.8.*

5.11. *Notation.* We denote by $\mathsf{Ho}(\mathsf{Sp}^\Sigma\text{-Cat})$ the homotopy category hence obtained.

In order to prove theorem 5.10, we will use a slight variant of theorem [17, X-4.1]. Notice that in the proof of lemma [17, X-4.4] it is only used the right properness assumption and in the proof of lemma [17, X-4.6] it is only used the following condition (A3). This allows us to state the following result.

5.12. **Theorem.** [17, X-4.1] *Let $\mathcal{C}$ be a right proper Quillen model structure, $Q : \mathcal{C} \to \mathcal{C}$ a functor and $\eta : Id \to Q$ a natural transformation such that the following three conditions hold:*

(A1) *The functor Q preserves weak equivalences.*
(A2) *The maps $\eta_{Q(A)}, Q(\eta_A) : Q(A) \to QQ(A)$ are weak equivalences in $\mathcal{C}$.*

(A3) *Given a diagram*

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\eta_A]{} & Q(A) \end{array}$$

with p a Q-fibration, the induced map $\eta_{\mathcal{A}_*} : A \underset{Q(A)}{\times} B \to B$ is a Q-weak equivalence.

Then there is a right proper Quillen model structure on $\mathcal{C}$ for which the weak equivalences are the Q-weak equivalences, the cofibrations those of $\mathcal{C}$ and the fibrations the Q-fibrations.

Proof of theorem 5.10. The proof will consist on verifying the conditions of theorem 5.12. We consider for $\mathcal{C}$ the Quillen model structure of theorem 4.8 and for Q and η, the functor and natural transformation defined in 5.5. The Quillen model structure of theorem 4.8 is right proper (4.17) and by lemma 5.9 the Q-weak equivalences are precisely the stable quasi-equivalences. We now verify condition (A1), (A2) and (A3):

(A1) Let $F : \mathcal{A} \to \mathcal{B}$ be a levelwise quasi-equivalence. We have the following commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & Q(\mathcal{A}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle Q(F)} \\ \mathcal{B} & \xrightarrow[\eta_{\mathcal{B}}]{} & Q(\mathcal{B}) \end{array},$$

with $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$ stable quasi-equivalences. Notice that since F satisfies condition L1), the spectral functor $Q(F)$ satisfies condition S1). The spectral categories $Q(\mathcal{A})$ and $Q(\mathcal{B})$ satisfy condition Ω) and so by lemma [16, 4.2.6] the spectral functor $Q(F)$ satisfies condition L1).

Observe that since the spectral functors $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$ induce the identity on sets of objects and F satisfies condition L2'), the spectral functor $Q(F)$ satisfies also condition L2').

(A2) We now show that for every spectral category $\mathcal{A}$, the spectral functors

$$\eta_{Q(\mathcal{A})}, Q(\eta_{\mathcal{A}}) : Q(\mathcal{A}) \to QQ(\mathcal{A})$$

are levelwise quasi-equivalences. Since the spectral functors $\eta_{Q(\mathcal{A})}$ and $Q(\eta_{\mathcal{A}})$ are stable quasi-equivalences between spectral categories which satisfy condition Ω), they satisfy by lemma [16, 4.2.6] condition L1). The functor Q induce the identity on sets of objects and so the spectral functors $\eta_{Q(\mathcal{A})}$ and $Q(\eta_{\mathcal{A}})$ clearly satisfy condition L2').

(A3) We start by observing that if $P : \mathcal{C} \to \mathcal{D}$ is a Q-fibration, then for all $x, y \in \mathcal{C}$ the morphism of symmetric spectra

$$P(x,y) : \mathcal{C}(x,y) \longrightarrow \mathcal{D}(Px, Py)$$

is a fibration in projective stable model structure [25, III-2.2]. In fact, by proposition 4.19, the functor

$$U : \mathsf{Sp}^{\Sigma} \longrightarrow \mathsf{Sp}^{\Sigma}\text{-Cat}$$

sends projective cofibrations to cofibrations. Since it sends also stable equivalences to stable quasi-equivalences the claim follows.

Now consider the diagram

$$\begin{tikzcd} \mathcal{A} \underset{Q(\mathcal{A})}{\times} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{B} \arrow[d, "P"] \\ \mathcal{A} \arrow[r, "\eta_{\mathcal{A}}"'] & Q(\mathcal{A}), \end{tikzcd}$$

with P a Q-fibration. The projective stable model structure on Sp^{Σ} is right proper and so, by construction of fiber products in Sp^{Σ}-Cat, we conclude that the induced spectral functor

$$\eta_{\mathcal{A}_*} : \mathcal{A} \underset{Q(\mathcal{A})}{\times} \mathcal{B} \longrightarrow \mathcal{B}$$

satisfies condition S1). Since $\eta_{\mathcal{A}}$ induces the identity on sets of objects so it thus $\eta_{\mathcal{A}_*}$, and so condition S2') is verified.

5.13. **Proposition.** *A spectral category is fibrant with respect to theorem 5.10 if and only if for all objects $x, y \in \mathcal{A}$, the symmetric spectrum $\mathcal{A}(x,y)$ is an Ω-spectrum.*

Proof. By corollary [17, X-4.12] $\mathcal{A}$ is fibrant with respect to theorem 5.10 if and only if it is fibrant (4.16), with respect to the model structure of theorem 4.8, and the spectral functor $\eta_{\mathcal{A}} : \mathcal{A} \to Q(\mathcal{A})$ is a levelwise quasi-equivalence. Observe that $\eta_{\mathcal{A}}$ is a levelwise quasi-equivalence if and only if for all objects $x, y \in \mathcal{A}$ the morphism of symmetric spectra

$$\eta_{\mathcal{A}}(x,y) : \mathcal{A}(x,y) \longrightarrow Q(\mathcal{A})(x,y)$$

is a level equivalence. Since $Q(\mathcal{A})(x,y)$ is an Ω-spectrum we have the following commutative diagrams (for all $n \geq 0$)

$$\begin{tikzcd} \mathcal{A}(x,y)_n \arrow[r, "\widetilde{\delta_n}"] \arrow[d] & \Omega\mathcal{A}(x,y)_{n+1} \arrow[d] \\ Q(\mathcal{A})(x,y)_n \arrow[r, "\widetilde{\delta_n}"'] & \Omega Q(\mathcal{A})(x,y)_{n+1}\,, \end{tikzcd}$$

where the bottom and vertical arrows are weak equivalences of pointed simplicial sets. This implies that

$$\widetilde{\delta}_n : \mathcal{A}(x,y)_n \xrightarrow{\sim} \Omega\mathcal{A}(x,y)_{n+1}, \ \ n \geq 0$$

is a weak equivalence of pointed simplicial sets and so we conclude that for all objects $x, y \in \mathcal{A}$, $Q(x,y)$ is an Ω-spectrum. $\surd$

5.14. *Remark.* Notice that proposition 5.7 and remark 5.4 imply that $\eta_{\mathcal{A}} : \mathcal{A} \to Q(\mathcal{A})$ is a functorial fibrant replacement of $\mathcal{A}$ in the model structure of theorem 5.10.

Appendix A. Non-additive filtration argument

In this appendix, we adapt Schwede-Shipley's non-additive filtration argument [26] to a 'several objects' context. Let $\mathcal{V}$ be a monoidal model category, with cofibrant unit $\mathbb{I}$, initial object 0, and which satisfies the monoid axiom [26, 3.3].

A.1. **Definition.** *Let*

$$U : \mathcal{V} \longrightarrow \mathcal{V}\text{-}\mathit{Cat}\,,$$

be the functor which sends an object $X \in \mathcal{V}$ to the $\mathcal{V}$-category $U(X)$, with two objects 1 and 2 and such that $U(X)(1,1) = U(X)(2,2) = \mathbb{I}$, $U(X)(1,2) = X$ and $U(X)(2,1) = 0$. Composition is naturally defined (the initial object acts as a zero with respect to $\wedge$ since the bi-functor $- \wedge -$ preserves colimits in each of its variables).

In what follows, by *smash product* we mean the symmetric product $- \wedge -$ of $\mathcal{V}$.

A.2. **Proposition.** *Let $\mathcal{A}$ be a $\mathcal{V}$-category, $j : K \to L$ a trivial cofibration in $\mathcal{V}$ and $F : U(K) \to \mathcal{A}$ a morphism in $\mathcal{V}$-Cat. Then in the pushout*

$$\begin{tikzcd} U(K) \arrow[r, "F"] \arrow[d, "U(j)"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, "R"] \\ U(L) \arrow[r] & \mathcal{B}\,, \end{tikzcd}$$

the morphisms

$$R(x,y) : \mathcal{A}(x,y) \longrightarrow \mathcal{B}(x,y), \quad x,y \in \mathcal{A}$$

are weak equivalences in $\mathcal{V}$.

Proof. Notice that $\mathcal{A}$ and $\mathcal{B}$ have the same set of objects and the morphism R induces the identity on sets of objects. The description of the morphisms

$$R(x,y) : \mathcal{A}(x,y) \to \mathcal{B}(x,y), \quad x,y \in \mathcal{A}$$

in $\mathcal{V}$ is analogous to the one given by Schwede-Shipley in the proof of lemma [26, 6.2]. The 'ideia' is to think of $\mathcal{B}(x,y)$ as consisting of formal smash products of elements in L with elements in $\mathcal{A}$, with the relations coming from K and the composition in $\mathcal{A}$. Consider the same (conceptual) proof as the one of lemma [26, 6.2]: $\mathcal{B}(x,y)$ will appear as the colimit in $\mathcal{V}$ of a sequence

$$\mathcal{A}(x,y) = P_0 \to P_1 \to \cdots \to P_n \to \cdots\,,$$

that we now describe. We start by defining a n-dimensional cube in $\mathcal{V}$, i.e. a functor

$$W : \mathcal{P}(\{1,2,\ldots,n\}) \longrightarrow \mathcal{V}$$

from the poset category of subsets of $\{1,2,\ldots,n\}$ to $\mathcal{V}$. If $S \subseteq \{1,2,\ldots,n\}$ is a subset, the vertex of the cube at S is

$$W(S) := \mathcal{A}(x,F(0)) \wedge C_1 \wedge \mathcal{A}(F(1),F(1)) \wedge C_2 \wedge \cdots \wedge C_n \wedge \mathcal{A}(F(1),y)\,,$$

with

$$C_i = \begin{cases} K & \text{if} \quad i \notin S \\ L & \text{if} \quad i \in S\,. \end{cases}$$

The maps in the cube W are induced from the map $j : K \to L$ and the identity on the remaining factors. So at each vertex, a total of $n+1$ factors of objects in $\mathcal{V}$, alternate with n smash factors of either K or L. The initial vertex, corresponding to the empty subset has all its C_i's equal to K, and the terminal vertex corresponding to the whole set has all it's C_i's equal to L.

Denote by Q_n, the colimit of the punctured cube, i.e. the cube with the terminal vertex removed. Define P_n via the pushout in $\mathcal{V}$

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A}(x, F(0)) \wedge L \wedge (\mathcal{A}(F(1), F(1)) \wedge L)^{\wedge(n-1)} \wedge \mathcal{A}(F(1), y) \arrow[d] \\ P_{n-1} \arrow[r] & P_n\,, \end{tikzcd}$$

where the left vertical map is defined as follows: for each proper subset S of $\{1, 2, \ldots, n\}$, we consider the composed map

$$W(S) \longrightarrow \underbrace{\mathcal{A}(x, F(0)) \wedge L \wedge \mathcal{A}(F(1), F(1)) \wedge \ldots \wedge L \wedge \mathcal{A}(F(1), y)}_{|S| \text{ factors } L}$$

obtained by first mapping each factor of $W(S)$ equal to K to $\mathcal{A}(F(1), F(1))$, and then composing in $\mathcal{A}$ the adjacent factors. Finally, since S is a proper subset, the right hand side belongs to $P_{|S|}$ and so to P_{n+1}. Now the same (conceptual) arguments as those of lemma [26, 6.2] shows us that the above construction furnishes us a description of the $\mathcal{V}$-category $\mathcal{B}$.

We now analyse the constructed filtration. The cube W used in the inductive definiton of P_n has $n+1$ factors of objects in $\mathcal{V}$, which map by the identity everywhere. Using the symmetry isomorphism of $- \wedge -$, we can shuffle them all to one side and observe that the map

$$Q_n \longrightarrow \mathcal{A}(x, F(0)) \wedge L \wedge (\mathcal{A}(F(1), F(1)) \wedge L)^{\wedge(n-1)} \wedge \mathcal{A}(F(1), y)$$

is isomorphic to

$$\overline{Q_n} \wedge \mathcal{Z}_n \longrightarrow L^{\wedge n} \wedge \mathcal{Z}_n\,,$$

where

$$\mathcal{Z}_n := \mathcal{A}(x, F(0)) \wedge L \wedge (\mathcal{A}(F(1), F(1)) \wedge L)^{\wedge(n-1)} \wedge \mathcal{A}(F(1), y)$$

and $\overline{Q_n}$ is the colimit of a punctured cube analogous to W, but with all the smash factors diferent from K or L deleted. By iterated application of the pushout product axiom, the map $\overline{Q_n} \to L^{\wedge n}$ is a trivial cofibration and so by the monoid axiom, the map $P_{n+1} \to P_n$ is a weak equivalence in $\mathcal{V}$. Since the map

$$R(x, y) : \mathcal{A}(x, y) = P_0 \longrightarrow \mathcal{B}(x, y)$$

is the kind of map considered in the monoid axiom, it is also a weak equivalence and so the proposition is proven. $\surd$

A.3. **Proposition.** *Let $\mathcal{A}$ be a $\mathcal{V}$-category such that $\mathcal{A}(x, y)$ is cofibrant in $\mathcal{V}$ for all $x, y \in \mathcal{A}$ and $i : N \to M$ a cofibration in $\mathcal{V}$. Then in the pushout*

$$\begin{tikzcd} U(N) \arrow[r, "F"] \arrow[d, "U(i)"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, "R"] \\ U(M) \arrow[r] & \mathcal{B}\,, \end{tikzcd}$$

the morphisms

$$R(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(x, y), \quad x, y \in \mathcal{A}$$

are cofibrations in $\mathcal{V}$.

Proof. The description of the morphisms

$$R(x,y) : \mathcal{A}(x,y) \longrightarrow \mathcal{B}(x,y), \quad x, y \in \mathcal{A}$$

is analogous to the one of proposition A.2. Since for all $x, y \in \mathcal{A}$, $\mathcal{A}(x,y)$ is cofibrant in $\mathcal{V}$, the pushout product axiom implies that in this situation the map

$$\overline{Q_n} \wedge \mathcal{Z}_n \longrightarrow L^{\wedge n} \wedge \mathcal{Z}_n$$

is a cofibration. Since cofibrations are stable under co-base change and transfinite composition, we conclude that the morphisms $R(x,y)$, $x, y \in \mathcal{A}$ are cofibrations. $\surd$

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