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1 Spectral Categories

1.1 Summary

1.1.1 Important definitions

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1.1.2 Notation

- Cat_S is the category of small spectral categories and spectral functors.
- Cat_T is the category of small simplicial categories and simplicial functors.

1.1.3 Key results

1. There's a Quillen adjunction

$$Cat_S \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} Cat_T$$

where $\Omega^\infty(F(A,B))$ is the zeroth space functor or equivalently the simplicial set $[n] \mapsto Hom_{Spt}(\mathbb{S} \otimes \Delta^n, F(A,B)) \cong Hom_{Spc}(\Delta^n, \Omega^\infty F(A,B)) \cong \Omega^\infty F(A,B)_n$. This is the main theorem of Tabuada's paper "Homotopy Theory of Spectral Categories" (see references). We can modify Cat_S up to weak equivalence to make this a simplicial Quillen adjunction.

Remark 1. It might also be helpful to recall that, given a monoidal functor from a monoidal category M to a monoidal category N, any category enriched over M can be reinterpreted as a category enriched over N. Furthermore, we recover the underlying category of an enriched category by considering the functor $M(I,-) : M \to Set$ where I is the unit of the monoidal structure.

2. There's a model category $\widehat{\mathcal{A}}$ of spectral $\mathcal{A}$-modules (these are defined as functors, so we put the projective model structure on the functor category). The referenced paper here is Stable model categories are categories of modules by Schwede and Shipley.

1.2 Clarifying Remarks

- The definition of A-module is a generalization of the ordinary one. Recall that a ring is equivalently a preadditive category (an Ab-enriched category) with one object, and a module is a functor from this ring to abelian groups. Along the same vein, a ring spectrum is a spectral category with one object, and a module over this ring spectrum is a functor from this category to spectra.
- A category of enriched functors is itself enriched. Recall that $Nat(F,G) = \int_C Hom(F-, G-)$. Via the theory of enriched ends we can define a mapping spectrum to be $\int_C A(F-, G-)$, where A denotes the mapping spectrum of two objects in our enriched category.
- Given a functor $F : \mathcal{A} \to \mathcal{B}$, we get an obvious restriction functor $F^* : modB := \widehat{B} \to \widehat{A}$ given by precomposition with F (recall the definition of a module as a functor!). It has a left adjoint functor $F_!$ given by an enriched coend. It sends an A-module N to the coequalizer of

$$\bigvee_{o,p \in A} N(p) \wedge A(o,p) \wedge B(-, F(o)) \rightrightarrows \bigvee_{o \in A} N(o) \wedge B(-, F(o)) .$$

Recall that representable objects are "free on one generator". Indeed, the usual definition of a free object is that maps out of it are determined by maps out of a basis. We know, by the Yoneda lemma, that maps out of representable objects are determined by maps out of the identity morphism.

- The paper omits the definition of a pretriangulated spectral category. It's in the Mandell-Blumberg paper in the references, definition 5.4.

1.3 Background Material

While the theory of infinity categories allows us to work "coordinate-free" in some sense, we first need some examples of meaningful infinity categories and functors between them. One way to get a slew of presentable ∞-categories is to take the underlying infinity category of a left proper, simplicial, combinatorial model category. We'll define the simplicial nerve after introducing combinatorial model categories.

1.3.1 Combinatorial Model Categories

Definition 2. An infinite cardinal κ is a *regular cardinal* if it satisfies the following property, which we think of as requiring the collection of sets of smaller cardinality to be "closed under union":

- no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ.

Example 3. $\aleph_0$, or the cardinality of $\mathbb{N}$, is a regular cardinal. This is because any set of cardinality less than $\aleph_0$ is finite, and no infinite set is a finite union of finite sets.

Example 4. Any successor cardinal is regular. For $\aleph_1$, this follows from the fact that a countable union of countable sets is countable (we need choice here).

Definition 5. Let κ be an infinite regular cardinal. Then a *κ-filtered category* is one such that any diagram $F : D \to C$, where D has fewer than κ morphisms admits an extension $\widetilde{F} : D^+ \to C$ (i.e. F has a cocone). Here D^+ is the category obtained by freely adjoining a terminal object to D.

Example 6. Let $\kappa = \omega$ (or $\aleph_0$ in the notation we've been using). Then we're requiring every FINITE diagram in C to have a cocone. This is equivalent to the usual definition of a filtered category: a nonempty category s.t. each pair of objects has a join, and for any parallel morphisms $f, g : c_1 \to c_2$ in C there exists a morphism $h : c_2 \to c_3$ such that $hf = hg$. In other words, we can build a cocone for any finite diagram from these cocones.

Example 7. A preorder (there exists a unique morphism between any two objects) is ω-filtered precisely when it is *directed*, i.e. any two objects have a join.

Definition 8. Let κ be a regular cardinal. Then an object X such that $C(X, -)$ commutes with κ-filtered colimits is called *κ-compact*.

Definition 9. An object X of a category is *small* if it is κ-compact for some regular cardinal κ.

Definition 10. A category C is *locally presentable* if

1. C is a locally small category
2. C has all small colimits
3. there exists a small set $S \hookrightarrow Obj(C)$ of λ-small objects that generates C under λ-filtered colimits.
4. every obect in C is a small object.

Definition 11. Let C be a category and $I \subset Mor(C)$. Let $cell(I)$ be the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in I.

Definition 12. A model category C is *cofibrantly generated* if there are small sets of morphisms $I, J \subset Mor(C)$ such that

- $cof(I)$, the set of retracts of elements in $cell(I)$, is precisely the collection of cofibrations of C.

- $cof(J)$ is precisely the collection of acyclic cofibrations in C; and
- I and J permit the small object argument.

Definition 13. (Smith) A model category C is *combinatorial* if it is

- locally presentable as a category, and
- cofibrantly generated as a model category.

Example 14. The category $sSet$ with the standand model structure on simplicial sets is a combinatorial model category.

Example 15. The category $sSet$ with the Joyal model structure (so that the quasi-categories are the fibrant objects) is combinatorial.

Question 1. Why should we care that a model category is combinatorial?

We'll define a left-proper model category, then give a fundamental result of Dugger's:

Definition 16. A model category is *left proper* if weak equivalence is preserved by pushout along cofibrations.

Example 17. A model category in which all objects are cofibrant is left proper. This includes the standard model structure on simplicial sets, as well the injective model structure on simplicial presheaves. This follows from the Reedy lemma, which allows us to calculate homotopy pushouts by considering diagrams s.t. the objects are cofibrant and one of the maps is a cofibration (the point is that replacing this diagram with a cofibrant diagram in the projective model structure on diagrams is an acyclic cofibration of diagrams, not just a weak equivalence).

Theorem 1. *(Dugger)*
Every combinatorial model category is Quillen equivalent to a left proper simplicial combinatorial model category.

Now, we have the following extremely useful result on Bousfield localizations:

Theorem 2. *If C is a left proper, simplicial, combinatorial model category, and $S \subset Mor(C)$ is a small set of morphisms, then the left Bousfield localization $L_S C$ does exist as a combinatorial model category. Moreover, the fibrant objects of $L_S C$ are precisely the S-local objects, and L_S is left proper and simplicial.*

In the context of infinity categories, we have some crucial results of Lurie. For these to make sense, however, we need to introduce the simplicial nerve construction. This is the generalization of the ordinary nerve construction to simplicially enriched categories.

Definition 18. We'll define a cosimplicial simplcially enriched category S. The objects of $S[n]$ are $\{0, 1, \ldots, n\}$; the hom objects $S[n]_{i,j} \in sSet$ for $i, j \in \{0, 1, \ldots, n\}$ are the nerves

$$S[n](i,j) = N(P_{i,j})$$

of the poset $P_{i,j}$ which is the poset of subsets of $[i,j]$ that contain both i and j with partial order given by inclusion.

Definition 19. The simplicial nerve of a simplicial category is the simplicial set characterized by

$$Hom_{sSet}(\Delta[n], N(C)) = Hom_{SSetCat}(S[n], C).$$

Theorem 3. *(HTT A.3.7.6) Let C be an ∞-category. The following conditions are equivalent:*

1. *The ∞-category C is presentable.*
2. *There exists a combinatorial simplicial model category A and an equivalence $C \cong N(A^\circ)$.*

Here A° is the underlying category of bifibrant objects.

Remark 20. (HTT A.3.7.7)

Let A and B be combinatorial simplicial model categories. Then the underlying ∞-categories $N(A^\circ)$ and $N(B^\circ)$ are equivalent iff A and B can be joined by a chain of simplicial Quillen equivalences.

Theorem 4. *(HTT 5.2.4.6) Let A and B be simplicial model categories, and let*

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

be a simplicial Quillen adjunction. This descends to an adjunction on the underlying ∞-categories.

2 Stable ∞-categories

2.1 Summary

2.1.1 Important definitions

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2.1.2 Notation

- Cat_∞ is the ∞-category of small ∞ categories.
- Cat_∞^{ex} is the ∞-category of small stable ∞-categories and exact functors.
- Cat_∞^{perf} is the ∞-category of small idempotent complete stable ∞-categories (and exact functors).

2.1.3 Key results

1. If C is a pretriangulated spectral category, then $\pi_0 C$ admits a triangulated category structure.
2. If C is a stable ∞-category, then hoC admits a triangulated category structure.
3. The inclusion $Cat_\infty^{perf} \to Cat_\infty^{ex}$ has a left adjoint called idempotent completion.
4. If C is a ∞-category with finite limits, there is a stable ∞-category $Stab(C)$ with a limit preserving functor $\Omega^\infty : Stab(C) \to C$. In particular, this is accomplished by taking $Stab(C) = Sp(C)$, the category of spectrum objects of C. If C is already stable, then Ω^∞ is an equivalence, with inverse $\Sigma^\infty : C \to Sp(C)$.
5. If C is a stable ∞-category, there is a *spectral Yoneda embedding*
$$Y : C \simeq Sp(C_*) \to Sp(Fun(C^{op}, N(T)^{cf})_*) \simeq Fun(C^{op}, S_\infty)$$
with an adjoint *mapping spectrum functor*
$$Map : C^{op} \times C \to S_\infty$$
6. If C is a stable ∞-category, a $C^{op} \to S_\infty$ is stably representable if and only if it is represented by the suspension spectrum $\Sigma^\infty z$ of some $z \in C$, and this object is unique upto equivalence.

2.2 Clarifying Remarks

1. Idempotent completion is unique upto equivalence by HTT Proposition 5.1.4.9. Existence is obtained by taking the full subcategory of $Pre(C)$ spanned by objects that are retracts of objects in the image of C under the Yoneda embedding.
2. For this paper we assume stable ∞-categories to be pointed, i.e. they have zero objects. In the description of spectral Yoneda embedding above, $*$ indicates the subcategory of pointed objects.

2.3 Background Material

2.3.1 Pretriangulated Spectral Categories

The material here is from Section 5 of Mandell-Blumberg (in the higher K-theory paper, they say Section 4 but that's either a typo or just not updated).

Definition 21. A spectral category C is *pretriangulated* if

1. There exists an object 0 of C such that $C(-,0)$ is homotopically trivial. This means that it is weakly equivalent to the constant functor $*$ at the one point symmetric spectrum.
2. Whenever a C-module M has the property that ΣM is weakly equivalent to a representable C-module $C(-,c)$, then M is weakly equivalent to some representable C-module $C(-,d)$.
3. Whenever the C-modules M and N are weakly equivalent to representables $C(-,a)$ and $C(-,b)$ respectively, the homotopy cofiber of any map of C-modules $M \to N$ is weakly equivalent to a representable C-module.

Remark 22. The first condition says that the homotopy category $\pi_0 C$ has a zero object. The second condition gives a desuspension functor on $\pi_0 C$ and the third condition gives a suspension functor on $\pi_0 C$.

Definition 23. A spectral functor between spectral categories $F : C \to D$ is a *DK-embedding* if for all objects a, b in C, the induced map of spectra $C(a,b) \to D(Fa, Fb)$ is a weak equivalence.

Remark 24. In Blumberg-Mandell, a DK equivalence is a DK embedding satisfying one of the following equivalent conditions

1. For all object d of D, there is an object c of C such that $D(-,d)$ and $D(-,Fc)$ are naturally isomorphic as D-modules.
2. The induced functor on the "graded homotopy categorie" $\pi_* C \to \pi_* D$ is an equivalence of categories.

In Blumberg-Gepner-Tabuada, this second condition is relaxed to $\pi_0 C \to \pi_0 D$ being an equivalence of categories. This is because a spectral functor between pretriangulated spectral categories is a DK equivalence if and only if it is a DK embedding and the induced functor on π_0 is an equivalence of categories.

Theorem 5. *(Blumberg-Mandell, Theorem 5.5) Any small spectral category C DK-embeds into a small pretriangulated spectral category.*

Remark 25. We should think of this as taking the closure of C under cofibration sequences and desuspensions in C-modules, via the Yoneda embedding. Note that this can be made functorial and it gives the "minimal pretriangulated closure" of C.

Definition 26. A *four-term Puppe sequence* in $\pi_0 C$ is a sequence of the form

$$a \to b \to c \to \Sigma a$$

if there exist a map of C-modules $f : M \to N$ such that the sequence

$$M \to N \to Cf \to \Sigma M$$

is isomorphic to the above sequence in the derived category of C-modules via the Yoneda embedding, and further the equvalence $\Sigma M \simeq C(-, \Sigma a) \simeq \Sigma C(-,a)$ is the suspension of the isomorphism $M \simeq C(-,a)$.

Theorem 6. *(Blumberg-Mandell, Theorem 5.6) Given a pretriangulated spectral category C, its homotopy category $\pi_0 C$ is triangulated with distinguished triangles the above four-term Puppe sequences.*

Proof. Proof of Theorem 5.6 is just observing that $\pi_0 C$ embeds as a full subcategory of the homotopy category of C-modules (with projective model structure) and checking that it is closed under suspensions, desuspensions and distinguished triangles. □

2.3.2 ∞-Categories

Definition 27. The **Joyal model structure** on simplicial sets is defined as follows :

- Cofibrations are levelwise monomorphisms.
- Weak equivalences are *(weak) categorical equivalence*. These are maps $f : A \to B$ such that for any ∞-category X, the map $X^B \to X^A$ induces an isomorphism on the fundamental category (the homotopy category) hoC.
- Fibrations are determined by the above.

Remark 28. All objects are cofibrant, and the fibrant objects are precisely ∞-categories.

Theorem 7. *(HA Theorem 1.1.2.15) Let C be a stable ∞-category. Then hoC has a triangulated category structure with distinguished triangles coming from cofiber sequences.*

Remark 29. There is a correspondance between pretriangulated spectral categories and stable ∞-categories. For example, if C is a pretriangulated spectral category, then $N((ModC)^{cf})$ is a stable ∞-category.

2.3.3 Idempotent completion

HTT 4.4.5 gives a good overview on the definition of idempotent completion for ∞-categories, while comparing it with the classical notion.

Remark 30. The notion of retracts between classical and ∞-categorical settings are a bit different. An ordinary category X is said to be *idempotent complete* if every idempotent map $X \to X$ comes from some retract Y of X. In such a situation Y can be determined uniquely as an equalizer (or a coequalizer). Hence, if C has finite limits or finite colimits, then C is idempotent complete.

This is not the case for ∞-categories. Consider the category $C_*(R)$ consisting of bounded chain complex of finite rank free R-modules and consider $N(C_*(R))$, which is actuallt a stable ∞-category. Hence it admits finite limits and colimits, but it is idempotent complete if and only if every finitely generated projective R-module is stably free.

The problem is that an idempotent in an ∞-category shouldn't be just a morphism e with $e \circ e \simeq e$ in hoC. It should specify homotopies on how to relate multiple compositions $e \circ e \circ ... \circ e \simeq e$. To achieve this, in HTT 4.4.5, simplicial sets called $Idem^+, Idem$ and Ret are introduced. Now idempotents, weak retractions, strong retractions in C are respectively defined to be maps of simplicial sets from $Idem, Ret, Idem^+$ to C. C is *idempotent complete* if every idempotent $F : Idem \to C$ has a colimit. This can be shown to be equivalent to the definition in the paper using results in HTT 5.1.4 and 5.1.5.