AN E_∞ STRUCTURE ON THE MOTIVIC SPECTRUM REPRESENTING HERMITIAN K-THEORY

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BY

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DISSERTATION

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Introduction

Hi I am a thesis.

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Equivariant Topologies

Notation 1. Throughout this section, G will be either a finite group or the group scheme over S associated to a finite group. Recall that to pass between finite groups and group schemes over S, we form the scheme $\coprod_G S$ with multiplication (using that fiber products commute with coproducts in Sch/S):

$$\textstyle\coprod_G S \times_S \coprod_G S \xrightarrow{\sim} \coprod_{(g_1,g_2)\in G\times G} S \xrightarrow{\mu} \coprod_G S$$

Whenever we write down a pullback square involving schemes, we'll tacitly be thinking of G as a group scheme, and $X \times Y$ will really mean $X \times_S Y$.

Definition 2. For a G-scheme X, the isotropy group scheme is a group scheme G_X over X defined by the cartesian square

$$\begin{tikzcd} G_X \arrow[r] \arrow[d] & G \times X \arrow[d, "(\mu_X, id_X)"] \\ X \arrow[r, "\Delta_X"] & X \times X \end{tikzcd}.$$

Definition 3. Let X be a G-scheme. The scheme-theoretic stabilizer of a point x in X is the pullback

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & G_X \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] & X. \end{tikzcd}$$

By the pasting lemma, this is the same as the pullback

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & G \times X \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] & X \times X. \end{tikzcd}$$

Definition 4. Let X be a G-scheme, and define the *set-theoretic* stabilizer S_x of $x \in X$ to be $\{g \in G \mid gx = x\}$ where we think of G as a finite group.

Remark 5. With notation as above, the underlying set of the scheme-theoretic stabilizer G_x can be described as

$$G_x = \{g \in S_x \mid \text{ the induced morphism } g : k(x) \to k(x) \text{ equals } id_{k(x)}\}.$$

The example below shows that set-theoretic and scheme-theoretic stabilizers need not agree.

Example 6. (Herrmann [1]) Let k be a field, and consider the k-scheme given by a finite Galois extension $k \hookrightarrow L$. Let $G = Gal(L/k)$ be the Galois group. The set-theoretic stabilizer of the unique point in $\operatorname{Spec} L$ is G itself, while the scheme-theoretic stabilzer is $\{e\} \subset G$.

Remark 7. Recall that if $Z \to X$ is a monomorphism of schemes, then the forgetful functor from schemes to sets preserves any pullback $Z \times_X Y$. The canonical examples of monomorphisms in schemes are closed embeddings, open immersions, and maps induced by localization.

Recall as well that the forgetful functor $GSch/S \to Sch/S$ is a right adjoint, hence preserves pullbacks.

Since the inclusion of a point $\operatorname{Spec} k(x) \hookrightarrow X \times_S X$ will be a closed embedding for any separated scheme, the difference between the set-theoretic and scheme-theoretic stabilizers is given by the fact that the underlying space of $X \times_S X$ is not necessarily the fiber product of the underlying spaces. Indeed, in the example above, $\operatorname{Spec} L \times_k \operatorname{Spec} L \cong \coprod_{g \in G} \operatorname{Spec} k$, whereas the pullback in spaces is just a single point.

2.0.1 The equivariant Étale topology

Notation 8. Let S be a G-scheme. The equivariant étale topology on Sm_S will denote the site whose covers are étale covers whose component morphisms are equivariant.

Definition 9. (Thomason) An equivariant map $f : Y \to X$ is said to be *isovariant* if it induces an isomorphism $G_Y \cong G_X \times_X Y$. A collection $\{f_i : X_i \to X\}_{i \in I}$ of equivariant maps called an isovariant etale cover if it is an equivariant etale cover such that each f_i is isovariant.

Remark 10. The isovariant topology is equivalent to the topology whose covers are equivariant, stabilizer preserving, étale maps. We'll use this notion more often in computations.

Remark 11. The points in the isovariant étale topology are schemes of the form $G \times^{G_x} \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}})$ where $\overline{x} \to x \to X$ is a geometric point, and $(-)^h$ denotes strict henselization.

Remark 12. If $G = C_2$, then $G_x = \{e\}$ or $G_x = C_2$ for all $x \in X$. If $G_x = \{e\}$, then $G \times^{G_x} \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}}) \cong C_2 \times \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}}) \cong \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}}) \coprod \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}})$ with a free action. If $G_x = C_2$, then $G \times^{G_x} \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}}) = \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}})$.

The following example shows that there are equivariant étale covers which are not isovariant:

Example 13. Fix a scheme X with trivial C_2-action, and consider the scheme $X \coprod X$ with the switch action. The map $X \coprod X \to X$ is an equivariant étale cover, but it is not stabilizer preserving. Indeed, the switch action on $X \coprod X$ is free, and the set-theoretic (hence scheme-theoretic) stabilizers are all trivial. On the other hand, the scheme theoretic stabilizers of the trivial action are all C_2.

Lemma 14. *Fix a ring R, and fix an ideal $I \subset R$, $J \subset R[x]$. Let $B = R[x]/J$. Then $B/IB \cong (R/I)[x]/\overline{J}$.*

Proof. First, recall that $R[x]/IR[x] \cong (R/I)[x]$ by the obvious map reducing the coefficients of a polynomial. Then $B/IB \cong R[x]/(IR[x] + J) \cong (R[x]/IR[x])/J \cong (R/I)[x]/\overline{J}$. □

Example 15. Let R be a commutative ring with 2 invertible and involution $- : R \to R$. Let $a \in R^\times$. Then $\operatorname{Spec} R[\sqrt{a}, \sqrt{\overline{a}}] \to \operatorname{Spec} R$ is an equivariant étale cover.

Proof. First, note that if a has a square root in R, so does $\overline{a}$, and the result is trivial. Assume that this is not the case. Give the ring $R[\sqrt{a}, \sqrt{\overline{a}}]$ an action by $r_0 + r_1\sqrt{a} + r_2\sqrt{\overline{a}} \mapsto \overline{r}_0 + \overline{r}_1\sqrt{\overline{a}} + \overline{r}_2\sqrt{a}$. The map $R \to R[\sqrt{a}, \sqrt{\overline{a}}]$ is clearly equivariant, so we need only check that it's an étale cover.

First, note that it is indeed a cover: because $R[\sqrt{a}, \sqrt{\overline{a}}]$ is a module-finite extension of R (hence integral), surjectivity after taking Spec follows from the injectivity of the map of rings by the lying over property for integral extensions.

Now, we claim that the map is étale. We'll prove that it's the composite of two étale maps, $R \to R[\sqrt{a}] \to R[\sqrt{a}, \sqrt{\overline{a}}]$. Since $\overline{a}$ must also be a unit, it's enough to show that $R \to R[\sqrt{a}]$ is étale. It's clearly flat because $R[\sqrt{a}]$ is a free module over R, so we just have to check that it's unramified. Let $B = R[\sqrt{a}]$. Fix a maximal ideal $m \subset B$, and let $I = R \cap m$. By the lemma above,

$$\frac{B}{IB} \cong (R/I)[x]/(x^2 - a) \cong (R/I)[\sqrt{a}].$$

Now if $a \neq 0$ in R/I, then $x^2 - a$ will be a separable polynomial. But because a is a unit, it's not contained in any prime ideal, and hence not contained in I.

An easy consequence of the going up theorem (recall that we have an integral extension), is that I is a maximal ideal in R; hence, $(R/I)[\sqrt{a}, \sqrt{\overline{a}}]$ is a finite separable field extension of R/I. Since localization commutes with taking quotients, it follows that the map is unramified. □

Example 16. A similar argument shows that $\operatorname{Spec} R[\sqrt{a}] \coprod \operatorname{Spec} R[\sqrt{\overline{a}}] \to \operatorname{Spec} R$ is an equivariant étale cover.

Lemma 17. *With notation as above, assume that a is a fixed point of the involution $- : R \to R$. There's an induced action on $R[\sqrt{a}]$ which fixes $\sqrt{a}$, and the map $\operatorname{Spec} R[\sqrt{a}] \to \operatorname{Spec} R$ is stabilizer preserving w.r.t. this action.*

Proof. Let $p \subset R[\sqrt{a}]$ be a prime ideal such that $\overline{p} = p$. Let g denote the non-trivial element of C_2. The induced map on stalks is (by abuse of notation) the inclusion $f : k(p \cap R) \hookrightarrow k(p \cap R)[\sqrt{a}]$. By equivariance, we have a commutative diagram

$$\begin{tikzcd} k(p \cap R) \arrow[r, "f"] \arrow[d, "\widetilde{g}"] & k(p \cap R)[\sqrt{a}] \arrow[d, "g"] \\ k(p \cap R) \arrow[r, "f"] & k(p)[\sqrt{a}]. \end{tikzcd}$$

Now if g induces the identity map $k(p \cap R) \to k(p \cap R)$, and hence is an element of the scheme-theoretic stabilizer, then $\widetilde{g}$ is an element of $Gal(k(p \cap R)[\sqrt{a}]/k(p \cap R)$. In other words, $\widetilde{g}$ is either the identity map, or is the map which sends $\sqrt{a} \to -\sqrt{a}$. By construction, the involution on $R[\sqrt{a}]$ sends $\sqrt{a} \mapsto \sqrt{a}$, so that $G_p = G_{f(p)}$.

If g doesn't fix $k(p \cap R)$, then since f is a monomorphism, clearly $\widetilde{g}$ can't fix $k(p \cap R)[\sqrt{a}]$, and again we have $G_p = G_{f(p)}$. It follows that f is an isovariant map. □

Lemma 18. *With notation as above, say $a - \overline{a} \in R^*$. The equivariant étale cover $f : \operatorname{Spec} R[\sqrt{a}] \coprod \operatorname{Spec} R[\sqrt{\overline{a}}] \to \operatorname{Spec} R$ is stabilizer preserving.*

Proof. The action on $\operatorname{Spec} R[\sqrt{a}] \coprod \operatorname{Spec} R[\sqrt{\overline{a}}] \to \operatorname{Spec} R$ is free, so that all the set-theoretic (and hence scheme-theoretic) stabilizers are trivial.

The assumption that $a - \overline{a}$ is not in any prime ideal implies that if p is a fixed point of the involution, $i : R_p/pR_p \to R_p/pR_p$ is not the identity map, so that the scheme-theoretic stabilizers of the action on $\operatorname{Spec} R$ are all trivial. □

Example 19. Even if a is a unit, it's certainly not true in general that $a - \overline{a} \in R^*$. Consider the ring $R = \mathbb{Z}[t, t^{-1}]$ with involution given by $t \mapsto -t$. Then $t - \overline{t} = 2t \notin R^*$. Furthermore, $(2t) = (2)$ is a prime ideal in R fixed by the involution. It's contained in the maximal ideal $(2, t-1)$. Note that this ideal is also fixed by the involution: $t-1 \mapsto -t-1$ and $-t-1 = 1-t-2 \in (2, t-1)$. The residue field at this maximal ideal is $\mathbb{Z}/2$. The only nonzero ring map of this field is the identity, so that the scheme-theoretic stabilizer of $(2, t-1)$ is C_2.

Note that if we wanted an example for a ring with 2 invertible, we could take $R = \mathbb{Z}[\frac{1}{2}][t, t^{-1}]$, and consider the element $\frac{3}{2}t$ and the maximal ideal $(3, t - \frac{3}{2})$. One also has to note that the induced map on the residue field $\mathbb{Z}/3$ is the identity, which follows simply because the involution is unital (and because it gives a well-defined map on the residue field!).

2.0.2 The equivariant Nisnevich topology

Similarly to the non-equivariant case, the equivariant Nisnevich topology is defined a particularly nice cd-structure. While there are a few different definitions of this topology in the literature which can give non-Quillen equivalent model structures, we use the definition from [2].

Definition 20. A distinguished equivariant Nisnevich square is a cartesian square in $\mathbf{Sm}_S^G$

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, hook, "i"] & X \end{tikzcd}$$

where j is an open immersion, p is étale, and $(Y - B)_{\mathrm{red}} \to (X - A)_{\mathrm{red}}$ is an isomorphism.

Definition 21. The equivariant Nisnevich *cd*-structure on $\mathbf{Sm}_S^G$ is the collection of distinguished equivariant Nisnevich squares in $\mathbf{Sm}_S^G$.

Remark 22. For finite groups G, any smooth G-scheme is Nisnevich-locally affine.

2.0.3 Computations with Equivariant Spheres

Because we'll be using equivariant spheres to index our spectra, we'll record some of their basic properties here. Though there are exotic elements of the Picard group even in non-equivariant stable motivic homotopy theory, we'll be concerned with the four building blocks $S^1, S^\sigma = \operatorname{colim}((C_2)_+ \to S^0), \mathbb{G}_m, \mathbb{G}_m^\sigma$.

Lemma 23. *Let $\mathbb{P}^\sigma$ denote $\mathbb{P}^1$ with the action defined by $[x : y] \mapsto [y : x]$. There is an equivariant Nisnevich square*

$$\begin{tikzcd} C_2 \times \mathbb{G}_m^\sigma \arrow[r] \arrow[d, "\pi_2"] & \mathbb{P}^1 - \{0\} \coprod \mathbb{P}^1 - \{\infty\} \arrow[d, "f"] \\ \mathbb{G}_m^\sigma \arrow[r, "i"] & \mathbb{P}^\sigma \end{tikzcd}$$

Proof. Here, we identify $\mathbb{G}_m^\sigma$ with $\mathbb{P}^\sigma - \{0, \infty\}$. The map i is clearly an open immersion. Its complement is $\{0, \infty\}$, and f maps $\pi^{-1}(\{0, \infty\})$ isomorphically onto $\{0, \infty\}$. Furthermore, f is a disjoint union of open immersions, and hence is (in particular) etale. $\square$

Lemma 24. *The following square is a homotopy pushout square:*

$$\begin{tikzcd}
(C_2)_+ \wedge (G_m^\sigma)_+ \arrow[r] \arrow[d, "\pi_2"] & (C_2)_+ \arrow[d, "f"] \\
(G_m^\sigma)_+ \arrow[r, "i"] & \mathbb{P}^1_+
\end{tikzcd}$$

Proof. The above square is equivalent to the square

$$\begin{tikzcd}
(C_2)_+ \wedge (G_m^\sigma)_+ \arrow[r] \arrow[d, "\pi_2"] & (C_2)_+ \wedge \mathbb{A}^1_+ \arrow[d, "f"] \\
(G_m^\sigma)_+ \arrow[r, "i"] & \mathbb{P}^1_+
\end{tikzcd}.$$

By the lemma above,

$$\begin{tikzcd}
(C_2 \times \mathbb{G}_m^\sigma)_+ \arrow[r] \arrow[d, "\pi_2"] & (C_2 \times \mathbb{A}^1)_+ \arrow[d, "f"] \\
(G_m^\sigma)_+ \arrow[r, "i"] & \mathbb{P}^1_+
\end{tikzcd}$$

is a homotopy pushout square. But adding a disjoint basepoint is a monoidal functor, so $X_+ \wedge Y_+ \cong (X \times Y)_+$ and this square is equivalent to the desired square. $\square$

Lemma 25. $\mathbb{P}^\sigma \approx S^\sigma \wedge \mathbb{G}_m^\sigma$.

Proof. Let Q denote the homotopy cofiber of $(C_2 \times \mathbb{G}_m^\sigma)_+ \to (\mathbb{G}_m^\sigma)_+$, and $\widetilde{Q}$ denote the homotopy cofiber of $(C_2 \times \mathbb{A}^1)_+ \to \mathbb{P}^\sigma_+$. Then the lemma above implies that $Q \approx \widetilde{Q}$.

Q is the homotopy cofiber of $(C_2)_+ \wedge (G_m^\sigma)_+ \to S^0 \wedge (G_m^\sigma)_+$, which is just $S^\sigma \wedge (G_m^\sigma)_+$. Recall that $\operatorname{colim}(* \leftarrow X \to X \wedge Y_+) \cong X \wedge Y$ since this is $X \wedge \operatorname{colim}(* \leftarrow S^0 \to Y_+)$. Thus the cofiber of $S^\sigma \to Q$ is $S^\sigma \wedge \mathbb{G}_m^\sigma$.

The diagram below in which the horizontal rows are cofiber sequences

$$\begin{tikzcd}
(C_2)_+ \arrow[r] \arrow[d, "id"] & S^0 \arrow[r] \arrow[d] & S^\sigma \arrow[d] \\
(C_2)_+ \arrow[r] \arrow[d] & \mathbb{P}^\sigma_+ \arrow[r] \arrow[d] & \widetilde{Q} \arrow[d] \\
\star \arrow[r] & \mathbb{P}^\sigma \arrow[r] & T
\end{tikzcd}$$

implies that the cofiber of $S^\sigma \to \widetilde{Q}$ is $\mathbb{P}^\sigma$.

The result now follows from the commutativity of the following diagram and homotopy invariance of homotopy cofiber:

$$\begin{tikzcd}
S^\sigma \arrow[r, "id"] \arrow[d] & S^\sigma \arrow[d] \\
Q \arrow[r, "\sim"] \arrow[d] & \widetilde{Q} \arrow[d] \\
S^\sigma \wedge \mathbb{G}_m^\sigma \arrow[r] & \mathbb{P}^\sigma
\end{tikzcd}.$$

□

Chapter 3

Hermitian Forms on Schemes

3.0.1 Definitions

Definition 26. Let R be a ring with involution $- : R \to R$. A *hermitian module over* R is a finitely generated projective module-R, M, together with a map

$$b : M \otimes_{\mathbb{Z}} M \to R$$

such that, for all $a \in R$,

1. $b(xa, y) = \overline{a} b(x, y)$,
2. $b(x, ya) = b(x, y)a$,
3. $b(x, y) = \overline{b(y, x)}$.

Definition 27. Let R be a ring with involution $-$. Given a right R-module M, define a left R-module, denoted $\overline{M}$ as follows: $\overline{M}$ has the same underlying abelian group as M, and the action is given by $r \cdot m = m \cdot \overline{r}$. If R is commutative, we can define an R-bimodule by $m \cdot r = m\overline{r}$ and $r \cdot m = m\overline{r}$.

Remark 28. Let R be a commutative ring. Given an involution $\sigma : R \to R$, and an $R-R$-bimodule M as above, we can identify $\overline{M}$ with $\sigma_* M$. Indeed, $\sigma_* M$ is an $R-R$-bimodule via the rule $r \cdot \overline{m} = \sigma(r)\overline{M}$, and since R is commutative, we can view this either as a left or right R-module.

Remark 29. Another way to define a Hermitian form over a ring R with involution σ is to give a finitely generated projective mod-R M together with an $R-R$-bimodule map

$$b : M \otimes_{\mathbb{Z}} M \to R$$

where we view R as a bimodule over itself just by $r_1 \cdot r \cdot r_2 = r_1 r r_2$, M as a left R-module via the involution, and such that $b(x, y) = \sigma(b(y, x))$. If we remove the final condition, we obtain a sesquilinear form.

By the usual duality, we have a third definition:

Definition 30. A hermitian module over a ring R with involution is a finitely generated projective R-module M together with an R-linear map $b : M \to \overline{M}^{\vee} = M^*$ such that $b = b^* can_M$, where $b^* : M^{**} \to M^*$ is given by $(b(f))(m) = f(b(m))$.

Now, we generalize the above definitions to schemes.

Definition 31. Let X be a scheme, and M a quasi-coherent (locally of finite presentation) $\mathcal{O}_X$-module. Define $\mathcal{O}_X{}^{\vee} = \underline{Hom}(M, \mathcal{O}_X)$.

Definition 32. Let X be a scheme with involution σ, and M a right $\mathcal{O}_X$-module. Note that there's an induced map $\sigma^{\#} : \mathcal{O}_X \to \sigma_*\mathcal{O}_X$. Define the right (note that we're working with sheaves of commutative rings, so we can do this) $\mathcal{O}_X$-module $\overline{M}$ to be σ_*M with $\mathcal{O}_X$ action induced by the map $\sigma^{\#}$. That is, if $m \in \sigma_*M(U)$, and $c \in \mathcal{O}_X(U)$, then $m \cdot c = m \cdot \sigma^{\#}(c)$. Note that this last product is defined, because $m \in \sigma_*M(U) = M(\sigma^{-1}(U))$, $c \in \sigma_*\mathcal{O}_X(U) = \mathcal{O}_X(\sigma^{-1}(U))$, and M is a right $\mathcal{O}_X$-module.

Remark 33. We have two choices for the definition of the dual M^*. We can either define $M^* = Hom_{mod-\mathcal{O}_X}(\sigma_*M, \mathcal{O}_X)$, or we can define $M^* = \sigma_* Hom_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)$. We claim that these two choices of dual are naturally isomorphic.

Proof. Let $f : \sigma_*M|_U \to \mathcal{O}_X|_U$ be a map of right $\mathcal{O}_X|_U$-modules. Post-composing with the map $\mathcal{O}_X|_U \to \sigma_*\mathcal{O}_X|_U$ yields a map $\overline{f} : \sigma_*M|_U \to \sigma_*\mathcal{O}_X|_U$, a.k.a. a map $M|_{\sigma^{-1}U} \to \mathcal{O}_X|_{\sigma^{-1}U}$. Note that $\sigma_* Hom_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(U) = Hom_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(\sigma^{-1}U)$, so that $\overline{f} \in \sigma_* Hom_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(U)$.

On the other hand, given $g \in \sigma_* Hom_{mod-\mathcal{O}_X}(M, \mathcal{O}_X)(U)$, so that $g : \sigma_*M|_U \to \sigma_*\mathcal{O}_X|_U$, we can postcompose with $\sigma_*(\sigma^{\#})$ to get a map $\widetilde{g} : \sigma_*M|_* \to \sigma_*\sigma_*\mathcal{O}_X|_U = \mathcal{O}_X|_U$. Since $\sigma^2 = id$, this is clearly the inverse to the map above.

It's clear that these assignments are natural, since they're just postcomposition with a natural transformation. □

Definition 34. Define the adjoint module M^* to be $Hom_{mod-\mathcal{O}_X}(\sigma_*M, \mathcal{O}_X)$. By the remark above, it doesn't really matter which of the two possible definitions we choose here.

Definition 35. Given a right $\mathcal{O}_X$-module M, we define the double dual isomorphism can : $M \to M^{**}$ as follows: given an open $U \subseteq X$, we define a map

$$M(U) \to Nat(\sigma_* Nat(\sigma_*M, \mathcal{O}_X)|_U, \mathcal{O}_X|_U) = Nat(Nat(\sigma_*M|_{\sigma(U)}, \mathcal{O}_X|_{\sigma(U)}), \mathcal{O}_X|_U)$$

by $u \mapsto \eta_u$, where for an open $V \subseteq U$,

$$(\eta_u)_V(\gamma) = (\sigma^{\#})_V^{-1}(\gamma_{\sigma(V)}(u|_V)).$$

Here $\gamma \in Nat(\sigma_*M|_{\sigma(U)}, \mathcal{O}_X|_{\sigma(U)})$ and $\sigma^{\#}$ is the morphism of sheaves $\sigma^{\#} : \mathcal{O}_X \to \sigma_*\mathcal{O}_X$. Note that $\gamma_{\sigma(V)}(u|_V)$ makes sense because $\sigma_*M(\sigma(V)) = M(V)$.

More globally, there's an evaluation map

$$ev_\sigma : M \otimes \sigma_* Nat(\sigma_*M, \mathcal{O}_X) \to \mathcal{O}_X$$

defined by the composition

$$M \otimes \sigma_* Nat(\sigma_*M, \mathcal{O}_X) \cong M \otimes Nat(M, \sigma_*\mathcal{O}_X) \xrightarrow{ev} \sigma_*\mathcal{O}_X \xrightarrow{(\sigma^{\#})^{-1}} \mathcal{O}_X$$

which under adjunction yields the above map.

Definition 36. Let X be a scheme with involution $- : X \to X$. A *hermitian vector bundle over* X is a locally free right $\mathcal{O}_X$-module V with an $\mathcal{O}_X$-module map $V \to V^*$.

Remark 37. Recall that there's an equivalence of categories between locally free coherent sheaves on X and geometric vector bundles given by $M \mapsto \mathbf{Spec} Sym(M^\vee)$ in one direction and the sheaf of sections in the other. For locally free sheaves, we have $M^\vee \otimes N^\vee \cong (M \otimes N)^\vee$ so that the functor is monoidal. We will use this to think of a hermitian form as a map of schemes $V \otimes V \to \mathbb{A}^1$.

Below we give the key example of a hermitian vector bundle.

Example 38. Define (diagonal) hyperbolic n-space over a scheme $(S, -)$ with involution to be $\mathbb{A}_S^{2n}$ with the hermitian form $(x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}) \mapsto \sum_{i=1}^n \overline{x}_{2i-1} y_{2i-1} - \overline{x}_{2i} y_{2i}$. Denote this hermitian form by h_{diag}.

As defined this way, the matrix of this hermitian form is

$$\begin{bmatrix} 1 & 0 & \cdots & & \\ 0 & -1 & 0 & \cdots & \\ \vdots & \vdots & \vdots & & \\ 0 & \cdots & \cdots & \cdots & -1 \end{bmatrix}$$

the diagonal matrix $\operatorname{diag}(1, -1, 1, \ldots, -1)$. For this definition to give a hermitian space isometric to other standard definitions of the hyperbolic form, it's crucial that 2 be invertible.

The isometries of $\mathbb{H}_{\mathbb{R}}$ (where we give it the hyperbolic form above) have the form

$$\begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix}$$

with $a = \pm\sqrt{1+b^2}, b \in \mathbb{R}$ (or $a^2 - b^2 = 1$). The usual identification with $\mathbb{R}^\times \rtimes C_2$ follows by considering the decomposition $a^2 - b^2 = 1 \iff (a+b)(a-b) = 1$.

Example 39. Similarly to above, we can define a hyperbolic form h by the matrix

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

This form is isometric to the above form, and we'll use both forms below.

Properties

Lemma 40. *Given a map of schemes with involution $f : (Y, i_Y) \to (X, i_X)$ and a (non-degenerate) hermitian vector bundle (V, ω) on X, $f^*(V)$ is a (non-degenerate) hermitian vector bundle on Y.*

Proof. The pullback of a locally free $\mathcal{O}_X$-module is a locally free $\mathcal{O}_Y$-module, so we just need to check that it's hermitian. Given the map $\omega : V \to V^*$, we get an induced map $f^*V \to f^*(V^*)$ which is an isomorphism if ω is. Thus we just need to check that $f^*(V^*) \cong (f^*V)^*$. But pullback commutes with sheaf dual for locally free sheaves of finite rank, so we just need to check that changing the module structure via the involution commutes with pullback; that is, we need to check that $f^*(\overline{V}) = \overline{f^*(V)}$. However, this is clear since the structure map on $f^*(\overline{V})$ is given by

$$O_Y \times f^*V \cong f^*\mathcal{O}_X \times f^*V \xrightarrow{f^*(-) \times id} f^*(O_X) \times f^*(V) \to f^*(V).$$

□

Theorem 41. *(Knus [3] 6.2.4) Let (M, b) be an ϵ-hermitian space over a division ring D. Then (M, b) has an orthogonal basis in the following cases:*

1. *the involution of D is not trivial*
2. *the involution of D is trivial, the form is symmetric, and char $D \neq 2$.*

Lemma 42. *(Knus) Let (M,b) be a hermitian module, and $(U, b|_U)$ be a non-degenerate f.g. projective Hermitian submodule. Then $M = U \oplus U^\perp$.*

Proof. Since $b|_U : U \to U^*$ is an isomorphism, given an $m \in M$, there exists $u \in U$ s.t. $b(m,-)|_U = b(u,-)|_U$. But then $b(m-u,-)|_U = 0$, so that $m - u \in U^\perp$, and $m = u + m - u$. Thus $M = U + U^\perp$. Since $\phi|_U$ is non-degenerate, $U \cap U^\perp = 0$, so we're done. □

3.0.2 Hermitian Forms on Semilocal Rings

Theorem 43. *Let R be a ring, and let E be a hermitian module over R. Let $I \subset Jac(R)$ be an ideal. For every orthogonal decomposition $\overline{E} = \overline{F} \perp \overline{G}$ of $\overline{E} = E/IE$ over R/I, where $\overline{F}$ is a free non-singular subspace of $\overline{E}$, there exists an orthogonal decomposition $E = F \perp G$ of E with F free and non-singular, and $F/IF = \overline{F}, G/IG = \overline{G}$.*

Proof. Write $\overline{F} = \langle \overline{x}_1 \rangle \oplus \cdots \oplus \langle \overline{x}_n \rangle$ with $\overline{x}_i \in \overline{F}$ and $\det(\overline{b}(\overline{x}_i, \overline{x}_j)) \in (R/I)^*$. Choose representatives $x_i \in E$ of $\overline{x}_i$, and let $F = Rx_1 + \cdots + Rx_n$. We claim that the x_i are independent, so that F is free: indeed, if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$, then we get n equations $\lambda_1 b(x_1, x_i) + \cdots + \lambda_n b(x_n, x_i) = 0$. But we know that $\det(b(x_i, x_j)) = t \in R^*$, since $1 - st \in I$ for some s by assumption, but then st cannot be contained in any maximal ideal, so $st \in R^* \implies t \in R^*$. It follows that the λ_i are zero, so that the x_i are independent as desired. The determinant fact also shows that F is regular, so by the lemma above, it has an orthogonal summand G. By construction $F/I = \overline{F}$, so that $\overline{G} = (\overline{F})^\perp = (F/I)^\perp = F^\perp/I = G/I$. □

Lemma 44. *Hermitian forms over $R_1 \times R_2$ (with trivial involution) are in bijection with $Herm(R_1) \times Herm(R_2)$.*

Proof. First, recall that modules over $R_1 \times R_2$ correspond to a module over R_1 and a module over R_2. Indeed, consider the standard idempotents $(1,0) = e_1, (0,1) = e_2$. Fix a module M over $R_1 \times R_2$. Then $M = e_1 M \oplus e_2 M$. Indeed, any $m \in M$ can be written as $e_1 m + e_2 m = (e_1 + e_2)m = m$. Furthermore, if $e_1 m_1 = e_2 m_2$, then $e_2 e_1 m_1 = e_2 e_2 m_2 \implies 0 = e_2 m_2$.

Now, a hermitian form $M \otimes M \to R_1 \times R_2$ is determined by two maps $M \otimes M \to R_1$ and $M \otimes M \to R_2$. Writing $M = e_1 M \oplus e_2 M$, we note that, by linearity, it must be the case that $e_1 M \otimes e_2 M \to R_1 \times R_2$ is the zero map; to wit, $b(e_1 m_1, e_2 m_2) = e_1 e_2 b(m_1, m_2) = 0$. Thus this hermitian form is determined completely by the maps $e_1 M \otimes e_1 M \to R_1 \times R_2$ and $e_2 M \otimes e_2 M \to R_1 \times R_2$. Finally, note that, again by linearity, we see that $e_1 M \otimes e_1 M \to R_2$ is the zero map: $b(e_1 m_1, e_1 m_2) = b(e_1^2 m_1, e_1 m_2) = e_1 b(e_1 m_1, e_1 m_2)$, and $e_1 R_2 = 0$. Similarly for the other map. Hence, at the end of the day, the hermitian form is completely determined by the maps $e_1 M \otimes e_1 M \to R_1$ and $e_2 M \otimes e_2 M \to R_2$. □

Corollary 45. Free hermitian modules diagonalize over rings with finitely many maximal ideals (semi-local rings).

Proof. By the Chinese Remainder Theorem, $R/(m_1 \cap \cdots \cap m_n) \cong R/m_1 \times \cdots \times R/m_n = F_1 \times \cdots \times F_n$. We claim that Hermitian forms over finite products of fields diagonalize, and then the result will follow from the above theorem. By induction and the lemma above, a hermitian module M is determined by hermitian modules M_i over F_i, $i = 1, \ldots, n$ as $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with action $(f_1, \ldots, f_n) \cdot (m_1, \cdots, m_n) = (f_1 m_1, \ldots, f_n m_n)$. Each M_i can be diagonalized into $M_i = \langle a_{1,i} \rangle \perp \cdots \perp \langle a_{m,i} \rangle$ (it's important to note here that since M is free, the rank of each M_i is the same). Thus a diagonalization of M is given by $\langle (a_{1,1}, \ldots, a_{1,n}) \rangle \perp \cdots \perp \langle (a_{1,m}, \ldots, a_{m,n}) \rangle$. □

Now, let R be a ring with involution, and $I \subseteq Jac(R)$ and ideal. Then $C_2 \cdot I \subseteq Jac(R)$ is an ideal fixed by the involution.

The following corollary has the same proof as the theorem above, the only subtlety is that we need the quotient ring to inherit the involution to make sense of an induced hermitian module.

Corollary 46. Let R be a ring with involution, and let E be a hermitian module over R. Let $I \subset Jac(R)$ be an ideal fixed by the involution. For every orthogonal decomposition $\overline{E} = \overline{F} \perp \overline{G}$ of $\overline{E} = E/IE$ over R/I, where $\overline{F}$ is a free non-singular subspace of $\overline{E}$, there exists an orthogonal decomposition $E = F \perp G$ of E with F free and non-singular, and $F/IF = \overline{F}, G/IG = \overline{G}$.

Corollary 47. Let R be a local ring with involution (necessarily a map of local rings). Then any Hermitian module (which is necessarily free) over R diagonalizes.

Lemma 48. *Let R be a ring, and consider the ring $R \times R$ with the involution that switches factors. Then any module M can be written as $e_1M \oplus e_2M$ as above. A non-degenerate hermitian form on this module is determined by a map $e_1M \otimes e_2M \to R \times R$, i.e. as a matrix it has the form*

$$\begin{bmatrix} 0 & A \\ \overline{A}^t & 0 \end{bmatrix}.$$

where A is invertible.

Proof. The first claim is just that $b(e_1x, e_1y) = 0 = b(e_2x, e_2y)$ for any $x, y \in M$. This follows because $b(e_1x, e_1y) = b(e_1^2x, e_1^2y) = \overline{e_1}e_1b(e_1x, e_1y) = e_2e_1b(e_1x, e_1y) = 0$. Similarly for $b(e_2x, e_2y)$. The statement about the matrix follows by identifying the map $M \otimes \overline{M} \to R \times R$ with an isomorphism $M \to \overline{M}^*$ and using the direct sum decomposition. □

Corollary 49. Let R be as in the lemma additionally with 2 invertible. Then $M \cong H(e_1M)$, where H denotes the hyperbolic space functor.

Proof. The assumption that 2 is invertible implies that M is an even hermitian space in the notation of Knus. Now by the corollary above $b|_{e_1M} = 0$, so M has a direct summand such that $e_1M = e_1M^\perp$. Now corollary 3.7.3 in Knus applies to finish the proof. □

Corollary 50. Let R be a semi-local ring with involution. Then any hermitian module over R diagonalizes.

Proof. Using the theorem above and reducing modulo the Jacobson radical (which is always stable under the involution), it suffices to prove the corollary for R a finite product of fields. Then $R = F_1 \times \cdots \times F_n$ is semi-simple, and hence we can index the fields in a particularly nice way (proof is by considering idempotents), writing $R = A_1 \times \cdots \times A_m \times B_1 \times \ldots B_{n-m}$ such that A_i is fixed by the involution, and $\sigma(B_{2i}) = B_{2i+1}$, $\sigma(B_{2i+1}) = B_{2i}$. Now, any finitely generated module M can be written as a direct sum $M = \bigoplus_{i=1}^m M_i \bigoplus_{i=1}^{\frac{n-m}{2}} N_{2i} \oplus N_{2i-1}$. By the two lemmas above, the form when restricted to each M_i or $N_{2i} \oplus N_{2i-1}$ is diagonalizable, so the form is diagonalizable (see the proof of the non-involution case). □

Corollary 51. Hermitian vector bundles are locally hyperbolic in the isovariant étale topology.

Proof. The points in the isovariant étale topology are either strictly henselian local rings with trivial involution or a ring of the form $\mathcal{O}_{X,x}^g \times \mathcal{O}_{X,x}^h$ with involution $(x, y) \mapsto (i(y), i(x))$. Via the map $(x, y) \mapsto (x, i(y))$, such rings are isomorphic to hyperbolic rings. If the ring is a local ring, the fact that all non-degenerate hermitian forms are trivial is well-known, since we have square roots. If the ring is hyperbolic, then all non-degenerate hermitian forms over the ring are hyperbolic by the corollary above. □

Corollary 52. Let $R \times R$ be a ring with the involution which switches factors. Fix a hermitian module M over R, and let $N = e_1 M$ (see above for notation). Then $O(M) \cong GL(N)$.

Proof. In corollary 49 above, we identified non-degenerate hermitian forms over such rings as hyperbolic. Thus it suffices to prove the statement for forms of the form

$$\begin{pmatrix} 0 & 1 \\ can & 0 \end{pmatrix}$$

□

Lemma 53. *Let (R, i) be a ring with involution with 2 invertible, and let (M, b) be a non-degenerate hermitian module over R. There exists an equivariant étale cover $\{U_i \to \operatorname{Spec} R\}$ of $\operatorname{Spec} R$ such that $(M, b)|_{U_i}$ is trivial.*

Proof. For a fixed prime p, consider the semilocal ring $R_{(p)} \times R_{i(p)}$. By the universal property of localization, there's an induced involution i on $R_{(p)} \times R_{i(p)}$ given by $(f_1, f_2) \mapsto (i(f_2), i(f_1))$. The restriction of M to this ring has a diagonalization $v(p)^* b v(p) = D$. Choose a greatest common denominator (f_1, f_2) for the entries of $v(p)^*$ and $v(p)$. By finding a common denominator and inverting the determinant, there's an element $(g_1, g_2) \in R - p_1 \times R - p_2$ s.t. $v(p)^* b v(p) = D$ is an equality in $R[g_1^{-1}, i(g_2)^{-1}] \times R[i(g_1)^{-1}, g_2^{-1}]$. By construction, the set of g s.t. we have such a diagonalization is not contained in any maximal ideal. Thus there exist $(g_1, g_2), \ldots, (g_{n-1} \cdot g_n)$ s.t. b diagonalizes over $R[g_i^{-1}, i(g_{i+1})^{-1}] \times R[i(g_i)^{-1}, g_{i+1}^{-1}]$ and s.t. $\prod R[g_i^{-1}, i(g_{i+1})^{-1}] \times R[i(g_i)^{-1}, g_{i+1}^{-1}] \to R$ is an equivariant Zariski cover. Now by adjoining square roots of the units corresponding to the diagonalization in each $R[g_i^{-1}, i(g_{i+1})^{-1}] \times R[i(g_i)^{-1}, g_{i+1}^{-1}]$ (and their images under the involution), if necessary, we obtain an étale cover $E_1 \times \cdots \times E_n$ of $\operatorname{Spec} R$ s.t. (M, b) is trivial when pulled back to each E_i. □

Lemma 54. *Let (V, ϕ) be a non-degenerate hermitian vector bundle over a scheme with trivial involution X, and let $(M, \phi|_M)$ be a (possibly degenerate) sub-bundle. Given a map of schemes $g : Y \to X$, there is a canonical isomorphism $g^*(M^\perp) \cong (g^* M)^\perp$.*

Proof. Recall that, by definition, $M^\perp = \ker(V \xrightarrow{\phi} V^* \to M^*)$. Equivalently, $M^\perp$ is defined by the exact sequence

$$0 \to M^\perp \to V \to M^* \to 0.$$

It follows that the composite map $g^*(M^\perp) \to g^* V \to g^*(M^*)$ is zero, and hence by universal property of kernel there's a canonical map

$$g^*(M^\perp) \to \ker(g^* V \to g^*(M^*) \cong (g^*(M))^*) = (g^*(M))^\perp$$

where we've used the canonical isomorphism $g^*(M^*) \cong (g^*(M))^*$ for locally free sheaves.

We claim that this map is an isomorphism. It suffices to check on stalks, where the map can be identified with a map

$$M^\perp_{g(y)} \otimes \mathcal{O}_{Y,y} \to \ker(V_{g(y)} \otimes \mathcal{O}_{Y,y} \to M^*_{g(y)} \otimes \mathcal{O}_{Y,y}).$$

But $V_{g(y)} \cong M^\perp_{g(y)} \oplus M^*_{g(y)}$, so the sequence

$$0 \to M^\perp_{g(y)} \otimes \mathcal{O}_{Y,y} \to V_{g(y)} \otimes \mathcal{O}_{Y,y} \to M^*_{g(y)} \otimes \mathcal{O}_{Y,y} \to 0$$

is split exact, and the canonical map is an isomorphism. $\square$

Chapter 4

Higher Grothendieck Witt Groups

In [8], the author works with coherent Grothendieck-Witt groups on a scheme. Because the negative K-theory of the category of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules with coherent cohomology vanishes (together with the pullback square relating the homotopy fixed points of K-theory to Grothendieck-Witt theory), there is no difference between the additive and localizing versions of Grothendieck-Witt spectra in this setting.

Therefore, we work instead with Grothendieck-Witt spectra of $\mathrm{sPerf}(X) = \mathrm{Ch}^b\,\mathrm{Vect}(X)$, the dg category of strictly perfect complexes on X. We review the relevant definitions from [6] now.

Definition 55. A *pointed dg category with duality* is a triple $(\mathcal{A}, \vee, \mathrm{can})$ where $\mathcal{A}$ is a pointed dg category, $\vee : \mathcal{A}^{op} \to \mathcal{A}$ is a dg functor called the duality functor, and $\mathrm{can} : 1 \to \vee \circ \vee^{op}$ is a natural transformation of dg functors called the double dual identification such that $\mathrm{can}_A^\vee \circ \mathrm{can}_{A^\vee} = 1_{A^\vee}$ for all objects A in $\mathcal{A}$.

Remark 56. A dg category with duality has an underlying exact category with duality $(Z^0\mathcal{A}^{\mathrm{ptr}}, \vee, \mathrm{can})$, where $Z^0\mathcal{A}^{\mathrm{ptr}}$ has the same objects as $\mathcal{A}^{\mathrm{ptr}}$ but morphism set the zero cycles in the morphism complex of $\mathcal{A}^{\mathrm{ptr}}$ and $\mathcal{A}^{\mathrm{ptr}}$ is the pretriangulated hull of $\mathcal{A}$ (see [6] definition 1.7).

Definition 57. A *dg category with weak equivalences* is a pair $(\mathcal{A}, w)$ where $\mathcal{A}$ is a pointed dg category and $w \subseteq Z^0\mathcal{A}^{\mathrm{ptr}}$ is a set of morphisms which saturated in $\mathcal{A}$. A map f in w is called a weak equivalence.

Definition 58. A *dg category with weak equivalences and duality* is a quadruple $\mathscr{A} = (\mathcal{A}, w, \vee, \mathrm{can})$ where $(\mathcal{A}, w)$ is a dg category with weak equivalences and $(\mathcal{A}, \vee, \mathrm{can})$ is a dg category with duality such that the dg subcategory $\mathcal{A}^w \subset \mathcal{A}$ of w-acyclic objects is closed under the duality functor $\vee$ and $\mathrm{can}_A : A \to A^{\vee\vee}$ is a weak equivalence for all objects A of $\mathcal{A}$.

Definition 59. Let $\mathscr{A} = (\mathcal{A}, w, \vee, \mathrm{can})$ be a dg category with weak equivalences and dualiy. The Grothendieck-Witt group $GW_0(\mathscr{A})$ of $\mathscr{A}$ is the abelian group generated by hermitian spaces $[X, \phi]$ in the underlying category with weak equivalences and duality $(Z^0\mathcal{A}^{ptr}, w, \vee, \mathrm{can})$, subject to the following relations:

1. $[X, \phi] + [Y, \psi] = [X \oplus Y, \phi \oplus \psi]$
2. if $g : X \to Y$ is a weak equivalence, then $[Y, \psi] = [X, g^\vee \psi g]$, and
3. if $(E_\bullet, \phi_\bullet)$ is a symmetric space in the category of exact sequences in $Z^0\mathcal{A}^{\mathrm{ptr}}$, that is, a map

$$\begin{tikzcd}
E_\bullet : \arrow[d, "\sim"', "\phi_\bullet"] & E_{-1} \arrow[r, tail, "i"] \arrow[d, "\sim"', "\phi_{-1}"] & E_0 \arrow[r, two heads, "p"] \arrow[d, "\sim"', "\phi_0"] & E_1 \arrow[d, "\sim"', "\phi_1"] \\
E_\bullet^\vee : & E_1^\vee \arrow[r, tail, "p^\vee"] & E_0^\vee \arrow[r, two heads, "i^\vee"] & E_1^\vee
\end{tikzcd}$$

of exact sequences with $(\phi_{-1}, \phi_0, \phi_1) = (\phi_1^\vee \mathrm{can}, \phi_0^\vee \mathrm{can}, \phi_{-1}^\vee \mathrm{can})$ a weak equivalence, then

$$[E_0, \phi_0] = \left[E_{-1} \oplus E_1, \begin{pmatrix} 0 & \phi_1 \\ \phi_{-1} & 0 \end{pmatrix}\right].$$

Chapter 5

Hermitian Grassmannians

Fix a separated base scheme S with trivial involution. The goal of this section is to define a sheaf on $\mathbf{Sm}_S^{C_2}$, denoted $\mathbb{R}\mathrm{Gr}_V$, which represents non-degenerate sub-bundles of a given hermitian vector bundle V.

5.0.1 The definition of $\mathbb{R}\mathrm{Gr}$

Lemma 60. *Let $\mathcal{F}$ be a presheaf on $\mathbf{Sm}_S$ and let $a : \mathcal{F} \implies \mathcal{F}$ be a natural transformation s.t. $a \circ a = id_{\mathcal{F}}$. Then there's an associated presheaf on $\mathbf{Sm}_S^{C_2}$ defined by the formula $(X, \sigma : X \to X) \mapsto \mathcal{F}(X)^{C_2}$ where the action of C_2 on $\mathcal{F}(X)$ is defined by $f \mapsto a_X \mathcal{F}(\sigma)(f)$.*

Proof. Note that this is indeed a C_2-action, since $a_X\mathcal{F}(\sigma)(a_X\mathcal{F}(\sigma)(f)) = \mathcal{F}(\sigma)a_X(a_X\mathcal{F}(\sigma)(f)) = \mathcal{F}(\sigma)(\mathcal{F}(\sigma)(f)) = f$ using naturality. $\square$

Fix a (possibly degenerate) hermitian vector bundle (V, ϕ) over the base scheme S (which has trivial involution).

We'll define a presheaf $\mathbb{R}\mathrm{Gr} : (\mathbf{Sm}_S^{C_2})^{op} \to \mathbf{Set}$ by first defining a presheaf on $\mathbf{Sm}_S$, showing that it's representable, equipping with an action, then taking the corresponding representable functor on $\mathbf{Sm}_S^{C_2}$.

- On objects, $\mathbb{R}\mathrm{Gr}(V)(f : X \to S)$ for an S-scheme $f : X \to S$ is a split surjection (p, s)

  $$\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] & W \arrow[l, dashed, bend right, "s"'] \end{tikzcd},$$

  where W is locally free.

  Here by an isomorphism of split surjections we mean a diagram

  $$\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] \arrow[d, equal] & W \arrow[l, dashed, bend right, "s"'] \arrow[d, "\phi"] \\ f^*V \arrow[r, two heads, "p'"'] & W' \arrow[l, dashed, bend right, "s'"'] \end{tikzcd}$$

  such that ϕ is an isomorphism satisfying $\phi \circ p = p'$ and $s = s' \circ \phi$.

- Given a morphism

  $$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "h"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

over S, define

$$\mathbb{R}\mathrm{Gr}_V(g)\left(\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] & W \arrow[l, bend right, dashed, "s"'] \end{tikzcd}\right) = \begin{tikzcd} h^*V \arrow[r, "can"] & g^*f^*V \arrow[r, two heads, "g^*p"'] & g^*W \arrow[l, bend right, dashed, "g^*s"'] \end{tikzcd}.$$

There's a natural action of C_2, on $\mathbb{R}\mathrm{Gr}_V$, whose non-trivial natural transformation will be denoted η. Define η as follows:

Fix an object $X \in \mathbf{Sm}_S$. Define

$$\eta_X\left(\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] & W \arrow[l, bend right, dashed, "s"'] \end{tikzcd}\right) = \begin{tikzcd} f^*V \arrow[r, two heads, "q"'] & (\ker p)^\perp \arrow[l, bend right, dashed, "t"'] \end{tikzcd}.$$

We claim that this is well-defined.

Recall that

$$W^\perp = \ker(f^*V \xrightarrow{f^*\phi} f^*(V^*) \xrightarrow{can} (f^*V)^* \xrightarrow{s^*} W^*).$$

Leaving out the *can* map for convenience, we get a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & W^\perp \arrow[r] & f^*V \arrow[r, "s^*"'] & W^* \arrow[r] \arrow[l, bend right, "p^*"'] & 0 \end{tikzcd}.$$

By the splitting lemma for abelian categories, $f^*V \cong W^\perp \oplus W^*$, and hence there's a split surjection $f^*V \twoheadrightarrow W^\perp$ with $W^\perp$ locally free.

Given an isomorphism

$$\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] \arrow[d, equal] & W \arrow[l, bend right, dashed, "s"'] \arrow[d, "\psi"] \\ f^*V \arrow[r, two heads, "p'"'] & W' \arrow[l, bend right, dashed, "s'"'] \end{tikzcd}$$

we get an isomorphism of (split) diagrams

$$\begin{tikzcd} f^*V \arrow[r, "f^*\phi"] \arrow[d, equal] & (f^*V)^* \arrow[r, "s^*"] \arrow[d, equal] & W^* \arrow[d, "(\psi^{-1})^*"] \\ f^*V \arrow[r, "f^*\phi"] & (f^*V)^* \arrow[r, "(s')^*"'] & (W')^* \end{tikzcd}$$

and hence an isomorphism of split surjections

$$\begin{tikzcd} f^*V \arrow[r, two heads, "q"'] \arrow[d, equal] & W^\perp \arrow[l, bend right, dashed, "t"'] \arrow[d, "\delta"] \\ f^*V \arrow[r, two heads, "q'"'] & (W')^\perp \arrow[l, bend right, dashed, "t'"'] \end{tikzcd},$$

so that η_X is a well-defined map of sets. Given a map of schemes $g : Y \to X$, such that $f \circ g = h$ and an element

$$\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] & W \arrow[l, dashed, bend right, "s"'] \end{tikzcd}$$

in $\mathbb{R}\mathrm{Gr}_V(X)$,

$$\begin{aligned} \mathbb{R}\mathrm{Gr}(g) \circ \eta_X\Big(\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] & W \arrow[l, dashed, bend right, "s"'] \end{tikzcd} \Big) &= \mathbb{R}\mathrm{Gr}(g)\Big(\begin{tikzcd} f^*V \arrow[r, two heads, "q"'] & (\ker p)^{\perp} \arrow[l, dashed, bend right, "t"'] \end{tikzcd} \Big) \\ &= \begin{tikzcd} h^*V \arrow[r, "can"] & g^*f^*V \arrow[r, two heads, "g^*q"'] & g^*((\ker(p))^{\perp}) \arrow[l, dashed, bend right, "g^*t"'] \end{tikzcd} \end{aligned}$$

while

$$\eta_Y \circ \mathbb{R}\mathrm{Gr}(g)\Big(\begin{tikzcd} f^*V \arrow[r, two heads, "p"'] & W \arrow[l, dashed, bend right, "s"'] \end{tikzcd} \Big) = \begin{tikzcd} h^*V \arrow[r, "can"] & g^*f^*V \arrow[r, two heads, "q'"'] & (g^*(\ker(p)))^{\perp} \arrow[l, dashed, bend right, "t'"'] \end{tikzcd}$$

By Lemma 54, there's a canonical isomorphism $g^*((\ker(p)^{\perp})) \to (g^*(\ker(p)))^{\perp}$, and under this isomorphism q' and t' correspond to g^*q, and g^*t, respectively. This concludes the check of naturality.

Now by Lemma 60, there's a presheaf $\mathbb{R}\mathrm{Gr} : \mathbf{Sm}_S^{C_2} \to \mathbf{Set}$. To determine its values on a C_2-scheme (X, σ), we note that a fixed point of the action of Lemma 60 is determined by an isomorphism of split surjections

$$\begin{tikzcd} f^*V \arrow[r, two heads, "q"'] \arrow[d, equal] & \sigma^*(\ker(p)^{\perp}) \arrow[l, dashed, bend right, "t"'] \arrow[d, "\psi"] \\ f^*V \arrow[r, two heads, "p"'] & \ker(p) \arrow[l, dashed, bend right, "s"'] \end{tikzcd}$$

Note that because σ is an involution, for any $\mathcal{O}_X$-module M, there's a canonical isomorphism of $\mathcal{O}_X$-modules $\sigma_*M \cong \sigma^*M$. Thus there's a natural isomorphism

$$\mathrm{Hom}_{mod-\mathcal{O}_X}(\sigma_* f^*V, -) \cong \mathrm{Hom}_{mod-\mathcal{O}_X}(\sigma^* f^*V, -) \cong \mathrm{Hom}_{mod-\mathcal{O}_X}(f^*V, -).$$

It follows that any Hermitian form

$$\phi : f^*V \to \mathrm{Hom}_{mod-\mathcal{O}_X}(f^*V, \mathcal{O}_X)$$

can be promoted to a Hermitian form

$$\widetilde{\phi} : f^*V \to \mathrm{Hom}_{mod-\mathcal{O}_X}(\sigma_* f^*V, \mathcal{O}_X)$$

compatible with an involution σ on X.

Let $(M, \phi|_M)$ be a hermitian sub-bundle of f^*V over the scheme X with trivial involution. We claim that $\sigma^*(M^{\perp})$ is the orthogonal complement of M viewed as a hermitian sub-bundle of f^*V with the promoted form $\widetilde{\phi}$. Said differently, we claim that

$$\sigma^*(\ker(f^*V \xrightarrow{\phi|_M} \mathrm{Hom}(M,\mathcal{O}_X)) \cong \ker(f^*V \xrightarrow{\widetilde{\phi}|_M} \mathrm{Hom}(\sigma_* M,\mathcal{O}_X)).$$

But using the natural isomorphism between σ^* and σ_*, together with the natural isomorphisms $\sigma^*\mathrm{Hom}(M,\mathcal{O}_x) \cong \mathrm{Hom}(M,\mathcal{O}_X)$ and $\sigma^* f^* V \cong f^* V$, this becomes a question of whether σ^* is left exact. In general it isn't, but because it's naturally isomorphic to σ_*, and σ_* is left exact, the claim follows.

5.0.2 Representability of $\mathbb{R}\mathrm{Gr}$

Fix a hermitian vector bundle (V,ϕ) over S where $\dim(V) = n$ and S is a scheme with trivial involution. Then the underlying scheme of $\mathbb{R}\mathrm{Gr}(V)$ is the pullback

$$\begin{tikzcd}
\mathbb{R}\mathrm{Gr}(V) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \times \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \arrow[d, "{\circ, id}"] \\
\underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \arrow[r, "\Delta"] & \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V) \times \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,V)
\end{tikzcd}$$

where the right vertical map sends $p \mapsto (p \circ p, p)$. In other words, the underlying scheme is the scheme of idempotent endomorphisms of V. The action corresponds to the map $p \mapsto p^\dagger$, where $p^\dagger$ is the adjoint of p with respect to the form ϕ.

Note that using this description, an equivariant map $(X,\sigma) \to \mathbb{R}\mathrm{Gr}(V)$ corresponds to an idempotent $p : V_X \to V_X$ such that $\phi^{-1}(\gamma^{-1}(\sigma^* p)\gamma)^*\phi = p$, where we're being cavalier and using $*$ to denote both dual (on the outside) and pullback (by σ). Here γ is the canonical isomorphism $V_X \xrightarrow{\gamma} \sigma^* V_X$; if the structure map of X is $f : X \to S$, then γ arises from the equality $\sigma \circ f = f$.

Note that the form on $V_{(X,\sigma)}$ is by definition the composite

$$\widetilde{\phi} : V_X \xrightarrow{\phi} V_X^* \xrightarrow{(\gamma^*)^{-1}} \sigma^* V_X^* \xrightarrow{(\eta^*)^{-1}} \sigma_* V_X^*,$$

and the adjoint of p is given by $\widetilde{\phi}^{-1}(\sigma_* p)^*\widetilde{\phi}$. Expanding, this is

$$\phi^{-1}(\gamma^*)(\eta^*)(\eta^*)^{-1}(\sigma^* p)^*(\eta^*)(\eta^*)^{-1}(\gamma^*)^{-1}\phi = \phi^{-1}(\gamma^{-1}(\sigma^* p)\gamma)^*\phi,$$

and so we recover the condition that $p^\dagger = p$, which corresponds to the fact that $V_X = \ker p \perp \operatorname{im} p$, and hence the restriction of the form on V_X to $\operatorname{im} p$ (and $\ker p$) is non-degenerate.

To summarize, the underlying scheme of $\mathbb{R}\mathrm{Gr}(V)$ represents idempotents, and equivariant maps pick out those idempotents which correspond to orthogonal projections.

Remark 61. Now fix a dimension d and a non-degenerate hermitian vector bundle (V,ϕ) over S. Recalling that the trace of an idempotent coincides with the rank of the image, define $\mathbb{R}\mathrm{Gr}_d(V)$ to be the closed subscheme of $\mathbb{R}\mathrm{Gr}(V)$ cut out by $\mathrm{tr}(p) = d$, where tr is the trace of an endomorphism. In other words, $\mathbb{R}\mathrm{Gr}_d(V)$ is the pullback

$$\begin{tikzcd}
\mathbb{R}\mathrm{Gr}_d(V) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{Gr}(V) \arrow[d, "\mathrm{tr}"] \\
\{d\} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

where $\mathbb{Z}$ is the locally constant sheaf on $\mathbf{Sm}_S^{C_2}$. The requirement that V be non-degenerate is necessary so that the action on $\mathbb{R}\mathrm{Gr}(V)$ sends rank d subspaces to rank d subspaces and hence induces an action on $\mathbb{R}\mathrm{Gr}_d(V)$.

5.0.3 The universal idempotent

Denote by $g : \mathbb{R}\mathrm{Gr}_d(V) \to S$ the structure map of $\mathbb{R}\mathrm{Gr}_d(V)$. Because $\mathbb{R}\mathrm{Gr}_d(V)$ is representable by a C_2-scheme, there's an idempotent $g^*(V) \to g^*(V)$ corresponding to the identity map $id : \mathbb{R}\mathrm{Gr}_d(V) \to \mathbb{R}\mathrm{Gr}_d(V)$. This idempotent is simply the idempotent which over a point of $\mathbb{R}\mathrm{Gr}_d(V)$ represented by an idempotent $p : V \to V$ restricts to p. There's an action σ on $\mathbb{R}\mathrm{Gr}_d(V) \times_S V$ induced by the action on $\mathbb{R}\mathrm{Gr}_d(V)$, and using the fact that $\sigma p \sigma = p^\dagger$ one can see that this idempotent is non-degenerate with respect to the promoted hermitian form on $g^*(V)$ compatible with the involution on $\mathbb{R}\mathrm{Gr}_d(V)$.

Remark 62. Since we've shown that $\mathbb{R}\mathrm{Gr}(V)$ represents non-degenerate hermitian subbundles of V, at this point we'll move away from explicitly referring to split surjections and just represent the sections of $\mathbb{R}\mathrm{Gr}(V)$ by non-degenerate subbundles.

Definition 63. Let $\mathbb{H}_S$ denote the hyperbolic plane. For $V \in \mathbb{H}^\infty$ a constant rank non-degenerate subbundle, let $|V|$ denote the rank of V. Order such subbundles of $\mathbb{H}^\infty$ by inclusion, and denote the resulting poset P. Given an inclusion $V \hookrightarrow V'$ of non-degenerate subbundles, denote by $V' - V$ the complement of V in V'. Let $\mathcal{H} : P \to \mathrm{Fun}(\mathbf{Sm}_S^{C_2,op}, Set)$ be the functor which on objects sends a subbundle V to $\mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^\infty)$. Given an inclusion $V \hookrightarrow V'$, the induced map $\mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^\infty) \to \mathbb{R}\mathrm{Gr}_{|V'|}(V' \perp \mathbb{H}^\infty)$ is given by $E \mapsto E \perp (V' - V)$. Note that because V is non-degenerate, $V \perp (V' - V) = V'$. Define

$$\mathbb{R}\mathrm{Gr}_\infty = \operatorname{colim} \mathcal{H}.$$

5.0.4 The Étale Classifying Space

Fix a scheme S with trivial involution and 2 invertible, and let (V, ϕ) be a (possibly degenerate) hermitian vector bundle over S. For a C_2-scheme $f : X \to S$, let

$$\mathcal{S}(V, \phi)(X)$$

be the category of non-degenerate hermitian sub-bundles of f^*V. A morphism in this category from E_0 to E_1 is an isometry not necessarily compatible with the embeddings $E_0, E_1 \subseteq V$. Using pullbacks of quasi-coherent modules, we turn $\mathcal{S}$ into a presheaf of categories on $\mathbf{Sm}_S^{C_2}$. For integer $d \geq 0$, define

$$\mathcal{S}_d(V, \phi) \subset \mathcal{S}(V, \phi)$$

to be the presheaf which on a C_2-scheme $f : X \to S$ assigns the full subcategory of non-degenerate hermitian sub-bundles of $(f^*V, f^*\phi)$ which have constant rank d. The associated presheaf of objects is $\mathbb{R}\mathrm{Gr}_d(V, \phi)$.

Note that the object $V = (V, 0) \in \mathcal{S}_{|V|}(V \perp H^\infty)$ has automorphism group $O(V)$. Thus we get an inclusion $O(V, \phi) \to \mathcal{S}_{|V|}(V \perp H^\infty)$, where $O(V)$ is the isometry group considered as a category on one object. After isovariant étale sheafification, this inclusion becomes an equivalence; this follows from our remarks above that on the points in the isovariant étale topology, hermitian vector bundles are isomorphic to $\mathbb{H}^n(P)$ for some n, where P is a hermitian vector bundle over a strictly henselian local ring. Since all hermitian vector bundles over strictly henselian local rings (with 2 invertible) are trivial, hermitian vector bundles over the points in the isovariant étale topology are completely determined by rank.

Upon applying the nerve, we get maps of simplicial presheaves $BO(V) \to B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$ which is a weak equivalence in the isovariant étale topology. Abusing notation, let $B_{isoEt}O(V)$ denote a global fibrant replacement of $B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$ in the isovariant étale topology so that we get a sequence of weak equivalences

$$BO(V) \to B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \to B_{isoEt}O(V).$$

Lemma 64. *Let (V,ϕ) be a non-degenerate hermitian vector bundle over a scheme S with trivial involution and $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Then for any affine C_2-scheme over S, $\operatorname{Spec} R$, the map*

$$B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)(R) \to B_{isoEt}O(V)(R)$$

is a weak equivalence of simplicial sets. In particular, the map

$$B\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \to B_{isoEt}O(V)$$

is a weak equivalence in the equivariant Nisnevich topology, and hence an equivalence after C_2 motivic localization.

Proof. Each Hermitian vector bundle $W \in S_{|V|}(V \perp \mathbb{H}^\infty)(R)$ gives rise to an $O(V)$-torsor via $W \mapsto Isom(V,W)$. Note that this is an $O(V)$-torsor because etale locally, $W \cong V$, so that etale locally $Isom(V,W) \cong Isom(V,V) \cong O(V)$. Because hermitian vectors bundles are isovariant etale locally determined by rank, the same proof as the vector bundle case shows that the category of $\mathcal{O}(V)$ torsors is equivalent to the category of Hermitian vector bundles. Because over an affine scheme, every hermitian vector bundle is a summand of a hyperbolic module, it follows that $S_{|V|}(V \perp \mathbb{H}^\infty)(R)$ is equivalent to the category of (etale) $\mathcal{O}(V)$ torsors.

Let $\mathcal{F} : \mathbf{Sm}_S^{C_2} \to Gpd$ be the sheaf which assigns to $f : X \to S$ the groupoid of $O(f^*V)$-torsors. The construction $W \mapsto Isom(f^*V, W)$ described above defines a functor $S_{|V|}(V \perp \mathbb{H}^\infty) \to \mathcal{F}$ which is an equivalence when evaluated at affine C_2-schemes. It follows that there's a sequence

$$BS_{|V|}(V \perp \mathbb{H}^\infty) \to B\mathcal{F} \to B_{isoEt}O(V)$$

where the first map is a weak equivalence of simplicial sets when evaluated at affine C_2-schemes, and by [4] Theorem 6, the second map is a weak equivalence of simplicial sets when evaluated at any C_2-scheme. □

Definition 65. Following [5], let

$$\mathcal{S}_\bullet = \operatorname*{colim}_{V \subset \mathbb{H}_S^\infty} \mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$$

where similarly to the definition of $\mathbb{R}\mathrm{Gr}$, for $V \subset V'$ the functor

$$\mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \to \mathcal{S}_{|V'|}(V' \perp \mathbb{H}^\infty)$$

is defined on objects by $E \mapsto E \perp V' - V$ and on morphisms by $f \mapsto f \perp 1_{V'-V}$.

Definition 66. Define

$$O = \operatorname*{colim}_{W \subseteq \mathbb{H}_S^\infty} O(W).$$

Because the nerve construction commutes with filtered colimits, and because filtered colimits of globally fibrant objects are globally fibrant (follows from the fact that filtered colimits of Kan complexes are Kan complexes), define by abuse of notation

$$B_{isoEt}O = \operatorname*{colim}_{W \subseteq \mathbb{H}_S^\infty} B_{isoEt}O(W).$$

Theorem 67. *Let R be a regular noetherian ring with involution which is either connected or hyperbolic. Then there's an equivalence of simplicial sets*

$$B\mathbb{R}\mathrm{Gr}_\bullet(\Delta R) \to |B\mathcal{S}_\bullet(\Delta R)|$$

where ΔR denotes the simplicial ring with involution $[n] \mapsto R[x_0,\ldots,x_n]/(\sum x_i - 1)$.

Lemma 68. *Let $\mathrm{Iso}_d(R)$ denote the set of isometry classes of finitely-generated, non-degenerate hermitian vector bundles over R. The map*

$$\operatorname*{colim}_{V \subset \mathbb{H}_R^\infty} \mathrm{Iso}_{|V|}(R) = \coprod_{V \subset \mathbb{H}^\infty} \mathrm{Iso}_{|V|}/\sim \cong \widetilde{GW}_{[0]}(R)$$

sending $(V,W) \in \mathrm{Iso}_{|V|}$ to $[V]-[W]$ is an isomorphism. Here $\widetilde{GW}_{[0]}(R)$ is the kernel of the rank map $GW_0(R) \to \mathbb{Z}$.

Proof. First, note that the map is well-defined. If there's an inclusion $V \hookrightarrow T \hookrightarrow \mathbb{H}_R^\infty$, then

$$(T, W \perp (T-V)) \mapsto [T] - [W + T - V] = [V] - [W].$$

Furthermore, by definition if $W \in \mathrm{Iso}_{|V|}(R)$, then $\mathrm{rk}(V) = \mathrm{rk}(W)$ and hence $[V]-[W] \in \ker(\mathrm{rk}) : GW_0(R) \to \mathbb{Z}$.

If $[V]-[W] = 0$ in GW_0, then there's a non-degenerate bundle $[K]$ such that $V \perp K \cong W \perp K$. It follows that $(V,W) \in \mathrm{Iso}_{|V|}(R) \sim (V \perp K, W \perp K) = (V \perp K, V \perp K) \sim (0, 0 \in \mathrm{Iso}_{|0|}(R))$ so that the map is injective. Surjectivity is clear because over a ring, every bundle is, up to isometry, a sub-bundle of $\mathbb{H}_R^\infty$. $\square$

Now, note that there are maps of sets

$$\mathbb{R}\mathrm{Gr}_d(V \perp \mathbb{H}_R^\infty) \to \mathrm{Iso}_d(R) : E \mapsto [E]$$

and (considering a set as a discrete category) maps of categories

$$\mathcal{S}_d(V \perp \mathbb{H}_R^\infty) \to \mathrm{Iso}_d(R) : E \mapsto [E].$$

These maps fit into cartesian squares

$$\begin{tikzcd}
\mathbb{R}\mathrm{Gr}_V(V \perp \mathbb{H}^\infty) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^\infty) \arrow[d] \\
\star \arrow[r, "V"] & \mathrm{Iso}_{|V|}(R)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{S}_V(V \perp \mathbb{H}^\infty) \arrow[r] \arrow[d] & \mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty) \arrow[d] \\
\star \arrow[r, "V"] & \mathrm{Iso}_{|V|}(R)
\end{tikzcd}$$

where $\mathbb{R}\mathrm{Gr}_V(V \perp \mathbb{H}^\infty_R)$ is the subset of $\mathbb{R}\mathrm{Gr}_{|V|}(V \perp \mathbb{H}^\infty_R)$ of bundles isometric to V, and similarly $\mathcal{S}_V(V \perp \mathbb{H}^\infty) \subseteq \mathcal{S}_{|V|}(V \perp \mathbb{H}^\infty)$ is the full subcategory whose objects correspond to the set $\mathbb{R}\mathrm{Gr}_V(V \perp \mathbb{H}^\infty_R)$.

Taking colimits over non-degenerate subspaces $V \subset \mathbb{H}^\infty_R$ and using the standard facts that the nerve functor commutes with filtered colimits and that filtered colimits of cartesian diagrams are cartesian, we get cartesian diagrams of simplicial sets

$$\begin{tikzcd} B\mathbb{R}\mathrm{Gr}_{[0]}(V \perp \mathbb{H}^\infty)(R) \arrow[r] \arrow[d] & B\mathbb{R}\mathrm{Gr}_\bullet(V \perp \mathbb{H}^\infty)(R) \arrow[d] \\ B\star \arrow[r, "V"] & B\widetilde{GW}_{[0]}(R) \end{tikzcd}$$

and

$$\begin{tikzcd} B\mathcal{S}_{[0]}(V \perp \mathbb{H}^\infty)(R) \arrow[r] \arrow[d] & B\mathcal{S}_\bullet(V \perp \mathbb{H}^\infty)(R) \arrow[d] \\ B\star \arrow[r, "V"] & B\widetilde{GW}_{[0]}(R) \end{tikzcd}$$

where the upper left corners are just defined as the respective colimits.

Lemma 69. *The diagrams*

$$\begin{tikzcd} B\mathbb{R}\mathrm{Gr}_{[0]}(V \perp \mathbb{H}^\infty)(\Delta R) \arrow[r] \arrow[d] & B\mathbb{R}\mathrm{Gr}_\bullet(V \perp \mathbb{H}^\infty)(\Delta R) \arrow[d] \\ B\star \arrow[r, "V"] & |B\widetilde{GW}_{[0]}(\Delta R)| \end{tikzcd}$$

and

$$\begin{tikzcd} |B\mathcal{S}_{[0]}(V \perp \mathbb{H}^\infty)(\Delta R)| \arrow[r] \arrow[d] & |B\mathcal{S}_\bullet(V \perp \mathbb{H}^\infty)(\Delta R)| \arrow[d] \\ B\star \arrow[r, "V"] & |B\widetilde{GW}_{[0]}(\Delta R)| \end{tikzcd}$$

are homotopy cartesian over any regular ring R with involution such that non-degenerate hermitian vector bundles have constant rank.

Proof. First, note that before applying the diagonal functor $|-|$, these diagrams are cartesian diagrams of bisimplicial sets. This follows simply because limits are computed object-wise in functor categories. For the same reason, we get a cartesian diagram after applying $|-|$. Thus to prove that the diagrams are homotopy cartesian (in the standard model structure on simplicial sets), it suffices to prove that the bottom horizontal map is a fibration. If R is a regular ring with involution, then $GW_0(R[t]) \cong GW_0(R)$ where the involution on $R[t]$ is on the coefficients of a polynomial. It follows that the reduced Grothendieck-Witt group is also homotopy invariant. It follows that the simplicial set in the bottom right corner of both diagrams is discrete. A map of discrete simplicial sets is a Kan fibration, since a map from a simplicial set to a discrete simplicial set is completely determined by the map on zero simplices, and the zero simplices of an n-horn and n-simplex for $n \geq 1$ agree. $\square$

Via inclusion of zero simplices, there is a map of homotopy fibrations

$$\begin{tikzcd} B\mathbb{R}\mathrm{Gr}_{[0]}(V \perp \mathbb{H}^\infty)(\Delta R) \arrow[r] \arrow[d] & B\mathbb{R}\mathrm{Gr}_\bullet(V \perp \mathbb{H}^\infty)(\Delta R) \arrow[r] \arrow[d] & |B\widetilde{GW}_{[0]}(\Delta R)| . \arrow[d, "id"] \\ |B\mathcal{S}_{[0]}(V \perp \mathbb{H}^\infty)(\Delta R)| \arrow[r] & |B\mathcal{S}_\bullet(V \perp \mathbb{H}^\infty)(\Delta R)| \arrow[r] & |B\widetilde{GW}_{[0]}(\Delta R)| \end{tikzcd} \tag{5.1}$$

In order to conclude that the map $B\mathbb{R}\mathrm{Gr}_\bullet(\Delta R) \to |B\mathcal{S}_\bullet(\Delta R)|$ is a weak equivalence of simplicial sets, it suffices to check two things:

- in diagram 5.1, the map on fibers $B\mathbb{R}\mathrm{Gr}_{[0]}(V \perp \mathbb{H}^\infty) \to |B\mathcal{S}_{[0]}(V \perp \mathbb{H}^\infty)(\Delta R)|$is a weak equivalence (the map on bases is the identity),
- in diagram 5.1, the maps are maps of E_∞-spaces.

Remark 70. Note that even over rings where $|B\widetilde{GW}_{[0]}(V \perp \mathbb{H}^\infty)(\Delta R)|$ is a constant simplicial set, $B\mathbb{R}\mathrm{Gr}_{[0]}(V \perp \mathbb{H}^\infty)$ will not be. This is simply because there are more hermitian vector bundles over $R[x]$ than over R when we don't mod out by isometry.

Example 71. For an explicit example that demonstrates why $|B\mathbb{R}\mathrm{Gr}_{[0]}(\Delta R)|$ has a hope of being connected (if it was discrete it would in general not be), let $R = \mathbb{R}$ with trivial involution, and consider the simplicial set

$$|B\mathbb{R}\mathrm{Gr}_{[0]}(\langle 1\rangle_{\mathbb{R}} \perp \mathbb{H}^\infty_{\mathbb{R}})(\Delta R)|.$$

Consider the two split surjections

$$X = \langle 1\rangle_{\mathbb{R}} \perp \mathbb{H}^\infty_{\mathbb{R}} \xrightarrow{\pi_{\langle 1\rangle}} \langle 1\rangle$$

and

$$Y = \langle 1\rangle_{\mathbb{R}} \perp \mathbb{H}^\infty_{\mathbb{R}} \xrightarrow{\pi_2 \oplus \pi_3} \mathbb{H} \xrightarrow{+} \mathbb{R}$$

where the second surjection is split by $\frac{1}{2}\Delta$. Now consider the split surjection over $R[x]$ given by

$$T = \langle 1\rangle_{\mathbb{R}} \perp \mathbb{H}^\infty_{\mathbb{R}} \xrightarrow{\pi_{\langle 1\rangle} \oplus \pi_2 \oplus \pi_3} \langle 1\rangle \oplus \mathbb{H} \xrightarrow{+} R[x]$$

where the last surjection is split by the map sending $1 \mapsto (x, \frac{1}{2}(1-x), \frac{1}{2}(1-x))$. We claim that under the two maps $R[x] \to R$, $x \mapsto 0, \overset{1}{\mapsto}$, the split surjection T restricts to X and Y. Indeed, this is just the fact that given an $R[x]$-module structure on R via the map $\eta_t : R[x] \to R$, $x \mapsto t$, as well as a map $R[x] \to R[x]$. $1 \mapsto x$, the induced map $R \cong R[x] \otimes_{R[x]} R \xrightarrow{x \otimes id} R[x] \otimes_{R[x]} R \cong R$ is multiplication by $\eta_t(x)$.

We proceed to prove that the map on fibers is a weak equivalence by presenting the domain and codomain as free quotients of contractible spaces. To set up the relevant group actions, we need the following lemma.

Lemma 72. *Let V be a nondegenerate hermitian vector bundle over a commutive ring with involution (R, σ) such that $\frac{1}{2} \in R$. Then the inclusion $\mathbb{H}^\infty \subset V \perp \mathbb{H}^\infty$ induces a homotopy equivalence of simplicial groups*

$$O(\mathbb{H}^\infty_{\Delta R}) \to O(V \perp \mathbb{H}^\infty_{\Delta R}) \qquad A \mapsto 1_V \perp A.$$

Proof. First, assume that $V = \mathbb{H}$. Consider the map $j : O(\mathbb{H}^n) \to O(\mathbb{H}^{2n+2})$ sending A to $1_H \perp A \perp 1_{\mathbb{H}^{n+1}}$. We claim that this is naively $\mathbb{A}^1$ homotopic to the inclusion $i : O(\mathbb{H}^n) \to O(\mathbb{H}^{2n+2})$, $i(A) = A \perp 1_{\mathbb{H}^{n+2}}$ which defines the colimit $O(\mathbb{H}^\infty)$. Let $g = \begin{pmatrix} 0 & I_{2n} & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_{2n+2} \end{pmatrix}$ where I_n denotes an $n \times n$ identity matrix. Then $i = gjg^{-1} = gjg^t$. Because g corresponds to an even permutation matrix, it can be written as a product of elementary matrices, each of which is naively $\mathbb{A}^1$ homotopic to the identity. It follow that g is naively $\mathbb{A}^1$ homotopic to the identity, and hence the induced maps $i, j : O(\mathbb{H}^n_{\Delta R}) \to O(\mathbb{H}^{2n+2}_{\Delta R})$ are simplicially homotopic via a base-point preserving homotopy. It follows that i, j induce the same map on homotopy groups, so that $j_* = i_* : \pi_k O(\mathbb{H}^\infty_{\Delta R}) = \operatorname{colim}_n \pi_k O(\mathbb{H}^n_{\Delta R}) \to \pi_k O(\mathbb{H}^\infty_{\Delta R})$ is the colimit of a map corresponding to a cofinal inclusion of diagrams, and hence is an isomorphism on all simplicial homotopy groups. Because simplicial groups are Kan complexes, it follows that j is a homotopy equivalence, and the claim is proved when $V = \mathbb{H}$.

Now a trivial induction shows that the lemma holds when $V = \mathbb{H}^n$. In general, choose an embedding $V \subseteq \mathbb{H}^n$, and consider the sequence of maps

$$O(\mathbb{H}^\infty_{\Delta R}) \to O(V \perp \mathbb{H}^\infty_{\Delta R}) \to O(\mathbb{H}^n \perp \mathbb{H}^\infty_{\Delta R}) \to O(\mathbb{H}^n \perp V \perp \mathbb{H}^\infty_{\Delta R}).$$

The composites $O(\mathbb{H}^\infty_{\Delta R}) \to O(\mathbb{H}^n \perp \mathbb{H}^\infty)$ and $O(V \perp \mathbb{H}^\infty_{\Delta R}) \to O(\mathbb{H}^n \perp V \perp \mathbb{H}^\infty_{\Delta R})$ are weak equivalences, so by 2 out of 6 the first map is a weak equivalence. Because it is a map of simplicial groups it is a homotopy equivalence.

□

For nondegenerate hermitian vector bundles $(V, \phi_V), (W, \phi_W)$ and a commutative R-algebra with involution (A, σ), let

$$\mathrm{St}(V, W)(A)$$

be the set of A-linear isometric embeddings $f : V_A \to W_A$. Given a map $A \to B$ of commutative R-algebras with involution, tensoring over R with B makes $\mathrm{St}(V, W)(-)$ a presheaf on commutative R-algebras with involution. There's a transitive left action of $O(V \perp \mathbb{H}^\infty)$ on $\mathrm{St}(V, V \perp \mathbb{H}^\infty)$ given by $(f, g) \mapsto f \circ g$. Let i_V denote the isometric embedding $V \hookrightarrow V \perp \mathbb{H}^\infty : v \mapsto (v, 0)$. The stabilizer of i_V is the subgroup $O(\mathbb{H}^\infty) \subset O(V \perp \mathbb{H}^\infty)$ where the inclusion map is $A \mapsto 1_V \perp A$.

It follows that there's an isomorphism of presheaves of sets

$$O(\mathbb{H}^\infty) \backslash O(V \perp \mathbb{H}^\infty) \cong \mathrm{St}(V, V \perp \mathbb{H}^\infty) \qquad f \mapsto f \circ i_V.$$

Now Lemma 72 shows that the map $O(\mathbb{H}^\infty_{\Delta R}) \to O(V \perp \mathbb{H}^\infty_{\Delta R})$ is an equivariant map which is a non-equivariant homotopy equivalence. The simplicial group $O(\mathbb{H}^\infty_{\Delta R})$ acts freely on both the domain and codomain, so that the quotients $O(\mathbb{H}^\infty_{\Delta R}) \backslash O(V \perp \mathbb{H}^\infty_{\Delta R})$ and $O(\mathbb{H}^\infty_{\Delta R}) \backslash O(\mathbb{H}^\infty_{\Delta R})$ are homotopy equivalent.

Together with the isomorphism of simplicial sets

$$O(\mathbb{H}^\infty_{\Delta R}) \backslash O(V \perp \mathbb{H}^\infty_{\Delta R}) \cong \mathrm{St}(V, V \perp \mathbb{H}^\infty_{\Delta R})$$

it follows that $\mathrm{St}(V, V \perp \mathbb{H}^\infty_{\Delta R})$ is a contractible for a commutative ring (R, σ) with involution and $\frac{1}{2} \in R$. Morever, this simplicial set is fibrant because G/H is fibrant for a simplicial group G and subgroup H.

Now we move to identifying $\mathbb{R}\mathrm{Gr}_d(V)$ as a quotient of a contractible space by a free group aciton. Let V be a non-degenerate hermitian vector bundle over a ring R with involution. Then the group $O(V)$ acts on the right on $\mathrm{St}(V,U)$ by precomposition. The map $\mathrm{St}(V,U) \to \mathbb{R}\mathrm{Gr}_V(U) : f \mapsto \mathrm{im}(f)$ factors through the quotient $\mathrm{St}(V,U)/O(V)$. The map is clearly surjective, and hence furnishes an isomorphism of sets

$$\mathrm{St}(V,U)/O(V) \cong \mathbb{R}\mathrm{Gr}_V(U) \qquad f \mapsto \mathrm{im}(f).$$

In particular, there's an isomorphism of presheaves of sets $\mathrm{St}(V, V \perp \mathbb{H}^\infty)/O(\mathbb{H}^\infty) \cong \mathbb{R}\mathrm{Gr}_V(U)$.

Now, for a non-degenerate hermitian vector bundle V over a ring with involution R, and let U be a possible degenerate hermitian form over R. Define $\mathcal{E}_V(U)$ to be the category whose objects are R-linear maps $V \to U$ of hermitian forms, and whose morphisms from two objects $a : V \to U$ and $b : V \to U$ are maps $c : \mathrm{im}(a) \to \mathrm{im}(b)$ making the diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[dr, "b"'] & \mathrm{im}(a) \arrow[d, "c"] \\ & \mathrm{im}(b) \end{tikzcd}$$

commute.

There's a natural right action of $O(V)$ on $\mathcal{E}_V(U)$ which on objects sends

$$\mathcal{E}_V(U) \times O(V) \to \mathcal{E}_V(U) : (a,g) \mapsto ag$$

and which morphisms is the trivial action.

Then clearly there's an isomorphism

$$\mathcal{E}_V(U)/O(V) \cong \mathcal{S}_V(U) \qquad a \mapsto \mathrm{im}(a).$$

Lemma 73. *The category $\mathcal{E}_V(V \perp \mathbb{H}^\infty)$ is contractible.*

Proof. The category is nonempty and every object is initial. □

Now we show that the map on fibers in 5.1 is a weak equivalence. The map of simplicial sets

$$\mathrm{St}(V, V \perp \mathbb{H}^\infty)(\Delta R) \to \mathcal{E}_V(V \perp \mathbb{H}^\infty)(\Delta R)$$

is $O(V_{\Delta R})$ equivariant and a weak equivalence after forgetting the action. Furthermore, $O(V_{\Delta R})$ acts freely on both sides, so that the induced map on quotients $\mathbb{R}\mathrm{Gr}_V(V \perp \mathbb{H}^\infty_{\Delta R}) \to \mathcal{S}_V(V \perp \mathbb{H}^\infty_{\Delta R})$ is also a weak equivalence.

As an aside, the inclusion $BO(V) \subset B\mathcal{S}_V(V \perp \mathbb{H}^\infty)$ is a weak equivalence since $\mathcal{S}_V(V \perp \mathbb{H}^\infty)$ is a connected groupoid.

Showing that digram 5.1 is a diagram in E_∞-spaces

For a commutative ring with involution (R,σ), let $\mathscr{E}(n)(R)$ be the set

$$\mathscr{E}(n)(R) = \lim_{V \subset \mathbb{H}^\infty_R} \mathrm{St}(V^{\perp n}, \mathbb{H}^\infty_R).$$

where limit is over non-degenerate subspaces of $\mathbb{H}^\infty$. The permutation group Σ_n acts by permuting the component subspaces. The maps in the limit are equivariant with respect to this free action, and hence there's an induced free action on the limit. Now if $V \subseteq W$, then the map $\mathrm{St}(V^n, \mathbb{H}^\infty_{\Delta R}) \to \mathrm{St}(W^n, \mathbb{H}^\infty_{\Delta R})$ is a Kan fibration, and hence

$$\mathscr{E}(n)(\Delta R) = \lim_k \mathrm{St}(\mathbb{H}^k \perp \cdots \perp \mathbb{H}^k, \mathbb{H}^\infty)(\Delta R)$$

is a tower of Kan fibrations with each object fibrant. It follows that this limit is a homotopy limit, and the Milnor sequence implies that $\mathscr{E}(n)(\Delta R)$ is contractible. The same reasoning as above implies that the action of Σ_n is free.

Now, define the structure maps of the operad by

$$\mathscr{E}(k) \times \mathscr{E}(j_1) \times \cdots \times \mathscr{E}(j_k) \to \mathscr{E}(j_1 + \cdots + j_k) : f, g_1, \cdots, g_k \mapsto f \circ (g_1 \perp \cdots \perp g_k).$$

It follows that $\mathscr{E}(\Delta R)$ is an E_∞-operad in the category of simplicial sets.

Proposition 74. *For any commutive ring with involution (R, σ) such that $\frac{1}{2} \in R$, the map given by inclusion of 0-simplices*

$$\mathbb{R}\mathrm{Gr}_\bullet(\Delta R) \to \mathcal{S}_\bullet(\Delta R)$$

is a map of group complete E_∞-spaces.

Proof. Write

$$\mathcal{S}_\bullet = \operatorname*{colim}_{V \subset \mathbb{H}^\infty} \mathcal{S}_{|V|}(V^- \perp V^+)$$

where V^- and V^+ are two copies of V and for $V \subset W$ the transition map is defined by

$$\mathcal{S}_{|V|}(V^- \perp V^+) \to \mathcal{S}_{|W|}(W^- \perp W^+) : E \mapsto (W - V)^- \perp E, g \mapsto 1_{(W-V)^-} \perp g.$$

Now, the action of $\mathscr{E}$ on $\mathcal{S}_\bullet$ is defined by

$$\mathrm{St}(V_1 \perp \cdots \perp V_k, W) \times \mathcal{S}_{|V_1|}(V_1^- \perp V_1^+) \times \cdots \times \mathcal{S}_{|V_k|}(V_k^- \perp V_k^+) \to \mathcal{S}_{|W|}(W^- \perp W^+)$$

where for $g \in \mathrm{St}(V_1 \perp \cdots \perp V_k, W)$, the functor

$$\mathcal{S}_{|V_1|}(V_1^- \perp V_1^+) \times \cdots \times \mathcal{S}_{|V_k|}(V_k^- \perp V_k^+) \to \mathcal{S}_{|W|}(W^- \perp W^+)$$

sends the object $(E_1, \ldots, E_k)$ to

$$(W - g(V_1 \perp \cdots \perp V_k))^- \perp g(E_1 \perp \cdots \perp E_k)$$

and the map $(e_1, \ldots, e_k) : (E_1, \ldots, E_k) \to (E_1', \ldots, E_k')$ to

$$1_{(W - g(V_1 \perp \cdots \perp V_k))^-} \perp g|_{E_1'} \circ e_1 \circ g^{-1}|_{E_1} \perp \cdots \perp g|_{E_k'} \circ e_k \circ g^{-1}|_{E_k}.$$

To see that the spaces are group complete, note that the homotopy fiber sequences above imply that the π_0 of both spaces is $\widetilde{G}W_0(R)$. Indeed, it's straightforward to check that $\pi_0(\mathbb{R}\mathrm{Gr}_{[0]}(\Delta R), x) = \{*\}$ for any choice of basepoint x, which implies that the right maps in 5.1 are an injection on π_0. The maps on zero simplices are clearly surjective, and hence the maps on π_0 must be surjective. $\square$

Now that we've checked that the maps in 5.1 are maps of E_∞-spaces and that they're weak equivalences on base and fiber, we can conclude that the map on total spaces is an equivalence.

Corollary 75. Let (R,σ) be a regular ring with involution such that non-degenerate hermitian vector bundles have constant rank, and such that $\frac{1}{2} \in R$. Then the map

$$\mathbb{R}\mathrm{Gr}_\bullet(\Delta R) \to \mathcal{S}_\bullet(\Delta R)$$

is a weak equivalence of simplicial sets.

5.0.5 The Grothendieck-Witt space

For a ring R with involution σ, there's an associated category $S(R)$ with duality given by vector bundles with their canonical duality and vector bundle morphisms as morphisms. The subcategory of hermitian objects and isometries is symmetric monoidal under $\perp$, and the translations $A \mapsto A \perp B$ are faithful. Quillen's $S^{-1}S(R)$ construction yields a symmetric monoidal category with objects pairs (A_0, A_1) in S and morphisms $(A_0, A_1) \to (B_0, B_1)$ equivalence classes $[C, a_0, a_1]$ with $a_i : C \perp A_i \to B_i$ an isometry. Two morphisms $[C, a_0, a_1], [C', a'_0, a'_1]$ are equivalent if there exists an isometry $f : C \cong C'$ such that $a'_i \circ (1_{A_i} \perp f) = a_i$. Unfortunately, this category is neither small (in general) nor strictly functorial in the underyling C_2-scheme.

Definition 76. Let (R,σ) be a ring with involution, and let

$$\mathcal{G}W(R,\sigma) \subset S^{-1}S(R,\sigma)$$

be the full subcategory whose objects are pairs (A,B) where $A \subset \mathbb{H}_R^\infty \perp \mathbb{H}_R^\infty$ and $B \subset (\mathbb{H}_R^\infty)^{\perp 3}$ are finitely genereated nondegenerate subspaces.

Let (X,σ) be a C_2-scheme, and let

$$\mathcal{G}W(X,\sigma) = \mathcal{G}W(\operatorname{Spec}\Gamma(X),\sigma).$$

Note that $\mathcal{G}W(R,\sigma) \hookrightarrow S^{-1}S(R,\sigma)$ is an equivalence because over a ring, every non-degenerate vector bundle is a summand of hyperbolic space. Thus by [6], Theorem A.1, there's an equivalence $\mathcal{G}W(R,\sigma) \cong \Omega^\infty GW(R,\sigma)$ for any ring with involution (R,σ).

Homotopy colimits of categories

Definition 77. Let $\mathcal{C}$ be a small category, and let $J : \mathcal{C} \to \mathbf{Cat}$ a functor into the category of small categories. The homotopy colimit

$$\operatorname*{hocolim}_{\mathcal{C}} J$$

is the category whose objects are pairs (X,A) with X an object of $\mathcal{C}$ and A an object of $J(X)$. A map from (X,A) to (Y,B) is a pair (x,a) where $x : X \to Y$ is a map in $\mathcal{C}$ and $a : J(x)(A) \to B$ is a map in $J(Y)$. Composition $(y,b) \circ (x,a)$ is the map $(y \circ x, b \circ J(y) \circ a)$.

We recall some notation from [7].

Definition 78. Let S be a symmetric monoidal category acting on another category X. The category $\langle S, X\rangle$ is by definition the category whose objects are the objects of X, and whose morphisms $F \to G$ are isomorphism classes of tuples $(F, G, A, A+F \to G)$ with $A \in S$ and F, G in X. An isomorphism of tuples is an isomorphism

$A \cong A'$ which makes the diagram

$$\begin{tikzcd} A+F \arrow[rr, "\sim"] \arrow[dr] & & A'+F \arrow[dl] \\ & G & \end{tikzcd}$$

commute.

Now consider the category $\mathcal{S}(\mathbb{H}_R^\infty)$ of finitely generated non-degenerate subspaces of $\mathbb{H}_R^\infty$. It's symmetric monoidal via $\perp$, and thus it acts on itself by translation. Then $\langle \mathcal{S}(\mathbb{H}_R^\infty), \mathcal{S}(\mathbb{H}_R^\infty)\rangle$ is the category whose objects are finitely generated non-degenerate subspaces of $\mathbb{H}_R^\infty$, and whose morphisms $W \to T$ are isomorphism classes of isometries $V \perp W \to T$.

We claim that the morphisms correspond to isometric embeddings $W \hookrightarrow T$ which don't necessarily commute with the embeddings into $\mathbb{H}^\infty$. First, given an isometry $\phi : V \perp W \to T$, $\phi|_W : W \to T$ is an isometric embedding. Given two isomorphic morphisms $W \to T$ (as defined above), they necessarily restrict to the same map on W so that there's a well-defined map of sets from the morphisms in $\langle \mathcal{S}(\mathbb{H}_R^\infty, \mathcal{S}(\mathbb{H}_R^\infty)\rangle$ to isometric embeddings. Given an isometric embedding $\phi : W \hookrightarrow T$, because W is non-degenerate there's a decomposition $T = \phi(W) \perp (\phi(W))^\perp$. It follows that there's an isometry $(\phi(W))^\perp \perp W \to T$, yielding a morphism in $\langle \mathcal{S}(\mathbb{H}_R^\infty), \mathcal{S}(\mathbb{H}_R^\infty)\rangle$.

Definition 79. Define a functor $\mathcal{I} : \langle \mathcal{S}(\mathbb{H}_R^\infty), \mathcal{S}(\mathbb{H}_R^\infty)\rangle \to \mathbf{Cat}$ which on objects is defined by $\mathcal{I}(V) = \mathcal{S}_{|V|}(V \perp \mathbb{H}_R^\infty)$ and given a morphism $g : V \hookrightarrow W$, $\mathcal{I}(g)$ is the functor

$$\mathcal{I}(g) : \mathcal{S}_{|V|}(V \perp \mathbb{H}_R^\infty) \to \mathcal{S}_{|W|}(W \perp \mathbb{H}_R^\infty),$$

$$\begin{aligned} E &\mapsto (W - g(V)) \perp (g \perp id)(E) \\ e &\mapsto id_{W-g(V)} \perp geg^{-1}. \end{aligned}$$

Now, let

$$\widetilde{\mathcal{GW}}(R) = \operatorname{hocolim} \mathcal{I}.$$

To spell this out, the objects of $\widetilde{\mathcal{GW}}(R)$ are pairs (V, W) with $V \subseteq \mathbb{H}_R^\infty$ a finitely generated non-degenerate subspace and $W \subset V \perp \mathbb{H}_R^\infty$ a finitely generated non-degenerate subspace of constant rank $|V|$.

A morphism $(V, W) \to (A, B)$ is a pair $(f : V \hookrightarrow A, g : (A - f(V)) \perp (f \perp id)(W) \xrightarrow{\sim} B)$.

To justify this definition, we need to describe the relationship between $\widetilde{\mathcal{GW}}(R)$ and $\mathcal{GW}(R)$.

Let $\mathbb{N}$ denote the discrete category on the natural numbers with its usual symmetric monoidal structure, and let $\mathbb{N}^{-1}\mathbb{N}$ denote Grayson's group completion of this symmetric monoidal category outlined above. There's a functor $\mathbb{N}^{-1}\mathbb{N} \to \mathbb{Z}$, where $\mathbb{Z}$ is the discrete category on the integers, defined on objects by $(n, m) \mapsto n - m$. This functor is non-canonically split by the functor $\mathbb{Z} \to \mathbb{N}^{-1}\mathbb{N}$, $z \mapsto (z, 0)$, and these two functors yield weak equivalences equivalences after application of the nerve.

Consider the map

$$Fr : \mathbb{N}^{-1}\mathbb{N} \to GW(R)$$

defined on objects by

$$(n, m) \mapsto (R^n, R^m)$$

where R^n, R^m have bilinear form corresponding to the identity matrix and

$$R^n \hookrightarrow \mathbb{H}_R^\infty \perp 0$$
$$R^m \hookrightarrow \mathbb{H}_R^\infty \perp 0 \perp 0$$

On morphisms an equivalence class $(k, a_0, a_1) : (n_0, n_1) \to (m_0, m_1)$ such that $a_i : n_i + k = m_i$ is sent to the isometry (R^k, a_0, a_1) where a_i is the canonical isometry $R^{n_i} \perp R^k \cong R^{m_i}$.

Consider as well the map

$$\iota : \widetilde{GW}(R) \to GW(R)$$

defined on objects by

$$(V, W) \mapsto (0 \perp V, 0 \perp W)$$

where

$$0 \perp V \hookrightarrow 0 \perp \mathbb{H}_R^\infty$$
$$0 \perp W \hookrightarrow 0 \perp V \perp \mathbb{H}_R^\infty.$$

For morphisms,note that given a morphism $(f, g) : (V, W) \to (A, B)$ in $\widetilde{\mathscr{G}W}(R)$, there are induced isometries

$$\widetilde{f} : A - f(V) \perp V \xrightarrow{id \perp f} A - f(V) \perp f(V) \xrightarrow{\sim} A$$
$$\widetilde{g} : A - f(V) \perp W \xrightarrow{id \perp f} A - f(V) \perp (f \perp id)(W) \xrightarrow{g} B.$$

Send such a pair (f, g) to the triple $(A - f(V), \widetilde{f}, \widetilde{g}) : (V, W) \to (A, B)$ in $\mathscr{G}W(R)$.

Now consider the composite functor

$$\mathbb{N}^{-1}\mathbb{N} \times \widetilde{\mathscr{G}W}(R) \xrightarrow{Fr \times \iota} \mathscr{G}W(R) \times \mathscr{G}W(R) \xrightarrow{\perp} \mathscr{G}W(R). \tag{5.2}$$

Lemma 80. *The functor* (5.2) *is an equivalence of categories over any ring R such that non-degenerate hermitian vector bundles have constant rank.*

Proof. Consider the functor $\mathscr{G}W(R) \to \mathbb{Z}$ defined on objects by $(V, W) \mapsto \mathrm{rk}(V) - \mathrm{rk}(W)$. By assumption, this is well-defined. Given a morphism $(C, a_0, a_1) : (V_0, V_1) \to (W_0, W_1)$, send it to the morphism $id_{\mathrm{rk}(W_0)}$. Now consider the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathscr{G}W}(R) \arrow[r, "\iota"] & \mathscr{G}W(R) \arrow[r, "\mathrm{rk}"] & \mathbb{Z}, \\
\widetilde{\mathscr{G}W}(R) \arrow[u, "id"] \arrow[r, "0 \times id"] & \mathbb{Z} \times \widetilde{\mathscr{G}W}(R) \arrow[u, "T"] \arrow[r, "\pi_{\mathbb{Z}}"] & \mathbb{Z} \arrow[u, "id"]
\end{tikzcd}$$

where T is the composite

$$\mathbb{Z} \times \widetilde{\mathscr{G}W}(R) \to \mathbb{N}^{-1}\mathbb{N} \times \widetilde{\mathscr{G}W}(R) \xrightarrow{(5.2)} \mathscr{G}W(R).$$

After applying the nerve, we get a diagram of fibrations of grouplike E_∞ spaces (and the maps are maps of E_∞ spaces) so that T is a weak equivalence by the 5-lemma. □

Corollary 81. The functor (5.2) is a weak equivalence in the equivariant Nisnevich topology, and hence an equivariant $\mathbb{A}^1$-equivalence.

Proof. The points in the equivariant Nisnevich topology have the form

$$R = C_2 \times^{S_x} \operatorname{Spec}(\mathcal{O}^h_{X,x}).$$

If $S_x = C_2$, then R is a local ring and hence connected. If $S_x = \{e\}$, then R is a hyperbolic ring, and non-degenerate hermitian vector bundles have the same rank over each connected component. □

Now that we've justified the definition of $\widetilde{\mathscr{G}W}(R)$, we produce maps

$$\mathbb{R}\mathrm{Gr}_\infty \to B_{et}O \to \widetilde{\mathscr{G}W}.$$

Recall that P is the poset of non-degenerate sub-bundles of $\mathbb{H}^\infty$. The inclusion $P \hookrightarrow \langle \mathcal{S}(\mathbb{H}^\infty), \mathcal{S}(\mathbb{H}^\infty)\rangle$ yields a natural transformation of functors $\mathcal{H} \to \mathcal{I}$.

Definition 82. Let

$$\mathbb{R}\mathscr{G}\mathrm{r}_\bullet(R) = \operatorname*{hocolim}_P \mathbb{R}\mathrm{Gr}_{|V|}(V_R \perp \mathbb{H}^\infty_R)$$
$$\mathscr{S}_\bullet(R) = \operatorname{hocolim} \mathcal{H}$$

Lemma 83. *Let $(\mathscr{P}, \leq)$ be a filtered poset, and let $\mathscr{F} : \mathscr{P} \to$ **Cat** be a functor from $\mathscr{P}$ into the category* **Cat** *of small categories. Then the canonical functor of categories*

$$\phi : \operatorname*{hocolim}_{\mathscr{P}} \mathscr{F} \to \operatorname*{colim}_{\mathscr{P}} \mathscr{F}$$

is a homotopy equivalence of simplicial sets after application of the nerve.

Proof. The tool for proving such results is Quillen's Theorem A. To use it to conclude that ϕ is a homotopy equivalence, we need to show that $N(d \downarrow \phi)$ is contractible for any object $d \in \operatorname{colim}_{\mathscr{P}} \mathscr{F}$. By definition, the comma category $d \downarrow \phi$ has as objects pairs

$$(c \in \operatorname*{hocolim}_{\mathscr{P}} \mathscr{F}, e \in \operatorname{Hom}_{\operatorname{colim}_{\mathscr{P}} \mathscr{F}}(d, \phi(c)))$$

and morphisms $(c, e) \to (c', e')$ are maps $t : c \to c'$ which make the square

$$\begin{tikzcd} d \arrow[r, "e"] \arrow[d, "id"'] & \phi(c) \arrow[d, "\phi(t)"] \\ d \arrow[r, "e'"] & \phi(c') \end{tikzcd}$$

commute. Given a morphism $P \leq Q$ in $\mathscr{P}$, and an object $A \in \mathscr{F}(P)$, denote by A_Q the object $\mathscr{F}(P \leq Q)(A)$. A fixed object $d \in \operatorname{colim}_{\mathscr{P}} \mathscr{F}$ is represented by a pair $[P, A]$ with $P \in \mathscr{P}$ and $A \in \mathscr{F}(P)$. Given such a pair, we claim that there is an equivalence of categories

$$\psi : \operatorname*{colim}_{P \leq Q \in \mathscr{P}} (id_{\operatorname{hocolim}_{\mathscr{P}} \mathscr{F}} \downarrow (Q, A_Q)) \cong (\phi \downarrow [P, A]).$$

Here for $Q \leq R$, the functor $(id \downarrow (Q, A_Q)) \to (id \downarrow (R, A_R))$ sends $t : (T, B) \to (Q, A_Q)$ to $c \circ t : (T, B) \to (R, A_R)$ with $c : (Q, A_Q) \to (R, A_R)$ the map in the homotopy colimit given by

$$(Q \leq R, id : A_R = \mathscr{F}(Q \leq R)(A_Q) \to A_R).$$

The functor ψ is defined on objects by $(c, e : c \to (Q, A_Q)) \mapsto (c, e)$. This is well-defined, because $\phi(Q, A_Q) = [P, A]$ by definition of colimit of categories. On morphisms, a map $t : c \to c'$ over (Q, A_Q) is sent to the corresponding map $t : c \to c'$ in $(\phi \downarrow [P, A])$.

Given such an equivalence, the colimit on the left is a filtered colimit of categories with initial objects given by $((Q, A_Q), id)$, and hence is a filtered colimit of contractible categories. Because the nerve commutes with filtered colimits, and simplicial homotopy groups commute with filtered colimits, it follows that the comma category $(\phi \downarrow [P, A])$ is contractible just as desired. ☐

Corollary 84. There are homotopy equivalences

$$\mathbb{R}\mathscr{G}\mathrm{r}_\bullet \xrightarrow{\sim} \mathbb{R}\mathrm{Gr}$$
$$\mathscr{S}_\bullet \xrightarrow{\sim} \mathcal{S}_\bullet.$$

Now, there's a sequence of maps

$$\mathbb{R}\mathscr{G}\mathrm{r}_\bullet \xrightarrow{\phi} \mathscr{S}_\bullet \xrightarrow{\psi} \widetilde{\mathscr{G}W}$$

where ϕ is induced by inclusion of objects and ψ is induced by inclusion of diagrams.

We now state the theorem that will set us up for a geometric model of $\widetilde{\mathscr{G}W}$ in the equivariant $\mathbb{A}^1$-homotopy category.

Theorem 85. *Let R be a commutative regular Noetherian ring which is either connected or hyperbolic, and with $\frac{1}{2} \in R$. Then there are weak equivalences*

$$B\mathbb{R}\mathscr{G}\mathrm{r}_\bullet(\Delta R) \xrightarrow{\phi} |B\mathscr{S}_\bullet(\Delta R)| \xrightarrow{\psi} |B\widetilde{\mathscr{G}W}(\Delta R)|$$

where $|-|$ denotes the diagonal of a bisimplicial set.

Remark 86. The theorem is evidently false if we replace ΔR by R, since $B\mathbb{R}\mathscr{G}\mathrm{r}_\bullet(R)$ is 0-truncated.

Proposition 87. *Let R be a regular noetherian ring with involution such that non-degenerate hermitian vector bundles over R have constant rank, and such that $\frac{1}{2} \in R$. Then inclusion of diagrams induces a weak equivalence of simplicial sets*

$$\mathscr{S}_\bullet(\Delta R) \xrightarrow{\sim} \widetilde{\mathscr{G}W}(\Delta R).$$

Proof. By the Group Completion Theorem, the map

$$\mathscr{S}_\bullet(R) \xrightarrow{\sim} \widetilde{\mathscr{G}W}(R)$$

is an isomorphism on integral homology groups. To write this out more explicitly, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{S}_\bullet(R) \arrow[r] \arrow[d] & \widetilde{\mathscr{G}W}(R) \arrow[d] & \\
\operatorname{colim}_{V\subset \mathbb{H}_R^\infty} \mathcal{S}(V \perp \mathbb{H}_R^\infty) \arrow[r] \arrow[d] & \operatorname{colim}_{\mathcal{I}} \mathcal{S}(V \perp \mathbb{H}_R^\infty) \arrow[r, "\sim"] \arrow[d] & GW(R) \\
\mathbb{Z} \arrow[r] & \mathbb{Z} &
\end{tikzcd}.$$

Using that homology commutes with filtered colimits of simplicial sets together we compute that the homology of $\operatorname{colim}_{V\subset \mathbb{H}_R^\infty} \mathcal{S}(V \perp \mathbb{H}_R^\infty)$ is the group completion of the homology of $\mathcal{S}(V \perp \mathbb{H}_R^\infty)$, and by the group completion theorem so is the homology of $GW(R)$.

Said another way, the map

$$\mathbb{Z}\mathscr{S}_\bullet(R) \xrightarrow{\sim} \mathbb{Z}\widetilde{\mathscr{G}W}(R)$$

is a weak equivalence. It follows that

$$\mathbb{Z}\mathscr{S}_\bullet(\Delta R) \xrightarrow{\sim} \mathbb{Z}\widetilde{\mathscr{G}W}(\Delta R)$$

is a level-wise weak equivalence of bisimplicial sets, and hence is a weak equivalence after taking the diagonal. It follows that the map in the proposition is an isomorphism on integral homology.

Now for regular rings with involution, we have $GW(\Delta R) \cong GW(R)$ and hence $\widetilde{\mathscr{G}W}(\Delta R) \cong \widetilde{\mathscr{G}W}(R)$ are group complete H-spaces.

Note that the E_∞-structure defined above on $\mathcal{S}_\bullet(\Delta R)$ gives an E_∞-structure on $\mathscr{S}_\bullet(\Delta R)$ simply by replacing all limits/colimits in the definition with homotopy limits/colimits. Now we have a map $\mathscr{S}_\bullet(\Delta R) \to \widetilde{\mathscr{G}W}(\Delta R)$ of group complete H-spaces which is a homology isomorphism. By uniqueness of group completions, It follows that the map is a homotopy equivalence. □

Corollary 88. There's an equivariant motivic equivalence $L_{mot}\mathbb{R}\mathrm{Gr}_\bullet \xrightarrow{\sim} L_{mot}\widetilde{\mathscr{G}W}$.

Chapter 6

An E_∞ structure on the Hermitian K-Theory Spectrum

6.0.1 A projective bundle formula for $\mathbb{P}^\sigma$

Consider the square

$$\begin{tikzcd} \mathcal{O}(-1) \arrow[r, "\frac{T+S}{2}"] \arrow[d, "\frac{T-S}{2}"] & \mathcal{O} \arrow[d, "\frac{T-S}{2}"] \\ \mathbf{Hom}(\sigma_*\mathcal{O},\mathcal{O}) \arrow[r, "\frac{T+S}{2}"] & \mathbf{Hom}(\sigma_*\mathcal{O}(-1),\mathcal{O}) \end{tikzcd} \tag{6.1}$$

where the map $\frac{T-S}{2} : \mathcal{O}(-1) \to \mathcal{O}$ is induced via the tensor-hom adjunction by the composition

$$\mathcal{O}(-1) \otimes \left\{\frac{T-S}{2}\right\} \otimes \sigma_*\mathcal{O} \xrightarrow{id\otimes i\otimes id} \mathcal{O}(-1)\otimes\mathcal{O}(1)\otimes\sigma_*\mathcal{O} \xrightarrow{id\otimes id\otimes(\sigma^\#)^{-1}} \mathcal{O}(-1)\otimes\mathcal{O}(1)\otimes\mathcal{O} \xrightarrow{\mu\otimes id} \mathcal{O}\otimes\mathcal{O} \xrightarrow{\mu} \mathcal{O}$$

and the map $\frac{T-S}{2} : \mathcal{O} \to \sigma_*\mathcal{O}(-1)$ is induced via the tensor-hom adjunction by the composition

$$\mathcal{O} \otimes \left\{\frac{T-S}{2}\right\} \otimes \sigma_*\mathcal{O}(-1) \xrightarrow{\sigma^\#\otimes\sigma^\#\circ i\otimes id} \sigma_*\mathcal{O}\otimes\sigma_*\mathcal{O}(1)\otimes\sigma_*\mathcal{O}(-1) \xrightarrow{id\otimes\sigma_*(\mu)} \sigma_*\mathcal{O}\otimes\sigma_*\mathcal{O} \xrightarrow{\sigma_*\mu} \sigma_*\mathcal{O} \xrightarrow{(\sigma^\#)^{-1}} \mathcal{O}$$

where μ denotes multiplication. Note that we're abusing notation here and using $\sigma^\#$ to also denote a map $\mathcal{O}(1) \to \sigma_*\mathcal{O}(1)$. We of course just mean the map induced by the graded automorphism of $k[S,T]$ given by $f(S,T) \mapsto f(T,S)$. This just yields the element $\frac{S-T}{2}$ as a global section of $\sigma_*\mathcal{O}(1)$. The map $\frac{T+S}{2}$: $\mathbf{Hom}(\sigma_*\mathcal{O},\mathcal{O}) \to \mathbf{Hom}(\sigma_*\mathcal{O}(-1),\mathcal{O})$ is induced by precomposition with $\sigma_*(\frac{T+S}{2})$, which is still multiplication by the global section $\frac{T+S}{2}$.

Note that these are both well-defined maps of $\mathcal{O}$-modules. We claim that diagram 6.1 commutes. Fix an open $U \subseteq \mathbb{P}^\sigma$ which need not be invariant, and open $V \subseteq U$. Going down then right yields the composite map

$$u \mapsto (v \mapsto \frac{T-S}{2}\cdot u \cdot (\sigma^\#)^{-1}\Big(\frac{T+S}{2}\cdot v\Big)).$$

Going right first then down yields the composite

$$u \mapsto (v \mapsto (\sigma^\#)^{-1}(\sigma^\#(\frac{T+S}{2}\cdot u)\cdot\frac{S-T}{2}\cdot v))$$

These are equal since $\frac{T+S}{2}$ is an invariant global section. Note that the diagram 6.1 is a map in $\mathrm{Fun}([1],\mathrm{Vect}(\mathbb{P}^\sigma))$ from

$$\mathcal{O}(-1) \xrightarrow{\frac{T+S}{2}} \mathcal{O}$$

to its dual,

$$\mathbf{Hom}(\sigma_*\mathcal{O},\mathcal{O}) \xrightarrow{\frac{T+S}{2}} \mathbf{Hom}(\sigma_*\mathcal{O}(-1),\mathcal{O}).$$

Thus this diagram defines a (not necessarily non-degenerate) form, which we denote by ϕ.

In order to show that thisϕ is symplectic, we have to check that $\phi^* \circ (-\,\mathrm{can}) = \phi$. To spell this out in detail, the dual and double dual are functors. Applying these two functors, we get the two objects

$$O^* \xrightarrow{\frac{T+S}{2}^*} O(-1)^*$$

and

$$O(-1)^{**} \xrightarrow{\frac{T+S}{2}^{**}} O^{**}$$

in $\mathrm{Fun}([1], \mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma))$.

Because can is a natural transformation $id \to **$, there's a commutative diagram

$$\begin{tikzcd}
O(-1) \arrow[r, "\frac{T+S}{2}"] \arrow[d, "\mathrm{can}"] & O \arrow[d, "\mathrm{can}"] \\
O(-1)^{**} \arrow[r, "\frac{T+S}{2}^{**}"] \arrow[d, "\frac{T-S}{2}^*"] & O^{**} \arrow[d, "\frac{T-S}{2}^*"] \\
O^* \arrow[r, "\frac{T+S}{2}^*"] & O(-1)^*
\end{tikzcd}$$

The goal is to show that the vertical maps in the large rectangle are the negative of the vertical maps in diagram 6.1.

Tracing through the definitions, we see that can is the map which sends $u \in \mathcal{O}(-1)(U)$ to the natural transformation

$$\gamma \mapsto (\sigma^\#)^{-1}(\gamma(u|_V)),$$

and $\phi^* \circ \mathrm{can}(u)$ is the natural transformation

$$v \mapsto (\sigma^\#)^{-1}\left(\frac{T-S}{2} \cdot v \cdot (\sigma^\#)^{-1}(u)\right)$$

which is the same thing as

$$v \mapsto \left(-\frac{T-S}{2} \cdot (\sigma^\#)^{-1}(v) \cdot u\right).$$

On the other hand, $\frac{T-S}{2} : \mathcal{O}(-1) \to \mathcal{O}^*$ is the map

$$u \mapsto (v \mapsto \frac{T-S}{2} \cdot u \cdot (\sigma^\#)^{-1}(v))$$

which is by what we calculated above equal to $-(\phi^* \circ \mathrm{can}) = \phi^* \circ (-\,\mathrm{can})$.

Now just as in [6], taking the mapping cone of ϕ via the functor

$$\mathrm{Cone} : \mathrm{Fun}([1], \mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma))^{[0]} \to \left(\mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma)\right)^{[1]}$$

yields a symplectic form $\beta^\sigma = \mathrm{Cone}(\phi)$.

We claim that there's an exact sequence

$$\mathcal{O}(-1) \xrightarrow{\begin{pmatrix} \frac{T+S}{2} \\ \frac{T-S}{2} \end{pmatrix}} \mathcal{O}\oplus\mathcal{O}^* \xrightarrow{\begin{pmatrix} \frac{T+S}{2} & -\frac{T-S}{2} \end{pmatrix}} \mathcal{O}(-1)^*$$

where the maps are the maps in diagram 6.1. The fact that the composite is zero follows from commutativity of that 6.1. To show that the kernel equals the image, note that any permutation of $(\frac{T+S}{2}, \frac{S-T}{2})$ is a regular sequence on $k[S,T]$. Thus if $\frac{T+S}{2}x + \frac{S-T}{2}y = 0$, reducing mod $\frac{T+S}{2}$ we see that $y \in (\frac{T+S}{2})$, and reducing mod $\frac{S-T}{2}$, we see that $x \in (\frac{S-T}{2})$. Proving that the kernel is equal to the image reduces to this fact.

It follows that the square defining ϕ is a pushout, and hence the induced map on mapping cones is a quasi isomorphism. It follows that β^σ is a well-defined, non-degenerate symplectic form in $\left(\mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^\sigma)\right)^{[1]}$.

Theorem 89. *Let X be a scheme with trivial involution, an ample famliy of line bundles, and $\frac{1}{2} \in X$, and denote by $p : \mathbb{P}^\sigma \to X$ the structure map of the equivariant projective line over X, with action $[x : y] \mapsto [y : x]$. Then for all $n \in \mathbb{Z}$, the following are natural stable equivalences of (bi-) spectra*

$$GW^{[n]}(X) \oplus GW^{[n-1]}(X, -\mathrm{can}) \xrightarrow{\sim} GW^{[n]}(\mathbb{P}^\sigma_X)$$
$$\mathbb{G}W^{[n]}(X) \oplus \mathbb{G}W^{[n-1]}(X, -\mathrm{can}) \xrightarrow{\sim} \mathbb{G}W^{[n]}(\mathbb{P}^\sigma_X)$$
$$(x,y) \mapsto p^*(x) + \beta^\sigma \cup p^*(y).$$

Proof. The proof of Theorem 9.10 in [6] can be easily adapted. Note that our Bott element β^σ is just a linear change of coordinates from the standard Bott element on $\mathbb{P}^1$. Keeping in mind that the involution only affects the duality and not the underlying derived category with weak equivalences, it's still true that $\beta^\sigma \otimes : \mathcal{T}\,\mathrm{sPerf}(X) \to \mathcal{T}\,\mathrm{sPerf}(\mathbb{P}^1_X)/p^*\mathcal{T}\,\mathrm{sPerf}(X)$ is an equivalence of triangulated categories. As in *loc. cit.*, if we denote by w the set of morphisms in $\mathrm{sPerf}(\mathbb{P}^1_X)$ which are isomorphisms in $\mathcal{T}\,\mathrm{sPerf}(\mathbb{P}^1_X)/p^*\mathcal{T}\,\mathrm{sPerf}(X)$, we get a sequence

$$(\mathrm{sPerf}(X), \mathrm{quis}) \xrightarrow{p^*} (\mathrm{sPerf}(\mathbb{P}^1_X), \mathrm{quis}) \longrightarrow (\mathrm{sPerf}(\mathbb{P}^1_X), w)$$

which is a Morita exact sequence of categories with duality. That is, the maps are maps of categories with duality, and the underlying sequence of categories is Morita exact. It follows that this sequence induces a homotopy fibration of $GW^{[n]}$ and $\mathbb{G}W^{[n]}$ spectra. As remarked above, these fibration sequences split via the exact dg form functors

$$(\mathrm{sPerf}(X), \mathrm{quis})\beta\otimes \longrightarrow (\mathrm{sPerf}(\mathbb{P}^1_X), \mathrm{quis}) \longrightarrow (\mathrm{sPerf}(\mathbb{P}^1_X), w)$$

so that the composite is an equivalence of triangulated categories. Finally, using that GW and $\mathbb{G}W$ are invariant under derived equivalences, we conclude the theorem. $\square$

Considering GW as a presheaf of spectra on $\mathbf{Sm}_S^{C_2}$ it follows from Theorem 89 that $GW^{[n]}(\mathbb{P}^\sigma, [1:1]) \cong GW^{[n-1]}(X, -\mathrm{can}) \cong GW^{[n+1]}(X)$, recovering one of the results of [8]. Hence

$$\mathbf{Hom}(\Sigma^\infty(\mathbb{P}^\sigma, [1:1]), GW^{[n]}) \cong GW^{[n+1]}$$

as presheaves of spectra on $\mathbf{Sm}_S^{C_2}$. In particular, by the projective bundle formula from [6] and the usual cofiber sequence

$$([1:1]\times \mathbb{P}^\sigma)\vee(\mathbb{P}^1\times[1:1]) \to \mathbb{P}^\sigma \times \mathbb{P}^1 \to \mathbb{P}^\sigma \wedge \mathbb{P}^1$$

we obtain the periodicity isomorphism

$$\mathbf{Hom}((\mathbb{P}^1,[1:1])\wedge(\mathbb{P}^\sigma,[1:1]), GW^{[n]}) \cong GW^{[n]}$$

induced by the map

$$\begin{aligned} GW^{[n]}(X) \to GW^{[n+1]}(\mathbb{P}^1_X) &\to GW^{[n]}(\mathbb{P}^\sigma_{\mathbb{P}^1_X}) \\ x \mapsto \beta \cup p^*(x) &\mapsto \beta^\sigma \cup q^*(k[-1]\otimes \beta \cup p^*(x)) \end{aligned}$$

where p is the projection $\mathbb{P}^1_X \to X$, and q is the projection $\mathbb{P}^\sigma_{\mathbb{P}^1_X} \to \mathbb{P}^1_X$. The analogous statements hold for the presheaf of spectra $\mathbb{G}W$.

As notation for later, let $\beta^{1+\sigma}$ denote the induced map

$$\beta^{1+\sigma} : (\mathbb{P}^1,[1:1])\wedge(\mathbb{P}^\sigma,[1:1]) \to GW. \tag{6.2}$$

Lemma 90. *The Bott element β^σ restricts to zero in $C_2 \times \mathbb{A}^\sigma = \mathbb{P}^\sigma - [1:0]\coprod \mathbb{P}^\sigma - [0:1]$.*

Proof. As in [6], because the Bott element is natural it suffices to prove that the bott element β^σ in $\mathbb{P}^\sigma_{\mathbb{Z}[\frac{1}{2}]}$ restricts to zero. From the definition of the Bott element, it's clear that it's supported on $[1:-1]$. There's a commutative diagram

$$\begin{tikzcd} GW^{[n]}(C_2\times\mathbb{A}^\sigma \text{ on } [1:-1]\coprod[1:-1]) \arrow[r] & GW^{[n]}(C_2\times \mathbb{A}^\sigma) \arrow[r] \arrow[d] & GW^{[n]}(C_2\times\mathbb{A}^\sigma - [1:-1]) \arrow[dl] \\ & GW^{[n]}(\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])) & \end{tikzcd}$$

where the vertical maps are induced by inclusion of the point $[1:1]$. Because $\mathbb{Z}[\frac{1}{2}]$ is regular, the results of [8] show that the middle vertical map is an isomorphism, hence the upper right map is an injection. By localization, the top row is exact, and it follows that the left horizontal map is the zero map. ☐

6.0.2 The periodization of GW

The idea behind the Bass construction in algebraic K-theory is that as a consequence of satisfying localization, there is a Bass exact sequence ending in

$$\cdots \to K_n(\mathbb{G}_m) \xrightarrow{\partial} K_{n-1}(X) \to 0$$

for all n. This comes from applying K-theory to the pushout square manifesting the usual cover of $\mathbb{P}^1$ together with the projective bundle formula. The map ∂ is split by $x \mapsto [T]\cup p^*(x)$ where p is the projection to the base scheme $p : \mathbb{G}_m \to X$. It follows that if K exhibits an exact Bass sequence in all degrees n, then $K_{n-1}(X)$ can be identified with the image of $\partial([T])\cup x$ (i.e. this map is an automorphism of $K_{n-1}(X)$. In fact, $\partial([T])\cup -$ is the idempotent endomorphism $(0,1)$ of $K_0(\mathbb{P}^1)\cong K_0(X)\oplus K_0(X)$. The Bass construction can be thought of as defining $K^B_n(X)$ so that there's an exact sequence $K^B_n(\mathbb{A}^1)\oplus K^B_n(\mathbb{A}^1) \to K^B_n(\mathbb{G}_m) \to K^B_{n-1}(X)$, then identifying $K^B_{n-1}(X)$ with $(0,1)\cdot K^B_{n-1}(\mathbb{P}^1)$. In other words, it can be constructed as the colimit

$$K^B = \operatorname{colim}(K \to \mathbf{Hom}(\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1, K) \to \dots)$$

where the pushouts are taken in presheaves and the maps are induced by applying $\mathbf{Hom}(-,K)$ in the category of K-modules to the composite

$$\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma\mathbb{G}_m \xrightarrow{T} K.$$

Here, loosely speaking, the first map in the composite represents the boundary ∂ while the second represents $[T]$, so that in the category of K modules this map represents cup product with $\partial([T])$.

We'll spell out an example a bit more explicitly to give a flavor for the constructions to come. Let $W = \mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1$, where we emphasize again that the pushout is in the category of presheaves. Because this is a (homotopy) pushout in the category of presheaves, applying $\mathbf{Hom}(-,K)$ gives us a homotopy pullback square, and hence a Mayer-Vietoris sequence. In particular, it gives us a map of presheaves of spectra (which can be promoted to a map of K-modules) $\Omega K(\mathbb{G}_m) \to K(W)$, where we abuse notation and write $K(W)$ for the internal hom of W into K. Because K^B satisfies Nisnevich descent, and $K_i(-) = K^B_i(-)$ for $i \geq 0$, it follows by the 5-lemma that $K_0(W) \cong K_0(\mathbb{P}^1) \cong K_0(X) \oplus K_0(X)$, and that the element $\partial([T]) \cup -$ represents projection onto the second factor as an endomorphism of $K_0(W)$.

Note that the image of $\partial([T])$ in $K(\mathbb{A}^1)$ is zero. Now, we claim that $\partial([T]) \cup K_{-1}(W) \cong K^B_{-1}(X)$. The important point is that from the Bass construction we know $K^B_{-1}(X) = \partial([T]) \cup K^B_{-1}(\mathbb{P}^1) = \partial(K^B_0(\mathbb{G}_m))$. Because $\partial([T])$ is zero in $K(\mathbb{A}^1)$, the image of $\partial([T]) \cup K_{-1}(W)$ in $K_{-1}(\mathbb{A}^1) \oplus K_{-1}(\mathbb{A}^1)$ is zero. By exactness, it follows that $\partial([T]) \cup K_{-1}(W) \subseteq \partial K_0(\mathbb{G}_m)$.

There's a map $\phi : K_{-1}(W) \to K^B_{-1}(W) \cong K^B_{-1}(\mathbb{P}^1)$ and a commutative diagram

$$\begin{tikzcd}
K_0(\mathbb{A}^1) \oplus K_0(\mathbb{A}^1) \arrow[r] \arrow[d] & K_0(\mathbb{G}_m) \arrow[r, "\partial"] \arrow[d] & K_{-1}(W) \arrow[d, "\phi"] \\
K_0(\mathbb{A}^1) \oplus K_0(\mathbb{A}^1) \arrow[r] & K_0(\mathbb{G}_m) \arrow[r, "\partial"] & K^B_{-1}(W)
\end{tikzcd}$$

which shows that $\partial(K_0(\mathbb{G}_m)) \cong \partial(K^B_0(\mathbb{G}_m))$. Now

$$\phi(\partial([T]) \cup \partial(K_0(\mathbb{G}_m)) = \partial([T]) \cup \phi(\partial K_0(\mathbb{G}_m)) = \partial([T]) \cup \partial K^B_0(\mathbb{G}_m) = \partial K^B_0(\mathbb{G}_m).$$

But $\partial([T]) \cup \partial(K_0(\mathbb{G}_m) \subseteq \partial(K_0(\mathbb{G}_m)$, and we know that $\phi|_{\partial K_0(\mathbb{G}_m)}$ is an isomorphism, hence $\partial K_0(\mathbb{G}_m) \subseteq \partial([T]) \cup K_{-1}(W)$. Together with what we argued above, this shows that

$$\partial([T]) \cup K_{-1}(W) = \partial K_0(\mathbb{G}_m) \cong \partial K^B_0(\mathbb{G}_m) \cong K^B_{-1}(X).$$

If we take pointed versions of the above sequences by pointing all the schemes in question at $[1:1]$ everything goes through as above with the extra benefit that $\partial([T]) \cup K^B_{-1}(W,1) = K^B_{-1}(W,1)$, and the map $K^B(X) \to K^B(W,1)$, $x \mapsto p^*(x) \cup \partial([T])$ is an isomorphism by the projective bundle formula. Now the map $p : (W,1) \to 1$ is split by inclusion of the base point, and thus $p^* : K(W,1) \to K((W,1) \otimes (W,1))$ is injective. Furthermore, $p^*(x \cup \partial([T])) = \partial([T]) \cup p^*(x)$, so that the image of $K_{-1}(W,1)$ in $K_{-1}((W,1) \otimes (W,1))$ is, by what we showed above, isomorphic to $K^B_{-1}(X)$.

Our goal is to mimic this construction for GW, so we begin by identifying the analogues of the Bass sequence and the splittings therein.

From [6], we know that there's a Bass sequence

$$0 \longrightarrow \mathbb{G}W_i^{[n]}(X) \longrightarrow \mathbb{G}W_i^{[n]}(\mathbb{A}_X^1) \oplus \mathbb{G}W_i^{[n]}(\mathbb{A}_X^1) \longrightarrow \mathbb{G}W_i^{[n]}(X[T,T^{-1}]) \longrightarrow \mathbb{G}W_{i-1}^{[n-1]}(X) \longrightarrow 0$$

where the last non-trivial map is split by cup product with (the pullback of) $[T]$ in $\mathbb{G}W_1^{[1]}(\mathbb{Z}[\frac{1}{2}][T,T^{-1}])$. This gives us a candidate map $\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma\mathbb{G}_m \xrightarrow{[T]} GW^{[1]}$.

Now, we want to find a candidate map $\Sigma^\sigma \mathbb{G}_m^\sigma \to GW^{[-1]}$ so that we can eventually invert

$$\Sigma^\sigma \mathbb{G}_m^\sigma \otimes \Sigma\mathbb{G}_m \to GW^{[-1]} \otimes GW^{[1]} \to GW^{[0]}.$$

Define W_σ by the pushout square in the category of presheaves

$$\begin{tikzcd} (C_2 \times \mathbb{G}_m^\sigma)_+ \arrow[r] \arrow[d] & (C_2 \times \mathbb{A}^\sigma)_+ \arrow[d] \\ (\mathbb{G}_m^\sigma)_+ \arrow[r] & W_\sigma \end{tikzcd} .$$

There's an associated homotopy pushout square

$$\begin{tikzcd} (C_2 \times \mathbb{G}_m^\sigma)_+/(C_2)_+ \arrow[r] \arrow[d] & (C_2 \times \mathbb{A}^\sigma)_+/(C_2)_+ \arrow[d] \\ (\mathbb{G}_m^\sigma)_+/S^0 \arrow[r] & W_\sigma/S^0 \end{tikzcd}$$

and taking the homotopy cofiber of the left vertical map yields $S^\sigma \wedge \mathbb{G}_m^\sigma$. It follows that the homotopy cofiber of the right vertical map is equivalent to $S^\sigma \wedge \mathbb{G}_m^\sigma$, and that there's a long exact sequence

$$\cdots \longrightarrow GW_i^{[n]}(S^\sigma \wedge \mathbb{G}_m^\sigma) \longrightarrow GW_i^{[n]}(W_\sigma/S^0) \longrightarrow GW_i^{[n]}((C_2 \times \mathbb{A}^\sigma)_+/(C_2)_+) \longrightarrow \cdots . \tag{6.3}$$

Here if $\mathbb{A}_S^\sigma \cong S$, then $(C_2 \times \mathbb{A}^\sigma)_+/(C_2)_+ \cong (C_2)_+ \wedge \mathbb{A}^\sigma$ is contractible and $W/S^0 \cong S^\sigma \wedge \mathbb{G}_m^\sigma$.

Working over the regular ring $\mathbb{Z}[\frac{1}{2}]$, $GW(W_\sigma/S^0) \cong GW(\mathbb{P}^\sigma/S^0)$, and

$$GW_i^{[n]}(W_\sigma/S^0) \cong GW_i^{[n]}(\mathbb{P}^\sigma/S^0) \cong GW_i^{[n+1]}(S)$$

by the projective bundle formula 89.

The maps in the sequence (6.3) are maps of $GW_*^{[0]}$-modules, and the sequence is natural in the base scheme. The induced map

$$GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma) \to GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$$

is an isomorphism of $GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$-modules, and hence the inverse is uniquely determined by a lift of the element $\langle 1 \rangle \in GW_0^{[0]}(\mathbb{Z}[\frac{1}{2}])$ to $GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma)$. We stress that this element $\langle 1 \rangle$ maps to $\beta^\sigma \cup k[-1] \cup \langle 1 \rangle$ in $\mathbb{P}^\sigma$, and in particular it isn't the unit of multiplication in $GW(\mathbb{P}^\sigma)$. We'll denote this element by $[T^\sigma]$ in analogy with the non-equivariant case.

Is there an ac
description of
element?

Over an arbitrary base scheme X, we denote by $[T^\sigma]$ the pullback of $[T^\sigma]$ to $GW_0^{[-1]}(S^\sigma \wedge \mathbb{G}_m^\sigma \times_{\mathbb{Z}[\frac{1}{2}]} X)$ using functoriality of GW.

Let $W = (\mathbb{A}^1 \coprod_{\mathbb{G}_m} \mathbb{A}^1)_+$. Now (by taking the pointed version of everything) we have a candidate map

$$\gamma : W_\sigma/S^0 \otimes W/S^0 \to S^\sigma \wedge \mathbb{G}_m^\sigma \otimes S^1 \wedge \mathbb{G}_m \xrightarrow{[T^\sigma]\otimes[T]} GW^{[-1]} \otimes GW^{[1]} \to GW$$

to invert.

Given a presentably symmetric monoidal ∞-category and a morphism $\alpha : x \to \mathbf{1}$ to the monoidal unit, define

$$Q_\alpha E = \operatorname{colim}(E \xrightarrow{\alpha} \mathbf{Hom}(x, E) \xrightarrow{\alpha} \mathbf{Hom}(x^{\otimes 2}, E) \xrightarrow{\alpha} \dots).$$

In general $Q_\alpha E$ is not the periodization of E with respect to α, one obstruction being that the cyclic permutation of α^3 can fail to be homotopic to the identity. This matters because defining the map $\mathbf{Hom}(x, E) \xrightarrow{\alpha} \mathbf{Hom}(x^{\otimes 2}, E)$ requires making a choice between using $\alpha \otimes id$ or $id \otimes \alpha$.

Lemma 91. *$\mathbb{G}W$ satisfies equivariant Nisnevich descent.*

Proof. Recall that the distinguished squares defining the equivariant Nisnevich cd-structure are cartesian squares in $\mathbf{Sm}_S^G$

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "j"] & X \end{tikzcd}$$

where j is an open immersion, p is étale, and $(Y - B)_{\text{red}} \to (X - A)_{\text{red}}$ is an isomorphism.

As in [6], Theorem 9.6, the cited theorem of Thomason as well as the localization sequence for $\mathbb{G}W$ allows us to conclude the result. □

Remark 92. As a matter of notation, we will frequently use the symbolts $\partial([T])$ and $\partial([T^\sigma])$ below. Since we use ∂ to denote every boundary map in a long exact sequence, we warn the reader that $\partial(([T])$ and $\partial([T^\sigma])$ are always (pullbacks of) the elements defined above.

Lemma 93. *The canonical map $\mathbb{G}W \to Q_\gamma \mathbb{G}W$ is an equivalence.*

Proof. We know by the projective bundle formulas that

$$\mathbb{G}W(\mathbb{P}^\sigma \times \mathbb{P}^1 = \mathbb{P}^\sigma_{\mathbb{P}^1}) \cong \mathbb{G}W(\mathbb{P}^1) \oplus \mathbb{G}W^{[1]}(\mathbb{P}^1) \cong \mathbb{G}W(X) \oplus \mathbb{G}W^{[n-1]}(X) \oplus \mathbb{G}W^{[n+1]}(X) \oplus \mathbb{G}W(X).$$

We claim that under this isomorphism, cup product with $\partial[T^\sigma]$ is projection onto $\mathbb{G}W^{[1]}(\mathbb{P}^1)$ and cup product with $\partial[T]$ on $GW^{[1]}(\mathbb{P}^1)$ is projection onto $\mathbb{G}W(X)$. The latter statement is already known from [6], so we show the former. It suffices to show that cup product with $\partial[T_X^\sigma] \cup - : \mathbb{G}W^{[n]}(X) \oplus \mathbb{G}W^{[n+1]}(X) \to \mathbb{G}W(X) \oplus \mathbb{G}W^{[n+1]}(X)$ is projection onto the second factor. But this is precisely how $[T^\sigma]$ is defined: it's a lift under ∂ of a generator of $\mathbb{G}W^{[1]}(X)$, so cup product with it is cup product with $\langle 1 \rangle$ on $GW^{[n+1]}(X)$ and it's necessarily zero on the other factor because it gives a well-defined element on the pointed $\mathbb{G}W^{[-1]}(\mathbb{P}^\sigma, [1 : 1])$.

Because $\mathbb{G}W$ satisfies equivariant Nisnevich descent, $\mathbb{G}W(W/S^0) \cong \mathbb{G}W(\mathbb{P}^1, [1 : 1])$, and $\mathbb{G}W(W_\sigma/S^0) \cong \mathbb{G}W(\mathbb{P}^\sigma, [1 : 1])$. Now we're essentially done. The maps in the colimit can be constructed as first identifying

$\mathbb{G}W_i^{[n]}(X)$ with $\partial([T]) \cup \mathbb{G}W_i^{[n]}(\mathbb{P}^1_X, [1:1])$, then identifying $\mathbb{G}W_i^{[n]}(\mathbb{P}^1_X)$ with $\partial([T^\sigma]) \cup \mathbb{G}W_i^{[n]}(\mathbb{P}^\sigma_{\mathbb{P}^1_X}, [1:1])$. By our remarks above, the image of $\mathbb{G}W_i^{[n]}(X)$ under these identifications is isomorphic to $\mathbb{G}W_i^{[n]}(X)$, and hence $\mathbb{G}W_i^{[n]}(Q_\gamma X) \cong \mathbb{G}W_i^{[n]}(X)$ as desired.

□

Lemma 94. *The canonical map* $Q_\gamma GW^{[m]} \to Q_\gamma \mathbb{G}W^{[m]} \cong \mathbb{G}W$ *induces isomorphisms* $\pi_n Q_\gamma GW^{[m]} \cong \pi_n \mathbb{G}W^{[m]}$ *for* $n \geq 0$ *and for all* m.

Proof. This follows from two out of three and the proof of lemma 93 since $\pi_n GW^{[m]} \cong \pi_n \mathbb{G}W^{[m]}$ for $n \geq 0$ and for all m. □

Lemma 95. *The canonical map* $Q_\gamma GW^{[m]} \to Q_\gamma \mathbb{G}W^{[m]} \cong \mathbb{G}W$ *induces an isomorphism* $\pi_n Q_\gamma GW^{[m]} \cong \pi_n \mathbb{G}W^{[m]}$ *for* $n <= 0$ *and for all* m.

Proof. Because homotopy groups commute with filtered (homotopy) colimits of spectra

$$\pi_n Q_\gamma GW^{[m]} = \operatorname{colim}(\pi_n GW \xrightarrow{\alpha} \pi_n \mathbf{Hom}(W/S^0 \otimes W_\sigma/S^0, GW) \xrightarrow{\alpha^{\otimes 2}} \cdots).$$

Fix $[m]$ for now and denote by F_n^i the image of π_n under the map

$$\alpha : GW_n^{[m]}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i}) \to GW_n^{[m]}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+1})$$

and note that $F_n^0 \cong GW_n^{[m]}$. Denote by FB_n^i the same construction as above with GW replaced by $\mathbb{G}W$.

For $i \geq -n$, we claim that there are exact sequences

$$F_n^i(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus F_n^i(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \longrightarrow F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$$

such that $\partial(F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0)) = \partial([T]) \cup \partial([T^\sigma]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$. We prove this in conjunction with the statement that $F_n^i \cong \mathbb{G}W_n^{[m]}$ for $i \geq -n$. The proof is induction in i, and we must show that $\partial(F_n^i(\mathbb{G}_m/1 \otimes W_\sigma/S^0)) = \partial([T]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$. For $n \geq 0$, the same argument that we gave for K-theory together with lemma 94 works. In more detail, there's an exact sequence

$$GW_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus GW_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \longrightarrow GW_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} GW_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0)$$

and because $n \geq 0$, the same argument we gave for K-theory above identifies $\partial(GW_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0))$ with $\partial([T]) \cup \partial([T^\sigma]) \cup GW_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0)$ and in turn with $\mathbb{G}W^{[m]}(X)$. Then we just use the fact that p^* is injective and a module map to conclude that $\partial([T]) \cup \partial([T^\sigma]) \cup p^*(GW_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0))$ is isomorphic to $\mathbb{G}W^{[m]}(X)$.

Now fix an i, and assume by induction that our claim holds for all $-n \leq i$. Then there's an exact sequence

$$\mathbb{G}W_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus \mathbb{G}W_n^{[m]}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \longrightarrow \mathbb{G}W_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$$

which identifies $\partial(\mathbb{G}W_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0))$ with $\partial([T]) \cup \partial([T^\sigma]) \cup F_{n-1}^i(W/S^0 \otimes W_\sigma/S^0)$, but we know that $\partial(\mathbb{G}W_n^{[m]}(\mathbb{G}_m/1 \otimes W_\sigma/S^0))$ is equal to $\mathbb{G}W_{n-1}^{[m]}(W/S^0 \otimes W_\sigma/S^0) \cong \mathbb{G}W_{n-1}^{[m]}(X)$. Thus, letting p denote the projection $W/S^0 \otimes W_\sigma/S^0 \to X$ to the basepoint,

$$\mathbb{G}W^{[m]}_{n-1}(X) \cong p^*(\partial([T]) \cup \partial([T^\sigma]) \cup F^i_{n-1}(W/S^0 \otimes W_\sigma/S^0)) = \partial([T]) \cup \partial([T^\sigma]) \cup p^*(F^i_{n-1}(W/S^0 \otimes W_\sigma/S^0)) = F^{i+1}_{n-1}$$

since p^* is split injective.

The meatier part of the argument is producing the exact sequence for $F^{[i+1]}_{n-1}$, though the proof is essentially the same as the proof of the base case.

First note that for all i and n, there's a chain complex

$$F^i_n(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus F^i_n(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \longrightarrow F^i_n(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \xrightarrow{\partial} F^i_{n-1}(W/S^0 \otimes W_\sigma/S^0)$$

which is just the image of the usual long exact sequence for GW under the map α. Depending on n, this sequence may or may not be exact, as the image of an exact sequence is in general not exact.

Consider the commutative diagram

$$\begin{tikzcd}
F^{i+1}_{n-1}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \oplus F^{i+1}_{n-1}(\mathbb{A}^1/1 \otimes W_\sigma/S^0) \arrow[r] \arrow[d] & F^{i+1}_{n-1}(\mathbb{G}_m/1 \otimes W_\sigma/S^0) \arrow[r, "\partial"] \arrow[d] & F^{i+1}_{n-2}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) \arrow[d, "\phi"] \\
\mathbb{G}W(\mathbb{A}^1/1 \otimes W_\sigma/S^0 \otimes (W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2})^{\oplus 2} \arrow[r] & \mathbb{G}W(\mathbb{G}_m/1 \otimes W_\sigma/S^0 \otimes (W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) \arrow[r, "\partial^B"] & \mathbb{G}W^{[m]}_{n-2}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+3})
\end{tikzcd}$$

where the left two vertical maps are isomorphisms by what we've already shown. We claim that the top row is exact. The composite is zero since it's a chain complex, and if $x \in \ker(\partial)$, then using the fact that the middle and left maps are isomorphisms we produce a lift of x.

Now it remains only to check that the image of ∂ coincides with $\partial([T]) \cup \partial([T^\sigma]) \cup F^{i+1}_{n-2}$. This is the part of the proof we adapt from the K-theory case. First, it's clear that $\partial([T]) \cup \partial([T^\sigma]) \cup F^{i+1}_{n-2} \subseteq \mathrm{im}(\partial)$, since $\partial([T])$ restricts to zero in $\mathbb{A}^1$. For the other containment, by exactness and the fact that the left two vertical arrows are isomorphisms, we know that $\mathrm{im}(\partial) \cong \mathrm{im}(\partial^B)$. Now since $\partial([T]) \cup \partial([T^\sigma]) \cup p^*(F^{i+1}_{n-2}) \subseteq \mathrm{im}(\partial)$, it is isomorphic to its image in $\mathbb{G}W^{[m]}_{n-2}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2})$. But ϕ is a map of modules, so that

$$\phi(\partial([T]) \cup \partial([T^\sigma]) \cup F^{i+1}_{n-2}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2})) \cong \partial([T]) \cup \partial([T^\sigma]) \cup \mathrm{im}(\phi)$$

But ϕ is necessarily surjective, and cup product with $\partial([T]) \cup \partial([T^\sigma])$ is an automorphism of $\mathbb{G}W$. It follows that

$$\partial([T]) \cup \partial([T^\sigma]) \cup F^{i+1}_{n-2}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) \cong \mathrm{im}(\phi) = \mathrm{im}(\partial^B) \cong \mathrm{im}(\partial)$$

so that $\partial([T]) \cup \partial([T^\sigma]) \cup F^{i+1}_{n-2}((W/S^0 \otimes W_\sigma/S^0)^{\otimes i+2}) = \mathrm{im}(\partial)$.

We've shown that if the inductive statement holds for i, n, then it holds for $i+1, n-1$. The fact that it holds for $i+1, m$ for any $m < n+1$ is clear by appealing to results for $\mathbb{G}W$. Now, the lemma follows from the explicit description for filtered colimits of groups.

□

Corollary 96. $\mathbb{G}W \cong Q_\gamma GW$.

Theorem 97. *$L_{\mathbb{A}^1}\mathbb{G}W$ is an E_∞-object in motivic spectra.*

Proof. GW is an E_∞ object in presheaves of spectra on $\mathbf{Sm}_S^{C_2}$ by results of Schlichting [6]. By [9] Lemma 3.3, together with corollary 96 above, $\mathbb{G}W$ is the periodization of GW with respect to γ. Now $\mathbb{G}W$ is Nisnevich excisive, so that $\mathbb{G}W(W/S^0 \otimes W_\sigma/S^0) \cong \mathbb{G}W(\mathbb{P}^\sigma)$, and $\mathbb{G}W$ is γ periodic if and only if it is Bott periodic with respect to the Bott element in the projective bundle formula. By [9], proposition 3.2, $\mathbb{G}W$ lifts to an E_∞ object in $GW_{mod}[(\mathbb{P}^\sigma))^{-1}]$. Because $\mathbb{A}^1$-localization preserves E_∞ objects, $L_{\mathbb{A}^1}\mathbb{G}W$ is an E_∞ object in the subcategory of Nisnevich excisive Bott periodic GW-modules. □

Recall the definition of $\beta^{1+\sigma}$ from equation 6.2.

Definition 98. A GW-module E is called *Bott periodic* if the map

$$\mathbf{Hom}(\beta^{1+\sigma}, E) : E \to \mathbf{Hom}((\mathbb{P}^1, [1:1]) \wedge (\mathbb{P}^\sigma, [1:1]), E)$$

is an equivalence.

There are zigzags

$$\mathbb{A}^1/\mathbb{G}_m \hookrightarrow \mathbb{P}^1/(\mathbb{P}^1 - [-1:1]) \twoheadleftarrow \mathbb{P}^1/[1:1]$$

and

$$\mathbb{A}^-/\mathbb{G}_m^- \hookrightarrow \mathbb{P}^\sigma/(\mathbb{P}^\sigma - [-1:1]) \twoheadleftarrow \mathbb{P}^\sigma/[1:1].$$

The maps $\beta : \mathbb{P}^1/[1:1] \to GW^{[1]}$ and $\beta^\sigma : \mathbb{P}^\sigma/[1:1] \to GW^{[-1]}$ lift to $\mathbb{P}^1/(\mathbb{P}^1 - [-1:1])$ and $\mathbb{P}^\sigma/(\mathbb{P}^\sigma - [-1:1])$ respectively.

Note that we've shown that $\mathbb{G}W$ is Bott periodic and Nisnevich excisive. Since it's the Bott periodization of GW, it is in fact the reflection of GW in the subcategory of Nisnevich excisive and Bott periodic GW-modules. Because C_2 is a finite group (scheme), making $\mathbb{A}^1$ contractible is equivalent to making the regular representation $\mathbb{A}^\sigma$ contractible. Thus by definition, $L_{\mathbb{A}^1}\mathbb{G}W$ is the reflection of GW in the subcategory of homotopy invariant, Nisnevich excisive, and Bott periodic GW-modules.

If one wants to work in a more general settings (i.e. work with stacks rather than C_2-schemes), then one can still define homotopy Grothendieck-Witt theory as As in [9], we can compare the definition of the $\mathbb{A}^1$-localization of homotopy Grothendieck-Witt theory to the periodization of $L_{mot}GW$ with respect to a certain Bott element.

Chapter 7

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