1 Initial Conditions

The initial mass profile is given by the equilibrium torus configuration a là Stone 1999:

$$\frac{P}{\rho} = \frac{GM}{(n+1)R_0}\left[\frac{R_0}{r} - \frac{1}{2}\left(\frac{R_0}{r\sin\theta}\right)^2 - \frac{1}{2d}\right]$$

and is derived in Papaloizou & Pringle 1984. In the above M is the mass of the central object, R_0 is the characteristic radius of the disk, n is the polytropic index and d is a parameter that describes the distortion of the disk. which apparently follows from a polytropic equation of state $p = a\rho^\gamma$, where $n = (\gamma-1)^{-1}$. If the polytropic equation of state $\rho = AP^\gamma$ is used with $\gamma = 1 + 1/n$, then one obtains the following initial density profile.

$$\rho_0 = \left\{\frac{GM}{(n+1)AR_0}\left[\frac{R_0}{r} - \frac{1}{2}\left(\frac{R_0}{r\sin\theta}\right)^2 - \frac{1}{2d}\right]\right\}^n \tag{1.1}$$

In the above there are four free(ish) parameters d, R_0, and A. We will want to control d and R_0 directly, and will select A by specifying a total initial disk mass $M_{d,0}$. The details on how this accomplished is present below.

1.1 Specifying the total mass

We can obtain $M_{d,0}$ by integrating (1.1) in the relevant regions:

$$M_{d,0} = 2\pi\left[\frac{GM}{(n+1)AR_0}\right]^n R_0^3 I(d,\gamma) = \alpha A^{-n} R_0^{3-n} \quad \text{with} \quad \alpha = 2\pi\left[\frac{GM}{(n+1)}\right]^n I(d,\gamma)$$

where $I(d,\gamma)$ is the following integral

$$I(d,\gamma) = \iint r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta \left[\frac{1}{r} - \frac{1}{2}\left(\frac{1}{r\sin\theta}\right)^2 - \frac{1}{2d}\right]^n \tag{1.2}$$

where the bounds of integration are set by the condition that $\rho_0 > 0$. The bounds are then set by the condition that

$$f(r,\theta) = \frac{1}{r^2}\left(\frac{1}{2\sin^2\theta}\right) - \frac{1}{r} + \frac{1}{2d} = 0.$$

From this, we obtain the following boundaries

$$r_\pm = \left[\sin^2\theta\,(1 \mp \sqrt{1 - 1/d\sin^2\theta})\right]^{-1} \quad \text{and} \quad \theta_\pm = \pm\arcsin d^{-1/2}.$$

and will allow us to evaluate (1.2) numerically. With this, one can find that

$$A = \left(\frac{M_{d,0}}{\alpha R_0^{3-n}}\right)^{-1/n}$$

NOTE: the α above is *not*!! the viscousity parameter.

1.2 Initial velocity profile

The equilibrium configuration (1.1) is one of constant specific angular momentum $h = \sqrt{GMR_0}$ (probably in the z direction), which leads to a velocity profile:

$$v_\phi = \frac{h}{r\sin\theta} = \frac{\sqrt{GMR_0}}{r\sin\theta}$$