JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 42, No. 7, July 2019

Engineering Notes

Two-on-One Pursuit

Meir Pachter*
Air Force Institute of Technology, Wright–Patterson Air Force Base, Ohio 45433
Alexander Von Moll,† Eloy Garcia,‡ and David W. Casbeer‡
U.S. Air Force Research Laboratory, Wright–Patterson Air Force Base, Ohio 45433
and
Dejan Milutinović§
University of California, Santa Cruz, Santa Cruz, California 95064

DOI: 10.2514/1.G004068

I. Introduction

PURSUIT and evasion have strong aerospace connotations. In surface-to-air missile (SAM) engagements, the standard procedure is to fire two SAMs to intercept a threat. It is conventional wisdom that the probability of intercept of a SAM is $P_K = 0.8$; therefore, when two SAMs are launched, the probability of intercept is enhanced and is $P_K = 0.96$. Note, however, that it is herein tacitly assumed that the SAMs are "independent," statistically speaking. Their effectiveness could be improved if they were cooperatively guided. Thus, in this Note, the foundational pursuit–evasion differential game in the Euclidean plane in which two pursuers P_1 and P_2, cooperatively chase an evader E is considered. The three players are holonomic, with the speeds of the pursuers each being greater than that of the evader. We are interested in point capture by either one or both: P_1 and/or P_2. The payoff of E and the cost of the P_1 and P_2 team is the time to capture. Thus, Isaacs's classical "two cutters and a fugitive ship" [1] differential game is revisited. Interestingly, the two cutters and a fugitive ship pursuit game was posed by Hugo Steinhaus back in 1925; his original paper was reprinted in 1960 in Ref. [2]. (Hugo Steinhaus was a contemporary of Borel and Von Neumann, who are credited with laying the foundations of game theory. Borel and Von Neumann mainly considered static games (also known as games in normal form) while referring to dynamic games as games in extensive form, believing that dynamic games can be easily transformed to static games [3]. The requirement of time consistency/subgame perfectness in dynamic games came to the attention of game theorists only in the 1970s. From the outset, Steinhaus was certainly attune to thinking about dynamic games, also known as differential games.) The solution, sans a proof, of the differential game was presented in Isaacs's ground-breaking book ([1] example 6.8.3, pp. 148–149], in which the players' optimal strategies were derived using a geometric method. Since then, several others have investigated the game, as well as other closely related games. In Ref. [4], the game of one fast pursuer against two evaders was solved. Ganebny et al. considered a two-pursuer/one-evader game on a line [5]. Most recently, the two–pursuer/one-evader scenario (in two dimensions) was investigated in Ref. [6], wherein evader strategies were derived for the case in which the evader knew the pursuers' strategies. In Ref. [7], a proof of the optimality of the three players' strategies proposed by Isaacs was undertaken. Reference [8] analyzed the two-pursuer/one-evader game with a finite capture radius, making use of the costate equilibrium dynamics to solve a boundary value problem backward in time. The extension of this game for any number of pursuers was explored in Ref. [9], wherein open-loop strategies were proposed with no proof of optimality. We also note the presence of a significant body of literature on both the two-on-one and multiple-on-one differential games with one or more of the following features: fixed duration, cost/payoff defined as terminal miss distance, various kinematic/dynamic models (e.g., inertial vs noninertial, bounded acceleration, bounded velocity, etc.), integral constraints, and/or a superior evader [10–16].

In this Note, some geometric features, perhaps overlooked by Isaacs but with a bearing on extensions, are addressed: The state-space regions in which pursuit devolves into pure pursuit (PP) by either P_1 or P_2, or into a pincer movement pursuit by the P_1 and P_2 team who cooperatively capture the evader, are characterized. Thus, in this Note, a complete solution of the game of kind is provided. Furthermore, in this Note, a three-dimensional reduced state space for analyzing the two-on-one pursuit–evasion differential game is introduced and the players' state feedback strategies as well as the value function are explicitly derived.

The Note is organized as follows. The geometric method employed by Isaacs to solve the two cutters and a fugitive ship differential game is expounded on in Sec. II. In Sec. III, a three-state reduced state-space reformulation of the two-on-one pursuit–evasion differential game is introduced, and Isaacs's geometric method is employed to yield the players' optimal state feedback strategies and the game's value function in closed form. Furthermore, the state-space regions in which either one of the pursuers captures the evader and the state-space region in which both pursuers cooperatively and isochronously capture the evader are characterized, thus solving the game of kind. Possible extensions are also discussed in Sec. III. Conclusions are presented in Sec. IV.

II. Geometric Method

We assume that the fast pursuers P_1 and P_2 have equal speed, which we normalize to one. The problem parameter is the speed of the evader E, which is $0 \le \mu < 1$.

There are three players in the Euclidean plane, and so the realistic state space is obviously $\mathbb{R}^6$; however, the state space could be reduced to $\mathbb{R}^4$ by collocating the origin of a nonrotating (x, y) Cartesian frame at the E instantaneous position. Because the players are holonomic, the dynamics A matrix is zero; there are no dynamics. This, and the fact that the performance functional is the time to capture, yields a Hamiltonian subject to the costates being all constant. This suggests that the optimal flowfield might consist of straight-line trajectories. Hence, geometry might come into play. To obtain the two cutters and fugitive ship differential game's solution, Isaacs employed [1] the geometric concept of an Apollonius circle to delineate the boundary of a safe region (BSR) for the evader: In pursuit–evasion differential games, an Apollonius circle is constructed based on the $E - P$ separation, and the speed ratio is $\mu < 1$. The Apollonius circle concept is conducive to the geometric solution of the two cutters and a fugitive ship differential game, as will be demonstrated. For a more

Received 9 October 2018; revision received 21 December 2018; accepted for publication 24 December 2018; published online 6 February 2019.

*Professor, Electrical Engineering Department.

†Research Engineer, Controls Science Center of Excellence (Corresponding Author).

‡Research Engineer, Controls Science Center of Excellence.

§Professor, Electrical and Computer Engineering Department.

thorough treatment of the Apollonius circle, see Ref. [17]; we include the main features here for reference. The Apollonius circle radius and circle center (along the ray $\rightarrow \boldsymbol{PE}$) are given by

$$\rho = \frac{\mu}{1-\mu^2}\overline{PE} \tag{1}$$

$$x_O = \frac{\mu^2}{1-\mu^2}\overline{PE}, \qquad y_O = 0 \tag{2}$$

We first present the solution of the two cutters and a fugitive ship differential game in the realistic plane using the geometric method. Two Apollonius circles C_1, for which the foci are at E and P_1, and the Apollonius circle C_2, for which the foci are at E and P_2, feature in this game. E is in the interior of both Apollonius disks, but the two Apollonius circles might or might not intersect. Concerning the calculation of the points of intersection (if any) of the Apollonius circles C_1 and C_2, subtracting the equation of circle C_1 from the equation of circle C_2 yields a linear equation in two unknowns: say, X and Y. One can thus back out Y as a function of X and insert this expression into one of the circle equations, thus obtaining a quadratic equation in X: The calculation of the two points of intersection of the Apollonius circles C_1 and C_2 boils down to the solution of a quadratic equation. The Apollonius circles intersect if, and only if, the quadratic equation has real solutions; in other words, the discriminant of the quadratic equation is positive. When the discriminant of the quadratic equation is negative, we are automatically notified that the Apollonius circles do not intersect; because E is in the interior of both Apollonius disks, we conclude that one of the Apollonius disks is contained in the interior of the second Apollonius disk. If $\rho_2 > \rho_1$, which is the case if, and only if, E is closer to P_1 than to P_2 [see Ref. (2)], circle C_2 is discarded, and vice versa. The geometry is illustrated in Fig. 1.

When the Apollonius circles do not intersect, the pursuer associated with the outer Apollonius circle is irrelevant to the chase. This is so because the configuration is subject to the following: should P_1 employ PP and E run for his life, player P_2 cannot reach E before the latter is captured by P_1 because he is too far away from the P_1/E engagement or is too slow to close in and join the fight. This renders player P_2 irrelevant. As far as the geometric method is concerned, the Apollonius disk associated with player P_1 is then contained in the interior of the bigger Apollonius disk associated with player P_2, as illustrated in Fig. 1. In this case, pursuer P_1 (on which the inner Apollonius circle is based) will singlehandedly capture the evader: He will optimally employ PP while the evader runs for his life and will be captured at I; the game with two pursuers devolves to the simple pursuit–evasion game with one pursuer and one evader in which P_1 employs PP and E runs away from P_1. Similarly, if the Apollonius disk associated with P_2 is contained in the interior of the bigger Apollonius disk associated with player P_1, player P_2 will employ PP while E runs for his life; P_1 is then redundant.

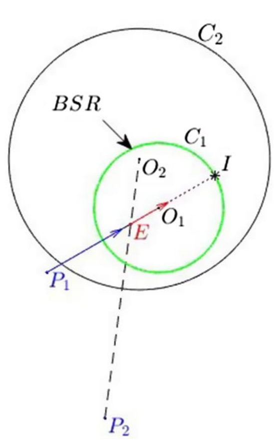

Fig. 1 One-cutter action.

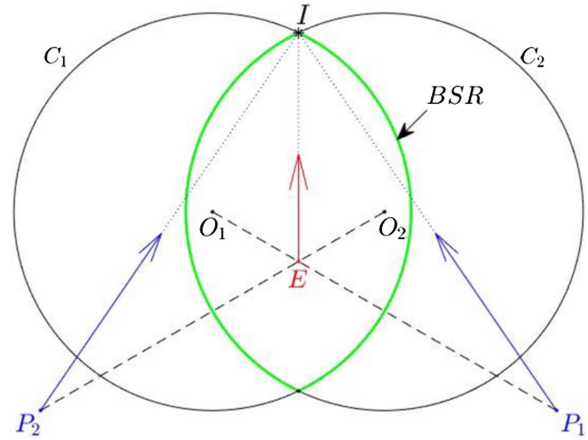

Fig. 2 Solution of the two cutters and a fugitive ship game.

The interesting case considered in Ref. [1], in which the Apollonius circles intersect, is illustrated in Fig. 2. Because there are two pursuers (similar to figure 6.8.5 in Ref. [1]), a lens-shaped BSR, delineated in green, is formed by the intersection of the two Apollonius circles. To calculate the aim point I, which is one of the two points where Apollonius circles C_1 and C_2 intersect, requires solving a quadratic equation; the quadratic equation has two real solutions and, among the two points of intersection of the Apollonius circles, the aim point I is the point farthest from E. Thus, E heads toward the most distant point I on the BSR, and so do P_1 and P_2. Both pursuers P_1 and P_2 will be active and cooperatively and isochronously capture the evader at point I; see Fig. 2.

When the discriminant of the quadratic equation is zero, the quadratic equation has a repeated real root. Geometrically, this means that one of the Apollonius circles is tangent from the inside to the second Apollonius circle. The following holds:

Proposition 1: Assume the Apollonius circles C_1 and C_2 are tangent; that is, the discriminant of the quadratic equation vanishes. The aim point of the three players is then the circles' point of tangency (say, T); that is, $I = T$ if, and only if, the three players E, P_1, and P_2 are collinear and E is sandwiched between P_1 and P_2. □

When the Apollonius circles C_1 and C_2 are tangent and their point of tangency T is subject to $T = I$, the points P_2, T, O_1, E, O_2, and P_1 are collinear and both pursuers employ PP to isochronously capture the evader. This is illustrated in Fig. 3.

Note, however, that when (as previously mentioned), P_1, P_2, and E are collinear and E is sandwiched between P_1 and P_2 but the Apollonius circles intersect, E will break out; see Fig. 4.

If the Apollonius circles C_1 and C_2 are tangent but E is not on segment $\overline{P_1P_2}$, the players' aim point I is not the circles' point of tangency T: If the tangent Apollonius circles are such that the Apollonius circle C_1 is contained in the Apollonius disk formed by the Apollonius circle C_2, optimal play then consists of the active player being P_1 and employing PP while E runs away from P_1 and player P_2 is redundant; and, if the Apollonius circle C_2 is contained in the Apollonius disk formed by the Apollonius circle C_1, optimal play

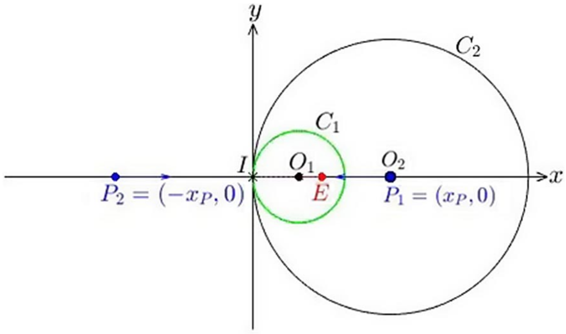

Fig. 3 PP by P_1 and P_2.

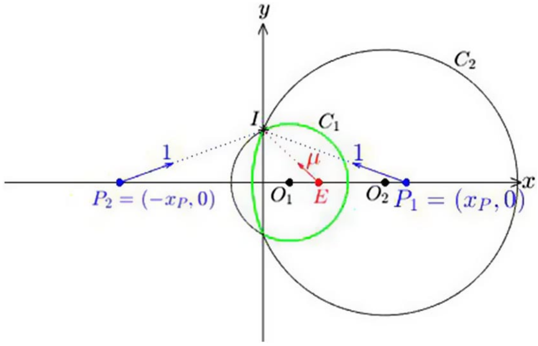

Fig. 4 Breakout of *E*.

then consists of the active player being P_2 and employing PP while E runs away from P_2, and now player P_1 is redundant; the circles' point of tangency T plays no role here. This should alert us to the fact that, even though the Apollonius circles intersect at their point of tangency (that is, $C_1 \cap C_2 \neq \varnothing$ and $T \in C_1 \cap C_2$), the players' aim point $I \ni C_1 \cap C_2$.

III. Geometric Solution in Reduced State Space

Similar to Isaacs's treatment of the homicidal chauffeur differential game, it is beneficial to analyze the two cutters and a fugitive ship differential game in a reduced state space. The dimension of the two cutters and a fugitive ship game's state space can be reduced to three using a noninertial, rotating reference frame by pegging the x axis to P_1 and P_2's instantaneous positions. The y axis is the orthogonal bisector of the $\overline{P_1 P_2}$ segment. In this rotating (x, y) reference frame, the states are E's x and y coordinates (x_E, y_E) and the x position x_P of P_1. In this reduced state space, the y coordinate of P_1 will always be zero and the position of P_2 will be $(-x_P, 0)$. Without loss of generality, we assume $x_E \geq 0$ and $y_E \geq 0$. The rotating reference frame (x, y) is shown overlaid on the realistic plane (X, Y) in Fig. 5, in which the P_1, E, and P_2 players' headings χ, ϕ, and ψ are also indicated. Without loss of generality, the rotating reference frame (x, y) is initially aligned with the inertial frame (X, Y).

Using the rotating reference frame (x, y), the state space of the two cutters and a fugitive ship differential game is reduced to the first quadrant of $\mathbb{R}^3$: that is, the set

$$\mathbb{R}^3_1 \equiv \{(x_P, x_E, y_E) | x_P \geq 0, y_E \geq 0\}$$

Symmetry allows us to confine our attention to the case where $x_E \geq 0$; that is, the state will evolve in the positive orthant of $\mathbb{R}^3$: that is, in

$$\mathbb{R}^3_+ = \{(x_P, x_E, y_E) | x_P \geq 0, x_E \geq 0, y_E \geq 0\}$$

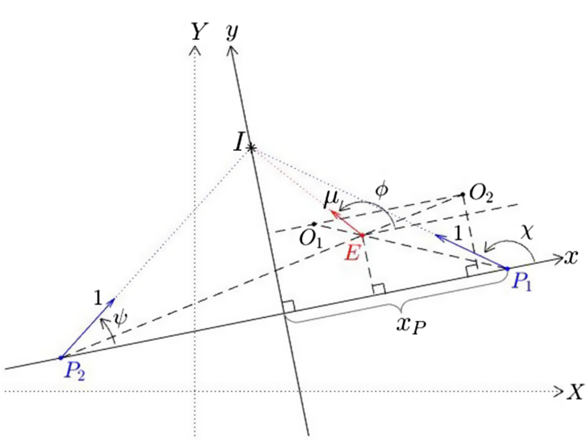

Fig. 5 Rotating reference frame.

where the three-state nonlinear dynamics of the two cutters and a fugitive ship differential game are

$$\dot{x}_P = \frac{1}{2}(\cos\chi - \cos\psi), \qquad x_P(0) = x_{P_0} \tag{3}$$

$$\dot{x}_E = \mu\cos\phi - \frac{1}{2}(\cos\chi + \cos\psi) + \frac{1}{2}\frac{y_E}{x_P}(\sin\chi - \sin\psi), \quad x_E(0) = x_{E_0} \tag{4}$$

$$\dot{y}_E = \mu\sin\phi - \frac{1}{2}(\sin\chi + \sin\psi) - \frac{1}{2}\frac{x_E}{x_P}(\sin\chi - \sin\psi), \quad y_E(0) = y_{E_0} \tag{5}$$

A. Game of Kind in Reduced State Space

The solution of the game of kind determines which pursuer actually captures the evader under optimal play: P_1, P_2, or simultaneous capture by both pursuers. More specifically, the game of kind partitions the state space into three regions that dictate the outcome of the differential game under optimal play. The solution of the game of kind in the reduced state space (x_P, x_E, y_E) using the geometric method proceeds as follows:

We have two Apollonius circles: C_1 is based on the instantaneous positions of E and P_1, and C_2 is based on the instantaneous positions of E and P_2. In the (x, y) frame (see Figs. 4 and 2, respectively), the center O_1 of the Apollonius circle C_1 is at

$$x_{O_1} = \frac{1}{1-\mu^2}(x_E - \mu^2 x_P), \qquad y_{O_1} = \frac{1}{1-\mu^2} y_E$$

Similarly, the center O_2 of the Apollonius circle C_2 is at

$$x_{O_2} = \frac{1}{1-\mu^2}(x_E + \mu^2 x_P), \qquad y_{O_2} = \frac{1}{1-\mu^2} y_E$$

Thus, using Eq. (1), the equation of the Apollonius circle C_1 is

$$\left[x - \frac{1}{1-\mu^2}(x_E - \mu^2 x_P)\right]^2 + \left(y - \frac{1}{1-\mu^2} y_E\right)^2 = \frac{\mu^2}{(1-\mu^2)^2}[(x_E - x_P)^2 + y_E^2] \tag{6}$$

and the equation of the Apollonius circle C_2 is

$$\left[x - \frac{1}{1-\mu^2}(x_E + \mu^2 x_P)\right]^2 + \left(y - \frac{1}{1-\mu^2} y_E\right)^2 = \frac{\mu^2}{(1-\mu^2)^2}[(x_E + x_P)^2 + y_E^2] \tag{7}$$

In the (x, y) reference frame, the y coordinate of the C_1 and C_2 Apollonius circles' centers is the same, and therefore the distance d between the circles' centers is

$$d = x_{O_2} - x_{O_1} = \frac{2\mu^2}{1-\mu^2} x_P$$

Hence, because the radii of the Apollonius circles are such that $\rho_1 < \rho_2$ if, and only if, $x_E > 0$, the Apollonius circles C_1 and C_2 intersect if, and only if,

$$d + \rho_1 > \rho_2$$

That is,

$$2\mu x_P + d_1 > d_2$$

In other words, the inequality holds:

$$2\mu x_P > \sqrt{(x_P + x_E)^2 + y_E^2} - \sqrt{(x_P - x_E)^2 + y_E^2}$$

which yields the algebraic condition: The Apollonius circles C_1 and C_2 intersect if, and only if,

$$\mu^2 y_E^2 + (1 - \mu^2)(\mu^2 x_P^2 - x_E^2) \geq 0 \tag{8}$$

In light of this, the reduced state space $\mathbb{R}_1^3$ is partitioned as follows:

$$\mathbb{R}_1^3 = R_1 \cup R_2 \cup R_{1,2}$$

During optimal play in R_1, E is captured solely by P_1 while P_2 is redundant; in R_2, E is captured solely by P_2 while P_1 is redundant; while in $R_{1,2}$, E is isochronously captured by P_1 and P_2. At this point, it appears that things stand as follows: If condition (8) does not hold and $x_E > 0$, the state is in R_1, in which E is captured solo by P_1. If condition (8) does not hold and $x_E < 0$, the state is in R_2, in which E is captured solo by P_2: From a kinematic point of view, the state is in R_1 if collision course (CC) guidance will not allow P_2 to capture E, who is running away from P_1, before P_1, using pure pursuit, captures E. Similarly, the state is in R_2 if CC guidance will not allow P_1 to capture E, who is running away from P_2, before P_2, using PP, captures E. As far as geometry is concerned, let D_i denote the disk that corresponds to the Apollonius circle C_i, $i = 1, 2$. In view of the aforementioned discussion, it would appear that set R_1 is characterized by $D_1 \subset D_2$ (see Fig. 1); similarly, set R_2 is characterized by $D_2 \subset D_1$ and, if condition (8) holds (see Fig. 2, in which the Apollonius circles C_1 and C_2 intersect), one might then be inclined to think that the state is in $R_{1,2}$ so that, during optimal play, E is isochronously captured by P_1 and P_2. And, as far as the characterization of the sets R_1 and R_2 is concerned, because $x_E \geq 0$ implies $\rho_1 \leq \rho_2$, disk D_2 cannot be contained in disk D_1, and so either $D_1 \subset D_2$ or the Apollonius circles C_1 and C_2 intersect. The geometric condition

$$D_1 \subset D_2 \Rightarrow d + \rho_1 < \rho_2$$

lets us recover algebraic condition (8):

$$C_1 \cap C_2 \neq \varnothing \Leftrightarrow d + \rho_1 > \rho_2 \Leftrightarrow \mu^2 y_E^2 + (1 - \mu^2)(\mu^2 x_P^2 - x_E^2) > 0$$

as expected. Algebraic condition (8) delineates the set in $\mathbb{R}_+^3$:

$$\mathcal{K}_1 = \{(x_P, x_E, y_E) | x_P \geq 0, x_E \geq 0, \mu^2 y_E^2 + (1 - \mu^2)(\mu^2 x_P^2 - x_E^2) < 0\}$$

This is a cone for which the x_E cross sections are arcs of ellipses. When the state is in the interior of the elliptical cone $\mathcal{K}_1$ or in its projection onto the plane, $y_E = 0$ and $D_1 \subset D_2$, and so E is captured by P_1 only. Thus, one is inclined to set $R_1 \equiv \mathcal{K}_1$. Similarly, when the state is in the interior of the elliptical cone,

$$\mathcal{K}_2 = \{(x_P, x_E, y_E) | x_P \geq 0, x_E \leq 0, \mu^2 y_E^2 + (1 - \mu^2)(\mu^2 x_P^2 - x_E^2) < 0\}$$

or, in its projection onto the plane, $y_E = 0$, $D_2 \subset D_1$; and so E is captured by P_2 only: the set $\mathcal{K}_2$ is the mirror image of the cone $\mathcal{K}_1$ about the plane $x_E = 0$, and one is inclined to set $R_2 \equiv \mathcal{K}_2$. The boundary of the elliptical cone $\mathcal{K}_1$ that is the set of states subject to the Apollonius circle C_1 is contained in the Apollonius disk formed by the bigger circle C_2 and is tangent to the Apollonius circle C_2; similarly, the boundary of the elliptical cone $\mathcal{K}_2$ that is the set of states subject to the Apollonius circle C_2 is contained in the Apollonius disk formed by the bigger circle C_1 and is tangent to the Apollonius circle C_1. When the state is on the boundary of the elliptical cones $\mathcal{K}_1$ or $\mathcal{K}_2$, the Apollonius circles C_1 and C_2 are tangent, say, at point T. According to Proposition 1, the players' aim point I is the point of tangency T of the Apollonius circles if, and only if, $y_E = 0$ and the tangent to the Apollonius circles at $T = I$ is the orthogonal bisector of the segment $\overline{P_1 P_2}$; from Eq. (8), we deduce $x_E = \mu x_P$; E is then isochronously captured by P_1 and P_2, who employ PP, as illustrated in Fig. 3. Note that, if $x_E = 0$, condition (8) holds, and so the quarter-plane

$$\{(x_P, x_E, y_E) | x_P \geq 0, x_E = 0, y_E \geq 0\} \subset R_{1,2}$$

and E is isochronously captured by P_1 and P_2. Obviously, E is also isochronously captured by P_1 and P_2 when $x_P = 0$. And, so far, it would appear that, during "optimal" play, when the state is outside the elliptical cones $\mathcal{K}_1$ and $\mathcal{K}_2$ where inequality (8) holds (that is, the state is in what appears to be $R_{1,2}$), E will be isochronously captured by the P_1 and P_2 team. Thus, at first blush, it would appear that Eq. (8) characterizes the set $R_{1,2}$. However, although in set $R_{1,2}$ inequality (8) holds, it also holds in subsets of R_1 and R_2: condition (8) does not characterize the set $R_{1,2}$. We must properly characterize the state-space regions R_1, R_2, and $R_{1,2}$ in $\mathbb{R}_1^3$. Inequality (8) does not provide the answer, and it will be replaced by an alternative condition.

In this respect, consider the following: In Fig. 1, let points E and P_2 be fixed while point P_1 is moved in a clockwise direction, keeping the $P_1 - E$ distance d_1 constant so that the Apollonius circles C_1 and C_2 will eventually intersect, whereupon inequality (8) will hold. The radius ρ_1 of the Apollonius circle C_1 is kept constant while it is approaching the Apollonius circle C_2 from the inside. The Apollonius circle C_1 first meets the Apollonius circle C_2 tangentially and, if the segment $\overline{P_1 E}$ rotates some more clockwise, the circles start intersecting. When this initially happens, point I in Fig. 1 is still in the interior of the disk formed by the Apollonius circle C_2. Thus, although the Apollonius circles intersect and condition (8) holds, E nevertheless flees toward point I with P_1 in hot pursuit, as if the configuration would have been as illustrated in Fig. 1, in which the Apollonius circle C_1 is in the interior of the Apollonius disk formed by the Apollonius circle C_2; it is only when point I on the extension of the segment $\overline{EO_1}$ meets the Apollonius circle C_2 and then exits the disk formed by the Apollonius circle C_2 that both pursuers, P_1 and P_2, cooperatively and isochronously capture E in a pincer movement maneuver. Thus, although the Apollonius circles do intersect, it nevertheless might be the case that neither one of their two points of intersection is the players' aim point I; as before, only one of the pursuers is active while the evader runs for his life from the active pursuer. The BSR then has the shape of a thick lens, and the evader's and the active pursuer's aim point I is the point on the thick lens-shaped BSR that is farthest away from E; it is on the circumference of the smaller Apollonius circle, on its diameter that runs though E, while at the same time, it is in the interior of the Apollonius disk formed by the bigger Apollonius circle. The critical configuration at which point $I \in C_2$ is illustrated in Fig. 6.

Because, without loss of generality, we have assumed $x_E \geq 0$ and $y_E \geq 0$, our universe of discourse will be confined to the positive orthant of $\mathbb{R}^3$: $\mathbb{R}_+^3$.

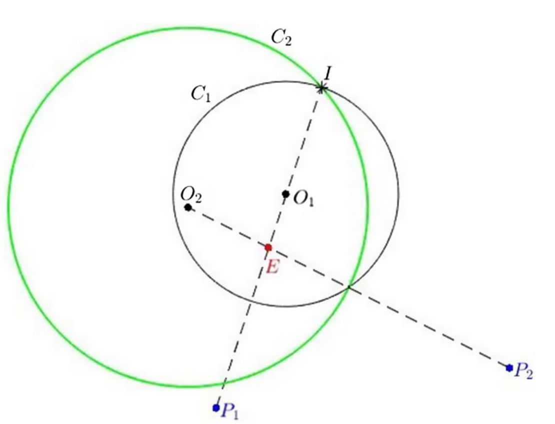

Fig. 6 Critical configuration.

Theorem 1: During optimal play, the evader is singlehandedly captured in PP by P_1 if the state is in the set R_1; the set R_1 is the wedge formed by the quarter-planes

$$\{(x_P, x_E, y_E) | x_P = 0, x_E \geq 0, y_E \geq 0\}$$

and

$$\{(x_P, x_E, y_E) | x_E = \mu x_P, x_P \geq 0, y_E \geq 0\}$$

The evader is singlehandedly captured in PP by P_2 if the state is in the set R_2; the set R_2 is the mirror image of R_1 about the plane $x_E = 0$. The evader is cooperatively and isochronously captured by P_1 and P_2 if the state is in the set

$$R_{1,2} = \{(x_P, x_E, y_E) | -\mu x_P \leq x_E \leq \mu x_P, x_P \geq 0, y_E \geq 0\}$$

Proof: To obtain a correct algebraic characterization of the sets R_1, R_2, and $R_{1,2}$ that will supersede condition (8), proceed as follows. Calculate the (x, y) coordinates of the critical point I on the circumference of the Apollonius circle C_1, which is antipodal to E, as shown in Fig. 6; see Fig. 7:

We have

$$\frac{x_P - x_I}{x_P - x_E} = \frac{\rho_1 + \overline{EO}_1 + d_1}{d_1}, \qquad \frac{y_I}{y_E} = \frac{\rho_1 + \overline{EO}_1 + d_1}{d_1}$$

where

$$\overline{EO}_1 = \frac{\mu^2}{1 - \mu^2} d_1, \qquad \rho_1 = \frac{\mu}{1 - \mu^2} d_1$$

Hence,

$$x_I = \frac{1}{1 - \mu}(x_E - \mu x_P), \qquad y_I = \frac{1}{1 - \mu} y_E \tag{9}$$

By construction, $I \in C_1$ and I is the critical aim point if, in addition, $I \in C_2$. To find the points of intersection (x_I, y_I) of the circles C_1 and C_2 boils down to the solution of a quadratic equation:

$$x_I = 0, \qquad y_I = \frac{y_E + \sqrt{\mu^2 y_E^2 + (1 - \mu^2)(\mu^2 x_P^2 - x_E^2)}}{1 - \mu^2} \tag{10}$$

Combining Eqs. (9) and (10), we obtain the result

$$x_E = \mu x_P$$

and the solution of the game of kind is as stated in this theorem. □

The cones $\mathcal{K}_1$ and $\mathcal{K}_2$ and/or condition (8) have no role to play here. The Apollonius circles C_1 and C_2 intersect if $-\mu x_P \leq x_E \leq \mu x_P$.

Remark: Proposition 1 is a corollary of Theorem 1.

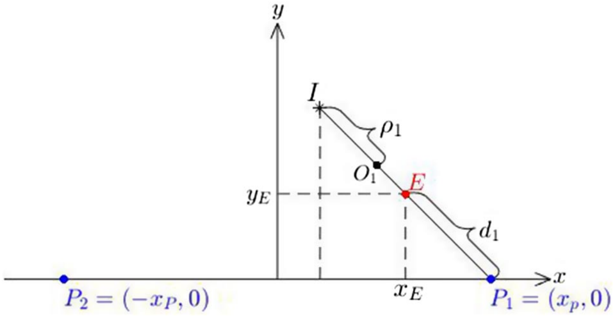

Fig. 7 Point *I*.

In summary, the reduced state space of the two cutters and a fugitive ship differential game is the first quadrant of $\mathbb{R}^3$; that is,

$$\mathbb{R}^3_1 = \{(x_P, x_E, y_E) | x_P \geq 0, y_E \geq 0\}$$

The state space $\mathbb{R}^3_1$ is symmetric about the plane $x_E = 0$; the region R_1 (and $\mathcal{K}_1$) resides in the positive orthant $\mathbb{R}^3_+$. Because point capture is desired, the terminal set in the R_1 subset of the $\mathbb{R}^3_+$ state space is the straight line

$$\{(x_P, x_E, y_E) | x_E = x_P, x_P \geq 0, y_E = 0\}$$

and the terminal set in the $R_{1,2}$ subset of the state space is the origin.

B. Game of Degree in Reduced State Space

1. *Game in R_1 and R_2*

In R_1, the active pursuer P_1 employs PP while the evader runs for his life. The actions of pursuer P_2 do not affect the outcome of the game; so, for exclusively illustrative purposes, we stipulate that P_2 mirrors the control of P_1. This ensures that the (x, y) frame will not rotate: it will just slide upward along the Y axis of the realistic plane, which then coincides with the y axis. The optimal trajectories in R_1 are the family of straight lines:

$$x_P(t) = x_{P_0} + \frac{x_{E_0} - x_{P_0}}{\sqrt{(x_{P_0} - x_{E_0})^2 + y_{E_0}^2}} t,$$

$$x_E(t) = x_{E_0} + \mu \frac{x_{E_0} - x_{P_0}}{\sqrt{(x_{P_0} - x_{E_0})^2 + y_{E_0}^2}} t,$$

$$y_E(t) = y_{E_0} - (1 - \mu) \frac{y_{E_0}}{\sqrt{(x_{P_0} - x_{E_0})^2 + y_{E_0}^2}} t$$

The state $y_E(t)$ is monotonically decreasing and, when parameterized by y_E, the optimal trajectories in R_1 are the family of straight lines:

$$x_P = \frac{1}{1 - \mu}\left(\frac{x_{P_0} - x_{E_0}}{y_{E_0}} y_E + x_{E_0} - \mu x_{P_0}\right),$$

$$x_E = \frac{1}{1 - \mu}\left(\mu \frac{x_{P_0} - x_{E_0}}{y_{E_0}} y_E + x_{E_0} - \mu x_{P_0}\right)$$

These trajectories terminate in the plane $y_E = 0$, and on the straight line $x_P = x_E$. The optimal flowfield in R_1 consists of the aforementioned family of straight-line trajectories which terminate on the straight line

$$\{(x_P, x_E, y_E) | x_E = x_P, y_E = 0\}$$

Similar considerations apply to R_2, for which the active pursuer is P_2. The optimal flowfield in R_2 is a mirror image of the optimal flowfield in R_1.

When $x_p = 0$, P_1 and P_2 are collocated. The half-plane

$$\{(x_P, x_E, y_E) | x_P = 0, y_E \geq 0\} \subset R_1 \cup R_2$$

2. *Game in $R_{1,2}$*

If the state is in

$$R_{1,2} = \{(x_P, x_E, y_E) | -\mu x_P \leq x_E \leq \mu x_P, x_P \geq 0, y_E \geq 0\}$$

E will be isochronously captured by the P_1 and P_2 team. The players' optimal headings are given in the next theorem.

Theorem 2: The players' optimal headings are constant in both the (x, y) and (X, Y) frames, and they are given by

$$
\begin{aligned}
\sin\psi^* &= \frac{y_{E_0} + \sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}{\sqrt{(1-\mu^2)(x_{P_0}^2 - x_{E_0}^2) + (1+\mu^2)y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}},\\
\cos\psi^* &= \frac{(1-\mu^2)x_{P_0}}{\sqrt{(1-\mu^2)(x_{P_0}^2 - x_{E_0}^2) + (1+\mu^2)y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}},\\
\chi^* &= \pi - \psi^*,\\
\sin\phi^* &= \frac{1}{\mu}\frac{\mu^2 y_{E_0} + \sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}{\sqrt{(1-\mu^2)(x_{P_0}^2 - x_{E_0}^2) + (1+\mu^2)y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}},\\
\cos\phi^* &= -\frac{1}{\mu}\frac{(1-\mu^2)x_{E_0}}{\sqrt{(1-\mu^2)(x_{P_0}^2 - x_{E_0}^2) + (1+\mu^2)y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}}
\end{aligned}
\tag{11}
$$

The initial state $(x_{P_0}, x_{E_0}, y_{E_0})$ can momentarily be viewed as the current state and, as such, Eq. (11) includes explicit state feedback optimal strategies, as provided by the geometric method; the attendant value function is given by

$$
t_f = \frac{1}{1-\mu^2}\sqrt{(1-\mu^2)(x_{P_0}^2 - x_{E_0}^2) + (1+\mu^2)y_{E_0}^2 + 2y_E\sqrt{\mu^2 y_{E_0}^2 + (1-\mu^2)(\mu^2 x_{P_0}^2 - x_{E_0}^2)}}
\tag{12}
$$

Proof: Because the $\Delta\ P_1P_2I$ in Fig. 2 is isosceles, the aim point $I = (0, y)$ is obtained upon setting $x = 0$ in Eq. (6) or Eq. (7), which yields a quadratic equation in y. The discriminant of the quadratic equation is positive if, and only if, the Apollonius circles C_1 and C_2 intersect, which is the case if, and only if, condition (8) holds; and it is certainly the case if $\mu x_P \le x_E \le \mu x_P$, whereupon

$$
y = \frac{1}{1-\mu^2}\left[y_E + \mathrm{sign}(y_E)\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_E^2)}\right]
$$

where the function

$$
\mathrm{sign}(x) \equiv \begin{cases} 1 & \text{if } x > 0,\\ 0 & \text{if } x = 0,\\ -1 & \text{if } x < 0 \end{cases}
$$

so

$$
y_I = \frac{1}{1-\mu^2}\left[y_E + \mathrm{sign}(y_E)\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_E^2)}\right]
\tag{13}
$$

Using the geometric method, the players' optimal state feedback strategies in $R_{1,2}$ are explicitly given by

$$
\sin\psi^* = \frac{y_I}{\sqrt{x_P^2 + y_I^2}}, \qquad \cos\psi^* = \frac{x_P}{\sqrt{x_P^2 + y_I^2}}
\tag{14}
$$

$$
\sin\chi^* = \frac{y_I}{\sqrt{x_P^2 + y_I^2}}, \qquad \cos\chi^* = -\frac{x_P}{\sqrt{x_P^2 + y_I^2}}
\tag{15}
$$

$$
\sin\phi^* = \frac{y_I - y_E}{\sqrt{(y_I - y_E)^2 + x_E^2}}, \qquad \cos\phi^* = -\frac{x_E}{\sqrt{(y_I - y_E)^2 + x_E^2}}
\tag{16}
$$

and the time-to-capture/value function is

$$
V(x_P, x_E, y_E) = \sqrt{x_P^2 + y_I^2}
\tag{17}
$$

where the function $y_I(x_P, x_E, y_E)$ is given by Eq. (13).

When the initial state $(x_{P_0}, x_{E_0}, y_{E_0}) \in R_{1,2}$ and P_1, P_2, and E play optimally, the closed-loop dynamics are

$$
\begin{aligned}
\dot{x}_P &= Gx_P, & x_P(0) &= x_{P_0}\\
\dot{x}_E &= Gx_E, & x_E(0) &= x_{E_0}\\
\dot{y}_E &= Gy_E, & y_E(0) &= y_{E_0}, \qquad 0 \le t
\end{aligned}
\tag{18}
$$

where

$$
G = -\frac{(1-\mu^2)}{\sqrt{(1-\mu^2)(x_P^2 - x_E^2) + (1+\mu^2)y_E^2 + 2y_E\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_E^2)}}}
$$

The solution of system (18) of strongly nonlinear differential equations is simply

$$
\begin{aligned}
x_P(t) &= \left(1 - \frac{t}{t_f}\right)x_{P_0},\\
x_E(t) &= \left(1 - \frac{t}{t_f}\right)x_{E_0},\\
y_E(t) &= \left(1 - \frac{t}{t_f}\right)y_{E_0}, \qquad 0 \le t \le t_f
\end{aligned}
\tag{19}
$$

where t_f is given by Eq. (12). Inserting Eq. (19) into Eqs. (14–16), we obtain the players' constant headings in both the (x, y) and (X, Y) frames. □

When the geometric method is applied and P_1 and P_2 play optimally, from Eq. (11), we deduce that in the (x, y) frame the headings of P_1 and P_2 are mirror images of each other: $\chi^* = \pi - \psi^*$. Therefore, the (x, y) frame does not rotate and the players' headings are also constant in the (inertial) (X, Y) frame of the realistic plane. Hence, in the realistic plane, the optimal trajectories are straight lines. Because, initially, the rotating (x, y) frame is aligned with the (X, Y) frame of the realistic plane, the y axis stays aligned with the Y axis, whereas the x axis stays parallel to the X axis moving in the upward direction at a constant speed. Therefore, the optimal trajectories are also straight lines in the (x, y) frame. Thus, when the state feedback strategies [Eq. (11)] synthesized using the geometric method are applied, the closed-loop system's optimal flowfield in the $R_{1,2}$ region of the reduced state space consists of the family of straight-line trajectories [Eq. (19)] that converge at the origin. Moreover, this flowfield, which was produced by the geometric method, covers the $R_{1,2}$ region of the reduced state space.

The following extensions are of interest. The cutters' speeds need not be equal. Furthermore, it is interesting to consider the case in which the speed of just one of the two cutters (say, P_1) is higher than the speed of the fugitive ship, whereas the speed of P_2 is equal to the speed of the fugitive ship. In this case, upon employing the geometric method, the Apollonius circle that is based on E and P_2 devolves into the orthogonal bisector of the segment $\overline{EP}_2$. It makes sense to also stipulate that the cutters P_1 and P_2 are endowed with circular capture sets with radii of $l_1 > 0$ and $l_2 > 0$, respectively. In this case, the elegant Apollonius circles will be replaced by Cartesian ovals and the boundary separating the R_1, R_2, and $R_{1,2}$ regions of the state space will not be planar and will be replaced by a more complex surface.

IV. Conclusions

In this Note, Isaacs's two cutters and a fugitive ship differential game has been revisited. The solution of the game of kind is provided; that is, the partition of the state space into regions in which, under optimal play, just one of the pursuers captures the evader and the state-space region in which both pursuers cooperatively capture the target has been characterized. The closed-form solution of the game of degree has been obtained using Isaacs's geometric method.

Acknowledgments

This Note is based on work performed at the Air Force Institute of Technology and the U.S. Air Force Research Laboratory's Control Science Center of Excellence. The views expressed in this Note are those of the authors and do not reflect the official policy or position of the U.S. Air Force, the U.S. Department of Defense, or the U.S. Government.

References

[1] Isaacs, R., *Differential Games: A Mathematical Theory with Applications to Optimization, Control and Warfare*, Wiley, New York, 1965, pp. 148–149.

[2] Steinhaus, H., and Kuhn, H. W., "Definitions for a Theory of Games and Pursuit," *Naval Research Logistics Quarterly*, Vol. 7, No. 2, 1960, pp. 105–108.
doi:10.1002/(ISSN)1931-9193

[3] Morgenstern, O., and Von Neumann, J., *Theory of Games and Economic Behavior*, Princeton Univ. Press, 1953, pp. 148–149.

[4] Breakwell, J. V., and Hagedorn, P., "Point Capture of Two Evaders in Succession," *Journal of Optimization Theory and Applications*, Vol. 27, No. 1, 1979, pp. 89–97.
doi:10.1007/BF00933327

[5] Ganebny, S. A., Kumkov, S. S., Le Ménec, S., and Patsko, V. S., "Model Problem in a Line with Two Pursuers and One Evader," *Dynamic Games and Applications*, Vol. 2, No. 2, 2012, pp. 228–257.
doi:10.1007/s13235-012-0041-z

[6] Makkapati, V. R., Sun, W., and Tsiotras, P., "Optimal Evading Strategies for Two-Pursuer/One-Evader Problems," *Journal of Guidance, Control, and Dynamics*, Vol. 41, No. 4, 2018, pp. 851–862.
doi:10.2514/1.G003070

[7] Garcia, E., Fuchs, Z. E., Milutinović, D., Casbeer, D. W., and Pachter, M., "A Geometric Approach for the Cooperative Two-Pursuer One-Evader Differential Game," *IFAC-PapersOnLine*, Vol. 50, No. 1, 2017, pp. 15,209–15,214.
doi:10.1016/j.ifacol.2017.08.2366

[8] Fuchs, Z. E., Garcia, E., and Casbeer, D. W., "Two-Pursuer, One-Evader Pursuit Evasion Differential Game," *2018 IEEE National Aerospace and Electronics Conference (NAECON)*, IEEE Publ., Piscataway, NJ, 2018, pp. 457–464.
doi:10.1109/NAECON.2018.8556827

[9] Von Moll, A., Casbeer, D. W., Garcia, E., and Milutinović, D., "Pursuit-Evasion of an Evader by Multiple Pursuers," *2018 International Conference on Unmanned Aircraft Systems (ICUAS)*, 2018, pp. 133–142.
doi:10.1109/ICUAS.2018.8453470

[10] Hagedorn, P., and Breakwell, J. V., "A Differential Game with Two Pursuers and One Evader," *Journal of Optimization Theory and Applications*, Vol. 18, No. 1, 1976, pp. 15–29.
doi:10.1007/BF00933791

[11] Ibragimov, G., and Hussin, N. A., "A Pursuit-Evasion Differential Game with Many Pursuers and One Evader," *Malaysian Journal of Mathematical Sciences*, Vol. 4, No. 2, 2010, pp. 183–194.

[12] Levchenkov, A., and Pashkov, A., "A Game of Optimal Pursuit of One Non-Inertial Object by Two Inertial Objects," *Journal of Applied Mathematics and Mechanics*, Vol. 49, No. 4, 1985, pp. 413–422.
doi:10.1016/0021-8928(85)90045-0

[13] Pashkov, A., and Terekhov, S., "On a Game of Optimal Pursuit of an Object by Two Others," *Journal of Applied Mathematics and Mechanics*, Vol. 47, No. 6, 1983, pp. 720–724.
doi:10.1016/0021-8928(83)90105-3

[14] Le Ménec, S., "*Linear Differential Game with Two Pursuers and One Evader*, " Advances in Dynamics Games, Birkhäuser, Boston, 2011, pp. 209–226, Chap. 11.
doi:10.1007/978-0-8176-8089-3_11

[15] Ibragimov, G. I., "A Game of Optimal Pursuit of One Object by Several," *Journal of Applied Mathematics and Mechanics*, Vol. 62, No. 2, 1998, pp. 187–192.
doi:10.1016/S0021-8928(98)00024-0

[16] Ibragimov, G., Ferrara, M., Kuchkarov, A., and Pansera, B. A., "Simple Motion Evasion Differential Game of Many Pursuers and Evaders with Integral Constraints," *Dynamic Games and Applications*, Vol. 8, No. 2, 2018, pp. 352–378.
doi:10.1007/s13235-017-0226-6

[17] Von Moll, A., Casbeer, D., Garcia, E., Milutinović, D., and Pachter, M., "The Multi-Pursuer Single-Evader Game: A Geometric Approach," *Journal of Intelligent and Robotic Systems*, Jan. 2019.
doi:10.1007/s10846-018-0963-9