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Toward a Solution of the Active Target Defense Differential Game

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Abstract A novel pursuit-evasion differential game involving three agents is considered. An Attacker missile is pursuing a Target aircraft. The Target aircraft is aided by a Defender missile launched by, say, the wingman, to intercept the Attacker before it reaches the Target aircraft. Thus, a team is formed by the Target and the Defender which cooperate to maximize the separation between the Target aircraft and the point where the Attacker missile is intercepted by the Defender missile, while at the same time the Attacker tries to minimize said distance. A long-range Beyond Visual Range engagement which is in line with current CONcepts of OPeration is envisaged, and it is therefore assumed that the players have simple motion kinematics á la Isaacs. Also, the speed of the Attacker is equal to the speed of the Defender and the latter is interested in point capture. It is also assumed that at all time the Attacker is aware of the Defender's position, i.e., it is a perfect information game. The analytic/closed-form solution of the target defense pursuit-evasion differential game delineates the state space region where the Attacker can reach the Target without being intercepted by the Defender, thus disposing of the Game of Kind. The target defense Game of Degree is played in the remaining state space. The analytic solution of the Game of Degree yields the agents' optimal state feedback strategies, that is, the instantaneous heading angles for the Target and the Defender team to maximize the terminal separation between Target and Attacker at the instant of

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interception of the Attacker by the Defender, and also the instantaneous optimal heading for the Attacker to minimize said separation. Their calculation hinges on the real-time solution of a quartic equation. In this paper we contribute to the solution of a differential game with three states—an additional example to the, admittedly small, repertoire of pursuit-evasion differential games in 3-D which can be solved in closed form.

Keywords Differential games · Missile guidance · Pursuit-evasion

1 Introduction

Multi-agent pursuit-evasion scenarios present important but challenging problems in aerospace guidance and control. In these types of problems one or more pursuers try to maneuver and reach a relatively small distance with respect to one or more evaders, which strive to escape the pursuers. This scenario should be considered in the context of dynamic games [10,16,24]. Thus, a dynamic Voronoi diagram has been used in scenarios with several pursuers in order to capture an evader within a bounded domain [1,16]. In contrast, [37] employed a receding-horizon formulation that provides evasive maneuvers for an Unmanned Autonomous Vehicle (UAV) assuming a known pursuer's input. In [8], a multi-agent scenario is considered where a number of pursuers are assigned to intercept a group of evaders and where the goals of the evaders are assumed to be known. Cooperation between two agents with the goal of evading a single pursuer has been addressed in [9]. Scott and Leonard [33] investigated a bioinspired scenario where two evaders coordinate their strategies to evade a single pursuer, but at the same time keep them close to each other.

The seminal work by Breakwell and Hagedorn [7] addressed the dynamic game of a fast pursuer trying to capture in minimal time two slower evaders in succession. Reference [19] extended this scenario and analyzed the case where the fast pursuer tries to capture several evaders. The slow evaders cooperate in order to maximize the total time from the beginning of the game until the last evader is captured. The obtained numerical solution shows that the optimal strategies of every agent, pursuer and evaders, consist of constant headings (the pursuer's heading is piecewise constant, and it changes at time instants when an evader is captured). The authors also discuss an iterative approach where the numerical solution is repeatedly computed in order for the evaders to increase the capture time when the pursuer does not play optimally.

In this paper we consider an active target defense scenario which is modeled as a three-agent zero-sum pursuit-evasion differential game. A two-agents team is formed which consists of a Target (T) and a Defender (D) who cooperate; the Attacker (A) is the opposition. The goal of the Attacker is to come as close as possible to the Target, possibly capturing the Target, while the Target cooperates with the Defender which tries to intercept the Attacker before the latter captures the Target, and as far away from the Target as possible. A long-range Beyond Visual Range (BVR) engagement which is in line with current CONcepts of OPeration (CONOPs) is envisaged, and it is therefore assumed that the agents/air vehicles have simple motion/holonomic kinematics á la Isaacs [17]. It is also assumed that the Attacker is aware of the Defender's position so that full state information is available and the differential game is a perfect information game.

The part of the state space where the Attacker can intercept the Target while evading the Defender is delineated, yielding the solution to the Game of Kind. Outside this part of the state space, the active target defense Game of Degree is addressed: The Target and

Defender team work to maximize the Target–Attacker separation at the time instant where the Defender intercepts the Attacker, whereas the latter strives to minimize said distance. This paper is dedicated to advancing the analytic/closed-form solution of the Active Target Defense Differential Game (ATDDG), which yields the players' state feedback optimal strategies, that is, the air vehicles' instantaneous optimal heading angles. Naturally, this is a state feedback strategy which is robust to unknown Attacker guidance laws, that is, when the Attacker pursues the Target using a conventional but non-optimal guidance law, such as Pure Pursuit (PP) or Proportional Navigation (PN), the Defender will intercept the Attacker further away from the Target than guaranteed by the value of the zero-sum differential game. In zero-sum games the players' saddle point strategies are security strategies.

The concept of active target defense was originally introduced in the context of cooperative *optimal control* [5,6], where the guidance laws of the A or D agents are *known*; see also [36,40]. Recent work has investigated different guidance laws for the agents A and D: In [27] the case is addressed where the Defender implements Command to Line-of-Sight (CLOS) guidance to pursue the Attacker, which requires the Defender to have at least the same speed as the Attacker. Rubinsky and Gutman [29,30] presented an analysis of the end-game Target–Attacker–Defender (TAD) scenario based on the Attacker/Target miss distance for *a non-cooperative* Target and Defender duo. The authors develop linearization-based Attacker maneuvers in order to evade the Defender and continue pursuing the Target. A different guidance law for the TAD scenario was given by Yamasaki et al. [38,39]. These authors investigated a guidance method they called Triangle Guidance (TG), where the objective is to command the defending missile to be on the line-of-sight between the attacking missile and the aircraft for all time, while the aircraft follows some predetermined trajectory. These formulations of the target defense differential game constrain and limit the level of cooperation between the Target and the Defender by implementing Defender guidance laws without regard to the Target's trajectory.

Different types of cooperation for the TAD scenario have also been recently proposed in [23,26,28,31,32,34,35]. Thus, in [32] optimal controls (lateral acceleration) for each agent are provided for the case of an aggressive Defender, that is, it is assumed that the Defender has a maneuverability advantage. A Linear Quadratic (LQ) *optimal control* problem is then posed where the Defender's control effort weight is driven to zero to model its aggressiveness. The work [28] provided a game theoretical analysis of the TAD encounter using a specified set of guidance laws for both the Attacker and the Defender. The cooperative strategies in [35] allow for a maneuverability disadvantage for the Defender with respect to the Attacker. Shaferman and Shima [34] implemented a Multiple Model Adaptive Estimator (MMAE) to identify the guidance law and parameters of the incoming missile and optimize the Defender's strategy to minimize its control effort. The recent paper [26] analyzes different types of cooperation assuming the Attacker is oblivious of the Defender and its guidance law is known. Two different one-way cooperation strategies are discussed: When the Defender acts independently, the Target knows its future behavior and cooperates with the Defender, and vice versa. Two-way cooperation where both Target and Defender communicate continuously to exchange their states and controls information is also addressed, and it is shown to have better performance than the other modes of cooperation—as expected. In this vein, our previous optimal control work [14,15] considered the case where the Attacker implements PP or PN guidance laws. In these papers, the Target–Defender team cooperatively solves an *optimal control* problem that returns the optimal control laws for the $T-D$ team so that the Defender intercepts the Attacker and at the same time the separation between Target and Attacker at the instant of interception of the Attacker by the Defender is maximized.

In this paper, the Active Target Defense scenario is naturally formulated as a zero-sum differential game with perfect (state) information. In this differential game an "intelligent" Attacker, which is aware of the Defender's position, strives to minimize the final separation between the pursued Target and itself at the instant of its interception by the Defender, while at the same time the Target–Defender team works to maximize the said distance. Having assumed that the Attacker's situational awareness is such that he also knows the instantaneous position of the Defender, this strategy provides better performance for the Attacker than him using the conventional PP or PN guidance laws. For a cooperative Target/Defender team, we showed in [11] that if the Attacker wishes to minimize the final separation between the Target's terminal position and the point where the Attacker is intercepted by the Defender, then the standard guidance laws do *not* achieve this goal. A comparison between the pursuit strategies PP, PN, and our solution of the differential game obtained in this paper shows that the PP and PN guidance laws yield a terminal separation that is greater than that provided by the solution of the Active Target Defense Differential Game (ATDDG), where the Attacker plays optimally.

We assume that the Attacker and Defender have the same speed, while the Target is slower than the Attacker and Defender. The Defender is interested in point capture of the Attacker. All three agents have simple motion á la Isaacs. Preliminary results concerning our ATDDG formulation were presented in [21]. The main contribution of the present paper is the analytic, closed-form, solution of the ATDDG. When the Target is initially closer to the Defender than to the Attacker and provided the Target and Defender play optimally, the Attacker will be intercepted by the Defender before reaching the escaping Target. In the case where initially the Target is closer to the Attacker than to the Defender we provide the solution to the Game of Kind: For a given Target–Attacker speed ratio $0 \leq \alpha < 1$, the state space region is delineated where the Attacker can reach the Target without being intercepted by the Defender, in which case the attacker wins. This disposes of the Game of Kind. The ATDDG is played outside this region of the state space: The Target and Defender team maneuver so that the Defender is able to intercept the Attacker before the latter reaches the Target. In the part of the state space where the Game of Degree is played out and which is fully covered by regular optimal trajectories—no singular surfaces here—the solution of the ATDDG hinges on the real-time solution of a quartic equation which is parameterized by the Target–Attacker speed ratio α.

The complete analysis of the novel ATDDG undertaken in this paper provides the air vehicles' optimal state feedback strategies, that is, their instantaneous headings, and the differential game's value function, which yields the Target–Attacker miss distance under optimal play as a function of the current positions of the agents. The value of the zero-sum ATDDG yields the Target–Defender team's guarantee of minimal final Attacker–Target separation, provided the Target and Defender played optimally; should the Attacker not play optimally, the terminal Attacker–Target separation will increase. At the same time, the value of the game is also the Attacker's guarantee of maximal Attacker–Target separation at the moment of its interception by the Defender, provided he played optimally; should the Target/Defender team not play optimally, the final Attacker–Target separation will decrease, and might even be reduced to zero, that is, the Target might be captured by the Attacker. From an operational point of view and from the Attacker's perspective, since the zero-sum game's value function specifies the minimal Attacker–Target separation at the instant of the Attacker's interception by the Defender, in principle, the Attacker could quantitatively, and ahead of time, assess the possible damage, if any, to be inflicted on the Target and decide whether to engage the Target. The same applies to the Target–Defender team, and in the event that the calculated maximal terminal Attacker–Target separation is too small, the Target might opt to also deploy passive defenses e.g., flares or chaff.

The paper is organized as follows. Following a brief discussion in Sect. 2 of Isaacs' method, the active target defense scenario with three agents, that is, Target, Attacker, and Defender, is described in Sect. 3. In Sect. 4 the active target defense pursuit-evasion differential game is formulated and solved. The interesting instance where the Target finds itself at equal distance from the Attacker and the Defender is analyzed in Sect. 5, concluding the analysis of the Game of Degree. The Game of Kind is addressed in Sect. 6, where the state space region where the Target cannot escape is delineated; the ATDDG is played in the remainder of the state space. The optimal flow field in the region of the state space where the ATDDG Game of Degree is played is provided. Illustrative examples are included in Sect. 7.1. A discussion of the novel ATDDG in the wider context of pursuit-evasion differential game theory is included in Sect. 8. Concluding remarks are made in Sect. 9, and detailed analysis of the crucial quartic equation is provided in "Appendix."

This paper is a contribution to the, admittedly small, repertoire of pursuit-evasion differential games which can be solved in closed form.

2 Isaacs' Method

In optimal control one is after the open-loop optimal control. The Pontryagin maximum principle (PMP) is used to calculate the optimal control time history; one refers to open-loop optimal control. However, the state feedback optimal control law can also be obtained, as discussed below. The concise statement of the PMP is included herein to make the exposition self-contained and, most importantly, to emphasize the connection of the two-sided PMP to Dynamic Programming and thus, Isaacs' method of solution of two-sided optimal control problems, a.k.a., differential games. In this respect, see Refs. [2] and [3] Chapter 8, Theorem 8.1.

In optimal control the dynamics are

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad 0 \le t \le t_f$$

Autonomous dynamics are considered where the terminal time t_f is free, the state $x, x_0 \in \mathbb{R}^n$, the control $u(t) \in \Omega \subset \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The cost functional is

$$J(u(\cdot); x_0) = \min_{u(t)\in\Omega} \int_0^{t_f} L(x, u)\mathrm{d}t$$

where the integrant in the cost functional

$$L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^1$$

The Hamiltonian is

$$H(x, u, \lambda) = -L(x, u) + \lambda^T f(x, u)$$

where the costate $\lambda \in \mathbb{R}^n$ and its dynamics are

$$\dot{\lambda} = -\frac{\partial H(x, u, \lambda)}{\partial x} \tag{1}$$

The Hamiltonian is pointwise maximized:

$$\max_{u \in \Omega} H(x, u, \lambda)$$

which yields

$$u^*(x, \lambda) = \arg\max_{u\in\Omega} H(x, u, \lambda)$$

Let $g : \mathbb{R}^n \to \mathbb{R}^1$, whereupon the terminal condition is specified by

$$g(x)\Big|_{t_f} = 0$$

and the terminal time is determined from the condition

$$H(x, u, \lambda)\Big|_{t_f} = 0$$

In fact

$$H(x, u^*(x, \lambda), \lambda) \equiv 0 \;\; \forall \;\; 0 \le t \le t_f$$

The transversality condition is

$$\lambda(t_f) = a \cdot \mathbf{n}(x(t_f)), \quad a > 0$$

where $\mathbf{n}$ is the normal to the terminal manifold $g(x) = 0$ and it points inward.

The PMP as stated is a necessary condition for optimality—it provides optimal trajectory candidates. It was initially conceived for obtaining a single optimal trajectory that emanates from a specified initial state x_0 and meets a terminal manifold, and it also provides the attendant open-loop optimal control time history.

Consider next the method of Dynamic Programming (DP) using the Hamiltonian formulation. Now the Hamiltonian

$$H(x, u, V_x) \equiv L(x, u) + V_x \cdot f(x, u)$$

where $V(x)$ is the value function, which is the optimal cost-to-go when starting out at the initial state $x \in \mathbb{R}^n$. Using the Hamiltonian formulation, the DP equation for autonomous control systems and a free terminal time t_f is

$$\min_u \; H(x, u, V_x) \equiv 0, \quad 0 \le t \le t_f,$$

that is

$$H(x, u^*(x, V_x), V_x) \equiv 0, \quad 0 \le t \le t_f \tag{2}$$

where

$$u^*(x, V_x) = \arg\min_u \; H(x, u, V_x)$$

Equation (2) is a hyperbolic PDE, and its solution by the method of characteristics requires the solution of the adjoint ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t} V_x = -\frac{\partial H}{\partial x}, \quad t_f \ge t \ge 0$$

The terminal condition for the adjoint vector $V_x \in \mathbb{R}^n$ is established as follows: At time $t = t_f$, where $x = x(t_f)$,

$$V(x) \equiv 0 \;\; \forall \, x \; s.t. \; g(x) = 0$$

Let the terminal manifold $g(x)$ be parameterized as follows:

$$x = \zeta(s), \;\; s \in \mathbb{R}^{n-1}$$

and since on the Usable Part (UP) of the terminal manifold the value function is 0, we have at t_f

$$V_x^T \cdot \zeta_s \equiv 0$$

Since the vectors ζ_s define the tangent hyperplane to the terminal manifold at $x(t_f)$, we conclude the vector V_x is normal to the terminal manifold at $x(t_f)$: We have obtained the DP's transversality condition at t_f

$$V_x(t_f) = a \cdot \mathbf{n},$$

where $a \in \mathbb{R}^1_+$ and $\mathbf{n}$ is an outward pointing vector which is normal to the terminal manifold.

The DP provided solution is the optimal control solution, and moreover, DP provides the optimal state feedback control law.

Although the PMP is a necessary condition for optimality—it provides optimal trajectory candidates—by virtue of its connection to the method of Dynamic Programming, it becomes a sufficient condition for optimality if the following holds. It is required that for each initial state in the state space the PMP provides a unique optimal trajectory candidate. In addition, if the trajectories of the constructed optimal flow field(s) do not intersect/no caustics, then the global optimal flow field has been synthesized using regular optimal trajectories and the DP's value function is C^1; when this is not the case, that is, the state space is not fully covered and/or optimal flow fields intersect, singular surfaces on some of which the value function might not be differentiable must be introduced, courtesy of the ingenuity of the analyst; think of shockwaves in supersonic flow where the governing equation is also a hyperbolic PDE. When the global optimal flow field has been successfully synthesized, the optimal control problem/differential game has been solved. This is so because the optimal flow field which covers the whole state space renders the solution of the Hamilton–Jacobi–Isaacs (HJI) hyperbolic PDE of DP—the value function: The global solution of the optimal control problem + the optimal state feedback control law have been obtained by the method of DP. The Euler–Lagrange equations of the PMP/Isaacs' method, used in the construction of the optimal flow field, are the equations of the characteristics of the HJI PDE—The method of characteristics has been used to solve the HJI hyperbolic PDE of DP, and solving the PDE of DP in the whole state space is a sufficient condition for optimality. Using the method of characteristics to solve the PDE of DP yields optimality, but this is precisely Isaacs' method for solving differential games with perfect information, a.k.a., two-sided optimal control problems, and, of course, also optimal control problems. At this point also note that the PMP costate λ and the partial derivatives V_x of the value function are related as follows: $\lambda = -V_x$; and thus, maximization of the Hamiltonian in the PMP is replaced by minimization in the DP. Indeed, the PMP—DP connection was already realized in [25] and [4], and is further evidenced in [2].

As far as two-sided optimal control problems/differential games are concerned, where the minimizing player's control is u and the maximizing player's control is v and the solution of the Game of Degree is undertaken: In a differential game setting the Hamiltonian is

$$H(x, u, v, V_x) \equiv L(x, u, v) + V_x^T \cdot f(x, u, v)$$

and, provided the Hamiltonian has a pointwise saddle point in the controls u and v, Eq. (2) is replaced by

$$H(x, u^*(x, V_x), v^*(x, V_x), V_x) \equiv 0, \quad 0 \le t \le t_f \tag{3}$$

where

$$u^*(x, V_x) = \arg\min_u \; H(x, u, v, V_x)$$

and

$$v^*(x, V_x) = \arg\max_v \; H(x, u, v, V_x)$$

Equation (3) is Isaacs' Main Equation 2 and the costate vector V_x satisfies differential system (2): If the construction of the optimal flow field is successful and the barrier surface—delimited part of the state space where the Game of Degree is played is fully covered, the differential game has been solved [17].

3 The Game

The Target (T), the Attacker (A), and the Defender (D) have "simple motion" à la Isaacs, are holonomic. Thus, the controls of T, A, and D are their respective headings θ_T, θ_A, and θ_D. In addition, T, A and D have constant speeds of V_T, V_A, and V_D, respectively. The positions of the Target, the Attacker, and the Defender are defined by their Cartesian coordinates $\tilde{\mathbf{x}}_T = (\tilde{x}_T, \tilde{y}_T)$, $\tilde{\mathbf{x}}_A = (\tilde{x}_A, \tilde{y}_A)$, and $\tilde{\mathbf{x}}_D = (\tilde{x}_D, \tilde{y}_D)$, respectively.[1] We assume that the Attacker and Defender missiles have similar capabilities, so $V_A = V_D$, while the Target–Attacker speed ratio $\alpha = V_T/V_A < 1$. We consider, without loss of generality, the normalized (by V_A) active target defense scenario. The complete state of the game is defined by $\tilde{\mathbf{x}} := (\tilde{x}_T, \tilde{y}_T, \tilde{x}_A, \tilde{y}_A, \tilde{x}_D, \tilde{y}_D) \in \mathbb{R}^6$. The Attacker's control variable is his instantaneous heading angle, $\mathbf{u}_A = \{\theta_A\}$. The $T - D$ team affects the state of the game by cooperatively choosing the instantaneous respective headings, θ_T and θ_D, of both the Target and the Defender, so the $T - D$ team's control variable is $\mathbf{u}_B = \{\theta_T, \theta_D\}$, where $\theta_A, \theta_T, \theta_D \in [-\pi, \pi]$. The dynamics in the realistic plane $\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}}, \mathbf{u}_A, \mathbf{u}_B)$ are specified by the system of ordinary differential equations

$$\begin{aligned}
\dot{\tilde{x}}_T &= \alpha \cos\theta_T \\
\dot{\tilde{y}}_T &= \alpha \sin\theta_T \\
\dot{\tilde{x}}_A &= \cos\theta_A \\
\dot{\tilde{y}}_A &= \sin\theta_A \\
\dot{\tilde{x}}_D &= \cos\theta_D \\
\dot{\tilde{y}}_D &= \sin\theta_D
\end{aligned} \tag{4}$$

where the speed ratio $\alpha = V_T/V_A < 1$ is the problem parameter. The initial state of the system is

$$\tilde{\mathbf{x}}_0 := (\tilde{x}_{T_0}, \tilde{y}_{T_0}, \tilde{x}_{A_0}, \tilde{y}_{A_0}, \tilde{x}_{D_0}, \tilde{y}_{D_0}) = \tilde{\mathbf{x}}(t_0)$$

Let us specify the order of preference in the ATDDG which will help in defining the admissible strategies of each player. Let m_t and n_t be numerical outcomes of terminating plays in the ATDDG, where $m_t < n_t$. Let $P = A$, that is, the Attacker is the Pursuer; it pursues the Target. Also, let the Evader E be the T & D team, that is, the Target, aided by the Defender, evades the Attacker.

Denote by Υ the outcome of any non-terminating play. A Type P order of preference [18] is such that the following holds

$$m_t \overset{P}{\succ} n_t \overset{P}{\succ} \Upsilon \quad \text{and} \quad \Upsilon \overset{E}{\succ} n_t \overset{E}{\succ} m_t.$$

The ATDDG has an order of preference of Type P. This means that player P, the Attacker, who aims at minimizing the numerical outcome of the game, considers non-termination to be

[1] Here, we have used $\tilde{x}$ to denote the state in the realistic plane. We reserve the use of x (without tilde) to denote the state in the reduced state space.

inferior to all other outcomes. Player E, the Target/Defender team, who aims at maximizing the numerical outcome of the game, considers non-termination to be superior to all other outcomes.

In this paper, we confine our attention to point capture, that is, the $D - A$ separation has to become zero in order for the Defender to intercept the Attacker. Hence, the ATDDG terminates when the state of the system satisfies the terminal condition

$$\tilde{x}_A = \tilde{x}_D, \quad \tilde{y}_A = \tilde{y}_D. \tag{5}$$

The terminal time t_f is the time instant when the state of the system satisfies (5), at which time the terminal state is $\tilde{\mathbf{x}}_f := (\tilde{x}_{T_f}, \tilde{y}_{T_f}, \tilde{x}_{A_f}, \tilde{y}_{A_f}, \tilde{x}_{D_f}, \tilde{y}_{D_f}) = \tilde{\mathbf{x}}(t_f)$. The cost/payoff functional is

$$\Phi(\tilde{\mathbf{x}}_f) := \frac{1}{2}[(\tilde{x}_{A_f} - \tilde{x}_{T_f})^2 + (\tilde{y}_{A_f} - \tilde{y}_{T_f})^2]. \tag{6}$$

The cost/payoff functional depends only on the terminal state

$$J(\mathbf{u}_A(t), \mathbf{u}_B(t), \tilde{\mathbf{x}}_0) = \Phi(\tilde{\mathbf{x}}_f). \tag{7}$$

We have a Mayer type zero-sum differential game. Its Value

$$V(\tilde{\mathbf{x}}_0) = \min_{\mathbf{u}_A(t)} \max_{\mathbf{u}_B(t)} J(\mathbf{u}_A(t), \mathbf{u}_B(t), \tilde{\mathbf{x}}_0). \tag{8}$$

subject to (4) and (5).

In the Active Target Defense Differential Game A strives to close in on T, while T and D form a team to defend from A: T and D maneuver such that D intercepts A before the latter reaches T and the A—T distance at interception time is maximized, while A strives to minimize the separation between A and T at the instant of interception.

In this paper, the Game of Degree is addressed first. In Sects. 4 and 5, we synthesize the players' optimal closed-loop strategies using Isaacs' method. The main contribution of the paper is the derivation of these closed-loop state feedback strategies that can be implemented in real-time upon solving a quartic equation. The $T - D$ team employs a cooperative optimal state feedback strategy $\theta_T^* = \theta_T^*(\tilde{\mathbf{x}})$ and $\theta_D^* = \theta_D^*(\tilde{\mathbf{x}})$ to maximize the separation between the Target and the Attacker at the time instant of the Defender-Attacker contact, while the Attacker devises his optimal state feedback strategy, that is, his instantaneous heading $\theta_A^* = \theta_A^*(\tilde{\mathbf{x}})$, to minimize the terminal $A - T$ separation/miss distance. The active target defense engagement in the realistic plane is illustrated in Fig. 1. We consider initial states such that the Defender is capable of intercepting the Attacker and the Target escapes. This state space region, R_e,

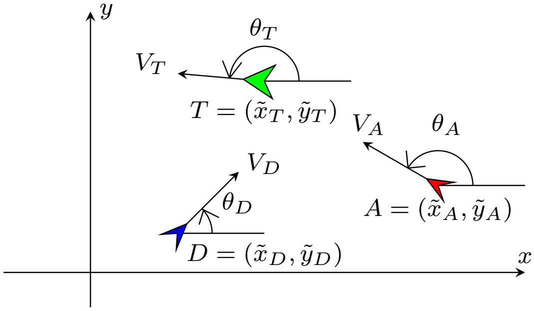


Fig. 1 The game in the realistic plane

where the Game of Degree is played out, is delineated in Sect. 6 where the Game of Kind is solved.

Concerning the Game of Degree in the state space region R_c where the Defender cannot help the Target by intercepting the Attacker before the latter is captured. When the initial state is in the state space region designated by the solution of the Game of Kind where the Target can be captured by the Attacker before the Defender can intervene, there a different Game of Degree is played, where the cost functional is given by

$$J_c(\mathbf{u}_A(t), \mathbf{u}_B(t), \tilde{\mathbf{x}}_0) := \frac{1}{2}[(\tilde{x}_{D_f} - \tilde{x}_{T_f})^2 + (\tilde{y}_{D_f} - \tilde{y}_{T_f})^2]$$

subject to the terminal condition $\tilde{x}_{A_f} = \tilde{x}_{T_f}$ and $\tilde{y}_{A_f} = \tilde{y}_{T_f}$, which signifies capture of the Target by the Attacker. In the Game of Degree in R_c the performance functional is maximized by the Attacker and minimized by the Target–Defender team. The Target–Attacker–Defender scenario belongs to the class of two termination set differential games which gives rise to two Games of Degree, one in region R_e and one in region R_c. We confine our attention to the Game of Kind and to the ATDDG, the Game of Degree in R_e. The pursuit-evasion game in R_c is discussed in reference [13].

4 Solution of the ATDDG Using Isaacs' Method

Isaacs' method, also known as, the "Two-sided" Pontryagin Maximum Principle (PMP) will be applied to the ATDDG Game of Degree; the method is also referred to as the "Two-person" extension of the PMP [3], Ch. 8, Theorem 8.1.

Admittedly, the PMP oftentimes, unjustifiably, is exclusively associated with necessary conditions for optimality and open-loop optimal control—this is why the "Two-sided" PMP moniker is in quotation marks. But, at the same time, by associating the co-states with the negative of the gradient of the value function, it is also the tool used for the synthesis of the closed-loop, state feedback, optimal controls/strategies in differential games—in this paper the complete field of regular optimal trajectories of the ATDDG is constructed using Isaacs' method. This, as first undertaken for the special case of optimal control in the pioneering work of Pontryagin [25], and further developed by Boltyanskii in reference [4]. This is also a viable and mathematically correct approach in differential games for the following reasons: The solution of zero-sum differential games with perfect information entails Dynamic Programming (DP). Only through DP can the solution of a differential game with perfect (state) information be obtained. This then requires the solution of the Hamilton–Jacobi–Bellman–Isaacs (HJBI) hyperbolic PDE for the value function. The HJBI equation is a first-order PDE of hyperbolic type, whence the possibility of shock waves (barrier surfaces), dispersal surfaces, switch envelopes, focal surfaces, and equivocal surfaces in differential games. Now, this PDE is solved by the method of characteristics. The characteristics are a system of ordinary differential equations which consist of the original dynamics equations and equations which propagate the partial derivatives of the value function—the latter being the negative of the co-states in the "two-sided" PMP in differential games. Indeed, as far as the theory is concerned, both the HJB PDE of optimal control and the HJBI PDE of differential games are PDEs of hyperbolic type. Applying the "Two-sided" PMP plus solving the state and co-state differential equations in a retrograde manner is equivalent to solving the PDE by the method of characteristics, which thus yields the solution of the HJBI PDE, and, in turn, the solution of the differential game: The solution of the ATDDG constructed herein consists of *primary* optimal trajectories only—no secondary and tributary optimal trajectories here. The optimal trajectories/characteristics do not intersect—have simple characteristics only. There are no holes left in the state space that need to be covered with optimal trajectories necessitating the introduction of singular surfaces. The full state space part where the ATDDG Game of Degree is solved, R_e, is covered by the *primary* optimal trajectories. Hence, no need for singular surfaces, e.g., equivocal surfaces do not arise. In other words, the value function is C^1—there are no corners. That is why the "two-sided" PMP is applicable here and it is equivalent to Isaacs' method [17]. So, our approach yields the solution of the Active Target Defense Differential Game (ATDDG) in the state space region R_e where T, aided by D, is able to escape. The derived strategies will satisfy, by construction, the necessary and sufficient conditions for optimality. The objective is to obtain, á la Isaacs, the players' optimal strategies.

The Target–Attacker–Defender engagement was previously addressed in [11] where the reduced 3-D state space R, r, and θ was used. The states R and r represent the separation between the Attacker and the Target and the separation between the Attacker and the Defender, respectively. The state θ represents the angle enclosed between these two radials. It is however convenient to consider the ATDDG in the realistic plane. The rationale behind this alternative formulation is that in the realistic plane the state and co-state dynamics are linear and the dynamic matrix is 0; this, because the players have simple motion. In addition, a Mayer type problem is considered. Consequently, the co-state variables are constant and therefore in the realistic state space the optimal trajectories are straight lines. Thus, the analysis of the ATDDG performed in the realistic plane and the attendant 6-dimensional state space is conducive to tractable mathematics, whereas the analysis of the ATDDG performed in the reduced 3-dimensional state space entails nonlinear dynamics, which complicates the solution. When the realistic plane is used the optimal state feedback strategies are readily obtained in closed form.

In this paper we solve the ATDDG using Isaacs' method, but we work in the original/realistic state space $\tilde{\mathbf{x}} = (\tilde{x}_A, \tilde{y}_A, \tilde{x}_D, \tilde{y}_D, \tilde{x}_T, \tilde{y}_T) \in R_e \subset \mathbb{R}^6$. Also, the co-state is $\tilde{\lambda} := (\tilde{\lambda}_{x_A}, \tilde{\lambda}_{y_A}, \tilde{\lambda}_{x_D}, \tilde{\lambda}_{y_D}, \tilde{\lambda}_{x_T}, \tilde{\lambda}_{y_T}) \in \mathbb{R}^6$. The dynamics are given by Eq. (4). Concerning the terminal condition, the cost/payoff function is evaluated when D intercepts A at time t_f.

Now, for certain initial conditions, under optimal play, A can capture T despite the best efforts of D. In Sect. 6.2, the solution to the Game of Kind is provided: It specifies the capture region R_c of the state space where A is guaranteed to be able to capture T. In the complement of R_c, in the escape region R_e, the ATDDG Game of Degree is played out, where D can intercept A before the latter reaches the fleeing Target. In this section, we address the Game of Degree. Note that if in the Game of Degree A should deviate from its optimal strategy then the final $A - T$ separation will increase, i.e., the payoff increases. Similarly, if either the Target or the Defender should deviate from their optimal strategy the $A - T$ separation will decrease and worse, the state might enter the Attacker's region of win, R_c.

Back to the Game of Degree, at the time instant where A is intercepted by D, the terminal time t_f is free and the terminal manifold $\mathcal{T}$ is the hyperplane in $\mathbb{R}^6$ defined by

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_A \\ \tilde{y}_A \\ \tilde{x}_D \\ \tilde{y}_D \\ \tilde{x}_T \\ \tilde{y}_T \end{bmatrix} = 0_{2\times 1}. \tag{9}$$

In order to obtain a compact representation of the closed-form solution of the ATDDG it is convenient to introduce the coordinate transformation

$$
\begin{aligned}
x_A &:= \frac{1}{2}\sqrt{(\tilde{x}_A-\tilde{x}_D)^2+(\tilde{y}_A-\tilde{y}_D)^2} \\
x_T &:= \frac{\left(\tilde{x}_T-\frac{1}{2}(\tilde{x}_A+\tilde{x}_D)\right)(\tilde{x}_A-\tilde{x}_D)+\left(\tilde{y}_T-\frac{1}{2}(\tilde{y}_A+\tilde{y}_D)\right)(\tilde{y}_A-\tilde{y}_D)}{\sqrt{(\tilde{x}_A-\tilde{x}_D)^2+(\tilde{y}_A-\tilde{y}_D)^2}} \\
y_T &:= \frac{\left(\tilde{y}_T-\frac{1}{2}(\tilde{y}_A+\tilde{y}_D)\right)(\tilde{x}_A-\tilde{x}_D)-\left(\tilde{x}_T-\frac{1}{2}(\tilde{x}_A+\tilde{x}_D)\right)(\tilde{y}_A-\tilde{y}_D)}{\sqrt{(\tilde{x}_A-\tilde{x}_D)^2+(\tilde{y}_A-\tilde{y}_D)^2}}.
\end{aligned}
\tag{10}
$$

Also, $x_D = -x_A$, $y_A = 0$, and $y_D = 0$. This is tantamount to using a rotating reference frame where the x-axis is the line through points A and D and the y-axis is the orthogonal bisector of the segment $\overline{AD}$—see Fig. 2. The reduced state is then $(x_A, x_T, y_T) \in \mathbb{R}^3$ and is obtained, using (10), in terms of the A, D, and T players' respective coordinates $(\tilde{x}_A, \tilde{y}_A)$, $(\tilde{x}_D, \tilde{y}_D)$, $(\tilde{x}_T, \tilde{y}_T)$ in the realistic plane.

The headings of T, A, and D in the reduced state space are denoted by ϕ, χ, and ψ, respectively, see Fig. 3; the players' headings in the realistic plane are given by $\theta_T = \phi + \omega$, $\theta_A = \chi + \omega$, and $\theta_D = \psi + \omega$, where $\tan\omega = \frac{\tilde{y}_A-\tilde{y}_D}{\tilde{x}_A-\tilde{x}_D}$.

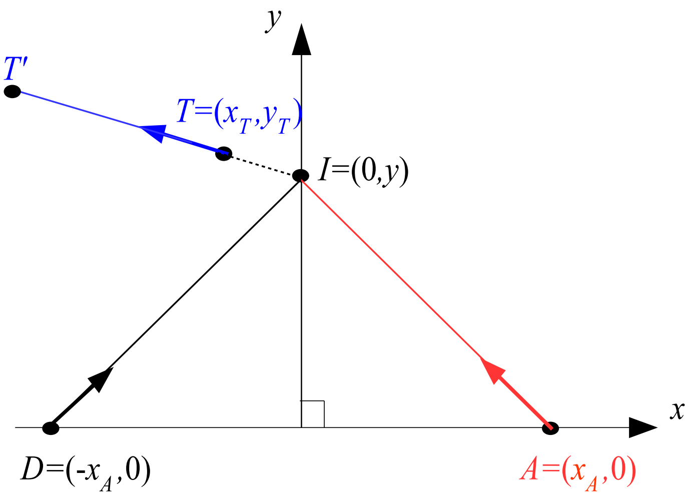


Fig. 2 Optimal play: $x_T < 0$

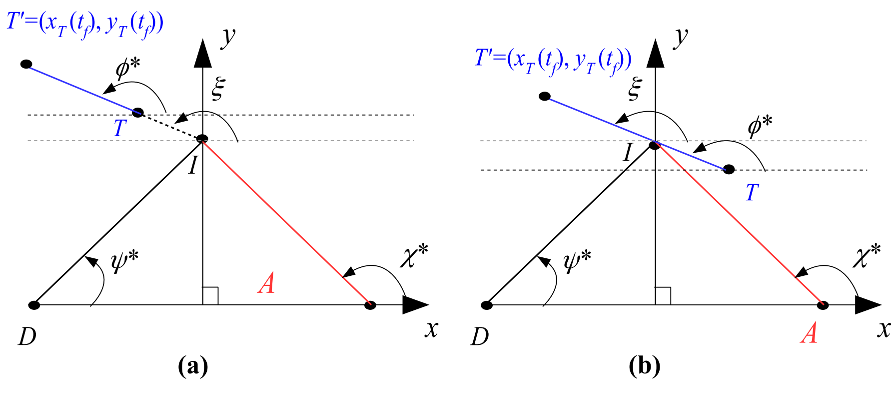


Fig. 3 The players' optimal headings

The reduced 3-D state space is the quadrant

$$\{(x_A, x_T, y_T) | x_A > 0, y_T \geq 0\} \subset \mathbb{R}^3 \tag{11}$$

and the Hamiltonian of the differential game in the reduced state space is

$$\mathcal{H} = \lambda_{x_A} \cos\chi + \lambda_{y_A} \sin\chi + \lambda_{x_D} \cos\psi + \lambda_{y_D} \sin\psi + \alpha\lambda_{x_T} \cos\phi + \alpha\lambda_{y_T} \sin\phi. \tag{12}$$

where $\lambda := (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}) \in \mathbb{R}^6$ is the corresponding co-state in the reduced state space. We note here that the Hamiltonian is separable in the controls ϕ, ψ and χ. We operate in the realistic state space, however in the following Theorem, the optimal state feedback strategies are given in compact form in terms of the reduced state (x_A, x_T, y_T) which, according to (10), are determined by the state $\tilde{\mathbf{x}}$ in the realistic plane.

The region of win R_c of the Attacker where the Target is captured by A despite D's best effort is characterized in Sect. 6.2. The solution of the Game of Degree in the remaining state space (the Target escape region, R_e) is given in the following Theorem. The proof of the Theorem follows the method for constructing regular solutions of differential games using the "Two-sided" or "Two-person" PMP to synthesize feedback strategies [3,17], [18]. In the process, the value function V of the ATDDG is obtained and then, it is shown that V and $\frac{\partial V}{\partial \mathbf{x}}$ are continuous in the region of the state space, region R_e, where the Target is guaranteed to escape.

Theorem 1 *Consider Active Target Defense Differential Game* (4), (6)–(9)*. The problem parameter is the speed ratio $0 \leq \alpha < 1$ and the reduced state space, which is determined by the instantaneous positions of A and D through transformation* (10)*, is (x_A, x_T, y_T) where, without loss of generality, $x_A > 0$ and $y_T \geq 0$. In the part of the state space where the Target can escape, the optimal headings of the Attacker, the Target, and the Defender are given by the state feedback control laws*

$$\begin{aligned}
&\cos\phi^* = \pm\frac{x_T}{\sqrt{x_T^2 + (y_T - y)^2}}, \quad \sin\phi^* = \pm\frac{y_T - y}{\sqrt{x_T^2 + (y_T - y)^2}} \quad \text{for } x_T \neq 0.\\
&\cos\chi^* = -\frac{x_A}{\sqrt{x_A^2 + y^2}}, \quad \sin\chi^* = \frac{y}{\sqrt{x_A^2 + y^2}}.\\
&\cos\psi^* = \frac{x_A}{\sqrt{x_A^2 + y^2}}, \quad \sin\psi^* = \frac{y}{\sqrt{x_A^2 + y^2}}.
\end{aligned} \tag{13}$$

where y is a real solution of the quartic equation

$$(1 - \alpha^2)y^4 - 2(1 - \alpha^2)y_T y^3 + \left[(1-\alpha^2)y_T^2 + x_A^2 - \alpha^2 x_T^2\right]y^2 - 2x_A^2 y_T y + x_A^2 y_T^2 = 0 \tag{14}$$

which is parameterized by the speed ratio $0 \leq \alpha < 1$. The quartic equation has two real solutions y_1 and y_2 and $y_1 \leq y_T \leq y_2$. In Eq. (13) *when $x_T \leq 0$ the solution y_1 is selected and when $x_T > 0$ the solution y_2 is used. Also, when $x_T < 0$ the sign in the Target heading in Eq.* (13) *is positive and when $x_T > 0$ the sign in the Target heading is negative. And when $x_T = 0$ the Target's optimal heading, ϕ^*, is given by*

$$\phi^* = \varphi^* + \pi/2 \tag{15}$$

where

$$\tan\varphi^* = \frac{\sqrt{x_A^2 + (1 - \alpha^2)y_T^2}}{\alpha y_T}. \tag{16}$$

The value function is explicitly given by

$$V(x_A, x_T, y_T) = \begin{cases} \alpha\sqrt{x_A^2 + y^{*2}} + \sqrt{(y_T - y^*)^2 + x_T^2} \;\; \forall\, x_T \le 0, \quad \textit{where } y^* \le y_T \\ \alpha\sqrt{x_A^2 + y^{*2}} - \sqrt{(y_T - y^*)^2 + x_T^2} \;\; \forall\, x_T > 0, \quad \textit{where } y^* > y_T \end{cases} \tag{17}$$

and $y^* \in \{y_1(x_A, x_T, y_T), y_2(x_A, x_T, y_T)\}$ is the relevant real solution of quartic equation (14) which is parameterized by x_A, x_T, y_T and the speed ratio α. Under optimal play, the agents' trajectories are straight lines. The value function is C^1, including on the plane $x_T = 0$.

Proof The optimal control inputs in terms of the co-state variables are obtained from Isaacs' Main Equation 1

$$\min_{\phi,\psi} \max_{\chi} \mathcal{H} = 0 \tag{18}$$

and they are characterized by

$$\cos\chi^* = \frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}}, \qquad \sin\chi^* = \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} \tag{19}$$

$$\cos\psi^* = -\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}}, \qquad \sin\psi^* = -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} \tag{20}$$

$$\cos\phi^* = -\frac{\lambda_{x_T}}{\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2}}, \qquad \sin\phi^* = -\frac{\lambda_{y_T}}{\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2}}. \tag{21}$$

Additionally, the co-state dynamics are

$$\dot{\lambda}_{x_A} = \dot{\lambda}_{y_A} = \dot{\lambda}_{x_D} = \dot{\lambda}_{y_D} = \dot{\lambda}_{x_T} = \dot{\lambda}_{y_T} = 0.$$

Hence, all co-states are constant and we have that $\chi^* \equiv$ constant, $\psi^* \equiv$ constant, and $\phi^* \equiv$ constant. In other words, the regular optimal trajectories are straight lines.

Concerning the solution of the attendant Two-Point Boundary Value Problem (TPBVP) on $0 \le t \le t_f$ in $\mathbb{R}^{12}$, we have 6 initial states specified by (4) and we need 6 more conditions for the terminal time t_f. In this respect, define the augmented Mayer terminal value function $\Phi_a : \mathbb{R}^6 \to \mathbb{R}^1$

$$\begin{aligned} \Phi_a(\mathbf{x}_f) := &\frac{1}{2}[(x_A(t_f) - x_T(t_f))^2 + (y_A(t_f) - y_T(t_f))^2] \\ &+ \nu_1(x_A(t_f) - x_D(t_f)) + \nu_2(y_A(t_f) - y_D(t_f)) \end{aligned} \tag{22}$$

where ν_1 and ν_2 are Lagrange multipliers. The PMP or Dynamic Programming yields the transversality/terminal conditions

$$\lambda(t_f) = -\frac{\partial}{\partial \mathbf{x}} \Phi_a(\mathbf{x}_f) \tag{23}$$

that is,

$$\lambda_{x_A} = x_T(t_f) - x_A(t_f) - \nu_1 \tag{24}$$

$$\lambda_{y_A} = y_T(t_f) - y_A(t_f) - \nu_2 \tag{25}$$

$$\lambda_{x_D} = \nu_1 \tag{26}$$
$$\lambda_{y_D} = \nu_2 \tag{27}$$
$$\lambda_{x_T} = x_A(t_f) - x_T(t_f) \tag{28}$$
$$\lambda_{y_T} = y_A(t_f) - y_T(t_f) \tag{29}$$

At this point, we have that Eqs. (24)–(29) plus Eq. (5) yield 8 conditions. Since we need only 6 conditions we eliminate the introduced Lagrange multipliers ν_1 and ν_2 from Eqs. (24)–(27) and we obtain

$$\lambda_{x_A} + \lambda_{x_D} = x_T(t_f) - x_A(t_f) \tag{30}$$
$$\lambda_{y_A} + \lambda_{y_D} = y_T(t_f) - y_A(t_f) \tag{31}$$

Thus, we have 6 relationships for the terminal time t_f: Eqs. (5) and (28)–(31). Finally, the time t_f is specified by the PMP requirement that the Hamiltonian $\mathcal{H}(\mathbf{x}(t), \lambda(t), \chi, \psi, \phi)|_{t_f} \equiv 0$ which, for this problem, takes the form of Isaacs' Main Equation 2

$$\lambda_{x_A} \cos\chi^* + \lambda_{y_A} \sin\chi^* + \lambda_{x_D} \cos\psi^* + \lambda_{y_D} \sin\psi^* + \alpha\lambda_{x_T} \cos\phi^* + \alpha\lambda_{y_T} \sin\phi^* \equiv 0. \tag{32}$$

So, using Isaacs' method, outside the domain of the solution of the Game of Kind, we can solve the differential game. The existence of a solution is predicated on the initial state—see Sect. 6.2 where the solution of the Game of Kind is given.

Because the optimal trajectories of A, D, and T are straight lines and $V_D = V_A$ we have that

$$x_A(t_f) = 0 \tag{33}$$
$$x_D(t_f) = 0 \tag{34}$$
$$y_A(t_f) = y_D(t_f) \tag{35}$$

Let $y := y_A(t_f) = y_D(t_f)$. Also, let $x_A = x_A(t')$, $x_T = x_T(t')$, and $y_T = y_T(t')$ be the instantaneous positions at some time $t' < t_f$. Hence, from Eq. (4) we obtain the following

$$x_T(t_f) = x_T + \alpha \cdot (t_f - t') \cos\phi, \tag{36}$$
$$y_T(t_f) = y_T + \alpha \cdot (t_f - t') \sin\phi, \tag{37}$$
$$0 = x_A + (t_f - t') \cos\chi, \tag{38}$$
$$y = (t_f - t') \sin\chi, \tag{39}$$
$$0 = -x_A + (t_f - t') \cos\psi, \tag{40}$$
$$y = (t_f - t') \sin\psi, \tag{41}$$

In addition, Eqs. (28)–(31) can be written as follows

$$\lambda_{x_T} = -x_T(t_f) \tag{42}$$
$$\lambda_{y_T} = y - y_T(t_f) \tag{43}$$
$$\lambda_{x_A} + \lambda_{x_D} = x_T(t_f) \tag{44}$$
$$\lambda_{y_A} + \lambda_{y_D} = y_T(t_f) - y \tag{45}$$

Consider Eqs. (21), (42) and (43). We calculate the following

$$\cos\phi^* = \frac{x_T(t_f)}{\sqrt{x_T^2(t_f) + (y - y_T(t_f))^2}}, \quad \sin\phi^* = \frac{y_T(t_f) - y}{\sqrt{x_T^2(t_f) + (y - y_T(t_f))^2}}. \tag{46}$$

Hence, in light of Eqs. (36) and (37), we have that the optimal heading angle of T is

$$\phi^* = \xi \tag{47}$$

where the angle ξ is shown in Fig. 3. The three points, T, T', and y are collinear and the optimal geometry is as shown in Fig. 3.

From the $\triangle ADy$ in Fig. 3 we conclude that $t_f - t' = \sqrt{x_A^2 + y^2}$. Without loss of generality, assume that $t' = 0$, then

$$t_f = \sqrt{x_A^2 + y^2}. \tag{48}$$

Thus, using Isaacs' method, we are now able to reduce the solution of the zero-sum differential Game of Degree to the optimization of a cost/payoff function of one variable. From (36)–(41), (46), and (48) we can write the cost (the terminal distance between A and T) as $J(y; x_A, x_T, y_T) = \overline{TT'} \pm \overline{TI}$, where $\overline{TT'} = \alpha\overline{AI} = \alpha t_f$. Depending on which side of the orthogonal bisector of the segment AD the Target is located we have the two cases:

For case (a) where $x_T < 0$ solve $\min_y J_1$ where

$$J_1(y; x_A, x_T, y_T) = \alpha\sqrt{x_A^2 + y^2} + \sqrt{(y_T - y)^2 + x_T^2}. \tag{49}$$

For case (b) where $x_T > 0$ solve $\max_y J_2$ where

$$J_2(y; x_A, x_T, y_T) = \alpha\sqrt{x_A^2 + y^2} - \sqrt{(y - y_T)^2 + x_T^2}. \tag{50}$$

Also, from Eqs. (46) and (47) we obtain:

$$\text{(a)} \qquad \cos\phi^* = \frac{x_T}{\sqrt{x_T^2 + (y_T - y)^2}}, \quad \sin\phi^* = \frac{y_T - y}{\sqrt{x_T^2 + (y_T - y)^2}}. \tag{51}$$

$$\text{(b)} \qquad \cos\phi^* = -\frac{x_T}{\sqrt{x_T^2 + (y - y_T)^2}}, \quad \sin\phi^* = \frac{y - y_T}{\sqrt{x_T^2 + (y - y_T)^2}}. \tag{52}$$

The A and D optimal headings are

$$\cos\chi^* = -\frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad \sin\chi^* = \frac{y}{\sqrt{x_A^2 + y^2}}. \tag{53}$$

$$\cos\psi^* = \frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad \sin\psi^* = \frac{y}{\sqrt{x_A^2 + y^2}}. \tag{54}$$

Let us now use Eqs. (20) and (54) to write the following relationships

$$-\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} = \frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} = \frac{y}{\sqrt{x_A^2 + y^2}}. \tag{55}$$

Similarly, from Eqs. (19) and (53) we obtain

$$\frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} = -\frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} = \frac{y}{\sqrt{x_A^2 + y^2}}. \tag{56}$$

We have four Eqs. (30), (31), (55), and (56) in the four unknowns λ_{x_A}, λ_{y_A}, λ_{x_D}, and λ_{y_D}. The solution is

$$
\begin{aligned}
\lambda_{x_A} &= \frac{1}{2}\left[x_T(t_f) - x_A(t_f) - \frac{x_A}{y}\left(y_T(t_f) - y_A(t_f)\right)\right] \\
\lambda_{y_A} &= \frac{1}{2}\left[y_T(t_f) - y_A(t_f) - \frac{y}{x_A}\left(x_T(t_f) - x_A(t_f)\right)\right] \\
\lambda_{x_D} &= \frac{1}{2}\left[x_T(t_f) - x_A(t_f) + \frac{x_A}{y}\left(y_T(t_f) - y_A(t_f)\right)\right] \\
\lambda_{y_D} &= \frac{1}{2}\left[y_T(t_f) - y_A(t_f) + \frac{y}{x_A}\left(x_T(t_f) - x_A(t_f)\right)\right]
\end{aligned}
$$

By substituting $y_A(t_f) = y$ and $x_A(t_f) = 0$ we obtain

$$\lambda_{x_A} = \frac{1}{2}\left[x_T(t_f) - \frac{x_A}{y}\left(y_T(t_f) - y\right)\right] \tag{57}$$

$$\lambda_{y_A} = \frac{1}{2}\left[y_T(t_f) - y - \frac{y}{x_A} x_T(t_f)\right] \tag{58}$$

$$\lambda_{x_D} = \frac{1}{2}\left[x_T(t_f) + \frac{x_A}{y}\left(y_T(t_f) - y\right)\right] \tag{59}$$

$$\lambda_{y_D} = \frac{1}{2}\left[y_T(t_f) - y + \frac{y}{x_A} x_T(t_f)\right] \tag{60}$$

which together with Eqs. (42) and (43) specify the co-states in terms of the states at time t_f. Now, Eqs. (36), (37), (48), and (51) yield

$$
\text{(a)} \quad \begin{cases}
x_T(t_f) = \left[1 + \alpha \dfrac{\sqrt{x_A^2+y^2}}{\sqrt{x_T^2+(y_T-y)^2}}\right] x_T \\[2ex]
y_T(t_f) = \left[1 + \alpha \dfrac{\sqrt{x_A^2+y^2}}{\sqrt{x_T^2+(y_T-y)^2}}\right] (y_T - y) + y
\end{cases} \tag{61}
$$

Similarly, Eqs. (36), (37), (48), and (52) yield

$$
b) \quad \begin{cases}
x_T(t_f) = \left[1 - \alpha \dfrac{\sqrt{x_A^2+y^2}}{\sqrt{x_T^2+(y-y_T)^2}}\right] x_T \\[2ex]
y_T(t_f) = \left[1 - \alpha \dfrac{\sqrt{x_A^2+y^2}}{\sqrt{x_T^2+(y-y_T)^2}}\right] (y_T - y) + y
\end{cases} \tag{62}
$$

Next, inserting Eqs. (61) and (62) into (42), (43), and (57)–(60) allows us to express the co-states in terms of y (or t_f):

$$
\begin{aligned}
\lambda_{x_T} &= -\left[1 \pm \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}}\right] x_T \\
\lambda_{y_T} &= -\left[1 \pm \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}}\right] (y_T - y) \\
\lambda_{x_A} &= \frac{1}{2}\left[1 \pm \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}}\right] \left(x_A + x_T - x_A \frac{y_T}{y}\right)
\end{aligned}
$$

$$\lambda_{y_A} = \frac{1}{2}\left[1 \pm \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}}\right]\left[y_T - \left(1 + \frac{x_T}{x_A}\right) y\right]$$
$$\lambda_{x_D} = \frac{1}{2}\Big[1 \pm \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}}\Big]\Big(x_T - x_A + x_A \frac{y_T}{y}\Big)$$
$$\lambda_{y_D} = \frac{1}{2}\Big[1 \pm \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}}\Big]\Big[y_T - \Big(1 - \frac{x_T}{x_A}\Big) y\Big] \tag{63}$$

So far we have not used Eq. (32) which, in view of Eqs. (19)–(21) is equivalent to

$$\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2} - \sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2} - \alpha \sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2} = 0. \tag{64}$$

It is however more convenient to use (32) in conjunction with Eqs. (51)–(54) for the optimal controls and Eq. (63) for the co-states. Doing so we obtain the following

$$\begin{aligned}
& -\frac{1}{2}\left(x_A + x_T - x_A \frac{y_T}{y}\right) \frac{x_A}{\sqrt{x_A^2 + y^2}} + \frac{1}{2}\left[y_T - \left(1 + \frac{x_T}{x_A}\right) y\right] \frac{y}{\sqrt{x_A^2 + y^2}} \\
& \quad + \frac{1}{2}\left(x_T - x_A + x_A \frac{y_T}{y}\right) \frac{x_A}{\sqrt{x_A^2 + y^2}} + \frac{1}{2}\left[y_T - \left(1 - \frac{x_T}{x_A}\right) y\right] \frac{y}{\sqrt{x_A^2 + y^2}} \\
& \quad + \frac{\alpha}{\sqrt{x_T^2 + (y_T - y)^2}} \left[x_T^2 + (y_T - y)^2\right] = 0 \\
& \Rightarrow \quad -x_A^2 + x_A^2 \frac{y_T}{y} + y_T y - y^2 + \alpha \frac{\sqrt{x_A^2 + y^2}}{\sqrt{x_T^2 + (y_T - y)^2}} \left[x_T^2 + (y_T - y)^2\right] = 0 \\
& \Rightarrow \quad \left(x_A^2 + y^2 - x_A^2 \frac{y_T}{y} - y_T y\right)^2 = \alpha^2 \left(x_A^2 + y^2\right) [x_T^2 + (y_T - y)^2] \\
& \Rightarrow \quad \left(1 - \frac{y_T}{y}\right)^2 \left(x_A^2 + y^2\right) = \alpha^2 [x_T^2 + (y_T - y)^2]
\end{aligned}$$

Finally, grouping terms in y we obtain (14), that is, the optimal aimpoint on the orthogonal bisector of $\overline{AD}$ specified by y^* is a real solution of quartic Eq. (14).

Equations (51), (52) do not directly provide the Target's optimal heading for the special case where $x_T = 0$. However, this case can be addressed by using (51), (52) together with (14) and analyzing the $\lim_{x_T \to 0} \phi^*$. From (51), (52) the Target's heading can be written as $\phi = \varphi + \pi/2$ where $\tan \varphi = \frac{\epsilon}{\delta}$, $\epsilon = -x_T$, and $\delta = y_T - y$. Figure 4 shows these relationships. In order to determine the optimal Target's heading φ^* when $x_T = 0$ we write (14) in the form

$$\begin{aligned}
& \left(1 - \frac{y_T}{y}\right)^2 \left(x_A^2 + y^2\right) = \alpha^2 \left[x_T^2 + (y_T - y)^2\right] \\
& \Rightarrow \quad (y - y_T)^2 \left(x_A^2 + y^2\right) = \alpha^2 y^2 \left[x_T^2 + (y_T - y)^2\right] \\
& \Rightarrow \quad \delta^2 \left[x_A^2 + (y_T - \delta)^2\right] = \alpha^2 (y_T - \delta)^2 (\epsilon^2 + \delta^2).
\end{aligned}$$

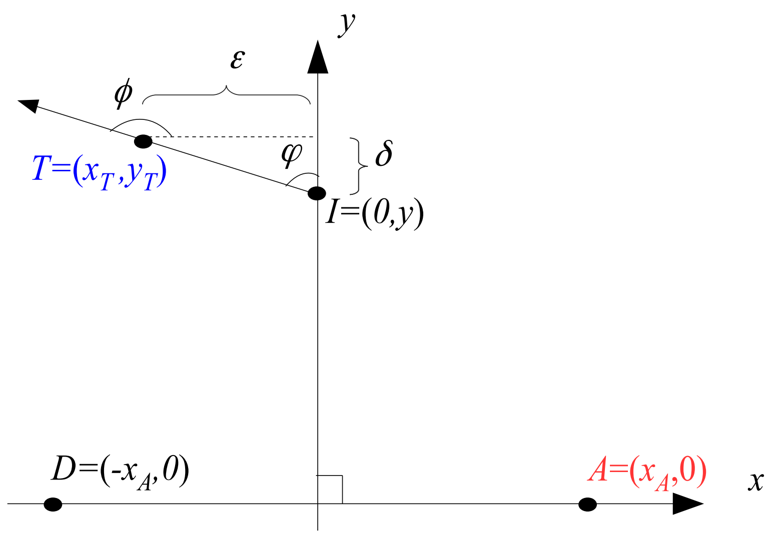


Fig. 4 Target's optimal heading when: $x_T \to 0$

By taking the limit $x_T \to 0$ we cancel higher order terms of the form $\epsilon^i \delta^j$ for $i + j \geq 3$ and we obtain

$$\delta^2 \left(x_A^2 + y_T^2\right) = \alpha^2 y_T^2 (\epsilon^2 + \delta^2)$$
$$\Rightarrow \quad \delta^2 \left(x_A^2 + (1 - \alpha^2) y_T^2\right) = \alpha^2 y_T^2 \epsilon^2$$

and the angle φ^* can be written as in (16). Section 5 provides an alternative analysis of the case $x_T = 0$ which results in an equivalent characterization of the angle φ^*.

Using Isaacs' method of solving differential games, the original problem was reduced to calculating y, so its solution hinges on the rooting in real time of quartic equation (14). This can be accomplished algebraically.

The value function is C^1 and is explicitly given by (17). The gradient of the value function is well defined for any triple (x_A, x_T, y_T) belonging to the escape region where the Game of Degree is played, however, the case where $x_T = 0$ requires special attention and is analyzed next.

When $x_T = 0$ the real solution of quartic equation (14) is $y = y_T, \forall\, x_A, y_T$.

1. The value function is continuous on the line $x_T = 0$. It is

$$V(x_A, 0, y_T) = \alpha \sqrt{x_A^2 + y_T^2}$$

2. Calculate $\left.\frac{\partial y}{\partial x_T}\right|_{x_T=0}$. To this end, note that quartic equation (14) can be written as

$$(y - y_T)^2 \left[(1 - \alpha^2) y^2 + x_A^2\right] - \alpha^2 x_T^2 y^2 = 0$$

Let $x_T = \epsilon$. The ensuing solution of Eq. (14) is then $y = y_T + \delta$ and we can write

$$\delta^2 \left[(1-\alpha^2)(y_T+\delta)^2 + x_A^2\right] = \alpha^2\epsilon^2(y_T+\delta)^2$$
$$\Rightarrow \quad \frac{\delta}{\epsilon} = \alpha \frac{y_T+\delta}{\sqrt{(1-\alpha^2)(y_T+\delta)^2 + x_A^2}}$$
$$\Rightarrow \quad \lim_{\epsilon\to 0}\frac{\delta}{\epsilon} = \alpha \frac{y_T}{\sqrt{(1-\alpha^2)y_T^2 + x_A^2}}$$

because of the continuous dependence of the polynomial's roots on the coefficients. Hence,

$$\left.\frac{\partial y}{\partial x_T}\right|_{x_T=0} = \alpha \frac{y_T}{\sqrt{(1-\alpha^2)y_T^2 + x_A^2}} \tag{65}$$

3. Calculate $\left.\frac{\partial V}{\partial \mathbf{x}}\right|_{x_T=0}$, where $\mathbf{x} = (x_A, x_T, y_T)$:

3.1. Set $x_T = \epsilon > 0$, whereupon the solution of quartic equation (14) is $y = y_T + \delta$, where $\delta > 0$. Calculate

$$\begin{aligned}
\varDelta V &= V(x_A, \epsilon, y_T) - V(x_A, 0, y_T)\\
&= \alpha\sqrt{x_A^2 + (y_T+\delta)^2} - \sqrt{\delta^2+\epsilon^2} - \alpha\sqrt{x_A^2+y_T^2}\\
&\approx \alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}\delta - \epsilon\sqrt{1+\frac{\delta^2}{\epsilon^2}}\\
&= \epsilon\left[\alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}\left(\frac{\delta}{\epsilon}\right) - \sqrt{1+\frac{\delta^2}{\epsilon^2}}\right].
\end{aligned}$$

Using (65) yields

$$\varDelta V = -\epsilon\sqrt{\frac{y_T^2(1-\alpha^2)+x_A^2}{y_T^2+x_A^2}}$$
$$\Rightarrow \quad \left.\frac{\partial V}{\partial x_T}\right|_{x_T=0^+} = -\sqrt{\frac{y_T^2(1-\alpha^2)+x_A^2}{y_T^2+x_A^2}}$$

Similarly, set $x_T = -\epsilon$, for $\epsilon > 0$, whereupon the solution of quartic equation (14) is $y = y_T - \delta$, where $\delta > 0$. Calculate

$$\begin{aligned}
\varDelta V &= V(x_A, 0, y_T) - V(x_A, -\epsilon, y_T)\\
&= \alpha\sqrt{x_A^2+y_T^2} - \alpha\sqrt{x_A^2 + (y_T-\delta)^2} - \sqrt{\delta^2+\epsilon^2}\\
&\approx \alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}\delta - \epsilon\sqrt{1+\frac{\delta^2}{\epsilon^2}}\\
&= \epsilon\left[\alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}\left(\frac{\delta}{\epsilon}\right) - \sqrt{1+\frac{\delta^2}{\epsilon^2}}\right].
\end{aligned}$$

Using (65) yields

$$\left.\frac{\partial V}{\partial x_T}\right|_{x_T=0^-} = -\sqrt{\frac{y_T^2(1-\alpha^2)+x_A^2}{y_T^2+x_A^2}}$$

Hence, the derivative $\frac{\partial V}{\partial x_T}$ is continuous at $x_T = 0$.

3.2. Calculate $\frac{\partial V}{\partial x_A}|_{x_T=0}$. Set $x_A := x_A + \epsilon$ and $x_T = 0$. When $x_T = 0$, $y = y_T$, $\forall x_A, y_T$. Calculate

$$\begin{aligned}\Delta V &= \alpha\sqrt{(x_A+\epsilon)^2+y_T^2} - \alpha\sqrt{x_A^2+y_T^2}\\ \Rightarrow \Delta V &= \alpha\frac{x_A}{\sqrt{x_A^2+y_T^2}}\epsilon\end{aligned}$$

Hence, the derivative

$$\left.\frac{\partial V}{\partial x_A}\right|_{x_T=0} = \alpha\frac{x_A}{\sqrt{x_A^2+y_T^2}}$$

3.3. Calculate $\frac{\partial V}{\partial y_T}|_{x_T=0}$. Set $y_T := y_T + \epsilon$ and $x_T = 0$. When $x_T = 0$, $y(y_T) = y_T$, $\forall x_A, y_T$. Calculate

$$\begin{aligned}\Delta V &= \alpha\sqrt{x_A^2+(y_T+\epsilon)^2} - \alpha\sqrt{x_A^2+y_T^2}\\ \Rightarrow \Delta V &= \alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}\epsilon\end{aligned}$$

Hence, the derivative

$$\left.\frac{\partial V}{\partial y_T}\right|_{x_T=0} = \alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}$$

Therefore, the value function is C^1 in the entire escape region R_e of the state space, including on the line $x_T = 0$. □

In summary, the optimal state feedback strategies were synthesized and the value function was explicitly obtained. The Target's escape region is fully characterized in Sect. 6. In the Target escape region of the state space the value function is C^1.

4.1 The Crucial Quartic Equation

To obtain the players' strategies quartic equation (14) must be solved in real-time. Now, same quartic equation (14) is obtained by solving the following equations in y: for case a) $\frac{\mathrm{d}J_1(y)}{\mathrm{d}y} = 0$; for case b) $\frac{\mathrm{d}J_2(y)}{\mathrm{d}y} = 0$. The cost/payoff $J_1(y)$ and $J_2(y)$ have the derivatives

$$\begin{aligned}\frac{\mathrm{d}J_1}{\mathrm{d}y} &= \alpha\frac{y}{\sqrt{x_A^2+y^2}} + \frac{y-y_T}{\sqrt{(y_T-y)^2+x_T^2}}\\ \frac{\mathrm{d}J_2}{\mathrm{d}y} &= \alpha\frac{y}{\sqrt{x_A^2+y^2}} + \frac{y_T-y}{\sqrt{(y-y_T)^2+x_T^2}}.\end{aligned} \tag{66}$$

For $\frac{\mathrm{d}J_1}{\mathrm{d}y}$ to vanish we need $y < y_T$ and for $\frac{\mathrm{d}J_2}{\mathrm{d}y}$ to vanish we need $y > y_T$, but when the derivatives are set equal to zero they both yield quartic equation (14) in y.

Quartic equations and thus Eq. (14) have an analytic closed-form solution. Nevertheless, the following analysis is illuminating. Consider $y_T > 0$ (when $y_T = 0$, Eq. (14) simplifies to $(1-\alpha^2)y^4 + (x_A^2 - \alpha^2 x_T^2)y^2 = 0$ and the solution is $y^* = 0$). Let $f(y)$ be given by the left-hand-side of (14) and consider $x_T \neq 0$ and $0 < \alpha < 1$. Note that $f(y) > 0$ for all $y >> 1$ and $f(y) > 0$ for all $y << -1$. Also $f(y_T) < 0$ and $f(0) > 0$. Hence, quartic equation (14) has a real solution $0 < y < y_T$ and a real solution $y > y_T$—just what we need.

In case a) we have that

$$\frac{\mathrm{d}^2 J_1}{\mathrm{d}y^2} = \alpha \frac{x_A^2}{\left(\sqrt{x_A^2 + y^2}\right)^3} + \frac{x_T^2}{\left(\sqrt{(y_T - y)^2 + x_T^2}\right)^3} > 0.$$

Then, $J_1(y)$ is strictly convex. $J_1(y)$ has exactly one minimum y^*, which is given by the real solution y^* of quartic equation (14), that satisfies $0 < y^* < y_T$.

In case b) we have from Fig. 3b that the headings of the Attacker and the Target are given by $\chi = \arctan \frac{y}{x_A}$ and $\phi = \pi - \arctan \frac{y - y_T}{x_T}$, respectively. Let $\phi_s = \pi - \phi = \arctan \frac{y - y_T}{x_T}$. The distance from point A to point I is $\overline{AI} = \frac{y}{\sin \chi}$ and the distance from point T to point I is given by $\overline{TI} = \frac{y - y_T}{\sin \phi_s}$. Then, $J_2'(y) = \frac{\mathrm{d}J_2(y)}{\mathrm{d}y}$ can be written as follows

$$J_2'(y) = \frac{\alpha y}{\overline{AI}} - \frac{y - y_T}{\overline{TI}} = \alpha \sin \chi - \sin \phi_s \tag{67}$$

Let us analyze the payoff function at $y = y_T$. In this case we have that $\phi_s = 0$ and $\chi = \arctan \frac{y_T}{x_A}$, where $y_T > 0$ and $x_A > 0$. The Attacker's heading satisfies $0 < \chi(y_T) < \frac{\pi}{2}$. Hence, $J_2'(y_T) = \alpha \sin \chi > 0$. Now, as $y \to \infty$ we have that $\chi \to \frac{\pi}{2}$, $\phi_s \to \frac{\pi}{2}$, and $J_2'(\infty) \to \alpha - 1 < 0$.

Because $J_2'(y_T) > 0$ and $J_2'(\infty) < 0$ the payoff $J_2(y)$ has at least one maximum at y^* that satisfies $y_T < y^* < \infty$. Let $\chi^* = \arctan \frac{y^*}{x_A}$ and $\phi_s^* = \arctan \frac{y^* - y_T}{x_T}$. Then, $J_2'(y^*) = \alpha \sin \chi^* - \sin \phi_s^* = 0$. We will now show the monotonicity of the payoff $J_2(y)$. Recall that the trigonometric function $\sin(\xi)$ is monotonically increasing in the interval $\xi \in [-\pi/2, \pi/2]$. Also, the inverse trigonometric function $\arctan(\cdot)$ is monotonically increasing. Let us find the values of y such that $\phi_s(y) = \chi(y)$, where $0 < \phi_s, \chi < \infty$. By obtaining the tangent of each side of the this equation we have that

$$\frac{y - y_T}{x_T} = \frac{y}{x_A}. \tag{68}$$

Define y_e the specific value of y that satisfies (68). Solving for y in this equation we obtain

$$y_e = \frac{x_A y_T}{x_A - x_T}. \tag{69}$$

Note that $y_e > y_T$ since $0 < x_T < x_A$. Also note that y_e is unique, that is, there is only one finite value of y such that $\phi_s(y) = \chi(y)$. Further, for $\epsilon > 0$ we have that

$$\frac{\epsilon}{x_T} > \frac{\epsilon}{x_A} \;\Rightarrow\; \frac{\epsilon}{x_T} + \frac{y_e - y_T}{x_T} > \frac{\epsilon}{x_A} + \frac{y_e}{x_A} \;\Rightarrow\; \frac{y_e + \epsilon - y_T}{x_T} > \frac{y_e + \epsilon}{x_A} \tag{70}$$

Let $y = y_e + \epsilon$, then the above inequality becomes

$$\frac{y - y_T}{x_T} > \frac{y}{x_A} \quad \Rightarrow \quad \frac{\pi}{2} > \phi_s(y) > \chi(y) > 0 \tag{71}$$

for any $y_e < y < \infty$ (because $y_e > y_T$ and $\epsilon > 0$).

Similarly, for $y_T \le y < y_e$ we can show that $0 \le \phi_s(y) < \chi(y) < \frac{\pi}{2}$. Additionally, note that $J_2'(y^*) = \alpha \sin \chi^* - \sin \phi_s^* = 0$ and $0 < \alpha < 1$, then $0 < \phi_s(y^*) < \chi(y^*) < \frac{\pi}{2}$. Altogether, $0 < y_T < y^* < y_e < \infty$.

(1) Case $y \ge y_e$. We have that $J_2'(y) = \alpha \sin \chi(y) - \sin \phi_s(y) < 0$ since $\phi_s(y) \ge \chi(y)$ and $0 < \alpha < 1$. Hence, $J_2(y)$ is monotonically decreasing in the interval $y \ge y_e$.

(2) Case $y^* < y < y_e$. Let us compute the second derivative of $J_2(y)$

$$J_2''(y) = \alpha \cos \chi \frac{\mathrm{d}\chi}{\mathrm{d}y} - \cos \phi_s \frac{\mathrm{d}\phi_s}{\mathrm{d}y} \tag{72}$$

where $\frac{\mathrm{d}\chi}{\mathrm{d}y} = \frac{x_A}{x_A^2+y^2} = \frac{\cos^2 \chi}{x_A}$ and $\frac{\mathrm{d}\phi_s}{\mathrm{d}y} = \frac{x_T}{(y-y_T)^2+x_T^2} = \frac{\cos^2 \phi_s}{x_T}$. Then, we can write (72) as follows

$$J_2''(y) = \alpha \frac{\cos^3 \chi}{x_A} - \frac{\cos^3 \phi_s}{x_T}. \tag{73}$$

Because $\phi_s(y) < \chi(y)$ for $y^* < y < y_e$, then $\cos \phi_s(y) > \cos \chi(y)$ and

$$\frac{\cos^3 \phi_s(y)}{x_T} > \alpha \frac{\cos^3 \chi(y)}{x_A}. \tag{74}$$

Hence, $J_2''(y) < 0$ for $y^* < y < y_e$ and $J_2(y)$ is monotonically decreasing for $y^* < y < y_e$.

(3) Case $y_T \le y \le y^*$. We have that $0 \le \phi_s(y) < \chi(y) < \frac{\pi}{2}$ for $y_T \le y \le y^*$ and $J_2'(y_T) > 0$ as it was previously shown. Also, since $\phi_s(y) < \chi(y)$ for $y_T \le y \le y^*$, we have that $J_2''(y) < 0$ for $y_T \le y \le y^*$. Hence, the payoff is monotonically increasing for $y_T \le y \le y^*$. Note that $J''(y^*) < 0$, as expected.

(4) Case $0 < y < y_T$. The Attacker's heading satisfies $0 < \chi(y) < \frac{\pi}{2}$ and the Target's heading satisfies $-\frac{\pi}{2} < \phi_s(y) < 0$. Then, $J_2'(y) > 0$ for $0 < y < y_T$ and $J_2(y)$ is monotonically increasing for $0 < y < y_T$.

In conclusion, the cost/payoff $J_2(y)$ defined in (50) is monotonically increasing for $0 < y < y^*$ and it is monotonically decreasing for $y > y^*$, and the unique maximizer y^*, which is a real solution of quartic equation (14), satisfies $y^* > y_T$.

The real solutions, y_κ, of quartic equation (14) provide the optimal solution y^*. It is now clear that the optimal solution is obtained as follows:

For case a), that is, $x_T \le 0$

$$y^* = \arg \min_{y_\kappa < y_T} \{J_1(y_\kappa)\}. \tag{75}$$

For case b), that is, $x_T > 0$

$$y^* = \arg \max_{y_\kappa > y_T} \{J_2(y_\kappa)\}. \tag{76}$$

The root locus analysis of quartic equation (14) is provided in "Appendix."

5 The Surface $x_T = 0$

When $x_T = 0$ the Target is at an equal distance from A and D. In the context of differential games such a configuration is interesting. Indeed, the plane in the reduced 3-D state space where $x_T = 0$ has a special place in the ATDDG. The instance where the state is such that $x_T = 0$ is important because the determination of the Target's optimal heading is particularly interesting.

Let us consider cost function (49). If $x_T = 0$, then y_T is a repeated real solution of quartic equation (14). In addition, Eq. (14) has two complex roots: $y = \pm i \frac{1}{\sqrt{1-\alpha^2}} x_A$. This is so because it can be factored as follows

$$(1-\alpha^2)y^4 - 2(1-\alpha^2)y_T y^3 + (1-\alpha^2)y_T^2 y^2 + x_A^2 \left(y^2 - 2y_T y + y_T^2\right)$$
$$= (y - y_T)^2\big((1-\alpha^2)y^2 + x_A^2\big). \tag{77}$$

It can be seen that the optimal point where the Attacker is intercepted by the Defender, $I = (0, y)$, is exactly the point $T = (0, y_T)$, the Target's initial position. The optimal strategy of the Target is to run away from point I and when $x_T \neq 0$ the choice is clear. But now, when $x_T = 0$ and Point I is collocated with the target's initial position, in what direction should the Target run?

If the Attacker' aim point is $I = (0, y) = (0, y_T)$ then any angle $0 < \varphi < \pi$ (refer to Fig. 6) is optimal for the Target. However, the Attacker will be able to redefine its aim point by knowing the Target's heading decision and reduce cost (49). Therefore there exists an optimal Target heading φ^* such that, for φ^*, the solution $y^* = y_T$ is a saddle point, that is, when $y^* = y_T$ the Attacker minimizes the separation between itself and the Target at its interception time instant, and the Target maximizes the said distance by choosing $\varphi = \varphi^*$. Contrary to the cases $x_T < 0$ and $x_T > 0$ considered in Sect. 4, quartic equation (14) by itself does not provide the optimal heading angle φ^* because it degenerates into Eq. (77) and the two points, T and I, that determine the Target's optimal heading are now one and the same.

5.1 Analysis

Consider the following:

1. The optimal paths of the Attacker, the Defender, and the Target are straight lines.
2. It is optimal for the Defender not to ever let the Attacker cross the orthogonal bisector of $\overline{AD}$; that is, the Attacker will always be intercepted by the Defender at $I = (0, y)$ on the orthogonal bisector of $\overline{AD}$.
3. It is optimal for the Target to be running away from I.
4. Because $T = (0, y_T)$ is initially on the orthogonal bisector of $\overline{AD}$ and since also I is on the orthogonal bisector of $\overline{AD}$, if $I \neq (0, y_T)$ it would be optimal for the Target to stay on the orthogonal bisector of $\overline{AD}$.
5. Should however the Target stay on the orthogonal bisector of $\overline{AD}$ and not head into the LHP, the faster Attacker could capture the Target by using Collision Course (CC) guidance.

We know that the Target starts out at $T = (0, y_T)$. The Target should be headed in a direction away from the interception point I. Hence, if $0 < y < y_T$, the Target should follow a slightly elevated trajectory compared to the Line-of-Sight (LOS) trajectory which is determined by the two points $T = (0, y_T)$ and $A = (x_A, 0)$. By the same token, if $y > y_T$, the

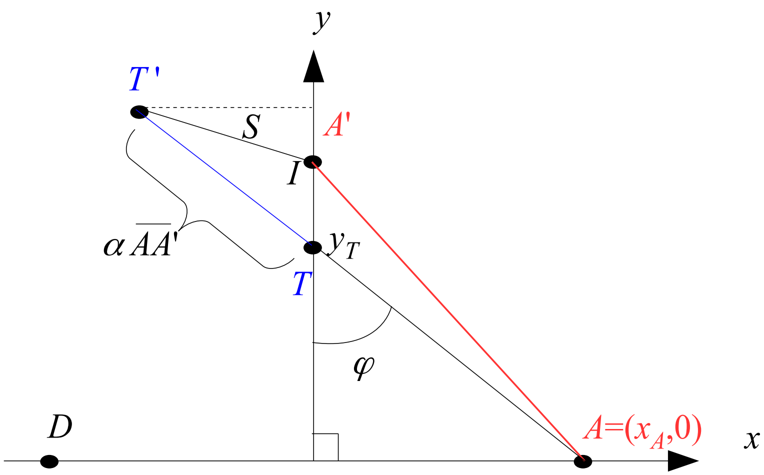


Fig. 5 The LOS strategy is not optimal

Target should follow a slightly depressed trajectory compared to the LOS trajectory. But the Target does not know the choice of the Attacker. Should the Target follow the LOS strategy? How should the Attacker react to the Target's LOS strategy?

If the Target uses the LOS strategy, the Attacker, who knows it, will be intercepted by the Defender on the orthogonal bisector of $\overline{AD}$, and hence he may find a better strategy for itself by trying to cut in front of the Target, as shown in Fig. 5, where $\overline{AA'} = \sqrt{x_A^2 + y^2}$.

The separation between the points A' and T' in Fig. 5 is s.t.

$$
\begin{aligned}
S^2(y) &= \left(\alpha\sqrt{x_A^2+y^2}\cos\varphi - (y-y_T)\right)^2 + \left(\alpha\sqrt{x_A^2+y^2}\sin\varphi\right)^2 \\
&= \alpha^2\left(x_A^2+y^2\right) + (y-y_T)^2 + 2\alpha(y_T-y)\sqrt{x_A^2+y^2}\cos\varphi \\
&= \alpha^2\left(x_A^2+y^2\right) + (y-y_T)^2 + 2\alpha(y_T-y)\sqrt{x_A^2+y^2}\frac{y_T}{\sqrt{x_A^2+y_T^2}}.
\end{aligned}
\tag{78}
$$

The first derivative of $S^2(y)$ is

$$
\begin{aligned}
\frac{\mathrm{d}S^2(y)}{\mathrm{d}y} &= 2\alpha^2 y + 2(y-y_T) + 2\alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}(y_T-y)\frac{y}{\sqrt{x_A^2+y^2}} \\
&\quad - 2\alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}\sqrt{x_A^2+y^2}.
\end{aligned}
\tag{79}
$$

The first derivative evaluated at $y = y_T$ is

$$
\left.\frac{\mathrm{d}S^2(y)}{\mathrm{d}y}\right|_{y=y_T} = 2\alpha(\alpha-1)y_T < 0. \tag{80}
$$

On the other hand, if $y \to \infty$, the derivative is

$$\begin{aligned}\left.\frac{\mathrm{d}S^2(y)}{\mathrm{d}y}\right|_{y>>1} &\approx 2y\Big(\alpha^2 - 2\alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}} + 1\Big) \\ &> 2y\Big(\alpha^2 - 2\alpha + 1\Big) \\ &= 2y\Big(1-\alpha\Big)^2 > 0.\end{aligned} \tag{81}$$

Hence, the minimal separation S is attained at $y^* > y_T$ and not at $y^* = y_T$. Hence, the Target should have stayed on the orthogonal bisector of $\overline{AD}$, which, according to point 5 above is not favorable for the Target. From this we conclude that the Attacker heads toward $I = (0, y_T)$.

The Target chooses the heading φ and the Attacker chooses y. It will be shown that there exists a heading φ^* such that the pair $\{\varphi^*, y_T\}$ is a saddle point, that is, if $\varphi = \varphi^*$, the Attacker minimizes the separation $\overline{A'T'}$ when $y^* = y_T$ and if $y = y^*$, the Target maximizes the separation $\overline{A'T'}$ by choosing $\varphi = \varphi^*$. The latter is trivially the case because any $0 < \varphi^* < \pi$ will do. Hence, we consider the $\max_\varphi \min_y S^2(\varphi, y)$ optimization problem and show that there exists a φ^* such that for this φ^*, $y = y_T$ minimizes the $\overline{A'T'}$ separation S.

Proposition 1 *When $x_T = 0$ the optimal Attacker's strategy is to head toward the point $I = (0, y_T)$, where it will be intercepted by the Defender, and the Target's optimal heading angle is specified by*

$$\cos\varphi^* = \alpha\frac{y_T}{\sqrt{x_A^2+y_T^2}}. \tag{82}$$

Proof Let us write cost function (78) as follows

$$S^2(\varphi, y) = \alpha^2(x_A^2+y^2) + (y-y_T)^2 + 2\alpha(y_T-y)\sqrt{x_A^2+y^2}\cos\varphi. \tag{83}$$

Its derivative is

$$\frac{\partial S^2(\varphi, y)}{\partial y} = 2\Big(\alpha^2 y + y - y_T + \alpha(y_T-y)\cos\varphi\frac{y}{\sqrt{x_A^2+y^2}} - \alpha\cos\varphi\sqrt{x_A^2+y^2}\Big) \tag{84}$$

and setting the derivative to zero yields

$$(1+\alpha^2)y - y_T + \alpha(y_T-y)\cos\varphi\frac{y}{\sqrt{x_A^2+y^2}} - \alpha\cos\varphi\sqrt{x_A^2+y^2} = 0. \tag{85}$$

Setting $y = y_T$

$$\alpha^2 y_T - \alpha\cos\varphi\sqrt{x_A^2+y_T^2} = 0 \tag{86}$$

and we obtain Eq. (82). To ascertain that we indeed have a minimum at $y = y_T$ when $\varphi = \varphi^*$, we evaluate the second derivative of S^2 at $y = y_T$:

$$\frac{\partial^2 S^2(\varphi^*, y)}{\partial y^2} = 2\left(\alpha^2 + 1 - 2\alpha\cos\varphi^*\frac{y}{\sqrt{x_A^2+y^2}} + \alpha(y_T-y)\cos\varphi^*\frac{x_A^2}{(x_A^2+y^2)^{3/2}}\right) \tag{87}$$

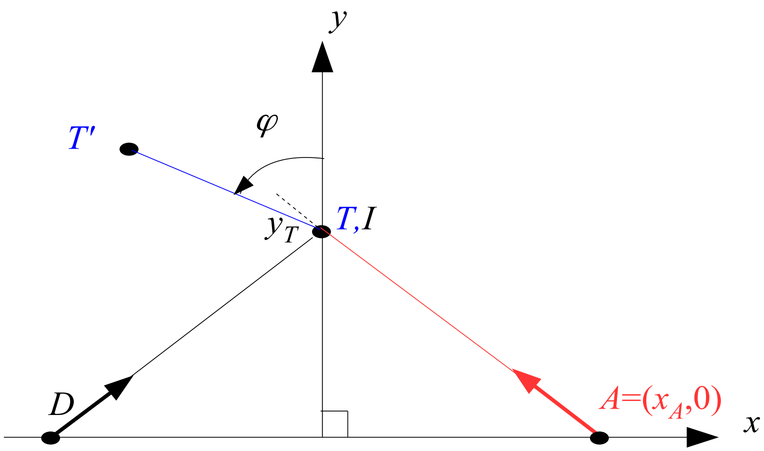


Fig. 6 Optimal trajectories: $x_T = 0$

and

$$\left.\frac{\partial^2 S^2(\varphi^*, y)}{\partial y^2}\right|_{y=y_T} = 2\left(\alpha^2 + 1 - 2\alpha^2 \frac{y_T^2}{x_A^2 + y_T^2}\right) > 2(\alpha^2 + 1 - 2\alpha^2) = 2(1 - \alpha^2) > 0. \tag{88}$$

By choosing $\varphi = \varphi^*$, the Target makes sure that the Attacker will not deviate from $y^* = y_T$. Also, by sticking with $y = y^* = y_T$, the Attacker is sure that his cost will not increase beyond $\alpha\sqrt{x_A^2 + y_T^2}$.

Indeed, recall that when $x_T = 0$, interestingly, quartic equation (14) can be factored as in (77) and it has a repeated root $y = y_T$ and two complex roots $y = \pm i \frac{1}{\sqrt{1-\alpha^2}} x_A$.

In summary, when $x_T = 0$, $y^* = y_T$ and $\varphi^* = \arccos\left(\alpha \frac{y_T}{\sqrt{x_A^2+y_T^2}}\right)$ yields a saddle point—see Fig. 6: When the Attacker plays $y = y^* = y_T$, it is guaranteed that its cost is not more than $\alpha\sqrt{x_A^2 + y_T^2}$. Also, when the Target plays $\varphi = \varphi^* = \arccos\left(\alpha \frac{y_T}{\sqrt{x_A^2+y_T^2}}\right)$, it is guaranteed that the Attacker will not deviate from $y = y^* = y_T$, otherwise, its cost (and the Target's payoff) will increase. Finally, the Defender intercepts the Attacker at $I = (0, y_T)$. □

Two equations have been provided for the angle φ^*, which specifies the optimal Target's heading $\phi^* = \varphi^* + \pi/2$ when $x_T = 0$. These equations are (16) and (82). One can show that these two characterizations are equivalent since

$$\sin\varphi^* = \cos\varphi^* \tan\varphi^* = \frac{\sqrt{x_A^2 + (1 - \alpha^2) y_T^2}}{\sqrt{x_A^2 + y_T^2}}$$

and verifying that

$$\sin^2\varphi^* + \cos^2\varphi^* = \frac{x_A^2 + (1 - \alpha^2) y_T^2}{x_A^2 + y_T^2} + \frac{\alpha^2 y_T^2}{x_A^2 + y_T^2} = \frac{x_A^2 + y_T^2}{x_A^2 + y_T^2} = 1.$$

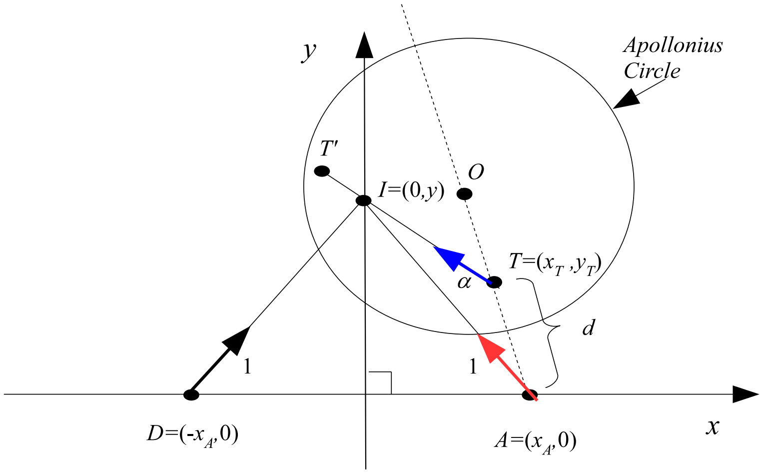


Fig. 7 Optimal play: $x_T > 0$

Remark The Target and the Defender will instantaneously take advantage of any non-optimal action of the Attacker to further increase the terminal $A - T$ separation. This is the innate nature of a saddle point solution of a zero-sum differential game.

6 Game of Kind

6.1 Critical Speed Ratio

In this section the scenario is considered where $x_T > 0$, that is, the Target is initially closer to the Attacker than it is to its Defender—see Fig. 7. The Target's escape is no longer guaranteed—it is then determined by the speed ratio α. We naturally assume that the Attacker is faster than the Target, that is, the speed ratio $\alpha = \frac{V_T}{V_A} < 1$. If the speed ratio $\alpha \geq 1$ the Attacker cannot capture the Target and there is no need for a Defender, and therefore no target defense differential game is played out. When $\alpha < 1$ the Target needs to be fast enough to break into the Left-Hand-Plane (LHP) without being intercepted by the Attacker, for the Defender to be able to assist the Target to escape, by intercepting, on the y-axis, the Attacker who is in route to the Target. Also recall that the orthogonal bisector of the segment $\overline{AD}$ separates the points in the realistic plane reachable by the Attacker before the Defender can get there from the points in the realistic plane the Defender can reach before the Attacker. Thus, the Target can escape from the Attacker and the Defender has a role to play in the target defense differential game if and only if the Apollonius circle, which is based on the segment $\overline{AT}$ and the speed ratio α, intersects the orthogonal bisector of the segment $\overline{AD}$—see Fig. 7. Given the current state in the reduced state space (x_A, x_T, y_T), this imposes a lower bound $\bar{\alpha}$ on the speed ratio, that is, the Target speed must be such that $\bar{\alpha} < \alpha < 1$. The critical speed ratio $\bar{\alpha}$ corresponds to the case where the Apollonius circle is tangent to the orthogonal bisector of $\overline{AD}$. The geometry of the Apollonius circle is as follows: The Attacker's initial position, the Target's initial position, and the center O of the Apollonius circle are collinear and lie on the dotted straight line in Fig. 7, whose equation is

$$y = -\frac{y_T}{x_A - x_T}x + \frac{x_A y_T}{x_A - x_T}.$$

The center of the circle, denoted by O, is at a distance of $\frac{\alpha^2}{1-\alpha^2}d$ from T and its radius is $\frac{\alpha}{1-\alpha^2}d$, where d is the distance between A and T and is given by

$$d = \sqrt{(x_A - x_T)^2 + y_T^2}. \tag{89}$$

Hence, the following holds

$$\left(\frac{x_T y_T}{x_A - x_T} - \frac{y_T}{x_A - x_T}x_O\right)^2 + (x_O - x_T)^2 = \frac{\alpha^4}{(1-\alpha^2)^2}\left[(x_A - x_T)^2 + y_T^2\right] \tag{90}$$

and we calculate the coordinates of the center of the Apollonius circle

$$x_O = \frac{1}{1-\alpha^2}x_T - \frac{\alpha^2}{1-\alpha^2}x_A, \quad y_O = \frac{1}{1-\alpha^2}y_T. \tag{91}$$

Consequently, the critical speed ratio $\bar{\alpha}$ is the positive solution of the quadratic equation

$$x_T - \alpha^2 x_A = \alpha\sqrt{(x_A - x_T)^2 + y_T^2} \tag{92}$$

and is given by

$$\bar{\alpha} = \frac{\sqrt{(x_A + x_T)^2 + y_T^2} - \sqrt{(x_A - x_T)^2 + y_T^2}}{2x_A}. \tag{93}$$

In the special case where $x_T = x_A$, the critical speed ratio is

$$\bar{\alpha} = \frac{\sqrt{4x_A^2 + y_T^2} - y_T}{2x_A},$$

as expected; and since $y_T > 0$, we have $\bar{\alpha} < 1$. In the special case when $y_T = 0$, we have $\bar{\alpha} = x_T/x_A < 1$ if $x_A > x_T$, and $\bar{\alpha} = 1$ if $x_A \leq x_T$. Also note that when $x_T = 0$, $\bar{\alpha} = 0$, as expected. In general, it can be seen from Fig. 7 that if $x_T < 0$ then $\bar{\alpha} = 0$ as well and the Target's escape is guaranteed.

Given the current reduced state (x_A, x_T, y_T), we will assume that the speed ratio α satisfies $\bar{\alpha}(x_A, x_T, y_T) < \alpha < 1$ (equivalently, the state space is in the region of win of the Target/Defender team). Then, the ATDDG can take place; otherwise, if the reduced state (x_A, x_T, y_T) is s.t. the speed ratio $\alpha \leq \bar{\alpha}(x_A, x_T, y_T)$, the Defender will not be able to help the Target by intercepting the Attacker before the latter inevitably captures the Target; and if $\alpha \geq 1$ then the Target can always escape from the Attacker and there is no need for a Defender. This argument shows the way to the solution of the Game of Kind.

6.2 Regions of Win

The Target's guaranteed escape region for a given speed ratio $\alpha = V_T/V_A$ (< 1) is characterized, and by doing so we obtain the solution of the Game of Kind.

When $x_T < 0$ the Target can escape. When $x_T > 0$ the state space is partitioned into two regions: R_e and R_c. The region R_e is the set of states such that if the Target's initial position (x_T, y_T) and the coordinate x_A are such that $(x_A, x_T, y_T) \in R_e$, then, the Target is guaranteed to escape the Attacker, provided the Target and the Defender team implement their optimal strategies ϕ^* and ψ^*. The Game of Degree, that is, the ATDDG, is played in

R_e. The region R_c specifies the states where under optimal Attacker play, notwithstanding the presence of the Defender, the Target cannot escape; in other words, the Target's escape is then possible only if A errs and the state (x_A, x_T, y_T) of the game enters the region R_e whereupon subsequent optimal play of the T & D team affords the Target's escape.

For a specified speed ratio $0 < \alpha < 1$ the equation

$$\alpha = \frac{\sqrt{(x_A + x_T)^2 + y_T^2} - \sqrt{(x_A - x_T)^2 + y_T^2}}{2x_A} \tag{94}$$

renders the boundary in the state space (x_A, x_T, y_T), $x_T, x_A > 0$, where the active target defense Game of Degree is played out, and as such this boundary constitutes the solution to the Game of Kind:

Proposition 2 *For a given speed ratio* $0 < \alpha < 1$ *the region of win of the Attacker*

$$R_c = \left\{(x_A, x_T, y_T)|x_A > 0,\ x_T > 0,\ y_T \geq 0,\ x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} < 0\right\}. \tag{95}$$

The manifold $\mathcal{B}$ *in* $\mathbb{R}^3$ *which is the boundary of the set* R_e *for which the Target is guaranteed to escape is*

$$\mathcal{B} = \left\{(x_A, x_T, y_T)|x_A > 0,\ x_T > 0,\ y_T \geq 0,\ x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} = 0\right\}. \tag{96}$$

and the region of win of the T & D *team where the ATDDG is played is*

$$R_e = \left\{(x_A, x_T, y_T)|x_A \geq 0,\ x_T \geq 0,\ x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} > 0\right\} \cup \{(x_A, x_T, y_T)|x_A \geq 0,\ x_T \leq 0\}. \tag{97}$$

Thus, for a fixed $x_A > 0$*, the* x_A*-cross section of* $\mathcal{B}$ *which divides the reduced state space into the two regions* R_e *and* R_c *is the right branch of the hyperbola (where* $x_T > 0$*)*

$$\frac{x_T^2}{\alpha^2 x_A^2} - \frac{y_T^2}{(1-\alpha^2)x_A^2} = 1. \tag{98}$$

Proof Equation (98) is obtained by noticing that solution (94) satisfies Eq. (92). Equation (92) can be written as follow

$$\frac{x_A^2}{(\frac{x_T}{\alpha})^2} + \frac{y_T^2}{\left(\frac{\sqrt{1-\alpha^2}}{\alpha}x_T\right)^2} = 1.$$

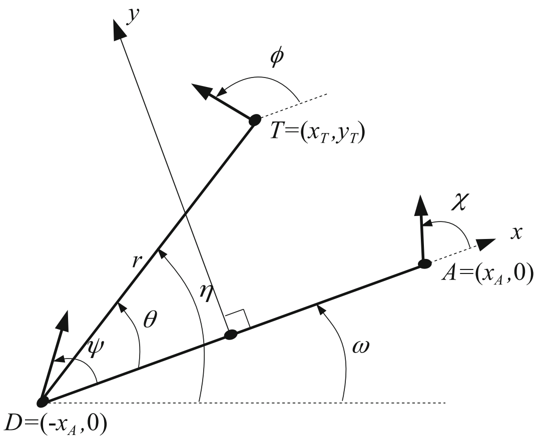
Letting $x = x_T$ and $y = y_T$ we obtain

$$\frac{\alpha^2 x_A^2}{x^2} = 1 - \frac{y^2}{\left(\frac{\sqrt{1-\alpha^2}}{\alpha}x\right)^2}$$

which is equivalent to (98). □

If $(x_A, x_T, y_T) \in R_e$, the Defender intercepts the Attacker before the latter could intercept the Target, provided the Target–Defender team plays optimally. If however $(x_A, x_T, y_T) \in R_c$, by heading toward the point on the circumference of the Apollonius circle closest to the y -axis, the Attacker can eventually capture the Target, unmolested by the Defender.

Fig. 8 Reduced state space



Hyperbola (98) is an x_A cross section of $\mathcal{B}$, that separates the two regions R_e and R_c. If the game starts on this surface, then the Target will be captured exactly at the same time when the Defender intercepts the Attacker, that is, the value of the ATDDG is zero on the barrier surface. Additionally, when $\alpha = 0$ the Target is static and hyperbola (98) becomes a straight line, the orthogonal bisector of the segment $\overline{AD}$, as expected. This can be shown from equivalent form (94)

$$\begin{aligned} 0 &= \frac{\sqrt{(x_A + x_T)^2 + y_T^2} - \sqrt{(x_A - x_T)^2 + y_T^2}}{2x_A} \\ \Rightarrow &\quad \sqrt{(x_A + x_T)^2 + y_T^2} = \sqrt{(x_A - x_T)^2 + y_T^2} \\ \Rightarrow &\quad 4x_A x = 0 \\ \Rightarrow &\quad x = 0 \end{aligned}$$

that is, the x_A cross section of the barrier separating the two regions is $x(y) = 0$, the orthogonal bisector of the segment $\overline{AD}$. From an operational point of view, if the state is in R_c, it then makes sense for the Target to employ passive defenses such as chaff, flares, jamming and spoofing. More about the target defense Game of Kind can be found in Ref. [12].

6.3 Semipermeability of the Barrier Surface

In order to show that barrier surface (96) is a semipermeable surface [17] we first obtain the dynamics of the ATDDG in the reduced state space (x_A, x_T, y_T). From Fig. 8 we have that

$$\begin{aligned} 2\dot{x}_A &= \cos\chi - \cos\psi \\ \dot{r} &= \alpha\cos(\phi - \theta) - \cos(\psi - \theta) \\ \theta &= \eta - \omega \\ \dot{\eta} &= \frac{1}{r}[\alpha\sin(\phi - \theta) - \sin(\psi - \theta)] \end{aligned}$$

$$\dot{\omega} = \frac{1}{2x_A}(\sin\chi - \sin\psi)$$
$$\dot{\theta} = \frac{1}{r}[\alpha\sin(\phi-\theta) - \sin(\psi-\theta)] - \frac{1}{2x_A}(\sin\chi - \sin\psi).$$

It also holds that

$$x_T = r\cos\theta - x_A, \quad y_T = r\sin\theta$$

and

$$r = \sqrt{(x_A + x_T)^2 + y_T^2}, \quad \theta = \arcsin\left(\frac{y_T}{r}\right).$$

We calculate the following

$$\begin{aligned}\dot{x}_T &= [\alpha\cos(\phi-\theta) - \cos(\psi-\theta)]\cos\theta - [\alpha\sin(\phi-\theta) - \sin(\psi-\theta)]\sin\theta \\ &\quad + \frac{r}{2x_A}(\sin\chi - \sin\psi)\sin\theta - \frac{1}{2}(\cos\chi - \cos\psi) \\ &= \alpha\cos\phi - \frac{1}{2}(\cos\chi + \cos\psi) + \frac{y_T}{2x_A}(\sin\chi - \sin\psi)\end{aligned}$$

and

$$\begin{aligned}\dot{y}_T &= [\alpha\cos(\phi-\theta) - \cos(\psi-\theta)]\sin\theta - [\alpha\sin(\phi-\theta) - \sin(\psi-\theta)]\cos\theta \\ &\quad - \frac{r}{2x_A}(\sin\chi - \sin\psi)\cos\theta \\ &= \alpha\sin\phi - \sin\psi - \frac{x_T + x_A}{2x_A}(\sin\chi - \sin\psi) \\ &= \alpha\sin\phi - \frac{1}{2}(\sin\chi + \sin\psi) + \frac{x_T}{2x_A}(\sin\psi - \sin\chi).\end{aligned}$$

Therefore, the dynamics in the reduced state space $(x_A, x_T, y_T) \in \mathbb{R}^3$ are given by:

$$\begin{aligned}\dot{x}_A &= \frac{1}{2}(\cos\chi - \cos\psi) \\ \dot{x}_T &= \alpha\cos\phi - \frac{1}{2}(\cos\psi + \cos\chi) - \frac{y_T}{2x_A}(\sin\psi - \sin\chi) \\ \dot{y}_T &= \alpha\sin\phi - \frac{1}{2}(\sin\psi + \sin\chi) + \frac{x_T}{2x_A}(\sin\psi - \sin\chi).\end{aligned} \tag{99}$$

Now, from (98) the capture region is as follows

$$R_c = \left\{(x_A, x_T, y_T) | x_A > 0, x_T > 0, y_T \geq 0, x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} < 0\right\} \tag{100}$$

and the barrier surface is given by

$$\mathcal{B} = \left\{(x_A, x_T, y_T) | x_A > 0, x_T > 0, y_T \geq 0, x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} = 0\right\}. \tag{101}$$

The normal to $\mathcal{B}$ which points into R_e is given by

$$\mathbf{n} = \left(x_A, -\frac{x_T}{\alpha^2}, \frac{y_T}{1-\alpha^2}\right)$$

The semipermeability condition of the barrier surface $\mathcal{B}$ is

$$\max_{\phi,\psi} \min_{\chi} (\mathbf{n} \cdot \mathbf{f})|_{\mathcal{B}} = 0$$

whereupon the state stays on $\mathcal{B}$; the latter being an invariant manifold; the function $\mathbf{f}(x_A, x_T, y_T)$ is given by the right-hand side of Eq. (99). We now calculate the following

$$\begin{aligned}
\mathbf{n} \cdot \mathbf{f} = & \frac{1}{2} x_A \cos\chi - \frac{1}{2} x_A \cos\psi - \frac{1}{\alpha} x_T \cos\phi + \frac{1}{2\alpha^2} x_T \cos\psi \\
& + \frac{1}{2\alpha^2} x_T \cos\chi + \frac{1}{2\alpha^2} \frac{x_T y_T}{x_A} (\sin\psi - \sin\chi) \\
& + \frac{\alpha}{1-\alpha^2} y_T \sin\phi - \frac{1}{2} \frac{1}{1-\alpha^2} y_T \sin\psi - \frac{1}{2} \frac{1}{1-\alpha^2} y_T \sin\chi \\
& + \frac{1}{2} \frac{1}{1-\alpha^2} \frac{x_T y_T}{x_A} (\sin\psi - \sin\chi) \\
= & \frac{1}{2\alpha^2(1-\alpha^2)} \frac{x_T y_T}{x_A} (\sin\psi - \sin\chi) + \frac{\alpha}{1-\alpha^2} y_T \sin\phi - \frac{1}{\alpha} x_T \cos\phi \\
& - \frac{1}{2} \frac{1}{1-\alpha^2} y_T \sin\psi + \frac{1}{2} \left(\frac{1}{\alpha^2} x_T - x_A \right) \cos\psi \\
& - \frac{1}{2} \frac{1}{1-\alpha^2} y_T \sin\chi + \frac{1}{2} \left(\frac{1}{\alpha^2} x_T + x_A \right) \cos\chi \\
= & \frac{\alpha}{1-\alpha^2} y_T \sin\phi - \frac{1}{\alpha} x_T \cos\phi + \frac{1}{2} \left(\frac{1}{\alpha^2} x_T - x_A \right) \left(\frac{1}{1-\alpha^2} \frac{y_T}{x_A} \sin\psi + \cos\psi \right) \\
& + \frac{1}{2} \left(\frac{1}{\alpha^2} x_T + x_A \right) \left(\cos\chi - \frac{1}{1-\alpha^2} \frac{y_T}{x_A} \sin\chi \right)
\end{aligned}$$

Hence, the optimal headings of each agent when $(x_A, x_T, y_T) \in \mathcal{B}$ are obtained from the following

$$\begin{aligned}
& \max_{\phi} \left(\frac{\alpha}{1-\alpha^2} y_T \sin\phi - \frac{1}{\alpha} x_T \cos\phi \right) + \frac{1}{2} \max_{\psi} \left(\left(\frac{1}{\alpha^2} x_T - x_A \right) \left(\frac{1}{1-\alpha^2} \frac{y_T}{x_A} \sin\psi + \cos\psi \right) \right) \\
& \quad + \frac{1}{2} \min_{\chi} \left(\left(\frac{1}{\alpha^2} x_T + x_A \right) \left(\cos\chi - \frac{1}{1-\alpha^2} \frac{y_T}{x_A} \sin\chi \right) \right) = 0
\end{aligned} \tag{102}$$

and they are given by

$$\sin\phi = \frac{\frac{\alpha}{1-\alpha^2} y_T}{\sqrt{\frac{1}{\alpha^2} x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2} y_T^2}}, \quad \cos\phi = -\frac{\frac{1}{\alpha} x_T}{\sqrt{\frac{1}{\alpha^2} x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2} y_T^2}}$$

$$\sin\psi = \operatorname{sign}(x_T - \alpha^2 x_A) \frac{\frac{1}{1-\alpha^2} \frac{y_T}{x_A}}{\sqrt{1 + \frac{1}{(1-\alpha^2)^2} \frac{y_T^2}{x_A^2}}}, \quad \cos\psi = \operatorname{sign}(x_T - \alpha^2 x_A) \frac{1}{\sqrt{1 + \frac{1}{(1-\alpha^2)^2} \frac{y_T^2}{x_A^2}}}$$

$$\sin\chi = \frac{\frac{1}{1-\alpha^2} \frac{y_T}{x_A}}{\sqrt{1 + \frac{1}{(1-\alpha^2)^2} \frac{y_T^2}{x_A^2}}}, \quad \cos\chi = -\frac{1}{\sqrt{1 + \frac{1}{(1-\alpha^2)^2} \frac{y_T^2}{x_A^2}}}$$

Now, on $\mathcal{B}$ (and for $y_T > 0$), it holds that $x_T - \alpha^2 x_A > 0$. Indeed, suppose $0 < x_T \leq \alpha^2 x_A$. Then,

$$x_A^2 - \frac{x_T^2}{\alpha^2} \geq x_A^2 - \alpha^2 x_A^2 = (1-\alpha^2)x_A^2 > 0 \quad \Rightarrow x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} > 0$$

so the state is in the interior of the region of win of T and D, in R_e, and not on the surface $\mathcal{B}$ which separates the R_c and R_e state space regions where $x_T > 0$. That $x_T > \alpha^2 x_A$ should come as no surprise. Consider the critical geometry which corresponds to the state $(x_A, x_T, y_T) \in \mathcal{B}$. The center of the Apollonius circle is given by (91) and, obviously, when the Apollonius circle contacts the y-axis $x_O > 0$. Hence,

$$x_O > 0 \;\;\Rightarrow\;\; x_T > \alpha^2 x_A.$$

Hence, the state feedback strategies of each agent which keep the state on the barrier surface $\mathcal{B}$ are explicitly given by

$$\begin{aligned}
\sin\hat{\phi} &= \frac{\frac{\alpha}{1-\alpha^2} y_T}{\sqrt{\frac{1}{\alpha^2}x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2} y_T^2}}, & \cos\hat{\phi} &= -\frac{\frac{1}{\alpha} x_T}{\sqrt{\frac{1}{\alpha^2}x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2} y_T^2}} \\
\sin\hat{\psi} &= \frac{y_T}{\sqrt{(1-\alpha^2)^2 x_A^2 + y_T^2}}, & \cos\hat{\psi} &= \frac{(1-\alpha^2)x_A}{\sqrt{(1-\alpha^2)^2 x_A^2 + y_T^2}} \\
\sin\hat{\chi} &= \frac{y_T}{\sqrt{(1-\alpha^2)^2 x_A^2 + y_T^2}}, & \cos\hat{\chi} &= -\frac{(1-\alpha^2)x_A}{\sqrt{(1-\alpha^2)^2 x_A^2 + y_T^2}}
\end{aligned} \tag{103}$$

and $\hat{\chi} = \pi - \hat{\psi}$. Hence, on $\mathcal{B}$, the three protagonists steer toward the point $I = (0, \frac{1}{1-\alpha^2} y_T)$ which is the point where the Apollonius circle is tangent to the orthogonal bisector of the segment $\overline{AD}$. When the state $(x_A, x_T, y_T) \in \mathcal{B}$, the saddle point strategy $\hat{\phi}$, $\hat{\psi}$ of the T/D team shown in (103) precludes the state from entering A's winning region R_c, despite A's best efforts to the contrary. At the same time, the saddle point strategy $\hat{\chi}$ of A shown in (103) precludes the state from entering the region R_e where T and D are the winners and the Game of Degree is played. Failure to do so by A will cause the state to cross the $\mathcal{B}$ boundary, from R_c to R_e, and vice versa, failure by T and/or D to play the saddle point strategies on $\mathcal{B}$ will cause the state to cross from region R_e to R_c and change the outcome of the game.

Finally, inserting strategies (103) into Eq. (102), we recover the surface $\mathcal{B}$:

$$\begin{aligned}
&\sqrt{\frac{1}{\alpha^2}x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2} y_T^2} + \frac{1}{2}\left(\frac{1}{\alpha^2}x_T - x_A\right)\sqrt{1 + \frac{1}{(1-\alpha^2)^2}\frac{y_T^2}{x_A^2}} \\
&\quad - \frac{1}{2}\left(\frac{1}{\alpha^2}x_T + x_A\right)\sqrt{1 + \frac{1}{(1-\alpha^2)^2}\frac{y_T^2}{x_A^2}} = 0 \\
&\Rightarrow \;\; \sqrt{\frac{1}{\alpha^2}x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2} y_T^2} - x_A\sqrt{1 + \frac{1}{(1-\alpha^2)^2}\frac{y_T^2}{x_A^2}} = 0
\end{aligned}$$

$$\Rightarrow \ \frac{1}{\alpha^2}x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2}y_T^2 = x_A^2 + \frac{1}{(1-\alpha^2)^2}y_T^2$$
$$\Rightarrow \ \frac{x_T^2}{\alpha^2} - \frac{y_T^2}{1-\alpha^2} = x_A^2$$

which is the equation of the semipermeable surface $\mathcal{B}$.

The closed-loop dynamics on $\mathcal{B}$ are

$$\begin{aligned}
\dot{x}_A &= -\frac{(1-\alpha^2)x_A}{\sqrt{(1-\alpha^2)^2x_A^2 + y_T^2}}, \quad x_A(0) = x_{A_0} \\
\dot{x}_T &= -\frac{x_T}{\sqrt{\frac{1}{\alpha^2}x_T^2 + \frac{\alpha^2}{(1-\alpha^2)^2}y_T^2}}, \quad x_T(0) = x_{T_0} \\
\dot{y}_T &= \left[\frac{\alpha^2}{\sqrt{\frac{(1-\alpha^2)^2}{\alpha^2}x_T^2 + \alpha^2 y_T^2}} - \frac{1}{\sqrt{(1-\alpha^2)^2x_A^2 + y_T^2}}\right]y_T, \quad y_T(0) = y_{T_0}
\end{aligned} \tag{104}$$

and on $\mathcal{B}$

$$x_A^2 + \frac{y_T^2}{1-\alpha^2} - \frac{x_T^2}{\alpha^2} \equiv 0$$

We insert $\frac{x_T^2}{\alpha^2} = x_A^2 + \frac{y_T^2}{1-\alpha^2}$ into the expressions for $\dot{x}_T$ and $\dot{y}_T$ in (104). On the semipermeable surface $\mathcal{B}$ where A, D, and T play the barrier strategy $\wedge$,

$$\begin{aligned}
\dot{x}_A &= -(1-\alpha^2)\frac{x_A}{\sqrt{(1-\alpha^2)^2x_A^2 + y_T^2}}, \quad x_A(0) = x_{A_0}\ (> 0) \\
\dot{x}_T &= -(1-\alpha^2)\frac{x_T}{\sqrt{(1-\alpha^2)^2x_A^2 + y_T^2}}, \quad x_T(0) = x_{T_0}\ (> 0) \\
\dot{y}_T &= -(1-\alpha^2)\frac{y_T}{\sqrt{(1-\alpha^2)^2x_A^2 + y_T^2}}, \quad y_T(0) = y_{T_0}\ (> 0)
\end{aligned} \tag{105}$$

for $0 \le t \le t_f$. We have that $\dot{x}_A(t) < 0$, $\dot{x}_T(t) < 0$, $\dot{y}_T(t) < 0$, $\forall$, $0 \le t \le t_f$; and $x_A(t_f) = x_T(t_f) = y_T(t_f) = 0$.

Furthermore, the trajectories $\left(\hat{x}_A(t; x_{A_0}), \hat{x}_T(t; x_{T_0}), \hat{y}_T(t; y_{T_0})\right)$, where $(x_{A_0}, x_{T_0}, y_{T_0}) \in \mathcal{B}$, are straight lines. Indeed,

$$\begin{aligned}
\frac{\mathrm{d}x_T}{\mathrm{d}x_A} &= \frac{x_T}{x_A} \ \Rightarrow \ x_T(t; x_{T_0}) = a x_A(t; x_{A_0}) \\
\frac{\mathrm{d}y_T}{\mathrm{d}x_A} &= \frac{y_T}{x_A} \ \Rightarrow \ y_T(t; y_{T_0}) = b x_A(t; x_{A_0})
\end{aligned}$$

where $a > 0$, $b > 0$ are constants

$$a = \frac{x_{T_0}}{x_{A_0}}, \quad b = \frac{y_{T_0}}{x_{A_0}}$$

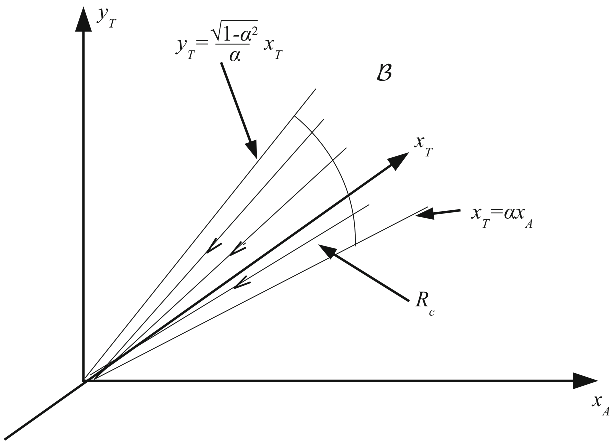


Fig. 9 Semipermeable surface $\mathcal{B}$ in the reduced state space; it separates the sets R_c and R_e

so

$$\frac{x_T(t;x_{T_0})}{x_{T_0}} = \frac{x_A(t;x_{A_0})}{x_{A_0}}$$
$$\frac{y_T(t;y_{T_0})}{y_{T_0}} = \frac{x_A(t;x_{A_0})}{x_{A_0}}$$

Consider the $\dot{x}_A$ equation

$$\dot{x}_A = -(1-\alpha^2)\frac{x_A}{\sqrt{(1-\alpha^2)^2x_A^2+y_T^2}} = -(1-\alpha^2)\frac{x_A}{\sqrt{(1-\alpha^2)^2x_A^2+b^2x_A^2}}$$
$$= -(1-\alpha^2)\frac{1}{\sqrt{(1-\alpha^2)^2+\left(\frac{y_{T_0}}{x_{A_0}}\right)^2}}$$

with $x_A(0)=x_{A_0}$. Hence,

$$x_A(t;x_{A_0}) = x_{A_0} - \frac{1-\alpha^2}{\sqrt{(1-\alpha^2)^2+\left(\frac{y_{T_0}}{x_{A_0}}\right)^2}}t$$

$$x_T(t;x_{T_0}) = x_{T_0} - \frac{1-\alpha^2}{\sqrt{(1-\alpha^2)^2+\left(\frac{y_{T_0}}{x_{A_0}}\right)^2}}\cdot\frac{x_{T_0}}{x_{A_0}}t$$

$$y_T(t;y_{T_0}) = y_{T_0} - \frac{1-\alpha^2}{\sqrt{(1-\alpha^2)^2+\left(\frac{y_{T_0}}{x_{A_0}}\right)^2}}\cdot\frac{y_{T_0}}{x_{A_0}}t$$

and $t_f = \frac{\sqrt{(1-\alpha^2)^2+(\frac{y_{T_0}}{x_{A_0}})^2}}{1-\alpha^2}x_{A_0}$.

$\mathcal{B}$ is a ruled surface - it is a quadrant of a cone in $\mathbb{R}^3$ with an elliptical cross section, with the vertex at the origin as shown in Fig. 9. On the surface $\mathcal{B}$, the value of the ATDDG is 0, which can be proved by noting that on $\mathcal{B}$ it holds that $y^* = y_O = \frac{1}{1-\alpha^2}y_T$. Then,

$$
\begin{aligned}
V &= \alpha\sqrt{x_A^2 + \frac{y_T^2}{(1-\alpha^2)^2}} - \sqrt{\left(y_T - \frac{y_T}{1-\alpha^2}\right)^2 + x_T^2} \\
&= \alpha\sqrt{\frac{x_T^2}{\alpha^2} - \frac{y_T^2}{1-\alpha^2} + \frac{y_T^2}{(1-\alpha^2)^2}} - \sqrt{\left(\frac{(1-\alpha^2)y_T - y_T}{1-\alpha^2}\right)^2 + x_T^2} \\
&= \sqrt{x_T^2 + \frac{\alpha^4 y_T^2}{(1-\alpha^2)^2}} - \sqrt{x_T^2 + \frac{\alpha^4 y_T^2}{(1-\alpha^2)^2}} = 0
\end{aligned}
$$

where (96) was used since the state is on the Barrier surface $\mathcal{B}$.

In conclusion, if in the reduced state space the initial state $(x_{A_0}, x_{T_0}, y_{T_0}) \in \mathcal{B}$ (where the parameter $c = 0$), A, T, and D will converge at the origin and the outcome can be considered a draw because the Value of the ATDDG is then zero. $\mathcal{B}$ being a semipermeable surface, should A fail to employ the barrier strategy $\hat{\chi}$, the state will enter the R_e zone whereupon, employing the Game of Degree optimal strategies ϕ^* and ψ^* the T & D team will secure T's escape. Conversely, should the T & D team fail to employ their barrier strategies $\hat{\phi}$ and $\hat{\psi}$, the state will enter the R_c zone and T will eventually be captured by A.

7 Optimal Flow Field in R_e

Having addressed the Game of Kind and delineated the winning regions of each team, the state space regions R_e and R_c, we conclude by returning to the Game of Degree in R_e. In the realistic plane we have

$$
\begin{aligned}
&\tilde{x}_A(t) = \tilde{x}_A(0) - t\frac{\tilde{x}_A(0)}{\sqrt{\tilde{x}_A^2(0) + y_0^2}}, \quad \tilde{y}_A(t) = \tilde{y}_A(0) + t\frac{y_0}{\sqrt{\tilde{x}_A^2(0) + y_0^2}} \\
&\tilde{x}_D(t) = \tilde{x}_D(0) + t\frac{\tilde{x}_A(0)}{\sqrt{\tilde{x}_A^2(0) + y_0^2}}, \quad \tilde{y}_D(t) = \tilde{y}_D(0) + t\frac{y_0}{\sqrt{\tilde{x}_A^2(0) + y_0^2}} \\
&\tilde{x}_T(0) < 0: \quad \tilde{x}_T(t) = \tilde{x}_T(0) + \alpha t\frac{\tilde{x}_T(0)}{\sqrt{\tilde{x}_T^2(0) + (\tilde{y}_T(0) - y_0)^2}}, \\
&\tilde{y}_T(t) = \tilde{y}_T(0) + \alpha t\frac{\tilde{y}_T(0) - y_0}{\sqrt{\tilde{x}_T^2(0) + (\tilde{y}_T(0) - y_0)^2}} \\
&\tilde{x}_T(0) > 0: \quad \tilde{x}_T(t) = \tilde{x}_T(0) - \alpha t\frac{\tilde{x}_T(0)}{\sqrt{\tilde{x}_T^2(0) + (\tilde{y}_T(0) - y_0)^2}}, \\
&\tilde{y}_T(t) = \tilde{y}_T(0) + \alpha t\frac{y_0 - \tilde{y}_T(0)}{\sqrt{\tilde{x}_T^2(0) + (\tilde{y}_T(0) - y_0)^2}} \\
&\tilde{x}_T(0) = 0: \quad \tilde{x}_T(t) = -\alpha t\sqrt{\frac{\tilde{x}_A^2(0) + (1-\alpha^2)\tilde{y}_T^2(0)}{\tilde{x}_A^2(0) + \tilde{y}_T^2(0)}} \\
&\tilde{y}_T(t) = \tilde{y}_T(0) + \alpha t\frac{\tilde{y}_T(0)}{\sqrt{\tilde{x}_A^2(0) + \tilde{y}_T^2(0)}}
\end{aligned}
$$

where y_0 is the solution of quartic equation (14) which is parameterized by $\tilde{x}_A(0)$, $\tilde{x}_T(0)$, $\tilde{y}_T(0)$, and α. and, without loss of generality, we have that $\tilde{x}_D(0) = -\tilde{x}_A(0)$, $\tilde{y}_D(0) = \tilde{y}_A(0) = 0$.

During optimal play the optimal trajectories in the reduced state space are

$$x_A(t) = \tilde{x}_{A_0}\big[1 - \frac{1}{\sqrt{\tilde{x}_A^2(0) + y_0^2}} \cdot t\big]$$

$$x_T(t) = \begin{cases} \big[1 - \alpha\frac{1}{\sqrt{\tilde{x}_T^2(0)+(\tilde{y}_T(0)-y_0)^2}} \cdot t\big]\tilde{x}_T(0) \text{ if } & \tilde{x}_T(0) > 0 \\ \big[1 + \alpha\frac{1}{\sqrt{\tilde{x}_T^2(0)+(\tilde{y}_T(0)-y_0)^2}} \cdot t\big]\tilde{x}_T(0) \text{ if } & \tilde{x}_T(0) < 0 \\ -\alpha\sqrt{\frac{\tilde{x}_A^2(0)+(1-\alpha^2)\tilde{y}_T^2(0)}{\tilde{x}_A^2(0)+\tilde{y}_T^2(0)}} \cdot t \qquad\qquad \text{ if } & \tilde{x}_T(0) = 0 \end{cases}$$

$$y_T(t) = \begin{cases} \tilde{y}_T(0) + \left[\alpha\frac{y_0-\tilde{y}_T(0)}{\sqrt{\tilde{x}_T^2(0)+(\tilde{y}_T(0)-y_0)^2}} - \frac{y_0}{\sqrt{\tilde{x}_A^2(0)+y_0^2}}\right]t & \text{if } \tilde{x}_T(0) > 0 \\ \tilde{y}_T(0) + \left[\alpha\frac{\tilde{y}_T(0)-y_0}{\sqrt{\tilde{x}_T^2(0)+(\tilde{y}_T(0)-y_0)^2}} - \frac{y_0}{\sqrt{\tilde{x}_A^2(0)+y_0^2}}\right]t & \text{if } \tilde{x}_T(0) > 0 \\ \tilde{y}_T(0) + \left[\alpha\frac{\tilde{y}_T(0)}{\sqrt{\tilde{x}_A^2(0)+\tilde{y}_T^2(0)}} - \frac{y_0}{\sqrt{\tilde{x}_A^2(0)+y_0^2}}\right]t & \text{if } \tilde{x}_T(0) = 0 \end{cases}$$

Here

$$0 \le t \le \sqrt{\tilde{x}_A^2(0) + y_0^2}$$

and

$$y_0 = y_0\big(\tilde{x}_A(0), \tilde{x}_T(0), \tilde{y}_T(0); \alpha\big)$$

is the solution of quartic equation (14). Hence, choose $\tilde{x}_A(0)$, $\tilde{x}_T(0)$, $\tilde{y}_T(0)$, and α such that

$$\frac{\tilde{x}_T^2(0)}{\big(\frac{\alpha}{\sqrt{1-\alpha^2}}\tilde{y}_T(0)\big)^2} - \frac{\tilde{x}_A^2(0)}{\big(\frac{\tilde{y}_T(0)}{\sqrt{1-\alpha^2}}\big)^2} = 1. \tag{106}$$

The optimal flow field in the reduced state space (x_A, x_T, y_T) is shown in Figs. 10 and 11. In Fig. 10, for a fixed x_A value, the slope of the lines tends to $1/\alpha$ as x_T decreases. In the particular case where $y_T = 0$, Fig. 11, the slope of each line is equal to $1/\alpha$.

7.1 Examples

The R_e set where the ATDDG is played and the players optimal strategies ϕ^*, ψ^*, and χ^* are employed is completely covered by the optimal flow field. It consists of straight line trajectories. Furthermore, we have obtained the value function in the whole of R_e and it is C^1—see Theorem 1. The optimal state feedback strategies rendered by the solution of the ATDDG in R_e are exercised. The optimal strategies are given by Theorem 1 and Proposition 1 in terms of the reduced state (x_A, x_T, y_T) and the solution $y^*(x_A, x_T, y_T)$ of quartic equation (14). In terms of the A, D, and T player's respective coordinates

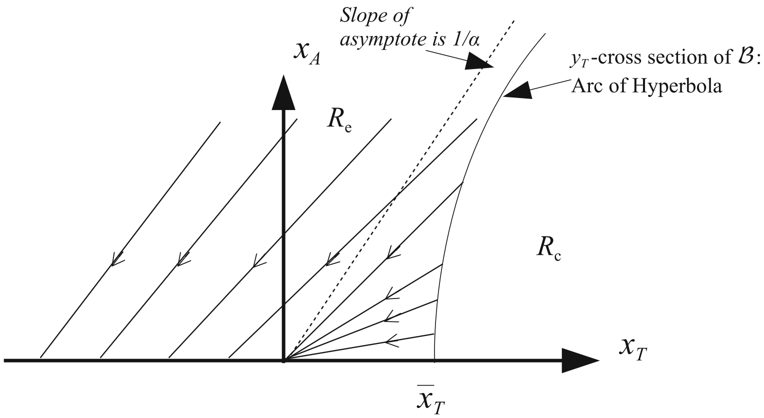


Fig. 10 Projection of the optimal flow field in R_e onto the (x_A, x_T) plane, $y_T > 0$, $\bar{x}_T = \frac{\alpha}{\sqrt{1-\alpha^2}} y_T$

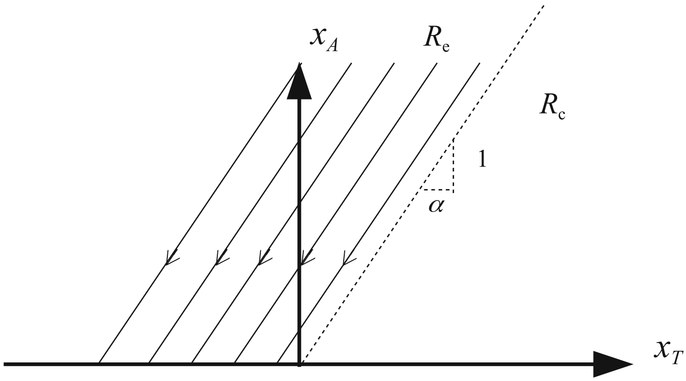


Fig. 11 The optimal flow field in the plane $y_T = 0$

$(\tilde{x}_A, \tilde{y}_A)$, $(\tilde{x}_D, \tilde{y}_D)$, $(\tilde{x}_T, \tilde{y}_T)$ in the realistic plane, the reduced state components are according to Eq. (10)

$$x_A := \frac{1}{2}\sqrt{(\tilde{x}_A - \tilde{x}_D)^2 + (\tilde{y}_A - \tilde{y}_D)^2}$$

$$x_T := \frac{\big(\tilde{x}_T - \frac{1}{2}(\tilde{x}_A + \tilde{x}_D)\big)(\tilde{x}_A - \tilde{x}_D) + \big(\tilde{y}_T - \frac{1}{2}(\tilde{y}_A + \tilde{y}_D)\big)(\tilde{y}_A - \tilde{y}_D)}{\sqrt{(\tilde{x}_A - \tilde{x}_D)^2 + (\tilde{y}_A - \tilde{y}_D)^2}}$$

$$y_T := \frac{\big(\tilde{y}_T - \frac{1}{2}(\tilde{y}_A + \tilde{y}_D)\big)(\tilde{x}_A - \tilde{x}_D) - \big(\tilde{x}_T - \frac{1}{2}(\tilde{x}_A + \tilde{x}_D)\big)(\tilde{y}_A - \tilde{y}_D)}{\sqrt{(\tilde{x}_A - \tilde{x}_D)^2 + (\tilde{y}_A - \tilde{y}_D)^2}}$$

so, in practice, the player's state feedback strategies are functions of the state in the realistic plane $(\tilde{x}_A, \tilde{y}_A, \tilde{x}_D, \tilde{y}_D, \tilde{x}_T, \tilde{y}_T) \in \mathbb{R}^6$. The plots of the optimal trajectories of T, A, and D are shown in Figs. 13, 14 and 15 in the realistic plane.

Example 1 The Target is initially on the Defender's side $x_T(0) < 0$, and as such is in R_e. The initial positions of the players are $T = (-0.25, 2.5)$, $A = (4, 0)$ and $D = (-4, 0)$. The Attacker and the Defender have unit speed and the Target's speed is $\alpha = 0.3$. The real solutions of quartic equation (14) are $y_1 = 2.541 > y_T$ and $y_2 = 2.46 < y_T$; the other

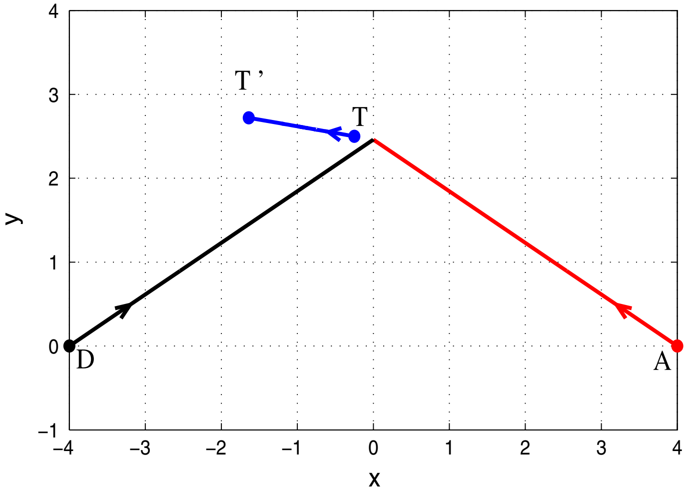


Fig. 12 Example 1: Optimal play, $x_T(0) < 0$

two solutions are complex. The costs are $J(y_1) = 1.6749$ and $J(y_2) = 1.662$. Hence, since $x_T < 0$, the interception point's coordinate is $y^* = 2.46$, the interception time is $t_f = 4.7$ s, and the final separation between Target and Attacker is $J(y^*) = 1.662$. The straight line optimal trajectories are shown in Fig. 12.

Example 2 The Target is initially on the Attacker's side $x_T(0) > 0$. Consider an Attacker and a Defender missile with unit speed and with initial positions given by $A = (7, 0)$ and $D = (-7, 0)$, respectively. The Target is initially located at $T = (3, 5)$. The speed of the Target is $\alpha = 0.55$, so the state is in R_e—see Eq. (100).

The four solutions of quartic equation (14) are the two real solutions $y_1 = 6.171 > y_T$, $y_2 = 4.127 < y_T$, and the complex pair $y_3 = -0.149 + 8.302i$, and $y_4 = -0.149 - 8.302i$. Complex solutions are discarded and the costs corresponding to the real solutions are $J(y_1) = 1.912$ and $J(y_2) = 1.345$. Thus, since $x_T > 0$, the optimal interception point's y-coordinate is $y^* = 6.171$, the interception time is $t_f = 9.332$ s, and the final Target–Attacker separation is $J(y^*) = 1.912$. The players' straight line optimal trajectories are illustrated in Fig. 13.

Example 3 $x_T = 0$. We consider the interesting case where initially the Target is on the orthogonal bisector of $\overline{AD}$ and Proposition 1 is exercised; the Target's initial position is $T = (0, 3.2)$. The initial positions of the Attacker and the Defender are $A = (4.5, 0)$ and $D = (-4.5, 0)$, respectively. The Attacker and the Defender have unit speed and the Target's speed is $\alpha = 0.35$ so the state is in R_e—see Eq. (100). The interception point I's y-coordinate is $y^* = 3.2$, which is the Target's initial y-coordinate. The optimal Target heading is given by Eq. (82) and results in $\varphi = 1.366$ ($\phi = 2.937$ in the realistic plane). The interception time is $t_f = 5.52\ sec$, and the final separation between Target and Attacker is $J(y^*) = 1.93$. The optimal trajectories are shown in Fig. 14.

The point of Examples 1–3: Quartic equation (14) has real solutions as long as the state is in R_e; and we have obtained the optimal solution of the ATDDG for the whole of R_e.

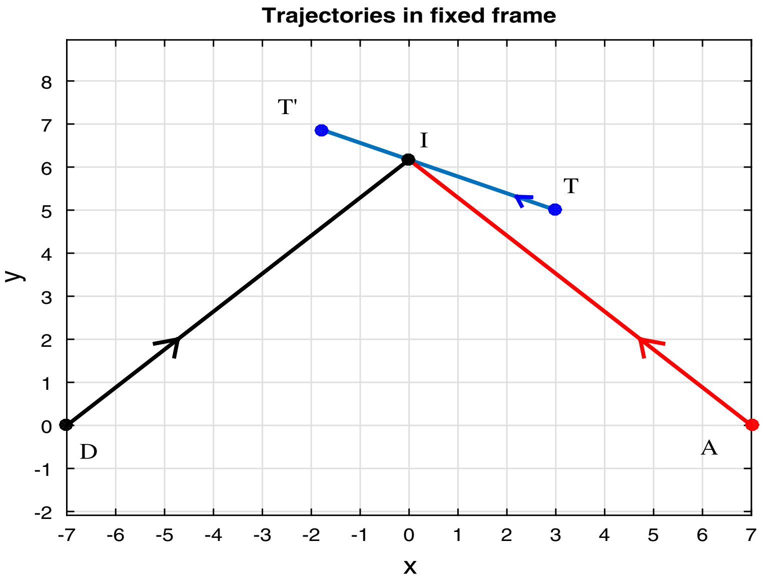


Fig. 13 Example 2: Optimal play, $x_T(0) > 0$

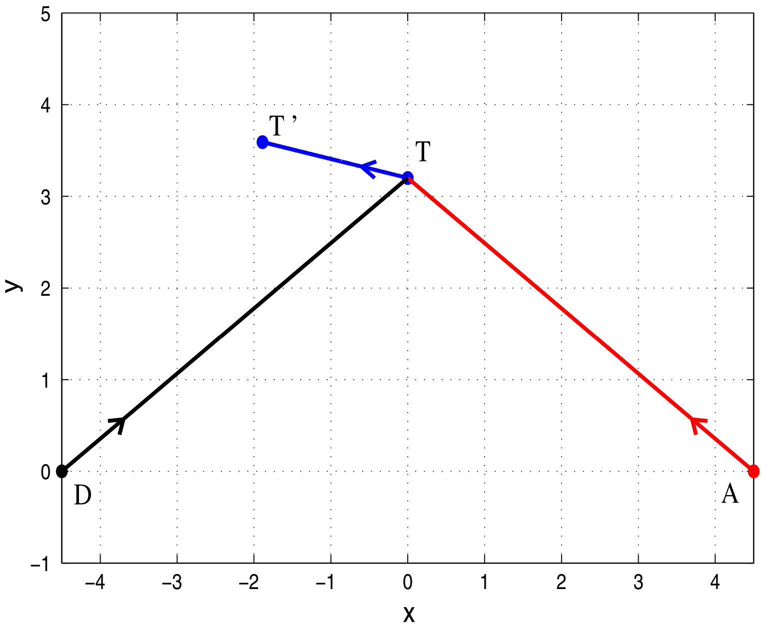


Fig. 14 Example 3: Optimal play, $x_T(0) = 0$

Example 4 Robustness to unknown Attacker guidance law. An important characteristic of the optimal solution of the cooperative T & D guidance laws in our zero-sum target defense differential game, which hinges on the solution of quartic Eq. (14) [and Eq. (82), when needed], is the fact that these are optimal state feedback strategies. As such, they naturally

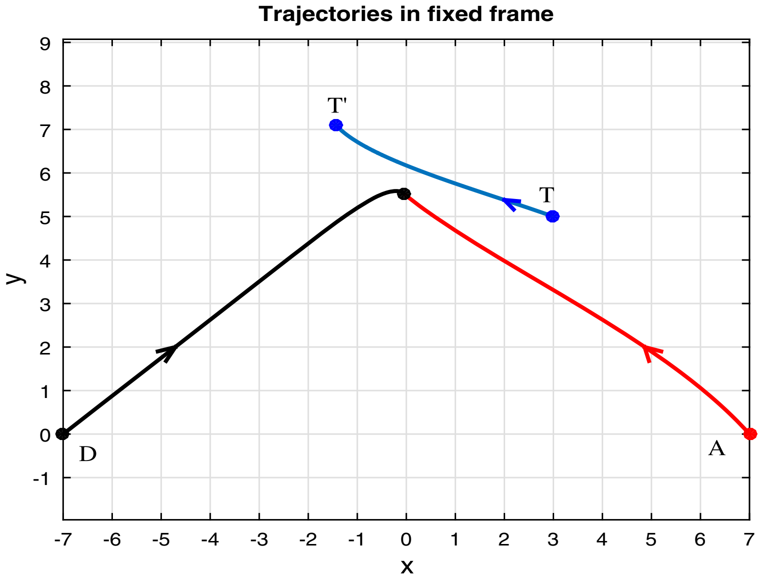


Fig. 15 The optimal T-D cooperative strategy is used in closed loop against a suboptimal A-strategy

render the target defense robust to unknown Attacker guidance laws. This means that if the Attacker does not employ its optimal strategy and instead uses a suboptimal guidance law that is completely unknown to the Target–Defender team then the Target and the Defender (having current measurements of the Attacker' position $(\tilde{x}_A, \tilde{y}_A)$) will at each time instant perform state transformation (10) to obtain (x_A, x_T, y_T), solve in real-time quartic equation (14), and by doing so implement their respective optimal strategies and continuously update their controls, thus increasing the $T - A$ separation at interception time above and beyond the guaranteed Value of the game.

Let us revisit the pursuit-evasion scenario from Example 2 where now the Attacker implements a PN guidance law with navigation constant $N = 3$. This information is not available to the Target–Defender team which however is able to measure the current position of the Attacker, $(\tilde{x}_A, \tilde{y}_A)$. By continuously updating their headings upon performing state transformation (10) and solving quartic equation (14), the Target/Defender team is able to obtain a better outcome—increase the final $A - T$ separation. The (non-rectilinear) trajectories are shown in Fig. 15. The coordinate frame (x_A, x_T, y_T) is now rotating with respect to the fixed, realistic, plane because one of the players, the Attacker, did not implement its saddle point strategy. The interception time is $t_f = 9.01$ *sec*. The final separation between Target and Attacker is $R(t_f) = 2.09 > J(y^*) = 1.912$ ($J(y^*)$ was obtained in Example 2), and, as expected, it is more than had the Attacker played optimally. Note however that if the T & D team would have known the Attacker's PN strategy they would have done even better, as exemplified in [11].

Value function. To illustrate the smoothness of value function (17) we plot its x_A-cross section; the T/A speed ratio is $\alpha = 0.5$ and the state component $x_A = 5$ is fixed. Figure 16 shows a 3-D plot of value function (17) for $x_T \in [-2, 2]$ and $y_T \in [0, 5]$. Figure 17 shows (x_A, x_T)-cross section of the value function for a few values of x_T and for $x_A = 5$. In particular the value function for $x_T = 0$ is shown for different values of y_T. Similarly,

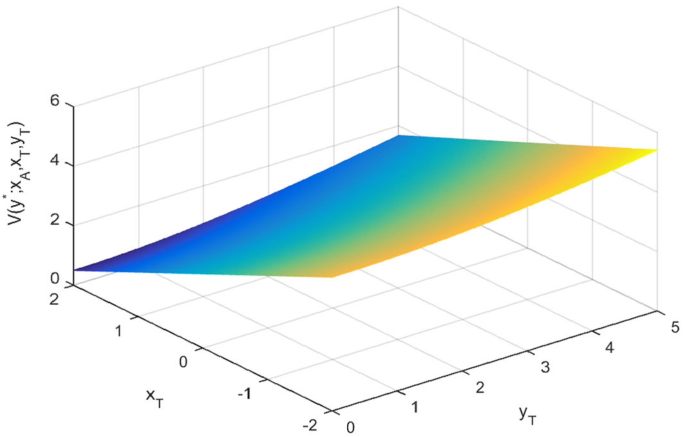


Fig. 16 $V(x_A, x_T, y_T; \alpha)$: x_A cross section, $x_A = 5, \alpha = 0.5$

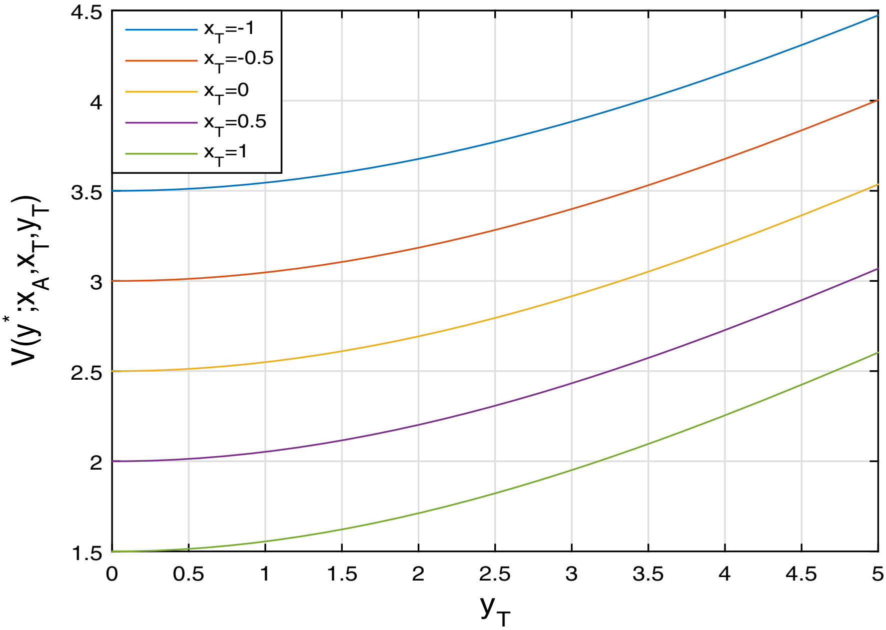


Fig. 17 $V(x_A, x_T, y_T; \alpha)$ for $x_A = 5$ and $-1 \leq x_T \leq 1; \alpha = 0.5$

Fig. 18 shows y_T cross sections of the value function. The value function is C^1 for every value of y_T, including on the plane $x_T = 0$.

8 Discussion and Extensions

The Active Target Defense Differential Game (ATDDG) which is played in the R_e part of the state space has military applications. Thus, in the ATDDG discussed in this paper the

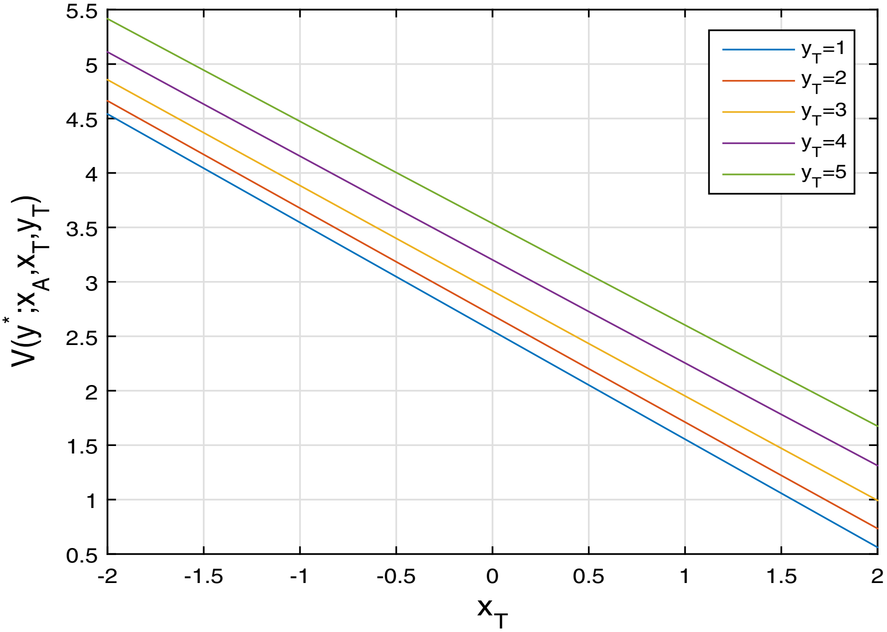


Fig. 18 $V(x_A, x_T, y_T; \alpha)$ for $x_A = 5$ and $1 \leq y_T \leq 5$; $\alpha = 0.5$

speed of the "pursuer," a.k.a., the Defender, is not less than the speed of the "evader," a.k.a., the Attacker; when the speeds of the Attacker and of the Defender are equal their respective instantaneous regions of dominance in the Euclidean plane are separated by a straight line, and if the Defender is faster than the Attacker the straight line will be replaced by an Apollonius circle; of course, if the Defender is only slightly faster than the Attacker, the circular boundary will be close to a straight line.

The assumption in the ATDDG that the speed of the Defender is not less than the speed of the Attacker is important. In pursuit-evasion differential games with simple motion, optimal trajectories which are not straight lines and singular equivocal surfaces occur when the evader—in the ATDDG the Attacker—is faster than the pursuer - in the ATDDG the Defender—and the latter comes in "safe contact" with the evader. In this context, a special case will arise when there are obstacles, which can be viewed as capture zone-endowed stationary "pursuers," a.k.a., stationary Defenders, and thus are indeed slower than the "evader"—the Attacker in the ATDDG. In this case the Attacker always captures the (slower) Target and his optimal trajectory consists of a polygonal path, and also a curved arc when the pursuer circumnavigates a convex obstacle—think of a circular disk. This brings us to the Obstacle Tag differential game which was posed by Isaacs and further investigated by Melikyan in [20]. In this differential game an intuitively optimal pursuer's path is a straight line which is tangent to the circular obstacle/stationary Defender, followed by a circumnavigation of the circular obstacle and then a straight line dash toward the Target, which all along follows a straight line path. However, Isaacs realized that this is not the end of the story. Indeed, there also are instances, as for example in a symmetric configuration, where the pursuer, the center of the circular obstacle, and the evader, are collinear. In this case there is a possibility that initially the pursuer will head straight toward the center of the circular obstacle and move on an equivocal surface before breaking off, going on a tangent, and then circumnavigate the circular obstacle, followed by a terminal dash to the fleeing Target; the Target will also initially move along an equivocal path and away from the obstacle. Speaking of symmetric configurations: In Breakwell and Hagedorn's differential game of point capture of two evaders in succession [7], in a generic configuration, the second evader runs away from the point of interception of the first evader and the motion is rectilinear, as in our ATDDG. However, similar to the Obstacle Tag differential game, a symmetric configuration where the pursuer finds himself equidistant from the two evaders might also arise and then a singular equivocal surface comes into play. Indeed, it is worthwhile to elucidate how the curved optimal trajectory and singular surface arise in the Obstacle Tag differential game—this , notwithstanding the protagonists simple motion. We emphasize here the role played by symmetry, which brings about the existence of an equivocal (singular) surface: There are configurations where taking the high road or the low road about the obstacle is isochronous, yet upon having taken for a while the high road it then becomes obvious that the low road yields a more direct straight line path to the evader and hence a shorter time to capture, and vice versa. This brings about a sliding motion along a possibly curved trajectory, and the existence of a singular surface. However, all this is not relevant to the ATDDG discussed in this paper for the following reasons.

All of this collapses in our ATDDG, where the Defender is *not* slower than the Attacker: The circular "obstacle" is mobile and is now the capture zone of a Defender which is of equal or higher speed than the Attacker. Should then the Attacker now come in contact with the faster Defender's circular capture zone, a.k.a., a mobile "obstacle" which has simple motion/is holonomic, he is instantaneously captured; and if the speed of the Defender happens to be equal to the speed of the Attacker, the latter then comes under the control of the Defender and will be pushed around by the Defender—in both instances the game is over and the Target is saved.

The relation to Isaacs' Obstacle Tag differential game where the pursuer and the evader have simple motion, like the protagonists in the ATDDG, is interesting. However, the correct association of the Obstacle Tag differential game considered in [20] and our ATDDG must be made. When making the connection one should correctly identify the pursuer and the evader: In the ATDDG the "pursuer" is the Defender and he strives to interpose himself between the Attacker, which is now the "evader," and the fleeing Target. In the ATDDG the Attacker is indeed faster than the Target which could even be stationary, but he is not faster than the Defender, who in the ATDDG fulfills the role of the "pursuer," whereas the Attacker is now the slower "evader." So, when viewing Melikyan's Obstacle Tag differential game as an ATDDG and by way of a correct analogy, the role of the Defender would then be assumed by the circular obstacle which stands in the way of the Attacker who is in pursuit of the Target. But in the ATDDG the Defender is the "pursuer," and in Melikyan's Obstacle Tag differential game he is stationary. Hence, when viewed as an ATDDG, the velocity of the "pursuer" in the Obstacle Tag game is less than the velocity of the "evader"/the Attacker—it is zero! This is conducive to a singular equivocal surface, as is indeed the case in the Obstacle Tag differential game where the Attacker/evader is faster than the stationary Defender. But in our ATDDG the Defender is not slower than the Attacker and is certainly not stationary. Therefore, in the ATDDG, there are no singular surfaces and curved optimal trajectories. And if, as in the ATDDG where point capture is required, the stationary Defender/obstacle is not endowed with a circular capture zone, there is no obstacle to speak of. The Obstacle Tag differential game does not inform on the ATDDG.

Since in the ATDDG the Defender is not stationary or slower than the Attacker, contact with the Defender is fatal to the Attacker—"safe contact" with an attendant curved trajectory is not possible. Also note that in the ATDDG we are interested in point capture of the Attacker by the Defender. Interestingly, similar to Breakwell and Hagedorn's [7] game of point capture of two evaders in succession, in the ATDDG the Target runs away from the point of interception of the Attacker by the Defender - this, when initially the Target is on the side of the Defender. However, when the Target is on the side of the Attacker it actually heads toward the point of interception of the Attacker by the Defender. So, as in Breakwell and Hagedorn's game, the generic solution of running away from the point of interception of the Attacker by the Defender does not always apply, but at no time do the players' paths become curved. Also, unlike the obstacle/tag and sequential pursuit games, in the ATDDG symmetric configurations do not arise. Hence, in the ATDDG, where the Attacker is not faster than the Defender, there are no singular surfaces. In our ATDDG there is however a surface $\mathcal{B}$ which separates the winning regions of the Attacker and the Defender, akin to a closed barrier in pursuit-evasion games. In the Obstacle Tag differential game no such surface exists because the pursuer's winning region is the whole Euclidean plane.

In the ATDDG the construction of the optimal paths in R_e is always possible, both when the Target is on the side of the Defender and when the Target is on the side of the Attacker: When the Target is on the side of the Defender, the Target–Defender team always wins. When the Target is on the side of the Attacker the Target–Defender team wins, that is, the Target gets away, provided the Apollonius circle based on the position of the Attacker and the Target intersects the orthogonal bisector of the segment connecting the positions of the Attacker and the Defender, that is, the ATDDG Game of Degree is played in R_e. The whole R_e region of state space, as specified in Theorem 1, is covered by the field of primary optimal trajectories. The geometric construction of the optimal trajectories and the synthesis of the optimal state feedback strategies is contingent on the integration of a set of 2n ODEs in order to solve the hyperbolic HJBI PDE of Dynamic Programming which the value function of the differential game satisfies. This is tantamount to solving the PDE of Dynamic Programming using the method of characteristics—as discussed in Sect. 3. Most importantly, the construction is globally applicable in the whole state space when $x_T \leq 0$ and when $x_T > 0$ in the R_e part of the state space specified by Proposition 1. Since in the ATDDG point capture of the Attacker by the Defender is considered and the Attacker, namely, the "evader," is not faster than the Defender, namely, the "pursuer," there are only primary optimal trajectories - no secondary optimal trajectories, no tributaries, and the characteristics do not intersect. Also, there are no holes in the state space to be filled with optimal trajectories and so singular surfaces are not required. And because there are no singular surfaces, patching together different regions of the state space is not required and the geometric construction of the optimal trajectories is globally applicable in the whole state space specified by the solution of the Game of Kind. The players' optimal state feedback strategies are given in closed form and the value function is C^1. While Isaacs' method has the looks of a "two-sided" PMP, the 2n differential equations are the same—our solution is valid not by virtue of the "two-sided" PMP, but because we have employed Isaacs' method for the solution of the HJBI PDE and only primary optimal trajectories are at work. We have a complete set of optimal trajectories filling the escape region R_e which is separated from the Attacker's winning region R_c by a 2-D surface $\mathcal{B}$.

In summary, in the ATDDG with simple motion where the Defender is not slower than the Attacker and point capture is considered, our solution is complete because we have applied Isaacs' method. There are no singular surfaces, there are no curved optimal trajectories—the players' optimal trajectories are rectilinear and the synthesized value function is C^1. The Obstacle Tag differential game [20] has no bearing on our ATDDG. The supposition that in the ATDDG there might be singular surfaces and curved trajectories appears to be unfounded.

As for the future, the critical hypothesis that the speed of the Defender is not less than the speed of the Attacker will be relaxed. In future work the case where the Attacker is faster than the Defender will be addressed. Then, in the special case where the Defender is stationary, but he is endowed with a capture circle of radius $l > 0$, we are back to the Obstacle Tag differential game and singular surfaces and curved optimal paths are possible, despite the simple motion kinematics. However, if the Defender is stationary but, as in our ATDDG, point capture is required, we have an Obstacle Tag differential game with a vanishing circular obstacle ($l = 0$). This entails trivial strategies—we refer to pure pursuit and evasion: The presence of a stationary Defender with no capture zone is tantamount to no obstacle being present. In general, when the Defender's speed is not zero but is less than the speed of the Attacker, the Defender cannot capture the Attacker: During a period of "safe contact" with the Defender the Attacker will circumnavigate the Defender's capture circle of radius $l > 0$ until he can safely break contact with the Defender and directly head straight toward the (slower) Target. However, when the Attacker is faster than the Defender, but the latter is endowed with a circular capture zone of radius $l > 0$, the ATDDG acquires some of the characteristics of Isaacs' Obstacle Tag differential game. The obstacle is no longer stationary and is controlled by a Defender. Some of the optimal trajectories will no longer be rectilinear and as in the classical Obstacle Tag differential game, singular surfaces of the equivocal type will show up—the Obstacle Tag differential game is now a special case of the ATDDG. This interesting scenario will be investigated in future work. However, if as in our ATDDG and in [7], point capture of the Attacker by the slower Defender is considered, then in case of contact the fast Attacker will in no time "circumnavigate" the Defender whose circular capture set radius is $l = 0$. Hence, when the Attacker is faster than the Defender and like in our ATDDG point capture is required, the Attacker can circumnavigate the Defender instantaneously and head straight toward the Target as in pursuit-evasion differential games with simple motion where Pure Pursuit (PP) is used. When the Attacker is faster than the Defender and point capture is required the presence of the Defender is inconsequential and the ATDDG is trivial; only a Defender capture radius $l > 0$ renders the game interesting.

In the case where the Defender is endowed with a capture zone of radius $l > 0$ and, in addition, has the same speed as the Attacker, the strategies are similar to the strategies in this paper. The orthogonal bisector of the segment $\overline{AD}$ is then replaced by an arc of a hyperbola. The calculation of the corresponding optimal interception point will then require rooting an 8-th-order polynomial equation.

Also the Attacker might be endowed with a circular capture zone. Interestingly, the symmetry responsible for singular equivocal surfaces and curved optimal trajectories in the Obstacle Tag differential game and also in the ATDDG where the Attacker is faster than the capture circle endowed Defender, is not of the type which causes a perpetual dilemma; the latter would be easily resolved by having the Attacker endowed with a small capture set. Unfortunately, this is not the case here.

If the Defender is slower than the Attacker, but he is endowed with a circular capture zone of radius $l > 0$, one might also consider the alternative defensive tactic where the Target runs toward the Defender to seek protection under his defensive umbrella. We will then be considering a Defender–Target rendezvous game where the question is posed whether the Target can come within range l of the Defender before being intercepted by the Attacker. Thus, an alternative target defense differential game where the Defender is now endowed with a non-vanishing circular capture zone which can serve as a defensive umbrella for the Target and where the Target and Defender collaborate to *rendezvous* before the Target is captured by the Attacker will be considered.

In this regard, one is also motivated to consider the following simpler differential game. One employs the geometry of the Obstacle Tag differential game where the circular "obstacle" of radius l, a stationary Defender's capture zone, now serves as the Target's defensive umbrella, i.e., the Target's safe haven. The pursuit-evasion differential game is considered where an evader—previously the Target and now the "Attacker"—strives to come as close as possible to the circular "obstacle"—now the safe haven - before being intercepted by the pursuer—previously the Attacker and now the "Defender" of the safe haven. The circular obstacle can be viewed as a Target set, the evader who tries to reach said Target can be considered an "Attacker" and the pursuer, whose job is to capture the evader before him reaching the Target, is now the "Defender." If the Attacker, namely, the previously fleeing Target, is not faster than the Defender, we have the differential game of guarding a (stationary) Target [22]. Recall that the fleeing Target is slower than the Attacker; should the "Attacker" be faster than the "Defender," he can always safely reach the Target. Finally, if also the Attacker is endowed with a circular capture zone and it is bigger than the Defender's, the rendezvous tactic is out of the question.

9 Conclusions

In this paper a differential game formulation for the active aircraft defense scenario is promulgated and progress toward its solution is reported. Simple motion kinematics á la Isaacs are assumed—this being compatible with the current Concept of Operations (CONOPs) of long-range, Beyond Visual Range (BVR) engagements. It is here assumed that at all time the Attacker is aware of the Defender's position, meaning that the target defense differential game is a perfect information game. In addition, the attacker is cognizant of the possible cooperation between the Target and Defender. Progress toward the analytic/closed-form solution of the novel ATDDG is demonstrated for the case of point capture and where the speed of the Defender is equal to (or is higher) than the speed of the Attacker. The solution of the Game of Kind yields the 2-D surface in the 3-D state space that delimits the state space region where the ATDDG Game of Degree is played. The optimal cooperative state feedback strategies for the Target and the Defender team to maximize the Target–Attacker separation at the time instant when the Defender intercepts the Attacker, and the optimal state feedback strategy for the Attacker to minimize same separation, are provided. The air vehicles' instantaneous headings are given in the form of state feedback control laws, albeit under *optimal* play the agents' trajectories in the realistic plane are straight lines. There are no singular surfaces and the explicitly obtained value function is C^1. The analytic results are highlighted in the examples included herein. In zero-sum differential games, the optimal strategies are security strategies—the calculated value of the zero-sum target defense differential game provides the players with a guarantee: If one of the Target–Defender team members does not employ his optimal strategy, an Attacker which uses the optimal state feedback strategy provided by the solution of the zero-sum differential game will have his cost reduced below the Value of the game, which is his upper limit guarantee; in other words, the Attacker will come closer to the Target where it is intercepted by the Defender, or might even be able to intercept the Target. Conversely, if the Attacker does not employ his optimal state feedback strategy but instead uses a conventional PP or PN guidance law, this will be beneficial to the Target–Defender team which employs the herein obtained state feedback strategy provided by the solution of the zero-sum differential game. This is so because the Target–Attacker separation at the latter's interception time will increase above the Value of the game, which is the Target–Defender team's lower limit guarantee. At the same time, the Target–Defender team could do even better if they knew ahead of time that the Attacker will use a particular suboptimal strategy, say, PP or PN—the ensuing one-sided optimization problem is then solved using the theory of optimal control.

In this paper the solution of a differential game with military applications has been advanced. In the reduced state space the ATDDG has three states. We contribute an additional example to the, admittedly small, repertoire of pursuit-evasion differential games in 3-D which can be solved in closed form.

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Appendix

Root locus analysis of the quartic equation

The problem parameter is $0 \leq \alpha \leq 1$. When $\alpha = 0$ the four roots of quartic equation (14) are: $y = y_T$, $y = y_T$, and $y = \pm i x_A$, for all $x_T \in \mathbb{R}^1$. When $\alpha = 1$ we have a quadratic equation whose roots are $y = \frac{x_A y_T}{x_A + x_T}$ and $y = \frac{x_A y_T}{x_A - x_T}$. If $x_T = \pm x_A$, then one of the roots is given by $y = +\infty$. This root takes a finite value as soon as α decreases, that is, $y < \infty$ for $\alpha = 1 - \epsilon$ for some small $\epsilon > 0$.

When the discriminant of the quartic equation $\Delta < 0$, the quartic equation has two real roots and a pair of complex roots. We already know that Eq. (14) has two real roots, $0 < y < y_T$ and $y > y_T$ as is indeed required by our theory for the case where $x_T < 0$ and $x_T > 0$, respectively. We will investigate quartic equation (14) using the continuation method, starting out from $\alpha = 0$ where we have a double real root $y = y_T$—we conduct a root locus investigation for $0 \leq \alpha < 1$.

To this end, differentiate quartic equation (14) in α

$$\begin{aligned}&\left\{2(1-\alpha^2)y^3 - 3(1-\alpha^2)y_T y^2 + \left[(1-\alpha^2)y_T^2 + x_A^2 - \alpha^2 x_T^2\right] y - x_A^2 y_T\right\} \frac{\mathrm{d}y}{\mathrm{d}\alpha}\\&\quad - \alpha\left[y^4 - 2y_T y^3 + \left(x_T^2 + y_T^2\right) y^2\right] = 0.\end{aligned} \tag{107}$$

Setting $\alpha = 0$ and $y = y_T$, yields the relationship $0 \cdot \frac{\mathrm{d}y}{\mathrm{d}\alpha} = 0$ and we cannot calculate $\frac{\mathrm{d}y}{\mathrm{d}\alpha}\big|_{\alpha=0}$. Hence, differentiate Eq. (107) in α again and set $\alpha = 0$

$$\begin{aligned}&\left[2y^3 - 3y_T y^2 + \left(y_T^2 + x_A^2\right) y - x_A^2 y_T\right] \frac{\mathrm{d}^2 y}{\mathrm{d}\alpha^2} + \left(6y^2 - 6y_T y + x_A^2 + y_T^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}\alpha}\right)^2\\&\quad - \left[y^4 - 2y_T y^3 + \left(x_T^2 + y_T^2\right) y^2\right] = 0.\end{aligned} \tag{108}$$

Set $y = y_T$ in Eq. (108)

$$0 \cdot \frac{d^2 y}{d\alpha^2} + \left(x_A^2 + y_T^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}\alpha}\right)^2 = x_T^2 y_T^2 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}\alpha}\Big|_{\alpha=0,\, y=y_T} = \pm\frac{x_T y_T}{\sqrt{x_A^2 + y_T^2}}. \tag{109}$$

Also, setting $\alpha = 0$ and $y = \pm i x_A$ in Eq. (107) yields

$$\frac{\mathrm{d}y}{\mathrm{d}\alpha}\Big|_{\alpha=0,\, y=\pm i x_A} = 0.$$

Note that setting $y = \pm i x_A$ in Eq. (108) yields

$$\left[2x_A^2 y_T + \left(y_T^2 - x_A^2\right) y\right] \frac{\mathrm{d}^2 y}{\mathrm{d}\alpha^2} = \left(x_A^2 - x_T^2 - y_T^2 + 2y_T y\right) x_A^2$$

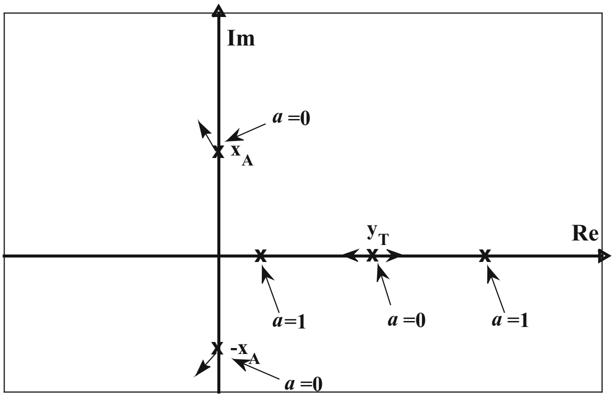

Fig. 19 Root locus of quartic equation (14)

Let $\frac{\mathrm{d}^2 y}{\mathrm{d}\alpha^2} = a + ib$. We calculate

$$\left[2x_A^2 y_T + i x_A \left(y_T^2 - x_A^2\right)\right](a+ib) = (x_A^2 - x_T^2 - y_T^2 + 2i x_A y_T)x_A^2$$
$$\Rightarrow \begin{bmatrix} 2x_A y_T & x_A^2 - y_T^2 \\ -\left(x_A^2 - y_T^2\right) & 2x_A y_T \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = x_A \begin{bmatrix} x_A^2 - x_T^2 - y_T^2 \\ 2x_A y_T \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\left(x_A^2 + y_T^2\right)^2} \begin{bmatrix} 2x_A y_T & -\left(x_A^2 - y_T^2\right) \\ x_A^2 - y_T^2 & 2x_A y_T \end{bmatrix} \begin{bmatrix} x_A^2 - x_T^2 - y_T^2 \\ 2x_A y_T \end{bmatrix} x_A$$
$$\Rightarrow$$
$$a = -2\left(\frac{x_A x_T}{x_A^2 + y_T^2}\right)^2 y_T < 0$$

$$b = \frac{\left(x_A^2 + y_T^2\right)^2 - \left(x_A^2 - y_T^2\right)x_T^2}{(x_A^2 + y_T^2)^2} x_A$$

Similarly,

$$\left[2x_A^2 y_T - i x_A \left(y_T^2 - x_A^2\right)\right](a+ib) = \left(x_A^2 - x_T^2 - y_T^2 - 2i x_A y_T\right)x_A^2$$
$$\Rightarrow \begin{bmatrix} 2x_A y_T & -\left(x_A^2 - y_T^2\right) \\ x_A^2 - y_T^2 & 2x_A y_T \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = x_A \begin{bmatrix} x_A^2 - x_T^2 - y_T^2 \\ -2x_A y_T \end{bmatrix}$$
$$\Rightarrow$$
$$a = -2\left(\frac{x_A x_T}{x_A^2 + y_T^2}\right)^2 y_T < 0 \quad \text{as before!}$$
$$b = -\frac{\left(x_A^2 + y_T^2\right)^2 - \left(x_A^2 - y_T^2\right)x_T^2}{\left(x_A^2 + y_T^2\right)^2} x_A; \quad \text{note sign inversion!}$$

So the two imaginary roots wander as a pair into the LHS of the complex plane, whereas the two real roots split off at $y = y_T$ and move in opposite direction toward $\frac{x_A y_T}{x_A \pm x_T}$. The root locus picture of the roots of quartic equation (14) as a function of the problem parameter

$0 \leq \alpha < 1$ is shown in Fig. 19. Importantly, our quartic Eq. (14) has two complex roots and two real roots $y_1 \geq y_T$ and $y_2 \leq y_T$, as required by the theory of the ATDDG.

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