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Robust Policies for a Multiple-Pursuer Single-Evader Differential Game

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Abstract

Analysis of the pursuit–evasion differential game consisting of multiple pursuers and single evader with simple motion is difficult due to the well-known curse of dimensionality. Policies have been proposed for this scenario, and we show that these policies are global Stackelberg equilibrium strategies. However, we also show that they are not saddle-point equilibria in the feedback sense. The argument is twofold: cases where the saddle-point condition is violated and cases where the strategy profiles are not time consistent (subgame perfect). The issue of capturability is explored, and sufficient conditions for guaranteed capture are provided. A new pursuit policy is proposed which guarantees capture while also providing an upper bound for capture time. The evader policy corresponding to the global Stackelberg equilibrium is shown to provide a lower bound for capture time. Thus, these policies are robust from the pursuer and evader perspectives, respectively, should they implement them. Several other interesting pursuit and evasion policies are explored and compared with the robust policies in a series of experiments.

Keywords Pursuit evasion · Differential game · Multiple pursuers · Robust

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1 Introduction

Pursuit–evasion games are, as a field of study, as old as differential games itself. They were the subject of many examples in the early works of Isaacs [16] and his later book [17]. The application of differential game theory to problems of military interest is immediately obvious in the context of pursuit–evasion games. One could imagine that the solutions to simple pursuit–evasion games in the plane may one day be expanded to equilibrium strategies for a real-world aerial dogfighting scenario. We see the problem of minimum time capture of a single evader by multiple pursuers as a stepping stone to the more difficult problem involving multiple evaders *and* multiple pursuers. The latter has clear applicability to large missile-on-missile engagements involving multiple missiles on either side—in particular, beyond-visual-range (BVR) missiles. Isaacs posed the single-pursuer single-evader pursuit–evasion differential game with simple motion—that is, the agents play on an unbounded plane in $\mathbb{R}^2$, have fixed velocities, and no turn constraints—and gave its solution for the case of a faster pursuer [17]. The solution, in this case, is quite intuitive: the pursuer should aim line of sight and the evader should similarly aim away from the pursuer along this line. A natural extension, then, is to consider the two-pursuer single-evader game (2P1E) which Isaacs referred to as the "two cutters and fugitive ship game" [17]. Apparently, the two-on-one game was also considered by Hugo Steinhaus toward the beginning of the development of game theory. Isaacs proposed the solution of the 2P1E game which was for all three agents to head to the further of the two Apollonius circle intersections, but he did not prove the result. Recently, Isaacs' geometrically intuitive policy was reanalyzed and verified to be the solution to the game by Garcia et al. [13].

Many other extensions to these two simple pursuit–evasion games have been explored. Some, like the immensely complex homicidal chauffeur game, have introduced different dynamics for one of the agents [3,17,22]. Others have flipped the numbers advantage to the evasion side and considered a single pursuer in the presence of multiple evaders [4,12,21]. Still others have considered obstacles and bounded environments [5,15,23,24].

Although Isaacs' work on differential games mostly focused on analytical methods, many have turned to numerical approaches for analysis. Notable categories for these numerical approaches include so-called viscosity solutions [2,8], decomposition methods [9,10], or just numerical integration of the backward partial differential equations (c.f. [12]). LQ game theory has also been used to skirt the issues inherent with coupled nonlinear partial differential equations (c.f [19]).

From a cooperative control standpoint, the multiple-pursuer and/or multiple-evader extensions are of primary interest. A full solution to the M-on-N (or MPNE) game is desired even for the case of simple motion on an unbounded plane; however, due to the curse of dimensionality, this is seemingly intractable. The same is still true for the case of MP1E, and much of the existing literature has focused on numerical techniques and/or sequential pursuit (as in [1,26]). Recently, along similar lines to Isaacs' arguments for the 2P1E game, a geometric policy has been proposed for the MP1E game which makes use of both the Apollonius circles of the pursuers as well as the Voronoi diagram created by the pursuers [27,28]. The reference [18] provides a survey of many works in the area of zero-sum differential games with many objects. That work discusses many of the same papers mentioned above but places the MP1E game of minimum time capture in a much broader context.

In this work, we highlight the fact that the full solution to the MP1E game is still at large. The geometric policy proposed by [27,28] is shown to be a global Stackelberg equilibrium strategy pair. For many games, it is the case that the global Stackelberg equilibrium is coincident with the feedback Nash equilibrium (i.e., the "full solution" of the game). However, we show that it is not the case for the MP1E game: the geometric policy violates the saddle-point condition necessary to be a feedback Nash equilibrium. This is shown through a counterexample which brings to light the deficiency of the geometric policy. One redeeming property of the geometric policy, though, is that it is robust from the evader's perspective. That is, the evader can do no worse if it implements this policy. A new pursuit policy is proposed which carries the same sort of robustness but from the pursuers' side. We address the deficiencies of the geometric policy to an extent; however, it is clear that the feedback Nash equilibrium strategies have yet to be discovered.

The remainder is organized as follows. Section 2 introduces the MP1E differential game as well as the evader's Safe Region, a geometric construction used throughout. Section 3 summarizes the geometric policy introduced by Von Moll et al. [27,28] and gives a proof that it corresponds to the global Stackelberg equilibrium. Capturability is discussed in Sect. 4 as it is a pertinent topic when discussing non-feedback-equilibrium strategies. Section 5 proposes a new pursuit policy which has robustness guarantees. Conclusions and some remarks about future research are contained in Sect. 8.

2 Technical Preliminaries

The pursuit–evasion scenario we consider is defined by the following kinematics

$$\begin{aligned} \dot{x}_E &= V_E \cos\phi, & x_E(t_0) &= x_{E_0}, \\ \dot{y}_E &= V_E \sin\phi, & y_E(t_0) &= y_{E_0}, \\ \dot{x}_i &= V_P \cos\psi_i, & x_i(t_0) &= x_{i_0}, \\ \dot{y}_i &= V_P \sin\psi_i, & y_i(t_0) &= y_{i_0}, \qquad i = 1, \ldots, M \end{aligned} \tag{1}$$

where E denotes the evader and the subscripts i denote the ith pursuer, of which there are M, $V_k,\ k = E, P$ are the speeds, ϕ is the evader's heading angle, and ψ_i are the pursuers' heading angles, $i = 1, \ldots, M$. Let

$$\mathbf{x} = (x_E, y_E, x_1, y_1, \ldots, x_M, y_M)\,,\ \mathbf{x} \in \mathbb{R}^{2M+2}$$

be the state of the system. The size of $\mathbf{x}$ may be reduced to $2M$ states by considering a relative coordinate system whose origin is fixed to the evader. This reduction is of little consequence: with $M = 3$, one still ends up with six states. Few differential games with more than two states have been solved analytically (see, e.g., [13,14,22]). Therefore, we retain the kinematics in Eq. (1) for the remainder. We are interested only in the case where $V_P > V_E$ since, as will be shown in the sequel, capture may be guaranteed for certain pursuit strategies. Let $\alpha = V_E/V_P < 1$ denote the speed ratio.

The game is over when one or more pursuers capture the evader; the capture time T is

$$T = \min\{t \mid \exists\, i\ s.t.\ (x_i(t), y_i(t)) = (x_E(t), y_E(t))\} \tag{2}$$

This capture criterion is commonly referred to as point capture, as opposed to scenarios where the pursuers have a nonzero capture radius. The pursuer team seeks to minimize the time to capture, while the evader seeks to maximize the time. We model the scenario as a two-player zero-sum differential game, wherein the pursuers, cooperating as a single entity, and evader seek to minimize/maximize the following cost/payoff, respectively,

$$J(\mathbf{u}_E(\mathbf{x}), \mathbf{u}_P(\mathbf{x})) = \int_0^T \mathrm{d}\tau \tag{3}$$

where $\mathbf{u}_E = \phi \in [0, 2\pi)$ is the evader's control policy and $\mathbf{u}_P = (\psi_1, \ldots, \psi_M)$, $\psi_i \in [0, 2\pi)$ is the pursuers' control policy in state-feedback form. The value function describes the minimax value of the cost/payoff, Eq. (3), when starting from some point $\mathbf{x}_0$ in the state space

$$V(\mathbf{x}_0) = \min_{\mathbf{u}_P} \max_{\mathbf{u}_E} \int_0^T \mathrm{d}\tau \tag{4}$$

Note that the min and max are interchangeable [17]. Such a function, if it exists, is continuous and continuously differentiable in $\mathbf{x}$ and satisfies the so-called Isaacs equation [3]:

$$\min_{\mathbf{u}_P} \max_{\mathbf{u}_E} \left[\frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}_P, \mathbf{u}_E) + g(\mathbf{x}, \mathbf{u}_P, \mathbf{u}_E) \right] = -\frac{\partial V}{\partial t} \tag{5}$$

where $\frac{\partial V}{\partial \mathbf{x}}$ is a vector of derivatives of the value function w.r.t. each state, $f(\mathbf{x}, \mathbf{u}_P, \mathbf{u}_E) = \dot{\mathbf{x}}$ are the kinematics (Eq. 1), and g is the integrand of the cost/payoff in Eq. (3). Thus, we have $g = 1$ and $\frac{\partial V}{\partial t} = 0$. Solving a differential game entails solving the Game of Kind (determining regions of the state space in which one or the other player wins) and solving the Game of Degree (i.e., determining the value of the game in each region of the state space). For pursuit–evasion, the Game of Kind asks whether capture is inevitable or escape is inevitable. Since we have restricted $\alpha < 1$, the evader cannot guarantee escape; however, this does not mean that capture is guaranteed for every pursuit strategy. In solving the Game of Degree, usually a candidate value function is derived and then shown to satisfy Eq. (5). This is not trivial in practice. The benefit of having a full solution is knowledge of the saddle-point strategies $\mathbf{u}_P^*$ and $\mathbf{u}_E^*$ which satisfy

$$J(\mathbf{u}_E, \mathbf{u}_P^*) \le J(\mathbf{u}_E^*, \mathbf{u}_P^*) \le J(\mathbf{u}_E^*, \mathbf{u}_P) \qquad \forall \mathbf{u}_E \in U_E,\ \mathbf{u}_P \in U_P \tag{6}$$

where U_E and U_P are the sets of admissable control strategies for the evader and pursuers, respectively. Condition (6) means that the saddle-point strategies are robust to *any* admissable opponent strategy. In this paper, we do not attempt to obtain $\mathbf{u}_E^*$ and $\mathbf{u}_P^*$ directly, but rather, in the sections to follow, we propose policies for the evader and pursuers which exhibit one-sided robustness.

We now define the Apollonius disk which is a geometric construct utilized in all of the policies described hereafter. For agents with simple motion and zero capture radius, the Apollonius disk represents the region in the realistic plane where the evader can reach before a particular pursuer. The Apollonius *circle* is the boundary of this region, and it is defined as the locus of points in which the evader and pursuer can reach simultaneously by taking straight-line paths at maximum speed, respectively. For each pursuer $i = 1, \ldots, M$, the Apollonius disk center, C_i, and radius, R_i are,

$$x_{C_i} = \frac{1}{1-\alpha^2} x_E - \frac{\alpha^2}{1-\alpha^2} x_i \tag{7}$$

$$y_{C_i} = \frac{1}{1-\alpha^2} y_E - \frac{\alpha^2}{1-\alpha^2} y_i \tag{8}$$

$$R_i = \frac{\alpha}{1-\alpha^2} \sqrt{(x_i - x_E)^2 + (y_i - y_E)^2} \tag{9}$$

The following set defines the Apollonius disk for pursuer i,

$$D_i = \left\{(x, y) \in \mathbb{R}^2 \mid \left(x - x_{C_i}\right)^2 + \left(y - y_{C_i}\right)^2 \leq R_i^2\right\} \tag{10}$$

The intersection of all the pursuers' Apollonius disks defines a region in which the evader can reach before *any* of the pursuers. We refer to this region as the Safe Region,

$$SR = \cap_{i=1}^{M} D_i \tag{11}$$

Remark 1 The Safe Region cannot be empty; the evader position, (x_E, y_E), is inside each Apollonius disk by definitions (7)–(10) and so, at the very least, the evader is inside SR.

Remark 2 The Safe Region collapses to a single point—the evader position—when one or more pursuers are coincident with the evader or the speed ratio is zero.

We will often make use of the Boundary of the Safe Region (or BSR) in the sequel. Let the BSR be parameterized by a set of circular arcs corresponding to segments of Apollonius circles, $\mathcal{C}$, and a set of vertices, $\mathcal{A}$, corresponding to endpoints of the arcs (which are Apollonius circle intersections),

$$\mathrm{BSR} = (\mathcal{C}, \mathcal{A}) \tag{12}$$

Whether we are referring to the BSR as a set of points or as its parameterization should be clear from context.

3 Robust Evader Policy

The robust evader policy presented here is based upon the work by Von Moll et al. [27,28]. We use the symbol G (for "geometric") to denote the policy, which is summarized in this section . Note that the G policy specifies an intercept point which is optimal, in some sense, for both the evader and pursuers, and thus, although we refer to it as a robust evader policy, it may also be implemented by the pursuers. The main idea in the work of Von Moll et al. [27,28] is that the evader can safely travel to any point inside SR (defined as above) and thus the agents should take straight-line paths to a point, I, which satisfies

$$\begin{aligned} I = \underset{(x,y)}{\arg\max} \; & \min_i (x_i - x)^2 + (y_i - y)^2 \\ \text{s.t. } & (x, y) \in SR, \qquad i \in 1, \ldots, M \end{aligned} \tag{13}$$

Should the point I fall somewhere in the interior of SR (i.e., not on the BSR), the evader would arrive at I before any pursuer and would need to stop to remain there. One may consider dealing with this issue by including the evader's velocity as one of its control variables, $V_E \in \left[0, V_{E_{max}}\right]$; however, this is not strictly necessary. The kinematics in Eq. (1) allow for instantaneous changes in heading. Therefore, the desired behavior of standing still may be achieved by modulating heading between $\phi = 0$ and $\phi = \pi$ (or any other pair of angles π radians apart) with a period of zero.

The max portion of Eq. (13) is done over the continuum of points which lie in the SR. However, the solution to Eq. (13) has strong geometric properties which allow the optimization over a continuous space to be supplanted with an optimization over a finite set of points (see [27], Theorem 1). These points correspond to capture by a single pursuer, simultaneous capture by two pursuers, and simultaneous capture by three or more pursuers.

Case 1 (Single Pursuer) Capture by a single pursuer occurs at the solution to the 1P1E game, solved by Isaacs [17], between a particular pursuer and the evader, which lies on the BSR

$$\begin{aligned} x_{S_i} &= \frac{R_i(1+\alpha)(x_E - x_i)}{\sqrt{(x_E - x_i)^2 + (y_E - y_i)^2}} \\ y_{S_i} &= \frac{R_i(1+\alpha)(y_E - y_i)}{\sqrt{(x_E - x_i)^2 + (y_E - y_i)^2}} \\ &\text{s.t. } (x_{S_i}, y_{S_i}) \in \text{BSR} \end{aligned} \tag{14}$$

Case 2 (Two Pursuers) Simultaneous capture can occur at the intersections of pursuers' Apollonius circles which lie on the BSR. The set $\mathcal{A}$, which are the vertices of the BSR, gives these points.

Case 3 (Three or More Pursuers) Simultaneous capture by three or more pursuers, wherein all of the agents take a straight-line path to the intercept point, must occur at a vertex of the pursuers' Voronoi diagram [27,28]. Let the Pursuers' Voronoi diagram be written as

$$\mathbb{V}_{\mathbb{P}} = (\mathcal{V}_P, \mathcal{E}_P, \{P_1, \ldots, P_M\}), \tag{15}$$

where $\mathcal{V}_P$ and $\mathcal{E}_P$ are the vertices and edges of the Voronoi diagram, respectively.

Now, let the set of all single-pursuer candidate solutions be

$$S_{\text{BSR}} = \left\{(x_{S_i}, y_{S_i}) \mid \text{s.t. } (x_{S_i}, y_{S_i}) \in \text{BSR},\ i = 1, \ldots, M\right\} \tag{16}$$

Similarly, the set of all candidate solutions involving three or more pursuers is

$$\mathcal{V}_{P_{\text{BSR}}} = \mathcal{V}_P \cap \text{BSR} \tag{17}$$

The main result of Von Moll et al. [27,28] states that the solution to (13) can be reduced to

$$\begin{aligned} I &= \arg\max_{(x,y)} \min_i (x_i - x)^2 + (y_i - y)^2 \\ &\text{s.t. } (x, y) \in S_{\text{BSR}} \cap \mathcal{A} \cap \mathcal{V}_{P_{\text{BSR}}} \end{aligned} \tag{18}$$

The G policy, then, is defined as taking the solution to (18), point I, to be the agent's instantaneous aim point. Note that both the evader and the pursuers may implement this policy; however, as will be shown in the following section, issues may arise when the pursuers implement it.

3.1 Properties of the *G* Policy

This section provides a more detailed characterization of the policy and identifies in what sense the policy is optimal. In order to do so, we first define the following:

Definition 1 *[Global Stackelberg equilibrium (GSE)]* An equilibrium over open-loop strategies wherein the leader selects a control action (i.e., a control trajectory from $t = 0$ through the game's termination) from a certain class of behaviors and announces the strategy to the follower.

The role of the follower is to compute its best response to the announced leader strategy. The leader can also compute the follower's best response and should therefore choose a strategy which maximizes its reward [7]. Also, the leader's control is purely a function of the initial conditions.

Definition 2 *[Feedback Stackelberg equilibrium (FSE)]* An equilibrium over feedback strategies in which, at each instant in time, the leader selects a control action and announces the strategy to the follower.

Note that the GSE is time consistent, which is not always the case for FSE [25]. In the open-loop case, if the leader is allowed to plan at $t_1 > 0$, then there is no benefit to adhere to the promised plan [25]. We will use the terms time consistent and subgame perfect interchangeably. If one were to apply the GSE policies over a finite timestep Δt, then as $\Delta t \to 0$ we expect to recover the FSE trajectories.

Definition 3 *[Feedback Nash equilibrium (FNE)]* An equilibrium over feedback strategies corresponding to the saddle point in (4).

The FNE is what is traditionally meant by the solution to the differential game. For detailed description of these equilibrium concepts, see Cruz [6].

In the context of the MP1E game, the evader may be thought of as the leader since the onus is on the pursuers to capture him. If the pursuers were the leader, then, upon announcing their strategy, the evader could choose from a myriad of trajectories which do not collide with the announced strategy. We define the class of leader behaviors as either a straight-line (constant heading) path, or a straight-line path to a point followed by stopping (dithering) at the point. In general, the follower's best response should satisfy the necessary conditions for optimality that are (usually) derived via the Hamiltonian. Thus, the pursuers' best response ought to consist of straight-line paths [27,28].

Theorem 1 *The G policy is a global Stackelberg equilibrium of the multiple-pursuer single-evader game under the kinematics in* (1)*, and the class of evader behaviors consisting of straight-line paths which may or may not terminate at a point.*

Proof Because of the restriction on the evader's behaviors, the evader selection of control strategy is equivalent to selecting a point $I \in SR$. For any point $(x, y) \notin SR$, there exists a pursuer strategy consisting of a straight-line path in which capture occurs on BSR. This follows from the definition of the Apollonius disk, (10). The point I, since it is in SR may be reached by one or more pursuers at or later than the evader. Thus, the pursuers' best response is to head directly to I; any deviation will delay their arrival to I and the capture of E. The objective in (13), which characterizes the G policy, is akin to capture time (evader reward) under such a pursuit policy. Therefore, (13) represents a maximization over the leader's reward given the follower's best response. □

Corollary 1 *The G policy is a robust evader policy. That is,*

$$J(\mathbf{u}_E^G, \mathbf{u}_P) \geq \max_{(x,y)\in SR} \min_i \frac{1}{V_P} \sqrt{(x_i - x)^2 + (y_i - y)^2} \qquad \forall \mathbf{u}_P \in U_P \tag{19}$$

Proof The result follows from the fact that the pursuers' best response is a straight-line path to the intercept point. The geometry of the problem prevents the pursuers from doing any better. □

Let the right-hand side of Eq. (19) be abbreviated as $LB(\mathbf{x})$ (for lower bound). Note, the G policy bears a striking resemblance to the so-called open-loop policy proposed by Liu et al. [21] for the single-pursuer multiple-evader game. There, the evaders jointly select their headings (assuming the worst case), announce their headings to the pursuer, and then commit to those headings for the duration of the game. Liu indicates that this policy is conservative from the evaders' perspective. Analogously, Corollary 1 indicates that the G policy is conservative from the evader's perspective for the MP1E game.

Rubio [25] investigated the criteria for which the FNE coincides with a Stackelberg equilibrium (GSE or FSE) and analyzed several cases where FNE and FSE coincide as well as cases where FNE and FSE do not coincide. The MP1E game falls under the latter category: the FNE does not coincide with the Stackelberg equilibria.

Theorem 2 *The G policy, although it is a global Stackelberg equilibrium, is not a feedback Nash equilibrium. That is,* $GSE \neq FNE$*, which means the G policy does not constitute a solution to the multiple-pursuer single-evader differential game under the kinematics in* (1).

Proof (by contradiction) Suppose $GSE = FNE$; that is, suppose that the G policy, implemented continuously in time by both teams, constitutes a feedback Nash equilibrium. Then the strategy pair $\mathbf{u}_E^G, \mathbf{u}_P^G$ must satisfy the saddle-point condition,

$$J(\mathbf{u}_E, \mathbf{u}_P^G) \leq J(\mathbf{u}_E^G, \mathbf{u}_P^G) \leq J(\mathbf{u}_E^G, \mathbf{u}_P), \qquad \forall \mathbf{u}_E \in U_E,\ \mathbf{u}_P \in U_P \tag{20}$$

Consider the case where $E = (0, 0)$, $P_1 = (0, 1)$, $P_2 = \left(\cos \frac{7\pi}{6}, \sin \frac{7\pi}{6}\right)$, $P_3 = \left(\cos -\frac{\pi}{6}, \sin -\frac{\pi}{6}\right)$, $V_P = 1$, with $\alpha = 0.8$. The solution to (18) is $(0, 0)$, the initial position of the evader and the Voronoi vertex v. The predicted capture time w.r.t. this solution is $J(\mathbf{u}_E^G, \mathbf{u}_P^G) = 1$. Let the evader strategy $\mathbf{u}_E^{\downarrow}$ be defined as $\mathbf{u}_E^{\downarrow}(t) = \frac{3\pi}{2}$. Under $\mathbf{u}_E^{\downarrow}$ the evader's heading angle, ϕ, is downward for all time, regardless of the state of the system. Note that $\mathbf{u}_E^{\downarrow}$ falls under the class of behaviors defined in Proposition 1 and is thus trivially in the set of admissable controls U_E. Let v and a represent the instantaneous position of the Voronoi vertex and lower Apollonius circle intersection, respectively. Let the pursuers implement the G policy—thus, the pursuers aim toward $(0, 0)$ initially. Because the evader is traveling downward (away from v and toward a) there comes a time $0 < t_1 < 1$ wherein the value (capture time) associated with all agents heading to a and all agents heading to v is equal. This situation is akin to a dispersal surface in differential games wherein the solution is non-unique. Let the distance between P_2 and v be written as $d_{P_2 v}$, and similarly for the point a. If the pursuers continue along their original trajectory for some small δt then the point a will become the solution to (18) and pursuers P_2 and P_3 have an incentive to switch their aim point to a. Figure 1 shows the configuration at this time. With the pursuers aimed at a, the distance $d_{P_2 v}$ changes as,

$$\begin{aligned} \dot{d}_{P_2 v} &= -V_v \cos \overline{\psi}_2 - V_P \sin\left(\frac{\pi}{2} - \overline{\psi}_2\right) \\ &= -V_v \cos \overline{\psi}_2 - V_P \cos \overline{\psi}_2 \end{aligned} \tag{21}$$

Similarly, the distance from P_1 to v changes as,

$$\dot{d}_{P_1 v} = V_v - V_P \tag{22}$$

Now, since v is the Voronoi vertex and the pursuers share the same speed, the distances to each of the pursuers must remain the same, and thus, the velocity at which each pursuer approaches v must also be equal. Setting (21) equal to (22) yields,

$$V_v - V_P = -V_v \cos \overline{\psi}_2 - V_P \cos \overline{\psi}_2 \tag{23}$$

Collecting terms and solving for V_v,

$$V_v = \frac{V_P(1 - \cos \overline{\psi}_2)}{(1 + \cos \overline{\psi}_2)} \tag{24}$$

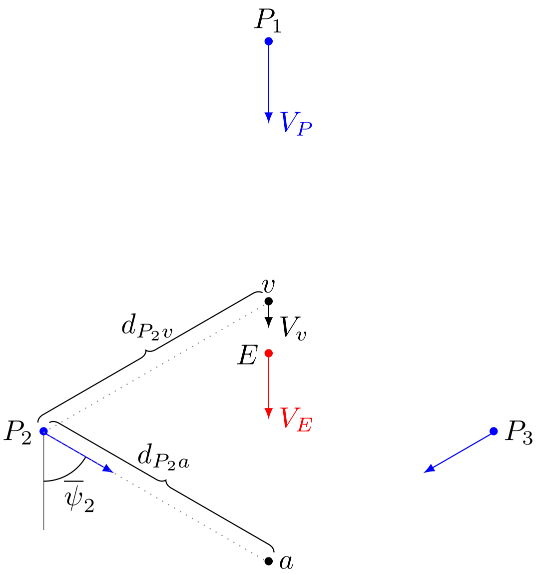


Fig. 1 Configuration at t_1 wherein the points a and v are equidistant from P_2

Here, we have $\overline{\psi}_2 = \pi/3$, and thus, $V_v = 1/3$ and $\dot{d}_{P_2v} = -2/3$. The distance d_{P_2a} changes as,

$$\dot{d}_{P_2a} = -V_P = -1 \tag{25}$$

because the point a is stationary when P_2, P_3, and E are aimed toward it. Therefore, we have,

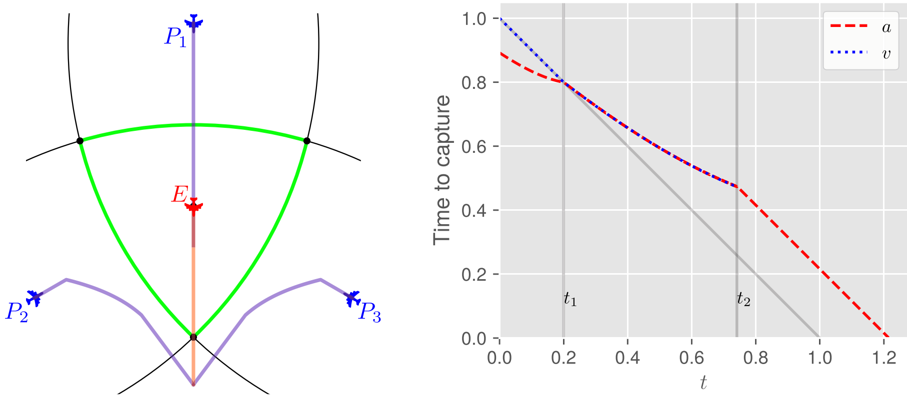
$$\dot{d}_{P_2v} > \dot{d}_{P_2a} \tag{26}$$

After some infinitesimally small amount of time δt, we have $d_{P_2v} > d_{P_2a}$ and thus pursuers P_2 and P_3 have incentive to switch their aim point *back* to the Voronoi vertex v. When P_2 and P_3 are aimed at v, (26) is reversed and thus the pursuers' aim point chatters between a and v until a time t_2 when v exits the Safe Region SR. This "fast switching" induced by non-optimal play by one player has been observed in the two-pursuer one-evader differential game[1] There, a dispersal surface is present when the three agents are collinear and the evader lies between the two pursuers. If the evader stands still, the pursuers' optimal behavior is to switch aim points between the two Apollonius circle intersections infinitely fast. In this case, the capture time is equal to the value of the game. Here, however, the two aim points a and v move at different rates; the point a is governed by the positions of P_2, P_3, and E, while the point v is solely a function of the pursuers' positions (see 24). Each point is stationary when the pursuers are aimed toward it. The result of the fast switching for $t_1 < t < t_2$ is that $t_f > 1$, that is, the evader increased its capture time under the $\mathbf{u}_E^{\downarrow}$ policy:

$$J(\mathbf{u}_E^{\downarrow}, \mathbf{u}_P^G) > J(\mathbf{u}_E^G, \mathbf{u}_P^G) \tag{27}$$

Equation (27) contradicts the saddle-point condition (6). Therefore, the strategy pair $\mathbf{u}_E^G, \mathbf{u}_P^G$ is not a feedback Nash equilibrium. □

[1] M. Pachter et al. *Two-on-One Pursuit*, submitted to the Journal of Guidance, Control, and Dynamics, Aug 2018.



(a) Trajectories **(b)** Time to capture

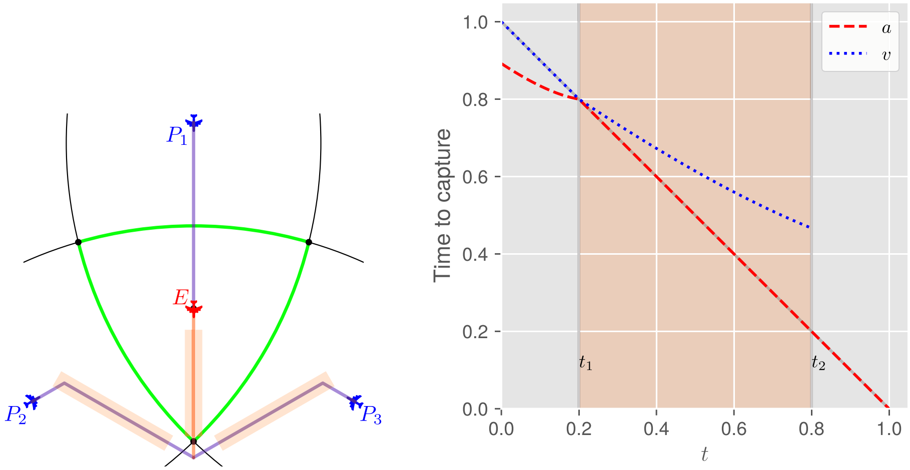
Fig. 2 Simulation of $\mathbf{u}_E^{\downarrow}$, $\mathbf{u}_P^{G}$ starting from a symmetric configuration

Remark 3 A full treatment of this scenario requires solutions to differential equations with discontinuous right-hand sides (i.e., in the sense of Filippov) due to the fast switching behavior of the pursuers [11].

The scenario described above is simulated numerically with a timestep of 0.005 wherein, at each time step, the pursuers aim at the current solution to (18). Figure 2 shows the trajectories generated from this strategy pair along with a plot of time to capture versus simulation time. If the G policy were truly the solution to the MP1E game, one would expect the time to capture to remain on or below the line from (0, 1) to (1, 0) in Fig. 2b if the pursuers implement G. In other words, under game-optimal play, one expects $\frac{\partial t_f}{\partial t} = -1$. From time t_1 to t_2 the points v and a remain nearly equidistant from P_2 and P_3 as their heading chatters back and forth. The actual capture takes place at $t = 1.2$ because there is a loss of pursuer performance during this window of time. This loss is caused by a dilemma of choosing between a and v which is perpetuated from t_1 to t_2. Each time P_2 and P_3 switch headings a small loss ε is incurred. In the limit as the timestep $\Delta t \to 0$ the pursuers incur an infinite number of these small losses. One may be tempted to think that the sum of these losses $\sum \varepsilon \to 0$ as $\Delta t \to 0$. However, even if we consider the trajectory of P_2 and P_3 in a Filippov sense in which they become smooth as opposed to piecewise continuous, it is clear that the trajectories will still be curved. Thus, the result in Fig. 2b changes negligibly for very small time steps.

Let us return to Fig. 1 which shows the position of all the agents at time t_1 when $d_{P_2 a} = d_{P_2 v}$. Under the evader strategy $\mathbf{u}_E^{\downarrow}$, it is clear that capture must occur on the line $x = 0$ at a $y < E_y(t_1)$. The only way to recover the initially predicted value of $t_f = 1$, P_2 and P_3 must either commit to heading to v or $a(t_1)$ for the remainder of the game. Clearly, committing to v is a poor choice since the evader is heading away from it. Figure 3 shows the case wherein at time t_1 pursuers P_2 and P_3 make a single switch and aim at a. The issue with this strategy is that at any time $t_1 < t < t_2$ the evader could switch to heading toward v and guarantee a capture time $t_f > 1$. This is evidenced by the fact that the line corresponding to v in Fig. 3b lies above the line corresponding to a.

The situation encountered when $d_{P_2 v} = d_{P_2 a}$, as mentioned previously, is something like a singular surface. Here, the pursuers can only recover the predicted capture time [associated with (18)] if they know which point, a or v, the evader will choose. Guessing wrong for



(a) Trajectories **(b)** Time to capture

Fig. 3 Simulation of $\mathbf{u}_E^{\downarrow}$, $\mathbf{u}_P^G$ until t_1, after which pursuers aim at a. Highlighted segments indicate portions of the trajectory which are not subgame perfect

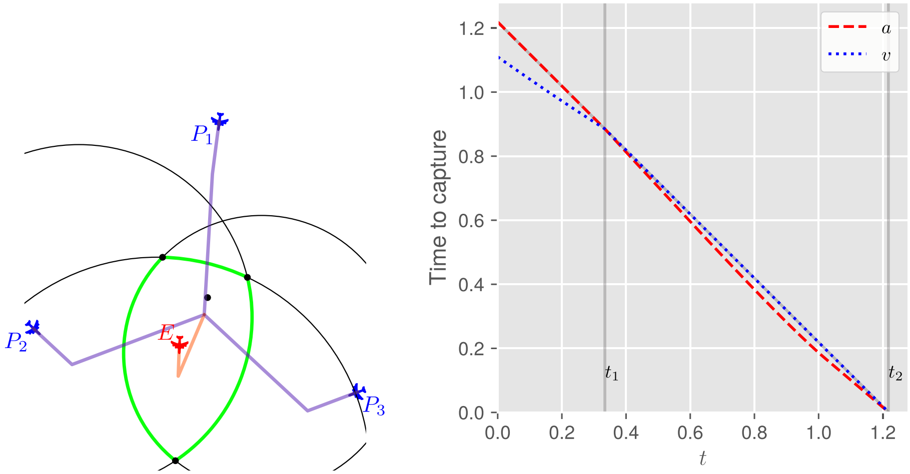
a short time does not eliminate this dilemma (see, e.g., Fig. 2b). Another case where the evader's control action is required for the pursuer to play optimally is found in the Homicidal Chauffeur (HC) game. There, using the classical parameters, an equivocal surface is present wherein the evader has the authority to stay or leave the surface and the pursuer must know the evader's choice in order to play optimally on the surface (see [22]). In HC, the pursuer can force the system off of the equivocal surface (not to return under optimal play) by choosing a suboptimal control for a short time. However, in MP1E, even if the pursuers take some suboptimal action when $d_{P_2v} = d_{P_2a}$ they cannot prevent the system from reentering such a configuration.

Consider another 3P1E scenario that is not symmetric and the solution to (18) is the point a, the lower Apollonius circle intersection. Figure 4 contains the trajectories and time to go for the points a and v under the strategy pair $\mathbf{u}_E^G$, $\mathbf{u}_P^G$. The capture time predicted by the G policy is 1.22. Interestingly, at t_1, when a and v are nearly equidistant from P_2 and P_3, all the agents switch to the current Voronoi vertex $v(t_1)$. Afterward, the agents have no further incentive to switch and capture occurs at precisely $t_f = 1.22$ as predicted, albeit not at the predicted location. Suppose the agents adhere to their initial headings, that is they aim toward a for the duration of the game. Figure 5 shows the results of this scenario; the trajectories in Fig. 5a correspond to the trajectories predicted by the G policy. The capture time t_f is unchanged; however, in Fig. 5b it is clear there is an incentive to switch to v at some time $t_1 < t < t_2$; thus, these trajectories are not subgame perfect. The behaviors shown in Figs. 2 and 3 are a symptom of the fact that the pair $\mathbf{u}_E^G$, $\mathbf{u}_P^G$ is not a feedback Nash equilibrium.

4 Capturability

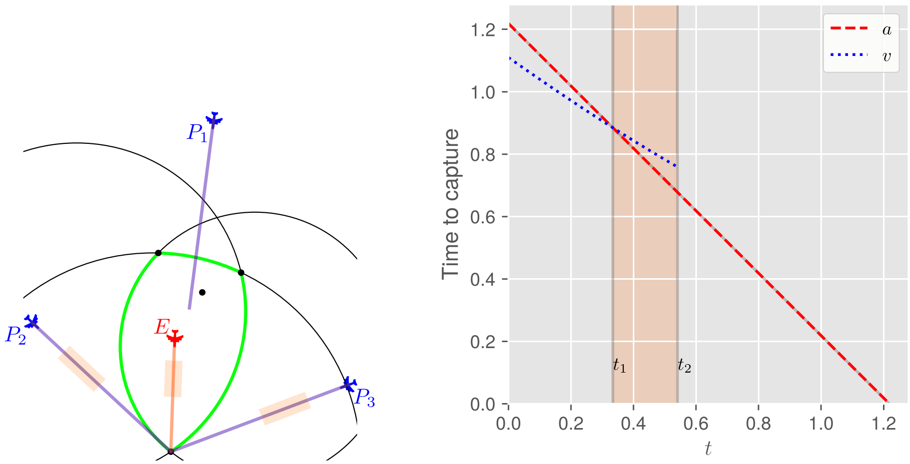
Let R be the distance between P and E. Then if we have $R(0) > 0$ and $\dot{R} < 0$ for all $t > 0$, then capture is guaranteed.

$$\dot{R} = -V_P \cos\psi + V_E \cos\phi \tag{28}$$



(a) Trajectories **(b)** Time to capture

Fig. 4 Simulation of $\mathbf{u}_E^G$, $\mathbf{u}_P^G$ starting from an asymmetric configuration; $t_f = 1.22$



(a) Trajectories **(b)** Time to capture

Fig. 5 Simulation with same initial conditions as Fig. 4 but the agents head toward a for the duration of the game; $t_f = 1.22$. Highlighted segments indicate portions of the trajectory which are not subgame perfect

where ψ and ϕ are defined relative to the line of sight (PE). Now, the evader can choose any heading; $\phi = 0$ maximizes (28); thus, we get,

$$\dot{R} \leq -V_P \cos \psi + V_E \tag{29}$$

The critical pursuer heading, ψ_c, thus occurs when the RHS is equal to zero,

$$-V_P \cos \psi_c + V_E = 0$$

yielding,

$$\psi_c = \cos^{-1} \alpha$$

Thus, $\dot{R} < 0$ when $\psi < \cos^{-1}\alpha$. In the limit as $\alpha \to 1$ the range of pursuer headings which guarantees capture collapses to $\psi = 0$ The line passing through P that is tangent to the Apollonius circle has an angle of $\sin^{-1}\alpha$ w.r.t. the line PE. When $\alpha < \frac{\sqrt{2}}{2}$ we have $\sin^{-1}\alpha < \cos^{-1}\alpha$ and thus $\dot{R} < 0$ for any pursuit policy in which the pursuer aims toward a point on the Apollonius circle. The canonical 2P1E pursuer policy [17] and the G policy both fall into this category. However, unlike the G policy, the 2P1E pursuer policy is a game-optimal policy (in the sense of FNE).

Proposition 1 *Obtaining regular solutions to the Game of Degree over the whole state space is sufficient to guarantee capturability in pursuit evasion differential games Isaacs* [16].

Garcia et al. [13] proved that the pursuit policy put forth by Isaacs is indeed the solution for the whole state space—thus, capture is guaranteed as long as $\alpha < 1$. Because the G policy is *not* the solution to the MP1E game (according to Theorem 2), rigorously proving that G guarantees capture for the case that $\alpha > \frac{\sqrt{2}}{2}$ is difficult. These two issues suggest the need for another Pursuit policy for MP1E with more desirable properties.

5 Robust Pursuit Policy

The difficulty in obtaining the solution to the MP1E differential game is, in part, due to the curse of dimensionality[2]. Li et al. [20] also note that the traditional process of obtaining the retrograde partial differential equations (c.f. [17]) is difficult because the terminal states are unknown. To combat this issue, Li et al. [20] propose a hierarchical approach for multiplayer pursuit–evasion differential games which is conservative from the pursuers' standpoint. The process is based on exploiting the solutions to games involving only a subset of the agents. Because the canonical 2P1E pursuit policy has been proven to be solution to the 2P1E differential game [13], we propose the following,

$$i^*, j^* = \underset{i,j \in \{1,\dots,M\}}{\arg\min} \; V^{2P1E}(E, P_i, P_j) \tag{30}$$

$$\psi_{i^*,j^*} = \psi^{2P1E}_{i^*,j^*} \tag{31}$$

$$\psi_k = \tan^{-1}\frac{y_E - y_k}{x_E - x_k}, \qquad k \notin \left\{i^*, j^*\right\} \tag{32}$$

where $V^{2P1E}(E, P_i, P_j)$ is the value function of the corresponding 2P1E game between E and the P_i, P_j pursuer team starting from their current positions. Let $\mathbf{u}_P^R$ denote the policy described in Eqs. (30)–(32); we will refer to this policy as the R (for robust pursuit) policy. In the R policy, the pursuers compare all possible 2P1E games and choose to play the game which yields the smallest capture time from the current positions. The headings for the chosen team, i, j, are given by the 2P1E game whereas all of the other agents aim line of sight (Pure Pursuit). The solutions to the 2P1E games are given by (18) (note $\mathcal{V}_{\mathcal{P}_{\mathrm{BSR}}} = \emptyset$). An explicit derivation of the 2P1E value function and heading angles is given by Garcia et al. [13]. In general, these solutions are either the further of the Apollonius circle intersections between P_i and P_j's circles or the 1P1E solution for one or the other pursuer. The idea is that as $\mathbf{u}_P^R$ is implemented continuously in feedback fashion the pursuer assignments (i.e., whether each pursuer is aiming line of sight or cooperating with another pursuer in the 2P1E game) may switch to whatever is most advantageous at that time according to Eq. (30).

[2] Von Moll et al. *The Multi-Pursuer Single-Evader Game: A Geometric Approach*, submitted to the Journal of Intelligent & Robotic Systems in September 2018.

Theorem 3 *The R policy is a robust pursuer policy. That is,*

$$J(\mathbf{u}_E, \mathbf{u}_P^R) \leq V^{2P1E}(E, P_{i^*}, P_{j^*}) \qquad \forall \mathbf{u}_E \in U_E \tag{33}$$

where i^*, j^* *are given by Eq.* (30).

Proof *Case 1 (No switches occur)* The team i^*, j^* selected at time $t = 0$ remains the best team according to Eq. (30) for all $0 \leq t \leq t_f$. Then the other pursuers, $k \notin \{i^*, j^*\}$ had no effect on the game and it is as if the scenario is an instance of the 2P1E game. In this case Eq. (33) is given by the fact that the headings in Eq. (31) are the saddle-point strategies for the pursuers P_{i^*}, P_{j^*} in the 2P1E game.

Case 2 (One or more switches occur) Let the initial team assignment be i_0^*, j_0^*. Let t_1 be the time in which the first switch occurs, $0 < t_1 < t_f$, and the new team assignment be i_1^*, j_1^*. At t_1 we have, from Eq. (30),

$$V^{2P1E}(E, P_{i_1^*}, P_{j_1^*}) < V^{2P1E}(E, P_{i_0^*}, P_{j_0^*})$$

otherwise, a switch would not have occurred. For the remainder of the game $t_1 < t \leq t_f$, the scenario falls into either of these two cases.
Thus, robustness of R is guaranteed by the robustness of the 2P1E solution and the method of assignment, Eq. (30). □

Theorem 4 *The value of the multiple-pursuer single-evader game under the kinematics in Eq.* (1) *and feedback Nash equilibrium strategies* $\mathbf{u}_E^*$, $\mathbf{u}_P^*$, *if they exist, satisfies*

$$LB(\mathbf{x}) \leq J(\mathbf{u}_E^G, \mathbf{u}_P^*) \leq J(\mathbf{u}_E^*, \mathbf{u}_P^*) \leq J(\mathbf{u}_E^*, \mathbf{u}_P^R) \leq V^{2P1E}(E, P_{i^*}, P_{j^*}) \tag{34}$$

where i^*, j^* *are given by Eq.* (30), *and* $LB(\mathbf{x})$ *is the right-hand side of Eq.* (19). *In other words, the value of the game is bounded.*

Proof The first inequality follows directly from Eq. (19) in Corollary 1. The middle two inequalities follow from Definition 3. The last inequality follows directly from Eq. (33) in Theorem 3. □

Although the true solution to the MP1E game is not known, the robustness properties of the G and R policies allow us to bound the game-optimal capture time. It is also true that if the evader implements G and the pursuers implement R the following is satisfied,

$$LB(\mathbf{x}) \leq J(\mathbf{u}_E^G, \mathbf{u}_P^R) \leq V^{2P1E}(E, P_{i^*}, P_{j^*}) \tag{35}$$

It is possible for the upper and lower bounds to be equivalent. The following theorem identifies the conditions under which this is true as well as the implications of such a scenario. Let the capture point associated with the 2P1E game between E, P_{i^*}, and P_{j^*} be denoted I^{2P1E}.

Theorem 5 *If* $I^{2P1E} \in SR$, *the strategy pair* $\mathbf{u}_E^G$, $\mathbf{u}_P^R$ *is a feedback Nash equilibrium of the multiple-pursuer single-evader game under the kinematics in* (1).

Proof Let i^*, j^* be the solution to Eq. (30) corresponding to I^{2P1E}. Pursuers P_{i^*} and P_{j^*} thus aim at I^{2P1E} under the R policy according to Eq. (31). Now we will show that I^{2P1E} is the solution to Eq. (18). This statement is predicated on the point I^{2P1E} being in SR. Suppose there is another point $I^\dagger = (x^\dagger, y^\dagger) \in SR$ such that,

$$\min_i (x_i - x^\dagger)^2 + (y_i - y^\dagger)^2 > \min_i (x_i - x^{2P1E})^2 + (y_i - y^{2P1E})^2$$

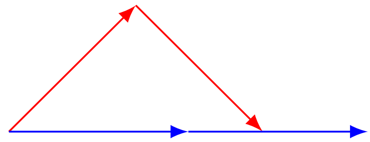


Fig. 6 Red path: slower effective speed due to chattering, blue path: smoothed trajectory obtained by taking a convex combination of headings

that is, the point $I^\dagger$ is further away from the nearest pursuer than I^{2P1E}. By virtue of the fact that $I^\dagger \in SR$, the evader can safely reach $I^\dagger$ by aiming directly at it and achieve a better capture time than if it had aimed at I^{2P1E} (i.e.,, the value of the 2P1E game, $V^{2P1E}(E, P_{i^*}, P_{j^*})$). Then $LB > V^{2P1E}$; but this statement contradicts Eq. (34) in Theorem 4. Therefore, since no point in SR can yield a better capture time for the evader, the point I^{2P1E} must be the solution to Eq. (18). Thus, the lower bound is equal to the upper bound,

$$LB(\mathbf{x}) = V^{2P1E}(E, P_{i^*}, P_{j^*}) \tag{36}$$

Under the G policy, then, the evader aims at the point I^{2P1E}. Thus, the strategies $\mathbf{u}_E^G$, $\mathbf{u}_P^R$ yield the same behavior for E, P_{i^*}, and P_{j^*} as under the game-optimal 2P1E strategies,

$$J(\mathbf{u}_E^G, \mathbf{u}_P^R) = J(\mathbf{u}_E^*, \mathbf{u}_P^*) \tag{37}$$

where i^*, j^* are given by Eq. (30). □

It is unclear, however, what the relationship between $J(\mathbf{u}_E^G, \mathbf{u}_P^R)$ and $J(\mathbf{u}_E^*, \mathbf{u}_P^*)$ is in general.

6 Convex Policy

In this section, we introduce another pursuit policy, the C policy (for convex) It may not have the same robustness properties of R, but is an attempt at smoothing out the chattering behavior of G in configurations like that of Fig. 1. The pursuers' loss observed in Fig. 2b is induced (1) by the curvature in P_2 and P_3's paths and (2) by an effective slowing down while the Voronoi vertex v and Apollonius circle intersection a are similar in value (i.e., the pursuers headings chatter). In the case of a discrete time implementation, this latter piece may be understood to be the result of vector addition. Figure 6 illustrates the idea that the effect of slowing down due to chattering can be combated by using a combination of *headings*. Thus, we propose the C policy for the pursuers as an augmentation of the G policy. Let $d(x, y)$ be the distance from the nearest pursuer to the point (x, y),

$$d(x, y) = \min_i \sqrt{(x_i - x)^2 + (y_i - y)^2} \tag{38}$$

and, let $d^* = d(x^*, y^*)$ be the distance from the nearest pursuer to the aim point suggested by the G policy be written where (x^*, y^*) is the solution to Eq. (18). Then, define the following set of points,

$$\mathcal{C} = \left\{(x_k, y_k) \mid (x_k, y_k) \in S_{\mathrm{BSR}} \cap \mathcal{A} \cap \mathcal{V}_{P_{\mathrm{BSR}}},\ |d(x_k, y_k) - d^*| < \varepsilon\right\} \tag{39}$$

which are the candidate solutions to Eq. (18) whose distance to the nearest pursuer is within some neighborhood ε of d^*. The C policy, then, for the ith pursuer is to take a convex combination of headings to each of the points in $\mathcal{C}$ in which the weight associated with a point (x_k, y_k) is inversely proportional to d_k.

$$\mathbf{u}_i^C = \frac{\sum_{k=1}^{|\mathcal{C}|} \frac{1}{d_k} \psi_{i,k}}{\sum_{k=1}^{|\mathcal{C}|} \frac{1}{d_k}}, \qquad i = 1, \ldots, M \tag{40}$$

where $\psi_{i,k}$ is the angle of the point $p_k \in \mathcal{C}$ relative to P_i,

$$\psi_{i,k} = \tan^{-1} \frac{y_{p_k} - y_i}{x_{p_k} - x_i}, \qquad i = 1, \ldots, M, \; k = 1, \ldots, |\mathcal{C}| \tag{41}$$

Remark 4 In practice it is simpler to apply the weighting in Eq. (40) to the points p_k directly to get the convex combination of *aim points*, and then converting to a heading.

The selection of ε is of some importance. As $\varepsilon \to 0$ only the solution to Eq. (18), (x^*, y^*), is considered, and all other candidates ignored; in this case, the C policy is identical to the G policy. It is possible for there to be more than one solution to Eq. (18); however, this almost never occurs in reality. For such a configuration to occur, the agents would either need to begin as such or implement their control in continuous time with infinite precision. For very large ε the set $\mathcal{C}$ is equivalent to the set of candidates $S_{\mathrm{BSR}} \cap \mathcal{A} \cap \mathcal{V}_{P_{\mathrm{BSR}}}$. pursuer performance in this case can be very poor, especially in the case when E is between and nearly collinear with two of the pursuers. Therefore, in order to improve upon the G policy, the neighborhood should generally be set such that,

$$0 < \varepsilon \ll 1 \tag{42}$$

For discrete time implementations, the size of ε should be larger for large Δt. This is because, in a single time step, the system state $\mathbf{x}$ may very well jump across the neighborhood wherein the solution to Eq. (18) is different. As previously mentioned, there are no obvious analytical properties of the C policy (e.g., in the way of robustness), but, as will be shown in the sequel, its performance is the best out of any of the pursuit policies presented herein for some particular evader behaviors.

7 Results

Because none of the policies presented in this work are feedback Nash equilibrium strategies, it is interesting to compare the performance of the different policies via simulation. The limitation of this approach is that one may never be certain whether a particular policy performs better than another for all cases. A framework for comparing the merits/demerits of non-FNE strategies via simulation is out of the scope of the present work, although it represents an interesting area of research in the case that the FNE strategies are not known (or do not exist). Here, in order to compare the G, R, and C policies we return to initial conditions like those in Fig. 2a and restrict the evader's control input ϕ to be constant over the course of a single playout:

$$\mathbf{u}_E^{\phi}(t) := \phi_c, \qquad \phi_c \in [0, 2\pi] \tag{43}$$

Then, we sweep ϕ from 0 to 2π and simulate the game for each constant evader heading. For all of the simulations in this section, the following settings are used: $V_P = 1$, $\alpha = 0.9$, and $\Delta t = 5e - 3$. Figure 7 depicts the location in which the evader is captured for each of the pursuit policies. The BSR shown (and hereafter mentioned) is the BSR corresponding to $t = 0$—the BSR changes as a function of the instantaneous position of the agents. Because the evader is implementing a constant heading policy, the best performance the pursuers can

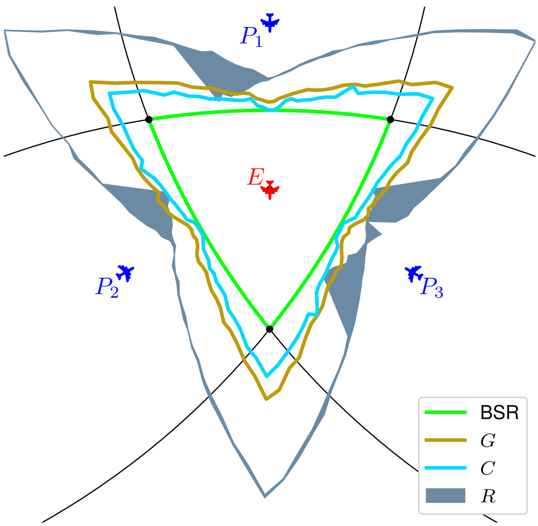


Fig. 7 Capture locations under evader constant heading policy $\mathbf{u}_E^\phi$ against pursuers' geometric policy $\mathbf{u}_P^G$, convex policy $\mathbf{u}_P^C$, and robust policy $\mathbf{u}_P^R$

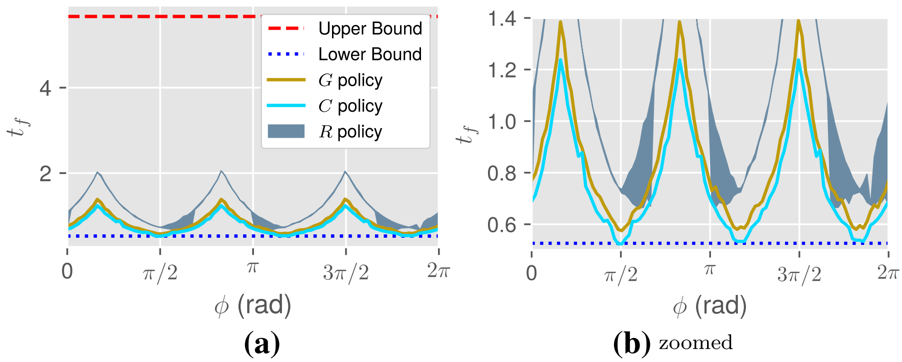


Fig. 8 Capture times under evader constant heading policy $\mathbf{u}_E^\phi$ against pursuers' geometric policy $\mathbf{u}_P^G$, convex policy $\mathbf{u}_P^C$, and robust policy $\mathbf{u}_P^R$

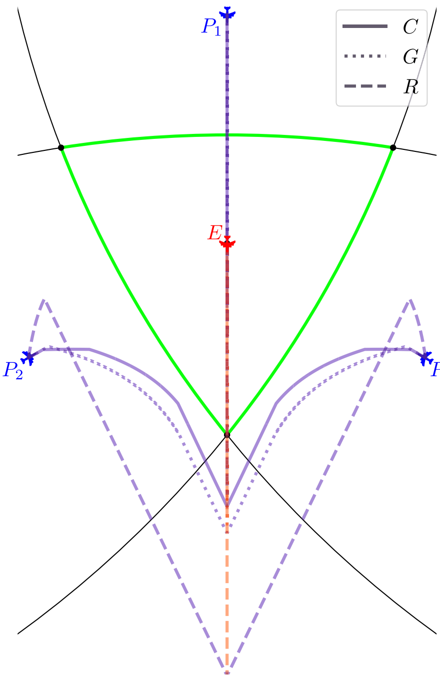
achieve is capture on the BSR. Capture on the BSR can only be achieved if the intercepting pursuer takes a constant heading path to the intercept point. In general, the pursuers would need to know ϕ_c from $t = 0$ in order to accomplish this. Nonetheless, the closer the intercept points are to the BSR the "better," in this case.

Figure 8 shows the capture times associated with the playouts in Fig. 7. The upper bound for the capture time is given by the two-on-one game with the smallest capture time, $V^{2P1E}(E, P_{i^*}, P_{j^*})$. The lower bound for the capture time is simply the smallest time required for a pursuer to each the evader if the evader were to take a collision course with that pursuer:

$$t = \min_i \frac{\overline{EP_i}}{V_E + V_P}$$

Note, *this* lower bound is not to be confused with the evader's robust bound, defined in (19). The value of the evader's robust bound is 1 and corresponds to the evader standing still at (0, 0), which is the Voronoi vertex at $t = 0$. If this bound were equal to the value of the game

Fig. 9 Comparison of pursuer trajectories under the robust (R), geometric (G), and convex (C) policies against an evader implementing a constant heading of $\phi_c = \frac{3\pi}{2}$



(i.e., if G were the evader's FNE strategy), then we would expect an FNE pursuit strategy to achieve capture times at or below 1 for these playouts. The fact that none of the pursuit policies meet this condition for all $\phi_c \in [0, 2\pi]$ means that G is not the evader's FNE strategy and/or none of the pursuit strategies presented herein are FNE pursuit strategies. It is likely that both are true and likely that the inequalities in (35) are strict equalities.

The thickness in the R line in the above plots is due to the fact that at $t = 0$ all three pursuer pairings yield the same two-on-one capture time. Whenever it is true that the pursuer team pairing i^*, j^* is not unique, the simulation selects a pairing from among the minima at random. Thus, for the R policy, the simulations were repeated five times and the minimum and maximum capture times recorded and displayed. From Fig. 8 it is clear that the C policy always out performs the G policy which almost always outperforms the R policy. This is, perhaps, due to fact that the G and C policies consider explicit cooperation among multiple (> 2) pursuers whereas the R policy addresses cooperation between 2 pursuers (since the other $M - 2$ pursuers operate independently). So although the G and C policies do not come with any robustness guarantees from the pursuers' perspective, they typically perform much better. This is especially comforting considering this configuration was used to disprove that the G policy is an FNE pursuit strategy (see Theorem 2)! Note that, for this particular scenario, the pursuers' robust (upper) bound is quite high compared to the capture times, even under the R policy. Thus, if one were to choose a pursuit policy, the relative benefit of robustness must be weighed against typical performance. One should expect that multi-pursuer cooperation will improve the performance of the pursuer team. However, analysis of cooperation among multiple (> 2) pursuers is more challenging and the FNE is not easy to obtain.

Table 1 Capture times for different pursuit policies

Policy	t_f	Penalty (%)
C	1.28	0
G	1.40	10
R	2.19	64

Lastly, Fig. 9 shows the agents' trajectories for the case where $\phi_c = \frac{3\pi}{2}$. The capture times corresponding to the trajectories in Fig. 9 are given in Table 1 along with the relative performance penalty w.r.t. the C policy. Notice the trajectories generated by the C policy and how P_2 and P_3 tend toward the y-axis much more so than G or R and thus capture E much sooner. Under the R policy, pursuers P_2 and P_3 begin by aiming to the intersection of their Apollonius circle intersections which is further away from the evader (far above the plot shown). They stay on this course until they become collinear with the evader. After this point, the game plays out as a normal two-on-one game between P_2, P_3, and E (as these three agents are now headed toward the furthest Apollonius circle intersection).

8 Conclusion

In this work, we presented the multiple-pursuer single-evader differential game comprised of agents with simple motion and a slow evader. It was shown that the recently proposed geometric policy based on comparing distances to relevant single-pursuer solutions, Apollonius circle intersections, and Voronoi vertices is a global Stackelberg (i.e., open loop) equilibrium when both the evader and pursuers implement it. We also showed that this policy is *not* a feedback Nash equilibrium strategy pair. The true feedback Nash equilibrium strategies correspond to the solution of this differential game; if they exist, they are not currently known. Despite the geometric policy not being a feedback Nash equilibrium strategy, we identified the fact that the policy is robust from the perspective of the evader. Similarly, we proposed a new pursuit strategy based on the solution to differential games between subsets of the agents which is robust from the Pursers' perspective. Alterations to the geometric policy were proposed and shown to improve performance for a particular test scenario. There is still much interest in obtaining the feedback Nash equilibrium strategies for this differential game; however, we also believe that the approach presented herein to obtain *robust* strategies has application in many other games whose solutions are not known. In particular, the bounds described by robust policies may be useful for making assignments (as in who ought to pursue who) in a multiple-pursuer multiple-evader scenario.

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