Journal of Guidance, Control, and Dynamics
Vol. 40, No. 11, November 2017

Engineering Notes

Differential Game of Guarding a Target

Meir Pachter*
Air Force Institute of Technology, Wright–Patterson Air Force Base, Ohio 45433
and
Eloy Garcia† and David W. Casbeer‡
U.S. Air Force Research Laboratory, Wright–Patterson Air Force Base, Ohio 45433

DOI: 10.2514/1.G002652

I. Introduction

SCENARIOS where dynamic agents are engaged in pursuit and evasion are correctly analyzed in the framework of dynamic games [1–4]. Approaches based on dynamic Voronoi diagrams have been used in scenarios with several pursuers in order to capture an evader within a bounded environment, as shown in [4,5]. In aerospace applications, the target–attacker–defender (TAD) problem has been extensively studied: The work in [6] considered a dynamic target and provided a game theoretical analysis of the TAD problem using classical guidance laws for both the attacker and the defender. On the other hand, [7] addressed the TAD game where the target was stationary. It is this stationary target defense differential game first formulated by Isaacs in [8] that is considered in this Note.

The game evolves in the Euclidean plane, and the target set T is assumed convex; for example, T is a circular disk, a convex polygon, or a point target. The protagonists are an attacker A and the defender D. Both have "simple motion" à la Isaacs; that is, their speed is constant and they can turn on a dime. The speed of A is V_A, and the speed of D is V_D where $V_D \geq V_A$. The nondimensional defender/attacker speed ratio is

$$\mu \equiv \frac{V_D}{V_A} \geq 1$$

and, in [8], it is assumed the speed ratio is $\mu = 1$. Player A's objective is to come as close as possible to the target T before being intercepted by the defender D. The defender's objective is to preclude A from reaching T. The target defense differential game is illustrated in Fig. 1. We denote the boundary of the target set by ∂T.

Because $\mu \geq 1$ once D comes in contact with A, the interception of A is effected. When the speed ratio is $\mu = 1$, once D comes in contact with A, the latter comes under the control of D; that is, A will be pushed around by D without being able to break off contact with D and, when $\mu > 1$, once the A–D separation becomes $l(\geq 0)$, A is captured by D and the game is over. Hence, the game terminates when A reaches T without having been intercepted by D, or A comes in contact with D as close as possible to the target set T: at which time, A is captured by D. In general, the defender could be endowed with a capture circle of radius l and the speed ratio μ could be greater than one. In the target defense differential game, the problem parameters are the speed ratio $\mu \geq 1$ and D's capture radius of $l \geq 0$. In this Note, we will assume that the capture range l is short; that is, it is $l \approx 0$: we are interested in point capture. Also, we will focus on the case where the speed ratio is $\mu = 1$.

The Note is organized as follows. A formal analysis of the differential game of guarding a target using optimal control theory and the theory of differential games is presented in Sec. II. This is the main contribution of the Note in that it provides a rigorous justification of Isaacs's geometric method analyzed in [8] to characterize the optimal state feedback strategies for players A and D as explained in Sec. III. We then turn to the solution of the game of kind: For a specified target set T, the boundary separating the winning regions of players A and D is constructed in Sec. IV. The optimal placement of the defender D such that the area of vulnerability of T is minimized is obtained in Sec. V. Extensions concerning the interesting scenarios where the defender/attacker speed ratio is $\mu > 1$, the requirement for point capture of A by D is relaxed, and the dynamic target defense differential game are briefly outlined in Sec. VI. Lastly, conclusions are given in Sec. VII.

II. Analysis

We solve the differential game of guarding a target using the method of optimal control and differential games. We work in the four-dimensional realistic state space $\zeta^T = (x_A, y_A, x_D, y_D)^T \in \mathbb{R}^4$. We first address the case where the stationary target is a point target at (x_T, y_T). The normalized dynamics are given by

$$\begin{aligned} \dot{x}_A &= \cos\chi, & x_A(0) &= x_{A_0} \\ \dot{y}_A &= \sin\chi, & y_A(0) &= y_{A_0} \\ \dot{x}_D &= \cos\psi, & x_D(0) &= x_{D_0} \\ \dot{y}_D &= \sin\psi, & y_D(0) &= y_{D_0} \end{aligned} \tag{1}$$

where χ and ψ are the respective heading angles of A and D. Concerning the terminal condition, the cost/payoff function is evaluated when D intercepts A at time t_f where

$$x_A(t_f) = x_D(t_f), \qquad y_A(t_f) = y_D(t_f) \tag{2}$$

The terminal cost/payoff function is

$$J(x_A(t_f), y_A(t_f); x_T, y_T) = \frac{1}{2}\left[(x_A(t_f) - x_T)^2 + (y_A(t_f) - y_T)^2\right] \tag{3}$$

We have a Mayer-type differential game. The terminal time t_f is free and the terminal manifold $\mathcal{T}$ is the hyperplane in $\mathbb{R}^4$ defined by

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} [\, x_A \quad y_A \quad x_D \quad y_D \,]^T = 0_{2\times 1} \tag{4}$$

The costate $\lambda^T = (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}) \in \mathbb{R}^4$ and $\lambda = -V_x$, where V is the value function of the differential game. The Hamiltonian of the differential game is

Received 1 December 2016; revision received 4 March 2017; accepted for publication 20 March 2017; published online 7 June 2017.

*Professor, Department of Electrical Engineering; meir.pachter@afit.edu. Associate Fellow AIAA.
†Electronics Engineer, Control Science Center of Excellence; eloy.garcia.2@us.af.mil. Member AIAA.
‡Research Engineer, Control Science Center of Excellence; david.casbeer@us.af.mil. Senior Member AIAA.

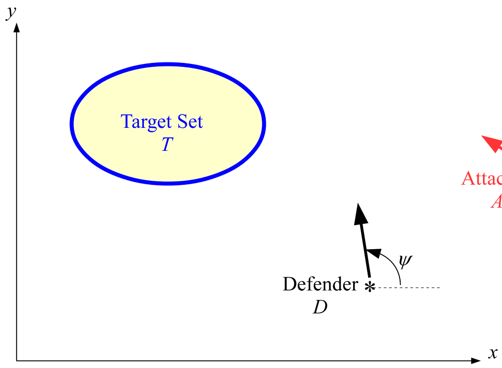


Fig. 1 Guarding a target.

$$\mathcal{H} = \lambda_{x_A} \cos\chi + \lambda_{y_A} \sin\chi + \lambda_{x_D} \cos\psi + \lambda_{y_D} \sin\psi$$

Theorem 1: Consider the differential game of guarding a target [Eqs. (1–3)]. The optimal headings of the defender and of the attacker are constant under optimal play. They are given, respectively, by the state feedback control laws:

$$\Phi(x_A, y_A, x_D, y_D; x_T, y_T) \triangleq \frac{1}{2}\Big[(x_A - x_T)^2 + (y_A - y_T)^2\Big] + \nu_1(x_A - x_D) + \nu_2(y_A - y_D)$$

where ν_1 and ν_2 are Lagrange multipliers. The Pontryagin maximum principle (PMP) or dynamic programming yields the transversality/terminal conditions $\lambda(t_f) = -(\partial/\partial x)\Phi(\boldsymbol{x}(t_f))$; that is,

$$\lambda_{x_A} = x_T - x_A(t_f) - \nu_1, \qquad \lambda_{y_A} = y_T - y_A(t_f) - \nu_2 \tag{10}$$

$$\lambda_{x_D} = \nu_1, \qquad \lambda_{y_D} = \nu_2 \tag{11}$$

At this point, we have Eqs. (10) and (11) plus Eq. (2), which yield six conditions. Because we need only four conditions, we use the additional two conditions to eliminate the introduced Lagrange multipliers ν_1 and ν_2 from Eqs. (10) and (11), and we obtain

$$\lambda_{x_A} + \lambda_{x_D} = x_T - x_A(t_f) \tag{12}$$

$$\lambda_{y_A} + \lambda_{y_D} = y_T - y_A(t_f) \tag{13}$$

$$\psi(x_A, y_A, x_D, y_D; x_T, y_T) = \frac{\pi}{2} - \arcsin\left(\frac{y_A - y_D}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}}\right) - \arctan\left(2\frac{(y_T - (1/2)(y_A + y_D))(x_A - x_D) - (x_T - (1/2)(x_A + x_D))(y_A - y_D)}{(x_A - x_D)^2 + (y_A - y_D)^2}\right) \tag{5}$$

and

$$\chi(x_A, y_A, x_D, y_D; x_T, y_T) = \frac{\pi}{2} - \arcsin\left(\frac{y_A - y_D}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}}\right) + \arctan\left(2\frac{(y_T - (1/2)(y_A + y_D))(x_A - x_D) - (x_T - (1/2)(x_A + x_D))(y_A - y_D)}{(x_A - x_D)^2 + (y_A - y_D)^2}\right) \tag{6}$$

In the region of win of D, the value function $V(x)$ is C^1 and is explicitly given by

$$V(x_A, y_A, x_D, y_D; x_T, y_T) = \frac{(x_A + x_D - 2x_T)(x_A - x_D) + (y_A + y_D - 2y_T)(y_A - y_D)}{2\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}} \tag{7}$$

Proof: The optimal control inputs (in terms of the costate variables) are obtained from $\min_\psi \max_\chi \mathcal{H}$, and they are given by

$$\cos\chi^* = \frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}}, \qquad \sin\chi^* = \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} \tag{8}$$

$$\cos\psi^* = -\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}}, \qquad \sin\psi^* = -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} \tag{9}$$

The costate dynamics are $\dot{\lambda}_{x_A} = \dot{\lambda}_{y_A} = \dot{\lambda}_{x_D} = \dot{\lambda}_{y_D} = 0$; hence, all the costates are constant, and therefore the optimal controls are $\chi^* \equiv$ constant and $\psi^* \equiv$ constant. In other words, in the realistic (x, y) plane, the optimal trajectories are straight lines.

Concerning the solution of the attendant two-point boundary value problem on $0 \le t \le t_f$ in $\mathbb{R}^8$, we have four initial states specified by Eq. (1) and we need four more conditions for the terminal time t_f. In this respect, introduce the augmented Mayer payoff/cost function Φ:$\mathbb{R}^4 \to \mathbb{R}^1$:

Thus, we have four relationships for the terminal time t_f: Eqs. (2), (12), and (13). Finally, the time t_f is specified by the PMP/dynamic programming requirement that the Hamiltonian $\mathcal{H}(\boldsymbol{x}(t), \lambda(t), \chi, \psi)|_{t_f} = 0$ that, for this problem, takes the form

$$\lambda_{x_A} \cos\chi^* + \lambda_{y_A} \sin\chi^* + \lambda_{x_D} \cos\psi^* + \lambda_{y_D} \sin\psi^* = 0 \tag{14}$$

So, we have enough relationships to solve the differential game. The existence of a solution is predicated on the initial state: If the initial positions of the attacker and the defender are such that the target is located on the defender's side of the orthogonal bisector of $\overline{AD}$, then a solution exists and the defender will be able to interpose itself between the target and the attacker to guard/protect the former. In view of the aforementioned and without loss of generality, assume that the state in the realistic plane is such that $x_D = -x_A$, $y_A = 0$, and $y_D = 0$, as shown Fig. 2. Because the optimal trajectories of A, D, and T are straight lines and $V_D = V_A$, we have that

$$x_A(t_f) = 0, \qquad x_D(t_f) = 0, \qquad y_A(t_f) = y_D(t_f)$$

Let $y \triangleq y_A(t_f) = y_D(t_f)$. Also, let $x_A = x_A(t')$ be the instantaneous positions at some time $t' < t_f$. Hence, from Eq. (1), we obtain the following:

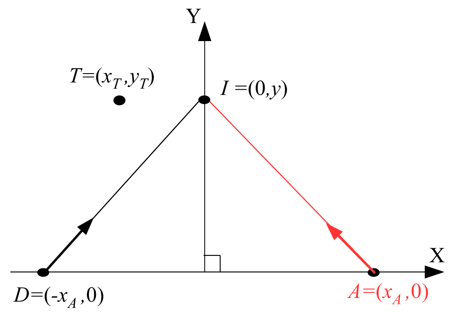


Fig. 2 Optimal strategies.

$$0 = x_A + (t_f - t')\cos\chi, \qquad y = (t_f - t')\sin\chi$$
$$0 = -x_A + (t_f - t')\cos\psi, \qquad y = (t_f - t')\sin\psi$$

In addition, Eqs. (12) and (13) can be written as follows:

$$\lambda_{x_A} + \lambda_{x_D} = x_T, \qquad \lambda_{y_A} + \lambda_{y_D} = y_T - y$$

From the ΔADI in Fig. 2, we conclude that $t_f - t' = \sqrt{x_A^2 + y^2}$. Without loss of generality, assume that $t' = 0$; then, $t_f = \sqrt{x_A^2 + y^2}$.

Thus, having used the theory of differential games, we are now able to reduce the solution of the zero-sum differential game of degree to the optimization of a cost/payoff function of one variable, namely,

$$J(y; x_A, x_T, y_T) = \sqrt{(y_T - y)^2 + x_T^2} \tag{15}$$

In terms of y, the A and D players' optimal headings are

$$\cos\chi^* = -\frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad \sin\chi^* = \frac{y}{\sqrt{x_A^2 + y^2}} \tag{16}$$

$$\cos\psi^* = \frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad \sin\psi^* = \frac{y}{\sqrt{x_A^2 + y^2}} \tag{17}$$

Using Eqs. (9) and (17) yields the following relationships:

$$-\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} = \frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} = \frac{y}{\sqrt{x_A^2 + y^2}} \tag{18}$$

Similarly, from Eqs. (8) and (16), we obtain

$$\frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} = -\frac{x_A}{\sqrt{x_A^2 + y^2}}, \qquad \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} = \frac{y}{\sqrt{x_A^2 + y^2}} \tag{19}$$

We have four equations [Eqs. (12, 13) and (18, 19)] in the four unknowns λ_{x_A}, λ_{y_A}, λ_{x_D}, and λ_{y_D}. The solution is

$$\lambda_{x_A} = \frac{1}{2}\left[x_T - x_A(t_f) - \frac{x_A}{y}(y_T - y_A(t_f))\right]$$
$$\lambda_{y_A} = \frac{1}{2}\left[y_T - y_A(t_f) - \frac{y}{x_A}(x_T - x_A(t_f))\right]$$
$$\lambda_{x_D} = \frac{1}{2}\left[x_T - x_A(t_f) + \frac{x_A}{y}(y_T - y_A(t_f))\right]$$
$$\lambda_{y_D} = \frac{1}{2}\left[y_T - y_A(t_f) + \frac{y}{x_A}(x_T - x_A(t_f))\right]$$

By substituting $y_A(t_f) = y$ and $x_A(t_f) = 0$, we obtain

$$\lambda_{x_A} = \frac{1}{2}\left[x_T - \frac{x_A}{y}(y_T - y)\right] \tag{20}$$

$$\lambda_{y_A} = \frac{1}{2}\left[y_T - y - \frac{y}{x_A}x_T\right] \tag{21}$$

$$\lambda_{x_D} = \frac{1}{2}\left[x_T + \frac{x_A}{y}(y_T - y)\right] \tag{22}$$

$$\lambda_{y_D} = \frac{1}{2}\left[y_T - y + \frac{y}{x_A}x_T\right] \tag{23}$$

which specify the costates in terms of the states at time t_f. Now, using Eq. (14) in conjunction with Eqs. (16) and (17) for the optimal controls and Eqs. (20–23) for the costates, we obtain the quadratic equation in y:

$$(y_T - y)\left(\frac{x_A^2}{y} + y\right) = 0$$

which has the real solution $y^* = y_T$ plus two imaginary roots, which are irrelevant to the game under consideration. Inserting $y = y_T$ into Eqs. (16) and (17) yields the optimal state feedback strategies of A and D:

$$\cos\chi^* = -\frac{x_A}{\sqrt{x_A^2 + y_T^2}}, \qquad \sin\chi^* = \frac{y}{\sqrt{x_A^2 + y_T^2}} \tag{24}$$

$$\cos\psi^* = \frac{x_A}{\sqrt{x_A^2 + y_T^2}}, \qquad \sin\psi^* = \frac{y}{\sqrt{x_A^2 + y_T^2}} \tag{25}$$

The value function is C^1 and is given by

$$V(x_A, x_T, y_T) = J(y^*; x_A, x_T, y_T) \quad \text{for all } x_T < 0 \tag{26}$$

where the function J is given by Eq. (15).

Using the theory of differential games, the original problem was simplified to an optimization problem in one variable y. The optimal headings χ^* and ψ^* are state feedback control strategies, and they are functions of the instantaneous values of the state $(x_A, y_A, x_D, y_D) \in \mathbb{R}^4$ and of the target location (x_T, y_T) in the realistic space; see Fig. 3. In the realistic space, the optimal headings are obtained as follows. Let

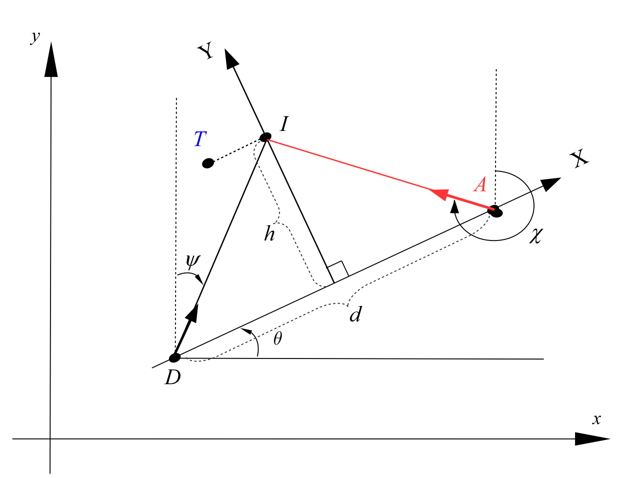


Fig. 3 Optimal heading angles in realistic space.

$$d_A \triangleq \sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}$$
$$\cos\theta \triangleq \frac{x_A - x_D}{d_A} \qquad \sin\theta \triangleq \frac{y_A - y_D}{d_A}$$
$$h \triangleq \left(y_T - \frac{y_A + y_D}{2}\right)\cos\theta - \left(x_T - \frac{x_A + x_D}{2}\right)\sin\theta$$
$$= \frac{(y_T - (1/2)(y_A + y_D))(x_A - x_D) - (x_T - (1/2)(x_A + x_D))(y_A - y_D)}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}}$$

The strategies of the players are determined by the coordinates (x_A, x_T, y_T) in a reduced state space and are therefore functions of d_A, h, and θ: From Fig. 3, the optimal heading of the defender is

$$\psi(x_A, y_A, x_D, y_D; x_T, y_T) = \frac{\pi}{2} - \theta - \arctan\left(\frac{2h}{d_A}\right) = \frac{\pi}{2} - \arcsin\left(\frac{y_A - y_D}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}}\right)$$
$$- \arctan\left(2\frac{(y_T - (1/2)(y_A + y_D))(x_A - x_D) - (x_T - (1/2)(x_A + x_D))(y_A - y_D)}{(x_A - x_D)^2 + (y_A - y_D)^2}\right)$$

and the optimal heading of the attacker is

$$\chi(x_A, y_A, x_D, y_D; x_T, y_T) = \frac{3\pi}{2} - \theta - \arctan\left(\frac{2h}{d_A}\right) = \frac{\pi}{2} - \arcsin\left(\frac{y_A - y_D}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}}\right)$$
$$+ \arctan\left(2\frac{(y_T - (1/2)(y_A + y_D))(x_A - x_D) - (x_T - (1/2)(x_A + x_D))(y_A - y_D)}{(x_A - x_D)^2 + (y_A - y_D)^2}\right)$$

Finally, in the D region of win, the value function of the target defense differential game is given by Eq. (7).

As illustrated in Fig. 4, a polygonal target is not strictly convex, and so there exist configurations in the state space where the geometric solution yields optimal strategies that are not unique: the attacker and the defender each choose a separate aimpoint (say I_A and I_D) on the segment $\overline{I_1 I_2}$ that runs parallel to side $\overline{AB}$ of the polygon; see Fig. 4. This, however, is not a show stopper. After a short time, the orthogonal bisector of the segment AD will cease being parallel to side $\overline{AB}$ of the polygon, and a unique aimpoint I will reappear, with the end result being that the distance V_0 is a lower bound for the value of the game and the defender can make the value arbitrarily close to V_0. This should come as no surprise because, in zero-sum games, the nonuniqueness of the optimal strategies is not an issue: the optimal strategies are interchangeable, and the value of the game is unique.

III. Geometric Solution

The aforementioned analysis is brought to bear on the target defense differential game as it evolves in the Euclidean plane. It is the justification for Isaacs's geometric solution [8], as discussed herein.

The Euclidean plane where the target defense differential game is played is partitioned into two sets. When the speed ratio is $\mu = 1$, the set of positions in the plane reachable by A before being possibly reached by D is the half-plane for which the boundary is the orthogonal bisector of the segment $\overline{AD}$ and which contains point A; we refer to A's half plane as H_A. Conversely, the set of positions in the plane reachable by D before being possibly reached by A is the half-plane for which the boundary is the orthogonal bisector of the segment $\overline{AD}$ and which contains point D; we refer to D's half plane H_D. Player A "wins" if, and only if, $H_A \cap T \neq \emptyset$. In other words, player A can reach the target set T without being intercepted by D. If, however, $H_A \cap T = \emptyset$, player A heads toward point I on the orthogonal bisector of segment $\overline{AD}$ that is closest to the convex target set T. It is the same for player D. Their optimal strategies are straight lines, as proved in Sec. II. The optimal strategies, as well as Isaacs's original geometric optimality proof from [8], are illustrated in Fig. 5. The analysis in Sec. II provides the justification for Isaacs's geometric solution [8]. Figure 5 compares the optimal and a non-optimal play. During optimal play, both A and D head toward point I, which is the point on the orthogonal bisector of segment AD closest to target set T. When one player plays nonoptimally (here, the Attacker), the opponent (here, the Defender) immediately exploits this by heading toward I', which is the point of the orthogonal bisector of the segment $\overline{A'D'}$ closest to the target set; the miss distance increases.

When the speed ratio is $\mu > 1$ (that is, the defender is faster than the attacker), the boundary of the set of positions in the plane reachable by A before being possibly reached by D is an Apollonius circle $\mathcal{C}$. The radius of the Apollonius circle is $R = (\mu/\mu^2 - 1)d_A$, where d_A is

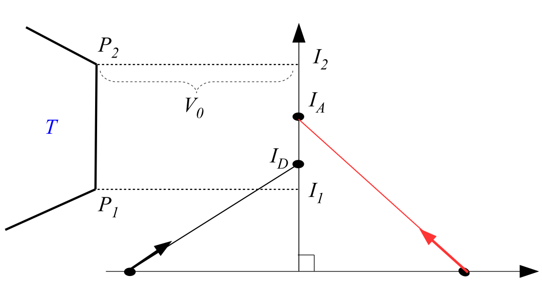


Fig. 4 Nonunique A and D strategies for a polygonal target.

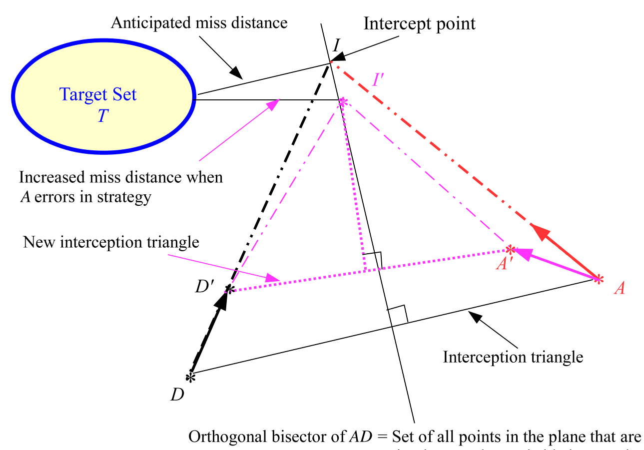


Fig. 5 Optimal and nonoptimal play.

the A–D separation. Its center O is on the straight line through points A and D, at a distance $(\mu^2/\mu^2 - 1)d_A$ from D; the length of the segment $\overline{OA}$ is $(1/\mu^2 - 1)d_A$. A wins if $\mathcal{C} \cap T \neq \emptyset$. If, however, $\mathcal{C} \cap T = \emptyset$, A heads toward the point $I \in \mathcal{C}$ that is closest to the convex target set T.

IV. Solution of the Game of Kind

Given the target T and defender D, the boundary of A's winning region, denoted by $\mathcal{B}$, is constructed for target sets defined by circular disks and polygonal sets.

A. Circular Disk

Consider the case where the target set T is a circular disk. The boundary $\mathcal{B}$ of A's winning region is illustrated in Fig. 6 and is constructed as follows:

1) Draw a radial emanating from T.
2) Draw a tangent to $\partial T \perp$ as the radial from step 1.[§]
3) Pick a point A such that the tangent from step 2 is the orthogonal bisector of $\overline{DA}$.

Note that the orthogonal bisector of $\overline{DA}$ is tangent to the target set T. Thus, the boundary $\mathcal{B}$ is the orthotomic of ∂T.

The boundary $\mathcal{B}$ of A's winning region is characterized by the optimality principle and the following holds;

Theorem 2: Consider the case where the target set T is a circular disk of radius r. In polar coordinates, the right-hand side of the (symmetric) boundary $\mathcal{B}$ of the winning region of A is

$$R(\theta) = 2(r + d\cos\theta), \qquad 0 \leq \theta \leq \theta_c \tag{27}$$

where

$$\theta_c = \arcsin\left(\frac{r}{d}\right) + \frac{\pi}{2}$$

for $d > r$, and $\theta_c = \pi$ otherwise. Thus, boundary $\mathcal{B}$ is a limaçon of Pascal. Also, the optimal placement of the defender with respect to a circular target of radius r is at the center of the target set, and then the minimal area of vulnerability is $S^* = 4\pi r^2$.

Proof: It is well known that the orthotomic of a circle is a limaçon of Pascal. Also, when $d < (1/2)r \rightarrow A$'s winning, the region is convex; when $d > (1/2)r \rightarrow$ the boundary, $\mathcal{B}$ has an indentation; and when $d = r \rightarrow$ the boundary, $\mathcal{B}$ is a cardioid: $R(\theta) = 2r(1 + \cos\theta)$, $0 \leq \theta \leq \pi$ (see Fig. 7a). Note that, when $d > r$, the limaçon has an inner loop; however, the inner loop is irrelevant to the differential game under consideration. Hence, in the corresponding figures, the inner loop is not drawn (see Fig. 7).

Furthermore, the area of A's winning region is

$$S = 2(2r^2 + d^2)\pi \quad \forall\ 0 \leq d \leq r \tag{28}$$

and, when $d \geq r$, the boundary $\mathcal{B}$ has a cusp and the area of A's winning region is

$$S = 2(2r^2 + d^2)\left[\pi - \arccos\left(\frac{r}{d}\right)\right] + 6r\sqrt{d^2 - r^2} \tag{29}$$

When $d > r$, the boundary $\mathcal{B}$ is not smooth and, as shown in Fig. 8, the indentation angle is given by the relation $r + d\cos\theta_c = 0$. Solving for θ_c, we obtain

$$\theta_c = \arcsin\left(\frac{r}{d}\right) + \frac{\pi}{2} \tag{30}$$

When $0 \leq d \leq r$, the vulnerable area as a function of the parameter d is $S(0) = 4\pi r^2$. In addition, $S(d) > S(0)$ for $0 < d \leq r$.

Now, when $d > r$, we can show that

$$\arg\min_{d \geq r} S(d) = r \tag{31}$$

[§]This tangent is labeled as the "orthogonal bisector" in Fig. 6.

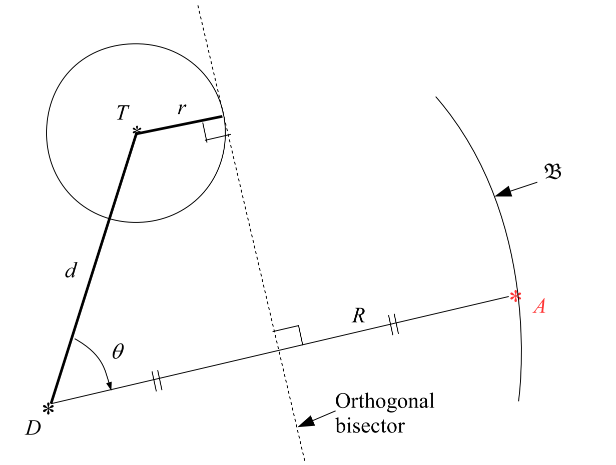


Fig. 6 Construction of $\mathcal{B}$.

This can be done by computing

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}d} &= 2\left((2r^2 + d^2)\frac{-r}{d\sqrt{d^2 - r^2}} + \frac{3rd^2}{d\sqrt{d^2 - r^2}} + 2d\left(\pi - \arccos\left(\frac{r}{d}\right)\right)\right) \\
&= 2\left(\frac{1}{d\sqrt{d^2 - r^2}}(3rd^2 - rd^2 - 2r^3) + 2d\left(\pi - \arccos\left(\frac{r}{d}\right)\right)\right) \\
&= 4\left(\frac{r}{d\sqrt{d^2 - r^2}}(d^2 - r^2) + d\left(\pi - \arccos\left(\frac{r}{d}\right)\right)\right) \\
&\geq 4\left(\frac{r}{d}\sqrt{d^2 - r^2} + \frac{\pi}{2}\right) \quad > 0
\end{aligned}$$

Thus, for $d \geq r$, $S(d)$ is an increasing function of its argument; therefore, the minimum value is attained at $d = r$. This minimum value is $S(r) = 6\pi r^2$ and $S(r) > S(0)$. Then, $d^* = 0$, and the defender should be placed at the center of the circular target set. The minimal ensuing vulnerable area is $S^* = 4\pi r^2$.

B. Polygonal Target

Consider the case where the target set T is a polygon. For this case, we can regard each vertex of the polygon as a circle of radius $r \rightarrow 0$. Then, the orthotomic curve is given by $R(\theta) = 2d\cos\theta$, which is the equation of a circle. When T is a convex $n$gon, the boundary $\mathcal{B}$ of the winning region of A consists of n circular arcs:

$$\mathcal{B} = \cup_{i=1}^{n} \mathcal{C}_i \tag{32}$$

where $\mathcal{C}_i$ are circular arcs of radius $\overline{DT_i}$ centered at the polygon's vertices T_i, where $i = 1, \ldots, n$.

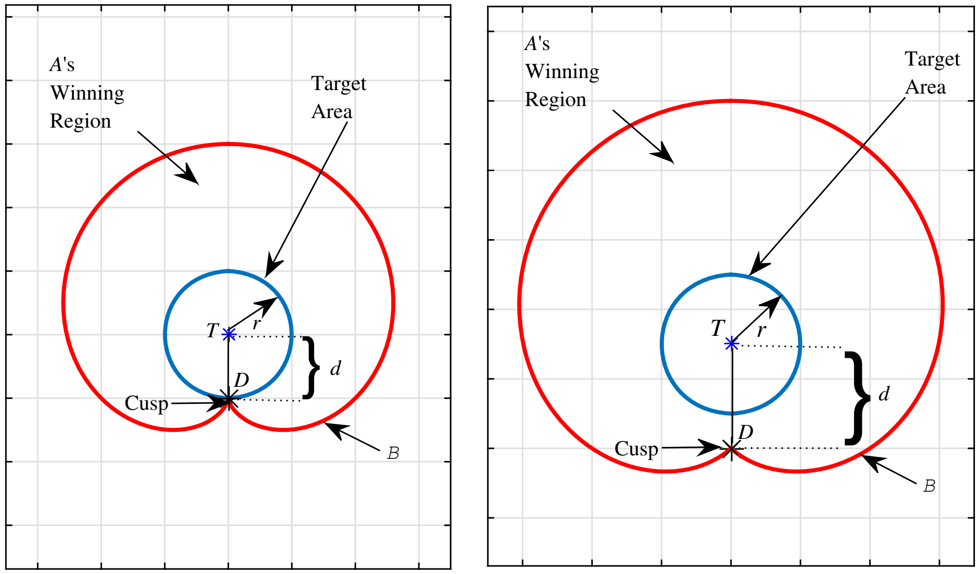
V. Optimization

Consider the case where T is a convex $n$gon for which the vertices are $T_i = (x_i, y_i)$, where $i = 1, \ldots, n$. From the geometry in Figs. 9 and 10, we deduce that the area enclosed by the boundary $\mathcal{B}$ (that is, the area S of A's winning region) is

$$S = 2^*\text{Area of Polygon} + \sum_{i=1}^{n}(\pi - \angle T_i)d_i^2 \tag{33}$$

where $d_i \equiv \|\overline{DT_i}\|$.

From Eq. (33), we infer that S is akin to the moment of inertia of the planar figure formed by the polygon where point masses $\pi - \angle T_i/2\pi$ are attached to its vertices; $\angle T_i$ is the internal angle at vertex T_i of the polygon. From mechanics, we know that the moment of inertia of a planar rigid body is minimal about an axis through its center of mass



a) Circular target, $d = r$ **b) Circular target, $d > r$**

Fig. 7 Boundary $\mathcal{B}$ of the winning region of A for the circular target.

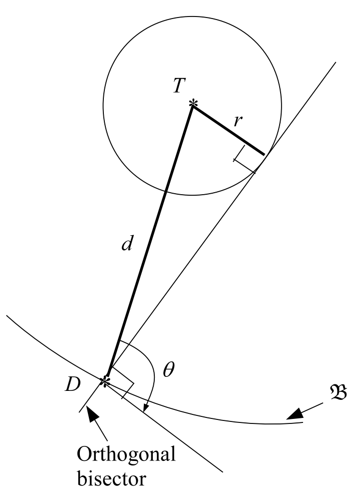


Fig. 8 Critical geometry.

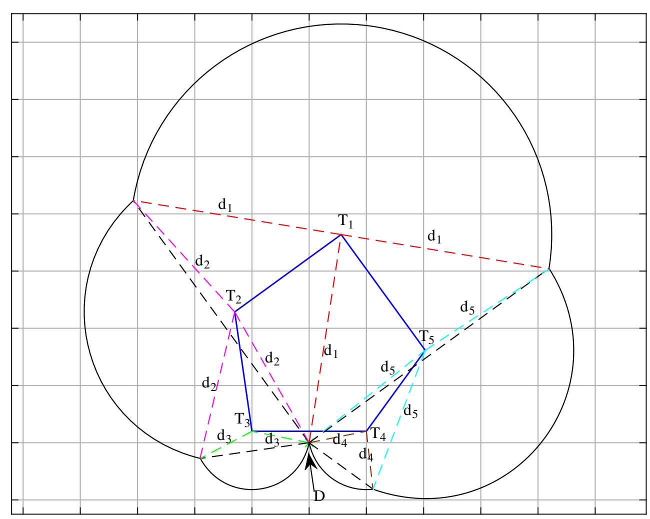


Fig. 10 Construction of $\mathcal{B}$; $D \notin T$.

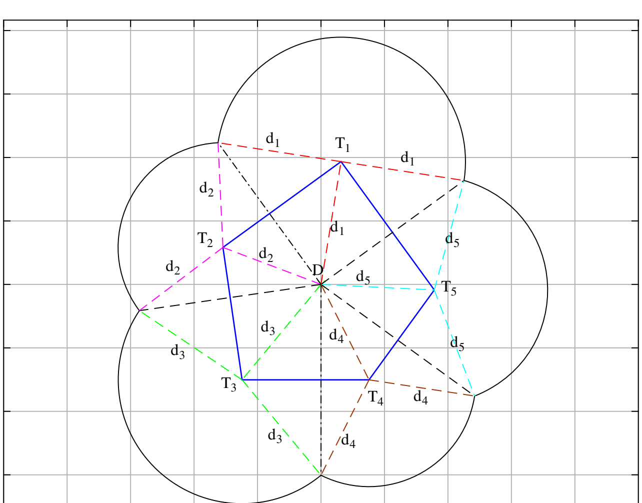


Fig. 9 Construction of $\mathcal{B}$; $D \in T$.

and the center of mass of a rigid body is calculated by averaging. Hence, the following holds:

Theorem 3: To minimize the polygonal target's area of vulnerability S, the defender D positions himself at the center of mass of the n-point configuration $T_1, \ldots, T_n$, where the "mass" of point T_i is $w_i = (\pi - \angle T_i / 2\pi)$ and $\angle T_i$ is the internal angle at the vertex T_i of the polygon. The optimal position of the defender D is $D^* = (x_D^*, y_D^*)$, where

$$x_D^* = \sum_{i=1}^{n} \frac{\pi - \angle T_i}{2\pi} x_i, \qquad y_D^* = \sum_{i=1}^{n} \frac{\pi - \angle T_i}{2\pi} y_i \tag{34}$$

Consequently, the minimal vulnerable area is

$$S^* = \sum_{i=1}^{n} \left[\det\left(\begin{bmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{bmatrix} \right) + (\pi - \angle T_i)[(x_i - x_D^*)^2 + (y_i - y_D^*)^2 \right] \tag{35}$$

where, in Eq. (35), $x_{n+1} \equiv x_1$ and $y_{n+1} \equiv y_1$.

Remark: Note that

$$\sum_{i=1}^{n} w_i = 1$$

as required.

A. Examples

Example 1: Consider a pentagon where the coordinates of each vertex and the corresponding internal angles in radians are given by

$$x = [0 \quad 4 \quad 11.056 \quad 10.12 \quad 5.52]^T$$
$$y = [0 \quad 0 \quad 2.54 \quad 7.45 \quad 7.45]^T$$
$$\angle T = [0.933 \quad 2.796 \quad 1.728 \quad 1.759 \quad 2.208]^T$$

The optimal defender position is $D^* = [5.754 \quad 3.318]$, with a corresponding minimal vulnerable area of $S^* = 309.99$. Figure 11 shows the polygonal target set, the optimal position to place the defender, and the boundary of the minimal area of vulnerability.

Example 2: Consider a polygon with seven vertices. The coordinates of each vertex and the corresponding internal angles are given by

$$x = [0 \quad 5.3 \quad 7.769 \quad 10.581 \quad 3.8 \quad 1.011 \quad -0.829]^T$$
$$y = [0 \quad 0 \quad 3.398 \quad 12.053 \quad 13.793 \quad 12.689 \quad 5.521]^T$$
$$\angle T = [1.72 \quad 2.2 \quad 2.83 \quad 1.51 \quad 2.51 \quad 2.2 \quad 2.74]^T$$

The optimal defender position is $D^* = [4.413 \quad 6.938]$ with a corresponding minimal vulnerable area of $S^* = 564.44$. Figure 12 shows the polygon target set, the optimal position to place the defender, and the boundary of the minimal vulnerable area.

B. Smooth Target Set

When the target's set boundary of $\mathcal{C} = \partial T$ is a smooth curve, it is convenient to think of the boundary $\mathcal{C}$ of the target set T as specified using intrinsic coordinates; that is, it is parameterized with respect to the arc length s, and $R(s)$ is the radius of curvature of $\mathcal{C}$ at s. The winning region of A is obtained as follows:

Approximate $\mathcal{C}$ by a polygonal curve, as shown in Fig. 13. The angle α in Fig. 13 is the internal angle of the polygon. The following holds:

$$\pi - \angle T = \theta(x + dx) - \theta(x) \tag{36}$$

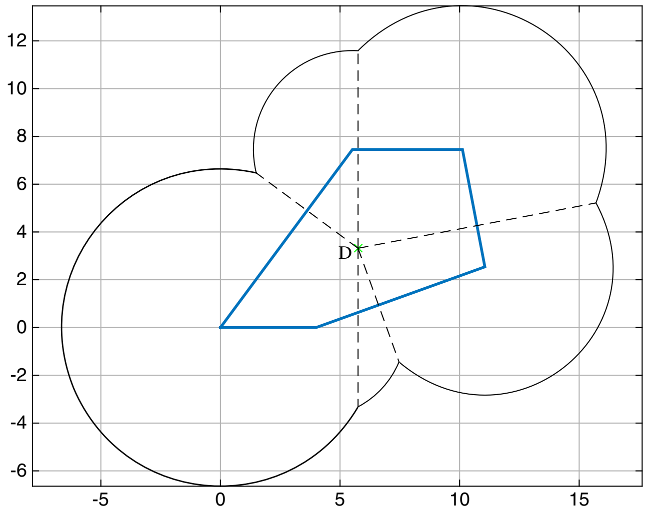


Fig. 11 Example 1: target (outer, curved lines) and $\mathcal{B}$ (inner, straight lines).

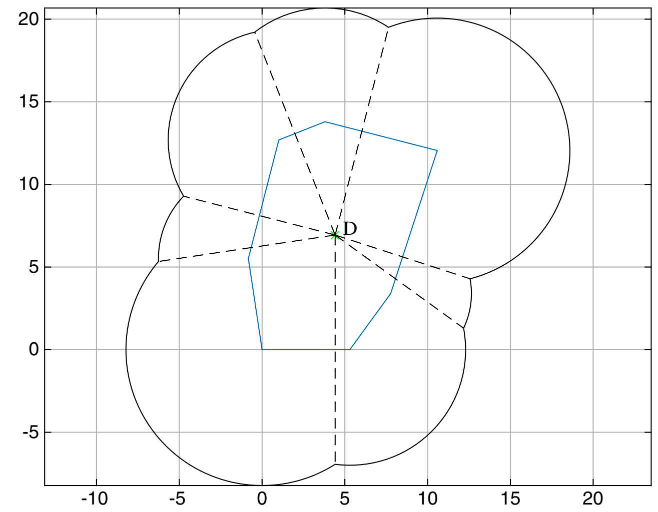


Fig. 12 Example 2: target (outer, curved lines) and $\mathcal{B}$ (inner, straight lines).

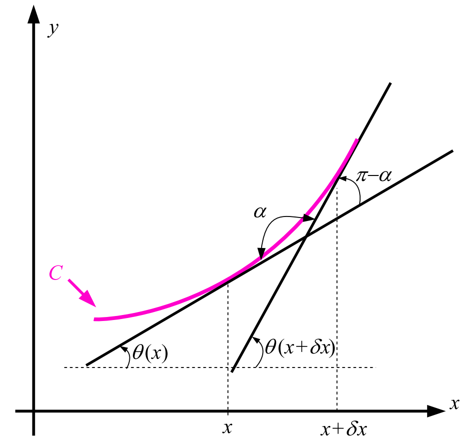


Fig. 13 Approximating $\mathcal{C}$ by a polygon line.

Consequently, let the weight

$$w \equiv \frac{1}{2\pi}[\theta(x + dx) - \theta(x)] = \frac{1}{2\pi}\frac{d\theta}{ds}ds = \frac{1}{2\pi}\frac{1}{R(s)}ds \tag{37}$$

Hence, in view of Theorem 3, we have the following:

Corollary 1: To minimize the area of vulnerability S, the defender D positions himself at $D = (x_D^*, y_D^*)$, where

$$x_D^* = \frac{1}{2\pi}\int_{\mathcal{C}} x(s)\frac{1}{R(s)}\,ds, \qquad y_D^* = \frac{1}{2\pi}\int_{\mathcal{C}} y(s)\frac{1}{R(s)}\,ds \tag{38}$$

VI. Extensions

When the attacker is endowed with a standoff weapon for which the range is R, we have to enlarge the defended target set accordingly; a circular disk target of radius r will become a circular disk target of radius $r + R$. When the target set T is not connected and the attacker $A \notin \mathrm{convhull}(T)$, set $T := \mathrm{convhull}(T)$. This yields a sufficient condition to be able to guarantee the defense of T.

One must also consider the case where the capture radius of D is $l > 0$. To analyze the target defense differential game in the case where the problem parameters are $\mu > 1$ and $l > 0$, we note that the winning regions of the attacker and the defender will be constructed along the lines discussed in Secs. IV and V on the following provisions:

1) When $\mu > 1$ and $l = 0$, the orthogonal bisector is replaced by an Apollonius circle defined by the segment $\overline{AD}$ and the speed ratio μ.

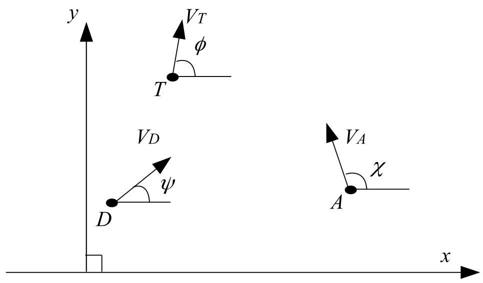


Fig. 14 Dynamic target defense differential game.

2) When $\mu = 1$ and $l > 0$, the orthogonal bisector is replaced by an arc of a hyperbola defined by the segment $\overline{AD}$ and l.

3) When $\mu > 1$ and $l > 0$, the orthogonal bisector is replaced by a Cartesian oval.

In this Note, the solution of the target defense differential game when the target is static is given. But, consider a dynamic point target: A scenario where a mobile target must be actively defended against a homing missile is described in [9]. The dynamic target defense differential game is illustrated in Fig. 14. Denote the speed of the fleeing target as V_T and the attacker/target speed ratio as $\nu = (V_A/V_T) > 1$. D and T form a team that plays against A. This differential game has been addressed in [10–13].

The case where the speed ratio is $\mu < 1$ has some of the attributes of the obstacle tag differential game where singular surfaces are present, and it is not considered herein.

VII. Conclusions

Isaacs's target defense differential game is revisited [8] and, in this Note, Isaacs's geometric method using the theory of differential games is justified. The target of a defense game of kind is solved: The winning regions of the attacker A and defender D are characterized. The optimal positioning of the defender so as to minimize the area of vulnerability of the target set T is calculated. It is also indicated how to address the target defense scenarios where the target set is not bounded, the requirement of point capture is relaxed, and the defender is faster than the attacker; the interesting case where the target is a point target, but is a dynamic target, has previously been addressed by the authors in [10]. Because the defender is not slower than the attacker and point capture is considered, there is a primary flowfield of only optimal trajectories and there are no singular surfaces. Thus, the results obtained in this Note are in closed form, which bodes well for their applicability to realistic target defense scenarios with military applications. In this Note, an additional example to the (admittedly small) repertoire of differential games that can be solved in closed form is provided.

References

[1] Cardaliaguet, P., "A Differential Game with Two Players and One Target," *SIAM Journal on Control and Optimization*, Vol. 34, No. 4, July 1996, pp. 1441–1460.
doi:10.1137/S036301299427223X

[2] Ganebny, S. A., Kumkov, S. S., Le Ménec, S., and Patsko, V. S., "Model Problem in a Line with Two Pursuers and One Evader," *Dynamic Games and Applications*, Vol. 2, No. 2, 2012, pp. 228–257.
doi:10.1007/s13235-012-0041-z

[3] Alpern, S., Fokkink, R., Lindelauf, R., and Olsder, G.-J., "The 'Princess and Monster' Game on an Interval," *SIAM Journal on Control and Optimization*, Vol. 47, No. 3, 2008, pp. 1178–1190.
doi:10.1137/060672054

[4] Huang, H., Zhang, W., Ding, J., Stipanovic, D. M., and Tomlin, C. J., "Guaranteed Decentralized Pursuit-Evasion in the Plane with Multiple Pursuers," *50th IEEE Conference on Decision and Control and European Control Conference*, IEEE Publ., Piscataway, NJ, 2011, pp. 4835–4840.
doi:10.1109/CDC.2011.6161237

[5] Bakolas, E., and Tsiotras, P., "Optimal Pursuit of Moving Targets Using Dynamic Voronoi Diagrams," *49th IEEE Conference on Decision and Control*, IEEE Publ., Piscataway, NJ, 2010, pp. 7431–7436.
doi:10.1109/CDC.2010.5717963

[6] Ratnoo, A., and Shima, T., "Guidance Strategies Against Defended Aerial Targets," *Journal of Guidance, Control, and Dynamics*, Vol. 35, No. 4, 2012, pp. 1059–1068.
doi:10.2514/1.56924

[7] Venkatesan, R. H., and Sinha, N. K., "The Target Guarding Problem Revisited: Some Interesting Revelations," *IFAC Proceeding Volumes*, Vol. 47, No. 3, 2014, pp. 1556–1561.
doi:10.3182/20140824-6-ZA-1003-02297

[8] Isaacs, R., *Differential Games*, Wiley, New York, 1965, pp. 8–20.

[9] Boyell, R. L., "Defending a Moving Target Against Missile or Torpedo Attack," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-12, No. 4, 1976, pp. 522–526.
doi:10.1109/TAES.1976.308338

[10] Pachter, M., Garcia, E., and Casbeer, D. W., "Active Target Defense Differential Game," *52nd Annual Allerton Conference on Communication, Control, and Computing*, IEEE Publ., Piscataway, NJ, 2014, pp. 46–53.
doi:10.1109/ALLERTON.2014.7028434

[11] Garcia, E., Casbeer, D. W., Pham, K., and Pachter, M., "Cooperative Aircraft Defense from an Attacking Missile," *53rd IEEE Conference on Decision and Control*, IEEE Publ., Piscataway, NJ, 2014, pp. 2926–2931.
doi:10.1109/CDC.2014.7039839

[12] Garcia, E., Casbeer, D. W., Pham, K., and Pachter, M., "Cooperative Aircraft Defense from an Attacking Missile Using Proportional Navigation," *AIAA Guidance, Navigation, and Control Conference*, AIAA Paper 2015-0337, 2015.
doi:10.2514/6.2015-0337

[13] Garcia, E., Casbeer, D. W., and Pachter, M., "Cooperative Strategies for Optimal Aircraft Defense from an Attacking Missile," *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 8, 2015, pp. 1510–1520.
doi:10.2514/1.G001083