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Asymptotic optimality for consensus-type stochastic approximation algorithms using iterate averaging

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Abstract: This paper introduces a post-iteration averaging algorithm to achieve asymptotic optimality in convergence rates of stochastic approximation algorithms for consensus control with structural constraints. The algorithm involves two stages. The first stage is a coarse approximation obtained using a sequence of large stepsizes. Then, the second stage provides a refinement by averaging the iterates from the first stage. We show that the new algorithm is asymptotically efficient and gives the optimal convergence rates in the sense of the best scaling factor and 'smallest' possible asymptotic variance.

Keywords: Stochastic approximation algorithm; Consensus; Iterate averaging; Asymptotic optimality

1 Introduction

This paper develops an iterate averaging algorithm for consensus-type controls of networked systems. The main idea is to first run a consensus type-algorithm with a stepsize going to zero slower than $\mathrm{O}(1/n)$, and then take an average of the resulting iterates. Consensus problems appear in many control systems that involve coordination of multiple agents with only limited neighborhood information to reach a global goal for the entire team. The applications are exemplified by multi-agents in robotics, flocking behavior in people and animals, wireless communication networks, sensor networks, platoon formation in ground and aerial vehicles, distributed computing, biological systems, etc. Our interest in this problem is motivated by cooperative and coordinated control of unmanned aerial vehicles (UAVs). Air force missions that require information extraction, like coordinated target search and tracking, are a prime example of the scenarios that require networked decision making with uncertainty for which stochastic optimization algorithms play a big role. Imagine you have a collection of UAV assets tasked with searching a forward operating location for the presence of targets. Decisions must be made about individual UAV task assignments, since the collections of UAVs might be heterogeneous with regard to capabilities. In addition, tracking potential targets over long distances may require 'target hand-off' that must be coordinated among teams of UAVs. In this type of scenario, the UAVs might even be required to take on 'human-level' decisions such as firing a weapon at a potential target or target classification. There are plenty of low level decisions that are part of this problem. These tasks include (but are not limited to): autonomous task assignment, coordinated rendezvous, autonomous team formation and realignment, optimal package (information) delivery, loiter and target revisit decisions for optimal information gain in target classification, and optimal vehicle routing. All of these types of tasks can be embedded within a scenario where the UAVs should be able to make some or all of these decisions with varying levels of stochastic uncertainties. Uncertainties are too numerous to list, but a partial list includes: weather (wind, shadows, clouds, rain), communication network delays and failures, true states of potential targets, true states of systems being controlled, human errors in model design and target classification, adversarial strategies, and a UAV's current capabilities.

Owing to the wide variety of applications, detailed system descriptions vary substantially and diversified methodologies are needed to treat such systems. Nevertheless, there is a common thread, the use of an online recursive stochastic approximation (SA) algorithm. There is extensive literature on consensus control in a variety of application areas, including load balancing in parallel computing [1–4], sensor networks [5–8], team formation [9], decentralized filtering, estimation, and data fusion [9–13], mobile agents [14–17], flocking behavior and swarms [18–21], and physics [21], etc. Applications of stochastic approximation algorithms and theoretical developments in related consensus control problems were reported in [22–24]. Switching network topologies were studied in [15, 25–26]. More recently, reference [27] employed a method on the convergence of products of stochastic matrices that uses randomly switching Laplacian matrices together with observation noises that may be state-dependent and Markovian based. In [28], we used a Markov model and treated a much larger class of noises, where the network graph is modulated by a discrete-time Markov chain. In addition to convergence and rates of convergence, a multiscale struc-

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ture, which captures differences between state adaptation speeds and topology switching frequencies, was explored fully. Related stochastic differential equations and switching stochastic equations were obtained.

Benefit of iterate averaging The core algorithms of these consensus control strategies are SA procedures. When the switching topology reduces to a fixed topology, our work in [28–29] provides the convergence and rates of convergence of the recursive algorithm. Once the convergence and the rate of convergence of the algorithm are ensured, an important question to address is: Can we further improve convergence rates to achieve efficiency? We take up this issue here. To motivate our work, let us illustrate the issues with a simple stochastic approximation algorithm. The problem is to find the zero θ_* of a function f based on noisy measurements y_n using

$$\theta_{n+1} = \theta_n + \mu_n y_n, \;\; y_n = f(\theta_n) + \xi_n,$$

where $\{\xi_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and covariance S_0, and $\{\mu_n\}$ is a sequence of stepsizes satisfying $\mu_n \geqslant 0$, $\mu_n \to 0$, and $\sum\limits_n \mu_n = \infty$. For simplicity of discussion in this section, we have assumed $\{\xi_n\}$ to be a sequence of i.i.d. random variables; more general noise sequences with substantially more complex dependence structure can be accommodated (see [30]). The main requirements for ξ_n are the first and second moment conditions; the precise distributional information need not be known. It is well known (see [30]) that under suitable conditions, $\theta_n \to \theta_*$ with probability one (w.p.1) or in some weak sense. To help understand the performance, consider the random variable $\hat{u}_n = (\theta_n - \theta_*)/\mu_n^\alpha$ and choose $\alpha > 0$ such that $\hat{u}_n$ converges in distribution to a nontrivial limit. The scaling factor α together with the asymptotic covariance of the scaled sequence, $\hat{u}_n$, gives the rate of convergence, and characterizes the dependence of the estimate on the stepsize and also the 'goodness' of the approximation. Herein lies the tradeoff; one must choose the largest α possible for fast convergence, while still maintaining an acceptable estimation error (small covariance).

The exploration on the improvement of efficiency began with [31], in which one considers $\theta_{n+1} = \theta_n + \dfrac{\hat{\Gamma}}{n}(f(\theta_n) + \xi_n)$, linearizing f near θ_* with $f(\theta) = f(\theta_*) + \hat{H}(\theta - \theta_*) + \mathrm{O}(|\theta - \theta_*|^2)$, where $\hat{H}$ is the Jacobian of f and is a stable matrix. It can be shown that $\sqrt{n}(\theta_n - \theta_*) \to N(0, \Sigma)$, where $\Sigma(\hat{\Gamma})$ satisfies the following Liapunov equation $(\dfrac{I}{2} + \hat{\Gamma}\hat{H})\Sigma + \Sigma(\dfrac{I}{2} + \hat{\Gamma}\hat{H})' = -\hat{\Gamma}S_0\hat{\Gamma}'$, and S_0 is the covariance matrix of the noise. By means of algebraic comparisons [32, Theorem 1], choosing $\hat{\Gamma} = -\hat{H}^{-1}$, the optimal covariance matrix Σ^* is obtained as $\Sigma^* = \hat{H}^{-1}S_0(\hat{H}^{-1})'$. Because $\hat{H}^{-1}$ is related to the gradient of f at θ_* and is unlikely to be known, an idea is then to replace $\hat{\Gamma}$ by $\hat{\Gamma}_n$ and construct another sequence of estimates $\hat{\Gamma}_n$. This, however, is not very practical, especially for large dimensional problems. In the late 1980s, Polyak [33] and Ruppert [34] independently proposed and analyzed an interesting computation scheme for recursive SA algorithms. The main idea of their approaches is the averaging of iterates obtained from a classical SA algorithm with slowly varying gains. They showed the desired optimality can be achieved by this averaging procedure. Further work on SA under more general settings can be found in [30, Chapter 11].

The benefits of the iterate averaging algorithm can be summarized by the following items: 1) The difficulty of selecting a good stepsize sequence $\{\mu_n\}$ in application is a handicap, and iterate averaging alleviates this difficulty by providing a systematic approach. 2) With the use of a large stepsize, i.e., one going to zero slower than $\mathrm{O}(1/n)$, the algorithm forces the estimates to move towards the true parameter more quickly. 3) Iterate averaging smoothes out the noise effect and reduces the 'variance' of the noise. As a result, it gives the best convergence rate with the best scaling factor and the 'smallest asymptotic covariance'. Further insight on this can be found in [30, Chapter 11]. It can also be shown that this optimality is related to the well-known Cramér-Rao lower bound (see [35]).

Using such an idea in this paper, we build algorithms using iterate averaging for the purpose of reaching consensus. Rather than dealing with the well-known consensus algorithms, we treat general classes of noise that can cover many communication schemes as an integrated part of networked systems. Nevertheless, neither the rate of convergence nor the optimality of a consensus-type algorithm can be obtained directly from existing results in SA theory. The matrix $\hat{H}$ in the above paragraph needs to be Hurwitz. However, for our consensus problem, the corresponding matrix M (to be precisely defined in the following section) is a generator of a continuous-time Markov chain, which has a zero eigenvalue that makes the existing results not applicable. To overcome this difficulty, we use the irreducibility of M, which indicates that apart from zero, all other eigenvalues have negative real parts. We use the ordinary differential equation (ODE) approach (see [30]) in our analysis. In lieu of working with the discrete iterates directly, we take a continuous-time interpolation of the iterates. Then, using compactness, we can show that the resulting sequence of functions converges to a solution to the ODE.

The rest of the paper is organized as follows. First, we propose the two-stage recursive algorithm in Section 2. Section 3 obtains convergence, rate of convergence, and asymptotic efficiency. Section 4 provides numerical examples to illustrate the asymptotes. Finally, Section 5 concludes the paper with further remarks.

2 Algorithms

We begin with a simple additive noise model. Then, we extend the idea to include more general cases of correlated noise.

2.1 Networked observation and control

Consider a networked system of r nodes

$$x_{n+1}^i = x_n^i + u_n^i, \;\; i = 1, \ldots, r, \tag{1}$$

where u_n^i is the node control for the ith node, or in a vector form

$$x_{n+1} = x_n + u_n$$

with $x_n = [x_n^1 \ \cdots \ x_n^r]'$, $u_n = [u_n^1 \ \cdots \ u_n^r]'$. The nodes are linked by a sensing network, represented by a directed graph $\mathcal{G}$ whose element (i, j) indicates estimation of the state x_n^j by node i via a communication link, and a permitted control v^{ij} on the link. For node i, $(i, j) \in \mathcal{G}$ is a departing edge and $(l, i) \in \mathcal{G}$ is an entering edge. The total number of communication links in $\mathcal{G}$ is $l_{\rm s}$. From its physical meaning, node i can always observe its own state, which will not be considered as a link in $\mathcal{G}$.

Here, we confine our attention to transportation control on the links among nodes permitted by $\mathcal{G}$. That is, the node control u_n^i is not an independent design variable, but is determined by the independent link control v_n^{ij}. Note that a positive transportation of quantity v_n^{ij} on (i, j) means a loss of v_n^{ij} at node i and a gain of v_n^{ij} at node j. The node control at node i is

$$u_n^i = -\sum_{(i,j)\in\mathcal{G}} v_n^{ij} + \sum_{(j,i)\in\mathcal{G}} v_n^{ji}. \tag{2}$$

The most relevant implication in this control scheme is that for all n,

$$\sum_{i=1}^r x_n^i = \sum_{i=1}^r x_0^i := \eta r,$$

for some $\eta \in \mathbb{R}$ that is the average of x_0. That is, $\eta = \sum_{i=1}^r x_0^i/r$. Consensus control seeks control algorithms such that $x_n \to \eta\mathbb{1}$, where $\mathbb{1}$ is the column vector of all ones.

A link $(i, j) \in \mathcal{G}$ entails an estimate $\hat{x}_n^{ij}$, of x_n^j by node i with estimation error d_n^{ij}. That is,

$$\hat{x}_n^{ij} = x_n^j + d_n^{ij}. \tag{3}$$

The estimation error d_n^{ij} is a function of the signal x_n^j itself and depends on communication channel noises ξ_n^{ij} in a nonadditive and nonlinear relation

$$d_n^{ij} = g(x_n^j, \xi_n^{ij}) \tag{4}$$

and can be spatially and temporally dependent. Most existing literature considers much simplified noise classes, e.g., $d_n^{ij} = \xi_n^{ij}$ with i.i.d. assumptions.

For simplification on system derivations, we use first $d_n^{ij} = \xi_n^{ij}$ here. Let $\tilde{\eta}_n$ and ξ_n be the $l_{\rm s}$ dimensional vectors that contain all $\hat{x}_n^{ij}$ and ξ_n^{ij} in a selected order, respectively. Then, (3) can be written as

$$\tilde{\eta}_n = H_1 x_n + \xi_n, \tag{5}$$

where H_1 is an $l_{\rm s} \times r$ matrix whose rows are elementary vectors such that if the ℓth element of $\tilde{\eta}_n$ is $\hat{x}^{ij}$ then the ℓth row in H_1 is the row vector of all zeros except for a one at the jth position. Each sensing link provides information $\delta_n^{ij} = x_n^i - \hat{x}_n^{ij}$, an estimated difference between x_n^i and x_n^j. This information may be represented, in the same arrangement as $\tilde{\eta}_n$, by a vector δ_n of size $l_{\rm s}$ containing all δ_n^{ij} in the same order as $\tilde{\eta}_n$, where δ_n can be written as

$$\delta_n = H_2 x_n - \tilde{\eta}_n = H_2 x_n - H_1 x_n - \xi_n = H x_n - \xi_n, \tag{6}$$

where H_2 is an $l_{\rm s} \times r$ matrix whose rows are elementary vectors such that if the ℓth element of $\tilde{\eta}_n$ is $\hat{x}^{ij}$ then the ℓth row in H_2 is the row vector of all zeros except for a one at the ith position, and $H = H_2 - H_1$. Due to network constraints, the information δ_n^{ij} can only be used by nodes i and j. When the control is linear, time invariant, and memoryless, we have $v_n^{ij} = \mu g_{ij} \delta_n^{ij}$, where g_{ij} is the link control gain and μ is a global scaling factor which will be used in state updating algorithms as the recursive step size. Let G be the $l_{\rm s} \times l_{\rm s}$ diagonal matrix that has g_{ij} as its diagonal element. In this case, the node control becomes $u_n = -\mu H' G \delta_n$. For convergence analysis, we note that μ is a global control variable and we may represent u_n equivalently as

$$\begin{aligned} u_n &= -\mu H' G(Hx_n - \xi_n) \\ &= -\mu(H'GHx_n - H'G\xi_n) \\ &= \mu(Mx_n + W\xi_n), \end{aligned}$$

with $M = -H'GH$ and $W = H'G$. The following assumption is intended to be an assumption on the network.

A0) 1) All link gains are positive, $g_{ij} > 0$.

2) $\mathcal{G}$ contains a complete tree.

Recall that a square matrix $\tilde{Q} = (\tilde{q}_{ij})$ is a generator of a continuous-time Markov chain if $\tilde{q}_{ij} \geqslant 0$ for all $i \neq j$ and $\sum_j \tilde{q}_{ij} = 0$ for each i. Also, a generator or the associated continuous-time Markov chain is irreducible if the system of equations

$$\begin{cases} \nu\tilde{Q} = 0, \\ \nu\mathbb{1} = 1 \end{cases} \tag{7}$$

has a unique solution, where $\nu = [\nu_1 \ \cdots \ \nu_r] \in \mathbb{R}^{1\times r}$ with $\nu_i > 0$ for each $i = 1, \ldots, r$ is the associated stationary distribution. Under assumption A0), we can show that 1) M has rank $r - 1$ and is negative semidefinite, and 2) M is a generator of a continuous-time Markov chain, and is irreducible; see [28] for a proof.

2.2 Algorithms

Based on the discussion of last section, we propose a class of stochastic approximation algorithms. In consideration of extensive early work on consensus control, we shall go to the algorithms directly. For previous work on such algorithms, we refer the reader to the references in [28]. Suppose $x \in \mathbb{R}^r$ and $W \in \mathbb{R}^{r\times r_1}$, $\hat{W} : \mathbb{R}^r \times \mathbb{R}^{r_1} \mapsto \mathbb{R}^r$. We begin by considering the following state updating algorithm:

$$x_{n+1} = x_n + \mu_n M x_n + \mu_n W \xi_n + \mu_n \hat{W}(x_n, \zeta_n), \tag{8}$$

together with the constraint

$$\mathbb{1}' x_n = \beta r, \tag{9}$$

where $\{\mu_n\}$ is a sequence of stepsizes, M is an irreducible generator of a continuous-time Markov chain (hence $\mathbb{1}'M = 0$ and $\mathrm{rank}M = r - 1$), $\{\xi_n\}$ and $\{\zeta_n\}$ are noise sequences taking values in $\mathbb{R}^{r_1}$, β is the team average, and consensus control aims to control each team member's state towards β. For example, in computer load balancing problems, β is the average perprocessor work load. Equal distribution of the total work load on multiple processors permits efficient utility of computing resources. In flight coordination of team UAVs, β may be the average speed of the team. In terms of the consensus control in this paper, the goal is to move the team in a uniform speed, without changing the team speed as a pack. The algorithm includes an additive noise as well as a non-additive noise. Therefore, the solution is sufficiently general to include many practical senarios in the setup. The stepsize satisfies the following conditions: $\mu_n \geqslant 0$, $\mu_n \to 0$ as $n \to \infty$, and $\sum_n \mu_n = \infty$. Some commonly used stepsize sequences include $\mu_n = a/n$ and $\mu_n = a/n^\gamma$ for $0 < \gamma \leqslant 1$. Since al-

gorithm (8) is a stochastic approximation procedure, we can use the general framework in [30] to analyze the asymptotic properties. Before proceeding further, we make a remark. If we assume that $W\mathbb{1} = 0$ and $\hat{W}(x,\zeta)\mathbb{1} = 0$ for each x and each ζ, then $\mathbb{1}'x_{n+1} = \mathbb{1}'x_n = r\beta$ hold for all n and for some $\beta \in \mathbb{R}$ (In the algorithms considered in the literature, one often begins with $\hat{W} = 0$ and W having the condition mentioned above). Thus, in this case, the constraint $\mathbb{1}'x_n = r\beta$ is always satisfied by the algorithm structure.

A1) The noise $\{\xi_n\}$ is a stationary, ϕ-mixing sequence such that $E\xi_n = 0$, $E|\xi_n|^{2+\Delta} < \infty$ for some $\Delta > 0$, and the mixing measure $\tilde{\phi}_n$ satisfies $\sum_{k=0}^{\infty} \tilde{\phi}_n^{\frac{\Delta}{1+\Delta}} < \infty$, where $\tilde{\phi}_n = \sup_{A\in\mathcal{F}^{n+m}} E^{\frac{1+\Delta}{2+\Delta}} \big|P(A|\mathcal{F}_m) - P(A)\big|^{\frac{2+\Delta}{1+\Delta}}$, $\mathcal{F}_n = \sigma\{\xi_n; k < n\}$, $\mathcal{F}^n = \sigma\{\xi_n; k \geqslant n\}$.

A2) i) The noise sequence $\{\zeta_n\}$ is a stationary sequence that is uniformly bounded and ϕ-mixing with mixing measure $\hat{\phi}_n$ such that for each $x \in \mathbb{R}^r$, $E\hat{W}(x,\zeta_n) = 0$, and the mixing rate condition holds with $\tilde{\phi}_n$ replaced by $\hat{\phi}_n$. ii) $\hat{W}(\cdot,\zeta)$ is a continuous function for each ζ and $|\hat{W}(x,\zeta)| \leqslant K(1+|x|)$ for each $x \in \mathbb{R}^r$ and ζ. iii) $\{\xi_n\}$ and $\{\zeta_n\}$ are mutually independent.

Recall that we have assumed that M is a generator of a continuous-time Markov chain and is irreducible. One of the consequences of this above assumption is that M has zero as an eigenvalue with multiplicity one and all other eigenvalues have negative real parts. Another distinct feature of M is that the null space of M is spanned by the vector $\mathbb{1} = (1,\ldots,1)' \in \mathbb{R}^r$. This characteristic is precisely why we can reach consensus. As a consequence of A2), ϕ-mixing implies that the noise sequences $\{\xi_n\}$ and $\hat{W}(x,\zeta_n)$ for each fixed x are strongly ergodic [36, pp. 488] implying that as $n \to \infty$, we have

$$\begin{cases} \dfrac{1}{n}\sum\limits_{j=m}^{m+n-1} \xi_j \to 0 \ \text{ w.p.1}, \\ \dfrac{1}{n}\sum\limits_{j=m}^{m+n-1} \hat{W}(x,\zeta_j) \to 0 \ \text{ w.p.1}. \end{cases} \tag{10}$$

If we are only interested in weak convergence, then we only need $\dfrac{1}{n}\sum\limits_{j=m}^{m+n-1} E_m\hat{W}(x,\zeta_j) \to 0$ in probability, where E_m denotes the conditioning on the σ-algebra $\mathcal{F}_m = \{\xi_{j-1},\zeta_{j-1} : j \leqslant m\}$.

Idea of technical development To study the convergence of the algorithm using the stochastic approximation methods developed in [30] instead of working with the discrete-time iterations, we examine sequences defined in an appropriate function space. This will enable us to get a limit ordinary differential equation (ODE). The significance of the ODE is that the stationary points are exactly the true parameters we wish to estimate. We define

$$t_n = \sum_{j=0}^{n-1} \mu_j, \ \ m(t) = \max\{n : t_n \leqslant t\}, \tag{11}$$

the piecewise constant interpolation $x^0(t) = x_n$ for $t \in [t_n, t_{n+1})$, and the shift sequence $x^n(t) = x^0(t+t_n)$. Due to the page limitation, we shall only outline the main steps involved below. We can first derive a preliminary estimate on the second moments.

Lemma 1 Under A1) and A2), for any $0 < T < \infty$,

$$\sup_{n\leqslant m(T)} E|x_n|^2 \leqslant K \text{ and } \sup_{0\leqslant t\leqslant T} E|x^n(t)|^2 \leqslant K, \tag{12}$$

for some $K > 0$, where $m(\cdot)$ is defined in (11).

Proof We only indicate the main ideas and leave most of the details out. Concerning the first estimate, because of the boundedness of the second moment $E|\xi_n|^2$, the condition $\sum\limits_{j=1}^{\infty} \mu_j^2 < \infty$, the boundedness of the nonadditive noise $\hat{W}(x,\zeta_n)$, and the linear growth of $\hat{W}(\cdot,\zeta)$ for each ζ, we can derive

$$E|x_n|^2 \leqslant K + K\sum_{j=1}^{n} \mu_j E|x_j|^2. \tag{13}$$

Here and henceforth, K is used as a generic positive constant, whose values may change for different usage. After an application of Grownwall's inequality to (13), and then taking the supremum over all $n \leqslant m(T)$, the first error bound is obtained. Likewise, we can obtain the second estimate. This completes the proof.

Theorem 1 Under assumptions A1) and A2), the iterates generated by the stochastic approximation algorithm (8) satisfy $x_n \to \beta\mathbb{1}$ w.p.1 as $n \to \infty$.

Proof We only present the main idea below. We show that $\{x^n(\cdot)\}$ is equicontinuous in the extended sense (see [30, pp. 102] for a definition) w.p.1. To verify this, we note that by the argument in the first part of the proof in [37, Theorem 3.1],

$$\sum_{j=1}^{\infty} \mu_j\xi_j \text{ converges w.p.1 and}$$
$$\sum_{j=1}^{\infty} \mu_j\hat{W}(x,\zeta_j) \text{ converges w.p.1 for a fixed } x.$$

Define $\Phi^0(t) = \sum\limits_{j=1}^{m(t)-1} \mu_j[\xi_j + \hat{W}(x,\zeta_j)]$ and $\Phi^n(t) = \Phi^0(t_n+t)$, where $m(\cdot)$ is defined in (11). Then, we can show that for each $T > 0$ and $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\limsup_n \sup_{0\leqslant|t-s|\leqslant\delta} |\Phi^n(t) - \Phi^n(s)| \leqslant \varepsilon \ \text{ w.p.1}.$$

The above estimate together with the form of the recursion then implies that $x^n(\cdot)$ is equicontinuous in the extended sense. Next, we can extract a convergent subsequence, which will be denoted by $x^{n_\ell}(\cdot)$. Then, the Arzela-Ascoli theorem concludes that $x^{n_\ell}(\cdot)$ converges to a function $x(\cdot)$ which is the unique solution (since the recursion is linear in x) of the ordinary differential equation (ODE)

$$\dot{x}(t) = Mx(t). \tag{14}$$

Owing to the law of large numbers, the noise is averaged out. What is the significance of the limit ODE? To answer this question, we set the right-hand side of (14) equal to zero ($Mx = 0$). We then obtain the stationary point of the ODE. Since M is a generator of a continuous-time Markov chain and is irreducible, the solutions to $Mx = 0$ constitute precisely the null space of M. The null space of M is spanned by the vector $\mathbb{1}$. That is, the set can be represented by $\Gamma = \{\gamma : \gamma = \gamma_0\mathbb{1}, \gamma_0 \in \mathbb{R}\}$. Moreover, from basic properties of Markov chains (see [38, Appendix A.1]), as

$t \to \infty$, the solution $x(t)$ to (14) satisfies that $x(t)$ converges to the set Γ. That is, $\mathrm{dist}(x(t), \Gamma) \to 0$ as $t \to \infty$, where $\mathrm{dist}(\cdot, \cdot)$ is the usual distance function defined by $\mathrm{dist}(x, \Gamma) = \inf_{y \in \Gamma} |x - y|$. Consequently, as $n \to \infty$ and $q(n_\ell) \to \infty$, $x^{n_\ell}(\cdot + q(n_\ell)) \to \Gamma$.

Furthermore, the algorithm (8) together with $x_n' \mathbb{1} = r\beta$ leads to the desired conclusion. The equilibria of the limit ODE (14) and this constraint lead to the following system of equations:

$$\begin{cases} Mx = 0, \\ \mathbb{1}' x = r\beta. \end{cases} \tag{15}$$

The irreducibility of M then implies that (15) has a unique solution $x_* = \beta \mathbb{1}$. In fact, by defining an augmented matrix $M_{\mathrm{a}} = \begin{pmatrix} M \\ \mathbb{1}' \end{pmatrix} \in \mathbb{R}^{(r+1)\times r}$, the above system may be written as

$$M_{\mathrm{a}} x = \begin{pmatrix} 0 \\ r\beta \end{pmatrix} := b_{\mathrm{a}} \in \mathbb{R}^{(r+1)\times 1}. \tag{16}$$

Note that $M_{\mathrm{a}}' M_{\mathrm{a}}$ has full rank owing to the irreducibility of M. Thus the solution to (16) can be written as $x_* = (M_{\mathrm{a}}' M_{\mathrm{a}})^{-1} M_{\mathrm{a}}' b_{\mathrm{a}} = \beta \mathbb{1}$.

3 Asymptotic efficiency

To improve the efficiency we average iterates, resulting in a two-stage stochastic approximation algorithm. The idea is that we first obtain a coarse approximation by using a relatively large stepsize, and then we refine the approximation by taking an iterate average. For definiteness and simplicity, we take $\mu_n = 1/n^\gamma$ for some $1/2 < \gamma < 1$. The algorithm is given as follows:

$$\begin{cases} x_{n+1} = x_n + \dfrac{1}{n^\gamma} M x_n + \dfrac{1}{n^\gamma} W \xi_n + \dfrac{1}{n^\gamma} \hat{W}(x_n, \zeta_n), \\ \bar{x}_{n+1} = \bar{x}_n - \dfrac{1}{n+1} \bar{x}_n + \dfrac{1}{n+1} x_{n+1}. \end{cases} \tag{17}$$

If we assume that $W\mathbb{1} = 0$ and $\hat{W}(x, \zeta)\mathbb{1} = 0$ for each x and each ζ, then $\mathbb{1}' \bar{x}_n = r\beta$.

Theorem 2 Suppose the conditions of Theorem 1 are satisfied. For iterates generated by algorithm (17) (together with the constraint $\mathbb{1}' x_n = r\beta$), $x_n \to \beta \mathbb{1}$ w.p.1 as $n \to \infty$.

Similar to what was alluded to in the introduction, the benefits of iterate averaging for the consensus algorithm include a faster approach to a neighborhood of the true parameter in its initial stage, a straightforward way of selecting the stepsize sequences, and the optimal convergence rate. To emphasize the dimension of the vector $\mathbb{1}$, we sometimes write $\mathbb{1}_\kappa$ for an integer κ in what follows. Since M has rank $r-1$, without loss of generality, assume that the first $r-1$ columns are independent. Partition the matrices M and W as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{18}$$

where $M_{11} \in \mathbb{R}^{(r-1)\times(r-1)}$, $M_{12} \in \mathbb{R}^{(r-1)\times 1}$, $M_{21} \in \mathbb{R}^{(r-1)\times 1}$, $M_{22} \in \mathbb{R}^{1\times 1}$, and similarly for W_{ij}. Then, M_{11} is nonsingular. Accordingly, we partition x_n, $\bar{x}_n$, and W as

$$x_n = \begin{pmatrix} \tilde{x}_n \\ x_{n,r} \end{pmatrix}, \quad \bar{x}_n = \begin{pmatrix} \Theta_n \\ \bar{x}_{n,r} \end{pmatrix}, \tag{19}$$

respectively, with compatible dimensions as those of M. We will assume another condition. This condition essentially is a linearization of $\hat{W}$ about the point x_*. Note that in A3) below, $\hat{W}_0 = \hat{W}_x(x_*, \zeta)$. Partition $\hat{W}_0$, ξ, and x_* similar to that of W and x, respectively. Our rate of convergence is a local analysis.

A3) $\hat{W}(x, \zeta) = \hat{W}(x_*, \zeta) + \hat{W}_0(x - x_*) + \mathrm{O}(|x - x_*|^2)$.

Note that $x_{n,r} = \beta r - \mathbb{1}_{r-1}' \tilde{x}_n$ and $\bar{x}_{n,r} = \beta r - \mathbb{1}_{r-1}' \Theta_n$. Using this together with the partition and A3), we can convert the constrained stochastic approximation to an unconstrained one. That is, we can concentrate on the first $r-1$ components of x_n. It follows from (17) that

$$\begin{cases} \tilde{x}_{n+1} = \tilde{x}_n + \dfrac{1}{n^\gamma} \tilde{M} \tilde{x}_n + \dfrac{1}{n^\gamma} [\hat{\xi}_n + \hat{W}_1(x_*, \zeta_n)] \\ \qquad + \dfrac{1}{n^\gamma} [\tilde{W}_0 (\tilde{x}_n - \tilde{x}_*) + M_{12} \beta r \\ \qquad + \mathrm{O}(|\tilde{x}_n - \tilde{x}_*|^2)], \\ \Theta_{n+1} = \Theta_n - \dfrac{1}{n+1} \Theta_n + \dfrac{1}{n+1} \tilde{x}_{n+1}, \end{cases} \tag{20}$$

where

$$\tilde{M} = M_{11} - M_{12} \mathbb{1}_{r-1}', \ \ \hat{\xi}_n = W_{11} \tilde{\xi}_n + W_{12} \xi_{n,r},$$
$$\tilde{W}_0 = \hat{W}_{0,11} - \hat{W}_{0,12} \mathbb{1}_{r-1}',$$

and $\hat{W}_1(x_*, \zeta)$ is an $(r-1)$-vector consisting of the first $(r-1)$ components of $\hat{W}(x_*, \zeta)$. Similar to Theorem 1, we can show that $\tilde{x}_n \to \tilde{x}_* = -\tilde{M}^{-1} M_{12} \beta r$. Furthermore, we can show that $\Theta_n \to \tilde{x}_*$ w.p.1 as $n \to \infty$. Note that when we define $\tilde{z} = \tilde{x} - \tilde{x}_*$ and substitute it into (20), the term involving $M_{12}\beta r$ will disappear. To study the rates of convergence of x_n, we need only examine that of $\tilde{x}_n$. To proceed, define

$$B_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} [\hat{\xi}_k + \hat{W}(x_*, \zeta_n)], \ \ t \in [0, 1], \tag{21}$$

where $\lfloor t \rfloor$ denotes the integer part of t. We have the following lemma.

Lemma 2 Under A2), $B_n(\cdot)$ converges weakly to $B(\cdot)$ an $\mathbb{R}^{r-1}$-dimensional Brownian motion such that $EB(t) = 0$ and covariance $\Sigma_0 t$, where

$$\Sigma_0 = E[\hat{\xi}_1 \hat{\xi}_1' + \hat{W}_1(1) \hat{W}_1'(1)] + \sum_{k=2}^{\infty} E[\hat{\xi}_1 \hat{\xi}_k' + \hat{\xi}_k \hat{\xi}_1'] + \sum_{k=2}^{\infty} E[\hat{W}_1(k) \hat{W}_1'(1) + \hat{W}_1(1) \hat{W}_1'(k)], \tag{22}$$

where $\hat{W}_1(k)$ is an abbreviation of $\hat{W}_1(x_*, \zeta_k)$.

Proof Note that $E\hat{\xi}_n = 0$, and it is also a mixing sequence satisfying the conditions of A2). The same observation holds for the sequence $\{\hat{W}_1(x_*, \zeta_n)\}$. Next, $B_n(t) = B_n^1(T) + B_n^2(t)$, where $B_n^1(t)$ and $B_n^2(t)$ are rescaled sequences of sums of $\hat{\xi}_k$'s and $\hat{W}(k)$, respectively. It can be shown that (see [39, Lemma 3.1]), $B_n^i(\cdot)$ converges weakly to a Brownian motion $B^i(\cdot)$. Next, $\{\hat{\xi}_n\}$ and $\{\hat{W}_1(x_*, \zeta_n)\}$ are independent. The sum of $B^1(t) + B^2(t)$ is again a Brownian motion and with the desired covariance given by (22). This completes the proof.

Working with (20), we obtain

$$\tilde{x}_{n+1}-\tilde{x}_* = [\tilde{x}_n-\tilde{x}_*]+\frac{\varGamma}{n^\gamma}(\tilde{x}_n-\tilde{x}_*) +\frac{1}{n^\gamma}[\hat{\xi}_n+\hat{W}_1(x_*,\zeta_n)]+\frac{1}{n^\gamma}(|\tilde{x}_n-\tilde{x}_*|^2), \tag{23}$$

where $\varGamma=\tilde{M}+\tilde{W}_0$.

Define

$$A_{nj}=\begin{cases}\prod\limits_{k=j+1}^{n}\Big(I+\dfrac{\varGamma}{k^\gamma}\Big), & n\geqslant j+1;\\ I, & n=j.\end{cases}$$

Then, for any $\kappa\geqslant 0$,

$$\begin{aligned}\tilde{x}_{n+1}-\tilde{x}_* =\ & A_{n,\kappa-1}[\bar{x}_\kappa-\tilde{x}_*]\\ &+\sum_{j=\kappa}^{n}\frac{1}{j^\gamma}A_{nj}\mathrm{O}(|\tilde{x}_n-\tilde{x}_*|^2)\\ &+\sum_{j=\kappa}^{n}\frac{1}{j^\gamma}A_{nj}[\hat{\xi}_j+\hat{W}_1(j)],\end{aligned}$$

and

$$\begin{aligned}&\sqrt{n+1}[\varTheta_{n+1}-\tilde{x}_*]\\ &=\frac{1}{\sqrt{n+1}}\sum_{k=1}^{\kappa-1}[\tilde{x}_k-\tilde{x}_*]\\ &\quad+\frac{1}{\sqrt{n+1}}\sum_{k=\nu}^{n}A_{k,\kappa-1}[\tilde{x}_\kappa-\tilde{x}_*]\\ &\quad+\frac{1}{\sqrt{n+1}}\sum_{k=\nu}^{n}\sum_{j=\kappa}^{k}\frac{1}{j^\gamma}A_{kj}\mathrm{O}(|\tilde{x}_j-\tilde{x}_*|^2)\\ &\quad+\frac{1}{\sqrt{n+1}}\sum_{k=\nu}^{n}\sum_{j=\kappa}^{k}\frac{1}{j^\gamma}A_{kj}[\hat{\xi}_j+\hat{W}_1(j)].\end{aligned}$$

Note that

$$|A_{nj}|\leqslant\exp\big(-\lambda\sum_{k=j+1}^{n}k^{-\gamma}\big)$$

for some $\lambda>0$. In what follows, we choose

$$\kappa=\kappa(n)=\Big[\big(\frac{1-\gamma}{\lambda}\big)\ln(\ln n)\Big]^{\frac{1}{1-\gamma}}.$$

To proceed, we define

$$\bar{B}_n(t)=\frac{\lfloor nt\rfloor}{\sqrt{n}}(\varTheta_{\lfloor nt\rfloor+1}-\tilde{x}_*). \tag{24}$$

We next show that asymptotically, the 'effective' term of the normalized error above is given by $-\varGamma^{-1}B_n(t)$.

Lemma 3 In addition to the assumptions of A1)– A3), assume $\varGamma$ is a stable matrix (all of its eigenvalues have negative real parts). Then, for $t\in[0,1]$,

$$\bar{B}_n(t)=-\varGamma^{-1}B_n(t)+o(1),\ \text{ where }\ o(1)\to 0$$

in probability uniformly in t as $n\to\infty$.

Remark 1 In the absence of the nonadditive noise, $\varGamma$ becomes $\tilde{M}$. The stability of $\tilde{M}$ is verified by using the irreducibility of the generator M.

We are now ready to present the following theorem.

Theorem 3 Under the conditions of Lemma 3, we have the following assertions:

- $\bar{B}_n(\cdot)$ converges weakly to $\bar{B}(\cdot)$, a Brownian motion with covariance $\varGamma^{-1}\varSigma_0(\varGamma^{-1})'t$;
- $\tilde{x}_n-\tilde{x}_*$ converges in distribution to a normal random variable with mean 0 and asymptotic covariance $\varGamma^{-1}\varSigma_0(\varGamma^{-1})'t$.

Proof We will be very brief. To prove the first part of the theorem, we need only to evaluate its covariance, which follows by the well-known Slutsky theorem. To obtain the second part, set $t=1$ in part one. Using Lemma 3 and part of the theorem, the desired result follows.

3.1 Matrix stepsize and optimality

The rate of convergence of algorithm (8) is equivalent to that of the first recursion in (20). This algorithm satisfies the sensing topology constraint and is strongly convergent, but the convergence speed of $\tilde{x}_n$ is usually not optimal. Then, what is the optimal convergence speed? How can the optimal convergence speed be achieved? To compute the optimal convergence rates, we consider matrix step sizes, rather than the scalar μ_n. Recall that the rates of convergence of stochastic approximation algorithms are determined jointly by the scaling factor in the centered and scaled estimation errors, and its asymptotic covariance. Among the step sizes of the order $\mathrm{O}(n^{-\gamma})$, $\gamma=1$ gives the best order of convergence. Then, we need to find the best covariance matrix. One may use a matrix step size sequence $\mu_n=\tilde{H}/n$, where $\tilde{H}$ is a matrix-valued parameter to be used as a variable to optimize the asymptotic covariance. It is known that by choosing the matrix $\tilde{H}$ suitably, it is possible to achieve optimal convergence speed [30, Chapter 10]. To study the rate of convergence, let us begin with

$$\tilde{x}_{n+1}-\tilde{x}_*=\tilde{x}_n-\tilde{x}_*+\mu_n[\varGamma(\tilde{x}_n-\tilde{x}_*)+\hat{\xi}_n+\hat{W}_1(n)],$$

with $\mu_n=\tilde{H}/n$. Recall that we used the notation $\hat{W}_1(n)=(\hat{W}_i(x_*,\zeta_n):i\leqslant r-1)$. We can take a continuous time interpolation of $v_n=n^{\frac{1}{2}}(\tilde{x}_n-\tilde{x}_*)$. Using the approach in [30, Chapter 10], we obtain the limit of the interpolated (and shifted) sequence of v_n denoted by $V^n(\cdot)$. The limit is a solution to the following stochastic differential equation

$$\mathrm{d}V=(\tilde{H}\varGamma+\frac{I}{2})V\mathrm{d}t+\tilde{H}\varSigma_0^{\frac{1}{2}}\mathrm{d}\tilde{B}(t),$$

where $\tilde{B}(\cdot)$ is a standard Brownian motion and $\varSigma_0$ is the error covariance as given in (22). The asymptotic covariance as a function of $\tilde{H}$ is then given by

$$\check{\varSigma}(\tilde{H})=\int_0^\infty\exp\big(D+\frac{I}{2}t\big)D\varSigma_0D'\exp\big(D'+\frac{I}{2}t\big)\mathrm{d}t,$$

where $D=\tilde{H}\varGamma$. This can be alternatively represented as a solution to a Liapunov equation (or algebraic Riccati equation). Thus, $n^{\frac{1}{2}}(\tilde{x}_n-\tilde{x}_*)$ is asymptotically normal with mean zero and asymptotic covariance given by $\check{\varSigma}(\tilde{H})$. To find the 'smallest' asymptotic covariance, we either minimize $\check{\varSigma}(\tilde{H})$ as a function of $\tilde{H}$ or minimize the trace of the covariance. The optimal asymptotic covariance is given by

$$\varSigma_*=\varGamma^{-1}\varSigma_0(\varGamma')^{-1}. \tag{25}$$

However, as far as implementation is concerned, the matrix step size approach is usually impractical. The iterate averaging provides a viable alternative; see Theorem 3.

3.2 Optimal convergence rates

We now illustrate the optimality of the algorithms from another angle. For convergence speed analysis, let $e_n=x_n-x_*$. Decompose $e_n=[\tilde{e}_n'\ \ e_{n,r}]'$ where $\tilde{e}_n=\tilde{x}_n-\tilde{x}_*$.

Remark 2 For simplicity, assume there is no nonadditive noise, i.e., $\hat{W}(x,\zeta)=0$. Suppose that $\{\xi_n\}$ is a sequence of i.i.d. random variables with mean zero and covariance $E\xi_n\xi_n'=\varSigma_0$. Then, the consensus errors satisfy that $\sqrt{n}(\tilde{x}_n-\tilde{x}_*)$ converges in distribution to a normal

random variable with zero mean and covariance given by $\Gamma^{-1}\Sigma_0(\Gamma^{-1})'$.

Note that the above result does not require any distributional information on the noise $\{\xi_n\}$, other than the zero mean and finite second moments. We now state the optimality of the algorithm when the density of ξ_1 is a smooth function.

Theorem 4 Suppose that the noise $\{\xi_n\}$ is a sequence of i.i.d. noise with a density f that is continuously differentiable. Then, the recursive sequence $\{\tilde{x}_n\}$ is asymptotically efficient in the sense of the Cramér-Rao lower bound on $E\tilde{e}_n'\tilde{e}_n$ being asymptotically attained, $nE\tilde{e}_n'\tilde{e}_n \to \mathrm{tr}(\Gamma^{-1}\tilde{\Sigma}_0(\Gamma^{-1})')$ as $n \to \infty$.

The convergence speed and optimality of the iterate x_n are directly related to those of $\tilde{x}_n$. Under the conditions of Theorem 4, the sequence $\{x_n\}$ from the algorithm (17) is asymptotically efficient in the sense of the Cramér-Rao lower bound on $Ee_n'e_n$ being asymptotically attained.

4 Examples

In this section, we use an example to illustrate the benefits of employing the post-iterate averaging technique. The main advantages include more consistent control accuracy and faster convergence speeds.

Example 1 Since our algorithm maintains the total average of the node states at every step of control, $\sum_{i=1}^{r} x_1^i/r = \beta$ is a constant. The consensus error at the index n will be plotted by using the error norm $[(x_n - \beta\mathbb{1})'(x_n - \beta\mathbb{1})]^{\frac{1}{2}}$.

In this example, we consider a networked system with 5 nodes. The initial states are $x_0^1 = 12$, $x_0^2 = 34$, $x_0^3 = 56$, $x_0^4 = 8$, $x_0^5 = 76$. The state average is $\beta = 37.2$, which will not change in the state update. Initial consensus error is $[(x_0 - \beta\mathbb{1})'(x_0 - \beta\mathbb{1})]^{\frac{1}{2}} = 57.94$.

The network interconnection is defined by the topology matrices

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The link control gain matrix is

$$G = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Consequently, from $H = H_2 - H_1$, we have

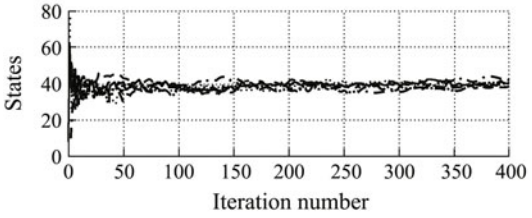
$$M = -H'GH = \begin{bmatrix} -6.4 & 2.0 & 0 & 2.0 & 2.4 \\ 2.0 & -2.6 & 0.6 & 0 & 0 \\ 0 & 0.6 & -3.0 & 2.4 & 0 \\ 2.0 & 0 & 2.4 & -6.4 & 2.0 \\ 2.4 & 0 & 0 & 2.0 & -4.4 \end{bmatrix}$$

and noise impact matrix $W = H' \times G$ (with $a = 0.9997$) is

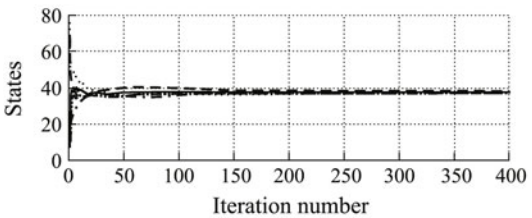
$$W = \begin{bmatrix} a & -a & 0 & 0 & 0 & 0 \\ 0 & a & -a & 0 & 0 & 0 \\ 0 & 0 & a & -a & 0 & 0 \\ 0 & 0 & 0 & a & -a & 0 \\ 0 & 0 & 0 & 0 & a & -a \end{bmatrix}.$$

The observations are corrupted by noises on each link, represented by the (vector) sequence $\{\xi_n\}$, whose elements are i.i.d. random variables with zero mean and variance $\sigma^2 = 40$. The noises are spatially independent, specifically observation noises on different links are independent. The SA algorithm is implemented with a fixed step size $\mu_n = 0.005$. The simulation runs for 400 steps.

Two algorithms are executed. The first one is the SA without post-iterate averaging. State trajectories of this algorithm are shown in the top plot of Fig. 1. The second algorithm adds the post-iterate averaging. The resulting state trajectories are illustrated in the bottom plot of Fig. 1. In both cases, the states converge to the team average. However, the SA with post-iterate averaging demonstrates improved convergence features with less volatility and faster convergence. This is consistent with our previous theoretical analysis. A further comparison of these two algorithms is shown in Fig. 2 by their respective consensus error trajectories. The SA with post-iterate averaging displays faster convergence with less fluctuations.



(a) State trajectories without post-iterate averaging



(b) State trajectories after post-iterate averaging

Fig. 1 State trajectories of the two SA algorithms.

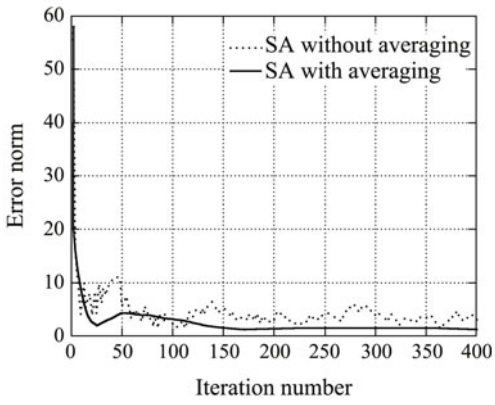


Fig. 2 Comparison of consensus errors of the standard SA and the SA with post-iterate averaging.

5 Conclusions

This work has been devoted to consensus type algorithms. We developed a two-stage averaging algorithm and demonstrated its asymptotic optimality. In the current setup, the topology is fixed. It would be a worthwhile effort to consider iterate averaging for algorithms with topology switching; see related references [40–42] and references therein. Here the methods developed in [40] will be useful.

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