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Decentralised event-triggered consensus of double integrator multi-agent systems with packet losses and communication delays

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Abstract: The event-triggered consensus problem with agents described by double integrator dynamics is addressed in this study. The authors consider the problem of non-consistent packet losses where the broadcast channel from one agent to its neighbours can drop the event-triggered packets of information, where the transmitting agent is unaware that the packet was not received and the receiving agents have no knowledge of the transmitted packet. They also consider the constraints associated with communication delays. In this study, they consider directed graphs, and they also relax the consistency on the packet dropouts and the delays. By relaxing the consistency they allow the dropouts and delays for a packet broadcast by one agent to be different for each receiving node. Under these constraints, an event-triggered consensus protocol is designed for the agents to achieve consensus asymptotically while reducing transmissions of measurements. In addition, positive inter-event times are obtained which guarantee that Zeno behaviour does not occur.

1 Introduction

In recent years, consensus problems over reliable and infinite bandwidth communication networks have been well studied due to their applications in sensor networks and multi-agent systems coordination [1–4]. Many consensus protocols rely on the assumption that continuous exchange of information among agents is possible. Since continuous communication is not possible in many applications, it becomes important to discern how frequently agents should communicate to preserve the system properties that stemmed from continuous information exchange. The sampled-data approach has been commonly used to estimate the sampling periods [5–7]. An important drawback of periodic transmission is that it requires synchronisation between the agents, that is, all agents need to transmit their information at the same time instants and, in some cases, it requires a conservative sampling period for worst case situations.

More recently, the event-triggered control paradigm has been used to design consensus protocols that account for limited bandwidth communication channels, by reducing the number of transmitted measurements by each agent in the network [8–10]. Different from periodic (or time-triggered) implementations, in event-triggered control information or measurements are not transmitted periodically in time, rather they are triggered by the occurrence of certain events. In event-triggered broadcasting [11–15], a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Event-triggered control strategies have also been applied to stabilise multiple coupled subsystems as in [16–20]. Guinaldo *et al.* [21] considered delays and packet losses in the stabilisation problem of coupled subsystems. Two event-triggered communication protocols were proposed in [21]. The first protocol preserves state consistency in the sense that all neighbours of a given node use the same version of the transmitted state by that node. In the presence of packet losses multiple retransmissions of the same measurement may be needed until it is guaranteed that all neighbours receive the update. At that point the transmitting node sends a permission message that allows neighbours to start using the new transmitted measurement. Due to disadvantages in retransmissions and extra delays a second protocol is presented where the state consistency is relaxed and the neighbours of a given node are allowed to use different versions of the state of that node. However, acknowledgment (ACK) messages are required. In this case, retransmissions are used for neighbours whose ACK message is not received by the transmitting node. Meanwhile, the neighbours that successfully received the measurement update can use it to recalculate their control inputs regardless of any remaining neighbour nodes that have not received the update.

Event-triggered control provides a more robust and efficient use of network bandwidth. Its implementation in multi-agent systems also provides a highly decentralised way to schedule transmission instants, which eliminates the need for the synchronisation required by periodic sampled-data approaches. Decentralised event-triggered consensus protocols give each agent the ability to broadcast information. These decisions are made locally by each agent based only on current, local information. Different authors have extended the event-triggered consensus approach, for instance, Chen and Hao [22] studied event-triggered consensus for discrete time integrators. The authors of [23] used event-triggered techniques for consensus problems involving a combination of discrete time single and double integrators. The authors of [24] studied event-triggered consensus of single integrator systems using non-linear consensus protocols. The authors of [25] investigated the event-triggered consensus of second order systems. The event-triggered consensus problem with general linear dynamics has been addressed in [26–30].

In this paper, we consider the event-triggered consensus problem of agents described by double integrator dynamics. In addition, we address the problems of non-consistent packet dropouts and non-consistent communication delays which to the best of our knowledge, have not been addressed before using the event-triggered control paradigm. Previous event-triggered consensus approaches such as [8–10] assumed that every event-based packet transmission, containing measurement updates, will arrive at its corresponding destination. In many practical scenarios however, this is not the case and packet dropouts occur. We consider the general case of packet dropouts in multi-agent systems where the same transmitted packet may arrive at some destinations and may be lost by other intended receiving agents. This type of unreliable communication is *not consistent* with respect to packet dropouts. Similarly, communication delays (when packets are successfully received) are not consistent if a particular broadcast packet could arrive at each receiving node at different time instants. The design of event-triggered consensus protocols in the presence of delays and packet losses represents an important challenge. Several event-triggered control methods [21, 31] demand the broadcasting and reception of long sequences of messages that include inquiries and ACKs for every single event that is generated. This type of controllers are undesired since each message is subject to delay or it may never be received at the destination nodes. The consensus protocol presented in this paper is only based on broadcasting (only one broadcasting per single event) and it does not require inquiry or acknowledgment messages.

This paper extends the results in [32] where agents with single integrator dynamics were considered. The extension to address systems with double integrator dynamics is not straightforward. In particular, we do not implement a zero-order hold (ZOH) approach. In order to provide better performance in terms of reduction of communication and in the presence of packet losses and communication delays, we implement a model-based approach for multi-agent consensus [26].

The remainder of this paper is organised as follows. Section 2 provides a brief background on graph theory and describes the problem and the consensus protocol. Section 3 analyses the problem of packet dropouts using event-triggered control. Asymptotic consensus for agents with double integrator dynamics and subject to packet losses and communication delays is shown in Section 4. Examples are given in Section 5 and Section 6 concludes the paper.

2 Preliminaries

Notation. The notations 1_N and 0_N represent column vectors of all ones and all zeros, respectively. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and the set of complex numbers, respectively. For any $s \in \mathbb{C}$, $\mathrm{Re}\{s\}$ represents the real part of s. $J_\nu^{\lambda_i}$ represents a Jordan block of size ν corresponding to eigenvalue λ_i and $\otimes$ denotes the Kronecker product. The boldface $\mathbf{e}^\lambda$ represents the exponential of the scalar λ and $\mathbf{e}^A$ represents the matrix exponential of matrix A.

2.1 Graph theory

For a team of N agents, the communication among them can be described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ denotes the agent set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set. An edge (i, j) in the set $\mathcal{E}$ denotes that agent j can obtain information from agent i, but not necessarily *vice versa*. For an edge $(i, j) \in \mathcal{E}$, agent i is a neighbour of agent j. The set $\mathcal{N}_j$ is called the set of neighbours of agent j, and N_j is its cardinality. We also define $\bar{N}_j = \max_j N_j$. A directed path from agent i to agent j is a sequence of edges in a directed graph of the form $(i, p_1), (p_1, p_2), \ldots, (p_{\kappa-1}, p_\kappa)(p_\kappa, j)$, where $p_\ell \in \mathcal{V}$, $\forall \ell = 1, \ldots, \kappa$. A directed graph has *a directed spanning tree* if there exists at least one agent with directed paths to all other agents.

The adjacency matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$ of a directed graph $\mathcal{G}$ is defined by $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ represents the degree matrix which is a diagonal matrix with entries $\mathcal{D}_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$. If a directed graph has a directed spanning tree, then the corresponding Laplacian matrix has only one eigenvalue equal to zero, $\lambda_1(\mathcal{L}) = 0$, and the following holds for the remaining eigenvalues [33]: $\mathrm{Re}\{\lambda_i(\mathcal{L})\} > 0$, for $i = 2, \ldots, N$.

2.2 Problem statement

Consider a group of N agents with double integrator dynamics which are interconnected by means of a directed communication graph. Each agent can be described by the following equation:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{1}$$

for $i = 1, \ldots, N$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The local control inputs $u_i(t)$ are given by

$$u_i(t) = cF \sum_{j \in \mathcal{N}_i} (y_i(t) - y_{ji}(t)) \tag{2}$$

for $i = 1, \ldots, N$. The variables y_i and y_{ji} represent decoupled models implemented by the local agent i; their dynamics can be described by

$$\dot{y}_i(t) = Ay_i(t), \; y_i(t_{k_i}) = x_i(t_{k_i}) \tag{3}$$

and

$$\dot{y}_{ji}(t) = Ay_{ji}(t), \; y_{ji}(t_{k_j^i} + d_{ji}) = f(x_j(t_{k_j^i}), d_{ji}) \tag{4}$$

The meaning of (3) and (4) is as follows. The variable y_i represents a model of the local agent dynamics x_i and it is updated at the event time instants denoted by t_{k_i}. The variables y_{ji} represent models of the states of agents j implemented by agent i, such that $j \in \mathcal{N}_i$, and they are not updated at every jth event time t_{k_j} because packet dropouts occur in the communication channels. The models y_{ji} are updated only when a measurement from agent j is successfully received by agent i. The event time instant associated with a successful arrival from agent j to agent i is denoted as $t_{k_j^i}$. The update rule in (4) is defined as $f(x_j(t_{k_j^i}), d_{ji}) = \mathbf{e}^{Ad_{ji}} x_j(t_{k_j^i})$, where d_{ji} represents the communication delay from agent j to agent i associated to the received packet $x_j(t_{k_j^i})$.

The implementation of dynamical models (3) and (4) represents a sharp difference with respect to [32]. In that paper, the ZOH approach is used where measurements received from neighbours are kept constant until new updates are received again. The use of models provides estimates of the real states of neighbours and reduce communication instants. However, the analysis and design of event-triggered controllers are more complicated due to the additional dynamics pertaining to (3) and (4).

We consider the event-triggered consensus problem of agents (1) in the presence of packet losses. This means that when any agent generates its own events and broadcasts its state measurement, there is no guarantee that all destination agents will receive the transmitted state measurement. In this paper, we consider a general or non-consistent type of packet dropouts [20]. In [20] a packet of information broadcasted by a subsystem is lost in some communication links, but it is not lost in other links, i.e. it may successfully arrive to a subset of nodes. Therefore, some agents may receive different sets of measurements from the same subsystem. This means that a broadcasted measurement $x_i(t_{k_i})$ may be successfully received by all, some, or none of the receiving (or destination) agents j, for $i \in \mathcal{N}_j$.

In addition, we consider non-consistent communication delays. In multi-agent systems consistent delays refer to the case where the delay $d_i(t_{k_i})$ associated with the transmitted state $x_i(t_{k_i})$ is the same for every receiving agent. By non-consistent delays we refer to the more general case where the delay associated to a transmitted state can be different to every receiving agent. In this case, we define $d_{ij}(t_{k_i})$ as the time it takes the measurement $x_i(t_{k_i})$ which is released at time t_{k_i} to arrive to agent j, for every j such that $i \in \mathcal{N}_j$. For instance, if agents 2 and 3 receive information from agent 1, then the measurement $x_1(t_{k_1})$ released at time t_{k_1} will arrive (if it is not dropped by either agent 2 or 3) to agent 2 at time $t_{k_1} + d_{12}(t_{k_1})$ and to agent 3 at time $t_{k_1} + d_{13}(t_{k_1})$, where, in general, $d_{12}(t_{k_1}) \neq d_{13}(t_{k_1})$. Also, the delay in the communication channel from i to j is time-varying, i.e. $d_{ij}(t)$ might not be equal to $d_{ij}(t')$ for $t \neq t'$ and for $i, j = 1, \ldots, N$.

3 Packet dropouts analysis

The main consequence of dealing with non-consistent packet dropouts and non-consistent communication delays is that agents j, for $i \in \mathcal{N}_j$, will generate different estimates y_{ij} of the state of agent i since each agent j will successfully receive different sets of updates from agent i. In this section we characterise the state error only in the presence of packet losses. In Section 4, we consider the joint effect of packet losses and communication delays.

Define the errors

$$e_i(t) = y_i(t) - x_i(t) \tag{5}$$

$$e_{ij}(t) = y_{ij}(t) - x_i(t)\,. \tag{6}$$

The error $e_i(t)$ represents agent's i local state error and it can be continuously measured in order for agent i to decide when to broadcast its state. On the other hand, the error $e_{ij}(t)$ represents the state error corresponding to agent i as seen by agent j, which cannot be measured by agent i, since agent i does not know the current variable $y_{ij}(t)$. Furthermore, error $e_{ij}(t)$ cannot be measured by agent j, since agent j does not have access to the state $x_i(t)$. Lastly, due to packet losses and delays, the error $e_{ij}(t)$ cannot immediately be reset to $e_i(t)$. How to manage and impose bounds on the error $e_{ij}(t)$ represents an important challenge. This problem is addressed in the remaining of this paper.

Let us assume that there exist a uniform Maximum Allowable Number of Successive Dropouts (MANSD) [19, 34], denoted as $\rho - 1$, where $\rho > 1$ is an integer. This means that if a measurement transmitted by agent i at time t_{k_i} is successfully received by agent j, then, at most $\rho - 1$ consecutive dropouts are allowed from i to j and, in the worst case, the measurement transmitted at time $t_{k_i+\rho}$ will be successfully received by agent j.

Due to presence of packet losses each agent will impose a maximum time between events $\bar{\delta} > 0$, i.e. an event is generated if $t - t_{k_i} \geq \bar{\delta}$. The need for this maximum inter-event time is explained at the end of Section 4.

Proposition 1: Assume that the MANSD is $\rho - 1$, for $\rho > 1$. If agent i's events, t_{k_i} for $i = 1, \ldots, N$, are generated according to the following condition:

$$t_{k_i+1} = \min\left\{t > t_{k_i} \,\middle|\, \| e_i(t) \| = \beta \mathbf{e}^{-\lambda t} \text{or } t = t_{k_i} + \bar{\delta}\right\} \tag{7}$$

where $\beta, \lambda > 0$, then, the error $e_{ij}(t)$ due to packet losses is bounded by

$$\| e_{ij}(t) \| \leq \gamma(\rho) \mathbf{e}^{-\lambda t}$$

for $t \in [t_{k_i}, t_{k_i+\rho})$ where

$$\gamma(\rho) = \beta \mathbf{e}^{\rho \lambda \bar{\delta}} \sum_{\mu=1}^{\rho} \| \mathbf{e}^{(\rho-\mu)\bar{\delta} A} \| \tag{8}$$

Proof: Let us consider the state error due to packet losses. Assume without loss of generality that the last update transmitted by agent i and successfully received by agent j takes place at time t_{k_i}. Hence, we have that $t_{k_i^j} = t_{k_i}$. Agent i will generate the next event at time t_{k_i+1} (and broadcast its current state $x_i(t_{k_i+1})$). This event is generated because the condition (7) is satisfied. This means that either $\| e_i(t_{k_i+1}) \| = \beta \mathbf{e}^{-\lambda t_{k_i+1}}$ or the time t_{k_i+1} is such that $t_{k_i+1} = t_{k_i} + \bar{\delta}$ (and $\| e_i(t_{k_i+1}) \| < \beta \mathbf{e}^{-\lambda t_{k_i+1}}$). In general, we have that the following holds $\| e_i(t_{k_i+1}) \| = \| y_i(t_{k_i+1}^-) - x_i(t_{k_i+1}) \| \leq \beta \mathbf{e}^{-\lambda t_{k_i+1}}$, where $y_i(t_{k_i+1}^-)$ represents the value of $y_i(t)$ just before the event at time t_{k_i+1} takes place. Note that we consider the state errors evaluated just before the event time instants t_k (which can be denoted as t_k^-); however, to simplify notation, we refer to the error $e_i(t)$ evaluated at time instants t_k^- simply as $e_i(t_k)$.

Assume that the update at time t_{k_i+1} is dropped, so $e_{ij}(t_{k_i+1}) \leq \beta \mathbf{e}^{-\lambda t_{k_i+1}}$. Similarly, agent i will generate the following event at time t_{k_i+2} (and broadcast its current state $x_i(t_{k_i+2})$) and we have that $\| e_i(t_{k_i+2}) \| \leq \beta \mathbf{e}^{-\lambda t_{k_i+2}}$ is satisfied. If the number of successive dropouts after the last successful update is the MANSD, $\rho - 1$, then we have that the error $e_{ij}(t)$ at time $t = t_{k_i+\rho}^-$, just before the update $x(t_{k_i+\rho})$ is successfully received by agent j, satisfies the following: (see (9)) From (3), we have that $y_i(t_{k_i+\mu}^-) = \mathbf{e}^{A(t_{k_i+\mu} - t_{k_i+\mu-1})} x_i(t_{k_i+\mu-1})$ for any $\mu = 1, \ldots, \rho$ and for any $t_{k_i} = 1, 2, \ldots$. For instance, the last term inside the norm brackets in (9) can be written as follows:

$$\begin{aligned}\mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})} y_i(t_{k_i+1}^-) &= \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})} \mathbf{e}^{A(t_{k_i+1} - t_{k_i})} x_i(t_{k_i}) \\ &= \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i})} x_i(t_{k_i})\,.\end{aligned}$$

Then we have that (see (10)) Similarly, from (4) we have that $y_{ij}(t_{k_i+\rho}^-) = \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i})} x_i(t_{k_i})$ and the first two terms inside the norm brackets of (10) cancel out. Then, we can write the following:

$$\begin{aligned}\left\| e_{ij}(t_{k_i+\rho}) \right\| = \Big\| & e_i(t_{k_i+\rho}) + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-1})} e_i(t_{k_i+\rho-1}) + \\ & \cdots + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})} e_i(t_{k_i+1}) \Big\|.\end{aligned}$$

Thus in the worst case, the error e_{ij} when the MANSD occurs is bounded by

$$\begin{aligned}\left\| e_{ij}(t_{k_i+\rho}) \right\| \leq & \beta \mathbf{e}^{-\lambda t_{k_i+\rho}} + \left\| \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-1})} \right\| \beta \mathbf{e}^{-\lambda t_{k_i+\rho-1}} + \\ & \cdots + \left\| \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})} \right\| \beta \mathbf{e}^{-\lambda t_{k_i+1}}\end{aligned}$$

Let $t = t_{k_i} + \delta$, for some $\delta \in [0, \delta_\rho)$, where $\delta_\rho \leq \rho \bar{\delta}$. Thus, the time instant $t = t_{k_i+1}$ can be represented by $t = t_{k_i} + \delta_1$. Similarly,

$$\begin{aligned}\left\| e_{ij}(t_{k_i+\rho}) \right\| &= || y_{ij}(t_{k_i+\rho}^-) - x_i(t_{k_i+\rho}) || \\ &= || y_{ij}(t_{k_i+\rho}^-) - x_i(t_{k_i+\rho}) + y_i(t_{k_i+\rho}^-) - y_i(t_{k_i+\rho}^-) \\ &\quad + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-1})} y_i(t_{k_i+\rho-1}^-) - \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-1})} y_i(t_{k_i+\rho-1}^-) \\ &\quad + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-2})} y_i(t_{k_i+\rho-2}^-) - \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-2})} y_i(t_{k_i+\rho-2}^-) \\ &\quad \vdots \\ &\quad + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})} y_i(t_{k_i+1}^-) - \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})} y_i(t_{k_i+1}^-) ||\,.\end{aligned} \tag{9}$$

$t_{k_i+2} = t_{k_i} + \delta_2$ and so on. We can write the following: (see equation below) Since, by definition, $0 \le \delta \le \rho\bar{\delta}$, we have that $\mathbf{e}^{-\lambda\delta_\mu} \le 1$, for $\mu = 1, \dots, \rho$, and we can write

$$\| e_{ij}(t_{k_i+\rho}) \| \le \beta \mathbf{e}^{\rho\lambda\bar{\delta}} \sum_{\mu=1}^{\rho} \| \mathbf{e}^{(\rho-\mu)\bar{\delta}A} \| \mathbf{e}^{-\lambda t}.$$

In addition, let us consider $\rho' < \rho$ and if some packet is received at $t_{k_i+\rho'}$ then we would have that

$$\mathbf{e}^{\rho'\lambda\bar{\delta}} \sum_{\mu=1}^{\rho'} \| \mathbf{e}^{(\rho'-\mu)\bar{\delta}A} \| < \mathbf{e}^{\rho\lambda\bar{\delta}} \sum_{\mu=1}^{\rho} \| \mathbf{e}^{(\rho-\mu)\bar{\delta}A} \|$$

since $\rho' < \rho$. Then, in general, we have that

$$\| e_{ij}(t) \| \le \gamma(\rho) \mathbf{e}^{-\lambda t}$$

for $t \in [t_{k_i}, t_{k_i+\rho})$, where $\gamma(\rho)$ is given by (8). □

Hereafter, we will refer to $\gamma(\rho)$ as simply γ with the understanding that γ depends on the design parameter ρ.

4 Decentralised event-triggered consensus protocol

Let us write (1) as follows:

$$\dot{x}_i(t) = Ax_i(t) + cBF \sum_{j \in \mathcal{N}_i} (x_i(t) + e_i(t) - x_j(t) - e_{ji}(t)) \tag{11}$$

Define the vector $x(t) = \left[x_1(t)^{\mathrm{T}} \dots x_N(t)^{\mathrm{T}}\right]^{\mathrm{T}}$. Then, we can write the overall system as follows:

$$\dot{x}(t) = (\bar{A} + \bar{B})x(t) + D\xi(t) \tag{12}$$

where $\bar{A} = I_N \otimes A$, $\bar{B} = c\mathcal{L} \otimes BF$, $D = cI_N \otimes BF$, and

$$\xi(t) = \begin{bmatrix} \sum_{j \in \mathcal{N}_1} (e_1(t) - e_{j1}(t)) \\ \sum_{j \in \mathcal{N}_2} (e_2(t) - e_{j2}(t)) \\ \vdots \\ \sum_{j \in \mathcal{N}_N} (e_N(t) - e_{jN}(t)) \end{bmatrix}. \tag{13}$$

Due to the pair (A, B) is controllable [35], we have that for $\alpha > 0$ there exists a (independent of the communication graph) symmetric and positive definite solution P to

$$PA + A^{\mathrm{T}}P - 2PBB^{\mathrm{T}}P + 2\alpha P < 0. \tag{14}$$

Let

$$F = -B^{\mathrm{T}}P \tag{15}$$

$$c \ge 1/\mathrm{Re}(\lambda_2). \tag{16}$$

Also, there exists a similarity transformation S_L such that $\mathcal{L}_J = S_L^{-1}\mathcal{L}S_L$ is in Jordan canonical form. Define $S = S_L \otimes I_2$ and $\hat{x} = S^{-1}x$. Thus, we can obtain the transformed system dynamics

$$\begin{aligned} \dot{\hat{x}}(t) &= S^{-1}\dot{x}(t) = S^{-1}(\bar{A} + \bar{B})x(t) + S^{-1}D\xi(t) \\ &= (\bar{A} + c\mathcal{L}_J \otimes BF)\hat{x}(t) + S^{-1}D\xi(t) \end{aligned} \tag{17}$$

Since $\lambda_1(\mathcal{L}) = 0$ we have

$$\mathcal{L}_J = \begin{bmatrix} 0 & 0_{N-1}^{\mathrm{T}} \\ 0_{N-1} & J \end{bmatrix}$$

where the matrix $J \in \mathbb{C}^{(N-1)\times(N-1)}$ contains Jordan blocks $J_\nu^{\lambda_i}$ corresponding to the eigenvalues $\lambda_i(\mathcal{L})$ for $i = 2, \dots, N$. Then, the transformed system dynamics can be expressed as

$$\begin{aligned} \dot{\hat{x}}_1 &= A\hat{x}_1 + \Delta_1\xi \\ \dot{\hat{x}}_{2:N} &= \hat{A}\hat{x}_{2:N} + \Delta\xi \end{aligned} \tag{18}$$

where $\hat{A} = I_{N-1} \otimes A + cJ \otimes BF$, $\Delta_1 \in \mathbb{C}^{2\times 2N}$ represents the first two rows of $S^{-1}D$ and $\Delta \in \mathbb{C}^{2(N-1)\times 2N}$ contains the remaining $2(N-1)$ rows of $S^{-1}D$. If the communication graph has a spanning tree then, by selection of the controller gains (15) and (16) we have that the matrix $\hat{A}$ is a Hurwitz matrix. Thus, there exist $\hat{\beta}, \hat{\lambda} > 0$ such that the relation $\| \mathbf{e}^{\hat{A}t} \| \le \hat{\beta}\mathbf{e}^{-\hat{\lambda}t}$ holds.

Define $\hat{x}_0 = \| \hat{x}_{2:N}(0) \|$. Let us also define $\Theta \in \mathbb{C}^{N\times(N-1)}$ as

$$\Theta = S_L \begin{bmatrix} 0_{N-1}^{\mathrm{T}} \\ J \end{bmatrix}.$$

This means that

$$S_L\mathcal{L}_J = S_L \begin{bmatrix} 0 & 0_{N-1}^{\mathrm{T}} \\ 0_{N-1} & J \end{bmatrix} = \begin{bmatrix} 0_N & \Theta \end{bmatrix}.$$

Due to communication delays, the update $x_i(t_{k_i+\rho})$ will not be received by agent j until time $t_{k_i+\rho} + d$, in the worst-case delay. Hence, we aim at finding a bound on the error $e_{ij}(t)$ within the extended time interval $t \in [t_{k_i}, t_{k_i+\rho} + d)$. Define the following:

$$z_i(t) = \mathbf{e}^{A(t - t_{k_i+\rho})}x_i(t_{k_i+\rho}) \tag{19}$$

and define the error due to delays as follows:

$$\begin{aligned} \| e_{ij}(t_{k_i+\rho}) \| = {} & || y_{ij}(t_{k_i+\rho}^-) - \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i})}x_i(t_{k_i}) + y_i(t_{k_i+\rho}^-) - x_i(t_{k_i+\rho}) \\ & + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-1})}(y_i(t_{k_i+\rho-1}^-) - x_i(t_{k_i+\rho-1})) \\ & + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+\rho-2})}(y_i(t_{k_i+\rho-2}^-) - x_i(t_{k_i+\rho-2})) \\ & \vdots \\ & + \mathbf{e}^{A(t_{k_i+\rho} - t_{k_i+1})}(y_i(t_{k_i+1}^-) - x_i(t_{k_i+1})) ||. \end{aligned} \tag{10}$$

$$\begin{aligned} \| e_{ij}(t_{k_i+\rho}) \| &\le \beta \mathbf{e}^{-\lambda t_{k_i}}(\mathbf{e}^{-\lambda\delta_\rho} + \| \mathbf{e}^{A(\delta_\rho - \delta_{\rho-1})} \| \mathbf{e}^{-\lambda\delta_{\rho-1}} + \dots + \| \mathbf{e}^{A(\delta_\rho - \delta_1)} \| \mathbf{e}^{-\lambda\delta_1}) \\ &\le \beta \mathbf{e}^{-\lambda(t-\delta)}(\mathbf{e}^{-\lambda\delta_\rho} + \| \mathbf{e}^{\bar{\delta}A} \| \mathbf{e}^{-\lambda\delta_{\rho-1}} + \dots + \| \mathbf{e}^{(\rho-1)\bar{\delta}A} \| \mathbf{e}^{-\lambda\delta_1}) \end{aligned}$$

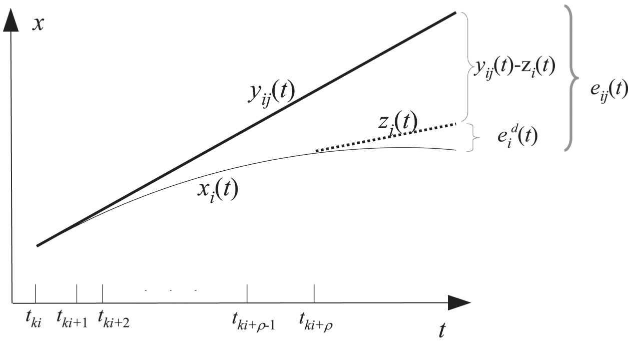

Fig. 1 *State errors: positions*

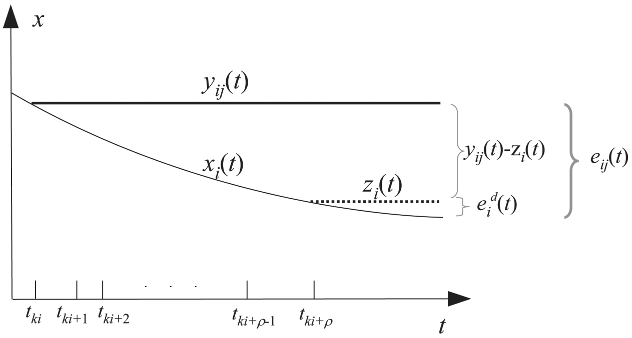

Fig. 2 *State errors: velocities*

$$e_i^{\mathrm{d}}(t) = z_i(t) - x_i(t) \tag{20}$$

for $t \in [t_{k_i+\rho}, t_{k_i+\rho} + d)$. Note that $e_i^{\mathrm{d}}(t_{k_i+\rho}) = 0$. Let us write the following:

$$e_{ij}(t) = y_{ij}(t) - x_i(t) = y_{ij}(t) - z_i(t) + e_i^{\mathrm{d}}(t). \tag{21}$$

We can also write $y_{ij}(t) = \mathbf{e}^{A(t - t_{k_i+\rho})} y_{ij}(t_{k_i+\rho})$. Hence, we have that

$$\begin{aligned} \| e_{ij}(t) \| &= \mathbf{e}^{A(t - t_{k_i+\rho})}(y_{ij}(t_{k_i+\rho}) - x_i(t_{k_i+\rho})) + e_i^{\mathrm{d}}(t) \\ &= \mathbf{e}^{A(t - t_{k_i+\rho})} e_{ij}(t_{k_i+\rho}) + e_i^{\mathrm{d}}(t) \end{aligned} \tag{22}$$

and the following expression is obtained:

$$\| e_{ij}(t) \| \le \left\| \mathbf{e}^{A(t - t_{k_i+\rho})} \right\| \gamma \mathbf{e}^{-\lambda t} + \left\| e_i^{\mathrm{d}}(t) \right\|. \tag{23}$$

These relationships are illustrated in Figs. 1 and 2 for each element of the state of agent i. Note that because of the model dynamics, the element of the state representing the position is modelled as a first-order hold while the velocity component is modeled as ZOH. These cases are captured in the two-dimensional states and state errors in (19)–(23). Let $d = t - t_{k_i+\rho}$. We now use the fact that

$$\mathbf{e}^{Ad} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

and the relation $\| \mathbf{e}^{Ad} \|_2^2 \le \| \mathbf{e}^{Ad} \|_1 \| \mathbf{e}^{Ad} \|_\infty = (1 + d)^2$ to obtain $\| \mathbf{e}^{Ad} \| \le 1 + d \le \mathbf{e}^d$ for $d \ge 0$.

In addition, given a $\gamma_{\mathrm{d}} > 0$ we can always guarantee that there exist some $d \ge 0$ such that $\| e_i^{\mathrm{d}}(t) \| \le \gamma_{\mathrm{d}} \mathbf{e}^{-\lambda t}$ for $t \in [t_{k_i+\rho}, t_{k_i+\rho} + d)$. The previous relation holds because $\gamma_{\mathrm{d}} > 0$, $e_i^{\mathrm{d}}(t_{k_i+\rho}) = 0$, and e_i^{d} is continuous in the interval $t \in [t_{k_i+\rho}, t_{k_i+\rho} + d)$. However, in Theorem 1, an estimate of the largest admissible delay will be obtained as a function of the design parameter γ_{d}.

We have obtained the following bound on the error $e_{ij}(t)$:

$$\| e_{ij}(t) \| \le \bar{\gamma} \mathbf{e}^{-\lambda t} \tag{24}$$

for $i, j = 1, \dots, N$, where $\bar{\gamma} = \mathbf{e}^d \gamma + \gamma_{\mathrm{d}}$.

Theorem 1: Assume that the communication graph has a spanning tree and MANSD is $\rho - 1$, for $\rho > 1$. Then, for some $\gamma_{\mathrm{d}} > 0$, the equation

$$(\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda} d})(\kappa_1 \mathbf{e}^{\lambda d} + \kappa_2 \mathbf{e}^{(1+\lambda)d}) - \gamma_{\mathrm{d}} = 0 \tag{25}$$

has only one solution d and this solution is positive, i.e. $d > 0$. Also, agents (1) with decentralised control inputs (2) achieve consensus asymptotically in the presence of packet losses and communication delays $d_{ij}(t_{k_i}) < d$ if the event time instants, t_{k_i} for $i = 1, \dots, N$, are generated according to condition (7) where $\beta > 0$, $0 < \lambda < \hat{\lambda}$,

$$\begin{aligned} \kappa_1 = {} & \frac{c\hat{\beta}\hat{x}_0 \| \Theta \| \| BF \| \mathbf{e}^{(\lambda - \hat{\lambda}) t_{k_i+\rho}}}{1 + \hat{\lambda}} + \frac{c N_i (\beta + \gamma_{\mathrm{d}}) \| BF \|}{1 + \lambda} \\ & + \frac{c\hat{\beta} \| \Theta \| \| \Delta \| \| BF \| (\beta \bar{N}_i \sqrt{N} + \gamma_{\mathrm{d}} (\sum_{i=1}^N N_i^2)^{1/2})}{\hat{\lambda} - \lambda} \Lambda \\ \kappa_2 = {} & \frac{c\hat{\beta} \| \Theta \| \| \Delta \| \| BF \| \gamma (\sum_{i=1}^N N_i^2)^{1/2}}{\hat{\lambda} - \lambda} \Lambda + \frac{c N_i \gamma \| BF \|}{1 + \lambda} \end{aligned} \tag{26}$$

and

$$\Lambda = \frac{1}{1 + \lambda} - \frac{\mathbf{e}^{(\lambda - \hat{\lambda}) t_{k_i+\rho}}}{1 + \hat{\lambda}}.$$

Proof: To prove Theorem 1, the following observation is required. Note that because of threshold (7), the error e_i is reset to zero at the event instants t_{k_i}, i.e. $e_i(t_{k_i}) = 0$. Thus, the error e_i satisfies $\| e_i(t) \| \le \beta \mathbf{e}^{-\lambda t}$ and we have that $\| e(t) \| \le \sqrt{N} \beta \mathbf{e}^{-\lambda t}$.

Let us define $e(t) = \left[e_1(t)^{\mathrm{T}} \dots e_N(t)^{\mathrm{T}} \right]^{\mathrm{T}}$ and write $\xi = \mathcal{D} e - \xi_d$, where

$$\xi_d(t) = \begin{bmatrix} \sum_{j \in \mathcal{N}_1} e_{j1}(t) \\ \sum_{j \in \mathcal{N}_2} e_{j2}(t) \\ \vdots \\ \sum_{j \in \mathcal{N}_N} e_{jN}(t) \end{bmatrix}. \tag{27}$$

Then we have that

$$\| \xi(t) \| \le \Gamma \mathbf{e}^{-\lambda t} \tag{28}$$

where $\Gamma = \bar{N}_i \sqrt{N} \beta + \bar{\gamma} (\sum_{i=1}^N N_i^2)^{1/2}$. We can write the following:

$$\hat{x}_1(t) = \mathbf{e}^{At} \hat{x}_1(0) + \int_0^t \mathbf{e}^{A(t-s)} \Delta_1 \xi(s) \, \mathrm{d}s. \tag{29}$$

The response of $\hat{x}_{2:N}$ can be bounded as follows:

$$
\begin{aligned}
\| \hat{x}_{2:N}(t) \| &= \left\| \mathbf{e}^{\hat{A}t}\hat{x}_{2:N}(0) + \int_0^t \mathbf{e}^{\hat{A}(t-s)}\Delta\xi(s)\,\mathrm{d}s \right\| \\
&\le \hat{\beta}\hat{x}_0\mathbf{e}^{-\hat{\lambda}t} + \hat{\beta}\Gamma \| \Delta \| \int_0^t \mathbf{e}^{-\hat{\lambda}(t-s)}\mathbf{e}^{-\lambda s}\,\mathrm{d}s \\
&\le \hat{\beta}\hat{x}_0\mathbf{e}^{-\hat{\lambda}t} + \frac{\hat{\beta}\Gamma \| \Delta \|}{\hat{\lambda}-\lambda}(\mathbf{e}^{-\lambda t} - \mathbf{e}^{-\hat{\lambda}t}).
\end{aligned}
$$

As t goes to infinity we have that

$$
\lim_{t\to\infty} \| \hat{x}_{2:N}(t) \| = 0. \tag{30}
$$

In order to show asymptotic consensus we first note that $\lim_{t\to\infty}\hat{x}(t) = [\lim_{t\to\infty}\hat{x}_1(t)\; 0\; \dots\; 0]^{\mathrm{T}}$, and then, we use the similarity transformation S to obtain the original state x from $\hat{x}$. Note that the first column of S contains the right eigenvector of $\mathcal{L}$ associated with $\lambda_1 = 0$.

$$
\lim_{t\to\infty} x(t) = S \lim_{t\to\infty}\hat{x}(t) = \alpha \begin{bmatrix} \lim_{t\to\infty}\hat{x}_1(t) \\ \lim_{t\to\infty}\hat{x}_1(t) \\ \vdots \\ \lim_{t\to\infty}\hat{x}_1(t) \end{bmatrix} \tag{31}
$$

and the agents achieve consensus asymptotically. Note that $\lim_{t\to\infty} x(t) \to \infty$, as it is expected, since the agents have double integrator dynamics.

We will now determine admissible delay d. Here, we want to determine the largest possible value of d such that $\| e_i^{\mathrm{d}}(t) \| \le \gamma_{\mathrm{d}}\mathbf{e}^{-\lambda t}$ holds, for $t \in [t_{k_i+\rho}, t_{k_i+\rho}+d)$ and for selected design parameter γ_{d}. The dynamics of the error $e_i^{\mathrm{d}}(t)$ can be written using (11), (19), and (20) as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}e_i^{\mathrm{d}}(t) &= \dot{e}_i^{\mathrm{d}}(t) = \dot{z}_i(t) - \dot{x}_i(t) \\
&= Ae_i^{\mathrm{d}}(t) - cBF\sum_{j\in\mathcal{N}_i}(x_i(t) - x_j(t)) \\
&\quad - cBF\sum_{j\in\mathcal{N}_i}(e_i(t) - e_{ji}(t))
\end{aligned} \tag{32}
$$

for $t \in [t_{k_i+\rho}, t_{k_i+\rho}+d)$. We have the following:

$$
\begin{aligned}
\| \dot{e}_i^{\mathrm{d}}(t) \| &\le \| A \| \| e_i^{\mathrm{d}}(t) \| + \| cBF \| \| \mathcal{L}_2 x(t) \| \\
&\quad + \| cBF(N_i e_i(t) - \sum_{j\in\mathcal{N}_i} e_{ji}(t)) \| \le \| e_i^{\mathrm{d}}(t) \| \\
&\quad + \| cBF \| (\| \mathcal{L}_2 x(t) \| + N_i(\beta + \bar{\gamma})\mathbf{e}^{-\lambda t})
\end{aligned}
$$

where $\mathcal{L}_2 = \mathcal{L} \otimes I_2$. Note that

$$
\begin{aligned}
\mathcal{L}_2 x &= \mathcal{L}_2 S\hat{x} = (\mathcal{L}\otimes I_2)(S_L \otimes I_2)\hat{x} = (\mathcal{L}S_L \otimes I_2)\hat{x} = (S_L\mathcal{L}_J \otimes I_2)\hat{x} \\
&= ([0_N\, \Theta] \otimes I_2)\hat{x}
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\| \dot{e}_i^{\mathrm{d}}(t) \| &\le \| e_i^{\mathrm{d}}(t) \| + c \| BF \| \Big[\| \Theta \| \big(\hat{\beta}\hat{x}_0\mathbf{e}^{-\hat{\lambda}t} \\
&\quad + \frac{\hat{\beta}\Gamma \| \Delta \|}{\hat{\lambda}-\lambda}(\mathbf{e}^{-\lambda t} - \mathbf{e}^{-\hat{\lambda}t}) \big) + N_i(\beta + \bar{\gamma})\mathbf{e}^{-\lambda t} \Big].
\end{aligned}
$$

Then, the error response during the time interval $t \in [t_{k_i+\rho}, t_{k_i+\rho}+d)$ can be bounded as follows: (see (33)) where

$$
\begin{aligned}
\hat{H} &= \hat{\beta} \| cBF \| \| \Theta \| \Big(\hat{x}_0 - \frac{\Gamma \| \Delta \|}{\hat{\lambda}-\lambda}\Big) \\
H &= \| cBF \| \Big(N_i(\beta + \bar{\gamma}) + \frac{\| \Theta \| \hat{\beta}\Gamma \| \Delta \|}{\hat{\lambda}-\lambda} \Big).
\end{aligned} \tag{34}
$$

We realise that the time $d > 0$ that it takes for the last expression in (33) to grow from zero, at time $t_{k_i+\rho}$, to reach the threshold $\gamma_{\mathrm{d}}\mathbf{e}^{-\lambda t} = \gamma_{\mathrm{d}}\mathbf{e}^{-\lambda(t_{k_i+\rho}+d)}$ is less or equal than the time it takes the error $\| e_i^{\mathrm{d}}(t) \|$ to grow from zero, at time $t_{k_i+\rho}$, to reach the same threshold. Hence, the admissible delay upper-bound d needs to satisfy

$$
(\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d})\Big(\frac{\hat{H}}{1+\hat{\lambda}}\mathbf{e}^{-\hat{\lambda}t_{k_i+\rho}} + \frac{H}{1+\lambda}\mathbf{e}^{-\lambda t_{k_i+\rho}}\Big) \le \gamma_{\mathrm{d}}\mathbf{e}^{-\lambda(t_{k_i+\rho}+d)} \tag{35}
$$

which can also be written as

$$
(\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d})\Big(\frac{\hat{H}}{1+\hat{\lambda}}\mathbf{e}^{(\lambda-\hat{\lambda})t_{k_i+\rho}} + \frac{H}{1+\lambda}\Big)\mathbf{e}^{\lambda d} - \gamma_{\mathrm{d}} \le 0 \tag{36}
$$

In order to show monotonicity of the previous expression and, ultimately, that the corresponding equation has only one root, it is convenient to write the left-hand side of (36) in terms of the coefficients κ_1 and κ_2; doing so we obtain

$$
(\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d})(\kappa_1\mathbf{e}^{\lambda d} + \kappa_2\mathbf{e}^{(1+\lambda)d}) - \gamma_{\mathrm{d}} \le 0 \tag{37}
$$

We first note that both κ_1 and κ_2 are positive for any $t_{k_i+\rho} \ge 0$. Thus, the term $\kappa_1\mathbf{e}^{\lambda d} + \kappa_2\mathbf{e}^{(1+\lambda)d} > 0$ for any $d \in (-\infty, \infty)$ and it is also monotonically increasing. Similarly, the term $\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d}$ is monotonically increasing. This can be shown simply by obtaining its derivative with respect to d which is given by $\mathbf{e}^d + \hat{\lambda}\mathbf{e}^{-\hat{\lambda}d}$. The term $\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d}$ is negative for $d < 0$ and it is positive for $d > 0$. Hence, we conclude that the expression $(\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d})(\kappa_1\mathbf{e}^{\lambda d} + \kappa_2\mathbf{e}^{(1+\lambda)d})$ is monotonically increasing and (25) has exactly one root. Due to the monotonicity property, we also conclude that the largest possible value of d such that (37) holds is obtained when the expression is satisfied with equality.

In the previous paragraph, it is important to note that we considered the case $d < 0$ only to show that (25) has only one root. This is practical in order to solve this equation numerically since it is guaranteed that the numerical solution represents the correct value of d. However, in terms of the admissible communication delays, negative values of d are meaningless. Finally, we can see that the solution of (25) is positive since the solution of

$$
\begin{aligned}
\| e_i^{\mathrm{d}}(t) \| &\le \int_{t_{k_i+\rho}}^t \mathbf{e}^{(t-s)}(\hat{H}\mathbf{e}^{-\hat{\lambda}s} + H\mathbf{e}^{-\lambda s})\,\mathrm{d}s \\
&\le \frac{\hat{H}}{1+\hat{\lambda}}(\mathbf{e}^{(t-t_{k_i+\rho}-\hat{\lambda}t_{k_i+\rho})} - \mathbf{e}^{-\hat{\lambda}t}) + \frac{H}{1+\lambda}(\mathbf{e}^{(t-t_{k_i+\rho}-\lambda t_{k_i+\rho})} - \mathbf{e}^{-\lambda t}) \\
&\le \frac{\hat{H}}{1+\hat{\lambda}}(\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d})\mathbf{e}^{-\hat{\lambda}t_{k_i+\rho}} + \frac{H}{1+\lambda}(\mathbf{e}^d - \mathbf{e}^{-\lambda d})\mathbf{e}^{-\lambda t_{k_i+\rho}} \\
&\le (\mathbf{e}^d - \mathbf{e}^{-\hat{\lambda}d})\Big(\frac{\hat{H}}{1+\hat{\lambda}}\mathbf{e}^{-\hat{\lambda}t_{k_i+\rho}} + \frac{H}{1+\lambda}\mathbf{e}^{-\lambda t_{k_i+\rho}}\Big)
\end{aligned} \tag{33}
$$

$(\mathbf{e}^{d} - \mathbf{e}^{-\hat{\lambda}d})(\kappa_1\mathbf{e}^{\lambda d} + \kappa_2\mathbf{e}^{(1+\lambda)d}) = 0$ is given by $d = 0$ and the parameter $\gamma_{\rm d} > 0$. □

Finally, as in every event-triggered control approach, it is necessary to avoid the presence of Zeno behaviour. The following theorem establishes a positive lower-bound $\tau > 0$ on the inter-event time intervals of every agent. The existence of some $\tau > 0$ guarantees that events are never triggered infinitely fast and, therefore, guarantees that Zeno behaviour does not occur.

Theorem 2: Given $\rho > 1$ and communication delays $d_{ij}(t_{k_i}) < d$ where d is given by the solution of (25), agents (1) implementing control inputs (2) do not exhibit Zeno behaviour if the event time instants, t_{k_i} for $i = 1, \dots, N$, are generated according to condition (7) where $\beta > 0$, $0 < \lambda < \hat{\lambda}$. Furthermore, the inter-event times $t_{k_i+1} - t_{k_i}$ for every agent $i = 1, \dots, N$ are bounded below by the *positive* time τ, i.e.

$$\tau < t_{k_i+1} - t_{k_i} \tag{38}$$

where

$$\tau = \frac{\ln(1 + \beta/\tilde{H})}{1 + \hat{\lambda}} \tag{39}$$

and

$$\tilde{H} = \frac{\hat{H}}{1 + \hat{\lambda}}\mathbf{e}^{(\lambda - \hat{\lambda})t_{k_i}} + \frac{H}{1 + \lambda}. \tag{40}$$

Proof: In order to establish a positive lower bound on the inter-event time intervals (and avoid Zeno behaviour) we obtain the following expression for the dynamics of the error $e_i(t)$ (see (41) and (42)) for $t \in [t_{k_i}, t_{k_i+1})$ with $e_i(t_{k_i}) = 0$. We have (see (42)) Then, we obtain the following

$$\begin{aligned} \| \dot{e}_i(t) \| \le\ & \| e_i(t) \| + \| cBF \| \Big[\| \Theta \| \Big(\hat{\beta}\hat{x}_0\mathbf{e}^{-\hat{\lambda}t} \\ & + \frac{\hat{\beta}\Gamma \| \Delta \|}{\hat{\lambda} - \lambda}(\mathbf{e}^{-\lambda t} - \mathbf{e}^{-\hat{\lambda}t})\Big) + N_i(\beta + \bar{\gamma})\mathbf{e}^{-\lambda t}\Big]. \end{aligned} \tag{43}$$

Further, the error response during the time interval $t \in [t_{k_i}, t_{k_i+1})$ can be bounded as follows:

$$\begin{aligned} \| e_i(t) \| &\le \int_{t_{k_i}}^{t} \mathbf{e}^{(t-s)}(\hat{H}\mathbf{e}^{-\hat{\lambda}s} + H\mathbf{e}^{-\lambda s})\,\mathrm{d}s \\ &\le \frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{(t - t_{k_i} - \hat{\lambda}t_{k_i})} - \mathbf{e}^{-\hat{\lambda}t}) + \frac{H}{1 + \lambda}(\mathbf{e}^{(t - t_{k_i} - \lambda t_{k_i})} - \mathbf{e}^{-\lambda t}) \\ &\le \frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{\tau} - \mathbf{e}^{-\hat{\lambda}\tau})\mathbf{e}^{-\hat{\lambda}t_{k_i}} + \frac{H}{1 + \lambda}(\mathbf{e}^{\tau} - \mathbf{e}^{-\lambda\tau})\mathbf{e}^{-\lambda t_{k_i}} \end{aligned} \tag{44}$$

where $\tau = t - t_{k_i}$.

Thus, the time $\tau > 0$ that it takes for the last expression in (44) to grow from zero, at time t_{k_i}, to reach the threshold $\beta\mathbf{e}^{-\lambda t} = \beta\mathbf{e}^{-\lambda(t_{k_i} + \tau)}$ is less or equal than the time it takes the error $\| e_i(t) \|$ to grow from zero, at time t_{k_i}, to reach the same threshold and generate the following event at time t_{k_i+1}, i.e. $0 < \tau \le t_{k_i+1} - t_{k_i}$. Thus, we wish to find a lower-bound $\tau > 0$ such that the following holds:

$$\begin{aligned} & \frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{\tau} - \mathbf{e}^{-\hat{\lambda}\tau})\mathbf{e}^{-\hat{\lambda}t_{k_i}} + \frac{H}{1 + \lambda}(\mathbf{e}^{\tau} - \mathbf{e}^{-\lambda\tau})\mathbf{e}^{-\lambda t_{k_i}} \\ & \le \beta\mathbf{e}^{-\lambda(t_{k_i} + \tau)} \end{aligned} \tag{45}$$

equivalently

$$\frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{\tau} - \mathbf{e}^{-\hat{\lambda}\tau})\mathbf{e}^{(\lambda - \hat{\lambda})t_{k_i}} + \frac{H}{1 + \lambda}(\mathbf{e}^{\tau} - \mathbf{e}^{-\lambda\tau}) \le \beta\mathbf{e}^{-\lambda\tau}. \tag{46}$$

An explicit solution for τ can be obtained as follows. For a given t_{k_i} we first obtain the admissible delay d and determine the values of H and $\hat{H}$ in (34). Note that $\bar{\gamma}$ and Γ depend on d. Then, since $0 < \lambda < \hat{\lambda}$, the following relation holds for any $\tau > 0$

$$\begin{aligned} & \frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{\tau} - \mathbf{e}^{-\hat{\lambda}\tau})\mathbf{e}^{(\lambda - \hat{\lambda})t_{k_i}} + \frac{H}{1 + \lambda}(\mathbf{e}^{\tau} - \mathbf{e}^{-\lambda\tau}) \\ & < \frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{\tau} - \mathbf{e}^{-\hat{\lambda}\tau})\mathbf{e}^{(\lambda - \hat{\lambda})t_{k_i}} + \frac{H}{1 + \lambda}(\mathbf{e}^{\tau} - \mathbf{e}^{-\hat{\lambda}\tau}). \end{aligned}$$

Thus, we solve for τ in the following inequality:

$$\frac{\hat{H}}{1 + \hat{\lambda}}(\mathbf{e}^{(1 + \hat{\lambda})\tau} - 1)\mathbf{e}^{(\lambda - \hat{\lambda})t_{k_i}} + \frac{H}{1 + \lambda}(\mathbf{e}^{(1 + \hat{\lambda})\tau} - 1) \le \beta$$

and the explicit solution for the lower bound on the inter-event time intervals is given by (39). By the selection $\hat{\lambda} > \lambda$, we have that $\mathbf{e}^{(\lambda - \hat{\lambda})t_{k_i}} \le 1$ for any $t_{k_i} \ge 0$, ensuring that $\tilde{H}$ remains bounded and that $\tau > 0$, i.e. the minimum inter-event time is positive. □

Remark 1: The time event given by the design parameter $\bar{\delta}$ guarantees that the sequence $t_{k_i} \to \infty$. In the presence of packet dropouts and using a pure event-triggered scheme we could have the case that an agent's local state error satisfies $e_i(t) = 0$, for $t \ge t_{k_i}$, i.e. after some update at time t_{k_i}, agent i's local control input is $u_i(t) = 0$, for $t > t_{k_i}$, and no further events are triggered; however, since packet dropouts are possible, some or all agents that receive information from agent i may never receive the final update. The addition of a time-event ensures that all agents are finally updated with the correct state values even if the error remains equal to zero after some update. Note that the time event is not needed if packet dropouts do not exist and every measurement update always arrives at all of their destinations. Finally, and no less important, is the fact that each agent can independently select its own local parameter $\bar{\delta}_i$. The results in this paper hold in the same way by defining $\bar{\delta} = \max_i \bar{\delta}_i$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}e_i(t) &= \dot{e}_i(t) = \dot{y}_i(t) - \dot{x}_i(t) \\ &= Ae_i(t) - cBF\sum_{j \in \mathcal{N}_i}(x_i(t) - x_j(t)) - cBF\sum_{j \in \mathcal{N}_i}(e_i(t) - e_{ji}(t)) \end{aligned} \tag{41}$$

$$\begin{aligned} \| \dot{e}_i(t) \| &\le \| A \| \| e_i(t) \| + \| cBF \| \| \mathcal{L}_2x(t) \| + \Big\| cBF\Big(N_ie_i(t) - \sum_{j \in \mathcal{N}_i} e_{ji}(t)\Big) \Big\| \\ &\le \| e_i(t) \| + \| cBF \|\big(\| \mathcal{L}_2x(t) \| + N_i(\beta + \bar{\gamma})\mathbf{e}^{-\lambda t}\big) \end{aligned} \tag{42}$$

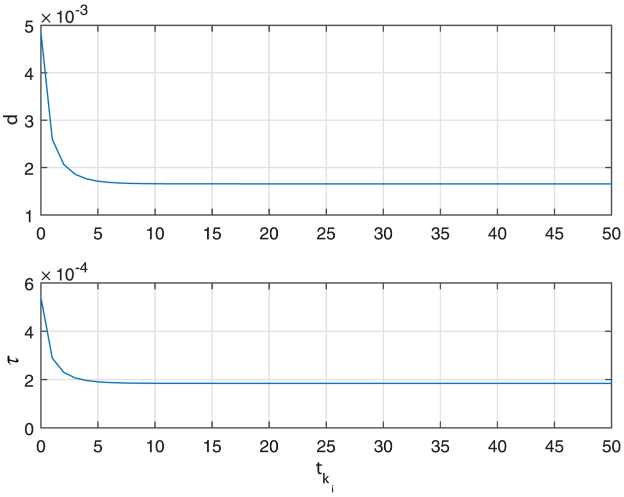

Fig. 3 *Admissible delays and lower-bound on inter-event time intervals*

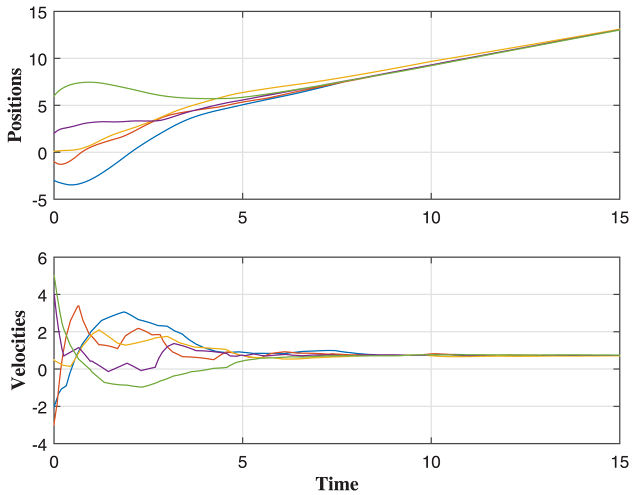

Fig. 4 *Positions (top) and velocities (bottom) of five agents*

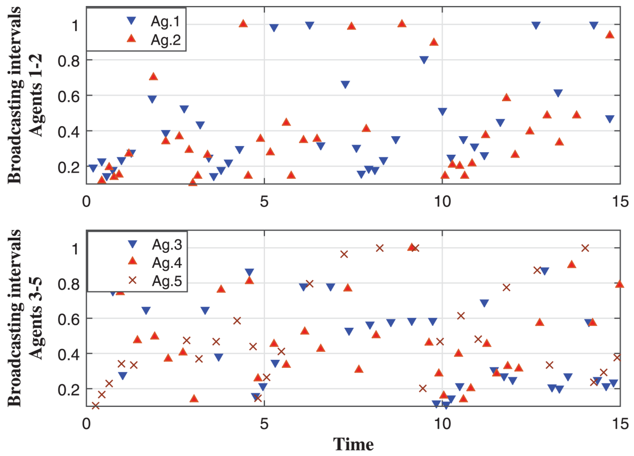

Fig. 5 *Broadcasting time intervals for each agent*

5 Example

Consider five agents connected using a directed graph. The entries of the adjacency matrix $\mathcal{A}$ are given by $a_{12} = a_{23} = a_{25} = a_{32} = a_{43} = a_{45} = a_{54} = 1$ and the remaining entries of $\mathcal{A}$ are equal to zero. The communication channel is subject to packet dropouts and the MANSD is $\rho - 1$, where $\rho = 4$. We choose the parameters $\beta = 1$, $\lambda = 0.4$, $c = 1.2$, $\bar{\delta} = 1$. The admissible delays and the minimum inter event times are shown in Fig. 3.

Fig. 4 shows the response of the five agents. Measurement updates generated by agent i based on its local events may be lost and may not be received by some or by all of the intended agents j, such that $i \in \mathcal{N}_j$, and for $i = 1, \ldots, N$. The maximum number of successive dropped packets is $\rho - 1$. It can be seen from Fig. 4 that the states of all agents converge in their corresponding dimension, i.e. consensus is reached. The time intervals between events are shown for each one of the agents in Fig. 5; however, some of these updates do not reach their destinations. Fig. 6 shows the receiving time intervals from agent 2 to agents 1 and 3; it also shows the receiving time intervals from agent 3 to agents 2 and 4. It can be clearly seen, for instance, that only a fraction of the measurement updates generated by agent 2 are able to reach agents 1 or 3. Thus, the corresponding receiving time intervals are much greater, in general, than the broadcasting time intervals. A similar situation occurs to every agent, but only two agents were selected to show this scenario due to space constraints.

An alternative to the event-triggered consensus protocol presented in this paper is the periodic or time-driven implementation (assuming synchronisation of sampling periods and update time instants is possible). The range of values of the update period for which the agents with double integrator dynamics will achieve consensus under a time-driven or periodic model-based implementation can be obtained from [36]. Necessary and sufficient conditions for consensus of double integrators using periodic updates are provide in [36] (although delays and dropouts are not considered in that reference). Two cases can be studied: the ZOH implementation and the model-based implementation. In any case, one can use these results to conclude that reduction of communication in the time-driven strategy is limited by the range of possible values one can choose for the update period (outside this range the overall system is unstable and the agents' states do not converge), while the back-up time event used in this paper can take any finite value. For the example in this section a periodic implementation free of delays and packet losses will require an update period in the range $(0, 0.207)$. In the presence of non-consistent communication delays and packet losses, it is expected that the upper limit on the update periods will be smaller and more communication will be needed. However, the periodic model-based consensus of multi-agent systems in the presence of non-consistent communication delays and packet losses seems to remain as an open problem.

Remark 2: The double integrator model is a simple model yet it captures some important dynamical aspects of multi-agent systems. An example with real world applications is given by the recovery of autonomous aerial vehicles. Consider a large UAV (or mother-ship) that needs to recover a group of smaller or micro-UAVs after they were deployed and they have performed their tasks. Only acceleration commands can be given to each one of the vehicles. Also, the large UAV cannot stop, but it needs to travel at a certain speed range. Hence, the single integrator model is not suitable whereas the double integrator model can capture these requirements. The consideration of directed graphs in this paper also makes this scenario feasible since the mother-ship speed profile is usually not affected by any of the micro-UAVs.

6 Conclusions

The consensus problem of agents with double integrator dynamics and with packet losses and communication delays was studied in this paper. A decentralised event-triggered consensus protocol was proposed and it was shown that the group of agents achieves consensus asymptotically when they are connected using a directed graph and that Zeno behaviour is avoided. We provided methods to obtain admissible delays and to determine the lower-bounds on the inter-event time intervals. The use of event-triggered control and communication techniques allows for further reduction of communication compared to periodic implementations. More importantly, the event-triggered consensus protocol presented in this paper provides a higher level of decentralisation since it is not necessary for agents to know a global sampling period and global communication time instants as in sampled-data approaches. The decentralised event-triggered consensus protocol provides each

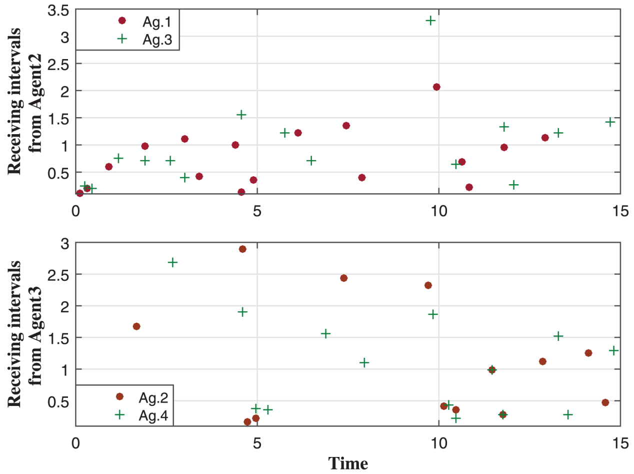

Fig. 6 *Receiving time intervals for agents* j *such that* $2 \in \mathcal{N}_j$ *(top) and* $3 \in \mathcal{N}_j$ *(bottom)*

agent the freedom to decide their own broadcasting instants independently of any other agent in the network.

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8 References

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