# Technical Notes and Correspondence

## Periodic Event-Triggered Synchronization of Linear Multi-Agent Systems With Communication Delays

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***Abstract*—Multi-agent systems' cooperation to achieve global goals is usually limited by sensing, actuation, and communication issues. At the local level, continuous measurement and actuation is only approximated by the use of digital mechanisms that measure and process information in order to compute and update new control input values at discrete time instants. Interaction with other agents takes place, in general, through a digital communication channel with limited bandwidth where transmission of continuous-time signals is not possible. This technical note considers the problem of consensus (or synchronization of state trajectories) of multi-agent systems that are described by general linear dynamics and are connected using undirected graphs. The proposed event-triggered consensus protocol not only avoids the need for continuous communication between agents but also provides a decentralized method for transmission of information in the presence of time-varying communication delays, where each agent decides its own broadcasting time instants based only on local information. This method gives more flexibility for scheduling information broadcasting compared to periodic and sampled-data implementations.**

***Index Terms*—Consensus, event-triggered control, multi-agent system.**

### I. Introduction

Cooperative control of multi-agent systems is an active research area with broad and relevant applications in commercial, academic and military areas [1]. The design of decentralized and scalable control algorithms provides the necessary coordination for a group of agents to outperform agents operating independently. However, in many scenarios, communication is limited and agents are not able to continuously broadcast information. The necessity for decentralized computation of the time instants at which agents broadcast relevant information arises. In addition, continuous actuation and continuous measurement of local states may also be restricted by particular hardware limitations.

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Consensus problems with limited communication have been studied using the sampled-data (periodic) approach [2] and [3]. An important drawback of periodic transmission is that it requires synchronization between the agents, that is, all agents need to transmit their information at the same time instants and, in some cases, it requires a conservative sampling period for worst case situations.

In the present technical note, in lieu of periodic approaches, we use an asynchronous communication scheme based on event-triggered control strategies, and we consider agents that are described by general linear dynamics which are subject to limited actuation update rates and also to limited local sensor measurement update rates. In addition, we consider the case where communication among agents is subject to communication delays. In event-triggered control, measurements are triggered by the occurrence of certain events. In event-triggered broadcasting [4]–[8], a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Consensus problems have been studied using these techniques [9]–[12]. Event-triggered control provides a more robust and efficient use of network bandwidth. Its implementation in multi-agent systems also provides a highly decentralized way to schedule transmission instants which does not require synchronization compared to periodic sampled-data approaches.

One important restriction related to event-triggered control techniques is that continuous measurement of state variables and continuous computation of state errors and time-varying thresholds is required. One solution explored by different researchers is self-triggered control [13]–[16]. The main difference with respect to event-triggered control is that a measure of the state is not constantly compared against a predefined threshold. Instead, the current state measurement is used to determine its next deadline, i.e. the next time that the sensor is required to send a measurement to the controller. A recent extension to event-triggered control is the so called *periodic event-triggered control* [17], [18], where measurements of states and computations of errors and thresholds occur, not continuously in time, but only at periodic time instants.

Consensus problems where all agents are described by general linear models have been considered by different authors [19]–[26]. In these papers it is assumed that continuous communication between agents is possible.

In the present technical note, decentralized event thresholds that guarantee practical consensus (where the difference between the states of any two agents is bounded) and strictly positive inter-event times are designed. The lower-bounds on the inter-event time intervals are independent of the particular system trajectories, therefore they hold for any two consecutive local events. The periodic event-triggered control technique automatically avoids the presence of Zeno behavior. However, and for completeness, we establish the relationship between a selected sampling period and the performance of the consensus protocol with respect to the bounds on the state disagreement.

Event-triggered consensus of agents with linear dynamics and limited communication was recently explored in [27] and [28]. In our previous work [29] and [30], we proposed a novel approach in which each agent implements models of the decoupled dynamics of each one of its neighbors and uses the model states to compute the local control input. The present technical note addresses the following problems that were not considered by [30]:

1) Time-varying communication delays: Transmitted event-based updates are subject to communication delays.
2) Constrained sensing and event computation rate: Each agent does not need to continuously measure its own state but only at finite time instants. The inability to continuously compute errors and events is also taken into account.
3) Constrained actuation rate: This event-based approach also provides sampled actuation time instants instead of continuous actuation.

Continuous measurement, actuation, and computation of events severely restrict the operation of the subsystems; therefore, the relaxations addressed in this technical note offer a significant advantage in terms of implementation and resource management.

The remainder of this technical note is organized as follows. Section II provides a brief background on graph theory and describes the problem and the consensus protocol. Section III gives a result assuming continuous communication which will be used in the main results of this technical note. Design of periodic decentralized event thresholds for multi-agent systems with limited sensing and actuation capabilities and with time-varying, but bounded, communication delays is addressed in Section IV. Section V concludes the technical note.

## II. Preliminaries

***Graph Theory:*** Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consisting of a set of vertices or nodes $\mathcal{V} = \{1, \ldots, N\}$ and a set of edges $\mathcal{E}$. An edge between nodes $i$ and $j$ is represented by the pair $(i, j) \in \mathcal{E}$. A graph $\mathcal{G}$ is called undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ and the nodes are called adjacent. The adjacency matrix $\mathcal{A}$ is defined by $a_{ij} = 1$ if the nodes $i$ and $j$ are adjacent and $a_{ij} = 0$ otherwise. If $(j, i) \in \mathcal{E}$, then $j$ is said to be a neighbor of $i$. The set $\mathcal{N}_i$ is called the set of neighbors of node $i$, and $N_i$ is its cardinality. A node $j$ is an element of $\mathcal{N}_i$ if $(j, i) \in \mathcal{E}$. A path from node $i$ to node $j$ is a sequence of distinct nodes that starts at $i$ and ends at $j$, such that every pair of consecutive nodes is adjacent. An undirected graph is connected if there is a path between every pair of distinct nodes. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ where $\mathcal{D}$ represents the degree matrix which is a diagonal matrix with entries $d_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$. For undirected graphs, $\mathcal{L}$ is symmetric and positive semi-definite. $\mathcal{L}$ has zero row sums and, therefore, zero is an eigenvalue of $\mathcal{L}$ with associated eigenvector $\mathbf{1}_N$ (a vector with all its $N$ entries equal to one), that is, $\mathcal{L}\mathbf{1}_N = \mathbf{0}_N$. If an undirected graph is connected then $\mathcal{L}$ has exactly one eigenvalue equal to zero and all its non-zero eigenvalues are positive; they can be set in increasing order $\lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \lambda_3(\mathcal{L}) \leq \cdots \leq \lambda_N(\mathcal{L})$, with $\lambda_1(\mathcal{L}) = 0$.

***Lemma 1:*** Let $\mathcal{L}$ be the symmetric Laplacian of an undirected and connected graph. Then, consensus is achieved if and only if

$$V = \chi^T \hat{\mathcal{L}} \chi = 0 \tag{1}$$

where $\hat{\mathcal{L}} = \mathcal{L} \otimes Q$, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\chi(t) = [\chi_1(t)^T \ \chi_2(t)^T \ \cdots \ \chi_N(t)^T]^T$, $\chi_i \in \mathbb{R}^n$, and $\otimes$ denotes the Kronecker product.

*Proof:* See [29].

***Problem Statement:*** Consider a group of $N$ agents with fixed communication graphs and fixed weights. Each agent's dynamics are described by the following:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t_\mu), \ i = 1, \ldots, N \tag{2}$$

$$u_i(t_\mu) = c_1 F \sum_{j\in\mathcal{N}_i} (x_i(t_\mu) - y_{ji}(t_\mu)), \ i = 1, \ldots, N \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_i \in \mathbb{R}^n$ is the state of agent $i$, and $u_i \in \mathbb{R}^m$ is the control input for agent $i$. $F \in \mathbb{R}^{m\times n}$ and $c_1 \in \mathbb{R}_+$ are design parameters that are defined below.

In this framework, every agent implements discretized models of itself and of its neighbors. Since the measurement updates will be delayed, the neighbors of agent $i$ will have a version of agent $i$'s model state that is different than agent $i$'s version. It is necessary to distinguish between the model state as seen by the local agent, itself, and as seen by its neighbors. Define the dynamics and update law of the model state of agent $i$ as seen by agent $i$ as

$$y_{ii}(t_{\mu+1}) = Gy_{ii}(t_\mu) \tag{4}$$

for $t_\mu \in [t_{k_i}, t_{k_i+1})$ where $G = e^{Ah}$, $h = t_{\mu+1} - t_\mu$, and $y_{ii}(t_{k_i}) = x_i(t_{k_i})$. The notation $t_{k_i}$ represents the local broadcasting time instants, i.e., when agent $i$ triggers an event and transmits its current state. Since we do not check for events continuously but only at sampling times $t_\mu$, then $t_{k_i}$ is equal to some sampling instant $t_\mu$ but not every time instant $t_\mu$ is a broadcasting instant $t_{k_i}$. The decision to trigger an event in order to broadcast the current state is given by the event-triggered schemes described in the Section IV.

The measurement $x_i(t_{k_i})$ is transmitted by agent $i$ at time $t_{k_i}$ and will arrive at agents $j$, for $j \in \mathcal{N}_i$, at time $t_{k_i} + d_i(t_{k_i})$. For a given update instant all receiving agents experience the same delay $d_i(t_{k_i})$. However, this is not a constraint and the communication delays can be generalized so, for the same update instant $t_{k_i}$, the neighbors are updated at different time instants.

Assume without loss of generality that $d_i(t_{k_i}) = hp_i(t_{k_i})$ and $p_i \geq 1$ is an integer, that is, the delay is an integer multiple of the sampling period $h$. If the delay is not an integer multiple of the sampling period the receiving agent uses the delayed measurement at the next sampling period to update the corresponding model, which effectively makes the delay to be an integer multiple of $h$. Define the dynamics of the model state of agent $i$ as seen by agent $j$ as

$$y_{ij}(t_{\mu+1}) = Gy_{ij}(t_\mu) \tag{5}$$

for $j \in \mathcal{N}_i$. The states of these models are updated when a delayed measurement of agent $i$ is received by agent $j$, for $j \in \mathcal{N}_i$. The state measurement $x_i(t_{k_i})$ transmitted by agent $i$ at time instant $t_{k_i}$ is received by agent $j$, for $j \in \mathcal{N}_i$, at time instant $t_{k_i} + hp_i(t_{k_i})$. Let us define the update law of the model state of agent $i$ as seen by agent $j$ as

$$y_{ij}\left(t_{k_i} + hp_i\left(t_{k_i}\right)\right) = f_d\left(x_i\left(t_{k_i}\right), p_i\left(t_{k_i}\right)\right) \tag{6}$$

for $j \in \mathcal{N}_i$, where $t_{k_i}$ represents the update instants triggered by agent $i$ and $hp_i(t_{k_i})$ represents the communication delay associated to the triggering instant $t_{k_i}$.

We refer to $h$ as the discretization or sampling period since it is used to obtain the discrete-time model $G$. Also, the state of each agent, $x_i$, $i = 1, \ldots, N$, is sampled every $h$ time units. The discretization period $h$ is constant but the communication intervals for each agent are not constant. At every time instant $t_\mu$ each agent samples its own state and updates its control input (3). It also uses its sampled state to compute its local state error and to determine if it is necessary or not to transmit the current state $x_i(t_\mu)$ to its neighbors.

The main advantage of using a periodic event-triggered control scheme is that measuring the state of each agent (and the associated computations that require evaluation of state errors and thresholds) is only performed at some periodic time instants (every $h$ time units) instead of doing it continuously as it is common in most event-triggered control schemes. Additionally, continuous actuation is not required and the operations related to implementing discrete-time models at every node are simplified with respect to implementing continuous-time models.

Note that the model state $y_{ii}(t_\mu)$ is not used by agent $i$ for control since the real state, $x_i(t_\mu)$, is locally available at every sampling time $t_\mu$. However, the local model state, $y_{ii}(t_\mu)$, is used to trigger local events. This way of defining the local control input (3) represents a difference with respect to the approach in [29] and [30] where the control input is a function of model variables, including $y_{ii}$. Also note that the local control input (3) is decentralized since it only depends on local information, that is, on the sampled state of the local agent and on the discretized model states of its neighbors. Continuous or even periodic access to the states of neighbors is not needed.

## III. Consensus With Continuous Measurements

In this section we consider the case where each agent is able to continuously update its control input. We also consider the case where agents exchange information continuously with their neighbors. In this case the local control inputs are given by

$$u_i(t) = cF \sum_{j\in\mathcal{N}_i} (x_i(t) - x_j(t)), \quad i = 1, \ldots, N. \tag{7}$$

Assume that the pair $(A, B)$ is controllable. Then, for $\alpha > 0$ there exists a symmetric and positive definite solution $P$ to

$$PA + A^T P - 2PBB^T P + 2\alpha P < 0. \tag{8}$$

Let

$$F = -B^T P; \; c \geq \frac{1}{\lambda_2}. \tag{9}$$

***Theorem 1:*** Assume the pair $(A, B)$ is controllable and the communication graph is connected and undirected. Define $F$ and $c$ as in (9). Then, the following symmetric matrix:

$$\bar{\mathcal{L}} = \hat{\mathcal{L}} A_c + A_c^T \hat{\mathcal{L}} \tag{10}$$

has only $n$ eigenvalues equal to zero and the rest of its eigenvalues are negative. In addition, the eigenvectors associated with its $n$ zero eigenvalues belong to the subspace spanned by the eigenvectors associated with the $n$ zero eigenvalues of $\hat{\mathcal{L}}$, where $\hat{\mathcal{L}} = \mathcal{L} \otimes P$, $A_c = \bar{A} + \bar{B}$, $\bar{A} = I_N \otimes A$, $\bar{B} = c\mathcal{L} \otimes BF$, and $I_N$ is an identity matrix of size $N$.

*Proof:* See [29].

***Lemma 2:*** Assume the pair $(A, B)$ is controllable and the communication graph is connected and undirected. Then, protocol (7)–(9) solves the consensus problem for agents described by (2). Furthermore, the Lyapunov function defined by $V = x^T \hat{\mathcal{L}} x$ has a time derivative along the trajectories of (2) with inputs (7) given by $\dot{V} = x^T \bar{\mathcal{L}} x$.

From Theorem 1 it can be seen that $\dot{V}$ is negative when the overall system is in disagreement and is equal to zero only when the corresponding states are in total agreement. In the latter case we also have $V = 0$, see Lemma 1. Different from consensus with single integrators, where the agents converge to a constant value, here it is only required that the difference between states of agents tends to zero, regardless of the particular response of the systems.

## IV. Decentralized Event Triggered Consensus With Communication Delays

We now address the case where agents use event-triggered communication strategies in order to reduce the frequency of transmissions in the presence of time-varying but bounded communication delays. We also consider the scenario where agents are only able to measure its own state and update its control input at discrete-time instants defined by $t_\mu$. Define

$$e_{ii}(t) = y_{ii}(t) - x_i(t), \; e_{ij}(t) = y_{ij}(t) - x_i(t) \tag{11}$$

and $x = [x_1^T \ldots x_N^T]^T$, where $y_{ii}(t)$ and $y_{ij}(t)$ represent hypothetical continuous-time models

$$\dot{y}_{ii}(t) = Ay_{ii}(t), \; \dot{y}_{ij}(t) = Ay_{ij}(t). \tag{12}$$

Definition (12) is only used for analysis of the event-triggered controller; it is not used for implementation of the event-triggered consensus algorithm.

Define the discretization error $\breve{x}_i(t) = x_i(t_\mu) - x_i(t)$ for $t \in [t_\mu, t_{\mu+1})$. Also define $\breve{x} = [\breve{x}_1^T \ldots \breve{x}_N^T]^T$.

Define a positive and constant upper bound on the communication delays by $d = ph < t_{k_i+1} - t_{k_i}$, that is, $d_i(t_{k_i}) \leq d$ for any triggering instant $t_{k_i}$ and for $i = 1, \ldots, N$. Later in this section we will define the design parameters that bound the inter-event times as a function of the delay $d$. Note that $p \geq p_i(t_{k_i})$, for $i = 1, \ldots, N$. Assume that the current delay, $p_i(t_{k_i})$, is known to the receiving agents, e.g., using time-stamps.

Since both models, (4) and (5), use the same state matrix to compute their response between their corresponding update instants, then we define

$$f_d(x_i(t_{k_i}), hp_i(t_{k_i})) \triangleq G^{p_i(t_{k_i})} x_i(t_{k_i}) \tag{13}$$

that is, the delayed measurement is propagated forward in time and the result is used to update the state of the model as shown in (6). By definition, we have that the following *local* triggering event will occur at time $t_{k_i+1} > t_{k_i} + d$, this means that $y_{ii}(t_{k_i} + d_i(t_{k_i})) = G^{p_i(t_{k_i})} x_i(t_{k_i})$ because no other local event has been triggered since time instant $t_{k_i}$. Therefore, we have that $y_{ii}(t_\mu) \neq y_{ij}(t_\mu)$ for $t_\mu \in [t_{k_i}, t_{k_i} + hp_i(t_{k_i}))$ and $y_{ii}(t_\mu) = y_{ij}(t_\mu)$ for $t \in [t_{k_i} + hp_i(t_{k_i}), t_{k_i+1})$.

Define the state errors

$$\begin{aligned} e_{ii}(t_\mu) &= y_{ii}(t_\mu) - x_i(t_\mu) \\ e_{ij}(t_\mu) &= y_{ij}(t_\mu) - x_i(t_\mu). \end{aligned} \tag{14}$$

Note that $e_{ii}(t_{k_i}) = 0$ and $e_{ij}(t_\mu) = e_{ii}(t_\mu)$, for $t_\mu \in [t_{k_i} + hp_i(t_{k_i}), t_{k_i+1})$.

Also define $\zeta(t_\mu) = [e_{11}^T(t_\mu) \ldots e_{NN}^T(t_\mu)]^T \in \mathbb{R}^{nN}$ and $\zeta_d(t_\mu) = [e_{1j_1}^T(t_\mu) \ldots e_{Nj_N}^T(t_\mu)]^T \in \mathbb{R}^{nN}$, where the components $e_{ij_i}$ in $\zeta_d$ represent the errors defined in (14), that is, the error of agent $i$ as seen by its neighbors $j_i$, $j_i \in \mathcal{N}_i$.

Define the discretization errors $\breve{y}_{ii}(t) = y_{ii}(t_\mu) - y_{ii}(t)$, $\breve{y}_{ij}(t) = y_{ij}(t_\mu) - y_{ij}(t)$, $\breve{e}_{ii}(t) = e_{ii}(t_\mu) - e_{ii}(t)$, $\breve{e}_{ij}(t) = e_{ij}(t_\mu) - e_{ij}(t)$, $\breve{y} = [\breve{y}_{11}^T \ldots \breve{y}_{NN}^T]^T$, $\breve{y}_d = [\breve{y}_{1j_1}^T \ldots \breve{y}_{Nj_N}^T]^T$, $\breve{\zeta} = [\breve{e}_{11}^T \ldots \breve{e}_{NN}^T]^T$, $\breve{\zeta}_d = [\breve{e}_{1j_1}^T \ldots \breve{e}_{Nj_N}^T]^T$.

The dynamics of every agent in (2) with communication delays can be written in compact form as follows:

$$\dot{x} = (A_c + \bar{B})x + \bar{B}_{\mathcal{A}} \zeta_d(t_\mu) + \bar{B}_1 \breve{x} \tag{15}$$

where $\bar{B}_{\mathcal{A}} = -c_1 \mathcal{A} \otimes BF$, $\bar{B}_1 = c_1 \mathcal{L} \otimes BF$, and the coupling strength $c_1 = 2c$ has been used, where $c \geq 1/\lambda_2$. The overall system dynamics (15) is written in terms of the closed-loop state $x$ plus two error variables that are introduced in the system dynamics because of the sampled inputs and the event-based communication strategies. The error $\zeta_d(t_\mu)$ is due to the fact that each agent only communicates its local state at some local event time instants and the transmissions are delayed. The error $\breve{x}$ is due to the fact that only discrete models of neighbors and periodic samples of the local state are used to compute the control input instead of continuous variables. In other words, the first error results from limiting the communication between agents and the time-delay that affects the transmitted measurements while the second error results from imposing a discrete-time actuation at each local node.

Before presenting Theorem 2, let us define the following:

$$\begin{aligned} z_i &= \sum_{j\in\mathcal{N}_i} (x_i(t_\mu) - y_{ji}(t_\mu)) \\ \breve{z}_i &= \sum_{j\in\mathcal{N}_i} (\breve{x}_i - \breve{y}_{ji}). \end{aligned} \tag{16}$$

Also define $\bar{\lambda} = \lambda_{\max}(\mathcal{A}^2 \otimes PBB^TP)$, $\Upsilon = \int_0^d \|e^{A(d-s)}cBF\|ds < 1/b_e$, $\Upsilon_h = \int_0^{d-h} \|e^{A(d-h-s)}cBF\|ds$, and $E = \int_0^h e^{A(h-s)}cBFds$ where $b_e = \sqrt{N_iN(N-1)((b/2)+(1/2b))}$ and $b>0$.

***Theorem 2:*** Assume the pair $(A,B)$ is controllable and the communication graph is connected and undirected. Then agents (2) with inputs (3) achieve, in the presence of communication delays $d_i < d$, a bounded consensus error where the difference between any two states is bounded by

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\|^2 \leq \frac{N\eta}{\beta\lambda_{\min}(P)} \tag{17}$$

for $i,j = 1,\dots,N$, if the events are triggered when

$$\delta_i > \sigma c_1 z_i^T PBB^TPz_i + \eta \tag{18}$$

where $0<\sigma<1$

$$\delta_i = c_1(1+b)^2 \breve{z}_i^T\left(t_\mu^-\right) PBB^TP\breve{z}_i\left(t_\mu^-\right) + \delta_{di} + \breve{\delta}_i \tag{19}$$

$$\begin{aligned} \delta_{di} = c_1 \left(1+\frac{1}{b}\right) \bar{\lambda} \big[& e_{ii}^T(t_\mu)(G^p)^TG^pe_{ii}(t_\mu) \\ &+ 2\|G^pe_{ii}(t_\mu)\|\Upsilon\bar{z}_i + (\Upsilon\bar{z}_i)^2\big] \end{aligned} \tag{20}$$

$$\begin{aligned} \breve{\delta}_i &= c_1(1+b)\left(1+\frac{1}{b}\right)\bar{\lambda} \\ &\times \Big[e_{ii}^T(t_\mu)\left((G-I)G^{p-1}\right)^T(G-I)G^{p-1}e_{ii}(t_\mu) \\ &+ 2\left\|(G-I)G^{p-1}e_{ii}(t_\mu)\right\|(\|G-I\|\Upsilon_h + \|E\|)\bar{z}_i \\ &+ \left((\|G-I\|\Upsilon_h + \|E\|)\bar{z}_i\right)^2\Big] \end{aligned} \tag{21}$$

and $\bar{z}_i = (\lambda_{\max}(\mathcal{L})/(1-b_e\Upsilon))\sqrt{V_M/\lambda_{\min}(\hat{\mathcal{L}})}$.

Furthermore, the agents do not exhibit Zeno behavior and the inter-event times $t_{k_i+1} - t_{k_i}$ for every agent $i=1,\dots,N$ are bounded by $d>0$, that is, $0 < d < t_{k_i+1} - t_{k_i}$ if

$$\begin{aligned} \eta &> c_1(1+b)^2\|PBB^TP\| \\ &\times \frac{\lambda_{\max}^2(\mathcal{L})}{\lambda_{\min}(\hat{\mathcal{L}})} V_M \left(\frac{1}{1-b_e\Upsilon} + \frac{1}{1-b_e\Upsilon_h}\right)^2 \\ &+ c_1\left(1+\frac{1}{b}\right)(\|G^p\|+1)^2\Upsilon^2\bar{z}_i^2 + c_1(1+b)\left(1+\frac{1}{b}\right) \\ &\times \left(\left\|(G-I)G^{p-1}\right\|\Upsilon + \|(G-I)\|\Upsilon_h + \|E\|\right)^2\bar{z}_i^2 \end{aligned} \tag{22}$$

*Proof:* Consider the candidate Lyapunov function $V = x^T\hat{\mathcal{L}}x$ and evaluate the derivative along the trajectories of systems (2) with inputs (3)

$$\begin{aligned} \dot{V} &= x^T\hat{\mathcal{L}}\left((A_c+\bar{B})x + \bar{B}_{\mathcal{A}}\zeta_d(t_\mu) + \bar{B}_1\breve{x}\right) \\ &\quad + \left((A_c+\bar{B})x + \bar{B}_{\mathcal{A}}\zeta_d(t_\mu) + \bar{B}_1\breve{x}\right)^T\hat{\mathcal{L}}x \\ &= x^T\bar{\mathcal{L}}x + 2 \\ &\quad\times\sum_{i=1}^N\left[\sum_{k\in\mathcal{N}_i}(x_i-x_k)^TPBB^TP\right. \\ &\quad\left.\times\left(-c\sum_{j\in\mathcal{N}_i}(x_i-x_j) + c_1\sum_{j\in\mathcal{N}_i}e_{ji}(t_\mu) - c_1\sum_{j\in\mathcal{N}_i}(\breve{x}_i-\breve{x}_j)\right)\right] \end{aligned}$$

We can write $\dot{V}$ in terms of the sampled state information $x_i(t_\mu)$, $x_j(t_\mu)$ and of the discretization errors $\breve{x}_i$, $\breve{x}_j$ as follows:

$$\begin{aligned} \dot{V} &= x^T\bar{\mathcal{L}}x + 2 \\ &\times\sum_{i=1}^N\left[\sum_{k\in\mathcal{N}_i}(x_i(t_\mu)-x_k(t_\mu))^T\right. \\ &\quad\times PBB^TP\left(-c\sum_{j\in\mathcal{N}_i}(x_i(t_\mu)-x_j(t_\mu)) + c_1\sum_{j\in\mathcal{N}_i}e_{ji}(t_\mu)\right) \\ &\quad+\sum_{k\in\mathcal{N}_i}(\breve{x}_i-\breve{x}_k)^TPBB^TP \\ &\quad\left.\times\left(c\sum_{j\in\mathcal{N}_i}(\breve{x}_i-\breve{x}_j) - c_1\sum_{j\in\mathcal{N}_i}e_{ji}(t_\mu)\right)\right] \end{aligned}$$

By using the state error definition (14), the definition of $z_i$ in (16), and the inequality $\|x^Ty\| \leq (b/2)x^Tx + (1/2b)y^Ty$, for $b>0$, we can write

$$\begin{aligned} \dot{V} &\leq x^T\bar{\mathcal{L}}x + 2 \\ &\times\sum_{i=1}^N\left[-cz_i^TPBB^TPz_i + c\left(1+\frac{1}{b}\right)\sum_{k\in\mathcal{N}_i}e_{ki}^T(t_\mu)PBB^TP\sum_{j\in\mathcal{N}_i}e_{ji}(t_\mu)\right. \\ &\quad\left.+ c(1+b)\sum_{k\in\mathcal{N}_i}(\breve{x}_i-\breve{x}_k)^TPBB^TP\sum_{j\in\mathcal{N}_i}(\breve{x}_i-\breve{x}_j)\right] \end{aligned}$$

where $e_{ji}(t_\mu)$ represents the state error, at time instants $t_\mu$, of agent $j$ as seen by agent $i$, for $j\in\mathcal{N}_i$. We can write the following:

$$\begin{aligned} \dot{V} &\leq x^T\bar{\mathcal{L}}x + \sum_{i=1}^N\Bigg[-c_1z_i^TPBB^TPz_i \\ &\quad+ c_1\left(1+\frac{1}{b}\right)\sum_{k\in\mathcal{N}_i}e_{ki}^T(t_\mu)PBB^TP\sum_{j\in\mathcal{N}_i}e_{ji}(t_\mu) \\ &\quad+ c_1(1+b)\Bigg(\breve{z}_i^TPBB^TP\breve{z}_i + 2\breve{z}_i^TPBB^TP\sum_{j\in\mathcal{N}_i}\breve{e}_{ji} \\ &\quad+ \sum_{k\in\mathcal{N}_i}\breve{e}_{ki}^TPBB^TP\sum_{j\in\mathcal{N}_i}\breve{e}_{ji}\Bigg)\Bigg]. \end{aligned}$$

Using the inequality $\|x^T y\| \leq (b/2)x^T x + (1/2b)y^T y$, for $b > 0$, once again, we obtain

$$\begin{aligned}\dot{V} \leq & x^T \bar{\mathcal{L}} x + \sum_{i=1}^{N} \left[ -c_1 z_i^T PBB^T P z_i + c_1 (1+b)^2 \breve{z}_i^T PBB^T P \breve{z}_i \right] \\ & + c_1 \left(1 + \frac{1}{b}\right) \zeta_d^T(t_\mu) (\mathcal{A} \otimes B^T P)^T (\mathcal{A} \otimes B^T P) \zeta_d^T(t_\mu) \\ & + c_1(1+b) \left(1 + \frac{1}{b}\right) \breve{\zeta}_d^T (\mathcal{A} \otimes B^T P)^T (\mathcal{A} \otimes B^T P) \breve{\zeta}_d^T.\end{aligned}$$

The variables $z_i$ and $\breve{z}_i$ can be computed locally by every node. Let us then focus on the delayed error terms $\zeta_d^T(t_\mu)$ and $\breve{\zeta}_d^T$. Define $\nu_i(t_\mu) = y_{ij}(t_\mu) - y_{ii}(t_\mu)$, then we have that

$$\begin{aligned} e_{ij}(t_\mu) &= y_{ij}(t_\mu) - x_i(t_\mu) \\ &= y_{ij}(t_\mu) - (y_{ii}(t_\mu) - e_{ii}(t_\mu)) \\ &= \nu_i(t_\mu) + e_{ii}(t_\mu). \end{aligned} \tag{23}$$

For the term $\nu_i(t_\mu)$ the following holds:

$$\nu_i(t_{\mu+1}) = \begin{cases} G\nu_i(t_\mu), & t_\mu \in [t_{k_i}, t_{k_i} + d_i(t_{k_i})) \\ 0, & t_\mu \in [t_{k_i} + d_i(t_{k_i}), t_{k_i+1}) \end{cases} \tag{24}$$

with $\nu_i(t_{k_i}) = e_{ii}(t_{k_i}^-)$. The notation $t_{k_i}^-$ represents the event time $t_{k_i}$ but just before the local error is reset to zero. This update of the variable $\nu$ at time $t_{k_i}$ is obtained by simply realizing that $\nu_i(t_{k_i}) = y_{ij}(t_{k_i}) - y_{ii}(t_{k_i}) = y_{ii}(t_{k_i}^-) - x_i(t_{k_i}^-) = e_{ii}(t_{k_i}^-)$ (the local error just before it resets to zero), since the local model $y_{ii}$ is updated using $x_i(t_{k_i})$ and $y_{ij}(t_{k_i}) = y_{ij}(t_{k_i}^-) = y_{ii}(t_{k_i}^-)$.

Define $\nu(t_\mu) = [\nu_1^T(t_\mu) \ldots \nu_N^T(t_\mu)]^T$ and we have that $\zeta_d(t_\mu) = \nu(t_\mu) + \zeta(t_\mu)$. Consider the worst case scenario (greatest difference between $\zeta_d(t_\mu)$ and $\zeta(t_\mu)$) given when all agents transmit at the same instant $t_k$ and the greatest possible delay $d\,(\geq hp_i)$ is present. Then, we have the following:

$$\begin{aligned} & \zeta_d^T(t_k + d)\zeta_d(t_k + d) \\ & = \zeta^T(t_k^-) (\bar{G}^p)^T \bar{G}^p \zeta(t_k^-) + 2\zeta^T(t_k + d)\bar{G}^p \zeta(t_k^-) \\ & \quad + \zeta^T(t_k + d)\zeta(t_k + d) \\ & = \sum_{i=1}^{N} \Big[ e_{ii}^T(t_{k_i}^-) (G^p)^T G^p e_{ii}(t_{k_i}^-) \\ & \qquad + 2e_{ii}^T(t_{k_i}+d) G^p e_{ii}(t_{k_i}^-) + e_{ii}^T(t_{k_i}+d) e_{ii}(t_{k_i}+d) \Big] \\ & \leq \sum_{i=1}^{N} \Big[ e_{ii}^T(t_{k_i}^-) (G^p)^T G^p e_{ii}(t_{k_i}^-) \\ & \qquad + 2\left\| G^p e_{ii}(t_{k_i}^-) \right\| \Upsilon z_{i,M} + (\Upsilon z_{i,M})^2 \Big] \end{aligned} \tag{25}$$

where $\bar{G} = I_N \otimes G$. The local error $e_{ii}^T(t_{k_i}^-)$ represents the error just before the update instant, i.e., before it is reset to zero because of the update at time $t_{k_i}$. On the other hand, $e_{ii}(t_{k_i} + d)$ represents the error *after* the update at time $t_{k_i}$ and it can only be estimated using

$$\begin{aligned} \|e_{ii}(t_{k_i} + d)\| & \leq \left\| e^{Ad} e_{ii}(t_{k_i}) \right\| + z_{i,M} \int_0^d \left\| e^{A(d-s)} cBF \right\| ds \\ & \leq \Upsilon z_{i,M} \end{aligned} \tag{26}$$

where $e_{ii}(t_{k_i}) = 0$ and $z_{i,M}$ represents an upper-bound on the local control input for the time interval $t_\mu \in [t_{k_i}, t_{k_i} + d]$, that is, $\|z_i(t_{k_i} + hp_i)\| \leq z_{i,M}$ for $p_i = 1, \ldots, p$.

Since the worst case is given by the maximum delay $d$ we can use (25) and the current local error $e_{ii}(t_\mu)$ to bound the delayed error $e_{ij}(t_\mu + d)$ for any sampling time $t_\mu > 0$, therefore the term (20) is used as a part of the overall threshold (19). In other words, we propagate the current error $e_{ii}(t_\mu)$ forward in time using the worst case delay $d$, as if we had an event at time $t_\mu$, then we check (18). If (18) holds then an event is triggered; otherwise, no event is needed and we repeat the same process at the following sampling time instant $t_{\mu+1}$.

Now, consider the discretization error $\breve{\zeta}_d$ corresponding to the delayed state errors $\zeta_d(t_\mu)$. The discretization errors are reset at every sampling time $t_\mu$; therefore, we only need to consider the effect of these errors at time $t_k + d$ for the worst case delay $d\,(\geq hp_i)$. In other words, we need to evaluate the difference $\breve{\zeta}_d(t_k + d) = \zeta_d(t_k + d - h) - \zeta_d(t_k + d)$ as a function of the state errors $\zeta(t_k^-)$. Doing so we obtain

$$\begin{aligned} & \breve{\zeta}_d(t_k + d)^T \breve{\zeta}_d(t_k + d) \\ & \leq \sum_{i=1}^{N} \Big[ e_{ii}^T(t_{k_i}^-) \left((G-I)G^{p-1}\right)^T (G-I)G^{p-1} e_{ii}(t_{k_i}^-) \\ & \qquad + 2\left\|(G-I)G^{p-1} e_{ii}(t_{k_i}^-)\right\| (\|G-I\|\Upsilon_h + \|E\|) z_{i,M} \\ & \qquad + \left((\|G-I\|\Upsilon_h + \|E\|)\, z_{i,M}\right)^2 \Big] \end{aligned} \tag{27}$$

since we want to guarantee (27) for any sampling time $t_\mu$ then we use the current error $e_{ii}(t_\mu)$ and we include (21) in the overall threshold (19). Then, the following holds:

$$\dot{V} \leq x^T \bar{\mathcal{L}} x + \sum_{i=1}^{N} \left[ -c_1 z_i^T PBB^T P z_i + \delta_i \right]. \tag{28}$$

Hence, the local thresholds can be defined based on the local errors $e_{ii}(t_\mu)$ as in (18) with $\delta_i$ given by (19). When an event is triggered the error $e_{ii}$ is reset to zero and the following holds:

$$\begin{aligned} \dot{V} & \leq x^T \bar{\mathcal{L}} x + c_1 \sum_{i=1}^{N} (\sigma - 1) z_i^T PBB^T P z_i + N\eta \\ & \leq x^T \bar{\mathcal{L}} x + N\eta \end{aligned} \tag{29}$$

Following the steps in [30] we can show that

$$\dot{V} \leq -\beta V + N\eta. \tag{30}$$

Solving (30) we have that

$$\begin{aligned} V(t) & \leq e^{-\beta t} V(0) + N\eta \int_0^t e^{-\beta(t-\tau)} d\tau \\ & \leq \left(V(0) - \frac{N\eta}{\beta}\right) e^{-\beta t} + \frac{N\eta}{\beta}. \end{aligned} \tag{31}$$

Expression (31) represents a bound on the consensus states as a function of the initial separation of the agents $V(0) = x(0)^T \hat{\mathcal{L}} x(0)$. We can express $V(t) = (1/2)\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} (x_i - x_j)^T P (x_i - x_j)$ and a direct bound on the difference between any two states $i, j$ can be obtained as follows. Since the graph is undirected the term $(x_i - x_j)^T P (x_i - x_j)$ appears twice in the summation $V(t)$, then we can write

$$\begin{aligned} \lambda_{\min}(P) \|x_i - x_j\|^2 & \leq (x_i - x_j)^T P (x_i - x_j) \\ & \leq \left(V(0) - \frac{N\eta}{\beta}\right) e^{-\beta t} + \frac{N\eta}{\beta}. \end{aligned} \tag{32}$$

Finally, the difference between any two states can be bounded as in (17). The rest of the proof can be found in the extended version of this technical note [31]. □

***Remark 1:*** The bound on the difference between states, given in (17), can be made less conservative by adjusting the parameters. For example, the sampling period $h$ could be decreased. However, there is a practical limit into how fast the computation and sampling can be made. Hence, the importance of these results is based on establishing convergence in the presence of discrete-time actuation, sensing, and computation of events. Different ways to obtain less conservative theoretical results are under current investigation such as implementing different types of event thresholds. Future work will consider time-dependent, instead of state-dependent, event thresholds. This choice usually yields greater lower-bounds on the inter-event time intervals and smaller upper-bounds on the consensus errors. It will also be useful in order to consider directed communication graphs as opposed to only undirected graphs.

## V. Conclusion

Synchronization of state trajectories of linear multi-agent systems was studied in this technical note. Multiple issues affecting the convergence to common trajectories were considered such as: limited sensing and actuation capabilities, limited communication, and time-varying communication delays. Event-triggered control schemes were proposed in this technical note which not only provide decentralized control inputs but also allow for decentralized design of transmission instants where each agent decides, based only on local information, when to broadcast its current measurements. The use of discretized and decoupled models and the implementation of periodic event-triggered techniques provides a formal framework that limits actuation and sensing update rates and reduces communication. This method also provides the necessary freedom to each agent in order to determine its own broadcasting instants.

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