Optimal Strategies for a Class of Multi-Player Reach-Avoid Differential Games in 3D Space

Eloy Garcia, *Senior Member, IEEE*, David W. Casbeer, *Senior Member, IEEE*, and Meir Pachter, *Fellow, IEEE*

***Abstract*—A multi-player reach-avoid differential game with autonomous aerial robots in the three dimensional space is studied. Two pursuers form a team to guard a target against an evader with the same speed as the pursuers. This letter provides the complete solution of this differential game that resides within a high-dimensional state space. The Barrier surface is characterized and the saddle-point strategies are synthesized and verified. Degeneration of the two-pursuer one-evader game into the one-on-one case is addressed and the corresponding strategies are obtained. Finally, several examples illustrate the robustness properties and the guarantees provided by the saddle-point strategies obtained in this letter.**

***Index Terms*—Differential games, aerial systems: applications, optimization and optimal control, cooperating robots.**

I. Introduction

DIFFERENTIAL game theory provides a framework to analyze pursuit-evasion scenarios and conflicts between adversarial teams. Pursuit-evasion scenarios and reach-avoid games are representative of many important but challenging problems in aerospace, control, and robotics. Interesting pursuit-evasion problems were formulated in the seminal work by Isaacs [1].

Recent authors have investigated several geometric approaches to solve for pursuit-evasion problems; these solutions generally rely on Voronoi partitioning [2]–[4]. Typically, pursuit-evasion games address a single set termination set where, for instance, the evader only tries to escape being captured by the pursuer without intentionally aiming at reaching a particular region of the game set [5]–[8].

The differential game framework is generally desired [9] in order to solve pursuit-evasion and reach-avoid games, but it is often avoided due to the perceived challenges in solving the Hamilton-Jacobi-Isaacs (HJI) equation [3]. In reach-avoid games [10]–[13], the HJI is solved numerically, and consequently computation time significantly degrades as the state dimension increases. Reach-avoid games extended to two-termination set differential games were introduced in [14]–[16], where an evader strives to reach a goal set while the pursuer tries to intercept the evader before reaching its goal. Reach-avoid games include the capture the flag game [17]–[19], defending a moving target [20]–[22], and assisting and rescuing teammates [23], [24].

Autonomous aerial robots in conflict scenarios require fast decision and guidance laws that can be implemented on-line and that offer robustness against adversarial action and strategies. The recent survey [25] laid out a number of missions requiring aggressive action and reaction by aerial robots, and the authors noted game theory as one promising technique to counter and defeat groups of adversarial aerial robots. Potential applications not only include military missions but also civilian ones, such as maintaining law and order and herding wildlife away from potential danger [25]–[28].

This paper offers a formal approach for enabling cooperation between autonomous aerial robots in an adversarial setup. This paper extends the differential game approach to reach-avoid scenarios in the 3D space. In addition, the game is solved in analytical form, which provides strategies that can be implemented on-line in order to take advantage of non-optimal behaviors by the opponent. The derived state-feedback cooperative guidance can potentially be extended to address reach-avoid games with large number of players [9], [29] and to enable task allocation in adversarial conflicts involving aerial swarms [25]. The main contribution of this paper is the use of Isaacs' method and in particular, the HJI equation, to determine and verify saddle-point strategies. Obtaining the solution via Isaacs' method is the ideal situation in differential games [9]. Verification is often overlooked due to challenges in solving the HJI equation; however, verification is of great importance in order to guarantee that the unique saddle-point solution of the game has been in fact synthesized. In differential games, verification is the process to obtain the continuously differentiable (or C^1) Value function and prove that it is the solution of the HJI equation.

Similar work addressing reach-avoid games has been reported in [29]–[32] where several scenarios with large number of agents, different speed players, and players with positive capture radii have been considered. In particular, the recent paper [33] considered a similar problem in the 3D space with three pursuers and one evader; the main difference is that the target set is a plane in the 3D space. The main result in [33] is to solve the Game of Kind by determining the Barrier surface. In this paper we solve the Game of Kind and, more importantly, we synthesize and verify the saddle-point strategies, thus, providing a complete solution to the Game of Degree as well.

In this paper, we consider the case of two pursuers cooperating to capture an evader who is trying to reach the target. First, we consider the one-on-one case. The strategies from the one-on-one problem are necessary to fully solve the two-on-one case. Depending on the initial conditions it may occur that the optimal play by the pursuers dictates that only one of them will

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Eloy Garcia and David W. Casbeer are with the Control Science Center of Excellence, Air Force Research Laboratory, Wright-Patterson AFB, OH 45433 USA (e-mail: eloy.garcia.2@us.af.mil; David.Casbeer@us.af.mil).

Meir Pachter is with the Department of Electrical Engineering, Air Force Institute of Technology, Wright-Patterson AFB, OH 45433 USA (e-mail: meir.pachter@afit.edu).

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single-handedly capture the evader. If this is the case, then it is counterproductive for the pursuers to try to force simultaneous capture when they select the active pursuer; implementing the individual optimal solution provides a better payoff.

In this paper, we consider the case of two pursuers cooperating to capture an evader who is trying to reach the target. The paper is organized as follows. The problem is formulated in Section II. Section III discusses the differential game with one pursuer and one evader. The operationally important case of two cooperative pursuers and one evader is analyzed in Section IV. Illustrative examples are shown in Section V and conclusions are drawn in Section VI.

II. The Differential Game in 3D

Consider two pursuers, P_1 and P_2, which cooperate in order to capture an evader E while defending a stationary target. The evader aims to reach the target, and the game is played in the Euclidean three-dimensional space. Without loss of generality, we assume that the target is located at the origin.

The state of E is specified by its Cartesian coordinates $\mathbf{x}_E = (x_E, y_E, z_E)$. Similarly, the states of P_1 and P_2 are respectively defined by $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_2 = (x_2, y_2, z_2)$ The complete state of the differential game is defined by $\mathbf{x} := (\mathbf{x}_E, \mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^9$. The control set is defined as $\mathcal{U} := \{\mathbf{u} \in \mathbb{R}^3 | \|\mathbf{u}\|_2 = 1\}$. We denote the controls of E, P_1, and P_2 respectively by $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{U}$. We consider the case where all players have the same speed. Hence, the dynamics/kinematics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ are specified by the system of ordinary differential equations

$$\begin{aligned} &\dot{x}_E = u_x, \qquad \dot{y}_E = u_y, \qquad \dot{z}_E = u_z \\ &\dot{x}_1 = v_x, \qquad \dot{y}_1 = v_y, \qquad \dot{z}_1 = v_z \\ &\dot{x}_2 = w_x, \qquad \dot{y}_2 = w_y, \qquad \dot{z}_2 = w_z \end{aligned} \tag{1}$$

with $x_E(0) = x_{E_0}, y_E(0) = y_{E_0}, z_E(0) = z_{E_0}, x_1(0) = x_{1_0}, y_1(0) = y_{1_0}, z_1(0) = z_{1_0}, x_2(0) = x_{2_0}, y_2(0) = y_{2_0}, z_2(0) = z_{2_0}$. Without loss of generality, the speeds have been normalized. The initial state of the system is $\mathbf{x}_0 := (x_{E_0}, y_{E_0}, z_{E_0}, x_{1_0}, y_{1_0}, z_{1_0}, x_{2_0}, y_{2_0}, z_{2_0}) = \mathbf{x}(0)$.

The game under consideration is a two-termination set differential game [14]–[16]. One terminal condition is capture of the evader by any of the pursuers, and the pursuer team wins the game. Alternatively, the evader wins if it can reach the target before being captured. Hence, the termination set is

$$\mathcal{T} := \mathcal{T}_p \bigcup \mathcal{T}_e \tag{2}$$

where

$$\mathcal{T}_p := \{\mathbf{x} \mid \|\mathbf{x}_1 - \mathbf{x}_E\|_2 = 0\} \cup \{\mathbf{x} \mid \|\mathbf{x}_2 - \mathbf{x}_E\|_2 = 0\} \tag{3}$$

represents the outcome where E is captured before reaching the target and

$$\mathcal{T}_e := \{\mathbf{x} \mid \|\mathbf{x}_E\|_2 = 0\} \tag{4}$$

represents E winning the game by reaching the target. The terminal time t_f is the time instant when the state of the system satisfies (2), at which time the terminal state is $\mathbf{x}_f := (x_{E_f}, y_{E_f}, z_{E_f}, x_{1_f}, y_{1_f}, z_{1_f}, x_{2_f}, y_{2_f}, z_{2_f}) = \mathbf{x}(t_f)$.

The concepts of Game of Kind and Game of Degree are fundamental in differential game theory [1]. The solution to the Game of Kind determines which team wins the game. Solving the Game of Degree provides the value of the game and the saddle-point strategies that realize the outcome prescribed by the Game of Kind. Because of the two different outcomes specified in (2), the Game of Kind needs to be solved in order to partition the state space into two winning regions, one for evader and one for the pursuer team. This partition of the state space $\mathbb{R}^9$ is denoted by the two sets: $\mathcal{R}_p$ and $\mathcal{R}_e$ which are defined as follows

$$\mathcal{R}_p := \{\, \mathbf{x} \mid B(\mathbf{x}) > 0 \,\}, \;\; \mathcal{R}_e := \{\, \mathbf{x} \mid B(\mathbf{x}) < 0 \,\}\,. \tag{5}$$

The Barrier surface, which separates the two sets $\mathcal{R}_p$ and $\mathcal{R}_e$, is specified by

$$\mathcal{B} := \{\, \mathbf{x} \mid B(\mathbf{x}) = 0 \,\}\,. \tag{6}$$

The Barrier function $B(\mathbf{x})$ is explicitly obtained in Sections III and IV for each of the two cases analyzed in this paper. In each winning region, $\mathcal{R}_p$ and $\mathcal{R}_e$, a different Game of Degree is played. Therefore it is essential for each team to determine which region the current state resides in. With this knowledge in hand, the appropriate optimal strategies are determined from the corresponding Game of Degree.

If $\mathbf{x} \in \mathcal{R}_p$ then the terminal performance functional is

$$J(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t); \mathbf{x}_0) = \Phi_p(\mathbf{x}(t_f)) \tag{7}$$

where $\Phi_p(\mathbf{x}(t_f)) := x_{E_f}^2 + y_{E_f}^2 + z_{E_f}^2$. The Value of the game is

$$V(\mathbf{x}_0) := \min_{\mathbf{u}(\cdot)} \max_{\mathbf{v}(\cdot), \mathbf{w}(\cdot)} J(\mathbf{u}(\cdot), \mathbf{u}(\cdot), \mathbf{w}(\cdot); \mathbf{x}_0) \tag{8}$$

subject to (1) and (3), where $\mathbf{u}(\cdot)$ and $\{\mathbf{v}(\cdot), \mathbf{w}(\cdot)\}$ are the teams' state-feedback strategies. When the solution of the Game of Kind prescribes that E is going to be captured before reaching the target, E strives to minimize its terminal separation with respect to the target at the time instant of capture. The pursuers aim at intercepting E while maximizing the terminal separation. This strategy provides a practical outcome in case the pursuers do not play optimally where E can further decrease its terminal distance with respect to the target, potentially winning the game by reaching the target.

If $\mathbf{x} \in \mathcal{R}_e$ then the terminal performance functional is

$$J(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t); \mathbf{x}_0) = \Phi_e(\mathbf{x}(t_f)) \tag{9}$$

where $\Phi_e(\mathbf{x}(t_f)) := \min_i\{(x_{i_f} - x_{E_f})^2 + (y_{i_f} - y_{E_f})^2 + (z_{i_f} - z_{E_f})^2\}$ for $i = 1, 2$. The Value of the game is

$$V(\mathbf{x}_0) := \min_{\mathbf{v}(\cdot), \mathbf{w}(\cdot)} \max_{\mathbf{u}(\cdot)} J(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t); \mathbf{x}_0) \tag{10}$$

subject to (1) and (4), where $\mathbf{u}(\cdot)$ and $\{\mathbf{v}(\cdot), \mathbf{w}(\cdot)\}$ are the teams' state-feedback strategies. In this case the pursuers are unable to capture the evader and they, therefore, try to come as close as possible to the evader at the terminal time. Since the target is stationary, the players' optimal strategies are to aim directly at the target.

Theorem 1: Consider the differential game (1), (7), (9). The regular optimal headings of E, P_1, and P_2 are constant under optimal play and their trajectories are straight lines.

Proof: Consider (1) and (7), the optimal control inputs (in terms of the co-state variables) can be immediately obtained from $\min_{\mathbf{u}} \max_{\mathbf{v},\mathbf{w}} \mathcal{H}$, where the Hamiltonian is

$$\mathcal{H} = \sum_j \lambda_{j_E} u_j + \sum_j \lambda_{j_1} v_j + \sum_j \lambda_{j_2} w_j, \;\; \text{for } j = x, y, z$$

where $\lambda^T = (\lambda_{x_E}, \lambda_{y_E}, \lambda_{z_E}, \lambda_{x_1}, \lambda_{y_1}, \lambda_{z_1}, \lambda_{x_2}, \lambda_{y_2}, \lambda_{z_2}) \in \mathbb{R}^9$ is the co-state. Additionally, the co-state dynamics are: $\dot{\lambda}_{x_E} =$

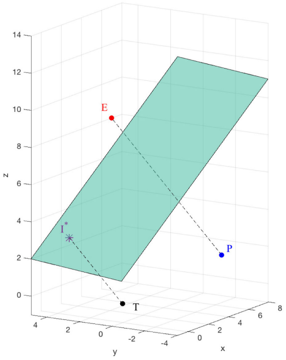

Fig. 1. Guarding a target in 3D with one pursuer and one evader.

$\dot{\lambda}_{y_E} = \dot{\lambda}_{z_E} = \dot{\lambda}_{x_1} = \dot{\lambda}_{y_1} = \dot{\lambda}_{z_1} = \dot{\lambda}_{x_2} = \dot{\lambda}_{y_2} = \dot{\lambda}_{z_2} = 0$; hence, all co-states are constant and the optimal headings are constant as well. Consequently, the optimal trajectories are straight lines. ■

III. 1 Pursuer vs. 1 Evader

This section considers the differential game in 3D with one pursuer and one evader; the state of the differential game is $\mathbf{x} := (\mathbf{x}_E, \mathbf{x}_1) \in \mathbb{R}^6$. Let us define

$$R_E = \sqrt{x_E^2 + y_E^2 + z_E^2}, \;\; R_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}. \tag{11}$$

Since both players have the same speed, their dominance regions in the 3D space is separated by a plane which is orthogonal to the segment $\overline{PE}$, see Fig. 1. P can win the game if the origin is located on its side of the orthogonal plane; similarly, E can win if it is located closer to the origin than P. The limiting case occurs when the origin is located on the orthogonal plane. Hence, the Barrier surface that separates the state space into the two regions $\mathcal{R}_p$ and $\mathcal{R}_e$ is specified by the function $B : \mathbb{R}^6 \to \mathbb{R}$

$$B(\mathbf{x}) = R_E^2 - R_1^2. \tag{12}$$

Thus, the Barrier surface is given by (6) and the winning regions are given by (5).

We now need to determine the optimal strategies of each player. For the case where $\mathbf{x} \in \mathcal{R}_p$ the saddle-point strategies are given in the following theorem.

Theorem 2: Consider the differential game with one pursuer and one evader and assume that $\mathbf{x} \in \mathcal{R}_p$. The Value function is C^1 and it is the solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation. The Value function is given by

$$V(\mathbf{x}) = \frac{1}{4}\frac{(R_E^2 - R_1^2)^2}{\Gamma^2} \tag{13}$$

where $\Gamma = \sqrt{(x_E - x_1)^2 + (y_E - y_1)^2 + (z_E - z_1)^2}$. The optimal state-feedback strategies are

$$\mathbf{u}^* = \frac{1}{\Gamma_E}[x^* - x_E, \; y^* - y_E, \; z^* - z_E]$$
$$\mathbf{v}^* = \frac{1}{\Gamma_1}[x^* - x_1, \; y^* - y_1, \; z^* - z_1] \tag{14}$$

and the coordinates of the optimal interception point, $I^* = (x^*, y^*, z^*)$, are

$$I^* = \frac{R_E^2 - R_1^2}{2\Gamma^2}[x_E - x_1, \; y_E - y_1, \; z_E - z_1] \tag{15}$$

where $\Gamma_E = \sqrt{(x^* - x_E)^2 + (y^* - y_E)^2 + (z^* - z_E)^2}$, $\Gamma_1 = \sqrt{(x^* - x_1)^2 + (y^* - y_1)^2 + (z^* - z_1)^2}$.

Proof: The dominance regions of each player in the 3D space are separated by the plane which is orthogonal to the segment $\overline{PE}$. The orthogonal plane is given by

$$(x_E - x_1)x + (y_E - y_1)y + (z_E - z_1)z = \frac{R_E^2 - R_1^2}{2} \tag{16}$$

Additionally, the normal vector of this plane is $\vec{r} = (x_E - x_1)\hat{i} + (y_E - y_1)\hat{j} + (z_E - z_1)\hat{k}$. Now, player E, who tries to minimize its terminal separation with respect to the origin of the coordinate frame, should aim at the point on the plane closest to the origin which we denote as $I^* = (x^*, y^*, z^*)$. Point I^* is determined by the line passing through the origin and which is also orthogonal to the plane. In 3D space, this line is characterized by

$$\frac{x}{x_E - x_1} = \frac{y}{y_E - y_1} = \frac{z}{z_E - z_1}. \tag{17}$$

Hence, the point of interception between the line (17) and the plane (16) is obtained by substituting (17) into (16) as follows

$$(x_E - x_1)x + \frac{(y_E - y_1)^2}{x_E - x_1}x + \frac{(z_E - z_1)^2}{x_E - x_1}x = \frac{(R_E^2 - R_1^2)}{2}.$$

Solving for x in the previous equation we obtain $x^* = \frac{R_E^2 - R_1^2}{2}\frac{x_E - x_1}{\Gamma^2}$. Finally, substituting x^* into (17) we obtain y^* and z^* as shown in (15).

The Value function can now be obtained by computing $V(\mathbf{x}) = (x^*)^2 + (y^*)^2 + (z^*)^2$ and it is explicitly shown in terms of the state in (13). The gradient of the Value function $V(\mathbf{x})$ can be directly obtained and it is given by

$$\frac{\partial V}{\partial x_E} = \frac{R_E^2 - R_1^2}{\Gamma^2}x_E - 2\frac{x_E - x_1}{\Gamma^2}V$$
$$\frac{\partial V}{\partial y_E} = \frac{R_E^2 - R_1^2}{\Gamma^2}y_E - 2\frac{y_E - y_1}{\Gamma^2}V$$
$$\frac{\partial V}{\partial z_E} = \frac{R_E^2 - R_1^2}{\Gamma^2}y_E - 2\frac{z_E - z_1}{\Gamma^2}V$$
$$\frac{\partial V}{\partial x_1} = 2\frac{x_E - x_1}{\Gamma^2}V - \frac{R_E^2 - R_1^2}{\Gamma^2}x_1$$
$$\frac{\partial V}{\partial y_1} = 2\frac{y_E - y_1}{\Gamma^2}V - \frac{R_E^2 - R_1^2}{\Gamma^2}y_1$$
$$\frac{\partial V}{\partial z_1} = 2\frac{z_E - z_1}{\Gamma^2}V - \frac{R_E^2 - R_1^2}{\Gamma^2}z_1 \tag{18}$$

where $\Gamma \neq 0$, otherwise, the evader has been captured by the pursuer and the game has ended.

The HJI equation is given in general by $-\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*) + g(t, \mathbf{x}, u^*, v^*)$. In this problem we have $\frac{\partial V}{\partial t} = 0$ and $g(t, \mathbf{x}, u^*, v^*) = 0$. In order to show that (13) is the solution of the HJI equation we write $\frac{\partial V}{\partial \mathbf{x}}$ as follows

$$\frac{\partial V}{\partial \mathbf{x}_E} = -\frac{R_E^2 - R_1^2}{\Gamma^2}[x^* - x_E, \; y^* - y_E, \; z^* - z_E]$$
$$\frac{\partial V}{\partial \mathbf{x}_1} = \frac{R_E^2 - R_1^2}{\Gamma^2}[x^* - x_1, \; y^* - y_1, \; z^* - z_1]. \tag{19}$$

and the HJI equation is

$$\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*)$$
$$= \frac{R_E^2 - R_1^2}{\Gamma^2} \cdot \frac{(x^* - x_1)^2 + (y^* - y_1)^2 + (z^* - z_1)^2}{\Gamma_1}$$
$$- \frac{R_E^2 - R_1^2}{\Gamma^2} \cdot \frac{(x^* - x_E)^2 + (y^* - y_E)^2 + (z^* - z_E)^2}{\Gamma_E}.$$

Since the point I^* is equidistant to the locations of both players, then $\Gamma_E = \Gamma_1$ and we have that

$$\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*) = \frac{R_E^2 - R_1^2}{\Gamma^2}(\Gamma_1 - \Gamma_E) = 0$$

Therefore, the C^1 Value function (13) is the solution of the HJI equation and (14)–(15) are the optimal strategies of the differential game. ■

IV. 2 Pursuers vs. 1 Evader

In this section, we consider the case where two pursuers cooperate to capture the evader. Acting together, it is possible for capture to occur farther from the target than if a single pursuer acted alone. This is operationally important because through cooperation the risk to the target could potentially be reduced. In order to improve the payoff, the pursuers need to determine the best way to cooperate to maximize the terminal separation between the evader and the target. In other words, better performance is not directly obtained when two pursuers independently implement their individual strategies. The cooperative optimal strategy, to be derived, will yield the best outcome.

Let us first determine the Barrier function $B : \mathbb{R}^9 \to \mathbb{R}$ for the two-on-one case. In addition to (11) let us also define

$$R_2 = \sqrt{x_2^2 + y_2^2 + z_2^2} \tag{20}$$

Define $B_1(\mathbf{x}) = R_E^2 - R_1^2$ and $B_2(\mathbf{x}) = R_E^2 - R_2^2$. The Barrier function is given by

$$B(\mathbf{x}) = \max\{B_1(\mathbf{x}), B_2(\mathbf{x})\} \tag{21}$$

where the winning regions and the Barrier surface are given, respectively, by (5) and (6). When $\mathbf{x} \in \mathcal{R}_p$ the evader is captured before reaching the target; however, several outcomes are possible. E can be captured only by P_1, only by P_2, or simultaneously by both pursuers. The latter represents the benefit of using two pursuers where the payoff improves with respect to the individual payoffs of each pursuer. Define $\mathcal{R}_{ps} \subset \mathcal{R}_p$ as the subregion where simultaneous capture occurs. Conditions to determine if $\mathbf{x} \in \mathcal{R}_{ps}$ holds will be given at the end of the section. The following theorem provides the saddle-point strategies for the differential game where two pursuers cooperate against one evader.

Theorem 3: Consider the differential game with two pursuers and one evader (1)–(2), (7) and assume that $\mathbf{x} \in \mathcal{R}_{ps}$. The Value function is C^1 and it is the solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation. The Value function is given by

$$V(\mathbf{x}) = x_L^2 + y_L^2 - \frac{(a x_L + b y_L)^2}{a^2 + b^2 + c^2} \tag{22}$$

where

$$x_L = \frac{1}{2}\frac{(y_E - y_1)(R_E^2 - R_2^2) - (y_E - y_2)(R_E^2 - R_1^2)}{(x_E - x_2)(y_E - y_1) - (x_E - x_1)(y_E - y_2)}$$
$$y_L = \frac{1}{2}\frac{(x_E - x_2)(R_E^2 - R_1^2) - (x_E - x_1)(R_E^2 - R_2^2)}{(x_E - x_2)(y_E - y_1) - (x_E - x_1)(y_E - y_2)} \tag{23}$$

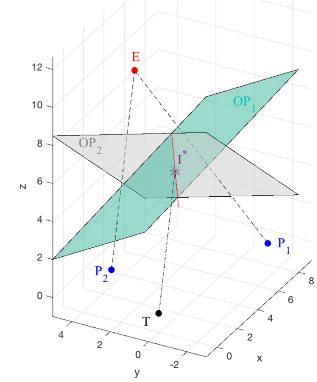

Fig. 2. Two pursuer and one evader case illustrating the two orthogonal planes, the intersection line, and the optimal capture point on the line.

and

$$a = (y_E - y_1)(z_E - z_2) - (y_E - y_2)(z_E - z_1)$$
$$b = (x_E - x_2)(z_E - z_1) - (x_E - x_1)(z_E - z_2)$$
$$c = (x_E - x_1)(y_E - y_2) - (x_E - x_2)(y_E - y_1). \tag{24}$$

The optimal state-feedback strategies are given by (14) and

$$\mathbf{w}^* = \frac{\mathbf{1}}{\mathbf{\Gamma_2}}[\mathbf{x}^* - \mathbf{x_2},\ \mathbf{y}^* - \mathbf{y_2},\ \mathbf{z}^* - \mathbf{z_2}] \tag{25}$$

where $\Gamma_2 = \sqrt{(x^* - x_2)^2 + (y^* - y_2)^2 + (z^* - z_2)^2}$. The coordinates of the optimal interception point are

$$x^* = aT + x_L,\ \ y^* = bT + y_L,\ \ z^* = cT \tag{26}$$

where $T = -\frac{a x_L + b y_L}{a^2 + b^2 + c^2}$.

Proof: When $\mathbf{x} \in \mathcal{R}_{ps}$ the outcome of the game is simultaneous capture of E by both P_1 and P_2. Capture occurs along the intersection of the two planes OP_1 and OP_2. The plane OP_1 is defined by (16) and the plane OP_2 is defined by the following

$$(x_E - x_2)x + (y_E - y_2)y + (z_E - z_2)z = \frac{R_E^2 - R_2^2}{2} \tag{27}$$

The intersection of two planes in the 3D space forms a line and the vector pointing in the direction of the line is $\vec{r_l} = a\hat{i} + b\hat{j} + c\hat{k}$. The vector components a, b, and c are obtained by computing the cross product of the normal vectors of the two planes and they are given explicitly in terms of the state in (24). A point on the line can be obtained by setting $z_L = 0$ and solving for x_L and y_L using the plane equations (16) and (27); this solution is given by (23), also in terms of the state. Hence, the line resulting from the intersection of the two planes OP_1 and OP_2 is parameterized by

$$\frac{x - x_L}{a} = \frac{y - y_L}{b} = \frac{z}{c} = T_L \tag{28}$$

where T_L is the line parameter. Player E, wishing to minimize its terminal separation with respect to the origin of the coordinate frame, aims at the point on the line closest to the origin which we denote as $I^* = (x^*, y^*, z^*)$. Point I^* can be obtained by computing the intersection of the line (28) and the orthogonal line to (28) that passes through the origin, see Fig. 2. The corresponding parameter is $T = -\frac{a x_L + b y_L}{a^2 + b^2 + c^2}$ and the coordinates of the optimal interception point are given by (26). The Value of the game is then $V(\mathbf{x}) = (x^*)^2 + (y^*)^2 + (z^*)^2$ and is explicitly given in terms of the state in (22). Both, (22) and (26), are given in terms of the state but in compact form using the notation of the line parameters.

In order to find the gradient of $V(\mathbf{x})$ we first compute $\frac{\partial x_L}{\partial \mathbf{x}}$ and $\frac{\partial y_L}{\partial \mathbf{x}}$. For instance

$$\begin{aligned}\frac{\partial x_L}{\partial x_E} &= \frac{y_1-y_2}{c}(x_E - x_L)\\ \frac{\partial x_L}{\partial y_E} &= \frac{\frac{1}{2}(R_2^2 - R_1^2) + (y_1-y_2)y_E - (x_2-x_1)x_L}{c}\\ \frac{\partial x_L}{\partial z_E} &= \frac{y_1-y_2}{c}z_E\end{aligned} \tag{29}$$

with similar expressions obtained for $\frac{\partial x_L}{\partial \mathbf{x}_1}$ and $\frac{\partial x_L}{\partial \mathbf{x}_2}$. Also,

$$\begin{aligned}\frac{\partial y_L}{\partial x_E} &= \frac{\frac{1}{2}(R_1^2 - R_2^2) + (x_2-x_1)x_E - (y_1-y_2)y_L}{c}\\ \frac{\partial y_L}{\partial y_E} &= \frac{x_2-x_1}{c}(y_E - y_L)\\ \frac{\partial y_L}{\partial z_E} &= \frac{x_2-x_1}{c}z_E\end{aligned} \tag{30}$$

and similar expressions are obtained for $\frac{\partial y_L}{\partial \mathbf{x}_1}$ and $\frac{\partial y_L}{\partial \mathbf{x}_2}$.

Now we are able to obtain the gradient of the Value function $V(\mathbf{x})$

$$\begin{aligned}\frac{\partial V}{\partial x_E} &= 2\left(\left[\frac{x_E-x_L}{c}x^* + Tz^*\right](y_1-y_2) + \left[\frac{\partial y_L}{\partial x_E} - T(z_1-z_2)\right]y^*\right)\\ \frac{\partial V}{\partial y_E} &= 2\left(\left[\frac{y_E-y_L}{c}y^* + Tz^*\right](x_2-x_1) + \left[\frac{\partial x_L}{\partial y_E} - T(z_2-z_1)\right]x^*\right)\\ \frac{\partial V}{\partial z_E} &= 2\left(\frac{z_E}{c} - T\right)[(y_1-y_2)x^* + (x_2-x_1)y^*]\end{aligned} \tag{31}$$

with similar expressions obtained for $\frac{\partial V}{\partial \mathbf{x}_1}$ and $\frac{\partial V}{\partial \mathbf{x}_2}$. Note that since they represent a vector, the scalars a, b, c are not all equal to zero; therefore, $a^2+b^2+c^2>0$. However, one or two of this vector components may be equal to zero. This case does not present any problem and it does not indicate any singularity. This particular case is addressed in Corollary 1, where the equation of the line is expressed using the appropriate representation.

Let us now prove that $V(\mathbf{x})$ is the solution of the HJI equation. First, we write $\frac{\partial V}{\partial x_E}$ in the following form

$$\begin{aligned}\frac{\partial V}{\partial x_E} &= 2\left(\frac{y_1-y_2}{c}x_E x^* + \frac{x_2-x_1}{c}x_E y^*\right)\\ &\quad + 2(y_1-y_2)\left(-\frac{x_L}{c}x^* + Tz^* - \frac{y_L}{c}y^*\right)\\ &\quad + 2\frac{y^*}{c}\left[\frac{1}{2}(R_1^2-R_2^2) - (z_1-z_2)z^*\right]\end{aligned} \tag{32}$$

where $\frac{\partial y_L}{\partial x_E}$ defined in (30) was substituted into the previous expression. Let us now add and subtract the terms $2\frac{y_1-y_2}{c}(x^*)^2$ and $2\frac{x_2-x_1}{c}x^*y^*$ into the right hand side of (32); doing so we obtain

$$\begin{aligned}\frac{\partial V}{\partial x_E} &= -\frac{2}{c}[(y_1-y_2)x^* + (x_2-x_1)y^*](x^*-x_E)\\ &\quad + 2(y_1-y_2)\left(-\frac{x_L}{c}x^* + Tz^* - \frac{y_L}{c}y^* + \frac{(x^*)^2}{c}\right)\\ &\quad + 2\frac{y^*}{c}\left[\frac{1}{2}(R_1^2-R_2^2) - (z_1-z_2)z^* + (x_2-x_1)x^*\right]\end{aligned} \tag{33}$$

We use the relations $x^* = aT + x_L$ and $z^* = cT$ in order to write the last term in (33) as follows

$$\begin{aligned}&\frac{1}{2}(R_1^2-R_2^2) - (z_1-z_2)z^* + (x_2-x_1)x^*\\ &= \frac{1}{2}(R_1^2-R_2^2) - (z_1-z_2)cT + (x_2-x_1)(aT + x_L).\end{aligned}$$

Substituting the expressions for a, c, and x_L from (23) and (24) into the previous expression we have that

$$\begin{aligned}&\frac{1}{2}(R_1^2-R_2^2) - (z_1-z_2)z^* + (x_2-x_1)x^* = (y_1-y_2)\\ &\times\left(\frac{1}{2}\frac{x_E(R_1^2-R_2^2) + x_2(R_E^2-R_1^2) + x_1(R_2^2-R_E^2)}{(x_E-x_1)(y_E-y_2) - (x_E-x_2)(y_E-y_1)}\right.\\ &\left.\quad - T[x_2(z_E-z_1) - z_2(x_E-x_1) + z_1x_E - x_1z_E]\right)\\ &= (y_1-y_2)(y_L + bT).\end{aligned} \tag{34}$$

Substituting (34) back into (33) we obtain

$$\begin{aligned}\frac{\partial V}{\partial x_E} &= -\frac{2}{c}[(y_1-y_2)x^* + (x_2-x_1)y^*](x^*-x_E) + 2\frac{y_1-y_2}{c}\\ &\quad\times\left(-x_Lx^* + (z^*)^2 - y_Ly^* + (x^*)^2 + y^*(y_L+bT)\right)\\ &= -\frac{2}{c}[(y_1-y_2)x^* + (x_2-x_1)y^*](x^*-x_E)\\ &\quad + 2\frac{y_1-y_2}{c}\left((aT - x^*)x^* + (z^*)^2 + (x^*)^2 + by^*T\right)\\ &= -\frac{2}{c}[(y_1-y_2)x^* + (x_2-x_1)y^*](x^*-x_E)\\ &\quad + 2\frac{y_1-y_2}{c}\left(aT(aT+x_L) + c^2T^2 + bT(bT+y_L)\right)\\ &= -\frac{2}{c}[(y_1-y_2)x^* + (x_2-x_1)y^*](x^*-x_E) + 2T\frac{y_1-y_2}{c}\\ &\quad\times\left(ax_L + by_L - (a^2+b^2+c^2)\frac{ax_L+by_L}{a^2+b^2+c^2}\right)\\ &= -\frac{2}{c}[(y_1-y_2)x^* + (x_2-x_1)y^*](x^*-x_E)\end{aligned} \tag{35}$$

Applying similar steps (32)–(35) to $\frac{\partial V}{\partial y_E}$, $\frac{\partial V}{\partial x_1}$, $\frac{\partial V}{\partial y_1}$, $\frac{\partial V}{\partial x_2}$, and $\frac{\partial V}{\partial y_2}$ we can rewrite the gradient of $V(\mathbf{x})$ in a compact and more convenient form

$$\begin{aligned}&\frac{\partial V}{\partial \mathbf{x}_E}\\ &= -2\frac{(y_1-y_2)x^* + (x_2-x_1)y^*}{c}[x^*-x_E,\ y^*-y_E,\ z^*-z_E]\\ &\frac{\partial V}{\partial \mathbf{x}_1}\\ &= -2\frac{(y_2-y_E)x^* + (x_E-x_2)y^*}{c}[x^*-x_1,\ y^*-y_1,\ z^*-z_1]\\ &\frac{\partial V}{\partial \mathbf{x}_2}\\ &= -2\frac{(y_E-y_1)x^* + (x_1-x_E)y^*}{c}[x^*-x_2,\ y^*-y_2,\ z^*-z_2]\end{aligned}$$

The HJI equation is given by $-\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*) + g(t, \mathbf{x}, u^*, v^*, w^*)$. Note that in this problem $\frac{\partial V}{\partial t} = 0$ and $g(t, \mathbf{x}, u^*, v^*, w^*) = 0$. Therefore, we compute

$$\begin{aligned}
&\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*) \\
&= -\frac{\frac{2}{c}[(y_1-y_2)x^*+(x_2-x_1)y^*][(x^*-x_E)^2+(y^*-y_E)^2+(z^*-z_E)^2]}{\Gamma_E} \\
&\quad -\frac{\frac{2}{c}[(y_2-y_E)x^*+(x_E-x_2)y^*][(x^*-x_1)^2+(y^*-y_1)^2+(z^*-z_1)^2]}{\Gamma_1} \\
&\quad -\frac{\frac{2}{c}[(y_E-y_1)x^*+(x_1-x_E)y^*][(x^*-x_2)^2+(y^*-y_2)^2+(z^*-z_2)^2]}{\Gamma_2}
\end{aligned}$$

Note that the interception point $I^* = (x^*, y^*, z^*)$ is equidistant to the location of every player. Thus, $\Gamma_E = \Gamma_1 = \Gamma_2$ and we have that

$$\begin{aligned}
&\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*) = -\frac{2\Gamma_E}{c}[(y_1 - y_2)x^* + (x_2 - x_1)y^* \\
&+(y_2 - y_E)x^*+(x_E - x_2)y^*+(y_E - y_1)x^*+(x_1 - x_E)y^*] \\
&= 0
\end{aligned}$$

Therefore the C^1 Value function $V(\mathbf{x})$ specified in (22) is the solution of the HJI equation and (25)–(26) are the saddle-point strategies of the differential game. ■

To conclude this analysis we consider the special case where, without loss of generality, $c = 0$. In this particular case it may appear that $V(\mathbf{x})$ and $\frac{\partial V}{\partial \mathbf{x}}$ are discontinuous. Fortunately, this is not the case and one only needs to provide the corresponding expressions for the optimal aimpoint and the Value function.

Corollary 1: Consider the differential game with two pursuers and one evader (1)–(2), (7). Assume that $\mathbf{x} \in \mathcal{R}_{ps}$ and that $c = 0$. The Value function is C^1 and it is the solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation. The Value function is given by

$$V(\mathbf{x}) = (z^*)^2 + \frac{a^2 y_L^2}{a^2 + b^2} \tag{36}$$

where

$$\begin{aligned}
y_L &= \frac{1}{2}\frac{(z_E - z_1)(R_E^2 - R_2^2) - (z_E - z_2)(R_E^2 - R_1^2)}{(y_E - y_2)(z_E - z_1) - (y_E - y_1)(z_E - z_2)} \\
z^* &= \frac{1}{2}\frac{(y_E - y_2)(R_E^2 - R_1^2) - (y_E - y_1)(R_E^2 - R_2^2)}{(y_E - y_2)(z_E - z_1) - (y_E - y_1)(z_E - z_2)}
\end{aligned} \tag{37}$$

and a and b are given in (24). The optimal state-feedback strategies are given by (25) where the coordinates of the optimal interception point $I^* = (x^*, y^*, z^*)$ are

$$x^* = aT, \;\; y^* = bT + y_L \tag{38}$$

and $T = -\frac{b y_L}{a^2+b^2}$.

Proof: In the case where the vector component $c = 0$, the equation of the line where the orthogonal planes intersect cannot be expressed in the form (28). The line is now parallel to the $x-y$ plane, that is, z is constant. Let the constant z-coordinate of the line be denoted by z^*. We can choose, without loss of generality, $x_L = 0$ and solve for z^* and y_L using the plane equations (16) and (27); this solution is given by (37), in terms of the state. The equation of the line is now given by

$$\frac{x}{a} = \frac{y - y_L}{b} = T_L \;\text{ and }\; z = z^*. \tag{39}$$

The point $I^* = (x^*, y^*, z^*)$ can be obtained by computing the intersection of the line (39) and the orthogonal line to (39) that passes through the origin. The corresponding parameter is $T = -\frac{b y_L}{a^2+b^2}$ and the coordinates of the optimal interception point are given by (37)–(38). Similarly, the Value of the game is $V(\mathbf{x}) = (x^*)^2 + (y^*)^2 + (z^*)^2$ and is explicitly given in terms of the state in (36).

The gradient of $V(\mathbf{x})$ is obtained and is as follows

$$\begin{aligned}
\frac{\partial V}{\partial x_E} &= -2\left[\frac{y_2 - y_1}{a}z^* + \frac{z_2 - z_1}{b}x^*\right](x^* - x_E) + 2\frac{y_2 - y_1}{a}x^* z^* \\
\frac{\partial V}{\partial y_E} &= -2\left[\frac{y_2 - y_1}{a}z^* + \frac{z_2 - z_1}{b}x^*\right](y^* - y_E) \\
&\quad + \frac{2z^*}{a}\left[\frac{1}{2}(R_1^2 - R_2^2) - (z_1 - z_2)z^* - (y_1 - y_2)y^*\right] \\
&\quad + 2\frac{z_2 - z_1}{b}x^*\left(\frac{a}{b}x^* + y^*\right) \\
\frac{\partial V}{\partial z_E} &= -2\left[\frac{y_2 - y_1}{a}z^* + \frac{z_2 - z_1}{b}x^*\right](z^* - z_E) \\
&\quad + \frac{2x^*}{b}\left[\frac{1}{2}(R_1^2 - R_2^2) - (z_1 - z_2)z^*\right. \\
&\quad \left. - (x_1 - x_2)x^* + \frac{a}{b}(y_1 - y_2)x^*\right]
\end{aligned}$$

with similar expressions obtained for $\frac{\partial V}{\partial \mathbf{x}_1}$ and $\frac{\partial V}{\partial \mathbf{x}_2}$. Note that only the nonzero terms a and b appear in the denominator of $V(\mathbf{x})$ and $\frac{\partial V}{\partial \mathbf{x}}$; the Value function is C^1. Finally, it can be shown that $V(\mathbf{x})$ is the solution of the HJI equation. It can be easily seen that the common terms in the HJI equation cancel out and the remaining expression is

$$\begin{aligned}
&\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*) \\
&= \frac{2}{\Gamma_E}\left(\frac{c x^* z^*}{a} + \frac{a x^*}{b}\left(\frac{a x^*}{b} + y^*\right)\right. \\
&\quad - \frac{z^*}{2a}[(R_1^2 - R_2^2)y_E + (R_2^2 - R_E^2)y_1 + (R_E^2 - R_1^2)y_2] \\
&\quad - \frac{x^*}{2b}[(R_1^2 - R_2^2)z_E + (R_2^2 - R_E^2)z_1 + (R_E^2 - R_1^2)z_2] \\
&\quad \left. + (z^*)^2 + (x^*)^2 + \frac{a^2}{b^2}(x^*)^2\right).
\end{aligned}$$

Using the expressions in (37) and noting that $c = 0$ we have that

$$\begin{aligned}
&\frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u^*, v^*, w^*) \\
&= \frac{2}{\Gamma_E}\left((z^*)^2 + \left(1 + \frac{2a^2}{b^2}\right)(x^*)^2 + \frac{a}{b}x^* y^*\right. \\
&\quad \left. - \frac{1}{2}\left(2\frac{z^*}{a}az^* - 2\frac{x^*}{b}ay_L\right)\right) \\
&= \frac{2x^*}{\Gamma_E}\left(\frac{a}{b}y^* + \frac{2a^2 + b^2}{b^2}x^* + \frac{a}{b}y_L\right) \\
&= \frac{2x^*}{b^2\Gamma_E}(2ab^2 T + 2a^3 T + 2aby_L) \\
&= \frac{4ax^*}{b^2\Gamma_E}[(a^2 + b^2)T + by_L]
\end{aligned}$$

$$= \frac{4ax^*}{b^2\Gamma_E}\left[(a^2+b^2)\left(-\frac{by_L}{a^2+b^2}\right)+by_L\right]$$
$$= 0$$

where (38) and $T = -\frac{by_L}{a^2+b^2}$ were used. Hence, the C^1 Value function $V(\mathbf{x})$ is the solution of the HJI equation. Similar analysis and expressions can be obtained in the cases where $a = 0$ or $b = 0$ or in the particular case where any two of a, b, c are equal to zero. ■

V. EXAMPLES

Determine type of capture. Given the initial positions of the players and assuming that $\mathbf{x} \in \mathcal{R}_p$, three different outcomes are possible: single capture of E by P_1 is optimal, single capture of E by P_2 is optimal, or simultaneous capture by both pursuers is the optimal outcome. It is important to determine the type of capture in order for each player to apply the optimal strategies. This problem can be solved by comparing the individual solutions of the one-on-one games as it is described next.

From Theorem 2 we can compute the solutions of the individual games between E and only P_1 and between E and only P_2. Let $I_1' = (x_1', y_1', z_1')$ denote the optimal capture point of the one-on-one game between E and P_1. Similarly, let $I_2' = (x_2', y_2', z_2')$ denote the optimal capture point of the one-on-one game between E and P_2. Then, we compute the following

$$\Upsilon_1 = (x_E - x_1)x_2' + (y_E - y_1)y_2' + (z_E - z_1)z_2' - \frac{R_E^2 - R_1^2}{2}$$
$$\Upsilon_2 = (x_E - x_2)x_1' + (y_E - y_2)y_1' + (z_E - z_2)z_1' - \frac{R_E^2 - R_2^2}{2}.$$

For a given point, such as I_2', the function Υ_1 determines in which side of the plane OP_1 that point is located; if $\Upsilon_1 > 0$ then I_2' is located on the E side of the OP_1 plane and if $\Upsilon_1 < 0$ then I_2' is located on the P_1 side of the OP_1 plane. The analogous reasoning applies to $\Upsilon_2 > 0$. Hence, simultaneous capture is the optimal outcome when both $\Upsilon_1 < 0$ and $\Upsilon_2 < 0$, and the capture point is given by (26).

Additionally, if $\Upsilon_1 < 0$ and $\Upsilon_2 > 0$, then P_1 single-handedly captures E under optimal play, P_2 is not active and $I^* = I_1'$. Also, if $\Upsilon_1 > 0$ and $\Upsilon_2 < 0$ then P_2 single-handedly captures E under optimal play, P_1 is not active and $I^* = I_2'$. In the case where both $\Upsilon_1 > 0$ and $\Upsilon_2 > 0$, then both points I_1' and I_2' are reachable to E and the optimal capture point is the one that provides the smaller cost to E, that is, the closest point to the target; the corresponding pursuer is the active pursuer. Note that determining the type of capture is a different problem that the distinction of active pursuers in [33]. Here we determine which pursuer(s) will capture the evader under optimal play while an active pursuer in [33] only has to do with construction of the boundary of the dominance region of the evader.

Example 1: Consider the initial positions of the players: $E = (4.2\ 2.1\ 6.7)$, $P = (5\ 3.2\ 1.5)$, and $Q = (4.9\ -1.5\ 2)$. It holds that $\mathbf{x} \in \mathcal{R}_p$; also, $\Upsilon_1 < 0$ and $\Upsilon_2 < 0$. The optimal trajectories are shown in Fig. 3 where each player implements the optimal strategies (25)–(26) and E is simultaneously captured by both pursuers. The optimal strategies are implemented in closed-loop manner: at each time instant all players update the state of the system, determine the type of capture by computing Υ_1 and Υ_2, and update the capture point and their controls. As it is expected, the capture point is time-invariant under optimal play and the resulting trajectories are straight lines. The capture point

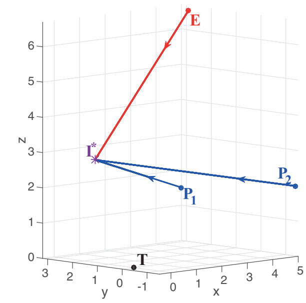

Fig. 3. Trajectories under optimal play.

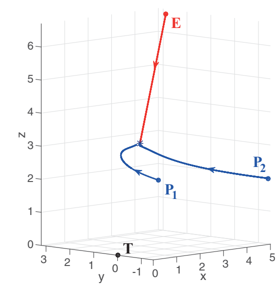

Fig. 4. Trajectories under non-optimal play by the evader.

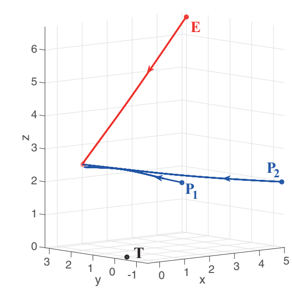

Fig. 5. Trajectories under non-optimal play by the pursuers.

is given by $I^* = (-0.452\ 1.097\ 2.994)$ and the Value of the game is $V = 10.374$.

Example 2: Consider the same initial positions of the players as in Example 1. In this case, however, E does not follow its optimal strategy and it aims directly at the target (the origin); however this guidance implemented by the evader is *unknown* to the pursuers. The resulting trajectories are shown in Fig. 4. We have that $R_{E_f}^2 = x_{E_f}^2 + y_{E_f}^2 + z_{E_f}^2 = 14.453$ and, as expected, $R_{E_f}^2 > V$. The evader, by disregarding the solution of the game and going directly after the target, is captured farther away from the target. The pursuers do not know the control applied by the evader and they only apply the saddle-point strategies in closed-loop state-feedback manner. They are able to react by updating the optimal capture point, hence, their trajectories curve, they capture E, and their payoff improves.

Example 3: We now consider a non-optimal play by the pursuers. In this example, the pursuers do not cooperate and they implement instead their individual optimal solutions. The resulting trajectories are shown in Fig. 5. The square of the terminal distance is $R_{E_f}^2 = x_{E_f}^2 + y_{E_f}^2 + z_{E_f}^2 = 9.8345$ and, as expected, $R_{E_f}^2 < V$. The evader implements its optimal strategy in closed-loop manner and is able to decrease its cost by being captured closer to the target than in Example 1, this, courtesy of the non-optimal actions of the pursuers.

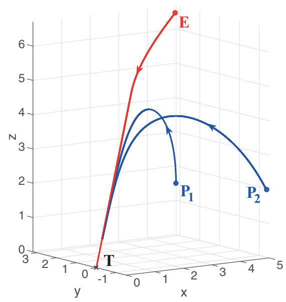

Fig. 6. Trajectories under non-optimal play by the pursuers. Pursuers implement PP guidance.

Example 4: Finally, we consider the case where the pursuers implement Pure Pursuit (PP) guidance. When using PP, the pursuers point towards the evader at each time instant. The resulting trajectories are shown in Fig. 6. The pursuers perform worse than in Example 3 and, consequently, worse than in Example 1, where they applied the cooperative optimal strategy. In this case, the pursuers, by implementing PP, allow the Evader to actually win the game since the Evader is able to reach the point target.

In each one of these examples the players *do not know* the control implemented by the opponent. Each player only implements the state-feedback optimal strategies derived in this paper and, by doing this, they are able to take advantage of any non-optimal control applied by the adversary. By verifying the the proposed solution, the Nash equilibrium is guaranteed since no player can gain by a unilateral change of strategy.

VI. Conclusion

The reach-avoid differential game in 3D of guarding a target was addressed. Two pursuers cooperate against an evader whose objective is to reach the target. The players' optimal saddle-point strategies were synthesized, and the Value function was obtained. It was shown that the Value function is continuous and continuously differentiable, and it is the solution of the Hamilton-Jacobi-Isaacs equation. Finally, examples were provided illustrating the guarantees and robustness properties of the obtained saddle-point strategies. That is, under non-optimal play by the evader, the pursuers are not only able to capture it, but they also improve their payoff by intercepting the evader farther away from the target. Similarly, under non-optimal play by the pursuers, the evader is able to lower its cost by reaching closer to the target. The paper highlights cooperative action where pursuers can increase their payoff by intercepting the evader further away from the target compared to their individual solutions. Future work will employ these fundamental results in order to analyze reach-avoid games with higher number of players.

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