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Distributed coestimation in heterogeneous sensor networks

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ABSTRACT
We propose a system-theoretical dynamic information fusion framework for heterogeneous sensor networks, where a sensor network with nonidentical node information roles and nonidentical node modalities is considered. Nonidentical node information roles allow nodes to be either active or passive in the sense that active nodes receive observations from a process, whereas passive nodes do not receive any information. Active and passive roles of nodes can be fixed or varying in time. Nonidentical node modalities allow active nodes to receive different classes of measurements from the process. For this class of sensor networks, we propose a distributed input and state coestimation architecture, where the time evolution of input and state updates of each node both depend on the local input and state information exchanges. Using Lyapunov theory and linear matrix inequalities, we establish stability and performance guarantees of the sensor network executing the proposed distributed coestimation architecture under local sufficient conditions.

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1. Introduction

1.1 Literature review

Dynamic information fusion in sensor networks supports a wide array of scientific, civilian, and military data-driven applications, which range from reconnaissance and surveillance to command and control of vehicle swarms.

Two common categories of dynamic information fusion are the Bayesian data fusion and the system-theoretical data fusion. While Bayesian methods regulate and fuse data according to probabilistic models (see, for example, Cunningham, Indelman, & Dellaert, 2013; Hollinger, Yerramalli, Singh, Mitra, & Sukhatme, 2015; Makarenko & Durrant-Whyte, 2004 and references therein), system-theoretical methods (see, for example, Olfati-Saber, 2005; Olfati-Saber & Shamma, 2005; Spanos, Olfati-Saber, & Murray, 2005 and references therein) focus on utilising dynamic motions of given processes and data exchange rules for controlling information fusion. This paper belongs to the latter category owing to attractive properties of system-theoretical data fusion in obtaining overall sensor network stability for real-world control, planning, and coordination applications.

Among several classes of heterogeneity in sensor networks, *nonidentical node information roles* and *nonidentical node modalities* are particularly important for numerous applications, which are considered in this paper. First, sensor networks are often associated with *nonidentical node information roles*. To elucidate this point, consider a representative application scenario shown in Figure 1. Here, the sensor network involves *active* and *passive nodes* in the sense that *active nodes* receive observations from the process whereas *passive nodes* do not receive any information. Note that the *active* and *passive node roles* also vary with respect to time in Figure 1; that is, active nodes can take passive roles during different time intervals and *vice versa*. Next, active nodes at any given time can practically have *nonidentical node modalities* in the sense that different measurements from the process can be observed.

There have been several papers in the literature that address heterogeneity in sensor networks, where the notable results can be listed as Olfati-Saber (2005), Olfati-Saber and Shamma (2005), Spanos et al. (2005), Freeman, Yang, and Lynch (2006), Demetriou (2009), Bai, Freeman, and Lynch (2010), Taylor, Beard, and Humpherys (2011), Chen, Cao, and Ren (2012), DeLellis, di Bernardo, and Russo (2010), Ustebay, Castro, and Rabbat (2011), Mu, Chowdhary, and How (2014), Casbeer, Cao, Garcia, and Milutinović (2015), Yucelen and Peterson (2014), Peterson, Yucelen, Chowdhary, and Kannan (2015), Peterson, Yucelen, and Pasiliao (2016), Yucelen and Peterson (2016), Cao, Casbeer, Garcia, and Zhang (2016), Mou, Garcia, and Casbeer (2017), Olfati-Saber (2007), Millán et al. (2013), Tran, Yucelen, Sarsilmaz, et al. (2017) and Tran, Yucelen, and Jagannathan (2019). In particular, the authors of Olfati-Saber (2005), Olfati-Saber and Shamma (2005), Spanos et al. (2005), Freeman et al. (2006), Demetriou (2009), Bai et al. (2010), Taylor et al. (2011), Chen et al. (2012) and DeLellis et al. (2010) concentrate on dynamic consensus algorithms relevant to sensor networks; however, they assumed that all nodes are being active at all times. The authors of Ustebay et al. (2011) and Mu et al. (2014) allow a subset of nodes being passive; however, these results are

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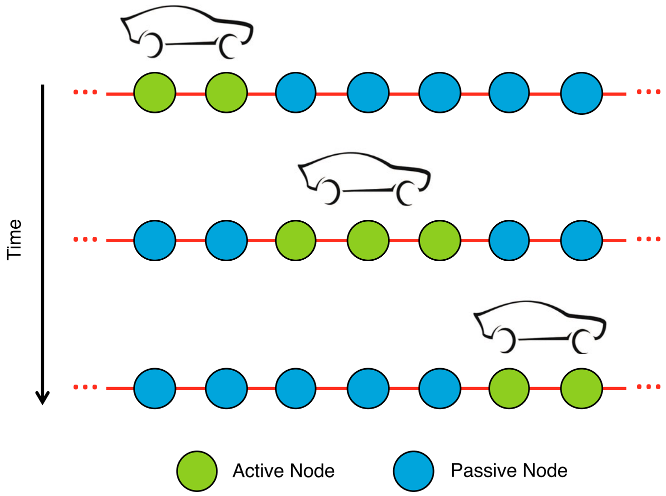

Figure 1. A dynamic information fusion scenario in a sensor network with time-varying active and passive node roles (lines and circles respectively denote communication links and nodes).

in the context of static consensus (that is, they are not suitable for dynamic data-driven applications). The authors of Yucelen and Peterson (2014), Peterson et al. (2015), Peterson et al. (2016), Yucelen and Peterson (2016), Casbeer et al. (2015), Cao et al. (2016) and Mou et al. (2017) focus on time-invariant (that is, fixed) and time-varying active and passive node roles; however, their results only consider nodes that are modelled as scalar integrator dynamics.

To address nonidentical node modalities in sensor networks, the authors of Olfati-Saber (2007), Millán et al. (2013), Tran, Yucelen, Sarsilmaz, et al. (2017) and Tran et al. (2019) concentrate on nonidentical node modalities with their proposed distributed information fusion algorithms. However, it should be noted that Olfati-Saber (2007) does not take into account the possibility of having passive nodes (that is, it requires all nodes are being active at all times). While fixed active and passive node roles are implicitly studied by the authors of Millán et al. (2013), the contribution documented in this work requires a *global sufficient stability condition*, which may not be suitable for practical sensor networks composed of a (sufficiently) large set of nodes. More recently, nodes with nonidentical modalities are considered by the authors of Tran, Yucelen, Sarsilmaz, et al. (2017) for both fixed and time-varying active and passive node roles under *local sufficient stability conditions* (we also refer to Tran, Yucelen, & Jagannathan, 2017b for a preliminary version of the results in Tran, Yucelen, Sarsilmaz, et al., 2017). However, as discussed in Section 4.3 of Tran, Yucelen, Sarsilmaz, et al. (2017), tuning the resulting distributed algorithm for satisfactory performance can be a challenge especially for the case when the active and passive node roles vary with respect to time. It should be also noted that the architecture proposed in Tran et al. (2019) requires the measurements to be passed through local observers to extract more information before sending over the network for fusion.

1.2 Contribution

As discussed above, existing methods (Cao et al., 2016; Casbeer et al., 2015; Millán et al., 2013; Mou et al., 2017; Mu et al., 2014; Olfati-Saber, 2007; Peterson et al., 2015, 2016; Tran et al., 2017b; Tran, Yucelen, Sarsilmaz, et al., 2017; Tran et al., 2019; Ustebay et al., 2011; Yucelen & Peterson, 2014, 2016) do not provide a general system-theoretical dynamic information fusion architecture for addressing heterogeneity in sensor networks due to nonidentical node information roles and nonidentical node modalities. Motivated by this standpoint, this paper's contribution is a new, general system-theoretical dynamic information fusion framework for *heterogeneous sensor networks*, where a sensor network with both *nonidentical node information roles* and *nonidentical node modalities* is considered. For this class of sensor networks, we propose a *distributed input and state 'coestimation'* architecture, where the time evolution of input and state updates of each node both depend on the *local* input and state information exchanges. Using tools and methods from Lyapunov theory and linear matrix inequalities, we establish *stability and performance guarantees* of the overall heterogeneous sensor network executing the proposed distributed coestimation architecture under *local sufficient conditions* for each node. We also consider stochastic extensions that capture the practical aspect when the process and the node observations both include noise. Finally, we present two numerical examples to demonstrate the efficacy of our theoretical contributions. As compared with the *distributed input and state 'estimation'* architecture in Tran, Yucelen, Sarsilmaz, et al. (2017) discussed above, where the time evolution of input (respectively, state) update of each node *only* depends the local input (respectively, state) unlike the *distributed input and state 'coestimation'* architecture of this paper, one of these examples further shows a substantially improved dynamic input and state fusion performance. Note that preliminary conference versions of this paper are appeared in Tran, Yucelen, and Jagannathan (2017a) and Tran, Yucelen, Jagannathan, and Casbeer (2018). The present paper *considerably* expands on Tran et al. (2017a, 2018) by providing the proofs of the results in Tran et al. (2017a, 2018); additional theoretical results including a generalisation in a stochastic setting; and additional informative discussions.

The remainder of this paper is organised as follows. The proposed distributed input and state coestimation architecture for fixed active and passive node roles subject to nonidentical active node modalities is presented in Section 2 with its system-theoretical stability analysis, where generalisations to the case of time-varying active and passive node roles is given in Section 3. For addressing practical situations when the process and the node observations both include noise, a stochastic extension is also presented in Section 4. Furthermore, the aforementioned illustrative numerical examples are given in Section 5 and concluding remarks are summarised in Section 6. Finally, for the *notation* used in this paper, we refer to the appendix.

2. Distributed coestimation: fixed active and passive node roles

2.1 Problem setup and proposed algorithm

Consider a process[1] with the dynamics given by

$$\dot{x}(t) = Ax(t) + Bw(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a process internal state vector and $w(t) \in \mathbb{R}^p$ is an input to this process. Here, we consider that $x(t)$ is not measurable. We also consider that $w(t)$ is an unknown but bounded signal with a bounded time rate of change. In addition, $A \in \mathbb{R}^{n\times n}$ is a Hurwitz system matrix and $B \in \mathbb{R}^{n\times p}$ is a system input matrix.

In this paper, a sensor network with N nodes is considered, where nodes exchange local measurements with their neighbours under an undirected and connected graph $\mathcal{G}$. Following the terminology from Yucelen and Peterson (2014), Peterson et al. (2015), Peterson et al. (2016) and Yucelen and Peterson (2016), a node i, $i = 1, \ldots, N$, is called an *active node* when it is subject to the observation of the process (1) given by

$$y_i(t) = C_i x(t). \tag{2}$$

Here, $y_i(t) \in \mathbb{R}^m$ and $C_i \in \mathbb{R}^{m\times n}$ respectively stand for a measurable process output and the system output matrix for an active node i, $i = 1, \ldots, N$. Moreover, a node i, $i = 1, \ldots, N$, is called a *passive node* when it has no observation from the process given by (1). For the results of this section, the active and passive roles of each node are considered to be fixed. Yet, recall from (2) that each active node can have nonidentical sensing modalities. We also practically consider for the well-posedness that each active node has complementary properties distributed over the sensor network to guarantee collective observability[2] while the pairs (A, C_i), $i = 1, \ldots, N$, may not be locally observable.

Based on the setup given above, we propose a distributed coestimation algorithm for sensor networks to estimate the unmeasurable state $x(t)$ and the unknown input $w(t)$ of the process (1). Here, we note that A is assumed to be Hurwitz to allow the employment of the passive nodes in the sensor network and this assumption does not result from the distributed coestimation approach proposed below.

Consider now a new input and state coestimation architecture for each node i, $i = 1, \ldots, N$, given by

$$\begin{aligned}
\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + B\hat{w}_i(t) + g_i L_i(y_i(t) - C_i\hat{x}_i(t)) \\
&\quad - \alpha M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \\
&\quad + \alpha S_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t)), \quad \hat{x}_i(0) = \hat{x}_{i0},
\end{aligned} \tag{3}$$

$$\begin{aligned}
\dot{\hat{w}}_i(t) &= g_i J_i(y_i(t) - C_i\hat{x}_i(t)) - \sigma_i K_i \hat{w}_i(t) \\
&\quad + \alpha T_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \\
&\quad - \alpha N_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t)), \quad \hat{w}_i(0) = \hat{w}_{i0},
\end{aligned} \tag{4}$$

where $\hat{x}_i(t) \in \mathbb{R}^n$ is the local estimate of $x(t)$ and $\hat{w}_i(t) \in \mathbb{R}^p$ is the local input estimate of $w(t)$. In addition, $L_i \in \mathbb{R}^{n\times m}$, $J_i \in \mathbb{R}^{p\times m}$, and $K_i \in \mathbb{R}^{p\times p}$ are the design gain matrices and $\alpha \in \mathbb{R}_+$ and $\sigma_i \in \mathbb{R}_+$ are the design coefficients. Finally, $M_i \in \mathbb{R}^{n\times n}$, $S_i \in \mathbb{R}^{n\times p}$, $T_i \in \mathbb{R}^{p\times n}$, and $N_i \in \mathbb{R}^{p\times p}$ are also the additional design gain matrices. Here, $g_i = 1$ when the node i is active and $g_i = 0$ when the node i is passive. Since the local state and input information exchange terms (i.e. the coupling terms '$\hat{x}_i(t) - \hat{x}_j(t)$' and '$\hat{w}_i(t) - \hat{w}_j(t)$') appear both in the state and input updates given by (3) and (4), the word '*coestimation*' is used to indicate the proposed distributed algorithm.

2.2 Analysis of proposed algorithm

For the main result of this section, first define

$$\tilde{x}_i(t) \triangleq x(t) - \hat{x}_i(t) \in \mathbb{R}^n, \tag{5}$$

$$\tilde{w}_i(t) \triangleq \hat{w}_i(t) - w(t) \in \mathbb{R}^p. \tag{6}$$

Now, the time derivative of (5) can be written as

$$\begin{aligned}
\dot{\tilde{x}}_i(t) &= Ax(t) + Bw(t) - A\hat{x}_i(t) - B\hat{w}_i(t) \\
&\quad - g_i L_i(y_i(t) - C_i\hat{x}_i(t)) + \alpha M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \\
&\quad - \alpha S_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t)) \\
&= (A - g_i L_i C_i)\tilde{x}_i(t) - B\tilde{w}_i(t) \\
&\quad - \alpha M_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{x}_j(t) - \alpha S_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{w}_j(t).
\end{aligned} \tag{7}$$

In (7), $\mathcal{L}_{ij}$ is the entry of the Laplacian matrix on the ith row and jth column. In addition, the time derivative of (6) can be written as

$$\begin{aligned}
\dot{\tilde{w}}(t) &= g_i J_i C_i \tilde{x}_i(t) - \sigma_i K_i(\tilde{w}_i(t) + w(t)) - \dot{w}(t) \\
&\quad - \alpha T_i \sum_{j=1}^{N} a_{ij}(\tilde{x}_i(t) - \tilde{x}_j(t)) \\
&\quad - \alpha N_i \sum_{j=1}^{N} a_{ij}(\tilde{w}_i(t) - \tilde{w}_j(t)) \\
&= g_i J_i C_i \tilde{x}_i(t) - \sigma_i K_i \tilde{w}_i(t) - \alpha T_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{x}_j(t) \\
&\quad - \alpha N_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{w}_j(t) - \sigma_i K_i w(t) - \dot{w}(t).
\end{aligned} \tag{8}$$

Next, let $z_i = [\tilde{x}_i^{\mathrm{T}}(t), \tilde{w}_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n+p}$. Now, (7) and (8) can be compactly written as

$$\begin{aligned}
\dot{z}_i(t) &= \underbrace{\begin{bmatrix} A - g_i L_i C_i & -B \\ g_i J_i C_i & -\sigma_i K_i \end{bmatrix}}_{\bar{A}_i} z_i(t) \\
&\quad - \alpha \sum_{j=1}^{N} \mathcal{L}_{ij} \underbrace{\begin{bmatrix} M_i & S_i \\ T_i & N_i \end{bmatrix}}_{H_i} z_j(t) + \underbrace{\begin{bmatrix} 0 \\ -\sigma_i K_i w(t) - \dot{w}(t) \end{bmatrix}}_{\phi_i(t)} \\
&= \bar{A}_i z_i(t) - \alpha \sum_{j=1}^{N} \mathcal{L}_{ij} H_i z_j(t) + \phi_i(t).
\end{aligned} \tag{9}$$

Here, we note that the local design terms L_i, J_i, K_i, and σ_i can be always chosen to ensure $\bar{A}_i$ being Hurwitz[3]. Therefore, $\bar{A}_i$ is implicitly considered to be Hurwitz for the following analysis (this is the *local sufficient stability* condition). We also note that for any given positive-definite matrix $Q_i \in \mathbb{R}^{(n+p)\times(n+p)}$, there exists a unique positive-definite matrix $P_i \in \mathbb{R}^{(n+p)\times(n+p)}$ satisfying

$$\bar{A}_i^{\mathrm{T}} P_i + P_i \bar{A}_i + Q_i = 0. \tag{10}$$

Now, let the aggregated vector be given by $z(t) \triangleq [z_1^{\mathrm{T}}(t), z_2^{\mathrm{T}}(t), \ldots, z_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{(n+p)N}$. To this end, (9) can be further written as

$$\begin{aligned}\dot{z}(t) &= \underbrace{\begin{bmatrix} \bar{A}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{A}_N \end{bmatrix}}_{\bar{A}} z(t) - \alpha \underbrace{\begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_N \end{bmatrix}}_{H} \\ &\quad \times \underbrace{\begin{bmatrix} \mathcal{L}_{11}\mathrm{I}_{n+p} & \mathcal{L}_{12}\mathrm{I}_{n+p} & \ldots & \mathcal{L}_{1N}\mathrm{I}_{n+p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{N1}\mathrm{I}_{n+p} & \mathcal{L}_{N2}\mathrm{I}_{n+p} & \ldots & \mathcal{L}_{NN}\mathrm{I}_{n+p} \end{bmatrix}}_{(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_{n+p})} z(t) \\ &\quad + \underbrace{\begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_N(t) \end{bmatrix}}_{\phi(t)} \\ &= \bar{A}z(t) - \alpha H(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})z(t) + \phi(t). \end{aligned} \tag{11}$$

In (11), $\mathcal{L}(\mathcal{G})$ is the Laplacian matrix. The following theorem presents the main result of this section.

Theorem 2.1: *Consider the process given by* (1) *and the distributed input and state estimation architecture given by* (3) *and* (4). *If the matrix H_i is selected as $H_i = P_i^{-1}$ and nodes exchange information using local measurements subject to an undirected and connected graph $\mathcal{G}$, then the error dynamics given by* (11) *is uniformly bounded.*

Proof: Consider the Lyapunov-like function candidate given by

$$V(z) = z^{\mathrm{T}} P z, \tag{12}$$

where $P = \mathrm{diag}([P_1, P_2, \ldots, P_N])$ is a positive-definite matrix. Note that $V(0) = 0$, and $V(z) > 0$ for all $z \neq 0$. Taking time-derivative of $V(z)$ along the trajectory of (11) yields

$$\begin{aligned}\dot{V}(z(t)) &= z^{\mathrm{T}}(t)(\bar{A}^{\mathrm{T}}P + P\bar{A})z(t) \\ &\quad - 2\alpha z^{\mathrm{T}}(t)PH(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})z(t) + 2z^{\mathrm{T}}(t)P\phi(t) \\ &= -z^{\mathrm{T}}(t)Qz(t) - 2\alpha z^{\mathrm{T}}(t)PH(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})z(t) \\ &\quad + 2z^{\mathrm{T}}(t)P\phi(t), \end{aligned} \tag{13}$$

where $Q = \mathrm{diag}([Q_1, Q_2, \ldots, Q_N])$ is also a positive-definite matrix. Since we choose M_i, S_i, T_i and N_i such that $H_i = \left[\begin{smallmatrix} M_i & S_i \\ T_i & N_i \end{smallmatrix}\right] = P_i^{-1}$ holds, we have $H = P^{-1}$ (i.e. $PH = \mathrm{I}_N \otimes \mathrm{I}_{n+p}$). Hence, (13) can be rewritten as

$$\begin{aligned}\dot{V}(z(t)) &= -z^{\mathrm{T}}(t)(Q + 2\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p}))z(t) + 2z^{\mathrm{T}}(t)P\phi(t) \\ &= -z^{\mathrm{T}}(t)\bar{Q}z(t) + 2z^{\mathrm{T}}(t)P\phi(t), \end{aligned} \tag{14}$$

where $\bar{Q} \triangleq Q + 2\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})$. Since Q is a positive-definite matrix and the Laplacian matrix $\mathcal{L}(\mathcal{G})$ is a positive-semidefinite matrix, then $\bar{Q}$ is a positive-definite matrix (Proposition 8.1.2, Bernstein, 2009). In addition, since $\|w(t)\|_2 \leq \bar{w}$ and $\|\dot{w}(t)\|_2 \leq \bar{\dot{w}}$, then $\phi_i(t)$ is bounded, i.e. $\|\phi_i(t)\|_2 \leq \bar{\phi}_i$ with

$$\bar{\phi}_i \triangleq \sigma_i \bar{w}\|K_i\|_2 + \bar{\dot{w}}. \tag{15}$$

As a result, $\|\phi(t)\|_2 \leq \bar{\phi}$ holds with

$$\bar{\phi} \triangleq \sqrt{\phi_1^2 + \phi_2^2 + \cdots + \phi_N^2}. \tag{16}$$

From (14), we can now write

$$\begin{aligned}\dot{V}(z(t)) &\leq -\lambda_{\min}(\bar{Q})\|z(t)\|_2^2 + 2\|z(t)\|_2\|P\|_2\bar{\phi} \\ &= \|z(t)\|_2(2\|P\|_2\bar{\phi} - \lambda_{\min}(\bar{Q})\|z(t)\|_2). \end{aligned} \tag{17}$$

Finally, by letting $\mu \triangleq \frac{2\|P\|_2\bar{\phi}}{\lambda_{\min}(\bar{Q})}$ and $\Omega \triangleq \{z(t) : \|z(t)\|_2 \leq \mu\}$, it follows that $\dot{V}(z(t)) < 0$ outside the compact set Ω. Therefore, the error dynamics given by (11) is uniformly bounded (Khalil, 2002). ■

The following corollary is now immediate with regard to the performance of the proposed estimation approach.

Corollary 2.1: *Consider the process given by* (1) *and the distributed input and state estimation architecture given by* (3) *and* (4). *If the matrix H_i is selected as $H_i = P_i^{-1}$ and nodes exchange information using local measurement subject to an undirected and connected graph $\mathcal{G}$, then the bound*

$$\|z(t)\|_2 \leq \sqrt{\frac{\lambda_{\max(P)}}{\lambda_{\min(P)}}}\mu, \tag{18}$$

holds for $t \geq T$.

Proof: In the proof of Theorem 1, we show that $V(z(t))$ cannot grow outside the compact set Ω. Thus, from $\lambda_{\min}(P))\|z(t)\|_2^2 \leq V(z(t)) \leq \lambda_{\max}(P))\|z(t)\|_2^2$, we have $\lambda_{\min}(P))\|z(t)\|_2^2 \leq \lambda_{\max}(P))\mu^2$; hence the bound given by (18) is immediate. ■

Remark 2.1: We now compare the new distributed input and state coestimation architecture given by (3) and (4) with its counterpart in Tran, Yucelen, Sarsilmaz, et al. (2017). For this purpose, recall the distributed input and state estimation law of Tran, Yucelen, Sarsilmaz, et al. (2017)

$$\begin{aligned}\dot{\hat{x}}_i(t) &= (A - \gamma P_i^{-1})\hat{x}_i(t) + B\hat{w}_i(t) + g_i L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - \alpha P_i^{-1}\sum_{i\sim j}(\hat{x}_i(t) - \hat{x}_j(t)), \quad \hat{x}_i(0) = \hat{x}_{i0}, \end{aligned} \tag{19}$$

$$\begin{aligned}\dot{\hat{w}}_i(t) &= g_i J_i(y_i(t) - C_i\hat{x}_i(t)) - (\sigma_i K_i + \gamma \mathrm{I}_p)\hat{w}_i(t) \\ &\quad - \alpha\sum_{i\sim j}(\hat{w}_i(t) - \hat{w}_j(t)), \quad \hat{w}_i(0) = \hat{w}_{i0}, \end{aligned} \tag{20}$$

where $P_i \in \mathbb{R}^{(n+p)\times(n+p)}$ is a positive-definite gain matrix satisfying the linear matrix inequality

$$R_i = \begin{bmatrix} \bar{A}_i^{\mathrm{T}} P_i + P_i \bar{A}_i & -P_i B + g_i C_i^{\mathrm{T}} K_i^{\mathrm{T}} \\ -B^{\mathrm{T}} P_i + g_i J_i C_i & -2\sigma_i K_i \end{bmatrix} \leq 0, \tag{21}$$

with $\bar{A}_i \triangleq A - g_i L_i C_i$. As discussed in Tran, Yucelen, Sarsilmaz, et al. (2017), the terms '$-\gamma P_i^{-1}\hat{x}_i(t)$' and '$-(\sigma_i K_i + \gamma \mathrm{I}_p)\hat{w}_i(t)$' appearing respectively in (19) and (20) are referred as leakage terms. In particular, if 'γP_i^{-1}' and '$\sigma_i K_i + \gamma \mathrm{I}_p$' in these terms are not small, they can lead to an unsatisfactory performance. Furthermore, '$\sigma_i K_i$' also appears in the linear matrix inequality given by (21). However, we may not be able to select this term as small while simultaneously satisfying (21), either due to the magnitude of the term '$-P_i B + g_i C_i^{\mathrm{T}} K_i$' being not small or a computational conservatism. To summarise, (19) and (20) of Tran, Yucelen, Sarsilmaz, et al. (2017) may not always yield to an acceptable performance (see also Remark 3.1).

In contrast to (19) and (20) of Tran, Yucelen, Sarsilmaz, et al. (2017) (Remark 2.1), the new input and state coestimation architecture given by (3) and (4) has only one leakage term '$-\sigma_i K_i \hat{w}_i(t)$' that appears in the input update (4). Additionally, as discussed in the next remark, '$\sigma_i K_i$' can be made judiciously small, and hence, the proposed approach in this section has the potential to achieve a better overall estimation performance as compared with the results in Tran, Yucelen, Sarsilmaz, et al. (2017).

Remark 2.2: The ultimate bound given by (18) can be used as a design metric in the sense that design parameters can be chosen to make (18) small. For instance, small values for σ_i and K_i can be selected such that (15) and (16) are small, where they appear on the ultimate bound (18) through the term μ.

3. Distributed coestimation: time-varying active and passive node roles

3.1 Problem setup and proposed algorithm

In this section, we generalise the result of Section 2 to the case when the active and passive roles of each node vary in time. For this purpose, we again consider the process of interest given by (1). We also have a sensor network with N nodes exchanging information among each other using their local measurements according to an undirected and connected graph $\mathcal{G}$.

In addition, a node is called active for some time instant when it is subject to the observation of the process given by (2) at that time instant. Likewise, a node is called passive when it has no observation at that time instant. Motivated by the case in (Figure 2(d), Zavlanos, Egerstedt, & Pappas, 2011) and without loss of much practical generality, a node is considered to have ability to smoothly change back and forth between active and passive node roles, where the role change is captured by the smooth function $g_i(t) \in [0, 1]$. Once again, we consider collective observability for well-posedness. In what follows, we first present the proposed distributed input and state coestimation algorithm below and then its analysis in Section 3.2.

In particular, consider now the proposed input and state coestimation architecture for each node i, $i = 1, \ldots, N$, given by

$$\begin{aligned} \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + B\hat{w}_i(t) + g_i(t)L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - \alpha M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \\ &\quad + \alpha S_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t)), \quad \hat{x}_i(0) = \hat{x}_{i0}, \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{\hat{w}}_i(t) &= g_i(t)J_i(y_i(t) - C_i\hat{x}_i(t)) - \sigma_i K_i \hat{w}_i(t) \\ &\quad + \alpha T_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \\ &\quad - \alpha N_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t)), \quad \hat{w}_i(0) = \hat{w}_{i0}, \end{aligned} \tag{23}$$

where $\hat{x}_i(t) \in \mathbb{R}^n$ is the local estimate of $x(t)$, and $\hat{w}_i(t) \in \mathbb{R}^p$ is the local input estimate of $w(t)$. In addition, $L_i \in \mathbb{R}^{n\times p}$, $J_i \in \mathbb{R}^{p\times m}$ and $K_i \in \mathbb{R}^{p\times p}$ are design gain matrices and $\alpha \in \mathbb{R}_+$ and $\sigma_i \in \mathbb{R}_+$ are the design coefficients. Finally, $M_i \in \mathbb{R}^{n\times n}$, $S_i \in \mathbb{R}^{n\times p}$, $T_i \in \mathbb{R}^{p\times n}$, and $N_i \in \mathbb{R}^{p\times p}$ are also the additional design gain matrices[4].

As discussed above, the smooth function $g_i(t) \in [0, 1]$, $i = 1, \ldots, N$ indicates whether a node is active or passive at time t.

3.2 Analysis of proposed algorithm

For the main result of this section, first define (5) and (6). Now, the time derivative of (5) can be written as

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= Ax(t) + Bw(t) - A\hat{x}_i(t) - B\hat{w}_i(t) - g_i(t)L_i(y_i(t) \\ &\quad - C_i\hat{x}_i(t)) + \alpha M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \end{aligned}$$

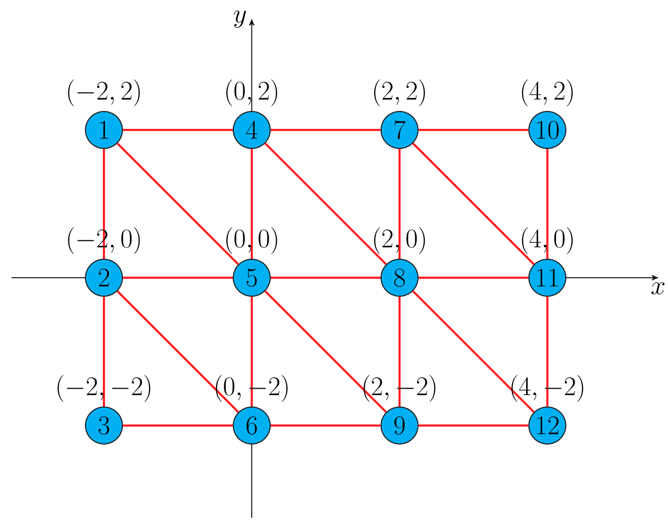

Figure 2. Communication graph of the time-varying heterogeneous sensor network with 12 nodes (lines denote communication links and circles denote nodes).

$$- \alpha S_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t))$$

$$= (A - g_i(t)L_iC_i)\tilde{x}_i(t) - B\tilde{w}_i(t) - \alpha M_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{x}_j(t)$$

$$- \alpha S_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{w}_j(t). \tag{24}$$

In (24), $\mathcal{L}_{ij}$ is the entry of the Laplacian matrix on the ith row and jth column. In addition, the time derivative of (6) can be written as

$$\dot{\tilde{w}}(t) = g_i(t)J_iC_i(x_i(t) - \hat{x}_i(t)) - \sigma_i K_i(\tilde{w}_i(t) + w(t))$$

$$+ \alpha T_i \sum_{j=1}^{N} a_{ij}(x(t) - \tilde{x}_i(t) - x(t) + \tilde{x}_j(t))$$

$$- \alpha N_i \sum_{j=1}^{N} a_{ij}(\tilde{w}_i(t) + w(t) - \tilde{w}_j(t) - w(t)) - \dot{w}(t)$$

$$= g_i(t)J_iC_i\tilde{x}_i(t) - \sigma_i K_i\tilde{w}_i(t) - \alpha T_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{x}_j(t)$$

$$- \alpha N_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{w}_j(t) - \sigma_i K_i w(t) - \dot{w}(t). \tag{25}$$

Next, let $z_i(t) \triangleq [\tilde{x}_i^{\mathrm{T}}(t), \tilde{w}_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n+p}$. Now, (24) and (25) can be compactly written as

$$\dot{z}_i(t) = \underbrace{\begin{bmatrix} A - g_i(t)L_iC_i & -B \\ g_i(t)J_iC_i & -\sigma_i K_i \end{bmatrix}}_{\bar{A}_i(g_i(t))} z_i(t)$$

$$- \alpha \sum_{j=1}^{N} \mathcal{L}_{ij} \underbrace{\begin{bmatrix} M_i & S_i \\ T_i & N_i \end{bmatrix}}_{H_i} z_j(t) + \underbrace{\begin{bmatrix} 0 \\ -\sigma_i K_i w(t) - \dot{w}(t) \end{bmatrix}}_{\phi_i(t)}$$

$$= \bar{A}_i(g_i(t))z_i(t) - \alpha \sum_{j=1}^{N} \mathcal{L}_{ij}H_i z_j(t) + \phi_i(t). \tag{26}$$

Here, the matrix $\bar{A}_i(g_i(t))$ can be rewritten in the form

$$\bar{A}_i(g_i(t)) = \underbrace{\begin{bmatrix} A & -B \\ 0 & -\sigma_i K_i \end{bmatrix}}_{\bar{A}_{i,0}} + g_i(t) \underbrace{\begin{bmatrix} -L_iC_i & 0 \\ J_iC_i & 0 \end{bmatrix}}_{\tilde{A}_i},$$

$$= \bar{A}_{i,0} + g_i(t)\tilde{A}_i, \tag{27}$$

where $g_i(t) \in [0, 1]$. Note that $\bar{A}_{i,0}$ and $\bar{A}_{i,1}$ are the matrices corresponding to $\bar{A}_i(g_i(t))$ at $g_i(t) = 0$ and $g_i(t) = 1$, respectively. Hence, $\tilde{A}_i = \bar{A}_{i,1} - \bar{A}_{i,0}$. The following lemma is needed for the stability analysis (Theorem 3.1) of the proposed distributed input and state coestimation algorithm.

Lemma 3.1: *If there exists a common positive-definite matrix $P_i \in \mathbb{R}^{(n+p)\times(n+p)}$ for node i, $i = 1, \ldots, N$, satisfying*

$$\bar{A}_{i,0}^{\mathrm{T}}P_i + P_i\bar{A}_{i,0} \leq -\epsilon \mathrm{I}_{n+p}, \tag{28}$$

$$\bar{A}_{i,1}^{\mathrm{T}}P_i + P_i\bar{A}_{i,1} \leq -\epsilon \mathrm{I}_{n+p}, \tag{29}$$

then the inequality given by

$$\bar{A}_i(g_i(t))^{\mathrm{T}}P_i + P_i\bar{A}_i(g_i(t)) \leq -\epsilon \mathrm{I}_{n+p}, \tag{30}$$

holds for all $g_i(t) \in [0, 1]$, where $\epsilon \in \mathbb{R}_+$.

Proof: First, note that the inequality given by (30) implies

$$\mathcal{S} \triangleq \xi^{\mathrm{T}}[\bar{A}_i(g_i(t))^{\mathrm{T}}P_i + P_i\bar{A}_i(g_i(t)) + \epsilon \mathrm{I}_{n+p}]\xi \leq 0, \tag{31}$$

for any arbitrary nonzero vector ξ. Next, using (28) and (29), one can write

$$\begin{aligned}
\mathcal{S} &= \xi^{\mathrm{T}}[(\bar{A}_{i,0} + g_i(t)\tilde{A}_i)^{\mathrm{T}}P_i + P_i(\bar{A}_{i,0} + g_i(t)\tilde{A}_i) + \epsilon \mathrm{I}_{n+p}]\xi \\
&= \xi^{\mathrm{T}}[(\bar{A}_{i,0} + g_i(t)(\bar{A}_{i,1} - \bar{A}_{i,0}))^{\mathrm{T}}P_i \\
&\quad + P_i(\bar{A}_{i,0} + g_i(t)(\bar{A}_{i,1} - \bar{A}_{i,0})) + \epsilon \mathrm{I}_{n+p}]\xi \\
&= \xi^{\mathrm{T}}[((1 - g_i(t))\bar{A}_{i,0} + g_i(t)\bar{A}_{i,1})^{\mathrm{T}}P_i + P_i((1 - g_i(t))\bar{A}_{i,0} \\
&\quad + g_i(t)\bar{A}_{i,1}) + \epsilon \mathrm{I}_{n+p}]\xi \\
&= \xi^{\mathrm{T}}[(1 - g_i(t))[\bar{A}_{i,0}^{\mathrm{T}}P_i + P_i\bar{A}_{i,0}] \\
&\quad + g_i(t)[\bar{A}_{i,1}^{\mathrm{T}}P_i + P_i\bar{A}_{i,1}] + \epsilon \mathrm{I}_{n+p}]\xi \\
&\leq \xi^{\mathrm{T}}[(1 - g_i(t))(-\epsilon \mathrm{I}_{n+p}) + g_i(t)(-\epsilon \mathrm{I}_{n+p}) + \epsilon \mathrm{I}_{n+p}]\xi \\
&= 0.
\end{aligned} \tag{32}$$

Thus, the proof is now complete. ■

Now, let the aggregated vector be given by $z(t) \triangleq [z_1^{\mathrm{T}}(t), z_2^{\mathrm{T}}(t), \ldots, z_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{(n+p)N}$. To this end, (26) can be written as

$$\dot{z}(t) = \underbrace{\begin{bmatrix} \bar{A}_1(g_1(t)) & & 0 \\ & \ddots & \\ 0 & & \bar{A}_N(g_N(t)) \end{bmatrix}}_{\bar{A}(g(t))} z(t)$$

$$- \alpha \underbrace{\begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_N \end{bmatrix}}_{H}$$

$$\times \underbrace{\begin{bmatrix} \mathcal{L}_{11}\mathrm{I}_{n+p} & \mathcal{L}_{12}\mathrm{I}_{n+p} & \cdots & \mathcal{L}_{1N}\mathrm{I}_{n+p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{N1}\mathrm{I}_{n+p} & \mathcal{L}_{N2}\mathrm{I}_{n+p} & \cdots & \mathcal{L}_{NN}\mathrm{I}_{n+p} \end{bmatrix}}_{(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_{n+p})} z(t)$$

$$+\underbrace{\begin{bmatrix}\phi_1(t)\\ \vdots \\ \phi_N(t)\end{bmatrix}}_{\phi(t)}$$

$$= \bar{A}(g(t))z(t) - \alpha H(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})z(t) + \phi(t). \quad (33)$$

In (33), where $\mathcal{L}(\mathcal{G})$ is the Laplacian matrix and $g(t) = [g_1(t), \ldots, g_N(t)]^{\mathrm{T}}$. In what follows, for each node i, $i = 1, \ldots, N$, we:

(i) Solve the linear matrix inequalities given by (28) and (29) for a common positive-definite matrix P_i (these are the *local sufficient stability* conditions).

(ii) Obtain the design gain matrices M_i, S_i, T_i, and N_i from the matrix equality given by

$$H_i = \begin{bmatrix} M_i & S_i \\ T_i & N_i \end{bmatrix} = P_i^{-1}. \quad (34)$$

Notice from (34) that $H = P^{-1}$ (i.e. $PH = \mathrm{I}_N \otimes \mathrm{I}_{n+p} = \mathrm{I}_{N(n+p)}$), where $P \triangleq \mathrm{diag}([P_1, P_2, \ldots, P_N])$. We are now ready to state the main result of this section.

Theorem 3.1: *Consider the process given by* (1) *and the distributed input and state coestimation architecture given by* (22) *and* (23). *If there exists a common positive-definite matrix P_i for each node i, $i = 1, \ldots, N$, satisfying* (28) *and* (29), *one selects H_i according to* (34), *and nodes exchange information according to an undirected and connected graph $\mathcal{G}$, then the error dynamics given by* (33) *is uniformly bounded.*

Proof: Consider the Lyapunov-like function candidate given by (12), where $P = \mathrm{diag}([P_1, P_2, \ldots, P_N])$ is a positive-definite matrix with each P_i obtained through solving the linear matrix inequalities given by (28) and (29). Note that $V(0) = 0$, and $V(z) > 0$ for all $z \neq 0$. Taking time-derivative of $V(z)$ along the trajectory of (33) and using Lemma 3.1 yields

$$\begin{aligned}\dot{V}(z(t)) &= z^{\mathrm{T}}(t)(\bar{A}^{\mathrm{T}}(g(t))P + P\bar{A}(g(t)))z(t) \\ &\quad - 2\alpha z^{\mathrm{T}}(t)PH(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})z(t) + 2z^{\mathrm{T}}(t)P\phi(t) \\ &\leq -\epsilon z^{\mathrm{T}}(t)z(t) - 2\alpha z^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})z(t) \\ &\quad + 2z^{\mathrm{T}}(t)P\phi(t) \\ &= -z^{\mathrm{T}}(t)\bar{Q}z(t) + 2z^{\mathrm{T}}(t)P\phi(t), \end{aligned} \quad (35)$$

where $\bar{Q} = \epsilon\mathrm{I}_{N(n+p)} + 2\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})$. Since $\epsilon\mathrm{I}_{N(n+p)}$ is a positive-definite matrix and the Laplacian matrix $\mathcal{L}(\mathcal{G})$ is a positive-semidefinite matrix, $\bar{Q}$ is a positive-definite matrix. In addition, $\|\phi\|_2 \leq \bar{\phi}$ with $\bar{\phi}$ defined by (16). Therefore, an upper bound to (35) can be found as $\dot{V}(z(t)) \leq -\lambda_{\min}(\bar{Q})\|z(t)\|_2^2 + 2\|z(t)\|_2\|P\|_2\bar{\phi} = \|z(t)\|_2(2\|P\|_2\bar{\phi} - \lambda_{\min}(\bar{Q})\|z(t)\|_2)$. Letting $\mu \triangleq \frac{2\|P\|_2\bar{\phi}}{\lambda_{min}(\bar{Q})}$ and $\Omega \triangleq \{z(t) : \|z(t)\|_2 \leq \mu\}$, it follows that $\dot{V}(z(t)) < 0$ outside the compact set Ω, and therefore, the error dynamics given by (33) is uniformly bounded (Khalil, 2002). ■

Note that Theorem 3.1 establishes the stability of the proposed distributed input and state coestimation architecture given by (22) and (23) in terms of uniform boundedness under *local sufficient stability conditions* (28) and (29) for each node. The following corollary is now immediate on the performance of the proposed architecture.

Corollary 3.1: *Consider the process given by* (1) *and the distributed input and state coestimation architecture given by* (22) *and* (23). *If there exists a common positive-definite matrix P_i for each node i, $i = 1, \ldots, N$, satisfying* (28) *and* (29), *one selects H_i according to* (34), *and nodes exchange information according to an undirected and connected graph $\mathcal{G}$, then the bound* (18) *holds for $t \geq T$.*

Proof: The result follows from the proof of Corollary 2.1. ■

Note that the discussion given in Remark 2.2 also holds for the results of this section. That is, since the ultimate bound in Corollary 3.1 depends on the design parameters of (22) and (23), this bound can be used as design metric in the sense that the design parameters can be judiciously selected to make (18) small.

Remark 3.1: Once again, we compare the proposed distributed input and state coestimation architecture of this section given by (22) and (23) with its counterpart in Tran, Yucelen, Sarsilmaz, et al. (2017). In particular, the related distributed estimation law of Tran, Yucelen, Sarsilmaz, et al. (2017) that allows time-varying active and passive roles have the form

$$\begin{aligned}\dot{\hat{x}}_i(t) &= (A - \gamma P_i^{-1})\hat{x}_i(t) + B\hat{w}_i(t) + g_i(t)L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - \alpha P_i^{-1}\sum_{i\sim j}(\hat{x}_i(t) - \hat{x}_j(t)), \quad \hat{x}_i(0) = \hat{x}_{i0},\end{aligned} \quad (36)$$

$$\begin{aligned}\dot{\hat{w}}_i(t) &= g_i(t)J_i(y_i(t) - C_i\hat{x}_i(t)) - (\sigma_iK_i + \gamma\mathrm{I}_p)\hat{w}_i(t) \\ &\quad - \alpha\sum_{i\sim j}(\hat{w}_i(t) - \hat{w}_j(t)), \quad \hat{w}_i(0) = \hat{w}_{i0},\end{aligned} \quad (37)$$

with $P_i \in \mathbb{R}^{(n+p)\times(n+p)}$ is a positive-definite gain matrix satisfying

$$R_{i1} \triangleq \begin{bmatrix} A^{\mathrm{T}}P_i + P_iA & -P_iB \\ -B^{\mathrm{T}}P_i & -2\sigma_iK_i \end{bmatrix} \leq 0, \quad (38)$$

$$R_{i2} \triangleq \begin{bmatrix} (A - L_iC_i)^{\mathrm{T}}P_i + P_i(A - L_iC_i) & -P_iB + C_i^{\mathrm{T}}J_i^{\mathrm{T}} \\ -B^{\mathrm{T}}P_i + J_iC_i & -2\sigma_iK_i \end{bmatrix} \leq 0. \quad (39)$$

Similar to discussion in Remark 2.1, (36) and (37) respectively contain the leakage terms '$-(\sigma_iK_i + \gamma\mathrm{I}_p)\hat{w}_i(t)$' in input and state updates. In particular, if the gains 'γP_i^{-1}' and '$\sigma_iK_i + \gamma\mathrm{I}_p$' are not small, then they can result in poor performance.

In contrast, the distributed coestimation architecture with time-varying active and passive node roles proposed this section has only one leakage term '$-\sigma_iK_i\hat{w}_i(t)$' appearing in the input update (23). Moreover, the proposed architecture of this section adds the coupling terms '$\hat{x}_i(t) - \hat{x}_j(t)$' and '$\hat{w}_i(t) - \hat{w}_j(t)$' to both input and state updates. Finally, the structure of the linear matrix inequalities given by (28) and (29) is simpler than the ones in (38) and (39). For these reasons, the proposed coestimation architecture here can be easily (i.e. better) tuned for an overall performance as opposed to the approach in Tran, Yucelen, Sarsilmaz, et al. (2017) (see also Section 4.3, Tran, Yucelen, Sarsilmaz, et al., 2017) for further details on the tuning challenges with regard to (36) and (37)).

Remark 3.2: The purpose of solving the linear matrix inequalities given by (28) and (29) is to find a common positive-definite solution P_i in order to select $H_i = P_i^{-1}$ according to (34). Note that the existence of such a common solution depends on many factors such as the characteristics of the system matrix A as well as the design gain matrices K_i, L_i, and J_i. To this end, the results in Bialas and Gora (2015), Ramos and Peres (2002), Narendra and Balakrishnan (1994), Shorten and Narendra (2003) and Mori, Mori, and Kuroe (1997) (also see references therein) can be utilised on the existence of a common positive-definite solution to linear matrix inequalities. Following the results in Ramos and Peres (2002), we can consider the matrix $\tilde{A}_{i,1}$ in (27) as a perturbation matrix and search for stability region, then design gain matrices accordingly and use convex programming tools to test the feasibility. In some special cases, where we can design $\bar{A}_{i,0}$ and $\bar{A}_{i,1}$ as commuting matrices or in companion form, the results of Narendra and Balakrishnan (1994) and Shorten and Narendra (2003), respectively, can be useful.

4. Distributed coestimation in a stochastic setting

4.1 Problem setup and proposed algorithm

In this section, we consider a stochastic case and generalise the results in Section 2. Note that this is without loss of generality as the theoretical content of this section can be similarly applied to the results in Section 3. Specifically, consider a process of interest with the dynamics given by

$$dx = (Ax(t) + Bw(t))dt + \nu(t)d\upsilon(t), \quad x(0) = x_0, \tag{40}$$

where $x(t) \in \mathbb{R}^n$ is a process internal state vector and $w(t) \in \mathbb{R}^p$ is an input to this process. Here, we consider that $x(t)$ is not measurable. We also consider that $w(t)$ is unknown but a bounded signal with a bounded time rate of change. In addition, $\nu(t) \in \mathbb{R}^n$ is a bounded external noise intensity function (i.e. $\|\nu(t)\|_2 \le \nu^*$) and $\upsilon(t)$ is a one dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with expectation $\mathbb{E}\{d\upsilon(t)\} = 0$ and variance $\mathbb{D}\{d\upsilon(t)\} = 1$. Furthermore, $A \in \mathbb{R}^{n\times n}$ is a Hurwitz system matrix and $B \in \mathbb{R}^{n\times p}$ is the system input matrix.

Consider a sensor network with N nodes exchanging information among each other using their local measurements through an undirected and connected graph $\mathcal{G}$. Following the terminology from previous sections, a node i, $i = 1, \dots, N$, is called an *active node* when it is subject to the observation of the process (40) given by

$$y_i(t) = C_i x(t) + h_i(t)\frac{ds_i(t)}{dt}. \tag{41}$$

Here, $y_i(t) \in \mathbb{R}^m$ and $C_i \in \mathbb{R}^{m\times n}$ respectively stand for a measurable process output and the system output matrix for an active node i, $i = 1, \dots, N$. In addition, $h_i(t) \in \mathbb{R}^m$ is a bounded external noise intensity function (i.e. $\|h_i(t)\|_2 \le h^*$), $s_i(t)$ is a one dimensional Brownian motion (independent of the process noise $\nu(t)$ and its neighbours $s_j(t)$) defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with expectation $\mathbb{E}\{ds_i(t)\} = 0$ and variance $\mathbb{D}\{ds_i(t)\} = 1$. Moreover, a node i is called a *passive node* when it has no observation from the process (40). Recall from (41) that each node can have nonidentical sensing modalities. We, once again, consider collective observability for well-posedness. Finally, we utilise here the distributed input and state architecture given by (3) and (4).

4.2 Analysis of proposed algorithm

To present our main results, we first define (5) and (6). Now, the stochastic differential of (5) is given by

$$\begin{aligned} d\tilde{x}_i(t) = \Bigg(& Ax(t) + Bw(t) - A\hat{x}_i(t) - B\hat{w}_i(t) - g_i L_i (y_i(t) \\ & - C_i\hat{x}_i(t)) + \alpha M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) \\ & - \alpha S_i \sum_{j=1}^{N} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t)) \Bigg) dt + \nu(t)d\upsilon(t) \\ = \Bigg(& (A - g_i L_i C_i)\tilde{x}_i(t) - B\tilde{w}_i(t) - \alpha M_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{x}_j(t) \\ & - \alpha S_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{w}_j(t) \Bigg) dt \\ & + \nu(t)d\upsilon(t) - g_i L_i h_i(t) ds_i(t). \end{aligned} \tag{42}$$

In addition, the stochastic differential of (6) is given by

$$\begin{aligned} d\tilde{w}(t) = \Bigg(& g_i J_i C_i \tilde{x}_i(t) - \sigma_i K_i \tilde{w}_i(t) - \alpha T_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{x}_j(t) \\ & - \alpha N_i \sum_{j=1}^{N} \mathcal{L}_{ij}\tilde{w}_j(t) - \sigma_i K_i w(t) - \dot{w}(t) \Bigg) dt \\ & + g_i J_i h_i(t) ds_i(t). \end{aligned} \tag{43}$$

Next, let $z_i(t) \triangleq [\tilde{x}_i^{\mathrm{T}}(t), \tilde{w}_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n+p}$. Now, (42) and (43) can be written in a compact form as

$$\begin{aligned} dz_i(t) = \Bigg(& \underbrace{\begin{bmatrix} A - g_i L_i C_i & -B \\ g_i J_i C_i & -\sigma_i K_i \end{bmatrix}}_{\bar{A}_i} z_i(t) \\ & - \alpha \sum_{j=1}^{N} \mathcal{L}_{ij} \underbrace{\begin{bmatrix} M_i & S_i \\ T_i & N_i \end{bmatrix}}_{H_i} z_j(t) \end{aligned}$$

$$+\underbrace{\begin{bmatrix}0\\-\sigma_i K_i w(t)-\dot{w}(t)\end{bmatrix}}_{\phi_i(t)}\Bigg)\mathrm{d}t$$

$$+\underbrace{\begin{bmatrix}\nu(t)\\0\end{bmatrix}}_{\bar{\nu}}\mathrm{d}\nu(t)+\underbrace{\begin{bmatrix}-g_i L_i h_i(t)\\g_i J_i h_i(t)\end{bmatrix}}_{r_i(t)}\mathrm{d}s_i(t)$$

$$=\left(\bar{A}_i z_i(t)-\alpha\sum_{j=1}^{N}\mathcal{L}_{ij}H_i z_j(t)+\phi_i(t)\right)\mathrm{d}t$$

$$+\bar{\nu}(t)\mathrm{d}\nu(t)+r_i(t)\mathrm{d}s_i(t). \tag{44}$$

Similar to the discussion in Section 2.2, one can always choose the local design terms L_i, J_i, K_i, and σ_i such that $\bar{A}_i$ is Hurwitz for each agent, and hence, there exists a unique positive-definite matrix $P_i\in\mathbb{R}^{(n+p)\times(n+p)}$ such that (10) holds for a given positive-definite matrix $Q_i\in\mathbb{R}^{(n+p)\times(n+p)}$.

Now, let the aggregated vector be given by $z(t)\triangleq[z_1^{\mathrm{T}}(t),z_2^{\mathrm{T}}(t),\ldots,z_N^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{(n+p)N}$. To this end, (44) can be further written as

$$\mathrm{d}z(t)=\left(\underbrace{\begin{bmatrix}\bar{A}_1 & & 0\\ & \ddots & \\ 0 & & \bar{A}_N\end{bmatrix}}_{\bar{A}}z(t)-\alpha\underbrace{\begin{bmatrix}H_1 & & 0\\ & \ddots & \\ 0 & & H_N\end{bmatrix}}_{H}\right.$$

$$\cdot\underbrace{\begin{bmatrix}\mathcal{L}_{11}\mathrm{I}_{n+p} & \mathcal{L}_{12}\mathrm{I}_{n+p} & \cdots & \mathcal{L}_{1N}\mathrm{I}_{n+p}\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{L}_{N1}\mathrm{I}_{n+p} & \mathcal{L}_{N2}\mathrm{I}_{n+p} & \cdots & \mathcal{L}_{NN}\mathrm{I}_{n+p}\end{bmatrix}}_{(\mathcal{L}(\mathcal{G})\otimes\mathrm{I}_{n+p})}$$

$$\left. z(t)+\phi(t)\right)\mathrm{d}t$$

$$+(\mathbf{1}_N\otimes\bar{\nu}(t))\mathrm{d}\nu(t)+R(t)(\mathrm{d}s(t)\otimes\mathbf{1}_{n+p})$$

$$=(\bar{A}z(t)-\alpha H(\mathcal{L}(\mathcal{G})\otimes\mathrm{I}_{n+p})z(t)+\phi(t))\mathrm{d}t$$

$$+(\mathbf{1}_N\otimes\bar{\nu}(t))\mathrm{d}\nu(t)+R(t)(\mathrm{d}s(t)\otimes\mathbf{1}_{n+p}), \tag{45}$$

where $\phi(t)\triangleq[\phi_1^{\mathrm{T}}(t),\phi_2^{\mathrm{T}}(t),\ldots,\phi_N^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{(n+p)N}$, $R(t)\triangleq\mathrm{diag}([r_1^{\mathrm{T}}(t)\ \cdots\ r_N^{\mathrm{T}}(t)]^{\mathrm{T}})=\mathrm{diag}(r(t))$, and $\mathrm{d}s(t)\triangleq[\mathrm{d}s_1(t)\ \ldots\ \mathrm{d}s_N(t)]^{\mathrm{T}}$. The following proposition presents the main result of this section.

Proposition 4.1: *Consider the process given by* (40) *and the distributed input and state estimation architecture given by* (3) *and* (4). *If the matrix H_i is selected as $H_i=P_i^{-1}$ and nodes exchange information using local measurements subject to an undirected and connected graph $\mathcal{G}$, then $z(t)$ evolving according to the dynamics given by* (45) *satisfies the bound*

$$\mathbb{E}\{z^{\mathrm{T}}Pz\}\leq \mathrm{e}^{-\kappa t}z^{\mathrm{T}}(0)Pz(0)+\kappa^{-1}\eta,\quad \forall\, t\geq 0, \tag{46}$$

where $\kappa\triangleq\frac{\lambda_{\min}(\bar{Q})}{2\lambda_{\max}(P)}$ and $\eta\triangleq\frac{2\bar{\phi}^2}{\lambda_{\min}(\bar{Q})}\|P\|_2^2+\lambda_{\max}(P)\bar{\psi}^2$.

Proof: Applying Ito formula (see, for example, Ladde & Ladde, 2013) to the Lyapunov-like function candidate $V(z)=\sum_{i=1}^{N}z_i^{\mathrm{T}}P_i z_i$, one can write

$$\mathrm{d}V=LV\mathrm{d}t+2\sum_{i=1}^{N}z_i^{\mathrm{T}}P_i(\bar{\nu}\mathrm{d}\nu(t)+r_i\mathrm{d}s_i(t)), \tag{47}$$

where L is a linear differential operator associated with (44)

$$LV=\sum_{i=1}^{N}\left(z_i^{\mathrm{T}}(P_i\bar{A}_i+P_i\bar{A}_i^{\mathrm{T}})z_i-2\alpha z_i^{\mathrm{T}}P_i\sum_{j=1}^{N}\mathcal{L}_{ij}H_i z_j\right.$$

$$\left.+2z_i^{\mathrm{T}}P_i\phi_i+\frac{1}{2}\mathrm{tr}(2P_i\bar{\nu}\bar{\nu}^{\mathrm{T}}+2P_i r_i r_i^{\mathrm{T}})\right)$$

$$=\sum_{i=1}^{N}\left(-z_i^{\mathrm{T}}Q_i z_i-2\alpha z_i^{\mathrm{T}}\sum_{j=1}^{N}\mathcal{L}_{ij}z_j\right.$$

$$\left.+2z_i^{\mathrm{T}}P_i\phi_i+\bar{\nu}^{\mathrm{T}}P_i\bar{\nu}+r_i^{\mathrm{T}}P_i r_i\right)$$

$$=-\sum_{i=1}^{N}z_i^{\mathrm{T}}Q_i z_i-2\alpha\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{L}_{ij}z_i^{\mathrm{T}}z_j+2\sum_{i=1}^{N}z_i^{\mathrm{T}}P_i\phi_i$$

$$+\sum_{i=1}^{N}\psi_i^{\mathrm{T}}(\mathrm{I}_2\otimes P_i)\psi_i, \tag{48}$$

where $\psi_i\triangleq[\bar{\nu}^{\mathrm{T}},r_i^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{2(n+p)}$. In addition, (48) can be written in a compact form as

$$LV=-z^{\mathrm{T}}Qz-2\alpha z^{\mathrm{T}}(\mathcal{L}(\mathcal{G})\otimes\mathrm{I}_{n+p})z+2z^{\mathrm{T}}P\phi+\psi^{\mathrm{T}}\bar{P}\psi$$

$$=-z^{\mathrm{T}}\bar{Q}z+2z^{\mathrm{T}}P\phi+\psi^{\mathrm{T}}\bar{P}\psi, \tag{49}$$

where $\psi\triangleq[\psi_1^{\mathrm{T}},\psi_2^{\mathrm{T}},\ldots,\psi_N^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{2(n+p)N}$; $P\triangleq\mathrm{diag}([P_1,P_2,\ldots,P_N])\in\mathbb{R}^{N(n+p)\times N(n+p)}$, $\bar{P}\triangleq\mathrm{diag}([\mathrm{I}_2\otimes P_1,\mathrm{I}_2\otimes P_2,\ldots,\mathrm{I}_2\otimes P_N])\in\mathbb{R}^{2N(n+p)\times 2N(n+p)}$, and $Q\triangleq\mathrm{diag}([Q_1,Q_2,\ldots,Q_N])\in\mathbb{R}^{N(n+p)\times N(n+p)}$ are positive-definite matrices; and $\bar{Q}\triangleq Q+2\alpha(\mathcal{L}(\mathcal{G})\otimes\mathrm{I}_{n+p})$. Since Q is a positive-definite matrix and the Laplacian matrix $\mathcal{L}(\mathcal{G})$ is a positive-semidefinite matrix, $\bar{Q}$ is a positive-definite matrix (Proposition 8.1.2, Bernstein, 2009). In addition, $\|\phi\|_2\leq\bar{\phi}$ with $\bar{\phi}$ defined in (16). Furthermore, since $\|\nu\|_2\leq\nu^*$ and $\|h_i\|_2\leq h^*$, $\bar{\nu}$ and r_i are bounded (i.e. $\|\bar{\nu}_i\|_2\leq\bar{\nu}^*$ and $\|r_i\|_2\leq r^*$). As a result, ψ_i is bounded (i.e. $\|\psi_i\|_2\leq\bar{\psi}_i$) with $\bar{\psi}_i\triangleq\sqrt{(\bar{\nu}^*)^2+(r^*)^2}$, $i=1,\ldots,N$. Therefore, we have $\|\psi\|_2\leq\bar{\psi}$ with $\bar{\psi}\triangleq\sqrt{\psi_1^2+\psi_2^2+\cdots+\psi_N^2}$.

By Young's inequality, one can now write $2\|z^{\mathrm{T}}\|_2\|P\phi\|_2 \leq \mu\|z\|_2^2 + \frac{1}{\mu}\|P\phi\|_2^2 \leq \mu\|z\|_2^2 + \frac{\bar{\phi}^2}{\mu}\|P\|_2^2$. Now, (49) becomes

$$LV \leq -(\lambda_{\min}(\bar{Q}) - \mu)\|z\|_2^2 + \frac{\bar{\phi}^2}{\mu}\|P\|_2^2 + \lambda_{\max}(P)\bar{\psi}^2. \quad (50)$$

From $V(z) = \sum_{i=1}^{N} z_i^{\mathrm{T}} P_i z_i = z^{\mathrm{T}} P z$, we also note that $\lambda_{\min}(P)\|z\|_2^2 \leq V(z) \leq \lambda_{\max}(P)\|z\|_2^2$. Thus, $\frac{V(z)}{\lambda_{\max}(P)} \leq \|z\|_2^2$. Using this fact and choosing $\mu = \frac{1}{2}(\lambda_{\min}(\bar{Q})$, we now have

$$\begin{aligned} LV &\leq -\underbrace{\frac{\lambda_{\min}(\bar{Q})}{2\lambda_{\max}(P)}}_{\kappa} V + \underbrace{\frac{2\bar{\phi}^2}{\lambda_{\min}(\bar{Q})}\|P\|_2^2 + \lambda_{\max}(P)\bar{\psi}^2}_{\eta} \\ &= -\kappa V + \eta. \end{aligned} \quad (51)$$

From Dynkin's formula (see, for example, Deng, Krstic, & Williams, 2001; Khasminskii, 2011), (46) is now immediate. ■

Remark 4.1: Proposition 4.1 also holds for the distributed input and state coestimation architecture given by (22) and (23), where the active and passive node roles are varying over time and the corresponding parameters are chosen as outlined in Section 3.

Remark 4.2: Considering (46), one can write $\lim_{t\to\infty} \mathbb{E}\{V(z(t))\} = \kappa^{-1}\eta$. Now, using the definitions of κ and η given in (51) along with the fact $\lambda_{\min}(P)\mathbb{E}\{\|z\|_2^2\} \leq \mathbb{E}\{V(z(t))\}$, this expression implies

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}\{\|z\|_2\} \\ &\leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(\mu^2 + \frac{2\lambda_{\max}(P)\bar{\psi}^2}{\lambda_{\min}(\bar{Q})})}, \quad \mu \triangleq \frac{2\|P\|_2\bar{\phi}}{\lambda_{\min(\bar{Q})}}. \end{aligned} \quad (52)$$

When we compare (52) with the deterministic (worst-case) bound given by (18), it can be seen that the only additional term '$2\lambda_{\max}(P)\bar{\psi}^2/(\lambda_{\min}(P)\lambda_{\min}(\bar{Q}))$' in (52) results from the bound of the external noise intensity functions of the process and sensors' measurements $\bar{\psi}$.

Remark 4.3: The error expression given by (45) can be rewritten as

$$\begin{aligned} \mathrm{d}z(t) &= (\mathcal{S}z(t) + \phi(t))\mathrm{d}t + (\mathbf{1}_N \otimes \bar{v}(t))\mathrm{d}\nu(t) \\ &\quad + R(t)(\mathrm{d}s(t) \otimes \mathbf{1}_{n+p}), \end{aligned} \quad (53)$$

where $\mathcal{S} \triangleq \bar{A} - \alpha H(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})$. Referring now to the steps taken in the proof of Theorem 2.1, one can write

$$\begin{aligned} \mathcal{S}^{\mathrm{T}}P + P\mathcal{S} &= (\bar{A} - \alpha H(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p}))^{\mathrm{T}}P \\ &\quad + P(\bar{A} - \alpha H(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})) \\ &= \underbrace{\bar{A}^{\mathrm{T}}P + P\bar{A}}_{-Q} - 2\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p}) \\ &= -(Q + 2\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_{n+p})) \triangleq -\bar{Q}. \end{aligned} \quad (54)$$

Hence, $\mathcal{S}$ is Hurwitz since both $\bar{Q}$ and P are positive-definite (Corollary 11.9.1, Bernstein, 2009). We note that a white noise process is the derivative of a Wiener process and its derivative is a generalised function (see, for example, Observation 1.2.11, Arnold, 1974; Ladde & Ladde, 2013). Recall that $\nu(t)$ and $s_i(t)$ are Brownian motion processes or normalised Wiener processes; thus, we can define the zero mean white noise processes $\zeta(t) \triangleq \frac{\mathrm{d}\nu(t)}{\mathrm{d}t} \in \mathbb{R}$ and $\xi(t) \triangleq \frac{\mathrm{d}s(t)\otimes\mathbf{1}_{n+p}}{\mathrm{d}t} \in \mathbb{R}^{N(n+p)}$. As a result, (53) can be rewritten as

$$\dot{z}(t) = \mathcal{S}z(t) + \phi(t) + (\mathbf{1}_N \otimes \bar{v}(t))\zeta(t) + R(t)\xi(t), \quad (55)$$

and its solution is given by

$$\begin{aligned} z(t) &= \mathrm{e}^{\mathcal{S}t}z(0) + \int_0^t \mathrm{e}^{\mathcal{S}(t-s)}\phi(s)\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{\mathcal{S}(t-s)}(\mathbf{1}_N \otimes \bar{v}(s))\zeta(s)\mathrm{d}s + \int_0^t \mathrm{e}^{\mathcal{S}(t-s)}R(s)\xi(s)\mathrm{d}s. \end{aligned} \quad (56)$$

Note that the error covariance of the system is given by $J(z(t)) \triangleq \mathbb{E}\{z(t)z^{\mathrm{T}}(t)\}$ and its differential equation is

$$\dot{J}(t) = \mathbb{E}\left\{\frac{\mathrm{d}z(t)}{\mathrm{d}t}z^{\mathrm{T}}(t)\right\} + \mathbb{E}\left\{z(t)\frac{\mathrm{d}z^{\mathrm{T}}(t)}{\mathrm{d}t}\right\}. \quad (57)$$

The first term of (57) can be distributed as

$$\begin{aligned} &\mathbb{E}\left\{\frac{\mathrm{d}z(t)}{\mathrm{d}t}z^{\mathrm{T}}(t)\right\} \\ &= \mathbb{E}\left\{(\mathcal{S}z(t) + \phi(t) + (\mathbf{1}_N \otimes \bar{v}(t))\zeta(t) + R(t)\xi(t))z^{\mathrm{T}}(t)\right\} \\ &= \mathcal{S}\mathbb{E}\{z(t)z^{\mathrm{T}}(t)\} + \mathbb{E}\{\phi(t)z^{\mathrm{T}}(t)\} \\ &\quad + \mathbb{E}\{(\mathbf{1}_N \otimes \bar{v}(t))\zeta(t)z^{\mathrm{T}}(t)\} + \mathbb{E}\{R(t)\xi(t)z^{\mathrm{T}}(t)\}. \end{aligned} \quad (58)$$

We note here that $(\mathbf{1}_N \otimes \bar{v}(t))\zeta(t)$ represents the process noise, $R(t)\xi(t)$ represents the measurement noise, $\phi(t)$ represents the leakage term containing the process input and its time rate of change, and $z(0)$ is the initial error of state and input estimation. Since these terms are unrelated, one can assume that they are mutually orthogonal. Using the solution given by (56) with the above assumption, one can calculate the cross-correlation matrices[5] $R_{\zeta z}(t,t) \triangleq \mathbb{E}\{(\mathbf{1}_N \otimes \bar{v}(t))\zeta(t)z^{\mathrm{T}}(t)\} = \frac{1}{2}(\mathbf{1}_N \otimes \bar{v}(t))(\mathbf{1}_N \otimes \bar{v}(t))^{\mathrm{T}}$. By substituting these matrices into (58), one can obtain

$$\begin{aligned} &\mathbb{E}\left\{\frac{\mathrm{d}z(t)}{\mathrm{d}t}z^{\mathrm{T}}(t)\right\} \\ &= \mathcal{S}J(t) + \phi(t)\mathbb{E}\{z^{\mathrm{T}}(t)\} \\ &\quad + \frac{1}{2}(\mathbf{1}_N \otimes \bar{v}(t))(\mathbf{1}_N \otimes \bar{v}(t))^{\mathrm{T}} + \frac{1}{2}R(t)R^{\mathrm{T}}(t). \end{aligned} \quad (59)$$

Notice that the second term of (57) is just the transpose of the first term; hence, from (59), (57) is equivalent to

$$\begin{aligned} \dot{J}(t) &= \mathcal{S}J(t) + J(t)\mathcal{S}^{\mathrm{T}} + (\phi(t)\mathbb{E}\{z^{\mathrm{T}}(t)\} + \mathbb{E}\{z(t)\}\phi^{\mathrm{T}}(t)) \\ &\quad + (\mathbf{1}_N \otimes \bar{v}(t))(\mathbf{1}_N \otimes \bar{v}(t))^{\mathrm{T}} + R(t)R^{\mathrm{T}}(t). \end{aligned} \quad (60)$$

From a practical standpoint, the overall coestimation performance gets better as the error covariance $J(t)$ gets smaller. Note that $J(t)$ can be obtained by solving (60) with the initial condition $J(0) = J_0$, where J_0 is the covariance of $z(0)$ or the uncertainty of the initial coestimate. However, solving (60) and determining the optimal design parameters would require global information. Here, we can observe from the last three terms on the right hand side of (60) that the error covariance $J(t)$ depends on the input of the process, its time rate of change, σ_i and K_i, $i = 1, \ldots, N$, via $\phi(t)$, the intensity of the process noise via $\bar{v}(t)$, and the intensity of the sensors' measurement noises as well as L_i and J_i, $i = 1, \ldots, N$, via $R(t)$. In addition, a necessary condition for $\mathcal{S}$ to be Hurwitz is that $\bar{A}$ or $\bar{A}_i$, $i = 1, \ldots, N$, is Hurwitz. Therefore, we can influence the coestimation performance without utilising global information by choosing σ_i, K_i, L_i, J_i such that $\bar{A}_i$ is Hurwitz for each agent as mentioned in Section 2.2 and the bound (52) is small, simultaneously.

5. Illustrative numerical examples

We now present two numerical examples to illustrate the proposed distributed input and state coestimation methodology. Specifically, the first example shows the behaviour of the sensor network in the absence of noise, while the second example shows the behaviour of the sensor network when the dynamics of the process of interest is a stochastic process and sensors contain noise.

Example 5.1: From Tran et al. (2017a, 2017b) and Tran, Yucelen, Sarsilmaz, et al. (2017), a process representing a linear target motion satisfying the dynamics given by (1) is considered here with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_{n1}^2 & -2\omega_{n1}\xi_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{n2}^2 & -2\omega_{n2}\xi_2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ \omega_{n1}^2 & 0 \\ 0 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix}, \tag{61}$$

where $\omega_{n1} = 1.2$, $\xi_1 = 0.9$, $\omega_{n2} = 1.3$, and $\xi_2 = 0.5$. This process composes of two decoupled systems in which the first and third states represent the target's positions in x and y directions, while the second and fourth states represent the target's velocities in x and y directions. The input of the process and the initial conditions of the states are respectively set to $w(t) = [2.5\sin(0.3t) + 1.5,\ \ 1.5\cos(0.5t)]^{\mathrm{T}}$ and $x_0 = [-2.5,\ 0.5,\ 2.5,\ 0.25]^{\mathrm{T}}$.

A sensor network with 12 nodes exchanging information according to an undirected and connected graph is utilised and arranged spatially as shown in Figure 2 for this example. Here, we consider the active and passive node roles are varying over time. Particularly, a sensor's sensing range is defined as a circle with radius $r = 2.5$ centred at each node. When the target (position) enters a sensor's sensing range, the sensor (smoothly) becomes active. Conversely, when the target leaves a sensor's sensing range, it (smoothly) becomes passive.

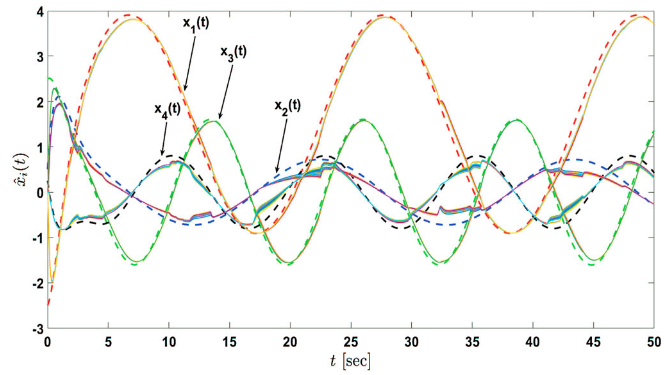

Figure 3. The time evolution of $\hat{x}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the proposed distributed 'coestimation' architecture given by (22) and (23) (the dashed lines denote the states of the actual process and the solid lines denote the state estimates of nodes).

Once again, the transition for $g_i(t)$ is adopted from (Figure 2(d), Zavlanos et al., 2011) with $g_i(t) = \mathrm{e}^{-\beta t}$ when node i is switching from 1 to 0, and $g_i(t) = 1 - \mathrm{e}^{-\beta t}$ when node i is switching from 0 to 1, where $\beta \in \mathbb{R}_+$.

The sensing capability of each node is given by (2) with $C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ for the odd index nodes and $C_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for the even index nodes. Similarly, σ_i is respectively set to 0.01 and 0.001 for odd and even index nodes. The pair (A, C_i) is observable for all $i = 1, \ldots, 12$ in this example; hence, collective observability holds. Furthermore, all nodes' estimations are set to zero initial conditions and gain matrices are chosen such that $J_i = \mathrm{diag}([20; 20])$, $K_i = \mathrm{diag}([10; 10])$ for $i = 1, \ldots, N$. The odd index nodes are subject to $L_i = \begin{bmatrix} 20.13 & 1.33 & 0.00 & 0.00 \\ 0.00 & 0.00 & 20.32 & 3.19 \end{bmatrix}^{\mathrm{T}}$, while the even index nodes are subject to $L_i = \begin{bmatrix} -40.17 & 57.54 & 4.53 & -6.45 \\ 4.14 & -5.88 & -40.20 & 60.42 \end{bmatrix}^{\mathrm{T}}$. For all nodes, we set $\alpha = 25$. In addition, we obtain the common P_i by solving the linear matrix inequalities (28) and (29) with $\epsilon = 0.000001$ for the odd nodes that results in

$$P_1 = \begin{bmatrix} 0.937 & 0.211 & 0.000 & 0.000 & 0.907 & 0.000 \\ 0.211 & 0.333 & 0.000 & 0.000 & 0.191 & 0.000 \\ 0.000 & 0.000 & 0.928 & 0.184 & 0.000 & 0.905 \\ 0.000 & 0.000 & 0.184 & 0.361 & 0.000 & 0.184 \\ 0.907 & 0.191 & 0.000 & 0.000 & 0.986 & 0.000 \\ 0.000 & 0.000 & 0.905 & 0.184 & 0.000 & 1.010 \end{bmatrix}, \tag{62}$$

and $\epsilon = 0.0001$ for the even nodes that results in

$$P_2 = \begin{bmatrix} 1.907 & 1.744 & -0.034 & 0.011 & 1.895 & -0.035 \\ 1.744 & 2.118 & -0.031 & 0.044 & 1.773 & -0.033 \\ -0.034 & -0.031 & 0.862 & 0.649 & -0.036 & 0.856 \\ 0.011 & 0.044 & 0.649 & 1.106 & 0.009 & 0.680 \\ 1.895 & 1.773 & -0.036 & 0.009 & 2.018 & -0.052 \\ -0.035 & -0.033 & 0.856 & 0.680 & -0.052 & 0.980 \end{bmatrix}. \tag{63}$$

That is, $P_1 = P_3 = P_5 = P_7 = P_9 = P_{11}$ and $P_2 = P_4 = P_6 = P_8 = P_{10} = P_{12}$. Based on the matrix P_i, $i = 1, 2, \ldots, 12$, we obtain H_i from (34) and the matrices M_i, S_i, T_i, and N_i are selected accordingly.

For the proposed distributed input and state 'coestimation' architecture given by (22) and (23), the process states and inputs are closely estimated as shown in Figures 3 and 4. Specifically, it

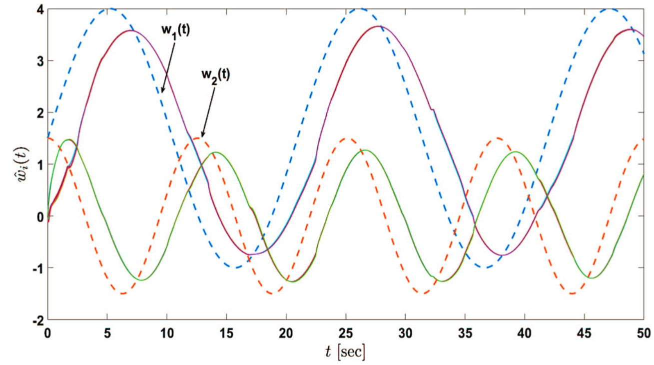

Figure 4. The time evolution of $\hat{w}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the proposed distributed 'coestimation' architecture given by (22) and (23) (the dashed lines denote the inputs of the actual process and the solid lines denote the input estimates of nodes).

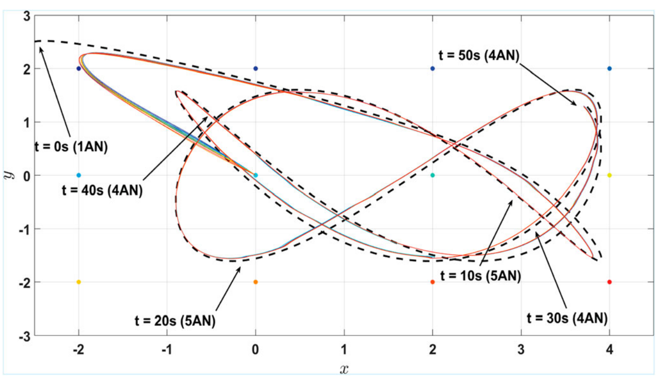

Figure 5. Position estimates (first and third states of the process) of the considered time-varying heterogeneous sensor network under the proposed distributed 'coestimation' architecture given by (22) and (23) (the dashed line denotes the trajectory of the actual process (i.e. the combination of the first and third state) and the solid lines denote the state estimates of nodes). Here, AN stands for the the active nodes.

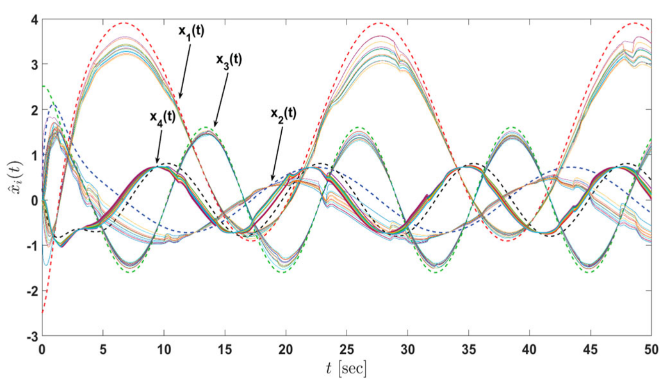

Figure 6. The time evolution of $\hat{x}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the recent distributed 'estimation' architecture in Tran, Yucelen, Sarsilmaz, et al. (2017) given by (36) and (37) (the dashed lines denote the states of the actual process and the solid lines denote the state estimates of nodes).

is illustrated in Figure 5 that the sensor network is able to estimate the trajectory of the target (the first and third states of the process). We note that the lag of the input estimate in Figure 4 can be reduced by increasing J_i. However, J_i may not be trivially selected as large while satisfying the linear matrix inequalities (28) and (29) simultaneously. In addition, the numerical

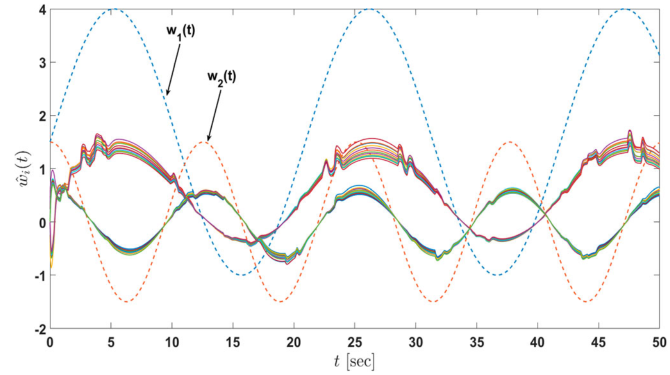

Figure 7. The time evolution of $\hat{w}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the recent distributed 'estimation' architecture in Tran, Yucelen, Sarsilmaz, et al. (2017) given by (36) and (37) (the dashed lines denote the inputs of the actual process and the solid lines denote the input estimates of nodes).

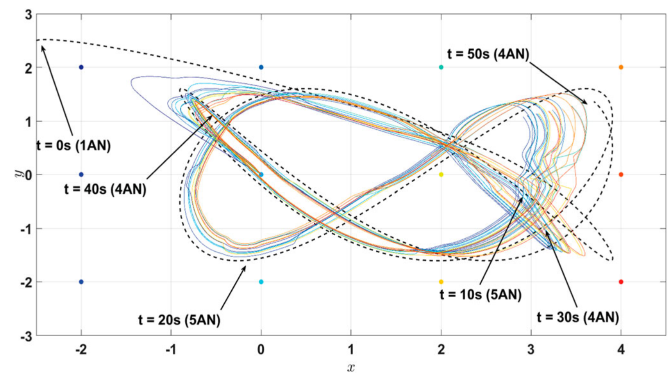

Figure 8. Position estimates (first and third states of the process) of the considered time-varying heterogeneous sensor network under the recent distributed 'estimation' architecture in Tran, Yucelen, Sarsilmaz, et al. (2017) given by (36) and (37) (the dashed line denotes the trajectory of the actual process (i.e. the combination of the first and third state) and the solid lines denote the state estimates of nodes). Here, AN stands for the the active nodes.

results in Figures 6–8 utilising the architecture in Tran, Yucelen, Sarsilmaz, et al. (2017) (i.e. utilising the distributed input and state 'estimation' law given by (36) and (37)) are included here for comparison purposes. These results are generated under the same scenario outlined above including the dynamics of the process, communication graph of nodes, and sensors' modalities. Figures 3–5 clearly highlight the substantially improved dynamic input and state fusion performance of the proposed distributed 'coestimation' architecture of this paper over the the distributed 'estimation' approach in Tran, Yucelen, Sarsilmaz, et al. (2017), which is depicted by Figures 6–8.

Example 5.2: In this example, we consider the same setup presented in Example 5.1 with the process subject to the stochastic dynamics given by (40), where the noise intensity function $\nu(t)$ is a constant vector with elements assigned randomly within the range $(0, 0.2)$. In addition, the observation of each sensor is given by (41) and the noise intensity function $h_i(t)$ is considered as a constant vector with elements assigned randomly within the range $(0, 0.5)$. All other design parameters are chosen as in Example 5.1.

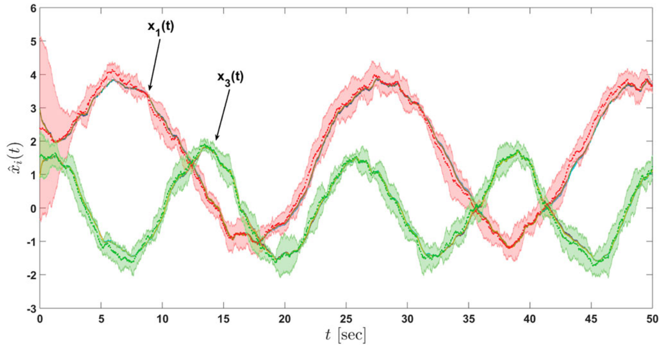

Figure 9. The time evolution of the first and third states of $\hat{x}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the proposed distributed 'coestimation' architecture given by (22) and (23) (the dashed lines denote the states of the actual process, the shaded areas denote the boundary of the process' states when repeatedly run five times, and the solid lines denote the state estimates of nodes).

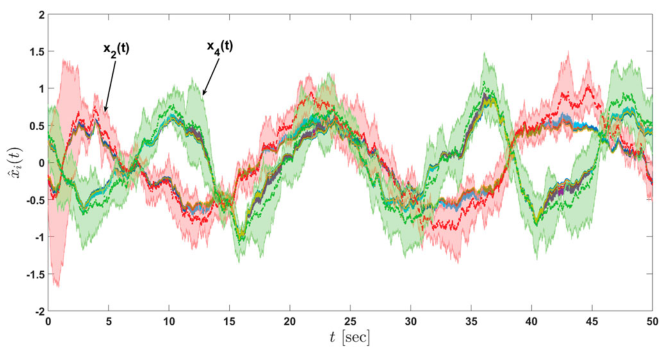

Figure 10. The time evolution of the second and the fourth states of $\hat{x}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the proposed distributed 'coestimation' architecture given by (22) and (23) (the dashed lines denote the states of the actual process, the shaded areas denote the boundary of the process' states when repeatedly run five times, and of the actual process and the solid lines denote the state estimates of nodes).

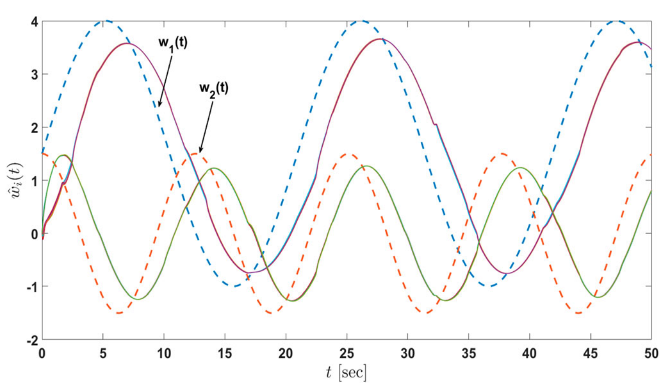

Figure 11. The time evolution of $\hat{w}_i(t)$, $i = 1, \ldots, N$, of the considered time-varying heterogeneous sensor network under the proposed distributed 'coestimation' architecture given by (22) and (23) (the dashed lines denote the inputs of the actual process and the solid lines denote the input estimates of nodes).

The simulations are run repeatedly five times to investigate the effect of the noises on the process and the performance of the sensor network under the proposed algorithm. For the proposed distributed input and state 'coestimation' architecture given by (22) and (23) (see Remark 4.1), sensor network nodes are able to closely estimate the process states and inputs as shown in Figures 9–11, respectively. Specifically, Figure 9 shows the estimates of the first and third states of the process, while Figure 10 shows the estimates of the second and fourth ones with the shaded areas being the boundaries of the process when the simulation repeated five times. Since estimates of the second and fourth states directly influenced by the input estimates, the noise intensities in these signals are greater comparing to estimates of the first and third states.

6. Conclusion

Considering an important practical class of *heterogeneous sensor networks* with both *nonidentical node information roles* and *nonidentical node modalities*, a new dynamic information fusion framework was documented in this paper. The proposed framework involved a *distributed input and state coestimation* algorithm for each node such that the time evolution of input and state updates *both* depend on the *local* input and state information exchanges. We first considered *fixed* active and passive node roles subject to nonidentical active node modalities (Section 2) and then provided generalisations to the case of *time-varying* active and passive node roles (Section 3) as well as to the stochastic case involving noise in the process and the node observations (Section 4). Furthermore, we analytically proved all the presented results using tools and methods from Lyapunov theory and linear matrix inequalities, where *local sufficient conditions* were also given for each node. Finally, illustrative numerical examples demonstrated that nodes executing the proposed coestimation architecture can closely estimate both states and inputs of the considered process of interest.

The results presented in this paper can be useful for several future research directions for dynamic data-driven applications including but not limited to: (a) From a practical standpoint, the system matrix and the input matrix for the considered process in this paper can be extended to the case when they are both subject to system uncertainties. (b) From another practical standpoint, one can consider processes involving nonlinear functions for further generalising the presented results of this paper. (c) Tools and methods from recent event-triggered control theory developments can be utilised in order to reduce the communication cost between sensor nodes. (d) For capturing cases when the process of interest leaves the sensing field of the network for certain time periods, tools and methods from filtering and estimation theories can be utilised with the presented results of this paper. (e) The presented setup can be extended to involve mobile sensor nodes. (f) The presented results can be applied to real-world sensor networks through experiments in order to bridge the gap between theory and practice.

Notes

1. We follow here the problem setup introduced in Tran et al. (2017b).
2. Collective observability is defined as the pair (A, C) is observable, where $C = [C_1^{\mathrm{T}}, C_2^{\mathrm{T}}, \ldots, C_N^{\mathrm{T}}]^{\mathrm{T}}$ (see, for example, Millán et al., 2013; Olfati-Saber, 2007; Tran, Yucelen, Sarsilmaz, et al., 2017).
3. As mentioned earlier, A is considered to be Hurwitz because of the existence of passive nodes in the sensor network. Hence, this argument follows from the upper diagonal structure of $\bar{A}_i$ when, for example, $L_i = 0_{n\times m}$, $J_i = 0_{p\times m}$, and K_i is any positive definite matrix with $\sigma_i > 0$. For a desirable performance, however, different values for

L_i, J_i and K_i should be judiciously selected such that $\bar{A}_i$ is Hurwitz (see Remark 2.2).
4. We refer to the steps *(i)* and *(ii)* given later in Section 3.2 on the selection of design gain matrices.
5. We refer to (Chapter 9.4, Lewis, Vrabie, & Syrmos, 2012) for a similar analysis.

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References

Arnold, L. (1974). *Stochastic differential equations*. New York: John Wiley & Sons.
Bai, H., Freeman, R. A., & Lynch, K. M. (2010). *Robust dynamic average consensus of time-varying inputs*. Conference on decision and control, Atlanta, GA, USA (pp. 3104–3109).
Bernstein, D. S. (2009). *Matrix mathematics: Theory, facts, and formulas*. Princeton, NJ: Princeton University Press.
Bialas, S., & Gora, M. (2015). On the existence of a common solution to the Lyapunov equations. *Bulletin of the Polish Academy of Sciences Technical Sciences*, *63*(1), 163–168.
Cao, Y., Casbeer, D., Garcia, E., & Zhang, X. (2016). *Towards cost-effective distributed information fusion with partially active sensors in directed networks*. IEEE conference on decision and control, Las Vegas, NV, USA (pp. 4308–4313).
Casbeer, D. W., Cao, Y., Garcia, E., & Milutinović, D. (2015). Average bridge consensus: Dealing with active-passive sensors. *ASME dynamic systems and control conference* (pp. V003T37A005–V003T37A005).
Chen, F., Cao, Y., & Ren, W. (2012). Distributed average tracking of multiple time-varying reference signals with bounded derivatives. *IEEE Transactions on Automatic Control*, *57*(12), 3169–3174.
Cunningham, A., Indelman, V., & Dellaert, F. (2013, May). *DDF-SAM 2.0: Consistent distributed smoothing and mapping*. 2013 IEEE international conference on robotics and automation, Karlsruhe, Germany (pp. 5220–5227). IEEE.
DeLellis, P., di Bernardo, M., & Russo, G. (2010). On quad, lipschitz, and contracting vector fields for consensus and synchronization of networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, *58*(3), 576–583.
Demetriou, M. A. (2009). Natural consensus filters for second order infinite dimensional systems. *Systems & Control Letters*, *58*(12), 826–833.
Deng, H., Krstic, M., & Williams, R. J. (2001). Stabilization of stochastic nonlinear systems driven by noise of unknown covariance. *IEEE Transactions on Automatic Control*, *46*(8), 1237–1253.
Freeman, R. A., Yang, P., & Lynch, K. M. (2006). *Stability and convergence properties of dynamic average consensus estimators*. IEEE conference on decision and control, San Diego, CA, USA.
Godsil, C. D., Royle, G., & Godsil, C. (2001). *Algebraic graph theory* (Vol. 207). New York: Springer.
Hollinger, G. A., Yerramalli, S., Singh, S., Mitra, U., & Sukhatme, G. S. (2015). Distributed data fusion for multirobot search. *IEEE Transactions on Robotics*, *31*(1), 55–66.
Khalil, H. K. (2002). *Nonlinear systems* (Vol. 3). Upper Saddle River, NJ: Prentice Hall.
Khasminskii, R. (2011). *Stochastic stability of differential equations* (Vol. 66). Berlin: Springer.
Ladde, A. G., & Ladde, G. S. (2013). *An introduction to differential equations: Stochastic modeling, methods and analysis* (Vol. 2). Singapore: World Scientific Publishing Company.
Lewis, F. L., Vrabie, D., & Syrmos, V. L. (2012). *Optimal control*. Hoboken, NJ: John Wiley & Sons.
Makarenko, A., & Durrant-Whyte, H. (2004). *Decentralized data fusion and control in active sensor networks*. Proceedings of the seventh international conference on information fusion, Stockholm, Sweden (Vol. 1, pp. 479–486).
Mesbahi, M., & Egerstedt, M. (2010). *Graph theoretic methods in multiagent networks*. Princeton: Princeton University Press.
Millán, P., Orihuela, L., Vivas, C., Rubio, F., Dimarogonas, D. V., & K. H. Johansson (2013). Sensor-network-based robust distributed control and estimation. *Control Engineering Practice*, *21*(9), 1238–1249.
Mori, Y., Mori, T., & Kuroe, Y. (1997). *A solution to the common Lyapunov function problem for continuous-time systems*. IEEE conference on decision and control, San Diego, CA, USA (Vol. 4, pp. 3530–3531).
Mou, S., Garcia, E., & Casbeer, D. (2017). *Distributed algorithms for the average bridge consensus*. IEEE conference on control technology and applications, Mauna Lani, HI, USA.
Mu, B., Chowdhary, G., & How, J. P. (2014). Efficient distributed sensing using adaptive censoring-based inference. *Automatica*, *50*(6), 1590–1602.
Narendra, K. S., & Balakrishnan, J. (1994). A common Lyapunov function for stable LTI systems with commuting a-matrices. *IEEE Transactions on Automatic Control*, *39*(12), 2469–2471.
Olfati-Saber, R. (2005). *Distributed Kalman filter with embedded consensus filters*. Proceedings of the 44th IEEE conference on decision and control, Seville, Spain (pp. 8179–8184). IEEE.
Olfati-Saber, R. (2007). *Distributed Kalman filtering for sensor networks*. IEEE conference on decision and control, New Orleans, LA, USA (pp. 5492–5498).
Olfati-Saber, R., & Shamma, J. S. (2005). *Consensus filters for sensor networks and distributed sensor fusion*. Proceedings of the 44th IEEE conference on decision and control, Seville, Spain (pp. 6698–6703).
Peterson, J. D., Yucelen, T., Chowdhary, G., & Kannan, S. (2015). *Exploitation of heterogeneity in distributed sensing: An active-passive networked multiagent systems approach*. American control conference, Chicago, IL, USA (pp. 4112–4117).
Peterson, J. D., Yucelen, T., & Pasiliao, E. (2016). *Generalizations on active-passive dynamic consensus filters*. American control conference, Boston, MA, USA (pp. 3740–3745).
Ramos, D. C., & Peres, P. L. (2002). An LMI condition for the robust stability of uncertain continuous-time linear systems. *IEEE Transactions on Automatic Control*, *47*(4), 675–678.
Shorten, R. N., & Narendra, K. S. (2003). On common quadratic Lyapunov functions for pairs of stable LTI systems whose system matrices are in companion form. *IEEE Transactions on Automatic Control*, *48*(4), 618–621.
Spanos, D. P., Olfati-Saber, R., & Murray, R. M. (2005). *Dynamic consensus on mobile networks*. IFAC world congress, Prague, Czech Republic (pp. 1–6). Citeseer.
Taylor, C. N., Beard, R. W., & Humpherys, J. (2011). *Dynamic input consensus using integrators*. 2011 American control conference, San Francisco, CA, USA (pp. 3357–3362).
Tran, D., Yucelen, T., & Jagannathan, S. (2017a). *A new result on distributed input and state estimation for heterogeneous sensor networks*. ASME dynamic systems and control conference, Tysons, VA, USA (pp. V002T14A003–V002T14A003).
Tran, D., Yucelen, T., & Jagannathan, S. (2017b). *On local design and execution of a distributed input and state estimation architecture for heterogeneous sensor networks*. American control conference, Seattle, WA, USA (pp. 3874–3879).
Tran, D. M., Yucelen, T., & Jagannathan, S. (2019). *Dynamic information fusion with the integration of local observers, value of information, and active-passive consensus filters*. AIAA scitech 2019 forum, San Diego, CA, USA (p. 2262).
Tran, D., Yucelen, T., Jagannathan, S., & Casbeer, D. (2018). *Distributed coestimation in heterogeneous sensor networks with time-varying active and passive node roles*. American control conference, Milwaukee, WI, USA (pp. 1033–1038).
Tran, D., Yucelen, T., Sarsilmaz, S. B., & Jagannathan, S. (2017). Distributed input and state estimation using local information in heterogeneous sensor networks. *Frontiers in Robotics and AI*, *4*, 30.

Ustebay, D., Castro, R., & Rabbat, M. (2011). Efficient decentralized approximation via selective gossip. *IEEE Journal of Selected Topics in Signal Processing*, *5*(4), 805–816.

Yucelen, T., & Peterson, J. D. (2014, December). *Active-passive networked multiagent systems*. 53rd IEEE conference on decision and control, Los Angeles, CA, USA (pp. 6939–6944). IEEE.

Yucelen, T., & Peterson, J. D. (2016). Distributed control of active–passive networked multiagent systems. *IEEE Transactions on Control of Network Systems*, *4*(4), 707–715.

Zavlanos, M. M., Egerstedt, M. B., & Pappas, G. J. (2011). Graph-theoretic connectivity control of mobile robot networks. *Proceedings of the IEEE*, *99*(9), 1525–1540.

Appendix. Notation

In this paper, $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R}^n$ stands for the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ stands for the set of $n \times m$ real matrices, $\mathbb{R}_+$ (respectively, $\overline{\mathbb{R}}_+$) stands for the set of positive (respectively, nonnegative-definite) real numbers, $\mathbb{R}_+^{n\times n}$ (respectively, $\overline{\mathbb{R}}_+^{n\times n}$) stands for the set of $n \times n$ positive-definite (respectively, nonnegative-definite) real matrices, $\mathbf{1}_n$ stands for the $n \times 1$ vector of all ones, I_n stands for the $n \times n$ identity matrix, and $\otimes$ stands for the Kronecker product operation. We also use $(\cdot)^{\mathrm{T}}$ for the transpose, $\lambda_{\min}(A)$ (respectively, $\lambda_{\max}(A)$) for the minimum (respectively, maximum) eigenvalue of a square matrix A, $\lambda_i(A)$ for the ith eigenvalue of a square matrix A, where the eigenvalues of A are ordered from least to greatest value, $det(\cdot)$ for the determinant, $\mathrm{diag}(a)$ for the diagonal matrix with the vector a on its diagonal, $[x]_i$ for the ith entry of the vector x, and $[A]_{ij}$ for the ith row and jth column entry of the matrix A.

Finally, we recall several graph-theoretical notions (see Godsil, Royle, & Godsil, 2001; Mesbahi & Egerstedt, 2010 for details). An undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_\mathcal{G} = \{1, \ldots, n\}$ of nodes and a set $\mathcal{E}_\mathcal{G} \subset \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$ of edges. If $(i, j) \in \mathcal{E}_\mathcal{G}$, then the nodes i and j are neighbours and $i \sim j$ indicates the neighbouring relation.

The number of a node's neighbours are its degree. Specifically, if we let d_i be the degree of node i, then $\mathcal{D}(\mathcal{G}) \triangleq \mathrm{diag}(d) \in \mathbb{R}^{N\times N}$ with $d = [d_1, \ldots, d_N]^{\mathrm{T}}$ is the degree matrix of a graph $\mathcal{G}$. A path $i_0 i_1 \cdots i_L$ is a (finite) sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$, and a graph $\mathcal{G}$ is said to be connected if a path exists between any distinct node pairs. $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N\times N}$ is the adjacency matrix of a graph $\mathcal{G}$ defined by $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $(i, j) \in \mathcal{E}_\mathcal{G}$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The Laplacian matrix of a graph, $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{R}}_+^{N\times N}$ is now defined by $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. For an undirected and connected graph $\mathcal{G}$, note that the spectrum of the Laplacian can be ordered as $0 = \lambda_1(\mathcal{L}(\mathcal{G})) < \lambda_2(\mathcal{L}(\mathcal{G})) \leq \cdots \leq \lambda_N(\mathcal{L}(\mathcal{G}))$ with the eigenvector $\mathbf{1}_N$ corresponds to the zero eigenvalue $\lambda_1(\mathcal{L}(\mathcal{G}))$ and $\mathcal{L}(\mathcal{G})\mathbf{1}_N = 0_N$ and both $e^{\mathcal{L}(\mathcal{G})}\mathbf{1}_N = \mathbf{1}_N$ hold.