# Cooperative Missile Guidance for Active Defense of Air Vehicles

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**In air combat, an active countermeasure against an attacking missile homing into a Target aircraft entails the launch of a defending missile. The Target is protected by the Defender, which aims to intercept the Attacker before the latter reaches the Target aircraft. A differential game is presented where the Target–Defender team strives to maximize the terminal separation between the Target and the Attacker at the time instant where the Attacker is intercepted by the Defender, whereas the Attacker strives to minimize the said separation. This paper discusses the case where the Defender is endowed with a positive capture radius. Optimal strategies for the three agents are derived and simulation examples illustrate the effectiveness of the proposed approach.**

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## I. INTRODUCTION

Two-player pursuit–evasion differential games have captured the interest of many researchers since the seminal work by Isaacs [1] first appeared. Scenarios involving several agents have also been considered in order to analyze team cooperation against an enemy and to better understand biological behaviors [2]–[4]. The important research by Breakwell and Hagedorn [5] studied the dynamic game of a fast pursuer trying to capture in minimal time two slower evaders in succession. Motivated by the work presented in [5], Liu *et al.* [6] analyzed the case where the fast pursuer tries to capture multiple evaders. The slow evaders cooperate in order to maximize the total time from the beginning of the game until the last evader is captured. The obtained numerical solution shows that the optimal strategies of the agents, pursuer and evaders, consist of constant headings (the pursuer's heading is piecewise constant and it changes at the time instants when an evader is captured). More recently, Oyler *et al.* [7] presented the Prey, Protector, and Predator game in order to model rescue missions in the presence of obstacles; dominance regions were provided for each agent in order to solve the game, that is, to determine if the Protector is able to rescue the Prey before the Predator captures it.

The target defense scenario including three agents, the Target ship, the Attacker missile, and the Defender missile (or counterweapon), was first introduced in the IEEE Transactions on Aerospace and Electronic Systems (see [8] and [9]). It was assumed that the Target $T$ holds a fixed course, whereas the Attacker $A$ and the Defender $D$ use collision course guidance. Cooperative missile strategies have been recently studied by different researchers: For instance, Jeon *et al.* [10] describe multimissile cooperative attacks on a single stationary target (ship). Cooperation to control the impact time in order to simultaneously hit the ship is implemented as an outer loop around the typical proportional navigation (PN) guidance law. Similar work was presented in [11] for moving targets. The work in [12] and [13] investigated an interception approach referred to as the triangle guidance (TG), where the objective is to command the defending missile to be on the line of sight (LOS) between the attacking missile and the aircraft for all time while the aircraft follows an arbitrary trajectory. The papers [14] and [15] considered the end-game Target–Attacker–Defender (TAD) scenario and focused on the $A-T$ miss distance where the Target and the Defender do not cooperate. The TAD problem where the Target does not cooperate with the Defender is also addressed [16]. In the aforementioned paper, the Target aircraft is not of the fighter class but it is a significantly less maneuverable air vehicle such as an airborne early warning and control system. A differential game approach for protection of airborne vehicles was also discussed in [17]. The work presented in [18] also provided a game theoretical method for missile interception and discussed the earliest intercept line guidance approach. Different types of cooperation between the Target and the Defender have been recently proposed in [19]–[25], where the Target is capable of performing maneuvers in order to evade the attacking missile. This situation usually occurs in air combat where the Target is a fighter aircraft. The TAD differential game discussed in this paper has significant military applications. The tactics described here can be employed in air combat and in suppression of enemy air defense and destruction of enemy air defense missions [26].

The practical application of the TAD problem for protection of valuable assets was also highlighted by Li and Cruz [27]. Li and Cruz [27] considered the game of defending an asset from an attacking intruder using an interceptor. The aforementioned paper represents another early contribution to the active target defense differential game (ATDDG) described in this paper. The authors considered three Target (asset) scenarios: when the Target is stationary; when it follows a known trajectory; and when it is evading the Attacker. This paper focuses on the latter case (a stationary Target being a particular case). Similar to the analysis of an evading Target in [27], we consider agents with simple motion á la Isaacs. The contributions of this paper can be better appreciated by following the analysis of the defense of an *evading* asset game in [27]. It is noteworthy that Li and Cruz [27] stated that an optimal evading strategy requires a particular Target heading that makes the Target cross into the reachable set of the Defender (the interceptor) while also accounting for the Attacker strategy—the Target also needs to head away from the Attacker.

The work in this paper picks up where [27] left. We provide a closed-form analytical solution of the ATDDG where the Defender is endowed with a capture circle of radius $r_c > 0$. Additionally, the closed-form optimal state feedback strategies obtained in this paper provide a robust, state-feedback controller for the $T/D$ team in order to intercept an Attacker which employs an unknown guidance law. We also obtain the solution of the Game of Kind; the set of initial conditions for which the Target is guaranteed to survive, provided the $T/D$ team plays optimally, this, regardless of any guidance law implemented by the Attacker.

In our previous work concerning the active target defense [28] and [29], we analyzed the cases where $A$ implements the simple guidance laws of pure pursuit (PP) and proportional navigation (PN), respectively. For these Attacker guidance laws, $T$ and $D$ cooperate and solve an optimal control problem that returns the optimal strategies for $T$ and $D$ so that $A$ is intercepted by $D$ and the separation between $T$ and $A$ at the interception time instant is maximized. In this paper, as in [30]–[33], the TAD problem is posed as a three-agent zero-sum pursuit–evasion differential game where $T$ and $D$ team up to defend from $A$ which is the opposition. Instead of implementing traditional guidance laws such as PP or PN, the Attacker strives to minimize the final separation between itself and the Target at interception time, whereas the $T/D$ team works to maximize the same. Assuming that the Attacker knows the instantaneous position of the Defender, this strategy provides better performance for the Attacker than using PP or PN guidance. Numerical solutions of this zero-sum differential game were obtained in [30], which required solving a two-point boundary value problem (TPBVP). The analytical solution of the ATDDG was derived in [31]–[33]; however, only the point capture case was considered ($r_c = 0$). In practice, a missile is able to intercept an aerial vehicle if it is able to reach the vehicle within a certain distance, or capture radius, $r_c > 0$. An analytical solution that explicitly considers a positive capture radius is desired as a computationally inexpensive alternative to solving the TPBVP [30]. In addition, the analytical solution can be implemented in a state-feedback fashion. It provides robustness to unknown Attacker guidance laws and the strategies of the Target and the Defender are security strategies.

Preliminary results concerning the ATDDG where the Defender possesses a positive capture radius $r_c$ were addressed in [34]. This paper provides complete proofs and extensions of the work presented in [34]. Specific extensions with respect to [34] include the following. We address the case where the Target is initially located on the boundary that separates the reachable sets of the Attacker and of the Defender. This is an important problem that cannot be fully addressed using the results in [34] and we provide an explicit solution of this problem in this paper. We also solve the Game of Kind; we provide a characterization of the initial positions of the agents such that a solution of the ATDDG exists, i.e., the Target survives, regardless of the strategy implemented by the Attacker. In addition, we establish the existence of solutions when the game starts in the Target's escape region by proving the existence of real solutions of the polynomial equation that returns the optimal strategies. Finally, this paper presents several examples where the optimal strategies are implemented in the three-dimensional (3-D) scenarios. We show that the solution of the planar ATDDG derived in this paper is optimal in the 3-D case under some assumptions on the agents speeds. We also show that the Target/Defender cooperative strategy is robust when those assumptions are relaxed, albeit optimality is not guaranteed. This means that the Defender is still able to intercept the Attacker.

This paper is organized as follows. Section II states the ATDDG where now $D$ is endowed with a circular capture set of radius $r_c > 0$. It also establishes that the agents' optimal headings are constant. Section III characterizes the $D - A$ boundary that separates the Attacker and Defender reachable regions. The closed-form solution of the differential game with a positive Defender capture radius is provided in Section IV. The analysis of the differential game for the important special case where the Target is exactly on the boundary separating the Attacker and Defender reachable sets is provided in Section V. The winning regions of the $T/D$ team and the Attacker are characterized in Section VI. Section VII presents illustrative examples, followed by concluding remarks in Section VIII.

## II. DIFFERENTIAL GAME

The Target $T$, the Attacker $A$, and the Defender $D$ have "simple motion" à la Isaacs. Thus, the controls of $T$, $A$, and $D$ are their respective headings $\phi$, $\chi$, and $\psi$, see Fig. 1.

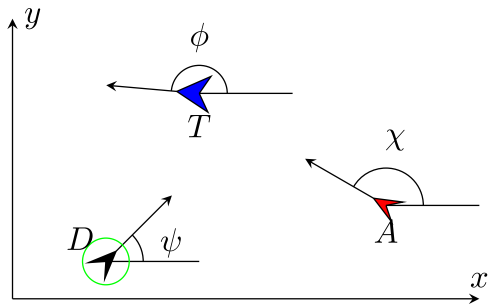


Fig. 1. TAD scenario.

In addition, $T$, $A$, and $D$ have constant speeds of $V_T$, $V_A$, and $V_D$, respectively. The states of the Target, the Attacker, and the Defender are defined by their Cartesian coordinates $\mathbf{x}_T = (x_T, y_T)$, $\mathbf{x}_A = (x_A, y_A)$, and $\mathbf{x}_D = (x_D, y_D)$, respectively. We assume that the Attacker and Defender have similar capabilities, so $V_A = V_D$, while the Target/Attacker speed ratio $\alpha = V_T/V_A < 1$. We consider, without loss of generality, the normalized (by $V_A$) active target defense scenario.

The complete state of the game is defined by $\mathbf{x} := (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$. The Attacker's control variable is his instantaneous heading angle, $\mathbf{u}_A = \{\chi\}$. The $T - D$ team affects the state of the game by choosing the instantaneous respective headings $\phi$ and $\psi$ of both the Target and the Defender, so the $T - D$ team control variable is $\mathbf{u}_B = \{\phi, \psi\}$. The dynamics in the realistic plane $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_A, \mathbf{u}_B)$ are defined by the system of ordinary differential equations as follows:

$$\dot{x}_T = \alpha \cos\phi, \qquad \dot{y}_T = \alpha \sin\phi \tag{1}$$

$$\dot{x}_A = \cos\chi, \qquad \dot{y}_A = \sin\chi \tag{2}$$

$$\dot{x}_D = \cos\psi, \qquad \dot{y}_D = \sin\psi \tag{3}$$

where the initial state of the system is $\mathbf{x}_0 := (x_{T_0}, y_{T_0}, x_{A_0}, y_{A_0}, x_{D_0}, y_{D_0}) = \mathbf{x}(t_0)$. In this game, the Attacker pursues the Target and tries to capture it. The Target and the Defender cooperate in order for the Defender to intercept the Attacker before the latter captures the Target. Define

$$r(t) := \sqrt{(x_A(t) - x_D(t))^2 + (y_A(t) - y_D(t))^2}. \tag{4}$$

The Defender is endowed with a capture circle of radius $r_c > 0$. Thus, the game terminates when the state of the system satisfies the terminal condition

$$r(t_f) = r_c. \tag{5}$$

The terminal time $t_f$ is defined as the time instant when the state of the system satisfies (5). The terminal state is defined as $\mathbf{x}_f := (x_{T_f}, y_{T_f}, x_{A_f}, y_{A_f}, x_{D_f}, y_{D_f}) = \mathbf{x}(t_f)$. Define

$$R(t) := \sqrt{(x_A(t) - x_T(t))^2 + (y_A(t) - y_T(t))^2} \tag{6}$$

the instantaneous distance between the Target and the Attacker. The Target/Defender team will employ a cooperative optimal strategy to maximize $R(t_f)$, the distance between the Target and the Attacker at the time instant $t_f$ when the Defender captures the Attacker. The Attacker's optimal strategy is geared to minimize $R(t_f)$.

REMARK The objective $R(t_f)$ represents an important operational cost measure. By maximizing $R(t_f)$, the $T/D$ team looks to avoid any possible damage to the Target aircraft since the Target will be the farthest away from the $A - D$ collision point. The same rationale applies to the Attacker who will try to minimize $R(t_f)$ in order to damage the Target by bringing the interception point as close as possible to the Target. Finally, there is no benefit for the Attacker in trying to evade the Defender because the $A - T$ separation will only keep increasing.

The ATDDG was addressed in [30] using Isaacs' method. In that paper, a reduced three state space is used, the state being the distances (4) and (6) and by the angle included between these radials. The resulting TPBVP is solved numerically. The numerical solution provides the optimal headings of the three agents; however, it is not a robust solution with respect to unknown Attacker guidance laws. This is important since the Attacker may decide not to play optimally and change to a conventional guidance law such as PP or PN. In this case, the $T/D$ team may not be able to recompute their controls in time and the Defender will not intercept the Attacker.

In this paper, we derive an analytical, state-feedback, optimal solution for the ATDDG for the case where the Defender is endowed with a capture circle of radius $r_c > 0$. We provide the optimal headings of the three agents. The controls (headings of $T$ and $D$) are continuously updated as a function of the positions of the agents, that is, the $T/D$ team only needs measurements of the Attacker's position. The optimal controllers designed in this paper guarantee interception of the Attacker even if the Attacker does not follow its optimal strategy and instead employs any other (unknown to $T$ and $D$) pursuit strategy. We prove that under optimal play, the headings are constant. This is a key step in order to transform the solution of the ATDDG into a static optimization problem that, when solved, provides the optimal aimpoint for each of the three agents.

## A. Analysis in the Realistic State Space

The cost/payoff functional is given by

$$J(\mathbf{u}_A(t), \mathbf{u}_B(t), \mathbf{x}_0) = \sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}. \tag{7}$$

We have a Mayer-type differential game where the dynamics matrix in (1)–(3) is zero. Its value

$$V(\mathbf{x}_0) = \min_{\mathbf{u}_A(t)} \max_{\mathbf{u}_B(t)} J(\mathbf{u}_A(t), \mathbf{u}_B(t), \mathbf{x}_0) \tag{8}$$

subject to (1)–(3) and (5). The costate is

$$\lambda := (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}) \in \mathbb{R}^6 \tag{9}$$

and the Hamiltonian of the differential game is

$$\begin{aligned}\mathcal{H} = {}& \lambda_{x_A}\cos\chi + \lambda_{y_A}\sin\chi + \lambda_{x_D}\cos\psi \\ &+ \lambda_{y_D}\sin\psi + \alpha\lambda_{x_T}\cos\phi + \alpha\lambda_{y_T}\sin\phi.\end{aligned} \tag{10}$$

THEOREM 1 Consider the ATDDG (1)–(8). The optimal headings of the Attacker, the Target, and the Defender are constant under optimal play and their trajectories are straight lines.

PROOF The optimal control inputs (in terms of the costate variables) can be obtained from

$$\min_{\phi,\psi} \max_{\chi} \mathcal{H} \tag{11}$$

and they are given by

$$\cos\chi^* = \frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2+\lambda_{y_A}^2}}, \qquad \sin\chi^* = \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2+\lambda_{y_A}^2}} \tag{12}$$

$$\cos\psi^* = -\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2+\lambda_{y_D}^2}}, \qquad \sin\psi^* = -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2+\lambda_{y_D}^2}} \tag{13}$$

$$\cos\phi^* = -\frac{\lambda_{x_T}}{\sqrt{\lambda_{x_T}^2+\lambda_{y_T}^2}}, \qquad \sin\phi^* = -\frac{\lambda_{y_T}}{\sqrt{\lambda_{x_T}^2+\lambda_{y_T}^2}}. \tag{14}$$

Additionally, the costate dynamics are: $\dot\lambda_{x_A} = \dot\lambda_{y_A} = \dot\lambda_{x_D} = \dot\lambda_{y_D} = \dot\lambda_{x_T} = \dot\lambda_{y_T} = 0$; hence, all costates are constant and we have that $\chi^* \equiv$ constant, $\psi^* \equiv$ constant, and $\phi^* \equiv$ constant. Consequently, the optimal trajectories are straight lines. ■

The constant heading characteristic of the optimal strategies allows us to transform the dynamic game to a geometric aim point selection problem: The Attacker and the Target/Defender team choose the optimal aim point on the curve which separates the states that are first reachable by $A$ before being reachable by $D$, and vice versa. This allows the reduction of the optimization problem at hand to the solution of an optimization problem in only one variable. The solution of the differential game in the realistic plane is validated by the solution á la Isaacs of the differential game in the reduced state space, which requires the numerical solution of the TPBVP described in [30].

### B. Analysis in the Reduced State Space

The following summary of the methodology used in [30] will be needed to characterize the interception boundary later in this section and to derive the players' optimal strategies in Section IV. The three-agent engagement is considered á la Isaacs in the reduced state space formed by the ranges $R$ and $r$ given in (4) and (6), and by the angle $\theta$ included between these two radials—see Fig. 2. The objective of the Target/Defender team is to determine their instantaneous optimal controls/heading angles, $\hat\phi$ and $\hat\psi$ in this reduced state space such that the distance $R(t_f)$ is maximized at the time instant $t_f$ where the separation $r(t_f) = r_c$. The objective of the Attacker is to determine its instantaneous optimal heading angle, denoted by $\hat\chi$, such that the terminal $A$–$T$ separation, $R(t_f)$, is minimized. The relative heading angles $\hat\phi$, $\hat\psi$, and $\hat\chi$ can be easily transformed to heading angles $\phi$, $\psi$, and $\chi$ in the realistic state

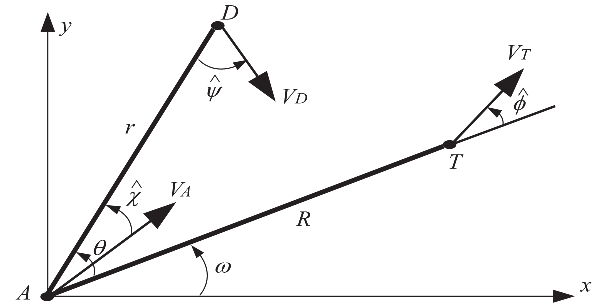


Fig. 2. Reduced state space.

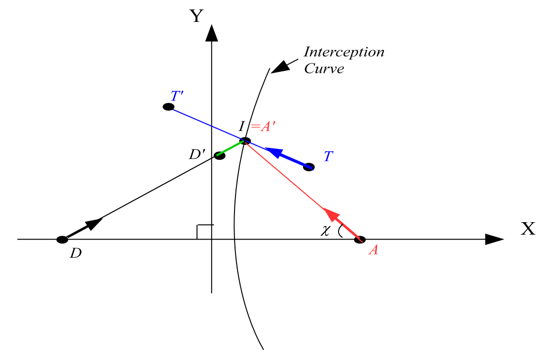


Fig. 3. Active target defense differential game.

space described in Section II-A using the LOS angle from the Attacker to the Target, denoted by $\omega$. In the reduced state space, the costate is $\hat\lambda = [\lambda_R\ \lambda_r\ \lambda_\theta]^T$. The terminal conditions for this free terminal time problem are as follows. The terminal state component $r(t_f)$ is fixed and equal to $r_c$. Because the terminal state components $R(t_f)$ and $\theta(t_f)$ are free, we have $\lambda_R(t_f) = \lambda_\theta(t_f) = 0$.

It was shown in [30] that the Target and Defender optimal headings in terms of the costate $\hat\lambda$ that maximize the separation between the Target and the Attacker and achieve $r(t_f) = r_c$ are given by

$$\sin\hat\psi^* = \frac{\lambda_\theta}{r\sqrt{\lambda_r^2+\lambda_\theta^2/r^2}} \tag{15}$$

$$\cos\hat\psi^* = \frac{\lambda_r}{\sqrt{\lambda_r^2+\lambda_\theta^2/r^2}} \tag{16}$$

$$\sin\hat\phi^* = \frac{\lambda_\theta}{R\sqrt{(1-\lambda_R)^2+\lambda_\theta^2/R^2}} \tag{17}$$

$$\cos\hat\phi^* = \frac{1-\lambda_R}{\sqrt{(1-\lambda_R)^2+\lambda_\theta^2/R^2}}. \tag{18}$$

## III. HYPERBOLA BOUNDARY

We consider, without loss of generality, the reference frame whose $X$-axis is anchored on the Attacker and Defender instantaneous positions. With respect to Fig. 3, the points $A$ and $D$ represent the positions on the $X$-axis of the Attacker and the Defender, respectively. The extension to infinity of the segment $\overline{AD}$ in both directions is the $X$-axis of the new Cartesian frame. The orthogonal bisector of $\overline{AD}$ is the $Y$-axis. The positions of the three agents in this frame are $T = (x_T, y_T)$, $A = (x_A, 0)$, and $D = (-x_A, 0)$.

The Attacker aims at minimizing the distance between the Target at the time instant when the Defender intercepts the Attacker, given by point $T'$, and the point where the Defender intercepts the Attacker which is point $I$. The initial and terminal positions of the Target are denoted by points $T$ and $T'$, respectively. Since the Defender is endowed with a positive capture radius $r_c > 0$, the interception point $I$ is determined by the position $A'$ of the Attacker at the time instant when the separation between the Attacker and the Defender is equal to $r_c$; $A$ is captured by $D$ at this point. In Fig. 3, the separation $\overline{A'D'} = r_c$ is shown in green color. The interception curve separates the reachable regions of the Attacker and the Defender. It is the locus of all points in the plane where the Defender could possibly capture the Attacker and it can be characterized in terms of the Defender capture radius $r_c$ and the Attacker's position $x_A$.

PROPOSITION 1 Consider an Attacker and a Defender with equal speeds. The interception curve where $D$ captures $A$ is characterized by the right branch, where $x > 0$, of the hyperbola as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{19}$$

where $a = \frac{r_c}{2}$ and $b = \frac{\sqrt{4x_A^2 - r_c^2}}{2}$. Additionally, point $A$ is a focus of the hyperbola, that is, $c^2 = a^2 + b^2 = x_A^2$.

PROOF For this proof, we use the basic results shown in Sections II-A and II-B. Both sections provide equivalent solutions. The difference is that in Section II-A the analysis is performed in the realistic and higher dimensional state space $\mathbf{x} \in \mathbb{R}^6$, whereas the analysis in Section II-B corresponds to the reduced state space $\hat{\mathbf{x}} \in \mathbb{R}^3$, where $\hat{\mathbf{x}} = [R\ r\ \theta]^T$. The important result in Section II-A is that the optimal headings $\phi^*$, $\psi^*$, and $\chi^*$ in the realistic state space are constant. The optimal headings $\hat{\phi}^*$, $\hat{\psi}^*$, and $\hat{\chi}^*$ are not constant; however, when these headings are transformed back to the realistic state space (using the LOS angle $\omega$), then the constant optimal headings $\phi^*$, $\psi^*$, and $\chi^*$ are obtained.

We now follow properties of both analyses: the constant property of the headings in the realistic state space and the relative properties of headings and positions at the terminal time $t_f$ provided by the reduced state-space solution. We evaluate the terminal configuration between $A$ and $D$ using the solution in Section II-B, that is, we evaluate (15) and (16) at time instant $t_f$. Recall that $\lambda_\theta(t_f) = 0$, then from (15) we have that $\sin\hat{\psi}^*(t_f) = 0$. Similarly, from (16), we obtain $\cos\hat{\psi}^*(t_f) = 1$ and we conclude that $\hat{\psi}^*(t_f) = 0$. This means that at the terminal time, the Defender is at point $D'$ and is heading directly to the Attacker whose terminal position is denoted by $A'$. Additionally, from Section II-A, we know that the headings in the realistic state space are constant, that is, the optimal trajectories are straight lines. Therefore, the configuration is as shown in Fig. 3 where the points $D$, $D'$, and $A'$ are collinear. We have the following relation:

$$(d + r_c)^2 = 4x_A^2 + d^2 - 4dx_A\cos\chi \tag{20}$$

where $d = \overline{AA'} = \overline{DD'} = \sqrt{(x_A - x)^2 + y^2}$ represents the distance traveled by the Attacker (and also by the Defender since these agents have the same speed). From (20), we obtain

$$\begin{aligned}
& r_c^2\left((x_A - x)^2 + y^2\right) = 4x_A^2x^2 + \frac{1}{4}r_c^4 - 2r_c^2x_Ax \\
\Rightarrow\quad & r_c^2(x_A^2 + x^2 + y^2) = 4x_A^2x^2 + \frac{1}{4}r_c^4 \\
\Rightarrow\quad & r_c^2\left(x_A^2 - \frac{1}{4}r_c^2\right) = (4x_A^2 - r_c^2)x^2 - r_c^2y^2. \qquad (21)
\end{aligned}$$

Expression (21) can ultimately be transcribed into the canonical form of the hyperbola equation (19). The distance from the origin to the focus $A$ of the hyperbola is $c = \sqrt{a^2 + b^2} = x_A$, which is the position of the Attacker. ■

## IV. CLOSED-FORM SOLUTION OF THE ATDDG

In Section II, it was shown that the agents' optimal trajectories are straight lines. In Section III, the hyperbola that separates the $A$ and $D$ players' reachable sets was constructed. In this section, we derive the players' optimal strategies by considering the following two cases. First, the case where the Target is initially on the left-hand side (LHS) of the right branch of the hyperbola (19) and, second, the case where the Target is initially on the right-hand side (RHS) of the right branch of the hyperbola (19). There is a subtle difference between these two cases concerning the simplification of the ATDDG to an optimization problem in one variable. These cases are analyzed next. We assume, without loss of generality, that $y_T > 0$. Also, since the left branch of the hyperbola (19) is irrelevant to the differential game under analysis, from now on, when we refer to (19), we specifically mean the right-hand branch of the hyperbola.

### A. Target is Initially on the LHS of the Interception Hyperbola

In this case, the Target chooses point $v$ on the hyperbola (19) in order to run away from that point and the Attacker chooses his aimpoint $u$ on the same curve. Additionally, the Defender tries to intercept the Attacker by choosing his aimpoint also on (19). Previously, in Section III, we showed that the Defender's aimpoint is the same as the Attacker's aimpoint, resulting in the three points: $D$, $D'$, and $A' = u$ being collinear. Thus, the Target and the Attacker are faced with the minmax optimization problem: $\min_u \max_v J(u, v)$, where $J(u, v)$ is the distance between the Target terminal position $T'$ and the point on the hyperbola (19) where the Attacker is intercepted by the Defender.

PROPOSITION 2 When initially the Target is on the LHS of the boundary (19), the Target's strategy is $v^*(u) = \arg\max_v J(u, v) = u$ so that it suffices to solve the opti-

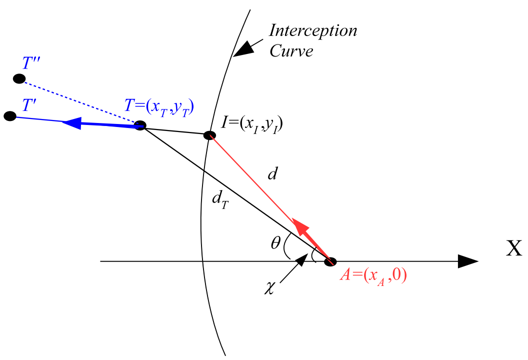


Fig. 4. Optimal strategies.

mization problem

$$\min_{x_I, y_I} J(x_I, y_I)$$
$$\text{subject to} \quad \frac{x_I^2}{a^2} - \frac{y_I^2}{b^2} = 1 \tag{22}$$

where

$$J(x_I, y_I) = \alpha\sqrt{(x_A - x_I)^2 + y_I^2} + \sqrt{(x_I - x_T)^2 + (y_I - y_T)^2}. \tag{23}$$

PROOF It can be seen from Fig. 4 that if the Target runs away from any point $v$ on the hyperbola different than $u = I$, then it will reach the position $T''$ instead of $T'$. Then, by the triangle inequality, the final separation $\overline{IT''}$ satisfies

$$\overline{IT''} < \overline{IT'}. \tag{24}$$

The Target maximizes the final distance between itself and the Attacker by running away from point $u$. Thus, we have that $v^*(u) = \arg\max_v J(u, v) = u$. The cost is now a function of only $u = I$. Let the coordinates of $I$ be given by $I = (x_I, y_I)$ and we have that $J(x_I, y_I) = \overline{TT'} + \overline{IT}$ is the final separation between the Target and the Attacker, which can be written in terms of the positions of the agents as in (23). ■

The Attacker chooses the coordinates $(x_I, y_I)$ of point $I$ that minimize the final separation $J(x_I, y_I) = \overline{IT} + \overline{TT'}$, see Fig. 4. The equality constraint in (22) can be used to write the cost function in terms of only one variable, either $x_I$ or $y_I$. However, we parameterize the coordinates of the interception point in terms of the hyperbolic angle $\mu$: $x_I(\mu) = a\cosh\mu$ and $y_I(\mu) = b\sinh\mu$. Using this parameterization, we obtain a more compact representation of the resulting polynomial equation that yields the solution of the underlying optimization problem. Define $d_T = \overline{AT}$ as the distance from $A$ to $T$ and let $\theta$ represent the angle included between $\overline{AT}$ and $\overline{AD}$, see Fig. 4.

THEOREM 2 The optimal interception point $I$ that minimizes (23) has coordinates $I = (a\cosh\mu^*, b\sinh\mu^*)$ where $\mu^* = \ln(\nu^*)$ and $\nu^*$ is a real solution of the eighth-order polynomial equation in $\nu$

$$\begin{aligned}
&\frac{x_A^2}{4}(1-\alpha^2)\nu^8 + (1-\alpha^2)(P-Q)\nu^7 \\
&+ \left(\frac{(P-Q)^2}{x_A^2} - \alpha^2 M\right)\nu^6 - ((1-\alpha^2)P + (3\alpha^2+1)Q)\nu^5 \\
&+ 2\left(\frac{Q^2 - P^2}{x_A^2} - \frac{x_A^2}{4}(1-\alpha^2) + \alpha^2 M\right)\nu^4 \\
&- ((1-\alpha^2)P - (3\alpha^2+1)Q)\nu^3 + \left(\frac{(P+Q)^2}{x_A^2} - \alpha^2 M\right)\nu^2 \\
&+ (1-\alpha^2)(P+Q)\nu + \frac{x_A^2}{4}(1-\alpha^2) = 0
\end{aligned} \tag{25}$$

which minimizes the cost

$$J(\mu) = \alpha d(\mu) + \sqrt{d_T^2 + d(\mu)^2 - 2d_T d(\mu)\cos(\chi(\mu) - \theta)} \tag{26}$$

and where $d(\mu) = \frac{x_A}{a}x_I(\mu) - a$, $\tan(\chi(\mu)) = \frac{y_I(\mu)}{x_A - x_I(\mu)}$, $P = a(d_T\cos\theta - x_A)$, $Q = bd_T\sin\theta$, and $M = d_T^2 + a^2 - 2x_A d_T\cos\theta$.

PROOF We express the cost function in terms of the hyperbolic angle $\mu$, that is, $J(\mu) = \overline{TT'}(\mu) + \overline{IT}(\mu)$ as it is shown in (26). The distance $d_T$ and the angle $\theta$ are constant, whereas the distance $d$, the angle $\chi$, and the coordinates of point $I$ are functions of the hyperbolic angle $\mu$. Hence, the optimal interception point is given by $(x_I(\mu^*), y_I(\mu^*))$, where $\mu^*$ minimizes (26).

Equation (26) can be written directly in terms of the hyperbolic angle $\mu$ as follows:

$$\begin{aligned}
J(\mu) = \Big(&d_T^2 + (x_A\cosh\mu - a)^2 \\
&- 2d_T\big(\cos\theta(x_A - a\cosh\mu) + b\sin\theta\sinh\mu\big)\Big)^{1/2} \\
&+ \alpha(x_A\cosh\mu - a).
\end{aligned} \tag{27}$$

The first derivative of (27) is

$$\frac{dJ(\mu)}{d\mu} = \alpha x_A\sinh\mu + \frac{(x_A\cosh\mu - a)x_A\sinh\mu - d_T(b\sin\theta\cosh\mu - a\cos\theta\sinh\mu)}{\sqrt{d_T^2 + (x_A\cosh\mu - a)^2 - 2d_T\big(\cos\theta(x_A - a\cosh\mu) + b\sin\theta\sinh\mu\big)}}. \tag{28}$$

Setting (28) shown at the bottom of this page, equal to zero, we obtain

$$\begin{aligned}&\alpha x_A \sinh\mu\Big(d_T^2 + (x_A\cosh\mu - a)^2\\&\quad - 2d_T\big(\cos\theta(x_A - a\cosh\mu) + b\sin\theta\sinh\mu\big)\Big)^{1/2}\\&= d_T(b\sin\theta\cosh\mu - a\cos\theta\sinh\mu)\\&\quad - (x_A\cosh\mu - a)x_A\sinh\mu. \end{aligned}\tag{29}$$

Both $\sinh\mu$ and $\cosh\mu$ can be expressed in terms of $e^{\mu}$: $\sinh\mu = \frac{1}{2}(e^{\mu} - e^{-\mu})$ and $\cosh\mu = \frac{1}{2}(e^{\mu} + e^{-\mu})$. Let $\nu = e^{\mu}$. We next insert the above expressions for $\sinh\mu$ and $\cosh\mu$ in (29), square both sides of this equation, and after some manipulation arrive at the eighth-order polynomial equation in $\nu$, in (25).

Some of the solutions $\mu = \ln(\nu)$ will be complex as it is also the case when $r_c = 0$; however, since the hyperbolic angle only takes real values, we only consider the real solutions using the cost function (26) to determine the optimal solution $\mu^*$ and the optimal interception point $I^*(\mu^*) = (x_I(\mu^*), y_I(\mu^*))$ on the hyperbolic arc. ∎

## B. Target is Initially on the RHS of the Interception Hyperbola

In the case where the Target is on the RHS of the right-hand branch of hyperbola (19), the Target chooses his aimpoint $v$ on the hyperbola (19) and the Attacker chooses his aimpoint $u$ on the same curve. Previously, we showed that the Defender's aimpoint is the same as the Attacker's aimpoint, resulting in the three points: $D$, $D'$, and $A' = u$ being collinear. Thus, the Target and the Attacker are faced with the optimization problem: $\max_v \min_u J(u, v)$, where $J(u, v)$ represents the distance between the Target terminal position $T'$ and the point on (19) where the Attacker is intercepted by the Defender.

PROPOSITION 3 When initially the Target is on the RHS of the hyperbola (19), the Attacker's strategy is $u^*(v) = \arg\min_u J(u, v) = v$ so that it suffices to solve the optimization problem

$$\max_{x_I, y_I} J(x_I, y_I) \quad \text{subject to } \frac{x_I^2}{a^2} - \frac{y_I^2}{b^2} = 1 \tag{30}$$

where

$$\begin{aligned}J(x_I, y_I) &= \alpha\sqrt{(x_A - x_I)^2 + y_I^2}\\&\quad - \sqrt{(x_I - x_T)^2 + (y_I - y_T)^2}.\end{aligned}\tag{31}$$

PROOF Based on the boundary conditions and the optimal heading $\hat{\phi}^*$ given by (17) and (18), we can see that $\sin\hat{\phi}^*(t_f) = 0$ and $\cos\hat{\phi}^*(t_f) = 1 \Rightarrow \hat{\phi}^*(t_f) = 0$. This means that the optimal trajectory for the Attacker is such that at the time of interception the point $A'$ lies on the line $\overline{TT'}$, see Fig. 3. In terms of aimpoint selection, the optimal Attacker's aimpoint is point $v$, the Target's aimpoint.

The Attacker minimizes the final separation between itself and the Target by aiming at point $v$ and we have that $u^*(v) = \arg\min_u J(u, v) = v$. Therefore, the cost is now a function of point $v = I$ only. Let the coordinates of $v = I$ be given by $I = (x_I, y_I)$. We have that $J(x_I, y_I) = \overline{TT'} - \overline{IT}$ is the final separation between the Target and Attacker, which can be written in terms of the positions of the agents, as it is shown in (31). ∎

The difference between the Target being on the LHS or on the RHS of the hyperbola arc (19) is not only the positive or negative sign in the corresponding cost functions (23) and (31), but also in the Target and Attacker strategies. In the case treated in this section, the Target chooses the coordinates $(x_I, y_I)$ that maximize the final separation $J(x_I, y_I)$ and the Attacker follows the Target's decision. Similar to the case where the Target is on the LHS of the hyperbola (19), we write the interception point's coordinates in terms of the hyperbolic angle $\mu$: $x_I(\mu) = a\cosh\mu$ and $y_I(\mu) = b\sinh\mu$.

THEOREM 3 The optimal interception point $I$ that maximizes (31) has coordinates $I = (a\cosh\mu^*, b\sinh\mu^*)$ where $\mu^* = \ln(\nu^*)$ and $\nu^*$ is the real solution of the eighth-order polynomial equation (25) that now maximizes the payoff

$$J(\mu) = \alpha d(\mu) - \sqrt{d_T^2 + d(\mu)^2 - 2d_T d(\mu)\cos(\chi(\mu) - \theta)} \tag{32}$$

where $d(\mu) = \frac{x_A}{a}x_I(\mu) - a$ and $\tan(\chi(\mu)) = \frac{y_I(\mu)}{x_A - x_I(\mu)}$. ∎

PROOF The proof is similar to the proof of Theorem 2.

## C. Existence of Real Roots of the Eighth-Order Polynomial Equation

In both cases discussed in this section, the optimal interception point is characterized by the hyperbolic angle $\mu^* = \ln(\nu^*)$, where $\nu^*$ is given by one of the real roots of the eighth-order polynomial equation (25). Define $H : (x_H, y_H)$ as the point on the right-hand branch of the hyperbola (19), which is the closest to point $T$. Define $\mu_H$ as the hyperbolic angle corresponding to point $H$, that is, $x_H = a\cosh\mu_H$ and $y_H = b\sinh\mu_H$. Note also that $\mu_H > 0$ since we consider, without losing generality, $y_T > 0$. In other words, the closest point on the right-hand branch of the hyperbola to the initial Target location $T : (x_T, y_T)$, $y_T > 0$ has coordinates $x_H > 0$, $y_H > 0$; hence, $\mu_H > 0$.

Given the location $T$, the value of $\mu_H$ can be obtained by solving

$$\frac{d}{d\mu}\left\{\sqrt{d_T^2 + d(\mu)^2 - 2d_T d(\mu)\cos(\chi(\mu) - \theta)}\right\} = 0. \tag{33}$$

PROPOSITION 4 Equation (25) has at least two real solutions. Let $\mu_m$ and $\mu_M$ be these two real solutions. It holds that $0 < \mu_m < \mu_H$ and $\mu_H < \mu_M < \infty$.

PROOF Let the polynomial function $F(\nu)$ be given by the LHS of (25) and consider $\mu = 0$ ($\nu = 1$). We have that

$$F(\mu = 0) = \frac{4}{x_A^2}b^2 d_T^2 \sin^2\theta > 0.$$

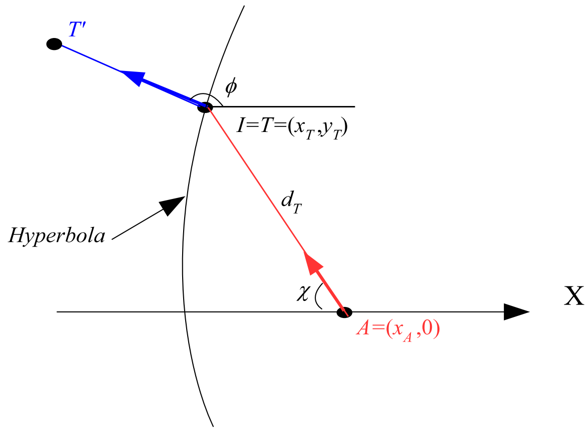


Fig. 5. Differential game when Target is on the interception hyperbola.

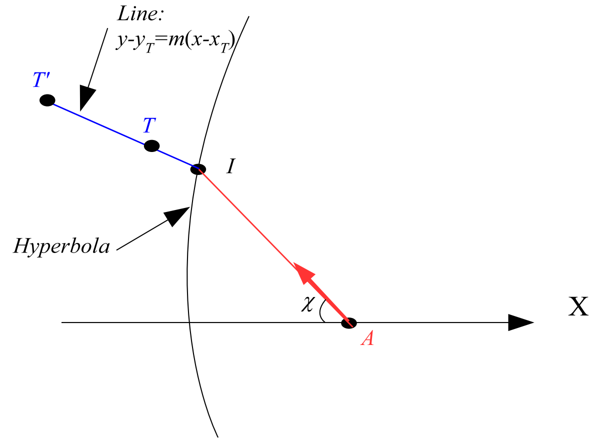


Fig. 6. Solution of differential game in terms of the Target slope $m$.

We also have that $F(\infty) > 0$. We now evaluate the eighth-order polynomial at $\mu = \mu_H$. This can be accomplished using (29) by noting that such polynomial is equivalent to the following expression:

$$\begin{aligned} F(\mu) = \frac{4e^{4\mu}}{x_A^2}\Big(&d_T(b\sin\theta\cosh\mu - a\cos\theta\sinh\mu) \\ &- (x_A\cosh\mu - a)x_A\sinh\mu\Big)^2 \\ &- 4\alpha^2 e^{4\mu}\sinh^2\mu\Big(d_T^2 + (x_A\cosh\mu - a)^2 \\ &- 2d_T\big(\cos\theta(x_A - a\cosh\mu) + b\sin\theta\sinh\mu\big)\Big). \end{aligned}$$

However, when evaluated at $\mu = \mu_H$, the first factor of the previous expression is equivalent to the LHS of (33), which means that $d_T(b\sin\theta\cosh\mu_H - a\cos\theta\sinh\mu_H) - (x_A\cosh\mu_H - a)x_A\sinh\mu_H = 0$ and we have that

$$F(\mu_H) = -(2\alpha e^{2\mu_H}\sinh\mu_H)^2 \cdot \overline{TH} < 0.$$

We conclude that since $F(\mu = 0) > 0$, $F(\mu = \infty) > 0$, and $F(\mu_H) < 0$, (25) has at least two real roots $\mu_m$ and $\mu_M$, which satisfy $0 < \mu_m < \mu_H$ and $\mu_H < \mu_M < \infty$, as required. ■

REMARK Following similar steps as in [32], it is also possible to establish uniqueness of the solution, that is, only one of the real solutions of (25) attains the optimal cost/payoff.

## V. OPTIMAL STRATEGIES WHEN THE TARGET IS ON THE INTERCEPTION HYPERBOLA

A particular, important, and interesting case is when the Target finds itself on the interception hyperbola (19). If the Target's initial position satisfies the hyperbola equation (19), then the optimal interception point is given by $x_I^* = x_T$ and $y_I^* = y_T$, see Fig. 5.

Contrary to the cases where the Target starts either on the LHS or on the RHS of the hyperbola, the solution of the eighth-order polynomial equation (25) by itself does not provide the optimal heading angle $\phi^*$. The Target's optimal heading was previously obtained based on two different points $T$ and $I$; however, the optimal solution is now $I^* = T$, that is, those two points now merge into one point and the Target's optimal heading cannot be obtained as before.

Let us assume that $x_T < x_A$ and, without loss of generality, consider $y_T \geq 0$. Note that for a capture radius $r_c > 0$, we have that $x_T > 0$ since the Target is on the hyperbola (19). Also note that when the Target is on the hyperbola, $x_A > r_c/2$ for otherwise the Attacker has been captured and the game is over. In order to obtain the optimal Target heading, we consider the straight line equation parameterized by the slope $m$ that passes through the Target location $(x_T, y_T)$—see Fig. 6. This equation can be written in the form

$$y - y_T = m(x - x_T). \tag{34}$$

PROPOSITION 5 When initially the Target is on the hyperbola, that is, its coordinates $(x_T, y_T)$ satisfy (19), the optimal Attacker's strategy is to head toward $I = (x_T, y_T)$, where it will be intercepted by the Defender. Also consider the case $x_T < x_A = c$. Then, the optimal Target's heading angle is given by $\phi^* = \arctan m^*$, where

$$m^* = \frac{-b^2 x_T y_T - \alpha c y_T\sqrt{b^4 + (1-\alpha^2)c^2 y_T^2}}{\frac{b^4}{a^2}x_T^2 - \alpha^2 c^2 y_T^2}. \tag{35}$$

PROOF In general, the optimal interception point can be characterized in terms of the slope $m$ of the straight line (34). This derivation is shown first in this proof and then the particular case when the Target is on the hyperbola is analyzed.

From (19) and (34), we obtain

$$\frac{x^2}{a^2} - \frac{(m(x - x_T) + y_T)^2}{b^2} = 1. \tag{36}$$

In general, (36) can be used to determine the intersection points of (19) and (34) as a function of $m$. Taking the total derivative of (36) with respect to the slope $m$, we obtain

$$\frac{2}{a^2}x\frac{dx}{dm} - \frac{2}{b^2}\big(m(x - x_T) + y_T\big)\left(m\frac{dx}{dm} + x - x_T\right) = 0 \tag{37}$$

and it follows that

$$\frac{dx}{dm}=\frac{x-x_T}{b^2}\frac{m(x-x_T)+y_T}{\frac{x}{a^2}-\frac{m}{b^2}(m(x-x_T)+y_T)}. \tag{38}$$

The cost/payoff function can be written in terms of $x$ and $m$ as follows:

$$J(x,m)=\alpha\left(\frac{c}{a}x-a\right)+(x-x_T)\sqrt{m^2+1}. \tag{39}$$

In order to determine the optimal slope $m^*$, we take the total derivative of (39) with respect to $m$ and set it equal to zero

$$\frac{dx}{dm}\left(\frac{\alpha c}{a}+\sqrt{m^2+1}\right)+\frac{m(x-x_T)}{\sqrt{m^2+1}}=0. \tag{40}$$

Substituting (38) into (40), we obtain

$$\frac{m(x-x_T)+y_T}{\frac{x}{a^2}-\frac{m}{b^2}(m(x-x_T)+y_T)}\left(\frac{\alpha c}{a}+\sqrt{m^2+1}\right)+\frac{b^2m}{\sqrt{m^2+1}}=0. \tag{41}$$

As an alternative to the eighth-order polynomial equation (25), one can now solve the system of two equations, (36) and (41), to obtain the optimal slope $m^*$ and the associated optimal interception point's coordinates $(x^*, y^*)$. In general, this alternative method does not offer computational advantages compared to rooting the polynomial (25). However, when the Target is on the hyperbola (19), we have that (41) can be simplified and the optimal slope can be directly determined. In this special case, we have that the interception point is $I^*=(x_T,y_T)$; thus, we set $x=x_T$ in (41) as

$$\frac{y_T}{\frac{x_T}{a^2}-\frac{my_T}{b^2}}\left(\frac{\alpha c}{a}+\sqrt{m^2+1}\right)+\frac{b^2m}{\sqrt{m^2+1}}=0$$

which results in the quadratic equation in the slope $m$

$$\left(\frac{b^4}{a^4}x_T^2-\frac{\alpha^2c^2}{a^2}y_T^2\right)m^2+2\frac{b^2}{a^2}x_Ty_Tm + y_T^2-\frac{\alpha^2c^2}{a^2}y_T^2=0. \tag{42}$$

Let $m_1$ and $m_2$ represent the two solutions of (42). In order to determine which one of these solutions is the optimal slope $m^*$, we first find the lower and upper bounds of the optimal slope. These bounds correspond to the cases when the speed ratio $\alpha=1$ and $\alpha=0$, respectively.

Let $\alpha=0$ in (42). We have that

$$\frac{b^4}{a^4}x_T^2m^2+2\frac{b^2}{a^2}x_Ty_Tm+y_T^2=0. \tag{43}$$

Equation (43) has a repeated solution. Let $\overline{m}$ be this solution, then the optimal slope when $\alpha=0$ is

$$\overline{m}=-\frac{a^2}{b^2}\frac{y_T}{x_T}. \tag{44}$$

Now, when $\alpha=1$, the solutions of (42) are given by

$$\begin{aligned} m_{1,2}&=\frac{-\frac{b^2}{a^2}x_Ty_T\mp\sqrt{\frac{b^4}{a^6}c^2x_T^2y_T^2+\frac{c^2}{a^2}y_T^4\left(1-\frac{c^2}{a^2}\right)}}{\frac{b^4}{a^4}x_T^2-\frac{c^2}{a^2}y_T^2}\\ &=\frac{-\frac{b^2}{a^2}x_Ty_T\mp\sqrt{\frac{b^4}{a^4}c^2y_T^2\left(1+\frac{y_T^2}{b^2}\right)+\frac{c^2}{a^2}y_T^4\left(1-\frac{c^2}{a^2}\right)}}{\frac{b^4}{a^4}x_T^2-\frac{b^2c^2}{a^2}\left(\frac{x_T^2}{a^2}-1\right)}\\ &=\frac{-x_Ty_T\mp cy_T}{-x_T^2+c^2}. \end{aligned} \tag{45}$$

Substituting $c=x_A$ and noting that $x_T<x_A$, we realize that $m_2=\frac{y_T}{x_A+x_T}>0$ does not qualify as an optimal solution. Thus, the optimal solution when $\alpha=1$ is the LOS strategy, as expected. Let $\underline{m}$ represent the LOS strategy which is given by

$$\underline{m}=\frac{-y_T}{x_A-x_T}. \tag{46}$$

Because the LHS of (42) is a continuous function of $\alpha$, for $0<\alpha<1$, we have that the following holds:

$$\underline{m}<m^*<\overline{m}\quad \text{for } 0<\alpha<1. \tag{47}$$

In order to prove that (35) is the optimal Target strategy we consider three cases of the speed ratio, that is, when $\alpha$ is greater than, less than, or equal to $\frac{r_c}{2x_A}$. Before that, we note that when $y_T=0$, the optimal slope is $m^*=0$ regardless of the value of $\alpha$. Therefore, in the following, we focus on the case $y_T>0$.

1) Case $\alpha>\frac{r_c}{2x_A}$: Consider the product of the two roots of (42), that is, $m_1m_2=k_3/k_1$, where $k_1$ and $k_3$ represent the first and the last coefficients of the quadratic equation (42), respectively. Let us write

$$\frac{k_3}{k_1}=\frac{y_T^2\left(1-\frac{\alpha^2c^2}{a^2}\right)}{\frac{b^4}{a^4}x_T^2-\frac{\alpha^2c^2}{a^2}y_T^2}=\frac{\left(\frac{x_T^2}{a^2}-1\right)\left(a^2-\alpha^2c^2\right)}{\frac{b^2}{a^2}x_T^2-\alpha^2c^2\left(\frac{x_T^2}{a^2}-1\right)}.$$

Since $\alpha>\frac{r_c}{2x_A}$, we have that

$$\frac{\alpha^2c^2}{a^2}>1.$$

Multiplying both sides of the previous inequality by the positive term $x_T^2-a^2$ and then subtracting $\frac{b^2}{a^2}x_T^2$ from both sides, we have the following relation:

$$\frac{\alpha^2c^2}{a^2}x_T^2-\alpha^2c^2-\frac{b^2}{a^2}x_T^2>x_T^2-a^2-\frac{b^2}{a^2}x_T^2. \tag{48}$$

At this point we note that the term $\frac{b^2}{a^2}x_T^2-\frac{\alpha^2c^2}{a^2}x_T^2+\alpha^2c^2$ is positive. This can be shown by writing the term in the following form:

$$\begin{aligned} &\frac{b^2}{a^2}x_T^2-\frac{\alpha^2\left(a^2+b^2\right)}{a^2}x_T^2+\alpha^2c^2\\ &=\frac{b^2}{a^2}\left(1-\alpha^2\right)x_T^2+\alpha^2\left(x_A^2-x_T^2\right)>0 \end{aligned} \tag{49}$$

because $x_A > x_T$. We now divide (48) by this term to obtain

$$\frac{x_T^2 - a^2 - \frac{b^2}{a^2}x_T^2}{\frac{b^2}{a^2}x_T^2 - \frac{\alpha^2c^2}{a^2}x_T^2 + \alpha^2c^2} < -1 \tag{50}$$

which is equivalent to

$$\frac{x_T^2 - \frac{\alpha^2c^2}{a^2}x_T^2 - a^2 + \alpha^2c^2}{\frac{b^2}{a^2}x_T^2 - \frac{\alpha^2c^2}{a^2}x_T^2 + \alpha^2c^2} < 0. \tag{51}$$

The LHS of (51) is equal to $k_3/k_1$. We have shown that the product $m_1m_2 < 0$, that is, the solutions are real and have different signs. However, the positive solution does not fulfill the requirement (47) for an optimal solution; hence, the negative solution of (42) which is given by (35) is the optimal slope.

2) Case $\alpha = \frac{r_c}{2x_A}$: In this case we have that the discriminant of (42) is equal to $k_2^2$, where $k_2$ is the second coefficient of (42). We have $m_1 < 0$ and $m_2 = 0$. The solution $m_2$ does not satisfy the constraint (47) and, therefore, the optimal solution is the negative solution given by (35).
3) Case $\alpha < \frac{r_c}{2x_A}$: The solutions of (42) are given by

$$m_{1,2} = \frac{-\frac{b^2}{a^2}x_Ty_T \mp \sqrt{\frac{b^4}{a^6}\alpha^2c^2x_T^2y_T^2 + \frac{\alpha^2c^2}{a^2}y_T^4\left(1 - \frac{\alpha^2c^2}{a^2}\right)}}{\frac{b^4}{a^4}x_T^2 - \frac{\alpha^2c^2}{a^2}y_T^2} \tag{52}$$

where the discriminant is positive because $1 - \frac{\alpha^2c^2}{a^2} > 0$. Previously, we showed that the denominator of (52) is positive for $x_A > x_T$. Let us consider the second solution of (42) as

$$\begin{aligned} m_2 &= \frac{-\frac{b^2}{a^2}x_Ty_T + \sqrt{\frac{b^4}{a^6}\alpha^2c^2x_T^2y_T^2 + \frac{\alpha^2c^2}{a^2}y_T^4\left(1 - \frac{\alpha^2c^2}{a^2}\right)}}{\frac{b^4}{a^4}x_T^2 - \frac{\alpha^2c^2}{a^2}y_T^2} \\ &> \frac{-\frac{b^2}{a^2}x_Ty_T}{\frac{b^4}{a^4}x_T^2 - \frac{\alpha^2c^2}{a^2}y_T^2} > \frac{-\frac{b^2}{a^2}x_Ty_T}{\frac{b^4}{a^4}x_T^2} = -\frac{a^2}{b^2}\frac{y_T}{x_T} = \overline{m}. \end{aligned} \tag{53}$$

It can be seen from (53) that $m_2$ does not satisfy the condition (47); thus, the optimal slope is the first solution $m_1$, which is given by (35). ■

## VI. GAME OF KIND

In the case where the Target starts on the RHS of the right-hand branch of the hyperbola (19), it needs to be able to cross this curve into the LHS. Then, the Defender will be able to help the Target escape by intercepting the Attacker. For a given speed ratio $\alpha$, it is possible to define the region or set of coordinates $(x_T, y_T)$ such that the Target is guaranteed to escape, regardless of the Attacker strategy. This is effectively the solution to the Game of Kind.

The hyperbola (19) can be parameterized as follows:

$$x = \frac{a}{\sqrt{1-t^2}}, \; y = \frac{bt}{\sqrt{1-t^2}} \tag{54}$$

where the parameter $t$ satisfies $0 \le t < 1$. Without loss of generality, we consider the case $y_T > 0$.

THEOREM 4 Given an initial Attacker–Defender separation $2x_A$, a Defender capture radius $r_c$, and the speed ratio $\alpha > 0$, the Target is guaranteed to escape from the Attacker if its initial coordinates $(x_T, y_T)$, $y_T > 0$, satisfy the following relationship:

$$\begin{aligned} & x_T^2 + y_T^2 + \frac{t^2-\alpha^2}{1-t^2}x_A^2 + \frac{r_c^2}{4}\left(1-\alpha^2\right) \\ & \quad \le \frac{1}{\sqrt{1-t^2}}\left(r_c(x_T - \alpha^2x_A) + 2ty_T\sqrt{x_A^2 - \frac{r_c^2}{4}}\right) \end{aligned} \tag{55}$$

where $0 < t(x_A, x_T, y_T) < 1$ is a real solution of the quartic equation in $t$

$$\begin{aligned} & r_c^2\left(x_T - \alpha^2x_A\right)^2t^4 + 4r_c\left(x_T - \alpha^2x_A\right)\sqrt{x_A^2 - \frac{r_c^2}{4}}y_Tt^3 \\ & + \left(4x_A^2\left[(1-\alpha^2)^2x_A^2 + y_T^2\right] - \left[y_T^2 + \left(x_T - \alpha^2x_A\right)^2\right]r_c^2\right)t^2 \\ & - 4r_c\left(x_T - \alpha^2x_A\right)\sqrt{x_A^2 - \frac{r_c^2}{4}}y_Tt + \left(r_c^2 - 4x_A^2\right)y_T^2 = 0. \end{aligned} \tag{56}$$

PROOF Let us consider first the particular case of the AT-DDG with a stationary Target. In this case, the problem is to find the aimpoint on the upper right branch of the hyperbola $I^*(t)$ such that the distance $\overline{IT}$ is minimized, that is,

$$\min_{0\le t<1}\left(\frac{a}{\sqrt{1-t^2}} - x_T\right)^2 + \left(\frac{bt}{\sqrt{1-t^2}} - y_T\right)^2. \tag{57}$$

Taking the derivative of (57) and setting it equal to zero, we obtain the following quartic equation in $t$:

$$\begin{aligned} & a^2x_T^2t^4 + 2abx_Ty_Tt^3 + \big((a^2+b^2)^2 \\ & -a^2x_T^2 + b^2y_T^2\big)t^2 - 2abx_Ty_Tt - b^2y_T^2 = 0. \end{aligned} \tag{58}$$

When $t = 0$, the LHS of (58) reduces to $-b^2y_T^2 < 0$. When $t = 1$, the LHS of (58) is given by $(a^2+b^2)^2 > 0$. When the Target is static, the quartic equation (58) has a unique real solution in the interval $0 < t < 1$. Since $x_A^2 = a^2 + b^2$, $a = r_c/2$, and $b = \sqrt{x_A^2 - (r_c/2)^2}$, the quartic equation is parameterized by the state $x = (x_A, x_T, y_T)$ as follows:

$$\begin{aligned} & r_c^2x_T^2t^4 + 4r_c\sqrt{x_A^2 - \frac{r_c^2}{4}}x_Ty_Tt^3 \\ & + \left(4x_A^2\left(x_A^2 + y_T^2\right)^2 - r_c^2\left(x_T^2 + y_T^2\right)\right)t^2 \\ & - 4r_c\sqrt{x_A^2 - \frac{r_c^2}{4}}x_Ty_Tt + r_c^2y_T^2 - 4x_A^2y_T^2 = 0. \end{aligned} \tag{59}$$

We will now use (59) to determine the point on the hyperbola, which is the closest to the center of the Apollonius circle of a moving Target with respect to the Attacker. The Apollonius circle based on the $\overline{AT}$ segment and speed ration $\alpha$ is the locus of points where the Target is captured by the Attacker if the latter is not first intercepted by the Defender. Thus, the Target is guaranteed to escape if the Apollonius circle intersects the right-hand branch of the hyperbola. This Apollonius circle is characterized by the separation

$\overline{AT}$ and the speed ratio $\alpha > 0$. The center $O = (x_0, y_0)$ of the Apollonius circle has the following coordinates:

$$x_0 = \frac{1}{1-\alpha^2}\left(x_T - \alpha^2 x_A\right), \quad y_0 = \frac{1}{1-\alpha^2} y_T. \tag{60}$$

We solve a new quartic equation in $t$ where the coordinates $(x_T, y_T)$ of point $T$ are replaced by the coordinates $(x_0, y_0)$ of the center of the Apollonius circle. This new quartic equation is given by

$$\begin{aligned} & r_c^2 \frac{1}{(1-\alpha^2)^2}\left(x_T - \alpha^2 x_A\right)^2 t^4 \\ & + 4 r_c \sqrt{x_A^2 - \frac{r_c^2}{4}} \frac{1}{(1-\alpha^2)^2}\left(x_T - \alpha^2 x_A\right) y_T t^3 \\ & + \left(4x_A^2\left[x_A^2 + \frac{1}{(1-\alpha^2)^2} y_T^2\right]\right. \\ & \left. - \frac{r_c^2}{(1-\alpha^2)^2}\left[y_T^2 + (x_T - \alpha^2 x_A)^2\right]\right) t^2 \\ & - 4 r_c \sqrt{x_A^2 - \frac{r_c^2}{4}} \frac{1}{(1-\alpha^2)^2}\left(x_T - \alpha^2 x_A\right) y_T t \\ & + \left(r_c^2 - 4x_A^2\right) \frac{1}{\left(1-\alpha^2\right)^2} y_T^2 = 0 \end{aligned} \tag{61}$$

which can also be written as shown in (56). By solving (56), we obtain the parameter $t$ that determines the point on the hyperbola which is closest to $(x_0, y_0)$. This point is then given by

$$x^*(t) = \frac{a}{\sqrt{1-t^2}}, \quad y^*(t) = \frac{bt}{\sqrt{1-t^2}}.$$

The escape region of the Target is characterized by

$$(x^*(t) - x_0)^2 + (y^*(t) - y_0)^2 \le R_A^2 \tag{62}$$

where $R_A$ is the radius of the Apollonius circle which is based on the segment $\overline{AT}$ and the speed ratio $\alpha$

$$R_A = \frac{\alpha}{1-\alpha^2}\sqrt{(x_A - x_T)^2 + y_T^2}. \tag{63}$$

Hence, the inequality (62) can be written as follows:

$$\begin{aligned} & \left(\frac{1-\alpha^2}{2\sqrt{1-t^2}} r_c - \left(x_T - \alpha^2 x_A\right)\right)^2 \\ & \quad + \left(\frac{t(1-\alpha^2)}{\sqrt{1-t^2}}\sqrt{x_A^2 - \frac{r_c^2}{4}} - y_T\right)^2 \\ & \le \alpha^2\left((x_A - x_T)^2 + y_T^2\right) \end{aligned} \tag{64}$$

which can be simplified as shown in (55). ■

The results in Theorem 4 can be used to find a parametric representation of the winning regions of $A$ and the $T/D$ team. In other words, it is possible to obtain the 2-D Barrier surface in the state space $(x_A, x_T, y_T)$ which delimits two regions, one in which $T$ is guaranteed to escape, denoted by $R_e$, and the other where, provided he plays optimally, $A$ will catch $T$ unmolested by $D$ and win the game. This region is denoted by $R_c$.

Equation (56) is a quadratic equation in $y_T$. This equation can be written in the following form:

$$\begin{aligned} & 4\left(1-t^2\right)\left(x_A^2 - \frac{r_c^2}{4}\right) y_T^2 \\ & \quad + 4 r_c \sqrt{x_A^2 - \frac{r_c}{4}}\left(x_T - \alpha^2 x_A\right) t\left(1-t^2\right) y_T \\ & \quad + \left(x_T - \alpha^2 x_A\right) r_c^2 t^2\left(1-t^2\right) - 4x_A^4\left(1-\alpha^2\right) t^2 = 0. \end{aligned} \tag{65}$$

Dividing both sides of (65) by $(1-t^2)$ and rearranging terms, we obtain the following equation:

$$\left(2\sqrt{x_A^2 - \frac{r_c^2}{4}} y_T + r_c(x_T - \alpha^2 x_A)t\right)^2 = 4\left(1-\alpha^2\right)\frac{t^2}{1-t^2} x_A^4. \tag{66}$$

Solving for $y_T$ in (66), we obtain the explicit solution

$$y_T = \frac{t}{\sqrt{x_A^2 - \frac{r_c^2}{4}}}\left(\frac{1-\alpha^2}{\sqrt{1-t^2}} x_A^2 - \frac{r_c}{2}\left(x_T - \alpha^2 x_A\right)\right). \tag{67}$$

Note that choosing $t$ yields the point $I(t) = (x(t), y(t))$ on the hyperbola where the Attacker is intercepted by the Defender. Also, replacing the inequality sign in (55) by equality yields the surface in $(x_A, x_T, t)$ which delimits the winning regions of the Attacker and of the Target/Defender team.

We now use (67) in (55), where the inequality sign has been replaced by equality. After rearranging terms, we obtain the following:

$$\begin{aligned} & \left(1 + \frac{r_c^2 t^2}{4x_A^2 - r_c^2}\right) x_T^2 \\ & - \left(\frac{r_c^2 \alpha^2 x_A t^2}{2x_A^2 - \frac{r_c^2}{2}} + \frac{1-\alpha^2}{\sqrt{1-t^2}} x_A^2 r_c \frac{t^2}{x_A^2 - \frac{r_c^2}{4}} + r_c\sqrt{1-t^2}\right) x_T \\ & + \left(\frac{(1-\alpha^2)^2}{1-t^2} x_A^4 + \frac{r_c^2}{4}\alpha^2 x_A^2 + \frac{1-\alpha^2}{\sqrt{1-t^2}} r_c \alpha^2 x_A^3\right) \frac{t^2}{x_A^2 - \frac{r_c^2}{4}} \\ & - \left(\frac{1-\alpha^2}{1-t^2} t^2 + \alpha^2\right) x_A^2 + r_c\sqrt{1-t^2}\alpha^2 x_A + \frac{r_c^2}{4}(1-\alpha^2) = 0 \end{aligned} \tag{68}$$

which is a quadratic function of $x_T$. Given a point $I(t)$ where $A$ is intercepted by $D$, we can obtain the corresponding initial position of $T = (x_T, y_T)$ such that at the interception time the Target $T$ is about to escape from $A$ because $D$ is able to interpose himself between $A$ and $T$.

In summary, given the Defender's capture radius $r_c$ and the positions of $A$ and $D$, the rotating frame of reference is determined—see Fig. 3. In this frame, we draw the hyperbola (19) with parameters $a = r_c/2$ and $b = \sqrt{x_A^2 - (r_c/2)^2}$. Then, we choose a point $I$ on the hyperbola, which is equivalent to choosing $t$ because given a point $I = (x, y)$ on the hyperbola we have $t = \frac{ay}{bx}$. Solving the quadratic equation (68), we obtain $x_T(x_A, t)$. Given $x_A$, $t$, and $x_T$, we use (67) to calculate $y_T(x_A, t)$. Then, we scan on $0 < t < 1$. Finally, note that this process can be

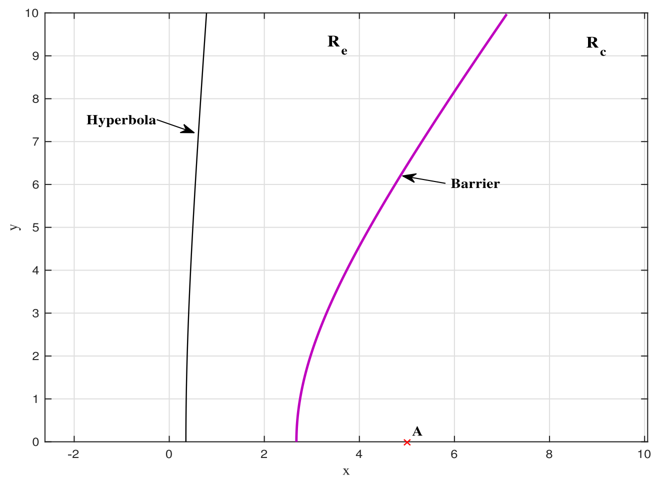


Fig. 7. Example of the solution of the Game of Kind.

repeated for different values of $x_A > r_c/2$, that is, we can obtain an offline parametric representation of the surface in the reduced state space $(x_A, x_T, y_T)$ which separates the $A$ and the $T/D$ team winning regions based only on the value of the two problem parameters, that is, the capture radius $r_c$ and the speed ratio $\alpha$.

Fig. 7 provides an example of the barrier surface that separates the Target escape region $R_e$ and the capture region $R_c$ for the following parameters: $\alpha = 0.5$ and $r_c = 0.7$. The figure illustrates the $x_A$ cross section where $x_A = 5$. Note that the Target can start on the RHS of the hyperbola (19) but only so far to the right. If it starts on the RHS of the Barrier surface then it will be captured by the Attacker.

### A. Three-Dimensional Implementation

The ATDDG was analyzed in the Cartesian plane $(x, y)$. However, air-to-air combat scenarios occur in the 3-D world. The ATDDG was conceived with beyond visual range operations in mind. Due to the long-range operations, the altitude differences are usually negligible. Nevertheless, we can address altitude differences among the agents using the optimal strategies derived in this paper. The standing assumption is that the positions in 3-D of the agents are known to all the agents in order to solve the game (determine the optimal strategies of every player in the game); it is a game of perfect information. Note that the Defender does not necessarily need to know the positions of the Target and the Attacker when it is guided by the Target, that is, when the Target can directly transmit the optimal heading (or the optimal aimpoint) to the Defender.

Assume that the speeds of each agent remain constant for all time (including when they climb or dive), then, the closed-form state-feedback strategies of the ATDDG derived in Sections IV and V are the optimal strategies in a 3-D engagement. The state $(x_A, x_T, y_T)$ (within the hyperplane formed by the Target, the Attacker, and the Defender) can be obtained from the 3-D positions of each agent. Let the 3-D coordinates of each agent in a fixed Cartesian frame be given by $T : (T_x, T_y, T_z)$, $D : (D_x, D_y, D_z)$, and $A : (A_x, A_y, A_z)$. Define $d_{AD} = \|A - D\|$, $d_{AT} = \|A - T\|$, $d_{DT} = \|D - T\|$, and $\omega = \arccos\left(\frac{d_{DT}^2 + d_{AD}^2 - d_{AT}^2}{2 d_{DT} d_{AD}}\right)$. Then,

$$\begin{aligned} x_A &= \frac{1}{2} d_{AD} \\ x_T &= d_{DT} \cos\omega - \frac{1}{2} d_{AD} \\ y_T &= d_{DT} \sin\omega. \end{aligned} \tag{69}$$

The optimal aimpoint $I^*$ in the hyperplane is computed using (69) and then the corresponding 3-D coordinates $I^*(x_I^*, y_I^*, z_I^*)$ are obtained by following the corresponding translation and rotations about the main orthogonal axes. The optimal strategies are such that the agents remain on the initial hyperplane during the duration of the engagement.

An appealing characteristic of the closed-form state-feedback optimal strategies presented in this paper is that they are robust with respect to unknown Attacker guidance laws. This means that the Defender intercepts the Attacker even when the Attacker guidance is unknown to the Target and the Defender. This includes the cases where the Attacker employs a classical pursuit strategy such as PP or PN. In the 3-D scenario, the robustness properties include the case where the Attacker pursues the Target in such a way that the Attacker does not remain on the initial hyperplane and climbs at an angle different than the one dictated by the optimal solution of the ATDDG. In this case, the Defender intercepts the Attacker and the terminal $A$–$T$ separation increases with respect to the value $J^*$ of the game.

The assumption in the 3-D scenario that the players' speeds remain constant can also be relaxed. Small variations in the speed of the agents due to climbing or descending can be implemented within the ATDDG framework. Optimality is not guaranteed in this case. However, the state-feedback strategies are robust with respect to speed variations, that is, the Defender is able to intercept the Attacker by updating the eighth-order polynomial's state-dependent coefficients on the fly and recomputing the new aimpoint.

## VII. EXAMPLES

EXAMPLE 1 Consider an initial TAD configuration in a 2-D plane where the positions of the agents are given by $T = (5.0, 11.0)$, $A = (12.71, -3.33)$, and $D = (-19.39, 7.85)$. The Target/Attacker speed ratio is $\alpha = 0.48$ and the Defender capture radius is $r_c = 0.62$. With this state and $\alpha, r_c$ parameter values, we obtain the optimal hyperbolic angle $\mu^* = 0.6774$, which determines the interception point $I^* : (x_I^*, y_I^*)$ and the value of the differential game is $J^* = 5.134$. The optimal interception point is given by $x_I^* = 1.114$ and $y_I^* = 13.85$. The optimal trajectories are shown in Fig. 8.

EXAMPLE 2 (Robustness to unknown Attacker guidance law): As mentioned earlier, the cooperative optimal guidance strategies for the ATDDG are robust control laws with respect to unknown Attacker guidance laws. This is a desirable property because of the inherent uncertainty concerning the opponent's choice of guidance law/strategy.

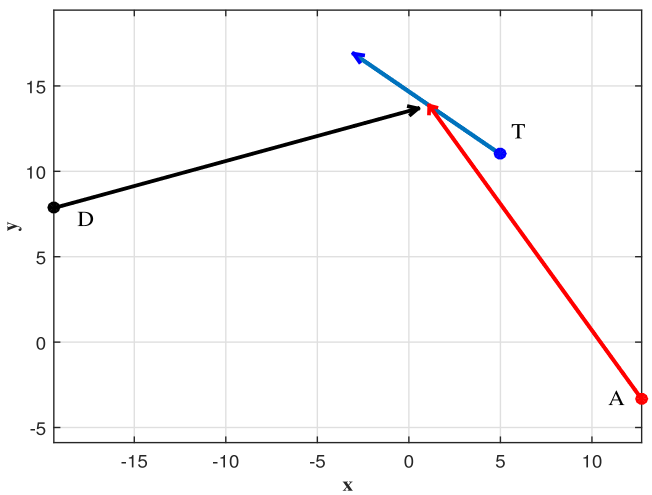


Fig. 8. Optimal trajectories in Example 1.

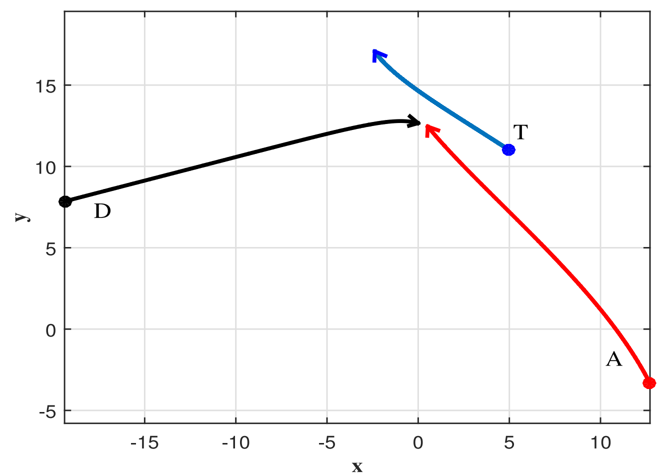


Fig. 9. Trajectories in Example 2 when the Attacker implements the PN guidance law.

Although each team is able to compute its own and its opponent's optimal headings, the opponent may choose a different, nonoptimal strategy. For instance, the Attacker may not follow its optimal policy and opt to use a different pursuit strategy that is unknown to the $T/D$ team. In such a case the Target and the Defender, having current measurements of the Attacker's position, are able to solve the polynomial (25) on the fly and update their cooperative interception strategy, thus increasing the $T$–$A$ separation at interception time, above and beyond the guarantee provided by the value of game.

Consider the same problem parameters and initial conditions as in Example 1, but the Attacker implements the PN guidance law with navigation constant $N = 3$. This information is *unknown* to the $T/D$ team. The $T/D$ team is, however, able to obtain measurements of the position of the Attacker. The $T/D$ team continuously updates the coefficients of the polynomial (25), solves it, and obtains the optimal aimpoint that dynamically changes as new measurements of $A$'s position become available.[1]

The heading of $A$ is no longer constant since it is using PN guidance instead of its optimal policy. The headings of $T$ and $D$ are not constant either since they react to the behavior of $A$ by updating (25) and constantly revising their aimpoint. $D$ intercepts $A$ and, additionally, the $T/D$ team is able to increase its payoff. The final separation in this example is $R(t_f) = 5.4601$, which is greater than the value $J^* = 5.134$ of the differential game realized in Example 1. In other words, $A$ is losing performance by not following its optimal strategy and instead implementing a standard guidance law, while the $T/D$ team's payoff increases. The resulting trajectories are shown in Fig. 9.

[1]Note that when all players follow their optimal headings, as in Example 1, the optimal aimpoint can be constantly recomputed (for robustness) using the updated positions of each agent. In such a case, the optimal aimpoint does not change and the agents' follow straight line trajectories as stated earlier in this paper.

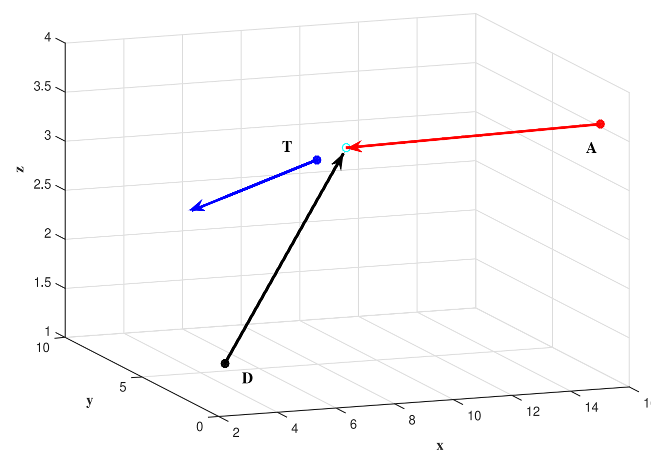


Fig. 10. Optimal trajectories in Example 3.

REMARK If the Target and the Defender would know the Attacker's guidance law, they could further improve their performance, that is, obtain an even larger final separation $R(t_f)$. Indeed, for the case when the Attacker implements the PN guidance law, a numerical solution of the attendant *optimal control* problem to obtain the optimal $T/D$ interception strategy was derived in [30].

*Example 3.* Consider the 3-D engagement where the initial locations of the players are $T = (9, 7, 2.9)$, $A = (15, 0, 3.7)$, and $D = (3, 1.5, 1.4)$. The speed ratio parameter is $\alpha = 0.45$ and the Defender's capture radius is $r_c = 0.25$. The value of the differential game is $J^* = 4.707$. The optimal interception point coordinates in the fixed 3-D frame are $I^* = (9.815, 6.597, 3.039)$ and the optimal trajectories are shown in Fig. 10.

*Example 4.* Consider the 3-D engagement where the initial locations of the players are $T = (7.8, 5, 4)$, $A = (12, 4, 2.9)$, and $D = (1.4, 2.2, 2.1)$. The speed ratio parameter is $\alpha = 0.59$ and the Defender capture radius is $r_c = 0.3$. Under the assumption that all players' speeds are constant, the value of the differential game is $J^* = 2.038$. The optimal interception point coordinates in the fixed 3-D frame are $I^* = (6.414, 5.053, 4.134)$.

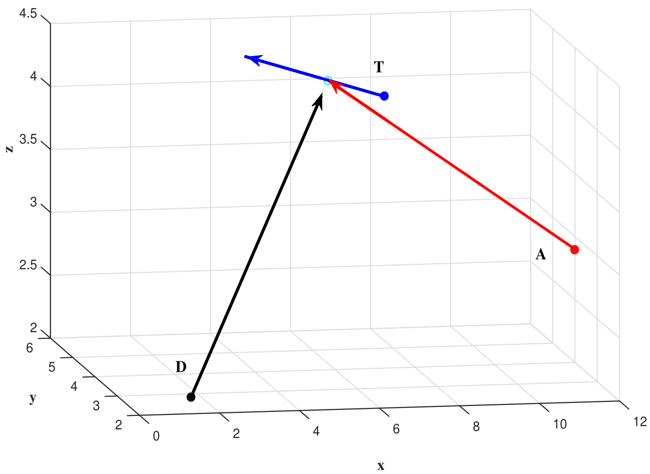


Fig. 11. Closed-loop trajectories in Example 4.

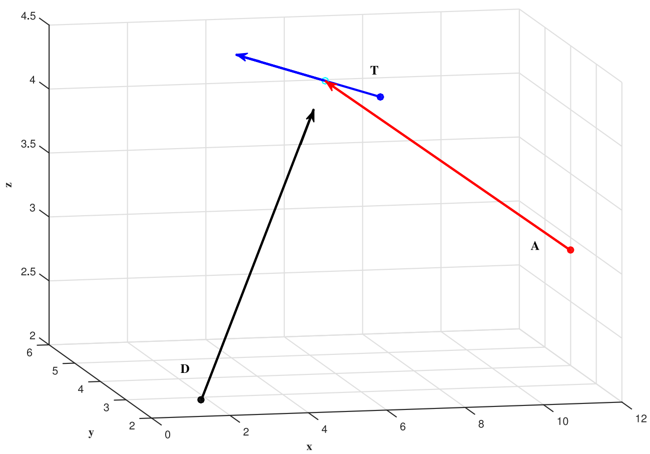


Fig. 12. Open-loop trajectories with uncertain speed change.

Now, consider the case where the agents' speeds do not remain constant and they change as a function of their corresponding elevation angle. Let us consider the case where the speed of agent $i$ is given by $v_i(t) = V_i - g_i\varphi_i(t)$, for $i = T, A, D$, where $V_i$ and $\varphi_i(t)$ represent, respectively, the constant nominal speed and the instantaneous elevation angle of agent $i$; $g_i > 0$ is an unknown constant gain representing the uncertainty related to the time-varying speed. In this example, we solve the ATDDG using the nominal speeds, which provides an optimal interception point with a corresponding elevation angle for each agent. Consider $g_i = 0.05$ for $i = A, D$ and $g_T = 0.14$. Every agent continuously recomputes the optimal interception point based on the current positions of every agent. It can be seen in Fig. 11that the Defender successfully intercepts the Attacker. The actual interception point $I = (6.393, 5.055, 4.137)$ is different than $I^*$; this is expected since the agents use the nominal speeds to compute the optimal strategies but the actual speeds are different because of the implemented nonzero elevation angles. The terminal separation is $R(t_f) = 2.021$, which in this example is less than the value of the game $J^* = 2.038$.

We compare the state-feedback solution to the open-loop solution of the problem using nominal parameters, but when implemented, the agents are subject to the same uncertain speed changing factor $g_i$. In such a case, the agents aim at point $I = (6.414, 5.053, 4.134)$—see Fig. 12. The trajectories at the time the Attacker reaches the open-loop interception point $I$, the distance between $A$ and $D$ is $r = 0.572 > r_c$. This means that the Defender fails to intercept the Attacker because it did not update its heading to respond to the time-varying speed of the agents.

## VIII. CONCLUSION

This paper addressed the ATDDG where the Defender missile is endowed with a capture circle of radius $r_c > 0$. A closed-form, state-feedback solution of this differential game was obtained. This solution only required the rooting on the fly of an eighth-order polynomial that yields the optimal interception point's coordinates, hence it provides the optimal instantaneous headings for the players and the solution of the differential game. The solution of the Game of Kind is also provided. The interesting particular case when the Target is initially located on the interception curve was also solved in this paper. The effectiveness of the analytical solution offered in this paper has been emphasized in the context of the active defense of valuable targets.

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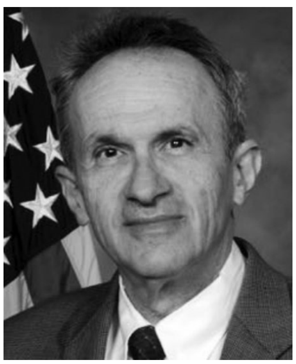

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