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Single Pursuer and Two Cooperative Evaders in the Border Defense Differential Game

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An interest in border defense, surveillance, and interdiction has recently increased for a variety of reasons related to issues of illegal immigration, terrorism, drug and human trafficking, and other potential threats. Unmanned aerial vehicles (UAVs) offer an attractive alternative to supporting and defending various threats at borders. This paper applies a differential game to define a border defense scenario where one UAV (pursuer) seeks to capture two intruders (evaders) before they reach a designated border. The evaders can be UAVs, marine or ground vehicles, or human agents, but they have a lower maximum speed than the pursuer throughout the game. Simple motion is assumed for the pursuer and evaders with complete state information shared across all agents. The game is played within a rectangular area with a parallel top and bottom border of length L and left and right borders with a length of W, for a game aspect ratio of L/W. The value of the game is the minimum distance to the bottom border achieved by the evaders at any time before capture of both evaders. Within the region where the pursuer wins, the game of degree is explored and the optimal policy for both the evaders and pursuer is derived using geometric properties.

Nomenclature

A_i	=	Apollonius circle of the ith evader
AR	=	aspect ratio of the game area defined as L/W
C_i	=	Apollonius circle center for the ith evader
d_i	=	distance between P and the ith evader
E_i	=	ith evader
F_i	=	capture point of the ith evader
H	=	barrier function separating $\Re_E$ from $\Re_P$
J	=	cost/payoff of the game
L	=	horizontal length of the game area
P	=	pursuer
$\Re_E$	=	region of win for evader
$\Re_P$	=	region of win for pursuer
r_i	=	Apollonius radius for ith evader
t_1	=	time at which the first evader is captured
t_2	=	time at which the second evader is captured
u_E	=	control variable for evader
u_P	=	control variable for pursuer
V	=	value function for the game
v_E	=	evader speed
v_P	=	pursuer speed
W	=	vertical width of the game area
x_i	=	x position of ith evader
x_P	=	x position of pursuer
$\boldsymbol{x}$	=	state of the game defined as $[x_P \quad y_P \quad x_1 \quad y_1 \quad x_2 \quad y_2]^T$
y_i	=	y position of ith evader
y_P	=	y position of pursuer
α	=	ratio of evader speed to pursuer speed

I. Introduction

UNFORTUNATELY, drug traffickers attempt to deliver thousands of pounds of narcotics and other drugs to the United States every year [1]. Likewise, thousands of attempts to illegally enter the United States are initiated daily [2]. Furthermore, suicide bombers, remotely controlled improvised explosive devices, and other threats attack U.S. buildings, territories, bases, and allies at irregular frequencies. The U.S. Department of Homeland Security keeps national borders safe and secure, and other military branches guard and protect various specific borders from infiltration or attack. Recently, these agencies and others have considered the use of unmanned aerial vehicles (UAVs) to assist with the surveillance and defense of the diverse border types throughout the world. Before full implementation, however, investigations into the best policies must be performed to guarantee success, reduce cost, and eliminate exploitation by the opposition.

To identify the optimal policies, a differential game for a border defense scenario is herein defined and analyzed. Differential games have been used extensively to explore military conflicts between agents with opposing objectives. Isaacs's foundational work [3] motivated a number of researchers to use differential games to analyze a variety of situations, including missile evasion [4], target defense [5], and naval applications [6]. In nonmilitary applications, differential games have been applied to air traffic management [7], spacecraft in low Earth orbit [8], and UAV formation flight [9].

The area of differential games has been extended to include pursuit–evasion games between one or more pursuers attempting to capture one or more evaders. Agent movement has been modeled using simple motion [10], orbital dynamics [11], and rigid-body motion [12]. Cases have been examined when players have differing levels of information [13–15]. Applications of pursuit–evasion games have been seen in constrained environments [16–18] and border defense scenarios [19,20]. In some cases, the goal is not capture, but rather confinement [21], escape [22], distance minimization [23], or maintaining visibility [24].

Many recent research efforts have focused on finding solutions to games with many players, using both analytical approaches [25,26] and computational solutions [27]. Numerous variants have been identified, such as games with obstacles [28,29], uncertain knowledge of the other players [30,31], and cases with superior evaders [32–35].

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Various formulations of the game with one pursuer and two evaders have been addressed by previous research [36–40]. The current study builds upon previous work by exploring the game of one pursuer and two cooperative evaders who strive to achieve their specific objectives. The results will form the foundation for the analytic solution to the pursuit–evasion game with M pursuers and N evaders in constrained border environments where cooperation between evaders is possible.

The game is first defined in Sec. II. Section III follows with a discussion of key definitions related to the single-pursuer, single-evader (1P1E) game in a constrained area pertinent to the solution of the single-pursuer, two-evader (1P2E) game. We then present in Sec. IV the optimal policies for the pursuer and evaders in the 1P2E game, including the starting positions and headings during the game, and then proceed to explore the game of degree. This work concludes in Sec. V with an analysis of the game of kind.

II. Game Definition

The game herein defined is played out on a rectangular area with unitless dimensions L for the length and W for the width as shown in Fig. 1, and $L \geq W$, with the bottom-left corner designated as the origin for convenience. Two or more intruders (red team) penetrate the top border at $y = W$ at time $t = 0$ and seek to reach anywhere along the line $y = 0$, at which they are assumed to escape, deliver a package, and the like. On the defending blue team, a single UAV defender (i.e., pursuer P) seeks to capture or prevent these intruders (i.e., evaders E_1 and E_2) from reaching the bottom border. Unless otherwise specified, the evaders are given complete information about each other's position $[(x_1, y_1)$ and $(x_2, y_2)]$ and the location of the pursuer (x_P, y_P), and they can cooperate to maximize their own or collective payoff at all times. The pursuer also has complete position information about the evaders throughout the game. Both the evaders and the pursuer move simultaneously, and thus no agent has a time advantage. Because the game's termination condition is when both evaders are captured or when one of the evaders has crossed the bottom border, the evaders are free to cooperate in any way possible, sacrificing one or the other evader, if necessary, in order to win the game.

The game assumes simple motion with the pursuer's speed v_P greater than the evaders' speed v_E and the speed ratio α defined as

$$\alpha = \frac{v_E}{v_P} \tag{1}$$

where $v_P > v_E > 0$ such that α is always less than one. Because the two evaders have identical speeds, their speeds will be referred to as v_E throughout this work rather than by their individual speeds, v_1 and v_2. The dynamics of the evaders and pursuer will be constrained to the two-dimensional plane and follow the simple motion of Eqs. (2) and (3), where $\boldsymbol{x} = [x_P \ y_P \ x_1 \ y_1 \ x_2 \ y_2]^T$ is the state vector comprising the three positions of the pursuer and both evaders. Speed changes can be simulated through a sequence of instantaneous heading changes over time; however, v_P and v_E are held constant during the whole game.

$$\boldsymbol{x}_{t+dt} = \boldsymbol{x}_t + \dot{\boldsymbol{x}} dt \tag{2}$$

$$\begin{aligned} \dot{x}_i &= v_E \cos\theta_i, \qquad i = 1, 2 \\ \dot{y}_i &= v_E \sin\theta_i, \qquad i = 1, 2 \\ \dot{x}_P &= v_P \cos\theta_P \\ \dot{y}_P &= v_P \sin\theta_P \end{aligned} \tag{3}$$

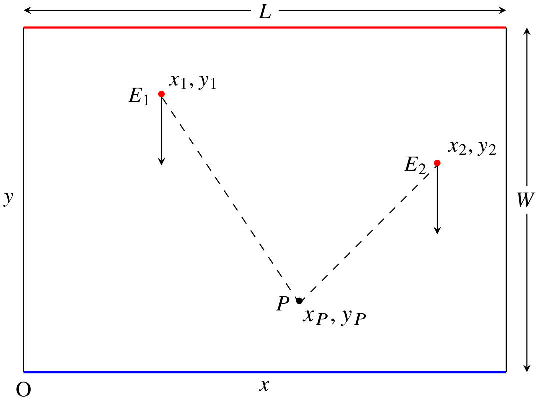


Fig. 1 Setup for game playing area for pursuit–evasion game with borders.

When the pursuer is able to succeed in point capture of both evaders before either can escape through the border, the game is played out in two stages. Stage 1 is the period required for P to capture either of the two evaders E_i, $i \in \{1, 2\}$, and concludes at time t_1. Stage 2 is the time period for P to capture the remaining evader E_j, $j \in \{1, 2\}$, $j \neq i$, and concludes at time t_2. The cost/payoff for the game is the minimum distance achieved by either evader at any point in the game. Because the bottom border is defined as the x axis, the distance to the border for each evader is given by y_1 and y_2, respectively, and thus the cost/payoff can be written as follows:

$$J = \min_t(y_1, y_2) \tag{4}$$

In the present work, we consider the Nash equilibrium of the game, such that the pursuer attempts to maximize the cost/payoff, whereas the evaders at the same time attempt to minimize it. We consider both the game of degree and the game of kind. Isaacs defined the game of degree in his seminal work as a game with a "continuum of possible payoffs" [3]. The game of kind was defined as one in which only two outcomes are possible (e.g., pursuer wins or evaders win).

Often, the consideration of initial conditions occurs in the game of kind analysis, such as in [19,22]. In other works, including [36,40], starting conditions alone have no significant influence on the outcome of the game. In the game of the lion and the man, in which a faster evader (the man) is constrained within a circular arena while attempting to avoid entering the capture radius of a pursuer (the lion), the game of kind is characterized by the speed ratio rather than by the initial positions of the agents [17], because the optimal strategy of the lion is defined by using the first portion of the game to achieve a particular starting position [16]. Likewise, in the bounded-border defense situation herein presented, the initial condition of the agents is closely connected to agent strategies and is therefore included as a control variable. The game of kind can then be defined in terms of other factors, similar to the lion and the man game.

The control variables u_P, u_1, and u_2 consist of the choice of initial x position as well as instantaneous headings throughout the game (i.e., $u_{(\cdot)}$:$\{x_{(\cdot),t_0}, \theta_{(\cdot)}\}$). The initial positions of the evaders can be at any point along the top border, but cannot extend past the left or right side borders of $x = 0$ and $x = L$, respectively. The pursuer can start at any location along the bottom border but is likewise constrained to remain within the game area $L \times W$ throughout the game. Capture of evaders can occur within the game area and on any of the four borders. Evaders and pursuer all choose their initial positions simultaneously at the beginning of the game with permissible cooperation between evaders.

III. Analysis of One Pursuer and One Evader in Closed Rectangular Borders

For the current game under investigation with two evaders, a few definitions from the 1P1E game with closed rectangular borders are needed, because the 1P2E game mathematically degenerates to 1P1E once one of the evaders has been captured.

A. Summary of Relevant Results from the Unconstrained 1P1E Game

In [19], the solution is derived for a non-closed-border defense scenario with one evader and one pursuer, and thus this section will briefly summarize the relevant results therein presented.

The cost/payoff function for the 1P1E border defense scenario was defined by Eq. (5) (taken from [19]) as the terminal distance, at the time of interception, between the evader and the border; that is,

$$J = \sqrt{(x_E - x_f)^2 + (y_E - y_f)^2} \tag{5}$$

where the point $F{:}(x_f, y_f)$ is the point on the border closest to the evader at capture. Applied to the present situation with a horizontal border, the value function derived in [19] from this cost/payoff function can be written as

$$V_1 = \frac{1}{1-\alpha^2}\left[y_E - \alpha^2 y_P - \alpha\sqrt{(x_E - x_P)^2 + (y_E - y_P)^2}\right] \tag{6}$$

For the constrained case, the value function in Eq. (6) is only valid in the region sufficiently far away from the edges of the environment, such that the constraining borders have no effect. To help define this region, which we shall denote as $\Re_1$, it is useful to discuss the 1P1E results for the game of kind.

B. Game of Kind for the Constrained 1P1E Game

In [19], the game of kind is described for a non-closed-border defense scenario by a barrier function that separates the region where an evader will win, $\Re_E$, from the region where the pursuer will win, $\Re_P$ (i.e., capture the evader), based on the current state $\boldsymbol{x}$, α, and the border definition.

In general, the function between $\Re_E$ and $\Re_P$ takes the form of a hyperbola with the following closed form when the border is the x axis (taken from [19] and applied to a horizontal border).

$$H_1 = \sqrt{\frac{\alpha^2 x^2 - 2\alpha^2 x_P x + \alpha^2(1-\alpha^2)(x_P^2 + y_P^2) + \alpha^4 x_P^2}{1-\alpha^2}} \tag{7}$$

Equation (7) is shown in Fig. 2 as a solid black line to the left of the point E_b and as a dashed line to the right of the point. The solid black line in Fig. 2 represents the locus of evader positions for which the pursuer will capture the evader exactly at the bottom border. For example, considering the notional evader E_a on the black line defined by Eq. (7), we note that the border line (i.e., $y = 0$) is tangent to E_a's Apollonius circle, implying that E_a will optimally be captured at point F_a on the border.

The Apollonius circle A_i is defined as the set of points at which P and E_i can reach simultaneously with constant headings and speed. The center C_i and radius r_i of an Apollonius circle are calculated as follows:

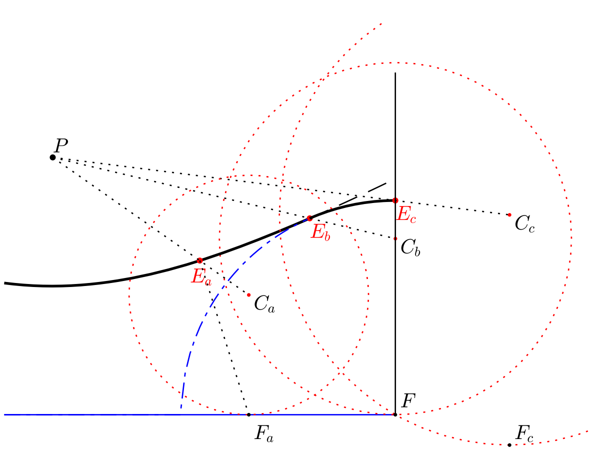


Fig. 2 Bottom-right corner of the game area showing two sections of the barrier function H dependent on the border constraints and $\alpha = 0.5$.

$$x_c = \frac{1}{1-\alpha^2}(x_i - \alpha^2 x_P)$$
$$y_c = \frac{1}{1-\alpha^2}(y_i - \alpha^2 y_P) \tag{8}$$

$$r_i = \frac{\alpha}{1-\alpha^2} d_i \tag{9}$$

where $d_i = \sqrt{(x_i - x_P)^2 + (y_i - y_P)^2}$ is the distance between P and E_i [30]. For the previously discussed evader E_a in Fig. 2, the center of E_a's Apollonius circle, C_a, is exactly one radius above the point F_a. Any evader position along H_1 up to the point defined by E_b will likewise have the bottom border tangent to their Apollonius circle.

The H_1 equation is conservative from P's perspective if extended to the right border, as evader positions to the right of E_b will be constrained by the border. For the evader positions on the dashed section between E_b and the right border, the tangent point of the Apollonius circle and $y = 0$ will be beyond the boundary created by the right border. In these cases, the Apollonius circle will intersect the right border above F, the evader will be captured above the bottom border, and therefore the states corresponding to these positions are considered well within the region $\Re_P$. However, at a certain point on the right border (i.e., E_c), the Apollonius circle will intersect the bottom-right corner and capture will occur at this corner. This point is therefore on the barrier separating $\Re_P$ and $\Re_E$ and is defined as

$$H_L = \alpha\sqrt{(x_P - L)^2 + y_P^2} \tag{10}$$

Between E_b and E_c, the barrier function H is defined as the evader positions that will result in capture at the bottom-right corner:

$$H_2 = \sqrt{\alpha^2((x_P - L)^2 + y_P^2) - (x - L)^2} \tag{11}$$

When on H_2, the evaders will have an Apollonius circle that intersects exactly at the bottom-right corner. The curve described by H_2 is a section of a circle created by rotating $\overline{E_c F}$ about F (shown in Fig. 2 with an extension down to the bottom border as a dash–dot line). For evader positions on H_2, the lowest point on the Apollonius circle, such as F_c in Fig. 2, will fall on or to the right of F, and therefore a heading toward F is optimal with capture at the bottom border at the corner. The point H_L defined above is a special case of H_2 when the evader is on the border $x = L$ and is shown at position E_c in Fig. 2. A similar function for the left border is calculated by replacing L with 0 in the equation for H_2 such that the equation becomes

$$H_0 = \sqrt{\alpha^2(x_P^2 + y_P^2) - x^2} \tag{12}$$

Together, these equations define a piecewise equation for the barrier function along the entire length of the game area, which can be separated into three regions, similar to the method used in [20]. These regions are defined by Eqs. (13a–13c), which are calculated by finding the values of x when the center of the Apollonius circle defined in Eq. (8) crosses the left and right boundaries of the game area.

$$\Re_0{:}\, x \le \alpha^2 x_P \tag{13a}$$

$$\Re_1{:}\, \alpha^2 x_P < x < L(1-\alpha^2) + \alpha^2 x_P \tag{13b}$$

$$\Re_L{:}\, x \ge L(1-\alpha^2) + \alpha^2 x_P \tag{13c}$$

Thus, the equation for the barrier function is given by

$$H(x_P, y_P, x_i, \alpha, L) = \begin{cases} H_0, & \boldsymbol{x} \in \Re_0 \\ H_1, & \boldsymbol{x} \in \Re_1 \\ H_2, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{14}$$

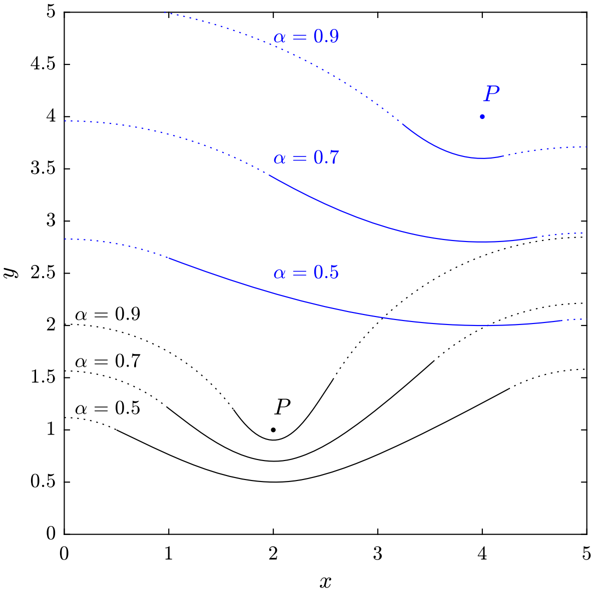


Fig. 3 Example H barriers at two P positions for three speed ratios. The piecewise H function is shown with solid lines for H_1 and dotted lines for H_0 and H_2.

Any evader with a positive y position between this H barrier and the x axis will win. On the other hand, E will be captured if it is on or above this piecewise continuous H function. A set of H barriers are visualized in Fig. 3 with example solutions of the above function with (x_P, y_P) at (2, 1) and (4, 4) each at three different speed ratios, $\alpha = 0.5, 0.7$, and 0.9. With lower speed ratios (i.e., P is much faster), the H barrier will lie closer to the x axis, and thus $\Re_P$ is larger.

C. Game of Degree for the Constrained 1P1E Game

The understanding of the game of kind and the regions involved can help us further define the value function for each region of the constrained 1P1E game, such that the value function can be written as

$$V(\alpha, \boldsymbol{x}) = \frac{1}{1-\alpha^2}\begin{cases} V_0, & \boldsymbol{x} \in \Re_0 \\ V_1, & \boldsymbol{x} \in \Re_1 \\ V_2, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{15a}$$

where

$$V_0 = y_E - \alpha^2 y_P - \sqrt{(\alpha d_E)^2 - (\alpha^2 x_P - x_E)^2} \tag{15b}$$

$$V_1 = y_E - \alpha^2 y_P - \alpha d_E \tag{15c}$$

$$V_2 = y_E - \alpha^2 y_P - \sqrt{(\alpha d_E)^2 - (L(1-\alpha^2) + \alpha^2 x_P - x_E)^2} \tag{15d}$$

and where

$$d_E = \sqrt{(x_E - x_P)^2 + (y_E - y_P)^2} \tag{15e}$$

is the distance from P to E.

Using the value function defined by Eq. (15) and limiting the region of analysis to $\Re_P$, we can make some observations about the optimal behavior of an evader in a constrained-border environment.

Theorem 1: The value of the 1P1E game is minimized when the evader starts in the corner farthest from the pursuer.

Proof: Taking the partial derivatives of the value function with respect to the evader position on the interval $x_i \in [0, L]$ and $y_i \in [0, W]$ results in Eqs. (16) and (17).

$$\frac{\partial V(\alpha, \boldsymbol{x})}{\partial x_E} = \begin{cases} \dfrac{x_E}{\sqrt{(\alpha d_E)^2 - (x_E - \alpha^2 x_P)^2}}, & \boldsymbol{x} \in \Re_0 \\ \dfrac{\alpha(x_P - x_E)}{(1-\alpha^2) d_E}, & \boldsymbol{x} \in \Re_1 \\ \dfrac{x_E - L}{\sqrt{(\alpha d_E)^2 - (x_E - \alpha^2 x_P + L(\alpha^2 - 1))^2}}, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{16}$$

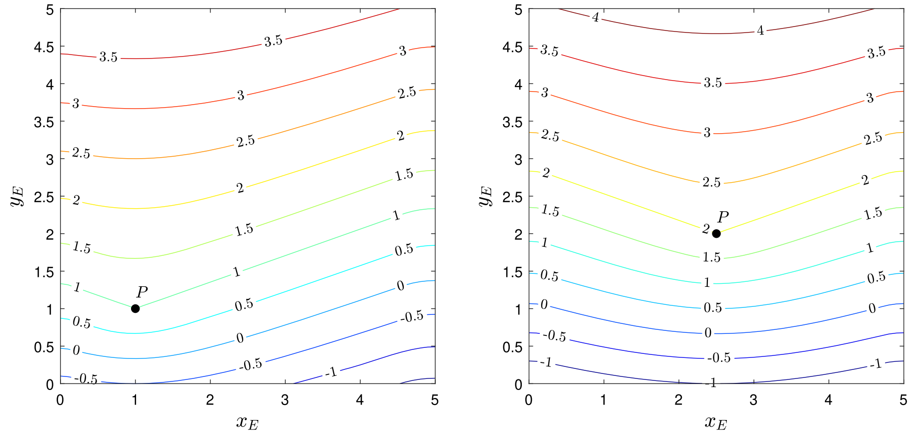
$$\frac{\partial V(\alpha, \boldsymbol{x})}{\partial y_E} = \begin{cases} \dfrac{\alpha^2 (y_P - y_E)}{(1-\alpha^2)\sqrt{(\alpha d_E)^2 - (x_E - \alpha^2 x_P)^2}} - \dfrac{1}{1-\alpha^2}, & \boldsymbol{x} \in \Re_0 \\ \dfrac{\alpha (y_P - y_E)}{(1-\alpha^2) d_E} - \dfrac{1}{1-\alpha^2}, & \boldsymbol{x} \in \Re_1 \\ \dfrac{\alpha^2 (y_P - y_E)}{(1-\alpha^2)\sqrt{(\alpha d_E)^2 - (x_E - \alpha^2 x_P + L(\alpha^2 - 1))^2}} - \dfrac{1}{1-\alpha^2}, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{17}$$

Setting Eq. (17) to zero results in no real-valued critical points. Evaluation of the equation at various points shows that it is exclusively positive on the interval $[0, W]$, and thus the value function monotonically increases with y. Setting Eq. (16) to zero results in real-valued critical points when $x_E = 0, x_P$, and L. Analysis of these x values reveals that, for all values of y, $x_E = x_P$ results in a local maximum of the value function, whereas $x_E = 0, L$ results in local minima. Evaluating Eq. (15) at these local minima and comparing the results, we find that $V_0(\alpha, \boldsymbol{x} | x_E = 0) = V_2(\alpha, \boldsymbol{x} | x_E = L)$ when $x_P = L/2$, $V_0(\alpha, \boldsymbol{x} | x_E = 0) > V_2(\alpha, \boldsymbol{x} | x_E = L)$ when $x_P > L/2$, and $V_0(\alpha, \boldsymbol{x} | x_E = 0) < V_2(\alpha, \boldsymbol{x} | x_E = L)$ when $x_P < L/2$. Examples of the value of the game with respect to evader position for two pursuer positions are given in Fig. 4. This result can be summarized by stating that the value of the game is minimized when the evader is on the side of the game area farthest from the pursuer. In the case where the evader is allowed to choose its starting position along the top border, the optimal starting points are $x_E = 0$ when $x_P \geq L/2$ and $x_E = L$ when $x_P \leq L/2$. □

Theorem 2: The evader should move down along the border the entire 1P1E game.

Proof: By Theorem 1, we know that the value function is minimized along the borders and that the optimal starting values for the evader are $x_E = 0$ or $x_E = L$ depending on the pursuer starting location. For any value of $0 \leq x_P \leq L$, it can be shown that $0 \leq \alpha^2 x_P$ and that $L \geq L(1-\alpha^2) + \alpha^2 x_P$; thus, for these starting values of x_E, the game occurs exclusively in regions $\Re_0$ and $\Re_2$. By definition of these regions, the lowest point of the Apollonius circle will be on or beyond the lower corners of the game area, and thus optimal capture will be constrained by one of the side borders. Because E optimally begins on a corner, the optimal policy will be to travel straight along the side of the game area to the capture point on the same side border. □

Taken together, Theorems 1 and 2 imply that the optimal control for the evader in the 1P1E game is $u_E = \{(0, W), (3\pi/2)\}$ for $x_P \geq L/2$ and $u_E = \{(L, W), (3\pi/2)\}$ for $x_P \leq L/2$. We note that this solution does not completely define the optimal evader strategy in a 1P1E game where agents must choose their starting points simultaneously; however, these conclusions have relevance in the 1P2E game.



a) $(x_p, y_p) = (1, 1)$ b) $(x_p, y_p) = (2.5, 2)$

Fig. 4 Contours of the game value with respect to evader position for two pursuer positions with $\alpha = 1/3$ and $L = W = 5$.

IV. Game of Degree for One Pursuer and Two Evaders in Closed Rectangular Borders

A. Optimal Policies

We present in this section geometric proofs for the optimal policies for E_1, E_2, and P based on the game of degree in $\Re_P$. We note that these optimal policies form the solution to the optimization problem described in the game definition, and given by the equation

$$\max_{u_P} \min_{u_1 u_2} \min_t (y_1, y_2) \tag{18}$$

In the following theorems, we begin with the assumption that P follows a defined capture order of first E_1 and then E_2.

Theorem 3: The value of stage 2 of the game is minimized when E_1 is captured at $(0, W)$ or (L, W).

Proof: Point capture of E_1 requires that $x_1 = x_P$ and $y_1 = y_P$ at t_1; therefore, we can think of E_1 as having the ability to choose the values of x_P and y_P at the beginning of stage 2 within the set of positions that E_1 can reach before capture (i.e., points within A_1). Because stage 2 involves only one evader, at $t \geq t_1$ the value function is that of the 1P1E game with closed rectangular borders and can be given by Eq. (15) defined in the previous section.

In a process similar to that of Theorem 1, we take the partial derivatives of Eq. (15) with respect to x_P and y_P on the interval $x_P \in [0, L]$ and $y_P \in [0, W]$, yielding

$$\frac{\partial V(\alpha, \boldsymbol{x})}{\partial x_P} = \begin{cases} \dfrac{-\alpha^2 x_P}{\sqrt{(\alpha d_2)^2 - (\alpha^2 x_P - x_2)^2}}, & \boldsymbol{x} \in \Re_0 \\ \dfrac{\alpha(x_2 - x_P)}{(1-\alpha^2) d_2}, & \boldsymbol{x} \in \Re_1 \\ \dfrac{\alpha^2 (L - x_P)}{\sqrt{(\alpha d_2)^2 - (L(1-\alpha^2) + \alpha^2 x_P - x_2)^2}}, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{19}$$

$$\frac{\partial V(\alpha, \boldsymbol{x})}{\partial y_P} = \frac{1}{1-\alpha^2} \begin{cases} \dfrac{\alpha^2 (y_2 - y_P)}{\sqrt{(\alpha d_2)^2 - (\alpha^2 x_P - x_2)^2}} - 1, & \boldsymbol{x} \in \Re_0 \\ \dfrac{y_2 - y_P - \alpha d_2}{d_2}, & \boldsymbol{x} \in \Re_1 \\ \dfrac{\alpha^2 (y_2 - y_P)}{\sqrt{(\alpha d_2)^2 - (L(1-\alpha^2) + \alpha^2 x_P - x_2)^2}} - 1, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{20}$$

The solutions for x_P from $(\partial V(\alpha, \boldsymbol{x}) / \partial x_P) = 0$ are

$$x_P = \begin{cases} 0, & \boldsymbol{x} \in \Re_0 \\ x_2, & \boldsymbol{x} \in \Re_1 \\ L, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{21}$$

The solutions for y_P can also be calculated by setting $[\partial V(\alpha, \boldsymbol{x}) / \partial y_P] = 0$ and are given by

$$y_P = \begin{cases} y_2 \pm \sqrt{\alpha^2 x_P^2 - x_2^2}, & \boldsymbol{x} \in \Re_0 \\ y_2 \pm \dfrac{\alpha(x_P - x_2)}{\sqrt{1-\alpha^2}}, & \boldsymbol{x} \in \Re_1 \\ y_2 \pm \sqrt{(\alpha(L - x_P))^2 - (L - x_2)^2}, & \boldsymbol{x} \in \Re_2 \end{cases} \tag{22}$$

The solutions for y_P in both border regions can be discounted, because they do not result in real solutions in the range for which they are valid. Including all boundary values reveals that the real-valued critical points of the equation occur at all combinations of $x_P \in \{0, (x_2/\alpha^2), x_2, ((x_2 - L(1-\alpha^2))/\alpha^2), L\}$ and $y_P \in \{0, y_2 \pm (\alpha(x_P - x_{2,t_1}) / \sqrt{1-\alpha^2}), W\}$. We exclude the critical points outside the reachability set of E_1 such as $y_P = 0$. Evaluation of Eq. (15) at the remaining critical points reveals that it is minimized at

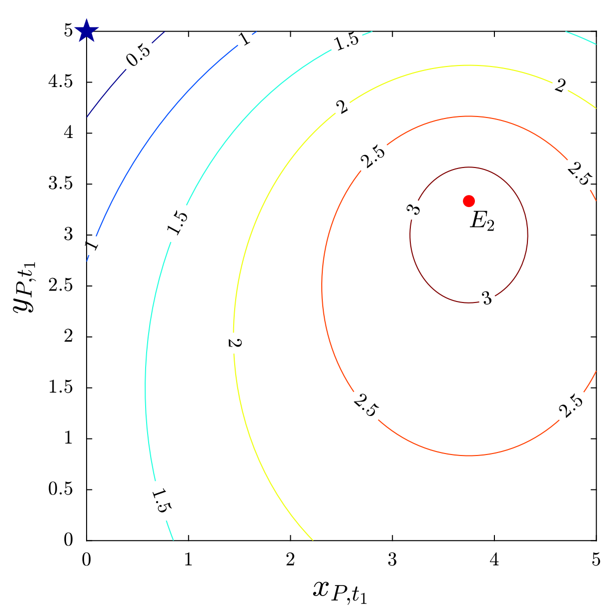


Fig. 5 Contours of the final cost/payoff for $L = W = 5$ and $\alpha = 1/3$ for various pursuer positions at $t = t_1$ given an evader at $(3L/4, 2W/3)$. The minimum value is marked with a star at $(0, W)$.

$(x_P, y_P) = (0, W)$ when $x_2 \geq L/2$ and at $(x_P, y_P) = (L, W)$ when $x_2 \leq L/2$. This minimum is illustrated in Fig. 5 for a single value of x_2. □

Remark: Because E_2 is allowed communication with E_1 throughout the game, including during the selection of starting positions, E_2 could choose a starting position in the corner opposite the anticipated capture point of E_1, thus allowing E_2 to minimize the value of the second stage of the game by Theorems 1 and 3.

Theorem 4: E_2 should move straight down along the border during the entire 1P2E game.

Proof: Assume that E_2 starts optimally in the corner opposite E_1's eventual capture point. Recall from Theorem 1 that the value function for the 1P1E game is monotonic with respect to y_E and decreases with lower values of y_E. Thus for a given pursuer position at the start of stage 2, the value of the game will be minimized if E_2 minimizes y_2 during stage 1. This minimization is done by moving in a straight line along the border with $\theta_2 = (3\pi/2)$. By Theorem 2, E_2 should maintain this heading during stage 2 of the game, and thus E_2 should maintain the heading $\theta_2 = (3\pi/2)$ during the entire 1P2E game. □

Proposition 1: When evaders play optimally, the cost/payoff of the 1P2E game with a defined capture order can be written in terms of the capture point of E_1, $(x_{1,c}, y_{1,c})$, and the initial positions of P, (x_P, y_P), and E_2, (x_2, y_2) as

$$J(\alpha, x) = \frac{1}{1-\alpha^2}\begin{cases} y_{2,t_1} - \alpha^2 y_{1,c} - \sqrt{(\alpha d_{2,t_1}) - (\alpha^2 x_{1,c} - x_2)^2}, & x_2 = 0 \\ y_{2,t_1} - \alpha^2 y_{1,c} - \sqrt{(\alpha d_{2,t_1}) - (\alpha^2 x_{1,c} - x_2)^2}, & x_2 = L \end{cases} \tag{23}$$

where

$$y_{2,t_1} = y_2 - \alpha\sqrt{(x_{1,c} - x_P)^2 + (y_{1,c} - y_P)^2} \tag{24}$$

and

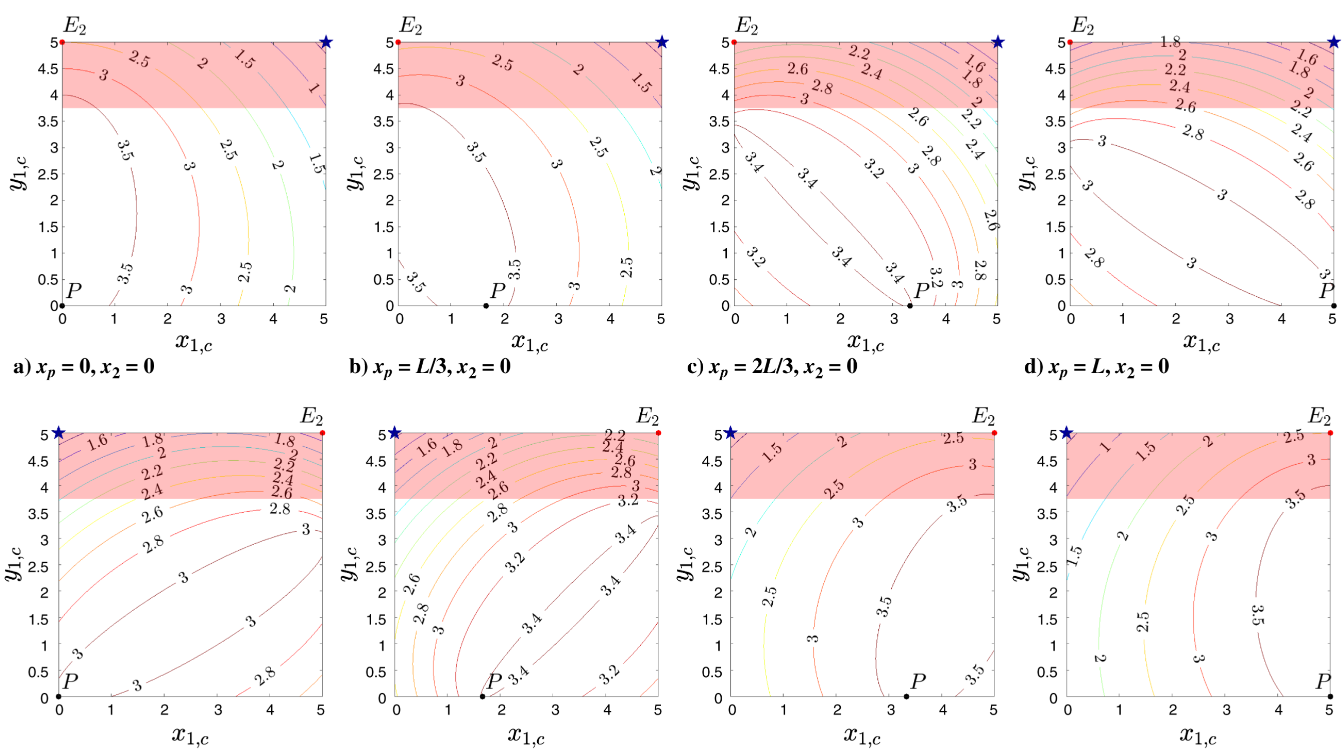
$$d_{2,t_1} = \sqrt{(x_2 - x_{1,c})^2 + (y_{2,t_1} - y_{1,c})^2} \tag{25}$$

Proof: The time for the pursuer to capture E_1 is proportional to the distance between P and the capture point of E_1 given by $\sqrt{(x_{1,c} - x_P)^2 + (y_{1,c} - y_P)^2}$. If moving in a single direction, the distance traveled by E_2 during stage 1 will be equal to $\alpha\sqrt{(x_{1,c} - x_P)^2 + (y_{1,c} - y_P)^2}$. Following the optimal policy described by Theorem 4, we have that E_2's position can be described at time t_1 by (x_2, y_{2,t_1}), where y_{2,t_1} is given by Eq. (24). At the start of stage 2 of the game, the distance between P and E_2 is the distance between the capture point of E_1, $(x_{1,c}, y_{1,c})$, and E_2's position, (x_2, y_{2,t_1}). Substituting these values into Eq. (15) results in Eq. (23). The value of the function in $\Re_1$ has been omitted because Theorem 4 states that E_2 will move along the sides of the game area throughout the game, which restricts the game to the regions of $\Re_0$ and $\Re_2$. □

Theorem 5: The cost/payoff of the 1P2E game is minimized when E_1 is captured at $(0, W)$ or (L, W).

Proof: Using the cost/payoff defined in Proposition 1, we show that the optimal capture points of E_1 defined in Theorem 3 continue to apply when both stages are taken into account. Although the derivatives Eq. (23) are not easily solved, it can be shown through simulation that the minimum value of the cost/payoff occurs in the corner opposite the starting position of E_2. Eight scenarios are given in Fig. 6 with combinations of four different starting positions for P and two different starting positions for E_2. These scenarios demonstrate that even when considering motion during the first stage of the game, it is still optimal for E_1 to be captured in the corner opposite E_2's starting position. □

For the following theorems, we relax the assumption of a defined capture order, such that P may begin by pursuing E_1 or E_2, and switch at any time. In this section, we will refer to the first evader captured as



a) $x_p = 0, x_2 = 0$ b) $x_p = L/3, x_2 = 0$ c) $x_p = 2L/3, x_2 = 0$ d) $x_p = L, x_2 = 0$

e) $x_p = 0, x_2 = L$ f) $x_p = L/3, x_2 = L$ g) $x_p = 2L/3, x_2 = L$ h) $x_p = L, x_2 = L$

Fig. 6 Contour plots of the final cost/payoff of the game with $L = W = 5$ and $\alpha = 1/3$ for all possible capture points of E_1 evaluated at various starting positions of P and E_2. The region of reachability for E_1 is highlighted in red, and the capture point corresponding to the minimum value is marked with a blue star.

E_i and the second evader captured as E_j, as explained in the game definition.

Theorem 6: Both evaders optimally move down during the first stage of the game at least to the point $y = W[1 - (\alpha/2)]$.

Proof: We begin by showing that by moving down to the point $y = W[1 - (\alpha/2)]$, the evaders do not increase the cost/payoff of the game. The minimum distance that must be traversed by the pursuer to reach one of the two optimal capture points of E_i on the top border is equal to W if the pursuer begins such that $x_P = x_i$. The distance each evader can move during this time is $\alpha\sqrt{(x_i - x_P)^2 + (y_i - y_P)^2} = \alpha W$. Therefore, if each evader were to move down a distance of $\alpha W/2$ to the point $y = W[1 - (\alpha/2)]$, the evader being pursued could still return to be captured at the optimal capture point of $(0, W)$ or (L, W).

We now show that this motion is necessary for the evaders to minimize the value of the game. We first assume that, in the case of an undefined capture order, Theorems 3 and 4 may be applied to the first and second evaders captured, respectively. At t_0 when starting points are selected, there is no way for the evaders to know which will be captured first. Therefore, in order to satisfy both Theorem 3 and Theorem 4, both evaders must choose starting positions on opposite corners of the game area.

Suppose that E_1 assumes that it will be the first evader captured and does not initially move downward, but rather moves in any other direction or remains stationary, knowing that the corner opposite E_2 is within A_1 and can be reached before capture. If P were to initially target E_2 rather than E_1, and capture E_2 in the corner opposite to E_1's starting position, E_1 will not have optimally minimized the cost/payoff as described in Theorem 4. □

The evader motion described in Theorem 6 is illustrated in Fig. 7, where the pursuer has chosen to pursue E_1 first, and E_1 has moved down past the point $y_1 = W[1 - (\alpha/2)]$. We note that this point is also where A_1 intersects $(0, W)$ when P starts at $(0, 0)$. The first evader to be captured is essentially acting as a decoy, enticing P to pursue it to the top-left corner. This action is done by acting in concert with the other evader by heading toward the bottom border in parallel paths.

Theorem 7: The pursuer optimally starts at $x_P = 0$ or $x_P = L$.

Proof: Because of the requirement of point capture, P begins stage 2 wherever capture of E_i has occurred. As described by Theorem 3, the pursuer has little influence over the capture point of E_i. Therefore, the primary method for the pursuer to maximize the cost/payoff of the 1P2E game is to minimize the progress toward the border of E_j during stage 1. This progress is proportional to the time taken to capture E_i, as described in Theorem 4. Because P knows that the optimal evader strategy is for both evaders to start in opposing corners such that one may be captured in the corner, P can capture E_i in the shortest amount of time by starting at either corner [i.e., $(0, 0)$ or $(L, 0)$]. This strategy allows P to capture E_i by traveling the shortest distance, thereby minimizing E_j's progress toward the bottom border during stage 1. To demonstrate this concept, Fig. 8 shows two potential paths for P to capture E_i and the resultant position of E_j at t_1, with $\alpha = 0.5$. □

Remark: To illustrate Theorem 7 in terms of the cost/payoff, Fig. 9 presents the game's value $V(\boldsymbol{x}, t_2)$ (i.e., y_{j,t_2}) for P's starting positions

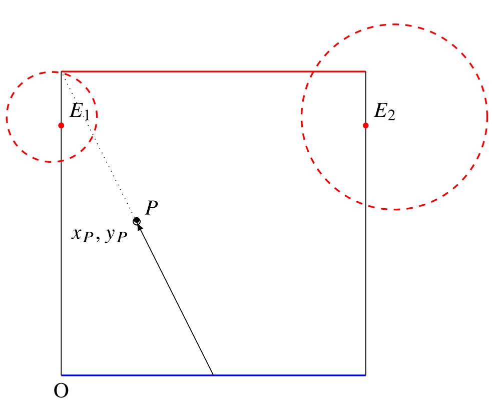


Fig. 7 Point at which E_1 turns back to be captured at (0, W).

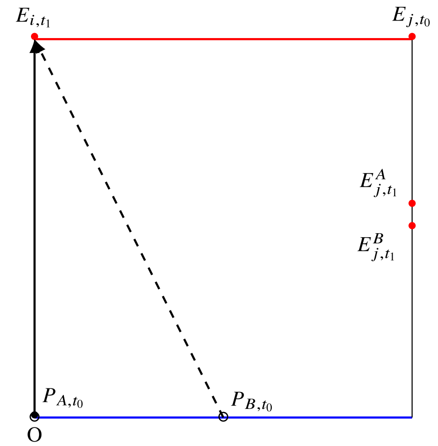


Fig. 8 Potential starting positions for P.

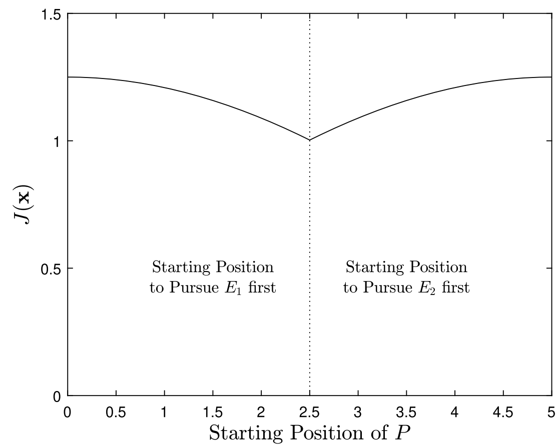


Fig. 9 $J(\alpha, x)$ of various starting positions for P when $W = L = 5$ and $\alpha = 1/3$.

between 0 and L (where $L = 5$, $W = 5$, and $\alpha = 1/3$), with optimal starting positions for E_1 and E_2 in the top-left and top-right corners, respectively. When P starts in the left half (i.e., $x_{P,t_0} < L/2$), P will capture E_1 first, and when starting on the right half, P will pursue and capture E_2 first. Interestingly, the worst policy for P's starting position is in the middle due to the largest distance required for capturing either E_1 or E_2 in their respective corners during stage 1.

Taken together, the theorems in this section define the optimal controls of the agents u_1, u_2, and u_P.

B. Exploration of Policies

This section will establish the intuition behind the optimal policies from the previous section, exploring deviations from the optimal strategies and their effects.

As shown in Sec. III.B, the H_0 and H_2 components of the H barrier are made up of circular segments centered on the bottom corners of the game area with local maxima on the edges. As was shown in Fig. 4, the H barrier is just the value contour of the 1P1E game corresponding to $V(\alpha, \boldsymbol{x}) = 0$, and thus the distance between an evader and the border is directly proportional to the value of the game. Because the barrier divides $\Re_P$ from $\Re_E$, if either one of the evaders crosses the H barrier, the game is over and the evaders win. Therefore, intuitively, it makes sense for the evaders to start above these local minima on the edges of the game area to be able to minimize the value of the game, as explained in Theorems 4 and 5. This principle is illustrated in Fig. 10.

Example: From the preceding analysis, it may be tempting for the pursuer to start in the middle of the game area in order to increase the initial distance from the evaders to the barrier. Theorem 7 shows that when P picks one evader to pursue first at the beginning of the game,

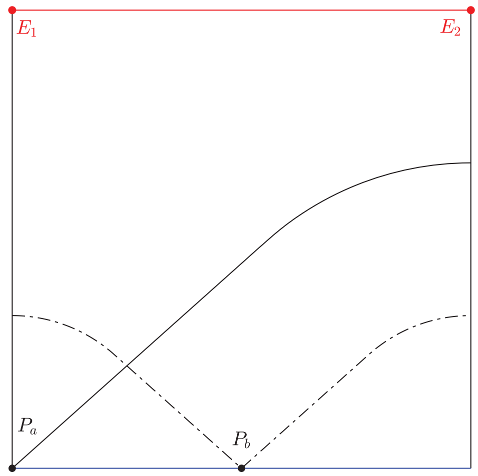


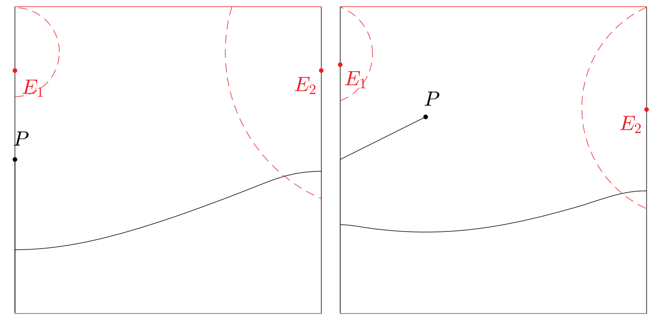
Fig. 10 Example hyperbola barriers for two potential pursuer starting positions. For both hyperbolas, at least one of the evaders is located directly above a local minimum.

this starting point is suboptimal. However, suppose that P decides to delay the decision of which evader to pursue and instead decides to maintain an equal distance from both evaders, similar to Breakwell's analysis of the 1P2E minimum time game [36], and then choose which evader to pursue at some time $t > 0$. The four images in Fig. 11 present different time points exploring this behavior with $\alpha = 0.4142$.

The sequence resulting from this strategy is visualized in Figs. 11a and 11b, where the pursuer is moving toward the bisection of $\overline{E_1 E_2}$. We note that at $t = 1.04$, both evaders are still making progress toward the bottom border. The evader strategy at this time has remained unchanged, because the optimal capture points are still within the Apollonius circles of both evaders. Because the pursuer has started in the middle, the evaders are able to move down further past the point described by Theorem 6 while still able to reach the corners at the same time as P. Because the capture point of the first evader remains unchanged from the optimal scenario, deciding to pursue one of the evaders at this point would clearly result in a lower value of the game than if the pursuer had begun on one of the corners.

As P continues to move toward the top border, the evaders can take advantage of the pursuer's strategy by continuing to move downward rather than back toward the optimal capture point described by Theorem 5, thus forcing the pursuer to choose one of the two evaders, as depicted in Fig. 11c. As soon as the pursuer changes trajectory, E_1 could respond by moving to the top of its Apollonius circle, thus drawing the pursuer away from E_2. We see in Fig. 11d that as P and E_1 move toward the capture point, E_2 is able to cross into $\Re_E$, thus guaranteeing an evader victory. We note that, had all agents played optimally with $\alpha = 0.4142$, the pursuer would have won with a final payoff of 0.

Example: Now suppose that the pursuer decides to start optimally in the corner, but at some point $t < t_1$ decides to switch which evader to pursue. If the pursuer makes this change before the evaders have reached the point $W[1 - (\alpha/2)]$, the evader strategy will remain unchanged, as explained by Theorem 6 and illustrated in the previous



a) $t = 1.04$ **b) $t = 1.7$**

Fig. 12 Relative positions of P, E_1, and E_2 at the beginning of a game where P decides to switch targets and pursue E_2.

example. Deviating from the straight line path before the evaders have reached this point will result in the same capture point of the first evader, but allow more time for the second evader to move down, and will therefore be worse for the pursuer. This scenario is shown at the point when $y_1 = W[1 - (\alpha/2)]$ in Fig. 12a.

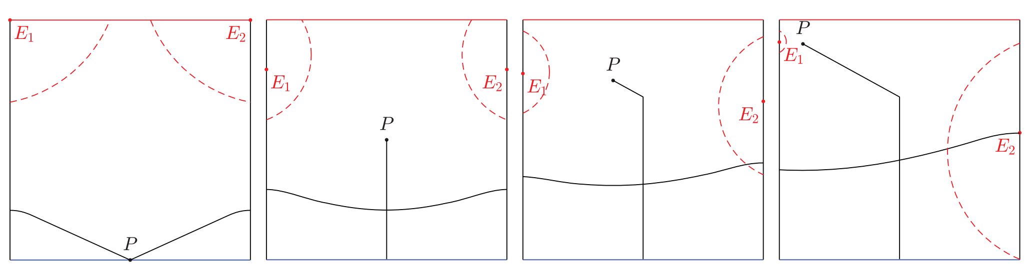
As P moves away from the left border, E_1's Apollonius circle will change, allowing E_1 to move only slightly to maintain the optimal capture point of $(0, W)$ within A_1 rather than retreating back to the corner as would be done if all agents had played optimally. E_1 can continue this motion until some time when both evaders' Apollonius circles will intersect the top corners, as shown in Fig. 12b. This evader's counterstrategy leaves P in a position similar to that of the previous example, where P must choose whether to return to pursuit of E_1 or continue in the pursuit of E_2.

Figure 13 shows three of many potential outcomes of this scenario: one where P continues pursuit of E_2, one where P returns to pursuit of E_1, and one where P attempts to confuse the evaders by switching back to E_1 and then again to E_2. Based on P's trajectory and with perfect information, the evaders are quickly able to respond to the pursuer's actions, and the evader not being actively pursued is able to continue downward toward the border. In each case, before capture of the first evader, one of the evaders is able to cross the barrier function, thus guaranteeing an evader victory. Because playing optimal strategies results in a win for the pursuer, these cases, where the pursuer attempts to switch targets, are clearly suboptimal.

As shown by these examples, there exist evader strategies such that switching targets is a suboptimal pursuer behavior. Although these examples do not provide an exhaustive exploration of suboptimal behavior, they are sufficient to show that players are penalized for deviating from optimal strategies. We leave a more complete characterization of the best responses to suboptimal strategies for future work.

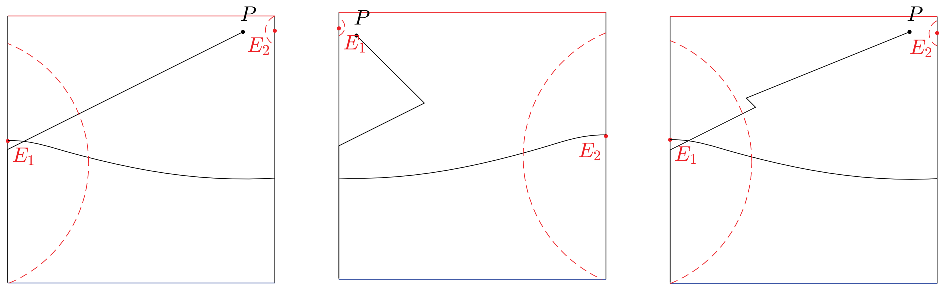
V. Game of Kind

Although the optimal policies for the pursuer and evaders remain the same as defined in Sec. IV.A, the outcome is highly dependent on both the aspect ratio ($AR = L/W$) of the game area and the speed ratio α. With a relatively low AR, such as $AR = 1$ shown in the top-left image of Fig. 14, the optimal evader policy is insufficient to win if α is 0.3 or 0.4. However, if $\alpha = 0.5$, E_2's position at t_1 will be below



a) $t = 0$ **b) $t = 1.04$** **c) $t = 1.7$** **d) $t = 2.36$**

Fig. 11 Relative positions of P, E_1, and E_2 at four different times during a game. P initially heads toward the point bisecting $\overline{E_1 E_2}$ before selecting one of the evaders to pursue at $t = 1.4$.



a) Choice 1 at $t = 3.08$ b) Choice 2 at $t = 2.52$ c) Choice 3 at $t = 3.24$

Fig. 13 Potential positions of the game described in Fig. 12 resulting from three different pursuer choices. In each case, evaders are guaranteed a victory due to their position below the barrier function.

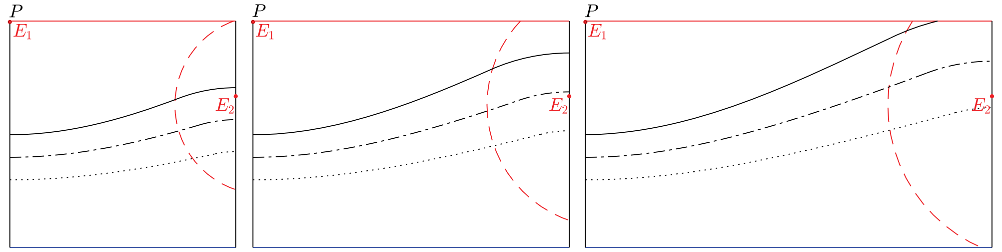


Fig. 14 Relative positions at t_1 for three aspect ratios ($AR = 1$, 1.4, and 1.9 from left to right) at three speed ratios ($\alpha = 0.5$ [solid], $\alpha = 0.4$ [dash–dot], and $\alpha = 0.3$ [dotted]).

the H barrier as shown by the solid line, thus guaranteeing an evader victory. When $\alpha = 0.5$, E_2 is even farther below H for the larger AR games of Fig. 14. Interestingly, for an $AR = 1.9$, E_2 could remain on the top border until t_1 and still win (due to the intersection of the H barrier and the top border). On the other hand, with $\alpha = 0.3$, the speed advantage to the pursuer is large enough to win even at large ARs. In all three subplots, the H barrier will be governed by x_P, y_P, α, and L according to Eq. (14).

The latest, or rather, lowest point from the x axis at which P can capture both E_1 and E_2 assuming all agents are playing optimally can be determined from Eq. (11). Under the situation that E_1 is captured at the top-left corner at t_1, then $x_P = 0$, $y_P = W$, and $x_2 = L$, and then the value for y_2 when E_2 is on the H barrier is

$$y_2 = \alpha\sqrt{L^2 + W^2} \tag{26}$$

confirming the expected result that the distance along the diagonal will be the worst-case situation, but still allow P to win by capturing E_2 at the bottom-right corner. We can define the game of kind on any combination of speed and aspect ratio to identify the regions $\Re_E$ and $\Re_P$ based on the value of y_2. For any particular combination of α and AR (with W set to 1), Fig. 15 indicates the value of y_2 at which capture will occur at the bottom corner. Thus, if y_2 is lower than this surface, illustrated as contours in Fig. 15, the evaders will win (i.e., E_2 is in $\Re_E$), whereas a y_2 equal to or greater than this surface will result in capture on or before the bottom border (i.e., E_2 is in $\Re_P$). The large region in the top-right corner ($y_2 = W = 1$) represents games with high aspect ratios, proving the intuitive result that, with an excessively long border section, E_2 can remain at its own border up until t_1 and still win because the H barrier will intersect the top border (as shown in the far-right game of Fig. 14). From another perspective, the speed difference must be considerably higher for a pursuer to protect and intercept all evaders with a game area comprising a relatively long border L. Also, the y axis in Fig. 15 is characterized by $v_E = 0$, and therefore all values of AR are considered to be in $\Re_P$.

Figure 16 summarizes the benefit for the pursuer and evaders when the AR changes assuming a constant α. A larger AR means that the distance between the evaders can be larger and the H function will extend closer to the top border, both of which benefit the evaders. Smaller ARs will have the opposite effects and benefit the pursuer.

The foregoing discussion has considerable relevance onto extensive borders spanning many miles that would necessitate the

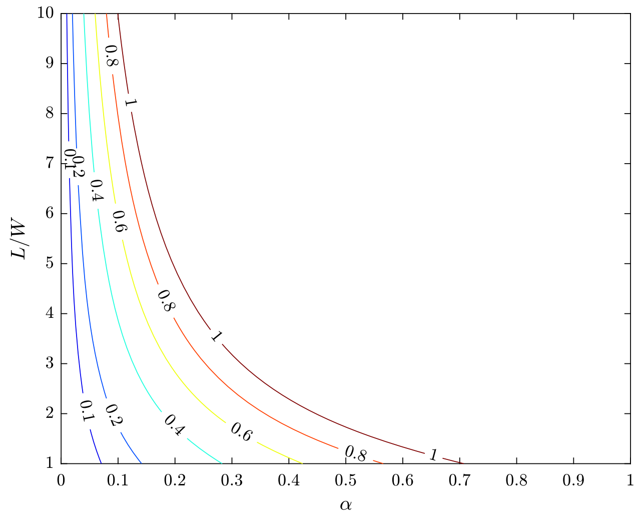


Fig. 15 Contour plot of the y_2 surface defining the game of kind for multiple speed and aspect ratio combinations. $\Re_E$ is below this surface while $\Re_P$ is on or above this surface.

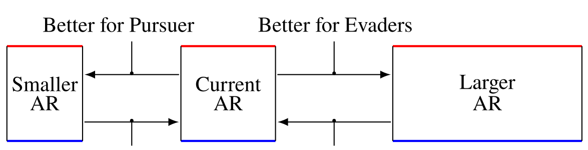


Fig. 16 Direction of AR change for benefit to evaders or pursuer.

conjoining of multiple smaller-border games together. Understanding the limits of these game area sections will be paramount to adequately protecting or defending a border with a unified strategy.

VI. Conclusions

The optimal policy for the differential border game defined in this paper is for the evaders to start in opposite corners at the top border and for the pursuer to start at one of the lower corners. This policy continues with the pursuer capturing the closer evader and then pursuing the other evader on the opposite border as in a constrained 1P1E game. The closer evader behaves as a decoy, moving to the top corner nearest the pursuer's starting position. The optimal policy defines the saddle point such that any deviation away from the starting condition and behavior will be beneficial to the other team. Aspect and speed ratios define the game of kind, with larger aspect ratios and smaller speed ratios favoring the evaders.

The foundation established in this study will enable future explorations into other similar border defense games. In particular, extensions to the game of M pursuers versus N evaders include the allocation of resources for capture, optimal pursuer cooperation, and multiple connected games along a border. Likewise, relaxing the assumption of perfect information for all agents yields variants such as delayed or nonexistent communication between agents, restricted visibility regions in which the pursuit–evasion is played, and time delays for multiple evaders entering the game. Other variants include arbitrary border shapes, agent motions constrained by guidance laws, and self-interested agents. For example, if it were possible for the evaders to avoid capture by escaping back through the top border, the game may become one of constraint rather than capture, as in [21]. Each of these game determinants has significant influence over the value of the game and ultimate implementation of a strategy at various border situations.

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