IET Control Theory & Applications

Research Article

Active target defence differential game: fast defender case

ISSN 1751-8644
Received on 4th April 2017
Revised 26th July 2017
Accepted on 22nd August 2017
E-First on 18th September 2017
doi: 10.1049/iet-cta.2017.0302
www.ietdl.org

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Abstract: A particular problem in aerospace control and applications is the active target defence differential game (ATDDG), where an attacker missile tries to capture a target aircraft which is significantly slower than the attacker. A third agent is a defender missile which is fired by the Target's wingman. The defender's mission is to intercept the attacker in time to enable a successful target evasion and its survival. The problem is naturally posed as a zero-sum DG. The target and the defender form a team; their objective is to determine a cooperative strategy to maximise the terminal separation between the target aircraft and the point where the Attacker missile is intercepted by the Defender missile. The Attacker is the opposition and its task is to devise its own strategy to minimise the same terminal distance. The solution of the ATDDG provides the instantaneous optimal heading angles for each agent playing the game in order to achieve their objectives. Most importantly, the agents' strategies are state feedback laws. The assumption of simple motion dynamics of the agents yields their optimal headings which is adequate in a beyond visual range engagement, as it is currently envisioned to be the pertinent operational scenario.

1 Introduction

Differential game (DG) theory provides the right tools for analysis of dynamic conflicts and its applications naturally deal with pursuit-evasion scenarios [1, 2]. Dynamic Voronoi diagrams have been used in [1, 3] to address dynamic conflicts among several players. The reference [4] presented a DG in which two agents cooperate and try to evade a single pursuer. A cost functional is proposed that accounts for the heading of the pursuer and the relative orientation of the three agents playing the game.

The paper [5] discussed a multiple player pursuit-evasion game that contains line segment obstacles labelled as the Prey, Protector, and Predator Game. Dominance regions were provided for each agent in order to solve the game, that is, to determine if the Protector is able to rescue the Prey before the Predator captures it.

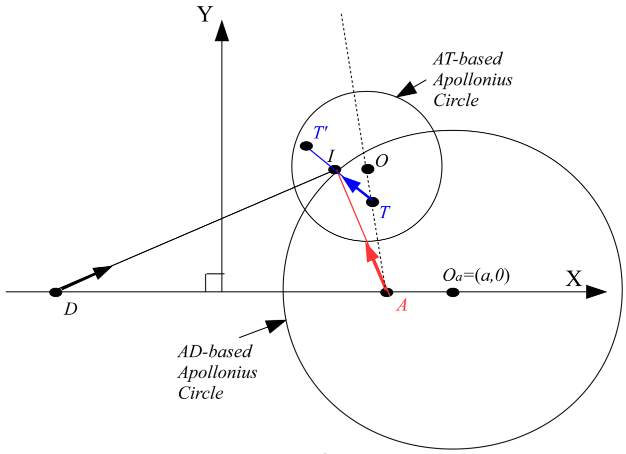
A practical problem with important applications in aerospace control of autonomous vehicles is active target defence (ATD) where three agents, the attacker (A), the defender (D), and the target (T), play a DG of pursuit, evasion, and interception. This novel problem has been studied as a cooperative optimal control setting in [6, 7]. Sensing capabilities in missiles and aircraft enable the implementation of complex pursuit and evasion strategies [8, 9], and recent references have proposed different guidance laws for the attacker and for the defender. For instance, the paper [10] considered the case where D implements command to the line-of-sight guidance to pursue the attacker. In this guidance method, the defender is restricted to have at least the same speed as the attacker. Rubinsky and Gutman [11, 12] analysed the end-game active defence scenario based on the attacker/target miss distance for a *non-cooperative* target/defender. Yamasaki and Balakrishnan and Yamasaki *et al.* of [13, 14]. proposed an interception method referred to as triangle guidance in which the objective is to position the defending missile to be on the line-of-sight between the attacker and the target for all time. However, the target is not given the freedom to manoeuvre and it follows a fixed trajectory. This is a significant limitation since the target may be able to cooperate with its own defender in order to jointly determine a strategy that defeats the attacker.

Different types of cooperation for the Target-Attacker-Defender (TAD) scenario have recently been proposed in [15–21]. These papers address different strategies such as the implementation of fixed guidance for the missiles, the analysis of mixed strategies, and some guidance laws that provide preliminary design and analysis of cooperative behaviours in the active defence scenario.

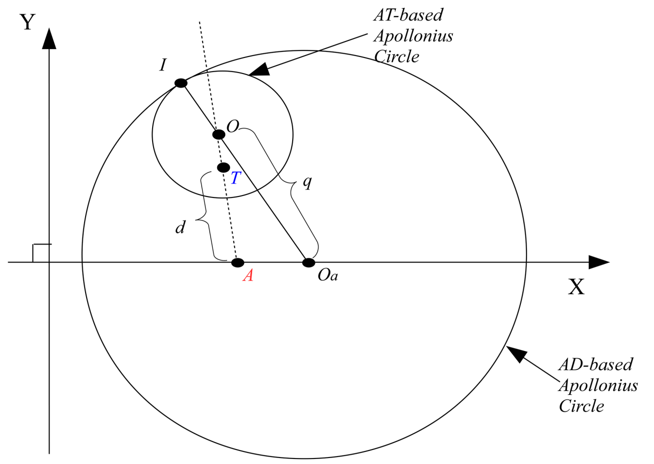
In the initial works [22–24], we addressed the optimal control problems where the target and the defender design cooperative optimal control laws in order to maximise the terminal separation between the target and the attacker for the particular case, where the attacker uses a fixed guidance law.

In the present paper, the ATD scenario is naturally formulated as a zero-sum three-agent pursuit-evasion DG (ATDDG). The protagonists are the target/defender team who seek cooperative control optimal strategies in order for the defender to successfully intercept the attacker and to maximise the attacker–target separation at the moment of interception. The attacker represents the opposing team which strives to minimise the final separation between itself and the target. The attacker's reasoning is as follows: it tries to place the defender-attacker interception point closer to the target's position in order to increase the likelihood of producing partial damage to the target, instead of running away from the defender. In such a case, the target is able to increase the terminal separation since the attacker is implementing its own evasion strategy without regard of accomplishing its mission of capturing the target in the first place. Assuming that the attacker knows the position of the defender, this strategy provides better performance for the attacker than using a fixed guidance law such as pure pursuit or proportional navigation (PN). The ATDDG for the special case where the attacker and the defender have the same speed was analysed in [25].

In this paper, we address the case where the attacker and the defender missiles have different speeds. The present paper focuses on the operationally relevant case where the defender is faster than the attacker. We provide the state feedback solution of the ATDDG. This is an important aspect compared with the numerical solution in [26]. Not only are optimal strategies computed online but the state feedback solution provides robustness against unknown attacker guidance laws. Under the simple motion assumption, the solution in this paper provides the saddle-point condition where the value of the game is a performance guarantee and it is realised when all agents play optimally. If an agent does not play optimally, it is to its own detriment and to the benefit of its adversary; this is a



a



b

Fig. 1 *Critical speed ratio for target survival*
(a) Target–attacker–defender scenario where $\gamma < 1$, ***(b)*** Determination of the critical speed ratio $\bar{\alpha}$

salient feature of zero-sum games. The players' tactics and their outcomes are thoroughly discussed in this paper.

Preliminary results concerning the ATDDG with a fast defender were obtained in [27]. This paper provides important extensions and complete proofs of relevant results compared with [27]. In this paper, we demonstrate the optimality of the saddle-point strategies of the players in the ATDDG for both cases, when the target is inside and when it is outside the *A,D b*ased Apollonius circle. We also provide a complete proof of the critical target's speed ratio to guarantee survival. Furthermore, we analyse the particular but important case where the target starts on the Apollonius circle which is defined by the defender and the Attacker speed ratio. We provide a formal design of the players' optimal strategies for such a case. All these important results extend and complement the preliminary work in [27].

This paper is organised as follows. Section 2 introduces the ATDDG. The minimum target speed ratio to evade capture by the attacker is derived in Section 3. The optimal strategies for the target, attacker, and defender are provided in Section 4. The results are illustrated in Section 5 and conclusions are made in Section 6.

2 Differential game

The players in the ATDDG have simple motion as it is typically encountered in the games of Isaacs [28]. The speeds of the target, attacker, and defender are denoted by V_T, V_A, and V_D, respectively, which are assumed to be constant. The dynamics/kinematics of the three vehicles in the inertial frame/realistic game space are given by

$$\dot{x}_T = V_T \cos\phi \quad \dot{y}_T = V_T \sin\phi \tag{1}$$

$$\dot{x}_A = V_A \cos\chi \quad \dot{y}_A = V_A \sin\chi \tag{2}$$

$$\dot{x}_D = V_D \cos\psi \quad \dot{y}_D = V_D \sin\psi \tag{3}$$

where ϕ, χ, and ψ represent the instantaneous headings of the target, the attacker, and the defender, respectively. Let $R(t)$ and $r(t)$ be the distance between the attacker and the target and between the attacker and the defender, respectively. In this game, the attacker seeks to capture the target. In addition, the target and the defender cooperate in such a way that the defender will intercept the attacker in time and help the target to survive. Therefore, the T/D team will compute a cooperative optimal strategy to maximise $R(t_f)$ which is the separation between the target and the attacker at the time instant t_f of interception. The attacker will compute its own optimal strategy to minimise $R(t_f)$.

The speed ratio problem parameter $\alpha \triangleq V_T/V_A$. In general, the target aircraft is slower than the attacker, so $\alpha < 1$. The additional speed ratio parameter $\gamma \triangleq V_A/V_D$. In this paper, we analyse the case where $\gamma < 1$, a faster defender. The co-state is

$$\lambda^T = (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}) \in \mathbb{R}^6 \tag{4}$$

and the Hamiltonian of the DG is

$$\begin{aligned} \mathscr{H} &= \lambda_{x_A}\cos\chi + \lambda_{y_A}\sin\chi + \lambda_{x_D}\cos\psi \\ &\quad + \lambda_{y_D}\sin\psi + \alpha\lambda_{x_T}\cos\phi + \alpha\lambda_{y_T}\sin\phi\,. \end{aligned} \tag{5}$$

Theorem 1: Consider the active target defence DG. The optimal headings of the attacker, the target, and the defender are constant under optimal play and their trajectories are straight lines.

Proof: The optimal control inputs (in terms of the co-state variables) can be immediately obtained from

$$\min_{\phi,\psi} \max_{\chi} \mathscr{H} \tag{6}$$

and they are given by

$$\cos\chi^* = \frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} \quad \sin\chi^* = \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} \tag{7}$$

$$\cos\psi^* = -\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} \quad \sin\psi^* = -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} \tag{8}$$

$$\cos\phi^* = -\frac{\lambda_{x_T}}{\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2}} \quad \sin\phi^* = -\frac{\lambda_{y_T}}{\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2}}\,. \tag{9}$$

Additionally, the co-state dynamics are: $\dot{\lambda}_{x_A} = \dot{\lambda}_{y_A} = \dot{\lambda}_{x_D} = \dot{\lambda}_{y_D} = \dot{\lambda}_{x_T} = \dot{\lambda}_{y_T} = 0$; hence, all co-states are constant and we have that $\chi^* \equiv$ constant, $\psi^* \equiv$ constant, and $\phi^* \equiv$ constant. Consequently, the optimal trajectories are straight lines. □

3 Critical speed ratio for target survival

Let us consider, without loss of generality, the relative coordinate frame which is shown in Fig. 1*a*. In this relative frame, the points A and D denote the positions of the attacker and the defender, respectively. The *X*-axis of this frame goes from D to A, then we have that $y_A = y_D = 0$. The *Y*-axis is given by the orthogonal bisector of the segment $\overline{AD}$. Therefore, we have that the following holds in the relative frame $T = (x_T, y_T)$, $A = (x_A, 0)$, and $D = (-x_A, 0)$.

The attacker strives to minimise the terminal distance between itself and the target. The points T and T′ represent the initial and terminal positions of the target, respectively. The point *I* is the final position of the attacker and also the final position of the defender because we consider point capture.

Owing to the optimal trajectories being straight lines, as it was shown in Theorem 1, the term Apollonius circle is a relevant tool to determine the critical speed ratio and, later on, the optimal strategies. An Apollonius circle is defined as the locus of points P that have a constant ratio of distances to two given points (also called foci), for instance A and D, i.e. $\gamma = \overline{AP}/\overline{DP}$. The circle defined in this way is referred to as the Apollonius circle and it is an important tool to analyse pursuit problems. In detail, A and D travel in straight lines at constant speeds $V_{\rm A}$ and $V_{\rm D}$, respectively, where the constant $\gamma = V_{\rm A}/V_{\rm D}$ is the speed ratio parameter and D strives to intercept A. D intercepts A at a point $I = (x_I, y_I)$ on the Apollonius circle and at that point the distance travelled by A is equal to γ times the distance travelled by D. Hence, an Apollonius circle can be constructed based on the speed ratio γ and on the distance between the attacker and the defender. The radius of the circle is denoted by $r_{\rm A}$ and its centre is denoted by O_a. The points A, D, and O_a are collinear – see Fig. 1*a*.

The coordinates of the centre, O_a, of the AD-based Apollonius circle are $(a, 0)$, where

$$a = \frac{1+\gamma^2}{1-\gamma^2}x_{\rm A}. \tag{10}$$

The radius is given by

$$r_{\rm A} = \frac{2\gamma}{1-\gamma^2}x_{\rm A}. \tag{11}$$

If the target is inside the AD-based Apollonius circle, it has to be fast enough to exit the circle to avoid capture. If it successfully exits the circle, then the defender will intercept the attacker in time; otherwise, the target is doomed.

Proposition 1: To guarantee the Target's escape, the target–attacker speed ratio must satisfy the inequality $\alpha > \bar{\alpha}$. Given the speed ratio $\gamma = V_{\rm A}/V_{\rm D} < 1$, the critical speed ratio $\bar{\alpha}$ is given by

$$\bar{\alpha} = \frac{\gamma\sqrt{(x_{\rm A}+x_{\rm T})^2 + y_{\rm T}^2} - \sqrt{(x_{\rm A}-x_{\rm T})^2 + y_{\rm T}^2}}{2\gamma x_{\rm A}}. \tag{12}$$

Proof: To determine $\bar{\alpha}$ we consider a second Apollonius circle which is defined by the attacker and the target using the target–attacker speed ratio α; this circle is called the AT-based Apollonius circle. The DG is well posed, meaning that the target is not captured by the attacker before the defender intercepts it if and only if the AT-based Apollonius circle intersects the AD-based Apollonius circle as it is shown in Fig. 1*a*. The case $\alpha = \bar{\alpha}$ corresponds to the case, where the AT-based Apollonius circle is tangent to the AD-based Apollonius circle. This is now shown in Fig. 1*b*. Note that if $\alpha \geq 1$ there is no need for a defender missile because the target can evade the attacker due to speed advantage.

The following points are collinear: attacker's initial position, the target's initial position, and the centre O of the AT-based Apollonius circle, see Fig. 1*b*. The dotted line in that figure satisfies the equation

$$y = -\frac{y_{\rm T}}{x_{\rm A}-x_{\rm T}}x + \frac{x_{\rm A}y_{\rm T}}{x_{\rm A}-x_{\rm T}}.$$

The centre of the circle AT-based Apollonius circle is O and it is located $(\alpha^2/(1-\alpha^2))d$ from T. The radius of the circle is $r_O = (\alpha/(1-\alpha^2))d$, where

$$d = \sqrt{(x_{\rm A}-x_{\rm T})^2 + y_{\rm T}^2}. \tag{13}$$

Hence, the following holds

$$\begin{aligned}&\Big(\frac{x_{\rm T}y_{\rm T}}{x_{\rm A}-x_{\rm T}} - \frac{y_{\rm T}}{x_{\rm A}-x_{\rm T}}x_0\Big)^2 + (x_0 - x_{\rm T})^2 \\ &\quad = \frac{\alpha^4}{(1-\alpha^2)^2}[(x_{\rm A}-x_{\rm T})^2 + y_{\rm T}^2].\end{aligned} \tag{14}$$

Therefore, we have that

$$x_O = \frac{1}{1-\alpha^2}x_{\rm T} - \frac{\alpha^2}{1-\alpha^2}x_{\rm A}, \quad y_O = \frac{1}{1-\alpha^2}y_{\rm T}. \tag{15}$$

From Fig. 1*b* we can see that the three points O_a, O, and I are collinear, where I represents the tangent point where both circles meet. Thus, we have the following relationship:

$$r_O = r_{\rm A} - q \tag{16}$$

where $q = \sqrt{(a-x_O)^2 + y_O^2}$. Equation (16) can be written as follows:

$$(a-x_O)^2 + y_O^2 = \Big(r_{\rm A} - \frac{\alpha}{1-\alpha^2}d\Big)^2. \tag{17}$$

Equation (17) can be expressed in terms of the known positions ($x_{\rm T}$, $y_{\rm T}$, and $x_{\rm A}$), the known speed ratio γ, and the variable we aim to solve for, which is the (critical) speed ratio α. After a few steps we obtain the following quartic equation in α

$$\begin{aligned}&\frac{4\gamma^2}{1-\gamma^2}x_{\rm A}^2\alpha^4 + \frac{4\gamma d}{1-\gamma^2}x_{\rm A}\alpha^3 - \Big(\frac{4\gamma^2}{1-\gamma^2}x_{\rm A}^2 + \frac{4\gamma^2}{1-\gamma^2}x_{\rm A}x_{\rm T} - d^2\Big)\alpha^2 \\ &\quad - \frac{4\gamma d}{1-\gamma^2}x_{\rm A}\alpha + \frac{4\gamma^2}{1-\gamma^2}x_{\rm A}x_{\rm T} - d^2 = 0\end{aligned} \tag{18}$$

which can be factored out to obtain the quadratic polynomials

$$(4\gamma x_{\rm A}\alpha(\gamma x_{\rm A}\alpha + d) + (1-\gamma^2)d^2 - 4\gamma^2 x_{\rm A}x_{\rm T})(\alpha^2 - 1) = 0.$$

The solutions $\alpha = \pm 1$ are irrelevant to the DG under analysis. Thus, $\bar{\alpha}$ is the positive solution of the following quadratic equation:

$$4\gamma^2 x_{\rm A}^2\alpha^2 + 4\gamma x_{\rm A}d\alpha + (1-\gamma^2)d^2 - 4\gamma^2 x_{\rm A}x_{\rm T} = 0$$

which is given by (12). □

4 Optimal strategies

We start by noting that all points outside the AD-based Apollonius circle can be reached by the defender before the attacker; similarly, all points inside this circle can be reached by the attacker before the defender.

4.1 Target starts outside of AD-based Apollonius circle

In the case where the target is initially outside the AD-based Apollonius circle, it is clear that the defender can help the target for any $0 < \alpha < 1$ because the attacker cannot reach the target before the defender does. It can be said that $\bar{\alpha} = 0$ in this case.

In this case, the strategies of the players are as follows: because the optimal trajectories are straight lines (Theorem 1), the target's strategy can be defined by the selection of a point v on the AD-based Apollonius circle in order to run away from that point. The target runs away from that point because it is already outside of the circle and it wishes to maximise the separation $\overline{AT}$. As the game is solved, the optimal point v^* for the target to maximise its terminal separation with respect to the attacker is obtained and it is shown, in Proposition 3, that any point other than v^* returns a lesser terminal distance.

Similarly, because optimal trajectories are straight lines, the attacker selects its aim point u on the $\overline{AD}$ Apollonius circle. Furthermore, the defender's aim point is w, also on the AD-based

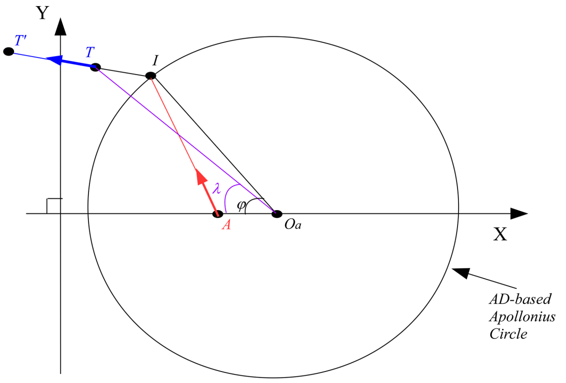


Fig. 2 *Optimal strategy*

Apollonius circle. The players are faced with the following minmax optimisation problem: $\min_u \max_{v,w} J(u,v,w)$, where $J(u,v,w)$ is the separation between point T′ and the point on the AD-based Apollonius circle where the attacker is intercepted by the defender.

To achieve interception, the defender's optimal policy is $w^*(u,v) = u$. This point is not guessed by the defender but is the solution of the DG. That is, by solving the game the players know the optimal aim point. Now, we have that the decision variables u and v jointly determine $J(u,v)$, which is the distance between the target terminal position T′ and the point $I = u$ on the circle where the defender intercepts the attacker.

Proposition 2: The solution u^* and v^* of the $\min_u \max_v J(u,v)$ optimisation problem is such that

$$u^* = v^*.$$

Moreover, when the target is outside the AD-based Apollonius circle, the target's strategy is $v^*(u) = \arg\max_v J(u,v) = u$, so that it suffices to solve the optimisation problem

$$\begin{aligned}&\min_{x_I, y_I} J(x_I, y_I)\\&\text{subject to } (a - x_I)^2 + y_I^2 = r_\mathrm{A}^2\end{aligned} \tag{19}$$

where

$$\begin{aligned}J(x_I, y_I) = &\sqrt{(x_I - x_\mathrm{T})^2 + (y_I - y_\mathrm{T})^2}\\&+\alpha\sqrt{(x_\mathrm{A} - x_I)^2 + y_I^2}.\end{aligned} \tag{20}$$

The optimality of the target and the defender choices, $v^* = u$ and $w^* = u$, will be proven in Proposition 3.

The target runs away from point $I = u$ which is chosen by the attacker. The attacker chooses the optimal coordinates (x_I, y_I) of point I that minimise the final separation $J(x_I, y_I) = \overline{IT} + \overline{TT'}$, see Fig. 2. One way to formulate this problem is as shown in Proposition 2. The constraint $(a - x_I)^2 + y_I^2 = r_A^2$ can be substituted to obtain $J(x_I)$, that is, to express J in terms of only one variable. A more compact form of the solution is as follows.

Theorem 2: The optimal interception location I that minimises (20) has polar coordinates $I = (\varphi^*, r_\mathrm{A})$ with respect to the centre of the AD-based Apollonius circle denoted by O_a, where φ^* is the solution of the sixth-order complex exponential equation

$$\begin{aligned}&\frac{Nr_\mathrm{A}}{l}\left(1 - \frac{N}{\alpha^2 Ml}\right)e^{6i\varphi} + \left(\left(\frac{N}{\alpha Ml}\right)^2(r_\mathrm{A}^2 + M^2) - r_\mathrm{A}^2 - N^2\right)e^{5i\varphi}\\&+Nr_\mathrm{A}\left(\frac{N}{\alpha^2 Ml^2}(2l^2 - 1) + l - \frac{2}{l}\right)e^{4i\varphi}\\&+2\left(r_\mathrm{A}^2 + N^2 - \left(\frac{N}{\alpha M}\right)^2(r_\mathrm{A}^2 + M^2)\right)e^{3i\varphi}\\&+Nr_\mathrm{A}\left(\frac{N}{\alpha^2 M}(2 - l^2) - 2l + \frac{1}{l}\right)e^{2i\varphi}\\&+\left(\left(\frac{Nl}{\alpha M}\right)^2(r_\mathrm{A}^2 + M^2) - r_\mathrm{A}^2 - N^2\right)e^{i\varphi} + Nr_\mathrm{A}l\left(1 - \frac{Nl}{\alpha^2 M}\right) = 0\end{aligned} \tag{21}$$

that minimises the cost

$$\begin{aligned}J(\varphi) = &\sqrt{r_\mathrm{A}^2 + N^2 - 2Nr_\mathrm{A}\cos(\varphi - \lambda)}\\&+\alpha\sqrt{r_\mathrm{A}^2 + M^2 - 2Mr_\mathrm{A}\cos\varphi}\end{aligned} \tag{22}$$

where $l = e^{i\lambda}$, $M = (2\gamma^2/(1 - \gamma^2))x_\mathrm{A} = \gamma r_\mathrm{A}$ is the distance between A and O_a, and $N = \sqrt{(a - x_\mathrm{T})^2 + y_\mathrm{T}^2}$ is the distance between O_a and T.

Proof: First, we write (20) in terms of the angle φ – see Fig. 2. As the angle φ changes, the point I moves along the circumference AD-based Apollonius circle. To obtain the corresponding expression, consider the triangles $\Delta O_a\mathrm{A}I$ and $\Delta O_a\mathrm{T}I$. The distance $\overline{TT'}$ is proportional to the distance $\overline{AI}$. The latter changes as a function of φ. However, the distance $\overline{O_aI} = r_\mathrm{A}$ and the distance $\overline{O_aA} = (2\gamma^2/(1 - \gamma^2))x_A$ are constant. Similarly, the distance $\overline{IT}$ changes as a function of φ but the distance $\overline{O_aI}$, the distance $\overline{O_aT}$, and the angle λ are constant. Hence, (20) is expressed in terms of φ as it is shown in (22). The first derivative of (22) is

$$\frac{\mathrm{d}J(\varphi)}{\mathrm{d}\varphi} = \frac{N\sin(\varphi - \lambda)}{\sqrt{r_\mathrm{A}^2 + N^2 - 2Nr_\mathrm{A}\cos(\varphi - \lambda)}} + \frac{\alpha M\sin\varphi}{\sqrt{r_\mathrm{A}^2 + M^2 - 2Mr_\mathrm{A}\cos\varphi}}. \tag{23}$$

Setting (23) equal to zero we have that

$$\frac{N^2\sin^2(\varphi - \lambda)}{r_\mathrm{A}^2 + N^2 - 2Nr_\mathrm{A}\cos(\varphi - \lambda)} = \frac{\alpha^2 M^2\sin^2\varphi}{r_\mathrm{A}^2 + M^2 - 2Mr_\mathrm{A}\cos\varphi} \tag{24}$$

which can be written in terms of the complex exponential $e^{i\varphi}$

$$\begin{aligned}&\frac{N^2}{4}(e^{i(\varphi - \lambda)} - e^{-i(\varphi - \lambda)})^2(r_\mathrm{A}^2 + M^2 - Mr_\mathrm{A}(e^{i\varphi} + e^{-i\varphi}))\\&= \frac{\alpha^2 M^2}{4}(e^{i\varphi} - e^{-i\varphi})^2(r_\mathrm{A}^2 + N^2 - Nr_\mathrm{A}(e^{i(\varphi - \lambda)} + e^{-i(\varphi - \lambda)}))\end{aligned} \tag{25}$$

and the sixth-order polynomial equation in $e^{i\varphi}$ (21) follows. The six solutions of (21) are complex, in general, of the form $e^{i\varphi} = \cos\varphi + i\sin\varphi$. Therefore, the corresponding angle φ can be directly obtained for each possible solution. Instead of a continuous search over φ, we only need to test a finite and very small number angles using the cost function (22) and obtain φ^*. Once φ^* is obtained, the optimal interception point is determined which provides the optimal headings of each agent. □

Proposition 3: Saddle-point equilibrium. Consider the case where the target is outside the *AD*-based Apollonius circle. The strategy φ^* of the attacker, where φ^* is the solution of the sixth-order equation (21) which minimises the cost (22), and the strategy of the defender of heading to the point $I(\varphi^*, r_\mathrm{A})$, which is determined by φ^* (and the radius r_A), together with the strategy of the target of running away from the point of interception of the attacker by the defender $I(\varphi^*, r_\mathrm{A})$, constitute a strategic saddle point, that is

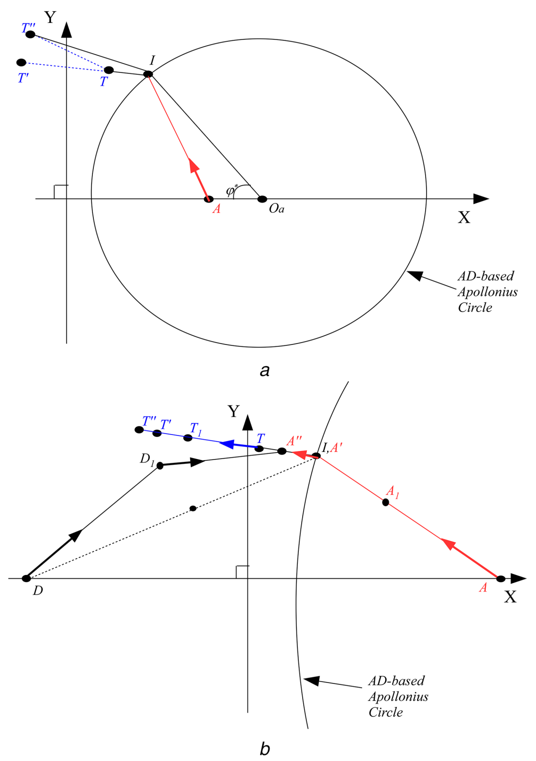


Fig. 3 *Target is outside the AD-based Apollonius circle*
(a) Case (2a): target does not employ optimal policy, ***(b)*** Case (2b): defender does not employ optimal policy

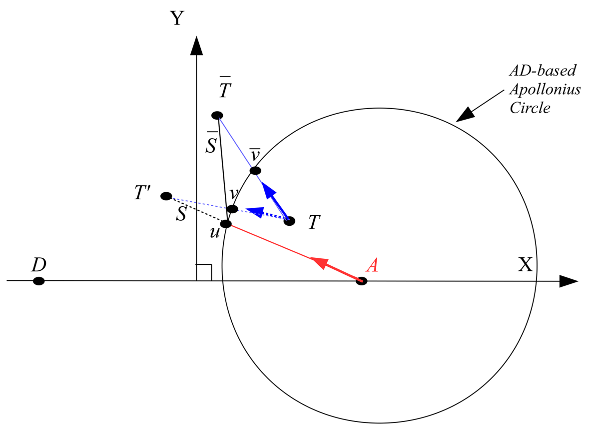


Fig. 4 *Maxmin optimisation*

$$\begin{aligned}\{J(u^*, v^*, w), J(u^*, v, w^*), J(u^*, v, w)\} \\ < J(u^*, v^*, w^*) < J(u, v^*, w^*)\,.\end{aligned} \tag{26}$$

Proof: Let us analyse the following three cases.

(1) If the attacker, the minimiser, deviates from his optimal strategy and chooses a different aim point $I'(\varphi)$, where $\varphi \neq \varphi^*$ but the target–defender team keeps its optimal strategy, then the defender intercepts the attacker at $I'(\varphi)$ and, in addition, the target runs away from $I'(\varphi)$, the cost $J(\varphi)$ increases since $\varphi \neq \varphi^*$, that is, $J(\varphi^*) < J(\varphi)$, equivalently, $J(u^*, v^*, w^*) < J(u, v^*, w^*)$. The performance of the attacker worsens.

(2a) If the attacker sticks with the optimal φ^* and the defender also plays optimally and heads toward $I(\varphi^*)$, where the attacker will be intercepted by the defender but the target deviates from its optimal strategy of running away from $I(\varphi^*)$ and heads toward T'' instead, then, in view of the triangle inequality, the separation

$$\overline{IT''} < \overline{IT'} = \overline{IT} + \overline{TT'}$$

which is detrimental to the target, that is, $J(u^*, v, w^*) < J(u^*, v^*, w^*)$. This situation is depicted in Fig. 3*a*. Here, $\overline{TT''} = \overline{TT'} = \alpha\overline{AI}$.

(2b) If the attacker sticks with the optimal φ^* and the defender does not head toward $I(\varphi^*)$ but instead ends up at some point D_1 that is not located on its optimal trajectory, then, the defender will not be able to intercept the attacker at point I since the attacker followed its optimal strategy and it is at point A_1 located on its own optimal trajectory. The attacker will reach the point $I(\varphi^*)$ and will be 'free'. Then, even if the target plays optimally and runs away from $I(\varphi^*)$, so that when the attacker reaches the point $I(\varphi^*)$ the target is at T' and the optimal (maximal) $\overline{IT'} = \overline{A'T'}$ separation is achieved by the target. Since the attacker is still 'free' at point $I(\varphi^*) = A'$, starting from that point, it will pursue the target using the pursuit strategy. By doing so the attacker relentlessly closes in on the target because $\alpha < 1$, until it is possibly captured by the defender. The attacker is intercepted by the defender using the collision course strategy at point A'', while the target reaches T''. Since the attacker has been closing in on the target, the final separation between target and attacker is $\overline{A''T''}$ and

$$\overline{A''T''} < \overline{IT'}$$

where $\overline{IT'}$ was the maximal separation achievable when the target and the defender team played optimally, equivalently, $J(u^*, v^*, w) < J(u^*, v^*, w^*)$. This is detrimental for the target–defender team. This situation is illustrated in Fig. 3*b*.

Finally, when the attacker plays optimally but neither the target nor the defender act optimally we can use cases (2a) and (2b) together to show that $J(u^*, v, w) < J(u^*, v^*, w^*)$.

□

4.2 Target starts inside of AD-based Apollonius circle

In the case where the target is initially inside the *AD*-based Apollonius, the target chooses point v on the *AD*-based Apollonius circle in order to run away from that point. The attacker selects the aim point u on the same circle. Furthermore, the defender's aim point is w, on the same circle. The players are faced with the following minmax optimisation problem: $\min_u \max_{v,w} J(u, v, w)$, where $J(u, v, w)$ represents the distance between the target final location T′ and the interception point I.

To achieve interception, the defender's optimal policy is $w^*(u, v) = u$. Similar to the previous case, this point is not guessed by the defender but it is the solution of the DG. That is, by solving the game the players know the optimal aim point. Now, we have that the decision variables u and v jointly determine $J(u, v)$, which is the distance between the target terminal position T′ and the point $I = u$ on the circle where the defender intercepts the attacker. If the target chooses v, the attacker will respond and choose u. If $u \neq v$, the target would correct his decision and choose some $\bar{v}$ such that $\bar{S} > S$ as shown in Fig. 4. However, choosing $u \neq v$ is detrimental to the attacker since the resulting cost will increase. Thus, it is clear that the attacker should aim at the point v which is chosen by the target.

Proposition 4: The solutions u^* and v^* of the $\max_v \min_u J(u, v)$ optimisation problem is such that

$$u^* = v^*\,.$$

Moreover, when the target is inside the *AD* Apollonius circle, the attacker's strategy is $u^*(v) = \arg\min_u J(u, v) = v$, so that it suffices to solve the optimisation problem

$$\begin{aligned}&\max_{x_I, y_I} J(x_I, y_I) \\ &\text{subject to } (a - x_I)^2 + y_I^2 = r_A^2\end{aligned} \tag{27}$$

where

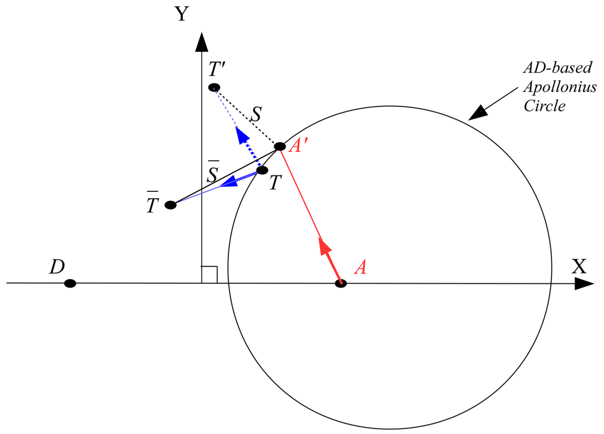


Fig. 5 *Maxmin optimisation problem when the target starts on the AD-based Apollonius circle*

$$J(x_I, y_I) = \alpha\sqrt{(x_A - x_I)^2 + y_I^2} - \sqrt{(x_I - x_T)^2 + (y_I - y_T)^2}\,. \tag{28}$$

The optimality of the defender and the attacker choices, $w^* = u$ and $u^* = v$, will be shown in Proposition 5.

In this case, the terminal separation between the target and the attacker is given by $J(x_I, y_I) = \overline{TT'} - \overline{IT}$. The target chooses the coordinates (x_I, y_I) that maximise the final separation $J(x_I, y_I)$.

Theorem 3: The optimal interception location I that maximises (28) has polar coordinates $I = (\varphi^*, r_A)$ with respect to the centre O_a of the AD-based Apollonius circle, where φ^* is the solution of the sixth-order complex exponential equation (21) that maximises the cost

$$J(\varphi) = \alpha\sqrt{r_A^2 + M^2 - 2Mr_A\cos\varphi} - \sqrt{r_A^2 + N^2 - 2Nr_A\cos(\varphi - \lambda)} \tag{29}$$

where $l = e^{i\lambda}$, $M = (2\gamma^2/(1-\gamma^2))x_A = \gamma r_A$ is the distance between A and O_a, and $N = \sqrt{(a - x_T)^2 + y_T^2}$ is the distance between O_a and T.

Proof: Equation (28) can be expressed as a function of φ as shown in (29). Then, the first derivative of (29) is

$$\frac{dJ(\varphi)}{d\varphi} = \frac{\alpha M\sin\varphi}{\sqrt{r_A^2 + M^2 - 2Mr_A\cos\varphi}} - \frac{N\sin(\varphi - \lambda)}{\sqrt{r_A^2 + N^2 - 2Nr_A\cos(\varphi - \lambda)}}\,. \tag{30}$$

When (30) is set equal to zero, (24) is obtained, hence the optimal angle φ^* is the solution of (21) that maximises (29). □

Proposition 5: Saddle-point equilibrium. Consider the case where the target is inside the *AD*-based Apollonius circle. The strategy φ^* of the target, where φ^* is the solution of the sixth-order equation (21) which maximises (29), and the strategy of the defender of heading to the point $I(\varphi^*, r_A)$, together with the strategy of the attacker of aiming at the point $I(\varphi^*, r_A)$, constitute a strategic saddle point, that is

$$\{J(u^*, v^*, w), J(u^*, v, w^*), J(u^*, v, w)\} < J(u^*, v^*, w^*) < J(u, v^*, w^*)\,. \tag{31}$$

Proof: Let us analyse the following three cases.

(1a) If the target, which is the maximiser, deviates from his optimal strategy and chooses $\varphi \neq \varphi^*$ but the attacker and the defender act optimally, that is, the attacker heads to the point $I'(\varphi)$ and is intercepted by the defender at that point, the cost $J(\varphi)$ decreases since $\varphi \neq \varphi^*$, that is, $J(\varphi) < J(\varphi^*)$, which is equivalent to $J(u^*, v, w^*) < J(u^*, v^*, w^*)$. This is detrimental to the target–defender team.

(1b) If the attacker sticks with the optimal heading obtained from φ^* and the defender does not head toward $I(\varphi^*)$ but instead ends up at some point D_1 that is not located on its optimal trajectory, then, the defender will not be able to intercept the attacker at point I since the attacker employed its optimal strategy and it is at point A_1 located on its own optimal trajectory. The attacker will reach the point $I(\varphi^*)$ and will be 'free'. Then, even if the target plays optimally and runs away from $I(\varphi^*)$, so that when the attacker reaches the point $I(\varphi^*)$ the target is at T' and the optimal (maximal) $\overline{IT'} = \overline{A'T'}$ separation is achieved by the target. Since the attacker is still 'free' at point $I(\varphi^*) = A'$, starting from that point, it will pursue the target using the Line-Of-Sight (LOS) pursuit strategy. By doing so the attacker relentlessly closes in on the target because $\alpha < 1$, until it is possibly captured by the defender. The attacker is intercepted by the defender using the collision course strategy at point A'', while the target reaches T''. Since the attacker has been closing in on the target, the final separation between target and attacker is $\overline{A''T''}$ and

$$\overline{A''T''} < \overline{IT'}$$

where $\overline{IT'}$ was the maximal separation achievable when the target and the defender team played optimally, that is, $J(u^*, v^*, w) < J(u^*, v^*, w^*)$. This is detrimental to the target–defender team. This situation is similar to the one illustrated in Fig. 3*b*, except that the target starts inside the *AD*-based Apollonius circle.

(2) If the target acts optimally, that is, the target chooses φ^* but the attacker chooses some $\varphi \neq \varphi^*$, then the target is able to correct his decision and choose a new heading in order to increase the cost and this is detrimental to the attacker, that is, $J(u^*, v^*, w^*) < J(u, v^*, w^*)$. This case was illustrated in Fig. 4.

Finally, when the attacker plays optimally but neither the target nor the defender act optimally, cases (1a) and (1b) can be used together to show that $J(u^*, v, w) < J(u^*, v^*, w^*)$. □

4.3 Target starts on the AD-based Apollonius circle

We now consider the particular and interesting case where the initial position of the target is such that T is a point on the circumference of the *AD*-based Apollonius circle. In this case, the attacker's optimal strategy is to aim at point T. Also note that the corresponding angle $\varphi = \lambda$ is a solution of (21). The optimal attacker's strategy can be analysed as follows. If the target aims at point T' – see Fig. 5 – and the attacker chooses its aim point at $A' \neq T$, where it will be intercepted by the defender, then the target will be able to redefine its aim point and choose some $\bar{T}$ and potentially increase the final separation $\bar{S} > S$. In general, choosing $A' \neq T$ is detrimental to the attacker since the resulting cost will increase. Thus, the attacker should aim at point T, the target's initial position.

It can be seen that the optimal point where the attacker is intercepted by the defender, $I(x, y)$, is exactly the point $T = (x_T, y_T)$, the target's initial position; choosing the point $I = T$ is equivalent to choosing $\varphi = \lambda$. The optimal strategy of the target is to run away from point I. If the attacker keeps its aim point at $I(x, y) = (x_T, y_T)$, then any angle $0 < \phi < \pi$, see Fig. 6, is optimal for the target; here, the angle ϕ is measured with respect to the AD-based Apollonius circle tangent line at point T. However, the attacker will be able to redefine its aim point by knowing the target's heading decision and potentially reduce the cost (22) compared with the cost that it will incur by aiming at $I(x, y) = (x_T, y_T)$. Therefore, there exists an optimal heading ϕ^* such that, for ϕ^*, the solution $\varphi = \lambda$ is a saddle point, that is, when $\varphi = \lambda$ the attacker minimises the distance between itself and the target at the interception time instant, and the target maximises the said distance by choosing $\phi = \phi^*$. Contrary to the cases where the target starts either inside or outside the

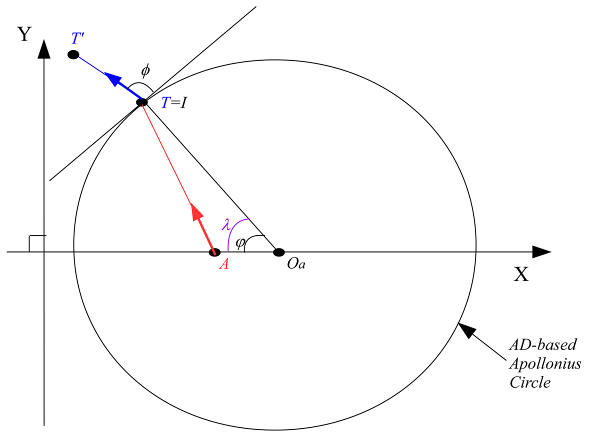


Fig. 6 *Optimal strategies when the target starts on the AD-based Apollonius circle*

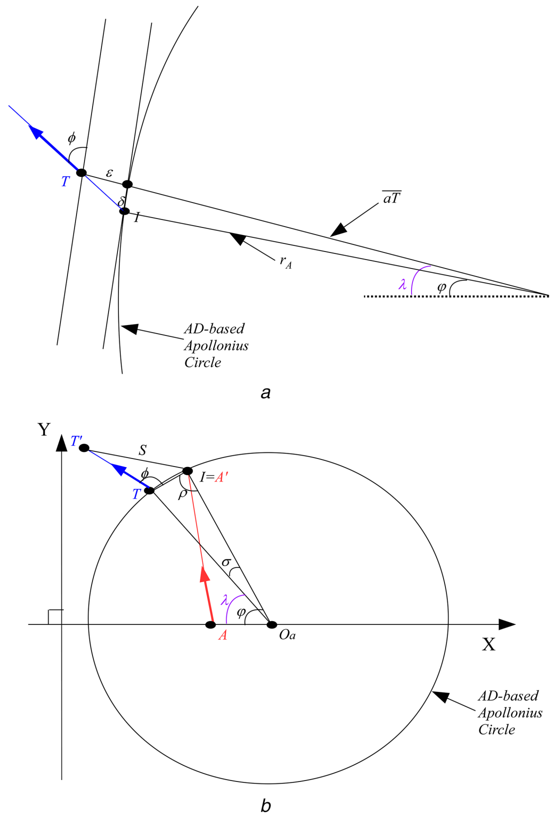


Fig. 7 *Target is on the AD-based Apollonius circle*
(a) Target's optimal strategy, ***(b)*** Target's optimal strategy – alternative derivation

Apollonius circle, the sixth-order equation (21) by itself does not provide the optimal heading angle ϕ^*. The target's optimal heading was previously obtained based on two different points T and I; however, the optimal solution $\varphi = \lambda$ results in $I = T$, that is, those two points are the same and the target's optimal heading cannot be obtained solely based on this information.

Proposition 6: When initially the target is on the circumference of the *AD*-based Apollonius circle, that is, $(a - x_T)^2 + y_T^2 = r_A^2$ the optimal attacker's strategy is to head toward $I = (x_T, y_T)$, where it will be intercepted by the defender, and the optimal target's heading angle is then given by

$$\tan\phi^* = \frac{2\sqrt{(1-\gamma^2)x_A x_T - ((\alpha/2)(1-\gamma^2)y_T)^2}}{\alpha y_T(1-\gamma^2)}. \tag{32}$$

Proof: Define $\sigma = \varphi - \lambda$. Let us consider the limit case where $N = \sqrt{(a - x_T)^2 + y_T^2} \to r_A$, that is, the case where the target starts on the *AD*-based Apollonius circle – see Fig. 7*a*. To obtain the target's optimal heading angle, we substitute $N = r_A + \epsilon$ in (24), [Note that (24) is equivalent to (21)] where $0 < \epsilon \ll 1$

$$\frac{(r_A+\epsilon)^2\sin^2\sigma}{r_A^2+(r_A+\epsilon)^2-2(r_A+\epsilon)r_A\cos\sigma} = \frac{\alpha^2M^2\sin^2(\lambda+\sigma)}{r_A^2+M^2-2Mr_A\cos(\lambda+\sigma)} \tag{33}$$

where $\sigma \simeq 0$ for any $\epsilon \simeq 0$. The small angle approximations $\sin\sigma$ and $\cos\sigma$ are given by: $\sin\sigma = \sigma$ and $\cos\sigma = 1 - \frac{\sigma^2}{2}$. Thus, we have

$$\frac{(r_A+\epsilon)^2\sigma^2}{r_A^2+(r_A+\epsilon)^2-2(r_A+\epsilon)r_A(1-\frac{\sigma^2}{2})} = \frac{\alpha^2M^2((1-\frac{\sigma^2}{2})\sin\lambda+\sigma\cos\lambda)}{r_A^2+M^2-2Mr_A((1-\frac{\sigma^2}{2})\cos\lambda-\sigma\sin\lambda)}. \tag{34}$$

Terms of the form $\epsilon^i\sigma^j \to 0$, for $i + j \geq 3$, and (34) is reduced to

$$(r_A^4 + r_A^2M^2 - 2Mr_A^3\cos\lambda)\sigma^2 = (\alpha M\sin\lambda)^2(\epsilon^2 + r_A^2\sigma^2). \tag{35}$$

The small angle σ generates a small arc δ

$$\delta = r_A\sigma \tag{36}$$

and (35) becomes

$$(r_A^2 + M^2 - 2Mr_A\cos\lambda - \alpha^2M^2\sin^2\lambda)\delta^2 = (\alpha M\epsilon\sin\lambda)^2. \tag{37}$$

Finally, the following relationship between ϵ and δ is obtained

$$\delta = \frac{\alpha M\sin\lambda}{\sqrt{r_A^2+M^2-2Mr_A\cos\lambda-\alpha^2M^2\sin^2\lambda}}\,\epsilon. \tag{38}$$

As it is shown in Fig. 7*a*, in the limit $\epsilon \to 0$ the relationship between ϵ and δ characterises the target's heading angle ϕ. The angle ϕ is given with respect to the *AD* Apollonius circle tangent line at the point (x_T, y_T). Thus, we have

$$\tan\phi^* = \frac{\epsilon}{\delta} = \frac{\sqrt{r_A^2+M^2-2Mr_A\cos\lambda-\alpha^2M^2\sin^2\lambda}}{\alpha M\sin\lambda} \tag{39}$$

which can be written in terms of the speed ratio γ and the target and attacker coordinates x_T, y_T, and x_A as in (32). □

Alternatively, the optimal target's heading angle ϕ can be obtained in the following way. We recall that when the target is on the *AD*-based Apollonius circle the optimal attacker strategy is $\varphi^* = \lambda$. The target chooses its heading ϕ and the attacker aims at point $I = (x, y)$, given by the coordinates (φ, r_A). It must be shown that there exists a ϕ^* such that the pair $\{\phi^*, \lambda\}$ is a saddle point, that is, if $\phi = \phi^*$, the attacker minimises the separation $\overline{A'T'}$ when $\varphi^* = \lambda$ and if $\varphi = \varphi^*$, the target maximises the separation $\overline{A'T'}$ by choosing $\phi = \phi^*$. The latter is trivially the case because any $0 < \phi^* < \pi$ will do. Hence, we must consider the $\max_\phi \min_\varphi S^2(\phi, \varphi)$ optimisation problem and show that there exists a ϕ^* such that for this ϕ^*, $\varphi^* = \lambda$ minimises the $\overline{A'T'}$ separation S. We show this in the following proposition.

Proposition 7: When $(a - x_T)^2 + y_T^2 = r_A^2$, the optimal attacker's strategy is to head toward $I = (x_T, y_T)$, where the defender will intercept it, and the optimal target's heading angle is then given by

$$\cos\phi^* = \frac{\alpha(1-\gamma^2)y_{\rm T}}{2\sqrt{(1-\gamma^2)x_{\rm A}x_{\rm T}}}. \tag{40}$$

Proof: Consider the separation $S(\phi, \sigma)$, where $\sigma = \varphi - \lambda$, between the points T' and I as shown in Fig. 7*b*. The distance that the target travels is given by

$$\overline{TT'} = \alpha\sqrt{r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos(\lambda+\sigma)}.$$

The separation between T and I is $\overline{TI} = r_{\rm A}(\sin\sigma/\sin\rho)$, where $\rho = (\pi-\sigma)/2$. The terminal $T - A$ separation is (see (41) and (42)) Next, we find the partial derivative of (41) with respect to σ and set the resulting expression equal to zero as follows: (see (42)) Since $\varphi = \lambda$, we set $\sigma = 0$ and (42) becomes

$$2\alpha^2 M r_{\rm A}\sin\lambda - 2\alpha r_{\rm A}\cos\phi\sqrt{r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos\lambda} = 0. \tag{43}$$

Note that as $\sigma \to 0$ the straight line joining T and A' becomes a segment of the *AD*-based Apollonius circle tangent line at point T. Thus, the angle ϕ becomes the target's heading angle with respect to that tangent line. Solving for $\cos\phi$ we obtain

$$\cos\phi^* = \frac{\alpha M\sin\lambda}{\sqrt{r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos\lambda}} \tag{44}$$

which can be written in terms of the speed ratio γ and the coordinates of the target and attacker x_T, y_T, and x_A as in (40). □

Remark 1: We have found two expressions concerning the target's optimal heading angle ϕ^* when it starts on the *AD*-based Apollonius circle: (32) and (40). Both expressions provide the target's optimal heading ϕ^* which is drawn from the tangent line of the *AD* Apollonius circle at point $T = (x_{\rm T}, y_{\rm T})$. One finds

$$\sin\phi^* = \cos\phi^*\tan\phi^* = \frac{\sqrt{(1-\gamma^2)x_{\rm A}x_{\rm T} - (\frac{\alpha}{2}(1-\gamma^2)y_{\rm T})^2}}{\sqrt{(1-\gamma^2)x_{\rm A}x_{\rm T}}} \tag{45}$$

and it can be checked that both (32) and (40) provide the same heading angle by calculating

$$\sin^2\phi^* + \cos^2\phi^* = \frac{4((1-\gamma^2)x_{\rm A}x_{\rm T} - \frac{\alpha^2}{4}(1-\gamma^2)^2 y_{\rm T}^2) + \alpha^2(1-\gamma^2)^2 y_{\rm T}^2}{4(1-\gamma^2)x_{\rm A}x_{\rm T}} = 1. \tag{46}$$

5 Examples

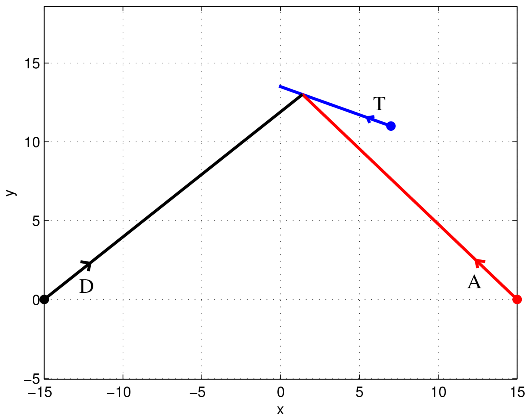
Example 1 (the Target is inside the AD-based Apollonius circle): Consider the conflict scenario of the target, the attacker, and the defender. The speed parameters are: $\alpha = 0.4$ and $\gamma = 0.9$. The initial positions are: $\rm A = (15, 0)$, $\rm D = (-15, 0)$, and $\rm T = (7, 11)$. In this case, we obtain $a = 142.895$ and $r_{\rm A} = 142.105$. In this example, the critical speed ratio $\bar{\alpha}$ is >0. Using (12), we determine that $\bar{\alpha} = 0.3161$. Hence, the value $\alpha = 0.4 > \bar{\alpha}$ guarantees successful evasion of the target. The optimal solution of the DG is obtained by solving the sixth-order equation (21) and evaluating the solutions using (29). The optimal solution is $\varphi^* = 0.0916$ rad, which translates into the coordinates $I^* = (1.385, 13.005)$. The optimal cost/payoff is $J^* = 1.57$. Fig. 8*a* presents the optimal trajectories.

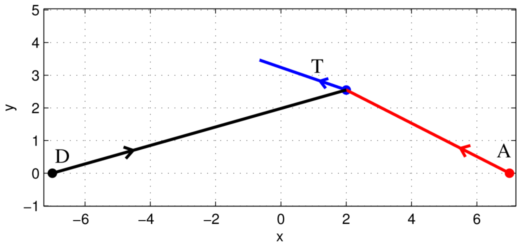
Example 2 (the Target is on AD-based Apollonius circle): Consider the speed parameters $\alpha = 0.5$ and $\gamma = 0.6$. The initial positions are: $\rm A = (7, 0)$, $\rm D = (-7, 0)$, and $\rm T = (2, 2.5495)$. The centre coordinate and the radius of the circle are given by $a = 14.875$ and $r_{\rm A} = 13.125$, respectively. In this example, we have that $\sqrt{(a - x_{\rm T})^2 + y_{\rm T}^2}$ is equal to $r_{\rm A}$, and the target lies exactly on the Apollonius circle. The optimal interception point is $I^* = \rm T = (2, 2.5495)$. These coordinates correspond to the angle $\varphi^* = 0.1955$ rad which is the solution of the sixth-order equation (21) that minimises (22). The optimal interception point $I^* = T$ provides the optimal aim point for the attacker and the defender; however, the target still needs to find its optimal heading angle ϕ^*. To compute the complete solution, the optimal heading angle of the target can be obtained from (32) or from (40) and we have $\phi^* = 1.4341$ rad (or 2.8094 rad with respect to the *X-axis*). The optimal cost/payoff is $J^* = 2.806$. The optimal trajectories are shown in Fig. 8*b*.

Example 3 (Robustness to unknown attacker guidance law): A very important characteristic of the cooperative guidance laws for the ATDDG as discussed here is that the solution given by the sixth-order equation (21) [and equations (32) and (40) when needed] is a state feedback interception strategy that is, therefore, robust with respect to unknown attacker guidance laws. This means that if the attacker does not follow its optimal policy and uses a different guidance law that is unknown to the target–defender team, then the target and the defender (having current measurements of the attacker' position) are able to solve (21) and continuously update their cooperative interception strategy, thus increasing the $T - A$ separation at interception time.
Let the initial positions be: $\rm A = (10, 0)$, $\rm D = (-10, 0)$, and $\rm T = (3, 7.5)$. The speed parameters are $\alpha = 0.6$ and $\gamma = 0.85$. However, the attacker's guidance in this example is fixed; in particular, it is a PN guidance law with navigation constant N=3. Furthermore, this information is unknown to the target–defender team and they are only able to measure the current position of the attacker, $A = (x_{\rm A}(t), y_{\rm A}(t))$. By continuously updating their headings, the target–defender team are able to defeat the attacker, that is, the defender intercepts the attacker and the target escapes being captured by the attacker. Fig. 8*c* shows the trajectories. The final T − A separation is $R(t_f) = 5.609 > J^*$. As expected, the final separation is more if the attacker played optimally: when the attacker plays optimally the cost/payoff is $J^* = 5.373$. Obviously, the attacker strategy in Example 4 is not an optimal strategy.

$$\begin{aligned} S^2(\phi, \sigma) &= \alpha^2(r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos(\lambda+\sigma)) + r_{\rm A}^2\frac{\sin^2\sigma}{\sin^2((\pi-\sigma)/2)} \\ &\quad -2\alpha r_{\rm A}\frac{\sin\sigma}{\sin((\pi-\sigma)/2)}\sqrt{r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos(\lambda+\sigma)}\cos\phi. \end{aligned} \tag{41}$$

$$\begin{aligned} \frac{\partial S^2(\phi,\sigma)}{\partial\sigma} &= 2\alpha^2 M r_{\rm A}\sin(\lambda+\sigma) + 2r_{\rm A}^2\frac{\sin^2((\pi-\sigma)/2)\sin\sigma\cos\sigma + \sin^2\sigma\sin((\pi-\sigma)/2)\cos((\pi-\sigma)/2)}{\sin^4((\pi-\sigma)/2)} \\ &\quad -2\alpha r_{\rm A}\cos\phi\Big(\sqrt{r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos(\lambda+\sigma)}\frac{\sin((\pi-\sigma)/2)\cos\sigma + \sin\sigma\cos((\pi-\sigma)/2)}{\sin^2((\pi-\sigma)/2)} \\ &\quad +\frac{\sin\sigma}{\sin((\pi-\sigma)/2)}\frac{Mr_{\rm A}\sin(\lambda+\sigma)}{\sqrt{r_{\rm A}^2 + M^2 - 2Mr_{\rm A}\cos(\lambda+\sigma)}}\Big) = 0. \end{aligned} \tag{42}$$





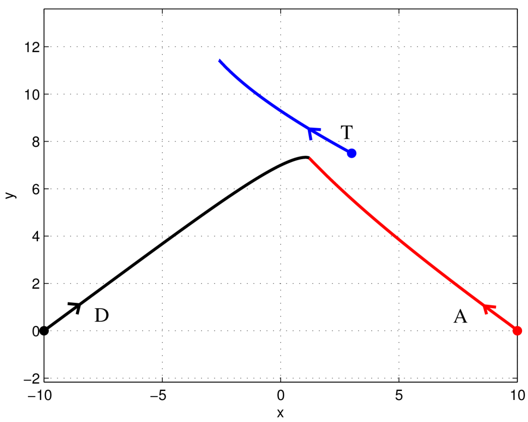


Fig. 8 *Trajectory examples*
(a) Target is initially inside the AD-based Apollonius circle; each agent implements its optimal heading, ***(b)*** Target is initially on the AD-based Apollonius circle; each agent implements its optimal heading, ***(c)*** Robustness of the optimal T/D team strategy with respect to unknown non-optimal attacker guidance

Remark 2: If the target and the defender would know the attacker guidance law, they could further improve their performance, that is, obtain an even larger final separation $R(t_f)$. Indeed, for the case when the attacker employs the PN guidance law, a numerical solution of the optimal control problem to obtain the optimal target–defender interception strategy was derived in [24].

6 Conclusions

This paper presented a complete analysis of the ATDDG with a defender which is faster than the attacker. Three cases were considered: when the attacker is outside a A,D-based Apollonius circle, when it is inside, and when it is exactly on said circle. The optimal strategies obtained by solving the ATDDG return the instantaneous optimal heading angles for the T/D team to maximise the final $T-A$ separation and it also provides the instantaneous optimal heading angle for the attacker to minimise the same distance. The approach employed herein is based on solving, in real time, a sixth-order polynomial equation which determines the optimal aim point. With this information on hand, the optimal headings for the players with simple motion are directly computed. This solution entails an expected increase in complexity compared with the particular case investigated in [25], where both A and D have the same speed. Most importantly, the agents' strategies are state feedback laws. The assumption of simple motion dynamics of the agents yield their optimal headings which are adequate in a beyond visual range engagement, as it is currently envisioned to be the pertinent operational scenario. Finally, the interesting case where the target is on the boundary of the reachable zones of A and D was addressed and the explicit solution was provided.

7 Acknowledgments

This work was supported in part by the AFOSR LRIR: 12RB07COR.

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