JOURNAL OF AEROSPACE INFORMATION SYSTEMS
Vol. 17, No. 8, August 2020

Cooperative Pursuit by Multiple Pursuers of a Single Evader

Meir Pachter*
Air Force Institute of Technology, Wright–Patterson Air Force Base, Ohio 45433
Alexander Von Moll,† Eloy Garcia,† and David Casbeer†
Air Force Research Laboratory, Wright–Patterson Air Force Base, Ohio 45433
and
Dejan Milutinović‡
University of California Santa Cruz, Santa Cruz, California 95064

https://doi.org/10.2514/1.I010739

This paper considers pursuit-evasion differential games in the Euclidean plane where an evader is engaged by multiple pursuers and point capture is required. The players have simple motion (i.e., holonomic) in the manner of Isaacs, and the pursuers are faster than the evader. The attention of this paper is confined to the case where the pursuers have the same speed, and so the game's parameter is that the evader/pursuers speed ratio is $0 < \mu < 1$. State feedback capture strategies and an evader strategy that yields a lower bound on his/her time-to-capture are devised using a geometric method. It is shown that, in group/swarm pursuit, when the players are in general position, capture is effected by one, two, or three critical pursuers, and this is irrespective of the size $N(> 3)$ of the pursuit pack. Group pursuit devolves into pure pursuit by one of the pursuers or into a pincer movement pursuit by two or three pursuers who isochronously capture the evader. The critical pursuers are identified. However, these geometric method-based pursuit and evasion strategies are optimal only in a part of the state space where a strategic saddle point is obtained and the value of the differential game is established. As such, these strategies are suboptimal. To fully explore the differential game's high-dimensional state space and get a better understanding of group pursuit, numerical experimentation is undertaken. The state space region where the geometric solution of the group pursuit differential game is the optimal solution becomes larger the smaller the speed ratio parameter is.

I. Introduction

WE CONSIDER pursuit-evasion differential games in the Euclidean plane where an evader (E) is cooperatively engaged by multiple pursuers $P_1, \ldots, P_N$, $N \geq 2$. All the players have simple motion in the manner of Isaacs; that is, they are holonomic. The pursuers are faster than the evader; they have different speeds or might all have the same speed V_P. The speed of the evader is V_E and we introduce the speed ratio parameters $\mu_i \equiv (V_E/V_{P_i})$, $i = 1, \ldots, N$. The pursuers might be endowed with capture circles of radius l and the individual pursuers might even have different capture radii; however, in this paper we are mainly interested in point capture where $l \to 0$. Evidently, the dimension of the state space is $2(N+1)$. By using a moving reference frame whose x and y axes are aligned with the x and y axes of the Euclidean plane but the origin is collocated with the instantaneous position of E, the dimension of the state space is reduced to $2N$. The dynamics in this state space are linear—in fact, there are no "dynamics," because the proverbial dynamics matrix $A = 0$. It is possible to further reduce the state space by using polar coordinates, the state variables being the E, P_i distances d_i, $i = 1, \ldots, N$, and the $N-1$ angles included between the radials to the pursuers emanating from point E, in total $2N-1$ states; this, however, renders the dynamics nonlinear. In this paper most of the discussion will be focused on the "simpler" case where $\mu_i = \mu$, $i = 1, \ldots, N$, and also, $N = 3$, whereupon the dimension of the reduced state space is 5. The game's only parameter is the speed ratio $0 < \mu < 1$.

There is strength in numbers, and many-on-one pursuit-evasion differential games have attracted considerable attention.§ We start from the beginning. The solution of the differential game of Two Cutters and a Fugitive Ship, where $N = 2$, the pursuers are faster than the evader, and point capture is required, was provided in [1]. Isaacs used a geometric method for the solution of pursuit games with simple motion, well aware that this might not always be possible. He amply demonstrated this with the obstacle tag chase differential game where the difficulty is caused by the violation of the requirement in dynamic games of time consistency/subgame perfectness. However, upon addressing the Two Cutters and Fugitive Ship differential game, Isaacs successfully employed the geometric concept of an Apollonius circle to determine the safe region (SR) and delineate the boundary of a safe region (BSR) for the evader (see Sec. III). An Apollonius circle is constructed based on the $E-P$ separation and the speed ratio $\mu < 1$—see Sec. II.A in the sequel. Thus, because there are two pursuers, a lens-shaped BSR is formed by the intersection of two Apollonius circles as shown in Fig. 1. E heads toward the most distant point on the BSR, that is, the BSR vertex farthest from E, and so do P_1 and P_2.

Interestingly, Hugo Steinhaus originally posed the Two Cutters and Fugitive Ship pursuit game back in 1925. Steinhaus's original paper on this topic was reprinted in 1960 in [2]. Some aspects of the two-on-one pursuit game were also investigated in [3]. In [4], the speed ratio parameter $\mu > 1$: E is faster than the two pursuers P_1 and P_2 but he/she seeks to pass between them. Obviously, if the fast E would just be interested in not being captured by the P_1 and P_2 team, he/she would head in the opposite direction. In [5], a differential game of approach with three pursuers and one evader, where the evader is faster than the pursuers, is considered. Pursuit and evasion is mentioned in the title, but this is a game of approach—a fast evader can always escape. In [6–9] many-on-one ($N \geq 3$) pursuit-evasion differential games are considered where all players have the same speed, that is, the speed ratio $\mu = 1$, and point capture is required. And in [10,11] discrete-time dynamics are considered in a continuous spatial domain; in [11] the players take turns. Obviously, if $E \notin \text{convhull}(\{P_1, \ldots, P_N\})$, E can escape. Hence, in Refs. [6–11] it is assumed that $E \in \text{convhull}(\{P_1, \ldots, P_N\})$ and it is proven that E can be captured. Capture by k pursuers with simple motion but endowed with capture disks of radius l_i, $i = 1, \ldots, k$, is discussed in Ref. [12]. However, it is therein uniformly assumed that the evader is obliged to broadcast his/her move/control ahead of time so that this information is available to the pursuers. Thus, the pursuers' strategies,

Received 27 March 2019; revision received 8 December 2019; accepted for publication 14 December 2019; published online 19 February 2020.

*Professor, Electrical Engineering.
†Research Engineer, Controls Science Center of Excellence.
‡Professor, Electrical and Computer Engineering.

§It has been said: "Quantity has a quality all its own" (I. V. Stalin).

rather than being straight state feedback strategies, are state feedback discriminatory/stroboscopic strategies. This state of affairs is not totally satisfactory—also according to Isaacs ([1] p. 149).[¶] A comprehensive bibliography of many-on-one pursuit-evasion differential games is included in the recent survey paper [13].

More recently, the many-on-one pursuit-evasion scenario with fast pursuers, simple motion, and point capture has been explored in [14,15] in the context of Voronoi diagrams—this requiring the assumption that the pursuers have equal speeds. The work reported in this paper is an enhancement over [14,15] in that we rigorously analyze the optimality of the Voronoi vertex between pursuers. Also, we identify analytical conditions to differentiate between the possible end-game scenarios. It is shown that group pursuit devolves into pure pursuit (PP) by one of the pursuers or into a pincer movement pursuit by two or three pursuers who isochronously capture the evader, a ménage à trois. The critical pursuers are identified and state feedback capture strategies are devised. However, the players' strategies are optimal only in a part of the state space, where a strategic saddle point is obtained and the value of the game is established. Although the pursuit strategy enforces global capturability, its optimality and the optimality of the evader's strategy are confined to a restricted part of the state space. As such, the players' strategies are suboptimal; at the same time, when playing against the pursuers' suboptimal strategy, the evader's strategy yields a lower bound on the time-to-capture. To fully explore the differential game's high-dimensional state space and get a better understanding of the group pursuit differential game, we employ numerical experimentation. The region of the state space where the devised strategies are optimal becomes larger the smaller the speed ratio parameter is.

The paper is organized as follows. Because the Apollonius circle construct is central to the geometric method employed herein, as well as to make the paper self-contained, its geometry is outlined in Sec. II. Following is the construction of the *SR* and the *BSR* in Sec. III, where the geometric method is extended to allow the treatment of pursuit games with $N > 2$ pursuers. Additional geometric considerations that critically impact the synthesis of optimal strategies when three or more pursuers are involved are discussed in Sec. IV. The pursuit-evasion game with three pursuers is analyzed in Sec. V. Section VI describes state feedback global capture strategies, and an evader strategy that yields a lower bound on his/her time-to-capture is synthesized. Examples are included in Sec. VII. The devised pursuit and evasion strategies are optimal; that is, they constitute a saddle point and the value of the game is obtained, only in a part of the state space. In this respect, the impact of the speed ratio parameter μ on the unfolding of the game is investigated in Sec. VIII. An optimal solution for the entire state space is not available, but if the speed ratio parameter $0 < \mu \ll 1$ the optimality of the players' strategies extends over bigger swathes of the state space. In this regard, a computational investigation of the group pursuit differential game is documented in Sec. IX. The case where the number of pursuers $N > 3$ is taken up in Sec. X, followed by a discussion of possible extensions in Sec. XI and concluding remarks in Sec. XII. The main results of the paper are stated in Proposition 1 in Sec. II.B, Proposition 2 in Sec. X, Algorithm 1 in Sec. VI, and Algorithm 2 in Sec. X.

We concede that only a suboptimal solution of the group pursuit differential game has been obtained. Capture is effected, and when playing against the devised pursuit strategy the evader's time-to-capture is bounded from below. However, the pursuit and evasion strategies constitute a saddle point and yield the value of the game only in part of the state space. When the number of pursuers $N > 2$ the game is significantly more complicated than the Two Cutters and Fugitive Ship analog—the dimension of the reduced state space is $2N - 1$. So already when the number of pursuers $N = 3$, the dimension of the reduced state space is 5. So far, no differential games whose state space dimension is higher than two, or perhaps three, and where singular surfaces (of focal or equivocal type) feature, have been completely solved.

II. Geometric Method

Geometry has a decisive role to play in the quest for solving multiplayer pursuit-evasion differential games, and, in particular, when the players have simple motion. The dynamics are simple motion dynamics and the cost/payoff is the time-to-capture. Thus, when the differential game is analyzed in the realistic plane, and also in the reduced state space of dimension $2N$, the Hamiltonian dictates the attendant $2N$ costates are all constant. Hence, the players' optimal controls are constant and the *primary* optimal flowfield will consist of straight lines/regular characteristics. The optimal trajectories being straight lines suggests that the geometric method is applicable. For the case where $N = 2$, the geometric solution was formally verified in [16,17]. The candidate Value function was provided by the geometric method, and it was shown that it is indeed the solution of Isaacs's Two Cutters and Fugitive Ship differential game because of the following reasons. 1) It is continuous over the entire capture set, which is the whole state space. 2) It is also continuously differentiable over the entire state space. 3) It satisfies the Hamilton–Jacobi–Isaacs (HJI) partial differential equation (PDE) in the whole state space. Isaacs's geometric solution was justified in [16] and the argument was further strengthened in [17,18]. Indeed, when the number of pursuers $N = 2$, the solution of this differential game with three-states consists of primary optimal trajectories/regular characteristics only—there are no "difficult" singular hypersurfaces of focal or switch envelope type. By extrapolation, one might then be tempted to believe that geometry reigns supreme and differential games with multiple pursuers and one evader are a tale of $N + 1$ points in the Euclidean plane, with computational geometry overtones. As will become clear in the sequel, when the number of pursuers $N > 2$ the geometric method does not yield the optimal solution of the differential game in the complete state space. In other words, there are parts of the state space where the optimal flowfield does not consist of primary trajectories/regular characteristics only and singular hypersurfaces of focal or switch envelope type may make their appearance.

A. Apollonius Circle

For the sake of completeness we provide the geometry of Apollonius circles that will prominently feature in the geometric approach to the solution of differential games with multiple pursuers and one evader. An Apollonius circle is the locus of all points in the plane s.t. the ratio of the distances to two fixed points in the plane, also referred to as foci, is constant. In our case the ratio in question is the pursuer/evader speed ratio parameter $0 \leq \mu < 1$ and the foci are the instantaneous positions of points E and P. The Apollonius circle is illustrated in Fig. 2.

The three points P, E, and the center O of the Apollonius circle are collinear, and E is located between P and O. Let the E-P distance be d. The radius of the Apollonius circle is then

$$\rho = \frac{\mu}{1 - \mu^2} d$$

and in Fig. 2 the coordinates of the center of the Apollonius circle are

$$x_O = \frac{\mu^2}{1 - \mu^2} d, \qquad y_O = 0$$

Thus, when the respective positions in the Euclidean plane of P and E are (x_P, y_P) and (x_E, y_E),

$$\rho = \frac{\mu}{1 - \mu^2} \sqrt{(x_P - x_E)^2 + (y_P - y_E)^2} \tag{1}$$

and

$$x_O = \frac{1}{1 - \mu^2} x_E - \frac{\mu^2}{1 - \mu^2} x_P, \qquad y_O = \frac{1}{1 - \mu^2} y_E - \frac{\mu^2}{1 - \mu^2} y_P \tag{2}$$

[¶]In the Homicidal Chauffeur differential game, on the equivocal line (EL), the pursuer is required to use a discriminatory/stroboscopic strategy. However, in the Homicidal Chauffeur game and at the cost of an infinitesimal reduction in optimality, the pursuer can make sure that the state will not reach the EL in the first place.

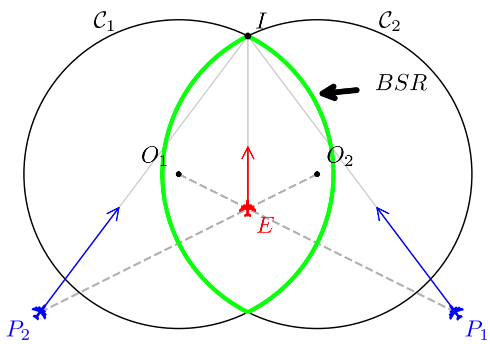


Fig. 1 Solution of the two cutters and fugitive ship game.

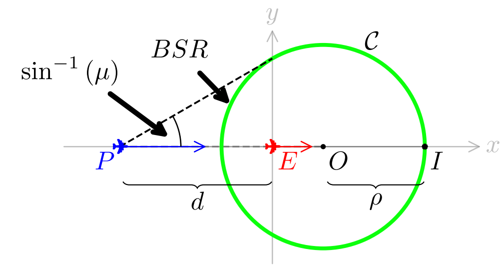
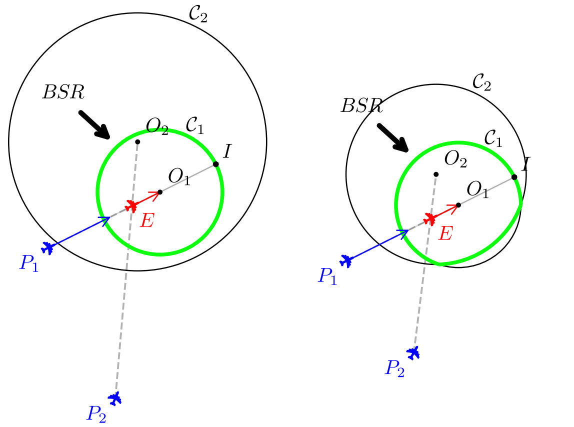


Fig. 2 Apollonius circle.



a) Apollonius circles do not intersect; *P*1 performs solo capture

b) Apollonius circles intersect, but still, *P*1 performs solo capture

Fig. 3 One cutter action.

Also, when $\cos(\sin^{-1}(\mu)) > \mu$, every straight line trajectory of the pursuer leading to the Apollonius circle is s.t. irrespective of the evader's constant heading, P closes in on E. This will always be the case if the speed ratio $0 \le \mu < (\sqrt{2}/2)$; this is a boon to capturability in many-on-one pursuit-evasion differential games.

The Apollonius circle construct provides the solution of the following elementary max min optimal control problem, which will play an important role in pursuit-evasion with simple motion. Assume that an evader (E) is constrained to choose a straight line path, and in addition he/she is discriminated against in that he/she is obliged to announce his/her course ahead of time. In this case the pursuer (P) will also hold course in order to in minimum time intercept the evader—he/she will employ collision course (CC) guidance. Capture will be effected on the circumference of the Apollonius circle with foci E and P, at the point where the evader's straight line path intersects the Apollonius circle. If there is only one pursuer, to maximize the time to capture the evader will obviously want to run away from the pursuer.

B. Geometric Construction

We will assume that the N pursuers and the evader are in general position; that is, no two Apollonius circles are tangent and no three Apollonius circles intersect at the same point. Not all pursuers are relevant to the chase, provided of course that the relevant pursuers and the evader play optimally [14]. In this respect, consider first the case $N = 2$. There are two Apollonius circles, C_1, whose foci are at E and P_1, and C_2, whose foci are at E and P_2. The two Apollonius circles are

$$(x - x_{O_i})^2 + (y - y_{O_i})^2 = \rho_i^2, \qquad i = 1, 2 \tag{3}$$

where x_{O_i}, y_{O_i}, and ρ_i are given by Eqs. (2) and (1). E is in the interior of both Apollonius disks. Thus, E is in the intersection of the two Apollonius disks, but the two Apollonius circles might or might not intersect. Concerning the calculation of the points of intersection, if any, of two Apollonius circles C_i and C_j, say C_1 and C_2: Subtracting the equation of circle C_1 from the equation of circle C_2—see Eq. (3)—yields the linear equation in the two unknowns x and y:

$$(x_{O_1} - x_{O_2})x + (y_{O_1} - y_{O_2})y = \frac{1}{2}[x_{O_1}^2 + y_{O_1}^2 - \rho_1^2 - (x_{O_2}^2 + y_{O_2}^2 - \rho_2^2)]$$

One can thus back out y as a function of x and insert this expression into one of the circle equations—see Eq. (3)—thus obtaining a quadratic equation in x. To calculate the two points of intersection, if any, of two Apollonius circles require the solution of a quadratic equation. The Apollonius circles intersect if and only if the quadratic equation has real solutions; in other words, the discriminant of the quadratic equation is positive. When the discriminant of the quadratic equation is negative we are automatically notified that the two Apollonius circles do not intersect; that is, one of the Apollonius disks is contained in the interior of the second Apollonius disk, with E in both Apollonius disks. In this case, if $\rho_2 > \rho_1$ the circle C_2 is discarded, and vice versa. The geometry is illustrated in Figs. 1 and 3. The *SR* in Fig. 1, where two pursuers are at work, is lens shaped and delimited by the two green circular arcs. The latter constitute the *BSR*.

When the Apollonius circles do not intersect, the pursuer associated with the outer Apollonius circle is irrelevant to the chase—see Fig. 3a. This is so because the configuration is s.t. should P_1 employ PP and E run for his/her life, player P_2 cannot reach E before the latter is captured by P_1 because either he/she is too far away from the P_1/E engagement, or is too slow to close in and join the fight. This renders player P_2 irrelevant, and as far as the geometric method is concerned, the Apollonius disk associated with player P_1 is then contained in the bigger Apollonius disk associated with player P_2. In this case the pursuer P_1 on which the inner Apollonius circle is based will single-handedly capture the evader: he/she will optimally employ PP while the evader runs for his/her life. The game with two pursuers collapsed to the iconic pursuit-evasion game with one pursuer and one evader where P_1 employs PP and E runs away from P_1. In Fig. 3b we see that it is not necessary for one of the circles to be contained in the other for this to be the case: capture will single-handedly be performed by pursuer P_1 if and only if the point on the circumference of the Apollonius circle $\mathcal{C}_1$, which is antipodal to E, is contained in the Apollonius disk $\mathcal{C}_2$. At this point (no pun intended), it is worthwhile to concisely restate the meaning of optimality in the context of zero-sum differential games, and, in particular, pursuit-evasion differential games. We are after state feedback strategies: the pursuers' strategy and the evader's strategy are optimal in the set of all possible pursuer and evader strategies if they constitute a saddle point. The following holds.

Proposition 1: In the two-pursuer single-evader pursuit-evasion game in the 2-D realistic plane wherein all agents have simple motion, optimal play results in isochronous capture of the evader by both pursuers iff

$$\frac{\sqrt{1 - \mu^2 \sin^2\theta} - \mu\cos\theta}{1 - \mu} > \frac{d_1}{d_2} > \frac{1 - \mu}{\sqrt{1 - \mu^2 \sin^2\theta} - \mu\cos\theta} \tag{4}$$

where $d_i = \|E - P_i\|_2$, $i = 1, 2$, $0 \le \mu < 1$ is the evader/pursuer speed ratio, and θ is the angle included between the radials EP_1 and EP_2. Furthermore, if the two pursuers have different speeds and the speed ratio parameters are $\mu_1 = (V_E/V_{P_1})(<1)$ and $\mu_2 = (V_E/V_{P_2})(<1)$, P_1 and P_2 will isochronously capture the evader iff

$$\frac{\sqrt{1-\mu_1^2\sin^2\theta}-\mu_1\cos\theta}{1-\mu_2} > \frac{\mu_1 d_1}{\mu_2 d_2} > \frac{1-\mu_1}{\sqrt{1-\mu_2^2\sin^2\theta}-\mu_2\cos\theta} \tag{5}$$

Proof: Consider $V_P = 1$ (which is akin to saying that all of the velocities are normalized by the pursuers' velocity; thus one may also understand this to mean that time has been scaled). First, without loss of generality, we consider the scenario depicted in Fig. 4, where E flees directly from P_2. Given these agents' headings, we seek to compare the time of flight for P_2 to reach E with the time of flight for P_1 to reach E. In general, the time-to-intercept can be expressed as

$$t = \frac{d}{|\dot{d}|}$$

Because the headings of P_2 and E are aligned with the line-of-sight (LOS) P_2E, the separation distance changes as

$$\dot{d}_2 = \mu - 1$$

For $\dot{d}_1$, let us first consider the contribution of P_1's motion. Note that for E and P_1 to meet simultaneously at point I, they must each travel the same distance in the vertical direction in the same amount of time. Thus we have

$$\dot{y}_{P_1} = \dot{y}_E = \mu\sin\theta$$

The velocity of the pursuers is fixed to 1; thus,

$$\dot{x}_{P_1} = \sqrt{1-\mu^2\sin^2\theta}$$

Because, without loss of generality, the x axis is aligned with the LOS P_1E, pursuer P_1's contribution to $\dot{d}_1$ is $-\dot{x}_{P_1}$. Thus $\dot{d}_1$ is the sum of the contributions due to P_1 and E's motion:

$$\dot{d}_1 = \mu\cos\theta - \sqrt{1-\mu^2\sin^2\theta}$$

If, in this configuration, the time-to-intercept for P_2 is less than that of P_1, then capture will be performed by P_2 alone. This is due to the construction of P_1's Apollonius circle (depicted in Fig. 4)—interception along the Apollonius circle yields the smallest possible time-to-intercept for the given evader heading θ. For isochronous capture, it is necessary that $t_2 > t_1$, which can be written as

$$\frac{d_2}{1-\mu} > \frac{d_1}{\sqrt{1-\mu^2\sin^2\theta}-\mu\cos\theta} \tag{6}$$

The above process may be repeated for the case where E flees directly from P_1. There, it is also necessary that the time-to-intercept

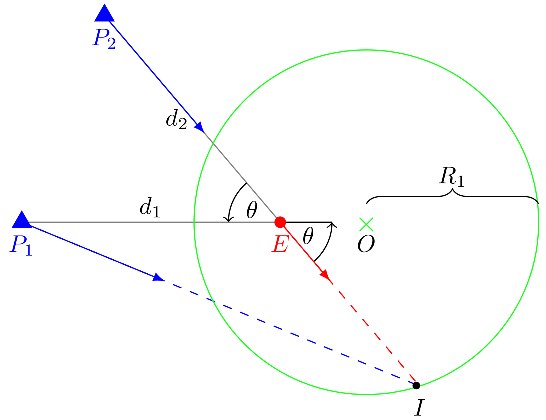


Fig. 4 Necessary condition for isochronous capture.

for P_1 is greater than that for P_2 for isochronous capture to be optimal:

$$\frac{d_1}{1-\mu} > \frac{d_2}{\sqrt{1-\mu^2\sin^2\theta}-\mu\cos\theta} \tag{7}$$

The satisfaction of both of the preceding equations is sufficient to guarantee the optimality of isochronous capture by both P_1 and P_2. Gathering the term d_1/d_2 onto one side, combining Eqs. (6) and (7), and reverting back to the speed ratio definition $\mu = (V_E/V_P)(<1)$ yield Eq. (4). □

Reference [17] gives a derivation of this condition in a reduced dimension state space, which, in a sense, is the solution of the Game of Kind for the two-on-one pursuit-evasion differential game.

Proposition 1 generalizes to the case when $N > 2$: if all the pursuers have equal speed, one need only check Eq. (4) for all pursuer pairs involving the closest pursuer, and if the pursuers have different speeds, one need only check Eq. (5) for all pursuer pairs involving the pursuer i^*, where

$$i^* = \arg\min_{1\le i\le N}\left(\frac{d_i}{1-\mu_i}\right)$$

III. SR and BSR

The solution of many-player pursuit-evasion differential games requires the construction of the *SR* of the evader and the boundary thereof (*BSR*). It is the evader's reachable set where he/she can proceed uncontested by the pursuers. The construction of the *SR* and *BSR* in the many-on-one pursuit-evasion differential game proceeds as follows. Apollonius circles C_i are formed based on the instantaneous positions of each pursuer P_i and the evader E, and the attendant speed ratio parameters $\mu_i(<1)$, $i = 1,\ldots,N$. N pursuers and one evader give rise to N Apollonius circles/disks C_i, $i = 1,\ldots,N$. When N (≥ 2) pursuers are present, first weed out the Apollonius disks that contain smaller Apollonius circles—the former are irrelevant to the pursuit [14,15]. In other words, the pursuers, which together with E gave rise to these outer Apollonius circles, will not participate in an optimal pursuit. The remaining, say, $M(\le N)$ Apollonius circles, are retained. The intersection of the retained M Apollonius disks is the *SR* [14]

$$SR = \cap_{i=1}^{N} C_i$$

It then all comes down to the construction of the intersection of M Apollonius disks, and computational geometry will render an efficient algorithm. The *SR* is convex [15], and $E \in SR$, by construction.

There are $(1/2)N(N-1)$ pairs of Apollonius circles to be considered, but not every pair of circles intersects. Hence, one is faced with solving $(1/2)N(N-1)$ quadratic equations, not all of which have real solutions. The real solutions, if they exist, yield the two intersection points of the pair of Apollonius circles. If the discriminant of the quadratic equation is negative the quadratic equation does not have real solutions, which means that one of the two Apollonius disks is contained in the second Apollonius disk. The Apollonius circle with the bigger radius ρ is discarded and the pursuer who gave rise to this Apollonius circle is out of the game. We end up with $M \le N$ active Apollonius circles. The *BSR* is a convex figure akin to a polygon whose sides are circular arcs. The vertices of the *BSR* are calculated as follows. Because E is contained in all of the M Apollonius circles, each Apollonius circle has two intersection points with each of the remaining $M-1$ Apollonius circles, and so on each Apollonius circle there are $2(M-1)$ intersection points. Hence, there are $M(M-1)$ intersection points in total. Now, to find the M vertices of the *BSR*, check each of the $M(M-1)$ intersection points whether it is in all the remaining $M-2$ Apollonius disks. This requires checking $M(M-1)(M-2) \approx O(M^3)$ quadratic inequalities—see Eq. (3). If so, this is a vertex of the *BSR*. If, for example, $M = 2$ no inequalities need to be checked, if $M = 3$ we must check 6 inequalities, and if $M = 4$ the number of inequalities to be checked is 24. However, the computational geometry aspect is perhaps not critical because of the overhead baggage efficient algorithms carry when the number M of pursuers is not too large. This is indeed the case in the herein considered pursuit scenarios. We focus on the case when there are three pursuers. Also, we note the availability of the EvaderCell algorithm in ([14] Sec. 5), which constructs *SR* by first placing the N pursuers in a queue.

The process of aimpoint selection is given as follows.

First calculate the points antipodal to E on the circumference of each of the identified M active Apollonius circles; this requires the solution of a quadratic equation—in total M quadratic equations are solved. Next, check whether such an antipodal point is in all the remaining $M - 1$ Apollonius disks; this requires checking $M(M - 1)$ quadratic inequalities. If the players are in general position, at most only one such antipodal point can exist [14]. This is so because the *SR* is, by construction, convex, and therefore a local maximum is the global maximum.

If such an antipodal point exists and, say, it resides on the circumference of the ith Apollonius circle, it is designated the aim point I of pursuer i and of the evader. The pursuit then degenerates into a tail chase where only one pursuer, pursuer i, is active; the aim point I of P_i and of E is on the *BSR*, but is not a vertex of the simpler *BSR*, which is now the $(E,\ P_i)$ Apollonius circle. The capture time is then $t_f = V(P_1, \ldots, P_N, E) = (1/\mu - 1)\mathrm{dist}(E, P_i)$.

If no such antipodal point exists, the point of the composite *BSR* farthest from E is always at a vertex of the *BSR*; the number of vertices of the *BSR* is $\leq N$. Among the vertices of the *BSR*, this critical vertex, a potential aim point, will be designated by I. Thus, among the *BSR*'s vertices, E potentially heads toward the most distant vertex, which then becomes the aim point I, and then so do the pursuers P_{i_1} and P_{i_2}. These two particular pursuers, P_{i_1} and P_{i_2}, are associated with the two Apollonius circles among the $M(\geq 2)$ Apollonius circles at whose intersection the vertex farthest from E, the potential aim point I, lies. Indeed, the optimality principle used in the two-on-one $(N = 2)$ pursuit-evasion game analyzed in [1,16,17] declared the three players' aim point I to be the *SR* point farthest from E. If the evader is not single-handedly captured by one of the two pursuer, that is, if $M > 1$, the *SR* is lens shaped and the *BSR* has two vertices. When only two pursuers partake in the chase and $M = N = 2$, the aim point I is then one of the two vertices/corners of the lens-shaped *BSR* and capture will be effected at the vertex, which is the one farther from E. At this vertex, which is the designated aim point I, the evader is isochronously captured by the two pursuers. During optimal play, when $N = 2$, only one pursuer or both pursuers will actively capture the evader. When capture is effected by the two pursuers the optimal strategies mandate that the two pursuers and the evader head in unison toward the aim point I where E is isochronously captured in a pincer movement maneuver.

So far, it would appear the optimality principle successfully used in [1,16,17], which would designate the *BSR* vertex, which is the farthest from E to be the aim point I of the evader and of the two pursuers associated with the Apollonius circles whose intersection gave rise to this *BSR* vertex in the first place, to also be applicable to the case where $N \geq M > 2$. Hence, similar to the two-pursuer case treated by Isaacs in [1,16,17], in the case of N pursuers, only two pursuers would always end up actively capturing the evader by executing a pincer movement maneuver. Exceptions include the cases where the number of active Apollonius circles comes down to $M = 1$, or it so happens that the point antipodal to E on the circumference of one of the Apollonius circles is contained in all the remaining $M - 1$ disks. Then and only then a single pursuer captures the evader in PP [14]. If this would be the case the optimal flowfield would consist of primary optimal trajectories/regular characteristics, the Value function would be C^1, and there would be no singular manifolds. Additionally, the geometric method, which yields the solution to the open loop max min optimal control problem, would automatically yield the solution of the differential game. However, as so often happens in differential games, there are surprises.

Already the differential game with three pursuers is more complex than the two-on-one pursuit-evasion game. An attempt at directly extending the geometric method used in the two-pursuer case in the fashion alluded to above does not yield a correct solution of the differential game when the number of pursuers $N \geq 3$. First, a new, extended optimality principle is needed.

IV. Operations Research Analogy

In this section an analogy from operations research (OR) is introduced, which is conducive to the determination of the evader's aimpoint in his/her quest to escape capture. Consider the case where there are just three pursuers. The analysis of the pursuit-evasion game with three pursuers, P_1, P_2, and P_3, naturally induces us to focus on the geometry of $\Delta P_1P_2P_3$ formed by the pursuers. It turns out that when there are more than two pursuers $(N > 2)$, the solution of the pursuit-evasion problem is intimately related to the OR problem of an optimal facility location in the Euclidean plane. When three pursuers engage the evader E, the following geometry is pertinent.

Consider a $\Delta P_1P_2P_3$ whose vertices are the three pursuers' instantaneous position in the Euclidean plane. If $\Delta P_1P_2P_3$ is acute its circumcenter is inside the triangle, and if the triangle is obtuse its circumcenter is outside the triangle. It is well known that an acute triangle's circumcenter is the solution of the classical optimal facility location problem. Suppose that there are three cities that are located in the Euclidean plane at P_1, P_2, and P_3, and it is required to determine where to locate, say, a warehouse, s.t. its maximal distance from one of these cities is minimal. The optimal location O^* of the facility/warehouse is the solution of the min max optimization problem:

$$O^* = \arg\min_{O\in R^2} \max_{1\leq i\leq 3} \mathrm{dist}(O, P_i) \tag{8}$$

When the $\Delta P_1P_2P_3$ is obtuse, the optimal location lies on the triangle's longest edge.

By way of an antithesis, consider now the obnoxious facility location problem [19] where it is required to place a noxious fumes-emitting facility s.t. the adverse effect on nearby cities is minimized, with the understanding that toxicity decreases the farther the source of the pollutant is. In the context of our triangle, the affected cities are located at its vertices P_1, P_2, P_3, and to find the optimal facility location P^* one must now solve the max min optimization problem

$$P^* = \arg\max_{P\in R^2} \min_{1\leq i\leq 3} \mathrm{dist}(P, P_i) \tag{9}$$

A solution in R^2 of the obnoxious facility location problem does not exist. In other words, if the cities are located at the vertices of $\Delta P_1P_2P_3$, the obnoxious facility should be located in the extended plane at infinity. Consequently, the impact of the pollutant would be infinitesimal. For a solution of the obnoxious facility location problem to exist, the search domain must be restricted to a compact set $\chi \subset R^2$.

However, when the $\Delta P_1P_2P_3$ is acute, its circumcenter O^* is nevertheless a local max min—whereas, as previously stated, the globally optimal facility location is still at infinity. To see that there is a local max min, momentarily consider small perturbations, in *all* possible directions, of the facility's location away from the circumcenter O^* of the acute $\Delta P_1P_2P_3$, say, from O^* to O'. Let $\mathscr{B}_{O^*}$ be a small neighborhood of the circumcenter O^*, so that $O' \in \mathscr{B}_{O^*}$. In an acute triangle, the distance from O' to at least one of the triangle's vertices/cities, say, vertex/city P_1, will decrease, $\overline{O'P_1} < \overline{O^*P_1}$, and this trend, where the distance to at least one city decreases, is true $\forall O' \in \mathscr{B}_{O^*}$. Obviously, the new obnoxious facility's location O' is worse than the current obnoxious facility's location at O^*; thus O' is not a good choice—this being true $\forall O' \in \mathscr{B}_{O^*}$. Hence, if the specified feasible set χ is compact and $O^* \in \chi$, the acute triangle's circumcenter O^* is a candidate for the optimal facility placement. Moreover, if the compact feasible set is the acute triangle proper, its circumcenter O^* *is* the optimum. This is so because we need not confine our attention to perturbations in the small neighborhood $\mathscr{B}_{O^*}$ of the circumcenter O^*. One can move the obnoxious facility along a radial emanating from O^* to new locations O' all the way to an edge of the triangle, and beyond. Outside the triangle, a *local* max min cannot exist. Suppose that the point P^*, $P^* \notin \Delta P_1P_2P_3$, is a local max min. But from any such location $P^* \notin \Delta P_1P_2P_3$ one can always move the obnoxious facility away from the triangle, in a direction normal to the side of the triangle that is close to the point P^*, to a new location P' s.t. $\overline{P'P_1} > \overline{P^*P_1}$, $\overline{P'P_2} > \overline{P^*P_2}$, and $\overline{P'P_3} > \overline{P^*P_3}$, and so the new location P' is an improvement on P^*—thus, P' is a better choice.

Hence, the circumcenter O^* of an acute triangle is the solution of the constrained obnoxious facility location problem (cOFLP), provided that 1) it is in the specified constraint set χ, and 2) the constraint set χ does not extend beyond the above-mentioned hexagon. Obviously, if the circumcenter O^* is then in the interior of the constraint set χ, the optimal location of the obnoxious facility at O^* will then be in the interior of the constraint set χ. If the circumcenter O^* of the $\Delta P_1P_2P_3$ is not in the specified constraint set χ, the optimal location of the obnoxious facility will be on the boundary $\partial\chi$ of the constraint set χ. Finally, if the circumcenter O^* is in the constraint set χ but the latter extends beyond the hexagon's boundary, the obnoxious facility's optimal location will be on the boundary of the set χ at a point $P^* \in \partial\chi$, which is outside the hexagon, provided that the following condition holds:

$$\min_{1\le i\le 3} \mathrm{dist}(P^*, P_i) > \mathrm{dist}(O^*, P_1)$$

where the point P^* is the solution of the max min optimization problem:

$$P^* = \arg\max_{P\in\partial\chi} \min_{1\le i\le 3} \mathrm{dist}(P, P_i) \tag{10}$$

Note that $\mathrm{dist}(O^*, P_1) = \mathrm{dist}(O^*, P_2) = \mathrm{dist}(O^*, P_3) = R$, where R is the radius of the circumcircle of the acute $\Delta P_1P_2P_3$, and so the condition is

$$\min_{1\le i\le 3} \mathrm{dist}(P^*, P_i) > R \tag{11}$$

If there does not exist a point P^* which 1) lies on the part of the boundary of χ outside the hexagon and 2) satisfies condition (11), then the optimal location of the obnoxious facility will be at the circumcenter O^*. Thus, if $\Delta P_1P_2P_3$ is acute, the solution of the cOFLP comes down to the determination of two points in R^2: 1) find the triangle's circumcenter O^* (this is a problem in geometry); 2) determine the point $P^* \in \partial\chi$, which requires the solution of the max min optimization problem (10). Finally, check whether condition (11) holds.

If, however, $\Delta P_1P_2P_3$ is obtuse, and if, without loss of generality, we assume that the angle at its vertex P_1 is s.t. $\angle P_2P_1P_3 > (\pi/2)$, the perturbation of its circumcenter to O' along the perpendicular bisector of P_2P_3 and away from P_1 will cause an increase in the distance from O' to each and all three vertices, P_1, P_2, and P_3, of the triangle. Hence, the circumcenter O^* of an obtuse triangle is not a local max min. When the triangle is obtuse, a local max min does not exist. Hence, if the configuration of the three cities is s.t. P_1, P_2, and P_3 are vertices of an obtuse triangle and a compact set χ where the facility should be located is specified, the facility's optimal location will always be on the boundary $\partial\chi$ of the constraint set χ. If $\Delta P_1P_2P_3$ is obtuse, it all comes down to the solution of the max min optimization problem (10). Thus, consider the following.

Example: $\Delta P_1P_2P_3$ is obtuse, with $\angle P_2P_1P_3 > (\pi/2)$, and the obnoxious facility must be placed within the confines of the triangle: the constraint set $\chi = \Delta P_1P_2P_3$. The geometric solution of the triangle-constrained max min optimization problem mandates that the optimal location P^* of the facility is on the base $\overline{P_2P_3}$ of the obtuse triangle, at the foot of the orthogonal bisector of the side $\overline{P_1P_3}$. Additionally, without loss of generality, the sides of $\Delta P_1P_2P_3$ are s.t. $\overline{P_1P_2} < \overline{P_1P_3} < \overline{P_2P_3}$. The max min optimization yields

$$P^* = \arg\max_{P\in\Delta P_1P_2P_3} \min_{1\le i\le 3} \mathrm{dist}(P, P_i)$$

This places the obnoxious facility at the point P^* on the base P_2P_3 of the triangle, at a distance $[\overline{P_1P_3}/\cos(\angle P_1P_3P_2)]$ from the P_3 vertex.

V. Pursuit and Evasion

In this section we focus on the pursuit-evasion differential game with three pursuers. We assume that the players are in general position, and so $M = N = 3$, and we also assume that the three pursuers have equal speed $V_p = 1$. For the remainder, we assume that condition (4) holds for all $N-1$ pairs (P_{i^*}, P_j) for $j = 1, \ldots M$, $j \ne i^*$, where P_{i^*} is the pursuer closest to E, initially, and thus the furthest point from E on the *BSR* is a vertex of the *BSR*. The objective is to determine the aim point of the evader and of the pursuers, and thus to characterize the players' "optimal" state feedback strategies using the geometric method. To this end, the constrained obnoxious facility location analogy is invoked.

Assume momentarily that three cities are located at the pursuers' positions. Thus, the three cities are at the vertices P_1, P_2, and P_3 of the $\Delta P_1P_2P_3$ formed by the pursuers. One is interested in finding a location for the obnoxious facility in the Euclidean plane s.t. its minimal distance from any of the three "cities" is maximal. The location of the "obnoxious facility" to be determined will be the future position of the evader, namely, the aim point I the evader will head to in the pursuit-evasion differential game. Finally, the constraint set χ in the Euclidean plane where the obnoxious facility/evader is to be located must be specified: it is the *SR* that was constructed in Sec. III. In this regard, note that whereas the circumcenter O^* and the radius R of the circumcircle of the $\Delta P_1P_2P_3$ are exclusively determined by the positions of the pursuers, the geometry of the constraint set *SR* is also influenced by the current position of the evader. The solution P^* of the above-formulated cOFLP, namely, the optimal location of the "obnoxious facility," will provide the aim point I for the players in the pursuit-evasion differential game with three pursuers:

$$I = P^*$$

From the analysis of the obnoxious facility location problem in Sec. IV and the *SR* construction in Sec. III, it follows that with the constraint set $\chi = SR$ the following holds. If $\Delta P_1P_2P_3$ is obtuse, or if $O^* \notin SR$, the optimal location P^* of the "obnoxious facility" will be at the *BSR* vertex farthest from E. Furthermore, the optimal location P^* of the "obnoxious facility" will also be at the farthest vertex from E of the *BSR*, provided that this point is outside the hexagon associated with $\Delta P_1P_2P_3$—this, irrespective of whether $\Delta P_1P_2P_3$ is acute or obtuse. If, however, $\Delta P_1P_2P_3$ is acute, its circumcenter $O^* \in SR$, and the *BSR* is contained in the hexagon associated with $\Delta P_1P_2P_3$, the optimal location P^* of the "obnoxious facility" will be at O^*. And if $\Delta P_1P_2P_3$ is acute but one or more of the *BSR*'s vertices are outside the hexagon, the optimal location P^* of the "obnoxious facility" will be at the vertex of the *BSR* farthest from E. Given the state of the game, that is, the instantaneous positions P_1, P_2, P_3, and E of the pursuers and the evader, the constrained obnoxious facility location analogy is herein used to determine the players' aim points using the geometric method. This, in turn, affords the synthesis of the players' "optimal" state feedback strategies.

We start with the *SR*. In the specific case of three pursuers, and in general position, meaning $M = N = 3$, there are three possible combinations of two out of three pursuers and therefore there are three elemental lens-shaped *SR* regions, each pertaining to a two-on-one pursuit-evasion differential game. Each such elemental lens-shaped figure is the *SR* in a two-on-one game with two pursuers, as investigated by Isaacs in [1,16,17]. And each such lens-shaped *SR* has two vertices, and in the two-on-one pursuit-evasion game the optimal strategies mandate that, among the two vertices of this lens-shaped *BSR*, the two pursuers and the evader head in unison toward the vertex farther from the evader E. The *SR* in our three-on-one pursuit-evasion differential game is the intersection of these three elemental lens-shaped *SR* figures. This is how the *SR* in the differential game with three pursuers is formed. With the four players in general position, this *SR*, formed by the intersection of the three lens-shaped *SR*s pertaining to the three elemental games with two pursuers only, has three vertices: V_1, V_2, and V_3. Thus, when there are three pursuers the shape of the *SR* is akin to that of a Reuleux triangle. Each vertex of the Reuleux-triangle-like composite *BSR* is formed by the intersection of a pair of Apollonius circles, each such Apollonius circle being associated with the evader and a member of the team of three pursuers. The vertex V_k of the Reuleux-triangle-shaped *BSR*, $1 \le k \le 3$, is formed by the intersection of the Apollonius circles of, say, pursuers $P_{i(k)}$ and $P_{j(k)}$, $1 \le i(k) \le 3$, $1 \le j(k) \le 3$, $i(k) \ne j(k)$.

All three vertices of the composite *SR* obviously are inherited vertices of the original lens-shaped *SR*s from the three two-on-one elemental games, but when $E \in \Delta P_1P_2P_3$, these vertices of the elemental lens-shaped *BSR*s are the ones that were closer to *E*. This is so for the following reason. If capture is effected at the vertex V_{k^*} of the *BSR*, that is, *E* is captured by the pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$, this invariably happens after *E*, who began in $\Delta P_1P_2P_3$ formed by the three pursuers, manages to break out from the encirclement. The breakout occurs between the two above-mentioned pursuers. In other words, *E* was initially running toward, and not away, from pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$. Thus, in contrast to the case where there are two pursuers only, capture is effected at the elemental lens-shaped *BSR*'s vertex closer to *E*—see the lens-shaped *BSR* in Fig. 1. However, in Fig. 1 capture is effected at the farther from *E* vertex; the vertex closer to *E* would not have qualified as aim point in the respective stand-alone two-on-one pursuit-evasion games. This is important because we cannot now directly fall back on the two-pursuer game to assert that capturability by the two designated pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$ is guaranteed. However, as will become apparent in the sequel, when the pursuers play "optimally," capturability is guaranteed. Also, if the speed ratio parameter $0 \le \mu < (\sqrt{2}/2)$, capturability is guaranteed right away by virtue of the fact that by heading to a point/any point on the circumference of the Apollonius circle, the pursuers relentlessly close in on the evader.

A. Optimality Principle

Consider the following. *E* is safe while holding course and staying in the *SR* [15]. Thus one could argue that *E* can be captured only on the *BSR*. Hence, to prolong his/her time-to-capture, *E* should head to point I^* of the *BSR*, which is farthest from him/her:

$$I^* = \underset{I \in BSR}{\arg\max}\, \mathrm{dist}(E, I)$$

and in view of the analysis in Sec. III the aim point I^* will be a vertex of the *BSR* or a point antipodal to *E* on the circumference of an Apollonius circle. This was the optimality principle successfully used in the two-on-one pursuit-evasion differential game in [1,16,17] where $N = 2$. Because any point on the *BSR* is, by construction, isochronously reachable by *E* and at least one of the three pursuers, the above equation is equivalent to

$$(I^*, j^*) = \underset{I \in BSR}{\arg\max} \min_{1 \le j \le 3} \mathrm{dist}(P_j, I) \tag{12}$$

Therefore Eq. (12) can be construed to be the optimality principle used in the two-on-one pursuit-evasion differential game. However, as it stands, the optimality principle statement (12) is lacking. There might also be a point $P^* \in \mathrm{Interior}(SR)$ s.t.

$$(P^*, i^*) = \underset{P \in \mathrm{Interior}(SR)}{\arg\max} \min_{1 \le i \le 3} \mathrm{dist}(P_i, P) \tag{13}$$

and the following holds:

$$\mathrm{dist}(P^*, P_{i^*}) > \mathrm{dist}(I^*, P_{j^*})$$

Now, $BSR \subset SR$ and this allows us to stipulate a new optimality principle for pursuit-evasion games with $N \ge 3$ pursuers. We combine Eqs. (12) and (13), to read [14]

$$(P^*, i_1^*, \ldots, i_l^*) = \underset{P \in SR}{\arg\max} \min_{1 \le i \le 3} \mathrm{dist}(P_i, P) \tag{14}$$

Comparing Eqs. (8) and (14) we see that Eq. (14) yields the solution of the cOFLP with the constraint set $\chi = SR$. The location P^* of the obnoxious facility maximizes the distance to the discrete point set $\{P_1, P_2, P_3\}$. Therefore, in the pursuit-evasion differential game with three pursuers and if $M = N = 3$, the aim point of the evader will be

$$I = P^*$$

and $l = 2$ or $l = 3$; that is, capture will be effected by two, or all three, pursuers.

The new optimality principle, which stipulates that the aim point is provided by the solution of the cOFLP with the constraint set being the *SR*, applies to pursuit-evasion games with three pursuers and also applies to pursuit-evasion games with $N \ge M > 3$ pursuers. At the same time, the optimality principle, as stated, also applies to the case where $N = 2$: if the point antipodal to *E* on the circumference of an Apollonius circle of one of the pursuers is not contained in the Apollonius disk of the second pursuer, the *SR* is lens shaped and it has two vertices, say, V_1 and V_2. Without loss of generality assume that the vertex farther from *E* is V_1. Because, by construction, $\mathrm{dist}(P_1, V_1) = \mathrm{dist}(P_2, V_1)$ and $\mathrm{dist}(P_1, V_1) > \mathrm{dist}(P_1, V_2) (= \mathrm{dist}(P_2, V_2))$, the solution of the cOFLP will place the aim point at the *SR*'s vertex V_1. This is why formulation (12) of the optimality principle used when $N = 2$ yielded the correct solution in [1]. Moreover, even when $N = 2$ and the pursuers' Apollonius circles intersect, the aim point/optimal obnoxious facility location need not be a vertex of the *SR*. This instance, foreseen in [17], arises when the point antipodal to *E* on the circumference of an Apollonius circle of, say, pursuer P_1, is contained in the Apollonius disk of P_2; this is equivalent to the analysis following Proposition 1. The *SR* is then no longer lens shaped, and the optimality principle stipulates that P^* is the antipodal point of *E* on the circumference of $\mathcal{C}_1$ and $i^* = 1$. Capture will single-handedly be effected by P_1 in a PP, which will end at the antipodal point of *E* on the circumference of $\mathcal{C}_1$.

Let the index

$$k^* = \underset{1 \le k \le 3}{\arg\max}\, \mathrm{dist}(E, V_k)$$

indicate the vertex V_{k^*} of the Reuleux-triangle-like *BSR* that is farthest from *E*. Note that the vertex that is farthest from *E* may not be unique (c.f. Fig. 5). From the construction of the *BSR* in Sec. III, we conclude that when $N = M = 3$ and $\Delta P_1P_2P_3$ is acute, the optimal solution of the cOFLP is as follows.

$$P^* = \begin{cases} O^* & \text{if } R > \mathrm{dist}(V_{k^*}, P_{i(k^*)}) \text{ and } O^* \in SR \\ V_{k^*} & \text{otherwise} \end{cases} \tag{15}$$

Thus, so far, according to the optimality principle, when $N = M = 3$, it would appear that the optimal strategy for the group pursuit differential game entails *E* heading to the aim point P^* given by Eq. (14). And if $\Delta P_1P_2P_3$ is acute, P^* is directly given by Eq. (15).

An interesting instance where the optimality-principle-mandated aim point is in the interior of the *SR*, that is, $P^* = O^*$, arises in a symmetric configuration where the three pursuers indicated by red triangles are located at the vertices of an equilateral triangle and the evader is smack at its center O^*—see Fig. 5.

The Reuleux-triangle-like *BSR* in the center of the figure is the intersection of the three Apollonius disks, C_1, C_2, and C_3, and its three vertices are all at an equal distance from *E*. The solution of the cOFLP is the circumcenter O^* of the equilateral $\Delta P_1P_2P_3$ and not a vertex of the Reuleux-triangle-like BSR: In the scenario illustrated in

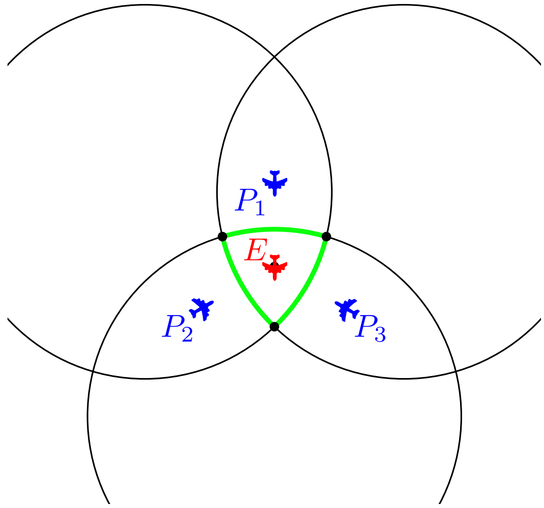


Fig. 5 ***E* is at circumcenter of equilateral $\Delta P_1 P_2 P_3$.**

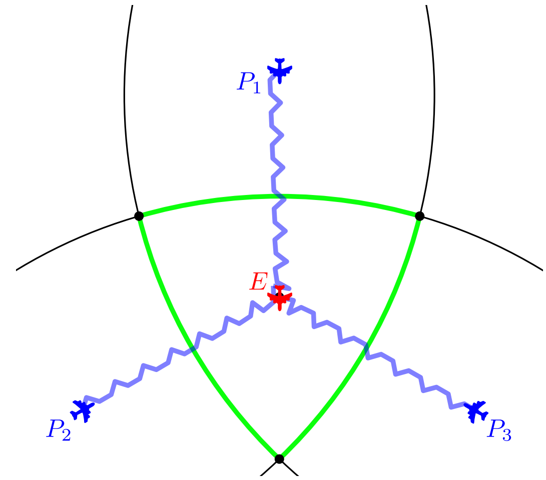


Fig. 7 ***E* is stationary, at circumcenter of equilateral $\Delta P_1 P_2 P_3$, and the pursuers aim at the furthest *BSR* vertex from *E*; $\Delta t = 0.06$.**

Fig. 5, E is at the center of the equilateral triangle; that is, E is at the circumcenter O^* from the get-go. In this initially symmetric configuration the three pursuers are all at an equal distance from the evader. However, the distance from P_2 and P_3 from their aforementioned *BSR* vertex potential aim point is *less* than their distance to the circumcenter O^* of the equilateral $\Delta P_1 P_2 P_3$—see Fig. 6 where $\overline{P_2 O^*} > \overline{P_2 I_{2,3}}$—and so, condition (11) does not hold. In view of this, Eq. (15) tells us that E should *not* be running to this, or any, vertex of the Reuleux-triangle-shaped *BSR*, and instead, by staying stationary at its initial position O^* at the triangle's circumcenter, the evader's time-to-capture will be extended:

$$\frac{\mathrm{dist}(P_2, I_{2,3})}{\mathrm{dist}(P_2, E)} = \frac{1}{2}\frac{1}{1-\mu^2}\left(\sqrt{4-3\mu^2}-\mu\right) \quad (< 1 \forall 0 < \mu < 1)$$

Hence, according to the optimality-principle-enabled geometric method the evader should stay put and he/she will isochronously be captured at O^* by the three pursuers. In this symmetric configuration/state, the strategy of remaining stationary afforded E an increase in his/her time-to-capture of $50 \cdot (\sqrt{4-3\mu^2}+\mu-2)\%$—this is courtesy of the optimality principle.

Suppose that E plays "optimally" and stays put at the circumcenter O^* of the equilateral $\Delta P_1 P_2 P_3$. Also suppose that the three pursuers deviate from the dictum of Eq. (15) and instead erroneously employ the old strategy of heading toward the farthest vertex of the *BSR*. The pursuers would zigzag/chatter their way toward the evader and capture him/her. As expected, this will take them longer, about 10% longer (for $\mu = 0.8$), than it would have taken them to capture E if they played according to the optimality principle—see Fig. 7, where the pursuers chatter. Note, this result is based on a discrete time simulation.

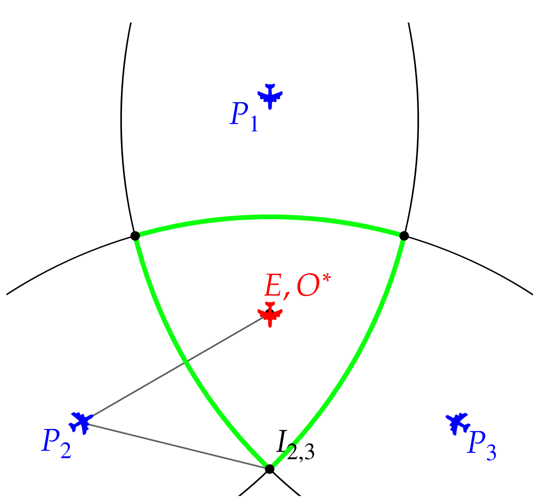


Fig. 6 **Solution of the cOFLP is O^*.**

Hence, when $N = M = 3$ and the players play "optimally" according to the optimality-principle-enabled geometric method and head to the aim point P^* provided by the solution of the cOFLP, this translates into the following:

Case 1 ($\Delta P_1 P_2 P_3$ is obtuse): If the configuration of the pursuers is s.t. the $\Delta P_1 P_2 P_3$ is obtuse, E heads to the vertex k^* of the BSR that is the farthest from him/her. This is so because the solution P^* of the constrained max min optimization problem (14) is on the boundary of the feasible set, that is, the BSR. The solution P^* of the max min optimization problem (14) yields the BSR vertex $V_{k^*} = P^*$. Because the vertex V_{k^*} of the BSR is generated by the intersection of the two Apollonius circles, $C_{i(k^*)}$ and $C_{j(k^*)}$, the capture of the evader will be effected by pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$; it is also possible that $M = 1$ whereupon just one pursuer effects the capture in PP. □

Case 2 ($\Delta P_1 P_2 P_3$ is acute): If the configuration of the pursuers is s.t. the $\Delta P_1 P_2 P_3$ is acute, it all boils down to the question of whether the circumcenter O^* of the triangle is in the *SR*.

Case 2.1: If $O^* \notin SR$, according to Eq. (15) E heads toward the vertex V_{k^*} of the BSR and as in case 1 the capture of E is effected by the pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$; if $M = 1$ or condition (4) does not hold, E is captured by just one pursuer. □

Case 2.2: If $O^* \in SR$ and $\mathrm{dist}(V_{k^*}, P_{i(k^*)}) < R$, according to Eq. (15), E heads toward the circumcenter O^* in the interior of the SR, and so do all three pursuers. Since $E \in \mathrm{Interior}(SR)$, he/she arrives at O^* before the pursuers. Upon arriving to O^*, E stays put, awaiting his/her isochronous capture by the three pursuers. The capture time is then R/μ. If, however, $\mathrm{dist}(V_{k^*}, P_{i(k^*)}) > R$, according to Eq. (15) E heads toward V_{k^*} and so do pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$, irrespective of whether the circumcenter O^* of $\Delta P_1 P_2 P_3$ is in the SR. □

When $E \notin \Delta P_1 P_2 P_3$ there obviously is one pursuer that is the farthest from E. The Reuleux-triangle-like *BSR*'s vertex V_{k^*} is formed by the intersection of the Apollonius circles $C_{i(k^*)}$ and $C_{j(k^*)}$, which correspond to the respective pursuers $P_{i(k^*)}$ and $P_{j(k^*)}$, the latter being the two pursuers that are closer to E. The Reuleux-triangle-like *BSR*'s vertex V_{k^*} is then inherited from the elemental lens-shaped *SR*'s vertex, which was the farther one from E. Hence, we are now back to the two-on-one pursuit-evasion differential game solved in [1,16,17], where the two pursuers are $P_{i(k^*)}$ and $P_{j(k^*)}$. When $E \notin \Delta P_1 P_2 P_3$, the third pursuer is redundant, and this is irrespective of whether $\Delta P_1 P_2 P_3$ is acute or obtuse. □

When $E \in \Delta P_1 P_2 P_3$, the triangle is acute, and condition (11) holds, the designated aim point according to Eq. (15) is V_{k^*}, whereupon capture is effected by the two pursuers, $P_{i(k^*)}$ and $P_{j(k^*)}$; we have yet to determine the strategy of the third pursuer. □

B. Third Pursuer

Concerning the third pursuer, consider the following. When $N = 2$ and there were only two pursuers, the *BSR* was lens shaped and it had two vertices. The evader's optimal strategy in [1,16,17] mandated that he/she run toward the *BSR* vertex farther from him/her. This is tantamount to running away from his/her two pursuers. If E had chosen to run to the lens-shaped *BSR*'s vertex that is closer to him/her, he/she would have been running toward the pursuers, which obviously is not the right thing to do when fleeing from the two pursuers—see Fig. 1. Now, when there are three pursuers, the *BSR* has three corners/vertices and is similar in shape to a Reuleux triangle. When there are three pursuers and E is inside $\Delta P_1P_2P_3$, he/she is encircled. If he/she decides to break out, he/she has the choice of three sectors: the slower E might try to pass between pursuers P_2 and P_3, or between P_1 and P_3, or between P_1 and P_2. Each sector is associated with one of the vertices of the *BSR*, and having chosen one of the escape sectors, E will be heading toward the respective *BSR* vertex, where he/she will be captured. This will determine E's time to capture. Naturally, E will choose the escape sector associated with the vertex that is farthest from him/her, thus maximizing his/her time-to-capture. The geometry is such that, E being inside the $\Delta P_1P_2P_3$, irrespective of the sector he/she chooses for his/her breakout attempt, E will start out running *toward* two pursuers. For example, in Fig. 8, E runs toward vertex V_1 and into the embrace of pursuers P_2 and P_3; this is courtesy of P_1, who is after him/her. Now, by construction, being in the Reuleux-triangle-like *SR* implies that E is also in the elemental lens-shaped *SR* pertaining to pursuers P_2 and P_3 only, but among the two vertices of the lens-shaped *SR*, E finds him-/herself heading toward the vertex, which is *closer* to him/her. And this is also true had the evader chosen to break out in one of the two other sectors associated with the two pursuer pairs (P_1, P_2) or (P_1, P_3), and their attendant elemental lens-shaped *SR*s formed by E and these pursuer pairs. Hence, the fact that in the case of three pursuers the Reuleux-triangle-like shaped *SR* is the intersection of the three elemental lens-shaped *SR*s that correspond to the three pairs of pursuers, (P_1, P_2), (P_1, P_3), and (P_2, P_3), allows us to conclude the following. When $E \in \Delta P_1P_2P_3$, each corner of the Reuleux-triangle-like shaped *SR* is the vertex that was closer to E in each of the three elemental lens-shaped *SR* pertaining to the two pursuers responsible for forming this vertex. In summary, E runs toward a close vertex of the elemental *BSR* *iff* $E \in \text{Interior}(\Delta P_1P_2P_3)$. Hence, if $E \in \Delta P_1P_2P_3$ and the circumcenter $O^* \in SR$, it might be beneficial for the evader to head into the interior of the *SR*, to the circumcenter O^* of $\Delta P_1P_2P_3$, and, once the circumcenter is reached, stay put and await his/her preordained demise. However, being encircled might be bad for the evader, and so the alternative is to break out. The strategy of heading to the Reuleux-triangle-like shaped *BSR* vertex that is farthest from him/her might then be his/her best choice. For example, in the scenario illustrated in Fig. 8 the evader will be isochronously captured during a pincer maneuver by the two pursuers, P_2 and P_3. As for the strategy of the third pursuer, P_1, the third pursuer could follow E in PP. P_1 then ensures the reduction of the distance $d = \text{dist}(P_1, E)$, thus shrinking the Apollonius circle $\mathcal{C}_1$ and moving its center O_1 closer to E. This, in turn, brings the two vertices of the Reuleux-triangle-like shaped *BSR* that, in the first place E chose not to use, is closer to him/her. Although the action of P_1 has no effect on the vertex V_1 of the Reuleux-triangle-like shaped *BSR* the evader is currently heading to, the choice of the remaining vertices of the *BSR* as an aim point makes them even less attractive to E. Thus, the PP strategy of P_1 discourages E to change the Reuleux-triangle-like shaped *BSR* vertex he/she is currently heading to/change his/her mind—time consistency/subgame perfectness is preserved.

Concerning capturability: In this positional game, when $E \in \text{Interior}(\Delta P_1P_2P_3)$, he/she is encircled. This is not good for the evader, because when according to our strategy the three pursuers are to head toward the circumcenter O^*, the triangle will be shrinking and the encirclement will tighten, bringing about capture. And when the state of the game is s.t. our strategy mandates that the encircled E needs to break out and should be heading toward a vertex of the Reuleux-triangle-shaped *BSR*, although the triangle is then not necessarily shrinking, the fact that the "redundant" third pursuer is employing PP cannot but help. Although initially E then finds him-/herself heading toward the closer vertex of the *BSR*, should E have been successful from getting out of the encirclement before being captured so that $E \notin \text{Interior}(\Delta P_1P_2P_3)$, our strategy mandates that E persists running toward this vertex. However, now this vertex of the Reuleux-triangle-like shaped *BSR* is the vertex of the elemental *BSR* that is farther from him/her. Hence, once $E \notin \text{Interior}(\Delta P_1P_2P_3)$ we are back to the two-on-one differential game where his/her capture by two active pursuers is guaranteed. In conclusion, the strategy of the three pursuers is s.t. as long as $E \in \text{Interior}(\Delta P_1P_2P_3)$ the triangle either keeps shrinking or at least one of the three pursuers is closing in—this is irrespective of what E does. And once E finds him-/herself outside the triangle, he/she is heading to the vertex of the elemental lens-shaped *BSR* that is farther from him/her while being tackled by two pursuers that are after him/her. This state of affairs is irreversibly leading to capture; we are back to the two-on-one pursuit game where capture is guaranteed. By breaking out, E jumped from the frying pan into the fire. E has no way out and capture is guaranteed.

VI. Strategies

The analysis from above is conducive to the synthesis of pursuit and evasion strategies.

Algorithm 1: In the pursuit-evasion differential game in the Euclidean plane with $N = 3$ pursuers and one evader, where all players have simple motion, the pursuers have equal speed $V_P = 1$, and the pursuers' capture range $l \to 0$, the players' "optimal" state feedback strategies are obtained as follows.

Based on the three pursuers' and the evader's instantaneous positions, one first calculates the centers and radii of the attendant three Apollonius circles/disks. *BSR* vertex candidates are the points of intersection of Apollonius circles, and this entails the solution of three sets of two bivariate equations (3) in two variables, x and y; as outlined in Sec. II.B, this boils down to the solution of three quadratic equations. The positivity of the quadratic equations' discriminants designates the set of $1 \le M \le 3$ active pursuers. Next, calculate the points antipodal to E on the circumference of the M Apollonius circles, each associated with an active pursuer. This requires the solution of a quadratic equation—in total, an additional M quadratic equations are solved. In addition, check whether such an antipodal point is in all the remaining $M - 1$ Apollonius disks; this requires checking $M(M - 1)$ quadratic inequalities. If such an antipodal point exists and, say, it resides on the circumference of the i^*th Apollonius circle, it is designated the aim point of pursuer i^* and of the evader—the pursuit degenerates into a tail chase/PP where only pursuer i^* is active. The two remaining pursuers are redundant. The generalization of Proposition 1 specifies when capture is effected by a single pursuer, that is, when condition (4) is violated for both pairs (P_{i^*}, P_j) and (P_{i^*}, P_k).

If no such antipodal point exists and $M = 2$, one reverts to the solution of the two-on-one differential game provided in [1,16,17]. The third pursuer that does not participate in the pincer movement maneuver to capture the evader is redundant. When $E \notin \Delta P_1P_2P_3$, capture is always effected by one, or at most, two pursuers; the remaining pursuers are redundant. In this case the value of the game is

$$V(P_1, P_2, P_3, E; \mu) = \frac{1}{1-\mu}\text{dist}(E, P_{i^*})$$

If no such antipodal point exists and $M = 3$, proceed as follows. Because a vertex $V_k = (x, y)$ of the *BSR* must be included in the Apollonius disks associated with each of the three pursuers, in order to determine the vertices of the *BSR*, the following six quadratic inequalities must be checked:

$$(x_i - x_{O_j})^2 + (y_i - y_{O_j})^2 \le \rho_j^2, \quad i \in \{1, 2\}, \quad j = 3; i \in \{3, 4\},$$
$$j = 2; \quad i \in \{5, 6\}, \quad j = 1$$

Only three inequalities will be satisfied, rendering a *BSR* with three vertices, which is similar to a Reuleux triangle. Let V_{k^*} be the *BSR*'s vertex farthest from E. The aim point P^* is decided on by the optimality principle, Eq. (14), and therefore the players' optimal strategies are specified according to Eq. (15): If $O^* \notin SR$ or condition (11) holds, the two pursuers, $P_{i(k^*)}$ and $P_{j(k^*)}$, head toward the *BSR* vertex V_{k^*} at the intersection of their respective Apollonius circles $C_{i(k^*)}$ and $C_{j(k^*)}$, which were responsible for forming the *BSR*'s vertex V_{k^*} in the first place. This identifies the two pursuers, $i(k^*)$ and $j(k^*)$, which cooperatively, by heading to their designated aim point $P^* = V_{k^*}$, will isochronously capture the evader. Ditto the evader, who also heads to the aim point $P^* = V_k^*$. The action comes down to a pincer movement pursuit with $P_{i(k^*)}$ and $P_{j(k^*)}$ on a CC with E. These two pursuers are designated to effect the capture of E. The third pursuer who does not participate in the pincer maneuver to capture E employs PP. The time-to-capture is

$$V(P_1, P_2, P_3, E; \mu) = \frac{1}{\mu}\,\mathrm{dist}(E, V_{k^*})$$

If $O^* \in SR$ and condition (11) does not hold the aim point $P^* = O^*$ is in the interior of the *SR*. All three pursuers head toward the circumcenter O^* of $\Delta P_1 P_2 P_3$. During optimal play also, E runs toward the aim point $P^* = O^*$ and upon arriving there, stays put, awaiting his/her isochronous capture by the three pursuers. The time-to-capture is

$$V(P_1, P_2, P_3, E; \mu) = R$$

where R is the radius of the circumcircle of $\Delta P_1 P_2 P_3$.

Symmetric states, that is, states on a dispersal surface (DS), where

$$\mathrm{dist}(E, V_{k^*}) = \mu R$$

might momentarily exist and they require special consideration. Thus, with reference to the control law (15), consider states where

$$\mathrm{dist}(E, V_{k^*}) \geq \mu R$$

Acknowledging now the possibility of symmetric states, a.k.a., the state is on a DS—see Fig. 9—the aim point I is specified as follows. Let $0 < \epsilon \ll 1$.

$$I = \begin{cases} O^* & \text{if } \mathrm{dist}(V_{k^*}, P_{i(k^*)}) < R - \epsilon \\ V_{k^*} & \text{if } \mathrm{dist}(V_{k^*}, P_{i(k^*)}) > R + \epsilon \\ E & \text{if } \mathrm{dist}(V_{k^*}, P_{i(k^*)}) \in [R - \epsilon, R + \epsilon] \end{cases}$$

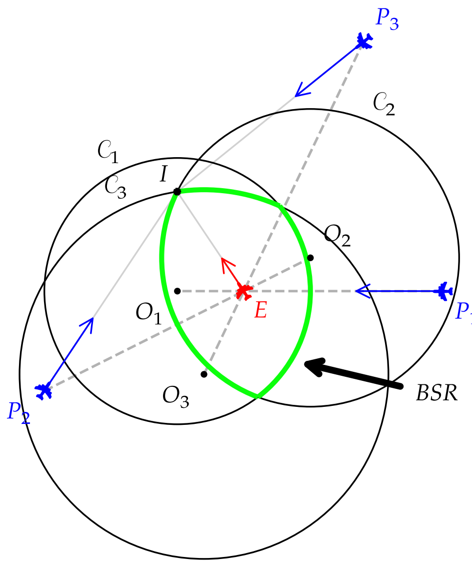


Fig. 8 Game with three pursuers.

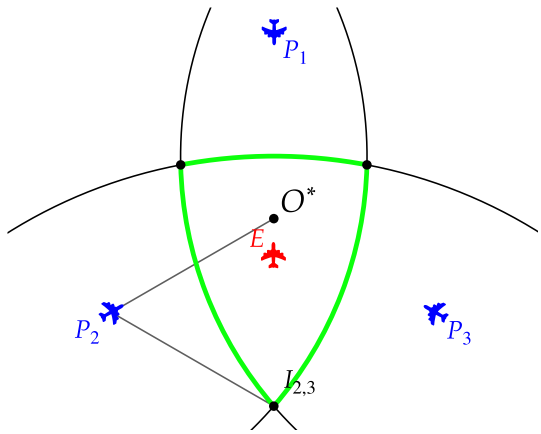


Fig. 9 Symmetric state: $\mathrm{dist}(P_2, I_{2,3}) = R$.

so if the current state is momentarily symmetric the three pursuers head straight toward E, as in PP. The time-to-capture is

$$V(P_1, P_2, P_3, E; \mu) = \begin{cases} \frac{1}{1-\mu}\mathrm{dist}(E, P_{i^*}) & \text{if (4) does not hold for } (P_i^*, P_j) \text{ or } (P_i^*, P_k) \\ \frac{1}{\mu}\mathrm{dist}(E, V_k^*) & \text{else if } O^* \notin SR \text{ or (11) holds} \\ R & \text{otherwise} \end{cases} \tag{16}$$

where P_{i^*} is the closest pursuer to E and P_j and P_k are the remaining two pursuers, V_k^* is the *BSR* vertex furthest from E, and R is the radius of the circumcircle of $\Delta P_1 P_2 P_3$. □

Figure 10 contains a flowchart for determining the equilibrium aim point, summarizing the logic in Algorithm 1.

A. Discussion

Consider the following: If E finds him-/herself at the circumcenter O^* of an acute triangle, $O^* \in SR$, and condition (11) does not hold, Algorithm 1 mandates that E should stay at O^*. Now, E could have achieved the same result/time-to-capture even if he/she declared ahead of time his/her intent of staying put. Using the cOFLP-solution-based optimality principle/Algorithm 1 we are certainly allowed to claim the solution of the open-loop max min optimal control problem of group pursuit with $N \geq 3$, where the discriminated evader is obliged to pre-announce, ahead of time, his/her planned control time history. But in a differential game the evader is allowed to employ state feedback control and yet, in this particular case, this would not have given him/her an advantage. This suggests that the geometric method might not always yield the optimal solution of the differential game. The optimality of the optimality principle/Algorithm 1–provided state feedback strategies is addressed next.

B. Optimality

1) It would appear that the solution of only a max min open-loop optimal control problem has been achieved, where the evader is discriminated— E is obliged to pre-announce his/her control time history ahead of time. Acting on this information, the pursuit pack will strive to intercept the evader in minimum time. Knowing this, the evader will choose his/her control time history s.t. the time-to-capture will be maximized. In the special case where the players have simple motion, the evader would gain no advantage by choosing a trajectory that is not a straight line; hence, the evader holds course and the pursuers' response will entail CC guidance. Capturability is guaranteed because the pursuers are faster than the evader.

2) Because all the protagonists have simple motion, the solution of the above-stated max min open-loop optimal control problem is provided by the geometric method with the Apollonius circle construct playing a central role.

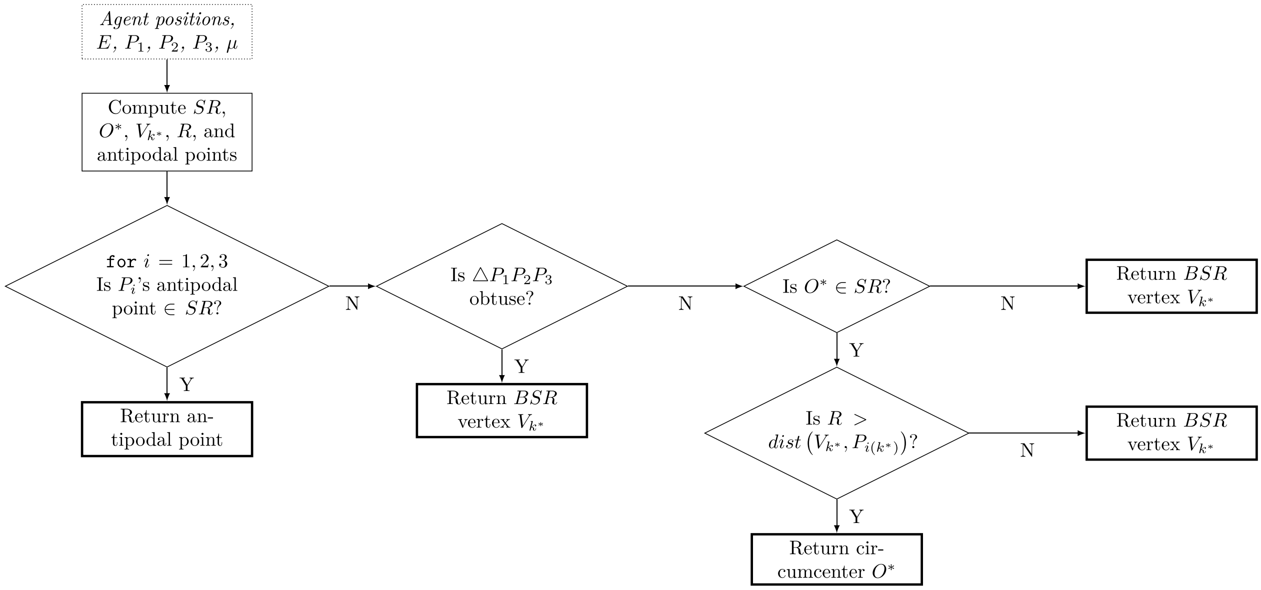


Fig. 10 Flowchart depicting the logic of Algorithm 1 for computing the optimal aim point.

3) It is important to emphasize that this is *not* yet the solution of the differential game. At the same time, in the case of the Two Cutters and Fugitive Ship differential game where $N = 2$ and the reduced state space dimension is 3, the geometric method did provide the optimal solution of the differential game.

4) Concerning the max min open-loop optimal control problem of group pursuit and its connection to the differential game of group pursuit, we note the following. The solution of max min open-loop optimal control pursuit problems using the two-sided Pontryagin maximum principle (PMP) is addressed in the seminal work ([20] pp. 226–237). We refer to Chap. 4, Sec. 28, entitled "A Pursuit Problem." We recognize that the Euler–Lagrange equations of the PMP are in fact the characteristics' equations of the HJI hyperbolic PDE that the Value function of the attendant differential game must satisfy. It is this very same HJI hyperbolic PDE that one must solve when embarking on the solution of the differential game proper using the method of dynamic programming. We here refer to Isaacs's method; of course, Isaacs's condition** must hold, this certainly being the case in group pursuit where the dynamics are separated. Hence, the solution of zero-sum differential games obtained by solving the max min open-loop optimal control problem using the two-sided PMP and synthesizing the players' state feedback optimal strategies in receding horizon optimal control fashion is valid, provided that the application of Isaacs's method results in primary optimal trajectories only, a.k.a., regular characteristics. In other words, if the solution of the two-sided optimal control problem using the PMP is s.t. the minimizing player enforces the termination of the game and a family of optimal trajectories that completely cover the state space, with no holes and, most important, *no* singular surfaces is obtained, the differential game has been solved. Because DSs can serve as anchor points for regular focal or a switch-envelope-type singular surfaces, their absence is conducive to an optimal flowfield consisting of regular optimal trajectories/no singular characteristics, whereupon an optimal solution is indeed achieved. The state feedback representation of the players' optimal control laws obtained by solving the max min open-loop optimal control pursuit problem are then the optimal strategies in the differential game. These are the strategies one would have correctly obtained by solving the differential game using the method of dynamic programming. It is dynamic programming that gives rise to the above-mentioned HJI hyperbolic PDE that the Value function of the differential game must satisfy. Because the solution of the max min optimal control problem entails straight-line trajectories only, the geometric method applies.
But, when the number of pursuers $N \geq 3$, there is the possibility that, in certain configurations, both the circumcenter O^*, along with a vertex V_{k^*} of the *BSR*, are candidate aim points. In other words, the state has reached a DS, which means that a regular focal or switch-envelope-type singular surface might be lurking. Optimal trajectories that start out from certain regions of the state space and reach these singular surfaces invalidate the contention that the realized flowfield consists of primary optimal trajectories/regular characteristics only. This invalidates the optimality of these trajectories. For the geometric method to provide the solution of the group pursuit differential game when the number of pursuers $N \geq 3$, no singular surfaces should be encountered. Therefore, for states in general position, the pursuit and evasion strategies provided by Algorithm 1 are suboptimal and "optimality" is henceforth set in quotation marks.

5) Concerning the introduction, at the price of a slight reduction in "optimality" (of $\mathcal{O}[(1/\mu)\epsilon]$), of the third aim point E, we allow for the possibility that a symmetric state has been reached; that is, the state is on the evader's DS where

$$\mathrm{dist}(E, V_{k^*}(P_1, P_2, P_3, E)) - \mu R(P_1, P_2, P_3) = 0$$

The three pursuers then in unison employ PP, heading toward E. This precludes the evader from wanting to adopt a strategy of vacillating between the aim points V_{k^*} and O^*, and by doing so cause the pursuers to vacillate; chatter would delay the pursuers more so than the evader and thus is anathema to optimality. Even if the evader does not vacillate, this strategy will cause the state to irreversibly leave the DS, and this is irrespective of the action of the evader, and then from this point on, the optimality-principle-provided strategies apply.

In conclusion, although in part of the state space Algorithm 1 yields optimal action, these strategies are not globally optimal. However, capturability is enforced and the evader is provided with a lower bound for the time-to-capture.

VII. Examples of "Optimal" Play

In this section we focus on examples of pursuit-evasion scenarios where three pursuers are at work. One first solves three quadratic equations to obtain the six *BSR* vertex point candidates. See Fig. 8 where the three pursuers are in general position, the speed ratio

**Interestingly, in the pursuit-evasion problem posed in [20] the dynamics are separated, that is, $\dot{x} = f(x, u) + g(x, v)$, where x is the state, and u and v are the respective controls of the pursuers and the evader. Thus, Isaacs's condition holds. Although the rather limited objective there was to solve a max min open-loop optimal control problem only, the foundation for the solution of zero-sum differential games with *no* singular surfaces was inadvertently laid—if only one would abandon the quest to solve the ensuing PMP's two-point boundary value problem (TPBVP) and instead adopt Isaacs's method of retrograde integration of the Euler–Lagrange/characteristics equations "starting" out from the terminal manifold.

$\mu = 0.5$, and $M = N = 3$. One checks six quadratic inequalities, only three of which are satisfied, to isolate the three vertices of the Reuleux-triangle-like shaped *BSR*. None of the points on the circumference of an Apollonius circle that are antipodal to E is contained in the *SR*. Hence, the *BSR*'s vertex farthest from E is designated the players' potential aim point I and the two Apollonius circles at whose intersection I lies then designate the two active pursuers that would isochronously capture the evader.

In the configuration illustrated in Fig. 8, the point I is the Reuleux-triangle-like *BSR*'s vertex farthest from E, and as such it would qualify to be the aim point of the evader. And being an intersection point of the Apollonius circles associated with pursuers P_2 and P_3, I qualifies to be the potential aim point of the pursuers P_2 and P_3. If only two pursuers were involved in the chase, this would be the end of the story. But because $E \in \Delta P_1P_2P_3$, the potential aim point I in Fig. 8 is not the vertex farthest from E among the two vertices of the elemental lens-shaped *BSR* formed by the Apollonius circles C_2 and C_3. The second, farther from E vertex of this lens-shaped *BSR*, was cut off from the *SR* by the Apollonius circle C_1 associated with the "inactive" pursuer P_1. But now $N = 3$ and the following additional geometric considerations come to the forefront.

The triangle $\Delta P_1P_2P_3$ is marginally acute; that is, $\angle P_2P_1P_3$ is almost 90 deg. The circumcenter O^*, which is barely inside the triangle, is close to the midpoint of its hypotenuse-like side P_2P_3. Because of this, the vertex I of the *BSR* that is quite close to the side P_2P_3 of $\Delta P_1P_2P_3$ is nevertheless outside the hexagon and can therefore serve as an aim point—see Fig. 11. Next, check condition (11):

$$\begin{aligned}\mathrm{dist}(P_1, O^*) &= \mathrm{dist}(P_2, O^*) = \mathrm{dist}(P_3, O^*)\\ &= R < \mathrm{dist}(P_2, I) = \mathrm{dist}(P_3, I)\end{aligned}$$

This and the optimality principle ensconced in Eq. (14) finally cement I's aim point status.

When dealing with three pursuers these new considerations are important, as opposed to the case of two pursuers only, previously considered by Isaacs in [1], and revisited in [16,17]. Had the last inequality not held, it would have been optimal for E to head toward the circumcenter O^* and for the three pursuers to converge on O^*. When condition (11) holds, the pursuer who does not participate in the pincer maneuver, the third pursuer P_1, who is the odd man out, employs PP. This puts pressure on E because the radius ρ of the Apollonius circle C_1 is directly proportional to the distance $d = \mathrm{dist}(E, P_1)$ and engaging E in PP works to quickly reduce d, and consequently the area of the Apollonius disk C_1, and so E's *SR* shrinks; this is not good for E. As long as $E \in \Delta P_1P_2P_3$ and the last inequality holds, the PP action of P_1 guarantees capturability. Conversely, if the last inequality did not hold, the shrinking $\Delta P_1P_2P_3$ would have guaranteed capturability.

The pursuit is illustrated in Fig. 8. When there were only two pursuers, things were much simpler: then there is no triangle, and no circumcenter of a triangle to worry about.

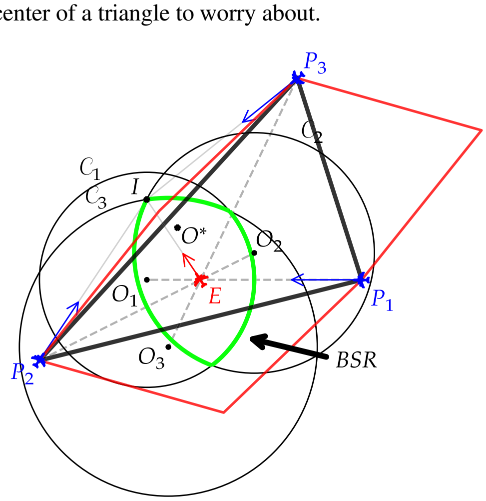


Fig. 11 Hexagon in game with three pursuers.

Next, consider the additional scenarios illustrated in Figs. 12–14, where the players' suboptimal strategies provided by Algorithm 1 are further exercised.

In Fig. 12 $\Delta P_1P_2P_3$ is marginally obtuse, and so its circumcenter $O^* \notin \Delta P_1P_2P_3$. The aim point $I_{2,3}$ is the farther from E vertex in the elemental lens-shaped *BSR* formed by the Apollonius circles $\mathcal{C}_2$ and $\mathcal{C}_3$: we are back to the two-on-one pursuit-evasion differential game considered in [1,16,17], where capturability is guaranteed.

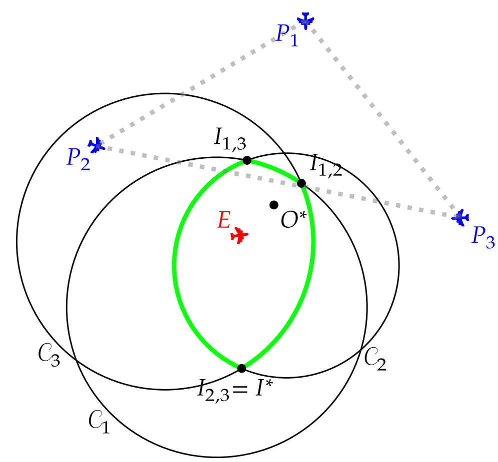


Fig. 12 P_2, P_3, and E head to $I_{2,3}$ and, for good measure, P_1 employs PP.

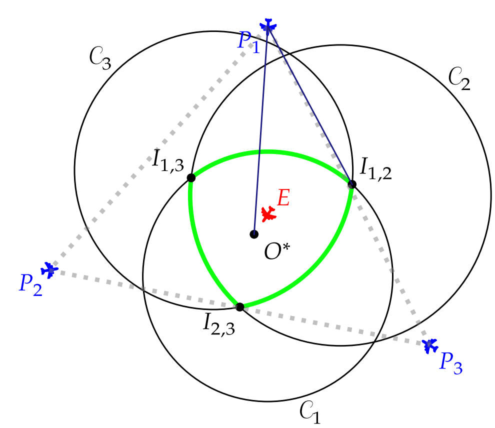


Fig. 13 P_1, P_2, P_3, and E head toward O^*.

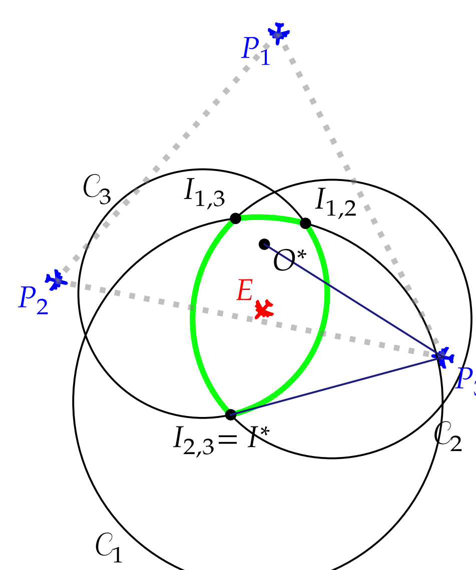


Fig. 14 The triangle is acute and $O^* \in SR$ but P_1, P_2, P_3, and E head toward I^*.

The pursuer P_1 is now truly redundant, but by employing PP he/she puts further pressure on the evader.

In Fig. 13 $\Delta P_1P_2P_3$ is acute. Condition (11) does not hold and therefore, according to Eq. (15), E heads toward the aim point O^*, where he/she will be isochronously captured by all three pursuers.

In Fig. 14 the situation is similar to the situation in Fig. 13, except that now condition (11) holds, which means that the point $I_{2,3}$ is further from its nearest pursuer than O^* is (i.e., the *BSR* vertex $I_{2,3}$ is outside of the hexagon).

In these examples a DS is not encountered during "optimal" play.

VIII. Critical Speed Ratio

The speed ratio parameter is of critical importance. In this section, its impact on the game's outcome is elucidated. Concerning the encounter of a DS during "optimal" play, consider the symmetric configuration illustrated in Fig. 5 and assume that the speed ratio parameter $\mu = 0.8$. Suppose that the evader strays from "optimality" and instead of staying put at O^* heads due south toward the *BSR*'s vertex $I_{2,3}$, while the pursuers employ the "optimal" state feedback strategy provided by the solution of the cOFLP/Algorithm 1. Since at time $t = 0$, $\mathrm{dist}(P_2, O^*) > \mathrm{dist}(P_2, I_{2,3})$—see[††] Fig. 6—the three pursuers initially head in unison toward the circumcenter O^* of $\Delta P_1P_2P_3$, and as long as they do so, O^*, who is exclusively determined by the instantaneous positions of the three pursuers, remains stationary. At the same time, the *SR*, whose shape is also influenced by the instantaneous position of the evader, changes—in particular, the *BSR*'s vertex $I_{2,3}$ moves along the y axis and at some point in time, say time $\bar{t}$, $\mathrm{dist}(P_2, O^*) = \mathrm{dist}(P_2, I_{2,3})$, while at the same time $O^*(t) \in SR\,\forall 0 \leq t \leq \bar{t}$—see Fig. 9. At time $\bar{t}$ the evader continues on his/her southward trek, whereupon at time $\bar{t}_+$, $\mathrm{dist}(P_2, O^*) < \mathrm{dist}(P_2, I_{2,3})$. This forces the pursuers, who adhere to Algorithm 1/ the cOFLP solution's provided "optimal" strategy, to change course and switch their heading to the new aim point, the dynamic *BSR*'s vertex $I_{2,3}$. In due course the evader will eventually be isochronously captured by P_2 and P_3 at the *BSR*'s vertex $I_{2,3}$, but by heading south he/she has extended his/her time-to capture by about 12%. The "optimal" pursuit strategy is not optimal.

That the cOFLP solution-based state feedback "optimal" strategies are not always optimal should not come as a surprise. We have not employed Isaacs's method for solving the group pursuit differential game and instead have relied on the geometric method to synthesize the pursuers' and the evader's strategies. This is due to the high dimension of the state space. Strictly speaking, relying on the geometric method is tantamount to assuming that the optimal flowfield will consist of primary optimal trajectories only—no singular hypersurfaces of focal or equivocal/switch envelope type here. It worked when the number of pursuers was $N = 2$ and the state space dimension was 3. Indeed, in [18] we applied Isaacs's method to the solution of the Two Cutters and Fugitive Ship differential game, and being able to fill the three-dimensional state space with an optimal flowfield consisting of primary optimal trajectories/no need for singular surfaces, we proved the geometric method—but this was for $N = 2$. Evidently, when $N = 3$, the speed ratio parameter $\mu = 0.8$, and the initial state is as depicted in Fig. 5, by going south the evader can cause the state to reach a dispersal hypersurface, which is "buried" in the state space whose dimension is now 5—see Fig. 9, where $V_{k^*} = I_{2,3}$ and $\mathrm{dist}(E_c, I_{2,3}) = \mu R$. The presence of a DS in and of itself would not invalidate our "optimal" solution, but the problem is that DSs are oftentimes harbingers of the more complex singular hypersurfaces of focal or equivocal type, this being the root cause of the loss of optimality; when the number of pursuers $N \geq 3$, the latter are endemic.

To better characterize the domain of optimality of the cOFLP-based state feedback strategies provided by Algorithm 1, we investigate the effect of the speed ratio parameter on the onset of the loss of optimality. Without loss of generality we take the speed of the pursuers to be unity, whereas the speed of the evader $0 < \mu < 1$, and we shall show that if the initial state is symmetric, as in Fig. 5, a speed ratio of $0 < \mu \leq (1/2)$ is conducive to the absence of a DS, thus eliminating the possibility of a singular surface. Also, let I be the Reuleux-triangle-like shaped *BSR*'s vertex farthest from E, and R is the radius of the circumscribing circle of $\Delta P_1P_2P_3$. In the remainder of this section we show why, with $0 < \mu \leq (1/2)$, the optimal solution of the cOFLP is always unique, that is, no *symmetric states* where

$$\mathrm{dist}(P_2, I) = \mathrm{dist}(P_3, I) = R$$

can arise, and thus an evader DS to which a regular focal singular surface or a switch envelope singular surface may abut is not possible. To this end, we revisit the utmost symmetric configuration where the three pursuers, P_1, P_2, and P_3, form an equilateral triangle whose side is, say, L_0, and the evader is initially at the circumcenter O^* of $\Delta P_1P_2P_3$—see Fig. 15. We are after symmetric states, and so, starting out from the most symmetric configuration possible, we allow E to move south from O^* a distance x along the equilateral triangle's altitude, all along still preserving a degree of symmetry of the configuration. As E moves south the pursuers exercise the strategy mandated by the optimality principle/Algorithm 1, all the while monitoring the changing shape of the *BSR*. This includes the calculation of the dynamic *BSR*'s vertex point that is farthest from E, $I(t)$, and the radius $R(t)$ of the circumscribing circle of $\Delta P_1P_2P_3$. As long as $\mathrm{dist}(P_2, I) < R$ and $O^* \in SR$, the three pursuers move in unison toward the triangle's center O^*, which remains stationary. In addition, the shrinking $\Delta P_1P_2P_3$ stays equilateral, and at time t its side $L(t) = L_0 - \sqrt{3}t$. We calculate the P_2-E distance

$$d_2 = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{1}{2\sqrt{3}}L - x\right)^2}$$

and from the properties of Apollonius circles ($\mathcal{C}_2$, $\mathcal{C}_1$) we have

$$\overline{EO}_2 = \frac{\mu^2}{1-\mu^2}d_2$$

$$\rho_2 = \frac{\mu}{1-\mu^2}d_2$$

$$\overline{EB} = \frac{\mu}{1+\mu}\left(\frac{1}{\sqrt{3}}L + x\right)$$

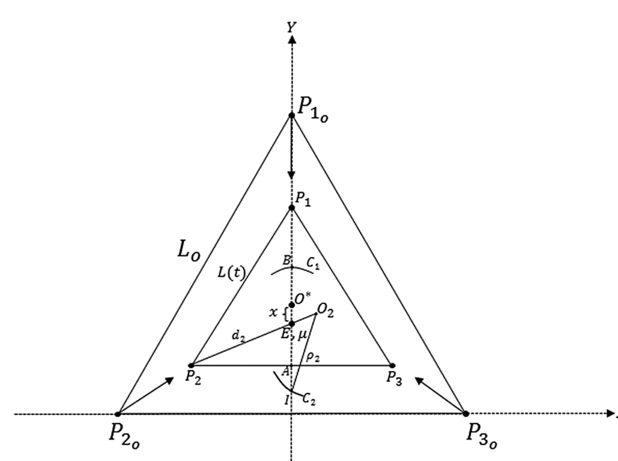


Fig. 15 Symmetric configuration.

[††]If $E = O^*$, always $\mathrm{dist}(P_2, O^*) > \mathrm{dist}(P_2, I_{2,3})\forall 0 < \mu < 1$, with equality only if $\mu = 1$.

The coordinates of the center O_2 of the Apollonius circle $\mathcal{C}_2$ are

$$O_{2_X} = \frac{1}{2}\frac{\mu^2}{1-\mu^2}L$$
$$O_{2_Y} = \frac{1}{1-\mu^2}\left(\frac{1}{2\sqrt{3}}L - x\right)$$

We also calculate

$$\overline{AI} = \sqrt{\rho_2^2 - O_{2_X}^2} - O_{2_Y}$$

which yields

$$\overline{AI}(x) = \frac{1}{1-\mu^2}\left[\mu\sqrt{x^2 - \frac{1}{\sqrt{3}}Lx + \left(\frac{1}{3} - \frac{1}{4}\mu^2\right)L^2} + x - \frac{1}{2\sqrt{3}}L\right]$$

Here, $L = L_0 - \sqrt{3}t$.

Now, as long as $\overline{AI} < (1/2\sqrt{3})L$, $\overline{P_2 I} < \overline{P_2 O^*}$. Hence O^* is stationary, provided that the circumcenter O^* of the equilateral $\Delta P_1 P_2 P_3$ is a valid solution of the cOFLP. The latter requires that $O^* \in SR$, that is, $\overline{EB} > x$; thus, we need

$$\overline{EB}(x) > x$$

Therefore, for $O^* \in SR$ we need

$$x < \frac{\mu}{\sqrt{3}}L$$

Since $x = x(t) = \mu t$ and $L = L(t) = L_0 - \sqrt{3}t$ we conclude that at time

$$\bar{t} = \frac{1}{2\sqrt{3}}L$$

the circumcenter O^* will exit the *SR* whereupon it will cease to be a valid solution of the cOFLP—in other words, the aim point will switch to the vertex I of the *BSR*. But for I to be the valid aim point, the following must hold.

$$\overline{AI}(x) \geq \frac{1}{2\sqrt{3}}L \tag{17}$$

We must solve this equation in x; that is, we must solve the equation

$$\mu\sqrt{x^2 - \frac{1}{\sqrt{3}}Lx + \left(\frac{1}{3} - \frac{1}{4}\mu^2\right)L^2} + x = (2-\mu^2)\frac{1}{2\sqrt{3}}L \tag{18}$$

This is reduced to solving the quadratic equation in x

$$x^2 - \frac{2}{\sqrt{3}}Lx + \frac{1}{3}(1-\mu^2)L^2 = 0$$

so

$$x(t) = (1-\mu)\frac{1}{\sqrt{3}}L(t)$$

As long as $0 \leq x < (1-\mu)(1/\sqrt{3})L$, $\overline{P_2 O^*} > \overline{P_2 I}$ and the unique aim point is O^*. The switch to the aim point I will happen at time

$$\bar{\bar{t}} = (1-\mu)\frac{1}{\sqrt{3}}L_0$$

When the state is symmetric and the evader heads south, at some point in time the solution of the cOFLP ceases to be unique; it is O^* *and* $I_{2,3}$—we have a bipolar situation. For a bipolar solution of the cOFLP not to be possible, we need

$$\bar{\bar{t}} \geq \bar{t}$$

which finally yields the condition on the speed ratio parameter

$$\mu \leq \frac{1}{2}$$

When the state is as in Fig. 5 but the speed ratio parameter is smaller than $\mu = 0.5$, heading south does no good to the evader. When the initial state is as shown in Fig. 5, to eliminate the possibility that at any point in time the state reaches a dispersal line (DL—pertaining to the two pursuer one evader game), the pursuers need to be at least twice as fast as the evader.

IX. Computational Investigation

The focus here is on the three-on-one pursuit evasion differential game. We document the results of numerical experiments to investigate the prevalence in its state space of initial states that are such that during "optimal" play a DS is reached and the role the speed ratio parameter μ plays. The existence of such a DS could presage the presence of singular surfaces of the focal or switch envelope type, which would invalidate the global optimality of the geometrically derived pursuit and evasion strategies: if the initial state was s.t. the "optimal" pursuit and evasion strategies bring the state to a focal or switch envelope surface, they do not provide a saddle point/are not security strategies, and the cost/payoff realized during "optimal" play is not the value of the game.

Concerning the initial state, the state space of dimension 5 of the 3P-1E pursuit-evasion differential game is partitioned into 5 regions according to the following taxonomy.

i) $E \notin \Delta P_1 P_2 P_3$
ii) $E \in \Delta P_1 P_2 P_3$, $\Delta P_1 P_2 P_3$ obtuse
iii) $E \in \Delta P_1 P_2 P_3$, $\Delta P_1 P_2 P_3$ acute, $O^* \notin SR$
iv) $E \in \Delta P_1 P_2 P_3$, $\Delta P_1 P_2 P_3$ acute, $O^* \in SR$, $\overline{P_2 O^*} \geq \overline{P_2 V_{k^*}}$
v) $E \in \Delta P_1 P_2 P_3$, $\Delta P_1 P_2 P_3$ acute, $O^* \in SR$, $\overline{P_2 O^*} < \overline{P_2 V_{k^*}}$

As described in [21], it is possible for the configuration of the pursuers and the evader to be such that both the circumcenter O^* of the $\Delta P_1 P_2 P_3$ formed by the three pursuers and the point on the *BSR* furthest from the evader are equidistant from their respective nearest pursuers. In such a configuration, the control action (agent headings, ϕ and ψ_i, $i = 1, \ldots, 3$) according to the policies presented in [14] are not well-defined—the prescribed control action is not unique. This is brought about by the solution to the *SR*-cOFLP not being unique. The cOFLP has two solutions, a *BSR* vertex (an Apollonius circles intersection), V_{k^*}, and the circumcenter O^* of $\Delta P_1 P_2 P_3$. The state is then a point on the evader dispersal surface (EDS). However, this is only possible when the players' configuration is s.t. 1) O^* is inside the *SR* and 2) the triangle $\Delta P_1 P_2 P_3$ is acute.

The EDS, described herein, is quite different from the "harmless" DL in the $2P - 1E$ game brought about by the two pursuers and the evader being collinear and in which the choice of optimal aim point is between the two Apollonius circles' intersections. There, the two candidate optimal intercept points are of the same *kind*—that is, they evolve (i.e., move in the plane) in the same manner when the two pursuers and the evader head toward the other point. In addition, if one of the parties momentarily deviates from its optimal strategy, collinearity is irrevocably destroyed and uniqueness of the solution of the cOFLP is restored. The same cannot be said of the two candidate optimal intercept points when the state is on the EDS in the many-on-one version of the game; the points O^* and V_{k^*} evolve quite differently when the agents head toward the opposite point [21]. More important, the DL in the $2P - 1E$ game is only encountered if the game *begin* on the surface and therefore is of little concern in general starting positions. Although it is possible, in some $2P - 1E$ plays to reach a DS under optimal play starting from general position, and we see from [21] that time-consistency/subgame perfectness can be violated when under "optimal" play the EDS is encountered in the many-on-one game. There is interest, then, in excluding the possibility of reaching the EDS (under "optimal" play) for some sufficiently small speed ratio μ and, in return, a large as possible set of initial states. However, we describe, below, cases in which during "optimal" play the possibility of reaching the EDS in the three-on-one pursuit-evasion differential game cannot be totally eliminated for any $0 < \mu < 1$. The set of initial states where the "optimal" strategies are not optimal shrinks as μ decreases, that is, when the speed ratio parameter $\mu \ll 1$.

For parties in which the players adhere to the "optimal" strategy of aiming at the solution to the cOFLP and the initial state was in regions (i) or (ii), the "optimal" pursuit and evasion state feedback strategies provided by our algorithm are optimal. That is, we have a strategic saddle point. The optimal flowfield consists of primary optimal trajectories only, there are no singular surfaces, and in this region of the state space the Value function is C^1. The "optimal" pursuit and evasion strategies are not globally optimal in the state space region (iii). In the state space region (iii) there are initial states wherefrom the "optimal" pursuit and evasion strategies lead the state to a DS whereupon the "optimal" strategies are not optimal. The loss of optimality is exacerbated in the state space region (iv), and more so in the state space region (v).

A. General effect of μ

Consider the following. For parties in which the players adhere to the "optimal" strategy of aiming at the solution to the cOFLP, the state of the system can reach the EDS from initial states in the state space where the following holds.

1) $V_{k^*}|_{t=0}$ is the solution to the cOFLP
2) $O^*|_{t=0} \in SR|_{t=0}$

This scenario, where the initial state is in region (v), is depicted (though without the *SR* drawn) in Fig. 16: there is another pursuer (P_3) whose Apollonius circle intersects with P_2's Apollonius circle at V_{k^*} and whose location is the reflection of P_2 about the line $O^* V_{k^*}$, but, for simplicity, this pursuer is not shown in the figure. Because P_2, P_3, and E aim toward V_{k^*}, the distance $\overline{P_2 V_{k^*}}$ shrinks faster than $\overline{P_2 O^*}$ and, at some point in time, say time t_{EDS}, the following might occur

$$\overline{P_2 V_{k^*}} = \overline{P_2 O^*} \tag{19}$$

which represents (in part) the EDS configuration and, pictorially, is the point in time at which V_{k^*} contacts the (dynamic) circle in Fig. 16. The reasoning in the preceding argument is similar to that presented in [21]. Another way of stating Eq. (19), in this case, is that the solution to the cOFLP is nonunique and that its two solutions correspond to an Apollonius circles intersection V_{k^*} and the circumcenter O^* of $\Delta P_1 P_2 P_3$. There are two possible cases with regard to the time in which Eq. (19) becomes true, t_{EDS}:

Case 1: The time at which O^* exits the SR (and is therefore no longer a candidate solution to the cOFLP), t_{exit}, is less than t_{EDS}:

$$t_{\text{exit}} < t_{\text{EDS}} \tag{20}$$

and thus the state never reaches the EDS.

Case 2: The time at which O^* exits the SR (and is therefore no longer a candidate solution to the cOFLP), t_{exit} is more than t_{EDS}. The candidate cOFLP's solution O^* remains in the SR up to the time t_{EDS}:

$$O^* \in SR \quad \forall\ t, \qquad 0 \le t \le t_{\text{EDS}} \tag{21}$$

and thus the state reaches the EDS at t_{EDS}.

Consider, then, the same initial state as in Fig. 16 but with a lower and lower speed ratio parameter μ.

Figure 17 depicts the movement of $V_{k^*}|_{t=0}$ as μ decreases. Note, while O^* stays in place, the *BSR*'s vertex V_{k^*} is constrained to move along the line $V_{k^*} O^*$ because, being at the intersection of P_2's and P_3's Apollonius circles, it lies on the orthogonal bisector of $P_2 P_3$ where also O^* is located. Decreasing μ pulls V_{k^*} toward E, and because in the configuration shown in Fig. 17, E is close to O^*, the *BSR* vertex V_{k^*} is being pulled toward O^* and will eventually reach the circumference of the circle in Fig. 17; the state will have reached the EDS. Reducing μ results in a smaller *SR*, hence the direction of V_{k^*}'s movement. In Fig. 17, there is a critical speed ratio, μ_{crit}, for which the point V_{k^*} slides inside the circle; that is, O^* is then the cOFLP's solution for all $\mu < \mu_{\text{crit}}$. Thus for the configuration/initial state shown in Fig. 17, for a sufficiently slow evader, all agents head toward the circumcenter O^* and there is no threat of encountering the EDS. The "optimal" strategies provided by the geometric method, which hinges on the real-time solution of the cOFLP, are indeed optimal.

In Fig. 18, however, a configuration/initial state is shown where, because $\angle P O^* V_{k^*} > \pi/2$, no speed ratio μ, however small, will cause $V_{k^*}|_{t=0}$ to contact the circle, and therefore O^* will not become

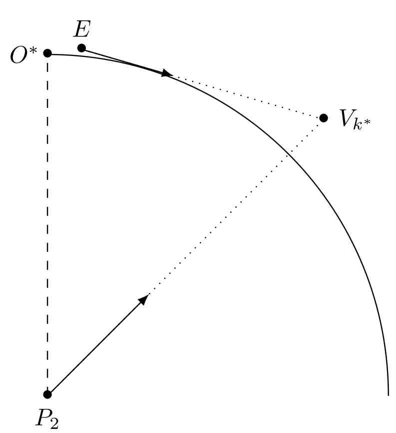


Fig. 16 Scenario satisfying necessary conditions for encountering EDS.

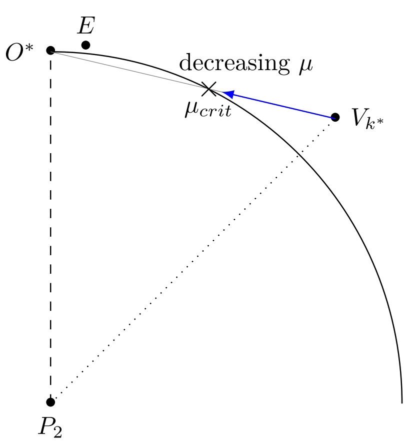


Fig. 17 Effect of lower μ on the location of V_{k^*} at $t = 0$, and the critical μ for which the circumcenter O^* becomes the optimal aim point.

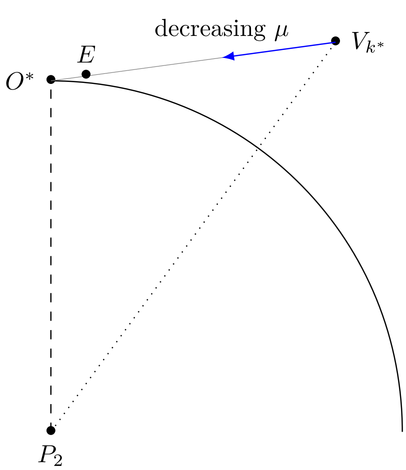


Fig. 18 Effect of lower μ on the location of V_{k^*} at $t = 0$.

the cOFLP's solution. Thus, in this configuration, both case 3 and case 4 are possible, and we cannot exclude the possibility of the state encountering the EDS during the partie, provided that $O^*|_{t=0} \in SR|_{t=0}$. As long as $E \neq O^*$, we can still find some μ_{crit} such that $O^*|_{t=0} \notin SR|_{t=0}$, which mends the situation. However, if $E = O^*$, then no speed ratio μ could cause O^* to lie outside the SR at $t = 0$ because always E is inside the SR, by construction.

B. Summary

In the state space region (v) and for the special case of $E = O^*$, the possibility that during "optimal" play the state reaches the EDS and the subsequent possible invalidation of the "optimal" solution cannot be excluded. For initial states in general positions, one can determine a μ_{crit} such that during "optimal" play, for any $\mu < \mu_{\text{crit}}$, the EDS will not be encountered when V_{k^*} is the cOFLP's solution at $t = 0$. Thus, consider the following:

Case 1 $\angle PO^*V_{k^*} < \pi/2$: Define μ_a as the speed ratio in which V_{k^*} contacts the circle centered on P_2 of radius $\overline{P_2O^*}$, that is, the μ_{crit} identified in Fig. 17. Define μ_b as the speed ratio that causes $O^* \in BSR$, that is, wherein the circumcenter of $\Delta P_1P_2P_3$ lies on the boundary of the SR. Then we have $\mu_{\text{crit}} = \min\{\mu_a, \mu_b\}$.

Case 2 $\angle PO^*V_{k^*} \geq \pi/2$: Here, μ_{crit} is simply the speed ratio at which $O^* \in BSR$.

The μ_{crit} determined as above applies only to a particular configuration of the agents/initial state. There the "optimal" solution is optimal. However, the point is that there is no fixed speed ratio parameter threshold μ_{crit} that will guarantee avoidance of an EDS for *all* initial states in the state space, as the speed ratio parameter may take on any value $0 < \mu_{\text{crit}} < 1$. In other words, there is no speed ratio parameter that is a sufficient condition for the "optimal" strategies to be the solution of the three-on-one pursuit-evasion differential game in the whole state space. But for a specified initial state and a given speed ratio parameter μ, one can check whether the geometrically derived "optimal" strategy provided by the solution of the cOFLP's solution is s.t. an EDS is avoided. This would render it optimal for the state space region close to this initial state. This is different from the situation in the Homicidal Chauffeur differential game, where if the parameters are s.t. the ratio of capture radius/turning radius of the car is 0.5 and the speed ratio is 0.05, there is no equivocal line. It has morphed into a switch line, as in optimal control, and the equivocal line singularity has been removed from the differential game's state space.

Figure 19 demonstrates the μ—dependency of the frequency of occurrences of EDS encounters in parties with all the agents playing "optimally," that is, aiming at the (instantaneous) cOFLP, starting in configurations in which the triangle $\Delta P_1P_2P_3$ is acute and $E \in \Delta P_1P_2P_3$. The computational results are for 1000 random initial conditions sampling the state space, but the same 1000 configurations are used for each setting of the speed ratio parameter μ. As each partie progresses, an EDS encounter is declared as

$$\frac{\overline{P_2O^*}}{\overline{P_2V_{k^*}}} \geq 1 \pm \varepsilon \tag{22}$$

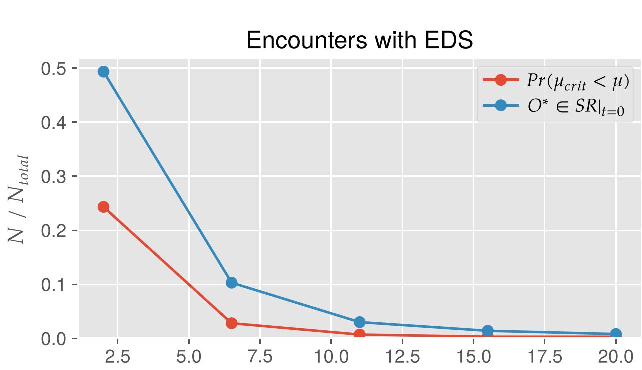


Fig. 19 Probability of encountering EDS as a function of μ.

where $\varepsilon = 1e-3$. The red line in Fig. 19 is the fraction of configurations in which the EDS is encountered at some point during the partie. The blue line is the fraction of configurations for which the circumcenter O^* of $\Delta P_1P_2P_3$ is inside the SR at $t = 0$, which is a necessary condition for EDS encounters. Thus, note the blue line–red line gap in Fig. 19.

Now, the red line is an estimate of the probability that during "optimal" play $\mu_{\text{crit}} < \mu$ for a random configuration/initial state (subject to $E \in \Delta P_1P_2P_3$ and $\Delta P_1P_2P_3$ is acute) so that the DES will be reached, and consequently the "optimal" strategies might not be optimal. The overall trend for EDS encounters follows the same trend, as the number of points O^* being initially inside the SR: as μ decreases, fewer points O^* remain inside the SR at $t = 0$ and thus there are fewer EDS encounters. These trends suggest that the circumcenter O^* of triangle $\Delta P_1P_2P_3$ being initially outside the SR is the primary reason why EDS encounters decrease with decreasing μ. If one were to constrain $O^* \in SR|_{t=0}$ for the lowest setting of μ (e.g., $\mu = 0.05$), then the trend may be much flatter. Indeed, the results in Fig. 20 are based on the same computational experiment; however, we enforce that initially the circumcenter O^* of triangle $\Delta P_1P_2P_3$ lies inside $SR|_{t=0}$ for all μ bigger than $\mu = 0.05$. Effectively, this forces the evader's initial position to be very close to O^*. By comparing Figs. 19 and 20 it is clear that most of the EDS encounters occur either because μ is high, a.k.a. μ is close to 1, or $\overline{EO^*}$ is small—all this, under the geometric method/cOFLP's solution mandated "optimal" play, which therefore is not optimal.

In the state space region (iv) and for the special case of $E = O^*$, that is, E is initially at the circumcenter of the acute triangle, when the players play "optimally," the evader stays put at O^* and the three pursuers converge on the evader. Thus, during "optimal" play a DS is not reached and consequently time consistency/subgame perfectness is preserved. But this does not imply optimality. Indeed, consider the very special case illustrated in Fig. 5, where initially $E = O^*$ and the acute $\Delta P_1P_2P_3$ is equilateral. Let the speed ratio parameter be $\mu = 0.8$. Now, when E does not play "optimally" and instead of staying put at O^* heads south to V_{k^*}, while the pursuers persist with the "optimal" strategy, the EDS is encountered. Furthermore, while continuing on its way south toward the BSR's vertex V_{k^*}, the three pursuers, P_1, P_2, and P_3, still head toward O^*. This of course brings them closer to the stationary O^* and at some point in time condition (22) holds. This then causes P_2 and P_3, who employ the "optimal" pursuit strategy to reverse course and also head toward the BSR vertex V_{k^*} where the evader is captured. By acting in this "rogue" way the evader manages to increase his/her time-to-capture by approximately 12%. The "optimality" of the geometric cOFLP's solution-based pursuit strategy is nixed. If, however, the speed ratio parameter $\mu < 0.5$, the evader should stay at the circumcenter O^* and optimality is preserved.

Thus, in the state space regions (i) and (ii), where $E \ni \Delta P_1P_2P_3$, or $\Delta P_1P_2P_3$ is obtuse, the players' "optimal" strategies are optimal. In the state space region (iii), where the circumcenter O^* of triangle $\Delta P_1P_2P_3$ is (initially) outside the SR, the players' "optimal" strategies might be optimal in a significant swath of region (iii). The "optimal" strategies also provide optimal performance in parts of

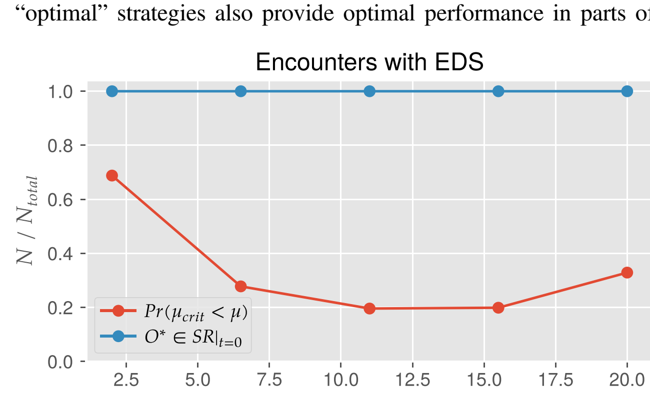


Fig. 20 Probability of encountering EDS as a function of μ with O^* always initially in SR.

the state space regions (iv) and (v), and more so the smaller the speed ratio parameter μ is. However if the initial state is in region (v) and the evader is at O^* from the get go, no matter how small the speed ratio is, the evader should move on—the players' strategies are not optimal.

Irrespective of optimality, the "optimal" pursuit strategy is conducive to capture and as such can be considered suboptimal. Also, when the "optimal" pursuit strategy is employed, the evader's "optimal" strategy provides a lower bound for the time-to-capture.

X. More Pursuers

We now consider the many-on-one pursuit-evasion differential game where the number of pursuers $N \geq M > 3$. First, note that the following holds.

Proposition 2: Assume that $N \geq M \geq 3$ and all the pursuers have equal speed. If $E \notin \text{convhull}(\{P_1, \ldots, P_M\})$ the evader will be captured single-handedly by the closest pursuer, or he/she will isochronously be captured by the closest pursuer and one additional pursuer, in a pincer maneuver. The latter being the case only if E is not in the cone of normals to $\text{convhull}(\{P_1, \ldots, P_M\})$ at the location of the pursuer closest to him/her. Moreover, when $E \notin \text{convhull}(\{P_1, \ldots, P_M\})$ and capture is effected by two pursuers, this will happen at one of the two points of intersection of their respective Apollonius circles. This point is the farther one from E, as also is the case in the two-on-one pursuit-evasion differential game where $N = M = 2$. Capturability is guaranteed, and under optimal play the additional $N - 2$ pursuers are redundant. When $E \in \text{convhull}(\{P_1, \ldots, P_M\})$ and capture is effected by two, but not three, pursuers, this will also happen at one of the two points of intersection of their respective Apollonius circles; however, this point of intersection was originally the one closer to E.

Proof: In general position, while E is on the run and is not stationary, he/she cannot isochronously be captured by more than two pursuers. This is so because when E plots his/her escape course he/she first and foremost contemplates running away from the most threatening pursuer, which is the pursuer closest to him/her. But while running on a straight line from the initially closest pursuer but before being captured by him/her, E could prematurely be intercepted by another, critical, pursuer who now employs CC guidance; among all the $M - 1$ additional pursuers, this is the pursuer who upon employing CC guidance can first intercept E while he/she is still running away from the pursuer who initially was closest to him/her. Hence, the evader must steer away from that second, now critical, pursuer. The new course will cause both the initially closest pursuer and the second, critical, pursuer to employ CC guidance: this will lengthen the time to capture of the second, critical, pursuer and shorten the time to capture of the initially closest pursuer. E will chose his/her new course to balance the threats posed by the two aforementioned pursuers, and so the chase ends in isochronous capture. In general position there exists such an optimal evader course where isochronous capture by exactly two pursuers will occur, provided of course that there was a pursuer in the first place that could interfere with the evader's plan of running away from the pursuer closest to him/her. If an additional, third "critical" pursuer could capture E while he/she is well on his/her way to isochronous capture by the aforementioned two pursuers, then obviously the first identified critical pursuer was not the critical pursuer after all. Moreover, when $E \notin \text{convhull}(\{P_1, \ldots, P_M\})$ the evader is running away from the pursuers. Now, when E is tackled by two pursuers and he/she is running away from them, his/her capture will occur at the farther point of intersection of the pursuers' respective Apollonius circles and we are back to the situation previously encountered in the Two Cutters and Fugitive Ship differential game where $N = 2$, that is, in the two-on-one pursuit-evasion differential game. However, should E find him-/herself running toward two pursuers, capture will occur at the point of intersection of the pursuers' respective Apollonius circles, but at their intersection point, which originally was closer to E. This can never be the case when $N = M = 2$ but will always be the case if $E \in \text{convhull}(\{P_1, \ldots, P_M\})$ and he/she is trying to break out of his/her encirclement. In both cases, whether originally E was outside or inside $\text{convhull}(\{P_1, \ldots, P_M\})$, if E is about to be isochronously captured by two pursuers, the two pursuers' and the evaders' aim point is the BSR's vertex farthest from him/her.

When $E \in \text{convhull}(\{P_1, \ldots, P_M\})$ then, as was the case when $N = M = 3$, also when $M > 3$ there is the possibility that it is best for E not to try to break out of the encirclement and instead stay in the interior of the *SR*, to be isochronously captured by three or more pursuers. If $M = 3$, according to Algorithm 1, capture will happen at the center O^* of the circle on whose circumference the three pursuers are located, provided that $O^* \in SR$ and condition (11) does not hold. Also if $M > 3$, at least in principle, the evader could be isochronously captured by more than three pursuers. If, for example, $N = M = 4$ and the four pursuers happen to be located at the vertices of a cyclic quadrilateral, the center O^* of the circumcircle is in *SR*, and its radius R does not satisfy condition (11), the evader will isochronously be captured by the four pursuers at the center O^* of the cyclic quadrilateral's circumscribing circle. However, whereas three pursuers are always located at the vertices of a triangle and the latter has a circumscribing circle, the same cannot be said about a quadrilateral in general. This only occurs when the quadrilateral formed by four pursuers located happens to be cyclic. But then, it could be argued that the four pursuers are not in general position.

Naturally, the geometry of triangles plays less of a role here and we state:

Algorithm 2: Consider the group/swarm pursuit-evasion differential game in the Euclidean plane with $N \geq M > 3$ pursuers and one evader where all players have simple motion; the pursuers are faster than the evader and have the same speed and the pursuers' capture range $l \to 0$. The results of Algorithm 1 generalize—the optimal state feedback strategies correspond to aiming at the solution of the SR-cOFLP. Here, we emphasize the possibility of nonuniqueness for the optimal aim point. A candidate aim point P^* might be provided by embarking on the actual solution of the cOFLP where the role of the cities is assumed by the pursuers and the constraint set, which is determined by the current positions of the pursuers and the evader, is the SR. The optimal location of the obnoxious facility will be the potential aim point P^*, to be provided by the solution of the static max min optimization problem

$$(P^*, i_1^*, \ldots i_l^*) = \arg\max_{P \in SR} \min_{1 \leq i \leq M} \text{dist}(P_i, P) \tag{23}$$

This is the solution of the cOFLP. The index $l \geq 1$ and

$$\text{dist}(P^*, P_{i_1^*}) = \text{dist}(P^*, P_{i_2^*}) = \ldots \text{dist}(P^*, P_{i_l^*}) = d^*$$

Now suppose that there is a point $Q \neq P^*$ such that

$$\min_{1 \leq i \leq M} \text{dist}(P_i, Q) \in [d^* - \epsilon, d^*] \tag{24}$$

for some sufficiently small neighborhood $0 < \epsilon \ll 1$. In this case, the points P^* and Q are sufficiently close in cost to consider this configuration to be on (or in the neighborhood of) an EDS. Based on the results of [21,22], it may be advantageous to aim directly at the evader because the pursuers may chatter between aiming at the points P^* and Q as the conflict evolves in time. The optimal aim point I is specified as follows:

$$I = \begin{cases} E & \text{if } \exists Q \neq P^* \text{s.t. (24) holds} \\ P^* & \text{otherwise} \end{cases}$$

The point P^* can be obtained in a process similar to Algorithm 1 except that the triangle $\Delta P_1 P_2 P_3$ is not used, and there may be more than one circumcenter that lies inside the *SR*. □[‡‡]

Figure 21 summarizes the logic contained in Algorithm 2. It is more vague than Fig. 10 in terms of computing the solution to the cOFLP, but more explicit in its treatment of the EDS, which of course is ever more likely or possible with larger numbers of pursuers.

[‡‡]We plan to release code implementations pending public release by the U.S. Government.

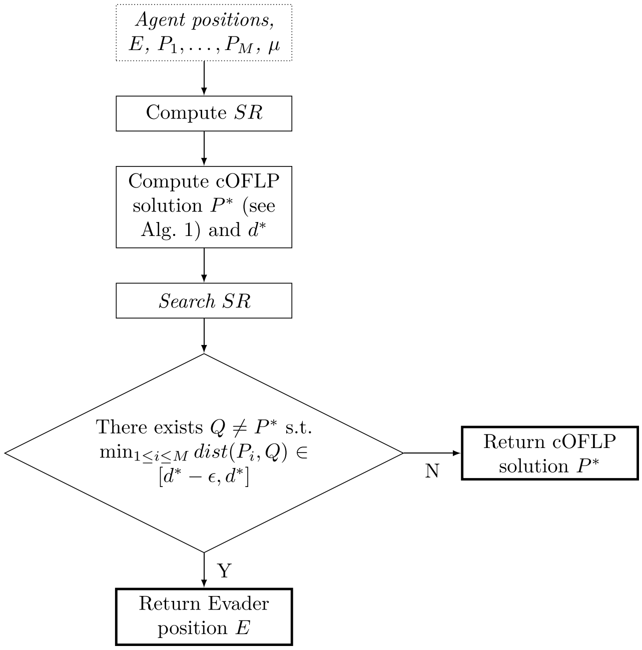


Fig. 21 Flowchart depicting the logic of Algorithm 2 for computing the optimal aim point for the general case.

XI. Extensions

The group pursuit differential game is a one parameter pursuit-evasion differential game where the speed ratio parameter $0 < \mu < 1$ features. When there are $N > 1$ pursuers the value of the game

$$V(P_1, \ldots, P_N, E; \mu) \le \min_{1\le i\le N} \frac{1}{1-\mu} \mathrm{dist}(E, P_i)$$

and when $M \ge 2$ and no antipodal to E point on the circumference of an Apollonius circle that is contained in the remaining $M - 1$ Apollonius disks exists [i.e., condition (4) holds for all pursuer pairs], the above becomes a strict inequality. Cooperative swarm/group action beats lone wolf performance. At the same time, multipursuer games in general position evolve in such a way that during optimal play one, two, or at most three strategically located pursuers are singled out to capture the evader in minimum time. There are configurations where just one pursuer is tasked with capturing the evader using PP while the evader runs away from the pursuer. When two pursuers are active, the evader is isochronously captured during a pincer maneuver where the two active pursuers use CC guidance. And when three pursuers are active, they isochronously converge on the aim point where the evader is bound to arrive in short order. In general position, no more than three pursuers actively engage the evader, leading to his/her isochronous capture. So, if the number of pursuers $N \ge 3$, $N - 3$, $N - 2$, or even as many as $N - 1$ pursuers are redundant. Now, bear in mind that the players' "optimal" strategies are state feedback strategies; that is, the game is continuously re-solved in real time. Should one of the active pursuers, or the evader, "err,"[§§] an "optimal" reassignment will automatically occur. However, during "optimal" play, when the active pursuers and the evader flawlessly execute their strategies, try as hard as they may, the redundant pursuers will not become active pursuers and the assignment of active pursuers will not change. As E flees on a straight-line path from the two, or one, active pursuers, none of the remaining redundant pursuers can embark on a CC with the course-holding evader, which will bring about his/her premature demise. This is by construction of the *BSR* and the "optimal" aim point P^*. Same is true if E is stationary and three pursuers converge on him/her—the redundant pursuers will stay nonactive during the duration of the game. Time consistency/subgame perfectness are not violated, in the sense that a pursuer that was active from the get go will stay active during the duration of the game and by the same token a "redundant" pursuer will not be activated. In wolfpack pursuit, when the active pursuers and the evader play according to the prescribed "optimal" strategies, the pursuers' assignment is stable and the remaining $N - 3$, $N - 2$, or $N - 1$ pursuers remain redundant. This should allow us to design a degree of robustness into the swarm/group pursuit strategy. The question is naturally posed: Foreseeing the possibility that one, two, or all three pursuers could "err," or allowing for the possibility that the evader might "err," and there is a departure from "optimal" play, how should the redundant $N - 3$, $N - 2$, or $N - 1$ pursuers position themselves during the game s.t. in the event that one, or more, active pursuers "err," the automatic optimal reassignment of one or more new active pursuers will bring about a minimal increase in the time-to-capture of the evader? Conversely, should the evader "err," upon the reassignment of the pursuers a maximal reduction in the time to capture will be achievable. Because there are additional pursuers in reserve, we address the question of how, during optimal play, should the redundant pursuers position themselves to be ready for such an eventuality.

We are analyzing a positional game and it stands to reason that during "optimal" play a redundant pursuer should move in a direction s.t. the comfort zone of the evader, namely, the area of his/her *SR*, decreases quickly; after all, also the redundant pursuers restrict the evader's *SR*. Bearing in mind that the radius ρ of an *E-P* Apollonius circle/disk $\mathcal{C}$ is directly proportional to the distance $d = \mathrm{dist}(E, P)$ and $\mathcal{C}^{SR}$, a good strategy for a redundant pursuer P is to engage E in PP. This works to quickly reduce d and consequently the area enclosed by the Apollonius circle $\mathcal{C}$ while assuming nothing about the evader's behavior (i.e., whether or not he/she will play according to the prescribed "optimal" strategy). The PP strategy of the redundant pursuer exacerbates the pressure on E and drives him/her to run toward his/her demise at P^*.

Moving away from point capture, when pursuers are endowed with capture disks of radius $l > 0$, the elemental Apollonius circles will be replaced by Cartesian ovals. Still, the herein developed geometry-based method of constructing the *BSR* and the players' attendant "optimal" state feedback strategies applies.

When the speed of the pursuers is not the same, the construction of the *SR* and the *BSR* does not change but the optimality principle must be amended. When solving the cOFLP, the distance from the obnoxious facility to pursuer P_i must be weighted with μ_i, $i = 1, \ldots, N$.

XII. Conclusions

In this paper swarm, or group pursuit, is investigated. The players have simple motion and the pursuers are faster than the evader. Point capture is stipulated. An algorithm for the online synthesis of sub-optimal state feedback pursuit and evasion strategies is provided. In general position and when $N \ge 3$, "optimal" group/swarm pursuit is fundamentally shaped by one, two, or three strategically placed pursuers, and this irrespective of the size N of the pursuit pack. The pursuit devolves into PP by one of the players or into a pincer maneuver by two or three players who chase the evader. If $N > 3$ it is possible to build robustness into the wolfpack pursuit strategy: during "optimal" play, the redundant pursuers will be positioned to be able to instantaneously take advantage of possible future pursuer "errors," and also better exploit evader "errors."

A globally optimal solution of the group pursuit differential game has not been obtained. However, in the state space regions (i) and (ii), where $E \notin \Delta P_1 P_2 P_3$, or $\Delta P_1 P_2 P_3$ is obtuse, the "optimal" strategies are optimal. In the state space region (iii) where the circumcenter O^* of triangle $\Delta P_1 P_2 P_3$ is (initially) outside the *SR*, the players' "optimal" strategies might be optimal in a significant swath. The "optimal" strategies also provide optimal performance in parts of the state space regions (iv) and (v), and more so the smaller the speed ratio parameter μ is. However, if the initial state is in region (v) and the evader is at O^* from the get go, no matter how small the speed ratio is, the players' "optimal" strategies are not optimal—the evader should move on.

[§§]We put *err* in quotation marks because the players' strategies are not globally optimal in the first place.

Thus, there are optimal strategies in part of the state space. The state space region where this is the case increases when the speed ratio parameter μ decreases. The smaller the speed ratio parameter is, the larger the state space swath where the "optimal" strategies are optimal. This is so because when $0 < \mu \ll 1$ a symmetric configuration does not arise, and so there is no DS. The latter is a good thing because a DS can serve as an anchor point for singular focal or switch envelope surfaces. The low speed ratio μ is thus conducive to an expanded region of the state space where the optimal flowfield consists of primary optimal trajectories only and no singular characteristics and singular surfaces of focal or switch envelope type. This is confirmed by extensive numerical experimentation.

Irrespective of optimality, the "optimal" pursuit strategy is conducive to capture and as such can be considered suboptimal. Also, when the "optimal" pursuit strategy is employed, the evader's "optimal" strategy provides a lower bound for the time-to-capture.

Indeed, if the number of pursuers $N > 2$, the group pursuit game is not at all simple. There is a marked increase in complexity when the number of pursuers is more than two. That the analysis of pursuit-evasion games with three pursuers or more is more complex than the analysis of pursuit-evasion games with two pursuers only should not come as a total surprise. A similar trend is discernible when one moves away from two-person games to games with three or more parties. In this respect, consider the Nash equilibrium concept in noncooperative games with three players/agents. The Nash strategy has the property that if all but one player use their Nash strategies, the deviating player could not decrease the value of his/her own cost function. Thus the Nash strategy safeguards against a single player deviating from the equilibrium strategy. However, when there are three or more players, two or more players could form a coalition and possibly the coalition could gain by deviating from the Nash strategy.

Concerning the group pursuit differential game, the game is of high dimension. Already when the number of pursuers $N = 3$, the dimension of the state space is 5. Never has a differential game with more than three states and with singular surfaces of focal or switch envelope type been solved. In fact, there is a dearth of complete solutions of differential games with three states only. This paper refers to the Game of Two Cars, the Isotropic Rocket Game, and the Obstacle Tag Chase Game. At the same time it is worth mentioning that some operationally interesting games with three states but with no singular surfaces, where the solution entails primary optimal trajectories only, have been solved. This paper refers to the differential game of Guarding a Target, provided that the defender is not slower than the attacker and the game of Two Cutters and a Fugitive Ship—also, the Active Target Defense Differential Game, where, again, the defender is not slower than the attacker and one is after point capture.

The geometric-method-based pursuit and evasion state feedback strategies promulgated in this paper yield suboptimal group pursuit action. But employing group pursuit is beneficial. There is strength in numbers, or put another way, quantity has a quality all its own.

Acknowledgments

The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the United States Government. This paper is based on work performed at the Air Force Research Laboratory (AFRL) Control Science Center of Excellence. Distribution Unlimited. 31 Jan 2019. Case #88ABW-2019-0465.

References

[1] Isaacs, R., *Differential Games: A Mathematical Theory with Applications to Optimization, Control and Warfare*, Wiley, New York, 1965, Chap. 6.

[2] Steinhaus, H., and Kuhn, H. W., "Definitions for a Theory of Games and Pursuit," *Naval Research Logistics Quarterly*, Vol. 7, No. 2, 1960, pp. 105–108.
https://doi.org/10.1002/(ISSN)1931-9193

[3] Melikyan, A. A., "Optimal Interaction of 2 Pursuers in a Game Problem," *Engineering Cybernetics*, Vol. 19, No. 2, 1981, pp. 49–56.

[4] Hagedorn, P., and Breakwell, J. V., "A Differential Game with Two Pursuers and One Evader," *Journal of Optimization Theory and Applications*, Vol. 18, No. 1, 1976, pp. 15–29.
https://doi.org/10.1007/BF00933791

[5] Pashkov, A., and Sinitsyn, A., "Construction of the Value Function in a Pursuit-Evasion Game with Three Pursuers and One Evader," *Journal of Applied Mathematics and Mechanics*, Vol. 59, No. 6, 1995, pp. 941–949.
https://doi.org/10.1016/0021-8928(95)00127-1

[6] Pshenichnyi, B. N., "Simple Pursuit by Several Objects," *Cybernetics*, Vol. 12, No. 3, 1976, pp. 484–485.
https://doi.org/10.1007/BF01070036

[7] Chikrii, A. A., and Prokopovich, P. V., "Simple Pursuit of One Evader by a Group," *Cybernetics and Systems Analysis*, Vol. 28, No. 3, 1992, pp. 438–444.
https://doi.org/10.1007/BF01125424

[8] Jin, S., and Qu, Z., "Pursuit-Evasion Games with Multi-Pursuer vs. One Fast Evader," *8th World Congress on Intelligent Control and Automation*, IEEE, New York, 2010.
https://doi.org/10.1109/WCICA.2010.5553770

[9] Ibragimov, G. I., "A Game of Optimal Pursuit of One Object by Several," *Journal of Applied Mathematics and Mechanics*, Vol. 62, No. 2, 1998, pp. 187–192.
https://doi.org/10.1016/S0021-8928(98)00024-0

[10] Kopparty, S., and Ravishankar, C. V., "A Framework for Pursuit Evasion Games in Rn," *Information Processing Letters*, Vol. 96, No. 3, 2005, pp. 114–122.
https://doi.org/10.1016/j.ipl.2005.04.012

[11] Alexander, S., Bishop, R., and Ghrist, R., "Capture Pursuit Games on Unbounded Domains," *L'Enseignement Mathématique*, Vol. 55, No. 1, 2009, pp. 103–125.
https://doi.org/10.4171/LEM

[12] Ivanov, R., and Ledyaev, Y. S., "Time Optimality for the Pursuit of Several Objects with Simple Motion in a Differential Game," *Trudy Matematicheskogo Instituta imeni VA Steklova*, Vol. 158, 1981, pp. 87–97.

[13] Kumkov, S. S., Le Ménec, S., and Patsko, V. S., "Zero-Sum Pursuit-Evasion Differential Games with Many Objects: Survey of Publications," *Dynamic Games and Applications*, Vol. 7, No. 4, 2017, pp. 609–633.
https://doi.org/10.1007/s13235-016-0209-z

[14] Von Moll, A., Casbeer, D. W., Garcia, E., and Milutinović, D., "Pursuit-Evasion of an Evader by Multiple Pursuers," *2018 International Conference on Unmanned Aircraft Systems (ICUAS)*, IEEE, New York, 2018, pp. 133–142.
https://doi.org/10.1109/ICUAS.2018.8453470.

[15] Von Moll, A., Casbeer, D., Garcia, E., Milutinović, D., and Pachter, M., "The Multi-Pursuer Single-Evader Game: A Geometric Approach," *Journal of Intelligent and Robotic Systems*, Vol. 96, No. 2, 2019, pp. 193–207.
https://doi.org/10.1007/s10846-018-0963-9

[16] Garcia, E., Fuchs, Z. E., Milutinović, D., Casbeer, D. W., and Pachter, M., "A Geometric Approach for the Cooperative Two-Pursuer One-Evader Differential Game," *IFAC-PapersOnLine*, Vol. 50, No. 1, 2017, pp. 15209–15214.
https://doi.org/10.1016/j.ifacol.2017.08.2366

[17] Pachter, M., Von Moll, A., Garcia, E., Casbeer, D., and Milutinović, D., "Two-on-One Pursuit," *Journal of Guidance, Control, and Dynamics*, Vol. 42, No. 7, 2019.
https://doi.org/10.2514/1.G004068

[18] Pachter, M., "Isaacs' Two-on-One Pursuit Evasion Game," *18th International Symposium on Dynamic Games and Applications (ISDG)*, 2018.

[19] Toussaint, G. T., "Computing Largest Empty Circles with Location Constraints," *International Journal of Computer & Information Sciences*, Vol. 12, No. 5, 1983, pp. 347–358.
https://doi.org/10.1007/BF01008046

[20] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., *The Mathematical Theory of Optimal Processes*, Wiley, New York, 1962, pp. 226–237, http://cds.cern.ch/record/234445.

[21] Von Moll, A., Pachter, M., Garcia, E., Casbeer, D., and Milutinović, D., "Robust Policies for a Multiple Pursuer Single Evader Differential Game," *Dynamic Games and Applications*, 2019.
https://doi.org/10.1007/s13235-019-00313-3

[22] Pachter, M., Von Moll, A., Garcia, E., Casbeer, D., and Milutinović, D., "Singular Trajectories in the Two Pursuer One Evader Differential Game," *2019 International Conference on Unmanned Aircraft Systems*, IEEE, New York, 2019.
https://doi.org/10.1109/ICUAS.2019.8798244.

H. Choi
Associate Editor