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Pursuit in the Presence of a Defender

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Abstract
A zero-sum pursuit-evasion differential game with three players, a Target, an Attacker, and a Defender, is considered. The Attacker pursues the Target aircraft, while the Defender strives to intercept the Attacker before he reaches the aircraft. In this paper, the game in the state space region where the Attacker prevails is analyzed. The state space region where the Target is vulnerable is characterized and the Attacker's strategy for capturing the Target despite the presence of the Defender is derived. The players' optimal strategies mesh with the previously obtained strategies in the state space region where the Active Target Defense Differential Game is played, and the Defender's presence by virtue of him intercepting the Attacker allows the Target to escape.

Keywords Differential games · Missile guidance · Pursuit-evasion

1 Introduction

Differential game theory provides the right framework to analyze pursuit-evasion scenarios and combat games. Pursuit-evasion scenarios involving more than two agents are important but challenging problems in aerospace control. They have also been used to analyze biologically inspired behaviors: for instance [19] addressed a scenario where two evaders employ coordinated strategies to evade a single pursuer, but also to keep them close to each other. The authors of [13] discussed a multi-player pursuit-evasion game with line segment obstacles labeled as the Prey, Protector, and Predator Game. Multi-agent cooperative behaviors have been considered in [2,4,11,20].

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The Target–Attacker–Defender (TAD) scenario considered in this paper is relevant in combat scenarios [16–18]. Recent work has employed one-sided optimal control formulations where one team assumes a known fixed guidance law, e.g., pure pursuit (PP) or proportional navigation (PN), employed by the opponent, such as in [22] and [6]. The work in [21] considers resource allocations in a TAD engagement where there are multiple targets, attackers, and defenders. Differential games involving three agents but not of the TAD variety were also studied in [3] and in [12].

The dynamical TAD scenario considered in this paper is naturally formulated as a zero-sum differential game. The TAD differential game provides an illustrative application of differential game theory with all of its possible outcomes and strategies. In our previous work [5,14,15], attention was given to the state space region where, under optimal play, the Target and Defender team can ensure the Target's escape despite the best efforts of the Attacker, this, by virtue of the Defender intercepting the Attacker. The focus of this paper is the synthesis of the players' optimal strategies when the state of the control system is in the Attacker's winning region where, under optimal play by the Attacker, the Target's capture is guaranteed despite the best efforts of the Defender. A geometric method is used to derive the players' strategies.

The paper is organized as follows. Section 2 introduces the TAD engagement in the context of differential games with two termination sets. Section 3 states the pursuit-evasion game of capture of the Target by the Attacker despite the involvement of the Defender. Preliminary results and properties of the optimal strategies are presented in Sect. 4. The Game of Kind is analyzed in Sect. 5. The main results are presented in Sect. 6 where, using a geometric method, the players' optimal state feedback strategies are obtained for the game in the Attacker's region of win. Illustrative examples are given in Sect. 7 followed by a discussion in Sect. 8 and concluding remarks in Sect. 9.

2 The TAD Differential Game

The players T (Target), A (Attacker), and D (Defender) have constant speed V_T, V_A, V_D, respectively, and have simple motion á la Isaacs [10]. Specifically, think of T as an aircraft, A is an Attacking missile and the counter-weapon D is also a missile fired to intercept A. The A and D missiles have similar capabilities, that is, we assume that $V_A = V_D$, whereas $V_T < V_A = V_D$. The states of T, A, and D are, respectively, specified by their Cartesian coordinates $\mathbf{x}_T = (x_T, y_T)$, $\mathbf{x}_A = (x_A, y_A)$, and $\mathbf{x}_D = (x_D, y_D)$. The state of the differential game is thus $\mathbf{x} := (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$. The universe of discourse is the entire space $\mathbb{R}^6$. The Attacker's control variable is his instantaneous heading angle, $\mathbf{u}_A = \{\chi\}$. The T & D team affects the state of the game by cooperatively choosing their instantaneous respective headings, ϕ and ψ. The T & D team's control variable is $\mathbf{u}_B = \{\phi, \psi\}$. The "dynamics" $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_A, \mathbf{u}_B)$ are specified by the system of linear differential equations

$$\begin{aligned}
\dot{x}_T &= \alpha \cos\phi, \quad & x_T(0) &= x_{T_0} \\
\dot{y}_T &= \alpha \sin\phi, \quad & y_T(0) &= y_{T_0} \\
\dot{x}_A &= \cos\chi, \quad & x_A(0) &= x_{A_0} \\
\dot{y}_A &= \sin\chi, \quad & y_A(0) &= y_{A_0} \\
\dot{x}_D &= \cos\psi, \quad & x_D(0) &= x_{D_0} \\
\dot{y}_D &= \sin\psi, \quad & y_D(0) &= y_{D_0}
\end{aligned} \tag{1}$$

$0 \leq t \leq t_f$, where the speeds are normalized by V_A and the speed ratio $\alpha = V_T/V_A < 1$ is the problem parameter. The admissible controls are the players' headings $\chi, \phi, \psi \in [0, 2\pi)$. Both, the state and the controls, are unconstrained. The initial state of the TAD system is

$$\mathbf{x}_0 := \left(x_{T_0}, y_{T_0}, x_{A_0}, y_{A_0}, x_{D_0}, y_{D_0}\right) = \mathbf{x}(t_0).$$

In this paper, we confine our attention to point capture: The game ends at time t_f when the separation between A and T becomes zero, that is, A captures T. An additional instance of termination entails the interception of A by D, that is, the game terminates if the $A - D$ separation becomes zero. Thus, the game's termination set is the union of the two four-dimensional subspaces in $\mathbb{R}^6$

$$\mathcal{S} := \mathcal{S}_1 \bigcup \mathcal{S}_2 \tag{2}$$

where the subspace

$$\mathcal{S}_1 := \left\{ \mathbf{x} \,|x_A - x_T = 0, \quad y_A - y_T = 0 \right\} \tag{3}$$

and the state entering $\mathcal{S}_1$ signals the capture of the Target by the Attacker. The state entering the subspace

$$\mathcal{S}_2 := \left\{ \mathbf{x} \,|x_A - x_D = 0, \quad y_A - y_D = 0 \right\} \tag{4}$$

indicates the opposite outcome where the Defender intercepts the Attacker so the Target can survive. The differential game with termination set (2) belongs to the class of two termination set pursuit-evasion differential games [7]. The two termination sets differential game concept was introduced in order to extend the applicability of classical pursuit-evasion games, where only one termination set is contemplated. The two termination set differential game concept is useful in the analysis of combat games [1,7,8], where the roles of pursuer and evader are not designated ahead of time; instead, each player is endowed with a capture set and he wants to defeat his opponent by terminating the game in its own capture set. This is so in the Active Target Defense Differential Game (ATDDG) where the Attacker strives to capture the Target while the Target and Defender team cooperatively maneuver such that the Defender intercepts the Attacker before the latter is able to capture the Target.

The two termination conditions in (2) give rise to the partition of the $\mathbb{R}^6$ state space into $\mathcal{R}_c$ and $\mathcal{R}_e$ sets, $\mathcal{R}_c \bigcup \mathcal{R}_e = \mathbb{R}^6$. The set $\mathcal{R}_e$ is the region of win of the T & D team, where the ATDDG is played and $\mathcal{R}_c$ is the region of win of A where the TAD game of capture is played. Thus, we need to first solve the Game of Kind. The solution of the Game of Kind in the TAD differential game yields a semipermeable Barrier surface $\mathcal{B}$ which partitions the state space $\mathbb{R}^6$ into two regions $\mathcal{R}_c$ and $\mathcal{R}_e$. So, if the initial state satisfies $\mathbf{x}_0 \in \mathcal{R}_c$ then, under optimal play of A, T is captured by A before D can intercept A and the game will terminate when the state $\mathbf{x}$ enters the target set $\mathcal{S}_1$. And, if the initial state $\mathbf{x}_0 \in \mathcal{R}_e$ then, under optimal play of the D & T team, D intercepts A before A is able to capture T and T can escape—the game terminates when the state $\mathbf{x}$ enters the target set $\mathcal{S}_2$. In Fig. 3, a cross section of the Barrier surface is illustrated: The positions of A and D are fixed, and it is the position of T in the plane (x, y) which determines the outcome under optimal play.

Two Games of Degree can then be formulated. The ATDDG is the differential game played when $\mathbf{x} \in \mathcal{R}_e$. It provides the saddle point strategies for the three players: the T & D team and A, such that at the time t_f of interception of A by D, when condition (4) holds, the $A-T$ separation is maximized by the T & D team and minimized by A. The ATDDG played in $\mathcal{R}_e$ was formulated in [14] and solved in [5] and [15].

The present paper addresses the Game of Degree in $\mathcal{R}_c$, the TAD differential game of capture. Solving the game of capture in $\mathcal{R}_c$ is important because, together with the solution of the ATDDG in $\mathcal{R}_e$, it provides the three players' optimal strategies everywhere in the state space $\mathbb{R}^6 = \mathcal{R}_c \cup \mathcal{R}_e$. This provides a complete analysis of the three-agent pursuit-evasion TAD differential game which comprises a Target, an Attacker, and a Defender. The optimal/saddle point strategies in the two different Games of Degree played in $\mathcal{R}_c$ and $\mathcal{R}_e$ agree on the Barrier surface boundary $\mathcal{B}$ that separates the two regions in $\mathbb{R}^6$. The TAD capture game is addressed in Sect. 3, where the players' cost/payoff functional for the case where $\mathbf{x} \in \mathcal{R}_c$ is introduced. It is designed to mesh with the cost functional in the ATDDG played in $\mathcal{R}_e$. It yields state feedback strategies for the players which possess the continuity property expected in two termination set differential games.

3 The TAD Differential Game of Capture

The TAD differential game of capture is played in the $\mathcal{R}_c$ set of the state space where under optimal play A will capture T irrespective of the strategy employed by the T & D team.

In the TAD differential game of capture, the termination condition is $\mathcal{S}_1$, that is,

$$x_A = x_T, \quad y_A = y_T. \tag{5}$$

The terminal manifold (5) is a four-dimensional hyperplane in the realistic space $\mathbb{R}^6$. The terminal time t_f is the time instant when the state of the system satisfies (5), at which time the terminal state is $\mathbf{x}_f := (x_{T_f}, y_{T_f}, x_{A_f}, y_{A_f}, x_{D_f}, y_{D_f}) = \mathbf{x}(t_f)$. We defer the characterization of the Attacker region of win $\mathcal{R}_c$ to the solution of the Game of Kind in Sect. 5 and now address the TAD capture Game of Degree. The terminal cost/payoff functional, which A strives to maximize while the T & D team work to minimize, is

$$J(\mathbf{u}_A(\cdot), \mathbf{u}_B(\cdot); \mathbf{x}_0) = \Phi(\mathbf{x}_f) \tag{6}$$

where

$$\Phi(\mathbf{x}_f) := \sqrt{\left(x_{D_f} - x_{T_f}\right)^2 + \left(y_{D_f} - y_{T_f}\right)^2}. \tag{7}$$

The cost/payoff functional depends only on the terminal state—the capture game is a terminal cost/Mayer type game. Its value is given by

$$V(\mathbf{x}_0) := \max_{\mathbf{u}_A(\cdot)} \min_{\mathbf{u}_B(\cdot)} J(\mathbf{u}_A(\cdot), \mathbf{u}_B(\cdot); \mathbf{x}_0) \tag{8}$$

subject to (1) and (5), where $\mathbf{u}_A(\cdot)$ and $\mathbf{u}_B(\cdot)$ are the opposing teams' state feedback strategies.

In the TAD game of capture, which is played in $\mathcal{R}_c$, the player A strives to capture T and maximize the terminal $T - D$ separation, while T and D cooperate in order to minimize their separation in the terminal instant when $A = T$.

Knowing that T will be captured by A despite his best efforts, the logical strategy for D is to try to reach a point at time t_f which is as close as possible to T. From a practical point of view, the strategy of T & D cooperating to minimize their terminal separation provides a good strategy if player A were to err and pursue T using a different guidance law other than the optimal saddle point strategy prescribed by the solution of the TAD differential game of capture. In such a case, where A does not play optimally and the T & D team which strives to minimize the terminal separation play optimally, the terminal $T - D$ distance will decrease with respect to the guaranteed Value of the game, which is attained when all players act optimally. And under further non-optimal play by the Attacker, the state can cross the Barrier surface and it will then hold that $\mathbf{x} \in \mathcal{R}_e$ where the strategies of the ATDDG, the Game of Degree in $\mathcal{R}_e$, will be used by T and D, and now the Target can escape.

From a theoretical point of view, when the state of the system $\mathbf{x}$ is exactly on the Barrier surface, both, the TAD capture and the ATDDG, Games of Degree have feasible solutions and simultaneous capture of T by A and interception of A by D is attained at termination. Given the nature of the terminal cost functionals in the TAD capture Game of Degree and in the ATDDG Game of Degree, the solution of the Game of Degree in $\mathcal{R}_c$ needs to provide the same strategies and the same Value of the game as the solution of the Game of Degree in $\mathcal{R}_e$ when simultaneous capture is realized, i.e., A captures T at the same time instant when D intercepts A; in other words, when the game terminates on the subspace $\mathcal{S}_1 \bigcap \mathcal{S}_2$. This will be corroborated in the sequel.

4 Strategies

Consider the game in the realistic plane. Let the co-state be represented by

$$\lambda^T = \left(\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}\right) \in \mathbb{R}^6. \tag{9}$$

The Hamiltonian of the differential game is

$$\mathcal{H} = \lambda_{x_A} \cos\chi + \lambda_{y_A} \sin\chi + \lambda_{x_D} \cos\psi + \lambda_{y_D} \sin\psi + \alpha\lambda_{x_T} \cos\phi + \alpha\lambda_{y_T} \sin\phi \tag{10}$$

where the problem parameter α is the V_T/V_A speed ratio.

Theorem 1 *Consider the TAD game of capture (1), (5)–(8). The headings of the players T, A, and D are constant under optimal play and the primary optimal trajectories are straight lines.*

Proof From the Hamiltonian (10), the Attacker's optimal heading χ^*, in terms of the co-state variables, is

$$\cos\chi^* = \frac{\lambda_{x_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}}, \qquad \sin\chi^* = \frac{\lambda_{y_A}}{\sqrt{\lambda_{x_A}^2 + \lambda_{y_A}^2}} \tag{11}$$

The Target and the Defender optimal headings are

$$\cos\phi^* = -\frac{\lambda_{x_T}}{\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2}}, \qquad \sin\phi^* = -\frac{\lambda_{y_T}}{\sqrt{\lambda_{x_T}^2 + \lambda_{y_T}^2}} \tag{12}$$

and

$$\cos\psi^* = -\frac{\lambda_{x_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}}, \qquad \sin\psi^* = -\frac{\lambda_{y_D}}{\sqrt{\lambda_{x_D}^2 + \lambda_{y_D}^2}} \tag{13}$$

Because the dynamics are simple motion dynamics and the cost functional is of Mayer type, the Hamiltonian is given by Eq. (10) and we have that $\dot{\lambda}_{x_A} = \dot{\lambda}_{y_A} = \dot{\lambda}_{x_D} = \dot{\lambda}_{y_D} = \dot{\lambda}_{x_T} = \dot{\lambda}_{y_T} = 0$, that is, all co-states are constant. We conclude that $\chi^* \equiv$ constant, $\phi^* \equiv$ constant, and $\psi^* \equiv$ constant and the regular optimal trajectories are straight lines. □

It is important to realize that Theorem 1 applies irrespective of the Mayer cost functional specification and the terminal condition (3) or (4). Hence, Theorem 1 applies to both the TAD game of capture and to the ATDDG [15]. Having characterized the regular optimal trajectories in Theorem 1, namely the optimal trajectories are straight lines, we proceed to

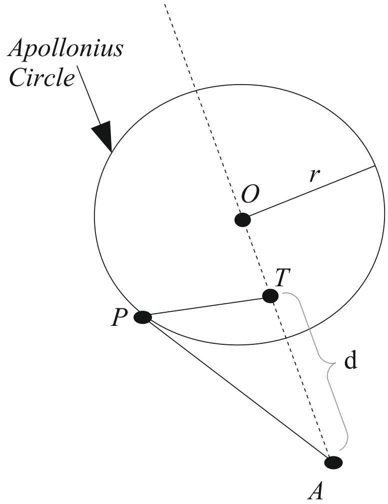

Fig. 1 Apollonius circle based on A and T where, for any point P on the circumference of the circle, $\alpha = \frac{\overline{TP}}{\overline{AP}} = \frac{V_T}{V_A}$ is constant

determine the optimal headings as a function of the current state $\mathbf{x}$. To this end, we introduce the geometric concept of the Apollonius circle—a useful tool in pursuit-evasion games with simple motion. The Apollonius circle is the locus of points P with a fixed ratio of distances to two given points which are referred to as foci. In our case, the foci are A and T, and the fixed ratio is the Target/Attacker speed ratio $\alpha = \frac{\overline{TP}}{\overline{AP}}$. Consider players A and T flying at constant speeds and with constant heading. The Apollonius circle divides the plane into two reachability regions: The interior of the circle is T's reachable region: T can reach any point inside the circle before A can do so; on the other hand, any point outside the circle can be reached by A before T does so. A aims at capturing T at a point $I = (x_{\mathrm{I}}, y_{\mathrm{I}})$ on the circumference of the Apollonius circle.

When both A and T travel to the same point I on the circle, the distance traveled by T is equal to α times the distance traveled by A, where α is the speed ratio parameter. The solution of the differential game yields the aim point I and thus determines the optimal strategies of the players. Each player, by independently solving the differential game, obtains its own and its opponent's optimal heading. When a saddle point state feedback optimal solution is obtained, actual implementation of non-optimal strategies is detrimental to the player which does not implement its optimal strategy and this benefits his adversary. This saddle point property will be illustrated for the TAD differential game of capture in Sect. 7.

In the TAD differential game of capture, we construct a dynamic Apollonius circle using the instantaneous separation between A and T and the speed ratio parameter α. Let O denote the center of the circle. The three points: A, T, and O are collinear, as can be seen in Fig. 1.

The distance $\overline{TO}$ between T, the Target position, and O, the center of the Apollonius circle is

$$\overline{TO} = \frac{\alpha^2}{1-\alpha^2} d \tag{14}$$

where $d = \sqrt{(x_A - x_T)^2 + (y_A - y_T)^2}$ is the instantaneous distance between agents A and T. The radius r of the Apollonius circle is

$$r = \frac{\alpha}{1-\alpha^2} d. \tag{15}$$

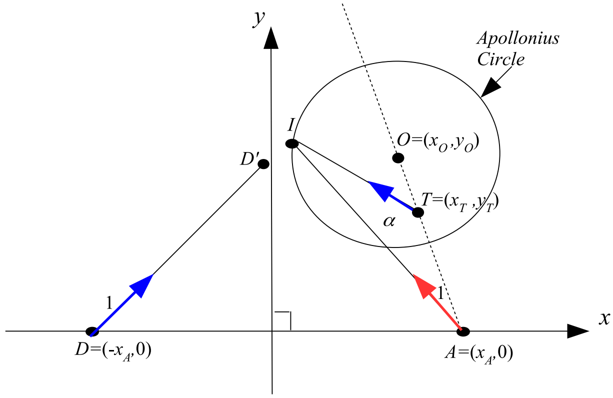

Fig. 2 Reduced state space

and the coordinates of the center of the Apollonius circle are $O = (x_O, y_O)$ where

$$x_O = \frac{1}{1-\alpha^2} x_T - \frac{\alpha^2}{1-\alpha^2} x_A, \qquad y_O = \frac{1}{1-\alpha^2} y_T. \tag{16}$$

In order to minimize the terminal $D - T$ separation, the Defender heads straight to the aim point I—see Fig. 2. Thus, the points D, D', and I are collinear.

Think of the analysis of the TAD game of capture in the realistic plane. In the realistic plane, the optimal primary trajectories are straight lines because the co-states are constants. Hence, geometry is at work: As long as the $A - T$ Apollonius circle does not intersect the orthogonal bisector of the segment AD, that is, the state is in R_c, these trajectories can always be constructed, as outlined in Sect. 6 in the sequel, and the state space region R_c will be covered with trajectories which do not intersect.

5 Game of Kind

Consider the rotating reference frame illustrated in Fig. 2: The points A and D are the instantaneous positions of the Attacker and the Defender, respectively. The x-axis goes from D to A which results in $y_A = y_D \equiv 0$. The y-axis is the orthogonal bisector of the segment $\overline{AD}$. Hence, we have that $x_D = -x_A$. Initially, the (x, y) frame of the reduced state space and the (X, Y) frame of the realistic plane are collocated. Using the rotating reference frame (x, y) yields the reduced state space $(x_A, x_T, y_T) \in \mathbb{R}^3$ where (x_T, y_T) is the position of the Target in the (x, y) reference frame. It is convenient to characterize A's winning region $\mathcal{R}_c$ in the reduced state space (x_A, x_T, y_T).

Reverting back to the realistic plane, we see that the illustration of the dynamics in Fig. 2 provides important insight into the differential game: if the Apollonius circle intersects the Y-axis, then there exists a strategy for T to escape and for D to intercept A. Then, $\mathbf{x} \in \mathcal{R}_e (= \overline{\mathcal{R}_c})$ and in $\mathcal{R}_e$ the ATDDG Game of Degree, solved in [14,15] is played. And if the Apollonius circle does no intersect the Y-axis, then A is able to capture T before D can reach A; in this case, the state $\mathbf{x} \in \mathcal{R}_c$ and in $\mathcal{R}_c$ a different Game of Degree, the TAD capture game, is played out. In this paper, the TAD capture Game of Degree is addressed. Finally, if the Apollonius circle is just tangent to the Y-axis, then the state $\mathbf{x} \in \mathcal{B}$ where $\mathcal{B}$ denotes the Barrier surface that separates the state space into the two regions $\mathcal{R}_e$ and $\mathcal{R}_c$.

The Game of Degree in $\mathcal{R}_e$, the ATDDG where the terminal condition is (4), was addressed in [5,14], and [15]. The solution in [5,14,15] presented in the reduced state space provided the optimal, saddle point, state feedback strategies for the T & D team to maximize the terminal $A-T$ separation and for A to minimize the same at the time instant t_f when D intercepts A. State feedback strategies based on the calculation of the instantaneous optimal interception point y^* on the Y-axis were obtained in [14,15]. The players' headings during optimal play are constant as is also the case in the TAD game of capture. Because $V_A = V_D$ the Y-axis separates the reachable regions of A and D, A is intercepted by D on a point located on the Y-axis. The Barrier surface $\mathcal{B}$ that separates the state space into the two regions $\mathcal{R}_e$ and $\mathcal{R}_c$ is characterized as follows.

The Barrier surface is obtained from the condition $x_O = r$, that is, the Apollonius circle is tangent to the Y-axis; using (15) and (16) such condition is

$$\begin{aligned} &\tfrac{1}{1-\alpha^2}(x_T - \alpha^2 x_A) = \tfrac{\alpha}{1-\alpha^2}\sqrt{(x_A - x_T)^2 + y_T^2} \\ \Rightarrow\ & x_T - \alpha^2 x_A - \alpha\sqrt{(x_A - x_T)^2 + y_T^2} = 0 \end{aligned} \tag{17}$$

where $y_A = 0$.

Theorem 2 *For a given speed ratio α, the Barrier surface $\mathcal{B}$ that partitions the reduced state space (x_A, x_T, y_T) into the two regions $\mathcal{R}_e$ and $\mathcal{R}_c$ is given by*

$$\mathcal{B} = \left\{(x_A, x_T, y_T) | x_A > 0, x_T > 0, x_A^2 + \tfrac{y_T^2}{1-\alpha^2} - \tfrac{x_T^2}{\alpha^2} = 0\right\}. \tag{18}$$

Proof From the condition $x_O = r$ we obtained (17) and it follows that

$$\begin{aligned} &(x_T - \alpha^2 x_A)^2 = \alpha^2\left[(x_A - x_T)^2 + y_T^2\right] \\ \Rightarrow\ & (1-\alpha^2)x_T^2 - \alpha^2(1-\alpha^2)x_A^2 - \alpha^2 y_T^2 = 0 \end{aligned}$$

dividing both sides of the previous equation by $-\alpha^2(1-\alpha^2)$ we obtain (18). □

Note that when $x_T < 0$ the state $\mathbf{x} \in \mathcal{R}_e$ since the Target is closer to D than to A. The *x_A-cross section of the Barrier surface* (18) is shown in Fig. 3, that is, for a fixed value of x_A the Barrier surface (18) is a (right-hand branch of a) hyperbola in the $X-Y$ plane. This figure helps to visualize the possible locations of the Target and the associated outcomes: If T is on the right-hand side of the hyperbola, then it will be captured, provided that A plays optimally. If T is on the left-hand side of the hyperbola, T can escape, provided the T & D team plays optimally.

We place especial emphasis on the Barrier surface. Since, in this paper, the optimal strategies in $\mathcal{R}_c$ will be derived, the complete solution for any state $\mathbf{x} \in \mathbb{R}^6$ (in the realistic state space) will have been obtained. While the optimal solution is conveniently presented and visualized in the reduced state space (x_A, x_T, y_T), it can be easily implemented in the realistic state space $\mathbb{R}^6$. Because the TAD differential game is a two-termination set game, we need to first determine in which region of the state space the current state is located, that is, use the solution of the Game of Kind given by Eq. (18). From that we are able to determine which strategy to implement: we will employ the saddle point solution of the corresponding Game of Degree, the TAD game of capture or the ATDDG, depending on wether $\mathbf{x} \in \mathcal{R}_c$ or $\mathbf{x} \in \mathcal{R}_e$.

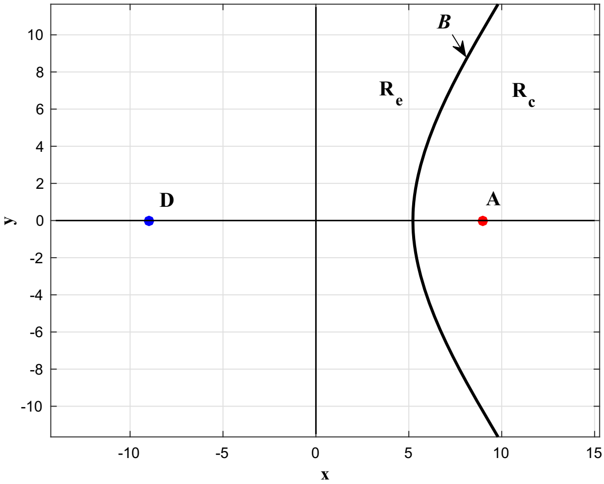

Fig. 3 x_A-cross section of the Barrier surface $\mathcal{B}$

6 TAD Capture Game of Degree in $\mathcal{R}_c$

The TAD capture Game of Degree is analyzed in the realistic plane (X, Y). Note, however, that the (x, y) frame of the reduced state space is initially aligned with the (x, y) frame of the realistic plane.

The coordinates of the center of the Apollonius circle are $O = (x_O, y_O)$ where (x_O, y_O) are given by (16). Let $\mathrm{Re}(\xi)$ and $\mathrm{Im}(\xi)$ be the real and the imaginary part of the complex number ξ. We provide the solution of the TAD game of capture, that is, we determine the optimal interception point I, i.e., the aimpoint of the players.

Theorem 3 *Consider the Target, Attacker, Defender differential game* (1), (5)–(8) *in* $\mathcal{R}_c$*, that is, the state* $\mathbf{x} \in \mathcal{R}_c$*. The optimal interception point given in terms of the coordinates* (x_A, x_T, y_T) *of the reduced state space is* $I^* = (r\cos\theta^* + x_O, r\sin\theta^* + y_O)$ *where* $\theta^* = \arccos Re(z^*) = \arcsin Im(z^*)$ *and* z^* *is the solution of the quartic equation with complex coefficients*

$$\begin{aligned} &\left[x_O y_O + \tfrac{i}{2}(x_O^2 - x_A^2 - y_O^2)\right] z^4 + r(y_O + i x_O) z^3 \\ &\quad + r(y_O - i x_O) z + x_O y_O - \tfrac{i}{2}(x_O^2 - x_A^2 - y_O^2) = 0 \end{aligned} \tag{19}$$

the so obtained θ^* *minimizes the function*

$$J(\theta) = \sqrt{(r\cos\theta + x_O + x_A)^2 + (r\sin\theta + y_O)^2} - \sqrt{(r\cos\theta + x_O - x_A)^2 + (r\sin\theta + y_O)^2}. \tag{20}$$

Proof Theorem 1 tells us that the players' regular optimal trajectories are straight lines. Thus, the point I where A intercepts T lies on

(i) The circumference of the Apollonius circle determined by points A and T and the speed ratio α.
(ii) On a hyperbola whose foci are A and D; the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{21}$$

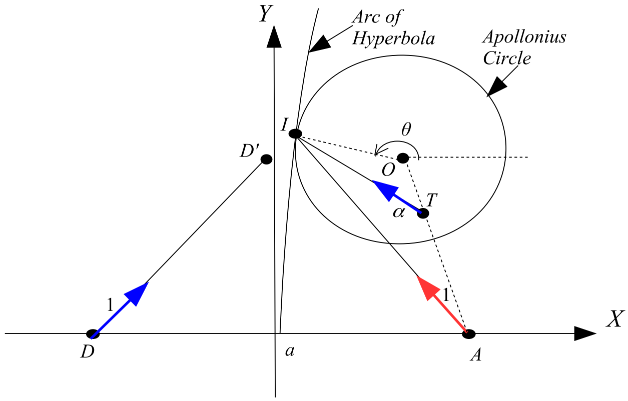

Fig. 4 TAD game of capture: optimality principle

that is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \tag{22}$$

where $c = x_A$ and a, which is $\frac{1}{2}\overline{D'I}$, is yet to be determined. This is illustrated in Fig. 4. Recall that hyperbolae are the locus of all points such that the difference in the distances from points on the hyperbola to A and D is constant.

Geometric optimality principle: The critical hyperbola is tangent to the Apollonius circle at $I = (x, y)$, because T strives to minimize a and A strives to maximize a.

The constitutive equations are

$$(x - x_O)^2 + (y - y_O)^2 = r^2 \tag{23}$$

$$\frac{x^2}{a^2} - \frac{y^2}{x_A^2 - a^2} = 1. \tag{24}$$

The derivative of the circle at I is

$$y'_c = -\frac{x - x_O}{y - y_O} \tag{25}$$

and the derivative of the hyperbola at I is

$$y'_h = \left(\frac{x_A^2}{a^2} - 1\right)\frac{x}{y}. \tag{26}$$

By the optimality principle $y'_c = y'_h$, that is,

$$a^2 = \frac{(y - y_O)x}{x_O y - y_O x} x_A^2. \tag{27}$$

Inserting Eq. (27) into Eq. (24) yields

$$\left(\frac{x}{y - y_O} + \frac{y}{x - x_O}\right)(x_O y - y_O x) = x_A^2. \tag{28}$$

We have two Eqs. (23) and (28), in the two unknowns x and y. We parameterize the circle Eq. (23) as follows

$$\begin{aligned} x &= x_O + r\cos\theta \\ y &= y_O + r\sin\theta \end{aligned} \tag{29}$$

and so Eq. (28) yields the equation in θ

$$x_O y_O(1-2\cos^2\theta) + (x_O^2 - y_O^2 - x_A^2)\sin\theta\cos\theta + r(x_O\sin\theta - y_O\cos\theta) = 0. \tag{30}$$

We use the complex exponential form of the angle θ,

$$\begin{aligned} &z = e^{i\theta} \\ &\Rightarrow \cos\theta = \tfrac{1+z^2}{2z}, \quad \sin\theta = \tfrac{1-z^2}{2z}i. \end{aligned} \tag{31}$$

Inserting those expressions into Eq. (30) yields the quartic equation in z, albeit with complex coefficients

$$\begin{aligned} &\left[2x_O y_O + i(x_O^2 - x_A^2 - y_O^2)\right]z^4 + 2r(y_O + ix_O)z^3 \\ &\quad + 2r(y_O - ix_O)z + 2x_O y_O - i(x_O^2 - x_A^2 - y_O^2) = 0. \end{aligned}$$

Therefore, the optimal solution θ^* is obtained by solving the quartic Eq. (19) whose solution θ minimizes function (20). The Value function $V(x_A, x_T, y_T)$ is obtained in terms of the reduced state space coordinates (x_A, x_T, y_T) by inserting (x_A, x_T, y_T) into Eq. (20). Note that $\theta = \text{ang}(z)$ is such that $\pi > \theta > \frac{\pi}{2}$. The Value of the game is $2a$. □

Remark In the ATDDG the (x, y) frame does not rotate whereas now, in the TAD capture game, this is no longer the case. In the reduced state space, the optimal flow field does not consist of straight lines because the (x, y) axes translate and rotate. Hence, the optimal flow field consists of regular/primary optimal trajectories only, provided that during the play the orthogonal bisector of the segment $\overline{AD}$ will not intersect the dynamic A, T Apollonius circle. That this is so is corroborated by the numerical results presented in Sect. 7.

Let us now consider the interesting case where the state of the system $\mathbf{x} \in \mathcal{B}$.

Corollary 1 *If the initial state $\mathbf{x} \in \mathcal{B}$, the solution of the quartic equation* (19) *is then $\theta^* = \pi$ and, consequently, the optimal interception point in the rotating coordinate frame is $I^* = (0, y_O)$ which is the same interception point I^* obtained by solving the ATDDG Game of Degree. Hence, on the Barrier surface, the Game of Degree in $\mathcal{R}_c$ and the Game of Degree in $\mathcal{R}_e$ provide the same solution.*

Proof Consider the quartic Eq. (19), which provides the optimal solution of the differential game. Multiply Eq. (19) by z^{-2} and write the equation in the following form

$$x_O y_O(z^2 + z^{-2}) + \tfrac{i}{2}(x_O^2 - x_A^2 - y_O^2)(z^2 - z^{-2}) + ry_O(z + z^{-1}) + irx_O(z - z^{-1}) = 0. \tag{32}$$

Performing the corresponding substitutions, Eq. (32) can be written in terms of the angle θ

$$x_O y_O(1 - 2\sin^2\theta) - (x_O^2 - x_A^2 - y_O^2)\sin\theta\cos\theta + ry_O\cos\theta - rx_O\sin\theta = 0. \tag{33}$$

When $\mathbf{x} \in \mathcal{B}$, the Apollonius circle is tangent to the Y-axis. It follows in this case that $x_O = r$ and the left-hand side of Eq. (33) can be expressed as follows

$$x_O(y_O - 2y_O\sin^2\theta - x_O\sin\theta) + \cos\theta\left[x_O y_O - \left(x_O^2 - x_A^2 - y_O^2\right)\sin\theta\right]. \tag{34}$$

Next, substitute $\theta = \pi$ ($\cos\theta = -1$, $\sin\theta = 0$) in (34) to obtain

$$x_O y_O - x_O y_O = 0. \tag{35}$$

It can be seen that $\theta^* = \pi$ is the optimal solution which corresponds to the interception point

$$\begin{aligned} x_{\mathrm{I}}^* &= r\cos\theta^* + x_O = -x_O + x_O = 0 \\ y_{\mathrm{I}}^* &= r\sin\theta^* + y_O = 0 + y_O = y_O \end{aligned} \tag{36}$$

where the coordinates $I^* = (0, y_O)$ correspond to the point where the Apollonius circle is tangent to the Y-axis. The Value of the game when $\mathbf{x} \in \mathcal{B}$ is then

$$\begin{aligned} V(\mathbf{x}|\mathbf{x} \in \mathcal{B}) &= \sqrt{(r\cos\theta + x_O + x_A)^2 + (r\sin\theta + y_O)^2} \\ &\quad - \sqrt{(r\cos\theta + x_O - x_A)^2 + (r\sin\theta + y_O)^2} \\ &= \sqrt{x_A^2 + y_O^2} - \sqrt{x_A^2 + y_O^2} \\ &= 0 \end{aligned} \tag{37}$$

as expected. This means that at the terminal time when A captures T the distance $\overline{DT} = 0$, i.e. all agents reach the interception point isochronously.

The solution of the Game of Degree in $\mathcal{R}_e$, the ATDDG, was presented in [14] and [15], and it is given by rooting the following quartic equation with real coefficients

$$(1-\alpha^2)y^4 - 2(1-\alpha^2)y_T y^3 + \left[(1-\alpha^2)y_T^2 + x_A^2 - \alpha^2 x_T^2\right]y^2 - 2x_A^2 y_T y + x_A^2 y_T^2 = 0. \tag{38}$$

The optimal value y^* is the point on the Y-axis where D intercepts A. We will now show that the solution of the Game of Degree in $\mathcal{R}_e$ when $\mathbf{x} \in \mathcal{B}$ is $I^* = (0, y_O)$, the same solution obtained by solving the TAD capture game.

The polynomial in the left-hand side of Eq. (38) can be written as follows

$$(y - y_T)^2\left[(1-\alpha^2)y^2 + x_A^2\right] - \alpha^2 x_T^2 y^2. \tag{39}$$

Since $\mathbf{x} \in \mathcal{B}$, then Eq. (18) holds, which is equivalent to the following expression

$$x_A^2 = \tfrac{x_T^2}{\alpha^2} - \tfrac{y_T^2}{1-\alpha^2}. \tag{40}$$

Substituting (40) into (39), we obtain

$$(y - y_T)^2\left[(1-\alpha^2)y^2 + \tfrac{x_T^2}{\alpha^2} - \tfrac{y_T^2}{1-\alpha^2}\right] - \alpha^2 x_T^2 y^2. \tag{41}$$

Finally, by substituting $y = y_O = \frac{1}{1-\alpha^2}y_T$ we have that

$$\alpha^4 \tfrac{y_T^2}{(1-\alpha^2)^2}\left[\tfrac{y_T^2}{1-\alpha^2} + \tfrac{x_T^2}{\alpha^2} - \tfrac{y_T^2}{1-\alpha^2}\right] - \alpha^2 x_T^2 \tfrac{y_T^2}{(1-\alpha^2)^2} = 0. \tag{42}$$

Hence, when $\mathbf{x} \in \mathcal{B}$, the solution of the Game of Degree in $\mathcal{R}_e$ is $I^* = (0, y_O)$, as expected. □

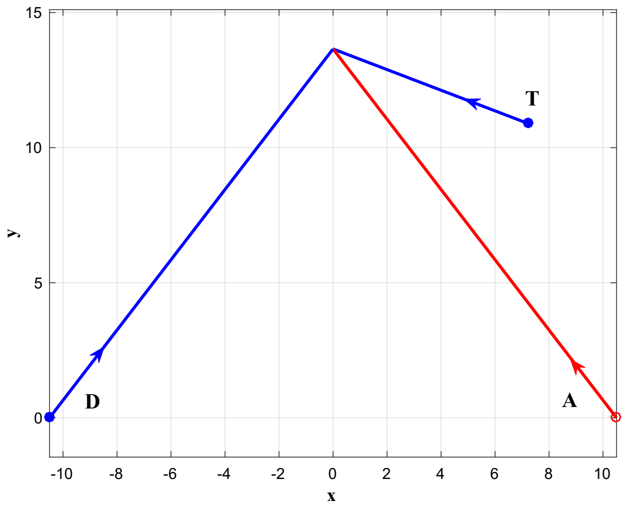

Fig. 5 Optimal play in Example 1. All players reach the interception point I^* isochronously

7 Examples

Example 1 Consider the players' initial positions in the realistic plane $A_0 = (10.5, 0)$, $D_0 = (-10.5, 0)$, and $T_0 = (7.24, 10.89)$. The speed ratio parameter is $\alpha = 0.45$. The initial state is chosen to satisfy $\mathbf{x}_0 \in \mathcal{B}$. Each agent implements its saddle point/optimal state feedback strategy. Since the initial state is on the Barrier surface, it is expected that the solutions of the two Games of Degree, the ATDDG and the TAD game of capture, will provide the same solution and the players reach the interception point at the same time. Indeed, optimal trajectories are shown in Fig. 5 where the solution of both the TAD game of capture and the ATDDG is computed at every time instant using the saddle point/optimal state feedback strategy. Both solutions provide the same static point I^* and, hence, the same headings. Figure 6a shows the separations $\overline{AT}(t)$ and $\overline{AD}(t)$ where $\overline{AT}(t_f) = \overline{AD}(t_f) = 0$, that is, D intercepts A at the same time instant that A captures T—the expected outcome when $\mathbf{x}_0 \in \mathcal{B}$ and each agent plays optimally. The function $\mathcal{B}(t)$, the right-hand side of Eq. (18), is also computed; it can be seen in Fig. 6b that, under optimal play, $\mathcal{B}(t) = 0$ for the duration of the game. This is expected since the Barrier surface is a semipermeable surface, that is, each player is able to hold the state on this surface by applying its saddle point strategy.

Example 2 Consider now the case where the players' initial states $A_0 = (8, 0)$, $D_0 = (-8, 0)$, $T_0 = (6, 2)$, and the speed ratio parameter $\alpha = 0.7$. Now the game's initial state $\mathbf{x}_0 \in \mathcal{R}_c$ and, under optimal play, the Attacker is able to capture the Target before the Defender can reach the Attacker, as shown in Fig. 7. The Value of the game is $V(\mathbf{x}) = 0.376$ which is the terminal distance between T and D when T is captured by A. The function $\mathcal{B}(t)$ is shown in Fig. 9a; note that $\mathcal{B}(t) > 0$ for the duration of the game, that is, the state of the game stayed in $\mathcal{R}_c$.

Consider the same initial condition in $\mathcal{R}_c$ but now the Attacker employs the standard but suboptimal PN guidance law with the standard navigation constant $N = 3$. Not only does the terminal distance between the Target and Defender decrease, but the Attacker is unable

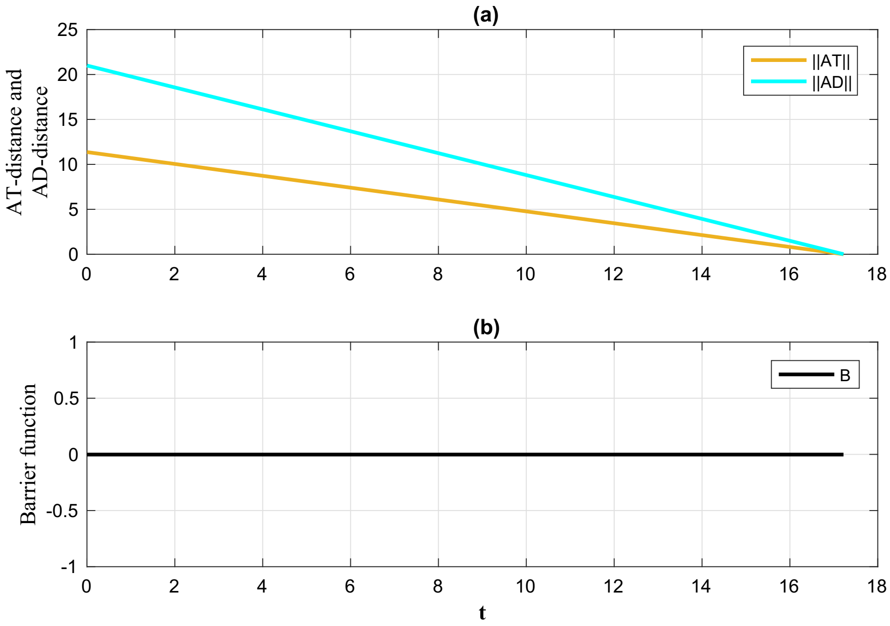

Fig. 6 Example 1. **a** $\overline{AT}(t)$ and $\overline{AD}(t)$, **b** $\mathcal{B}(t)$

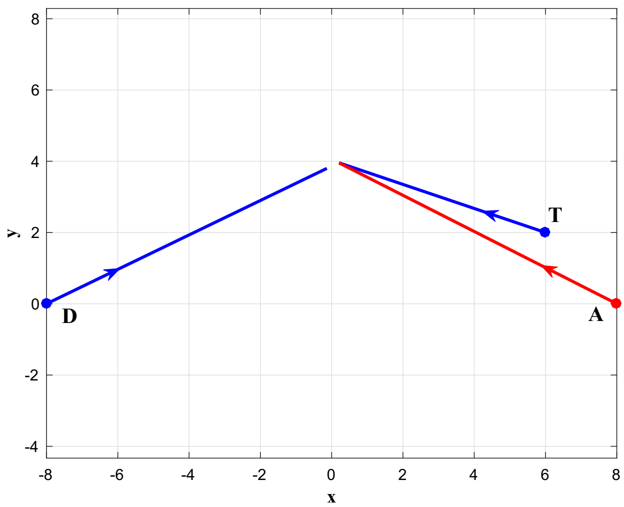

Fig. 7 Optimal trajectories in Example 2. $\mathbf{x}_0 \in \mathcal{R}_c$ and under optimal play A captures T

to hold the state in $\mathcal{R}_c$ because he is not using his saddle point strategy. The state transitions into $\mathcal{R}_e$ where the T & D team then implements the ATDDG strategy, that is, the saddle point strategy in $\mathcal{R}_e$. The resulting trajectories are shown in Fig. 8 where, although the game started in $\mathcal{R}_c$, D actually intercepts A and T escapes. The terminal separation between the Attacker and the Target is $\overline{AT} = 0.168$.

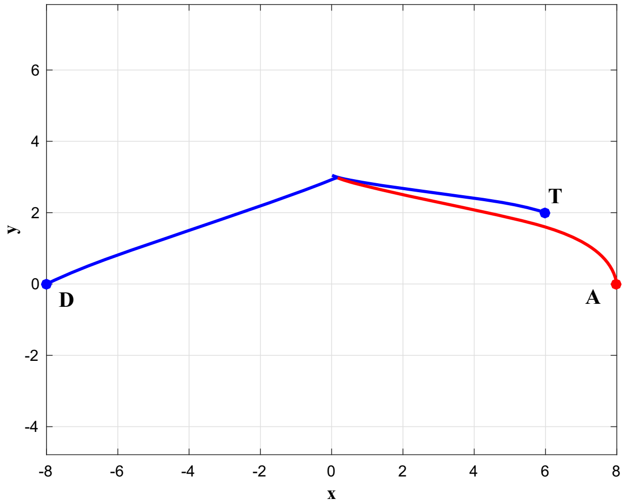

Fig. 8 Example 2. Attacker implements PN. State of the system crosses from $\mathcal{R}_c$ into $\mathcal{R}_e$ and T survives

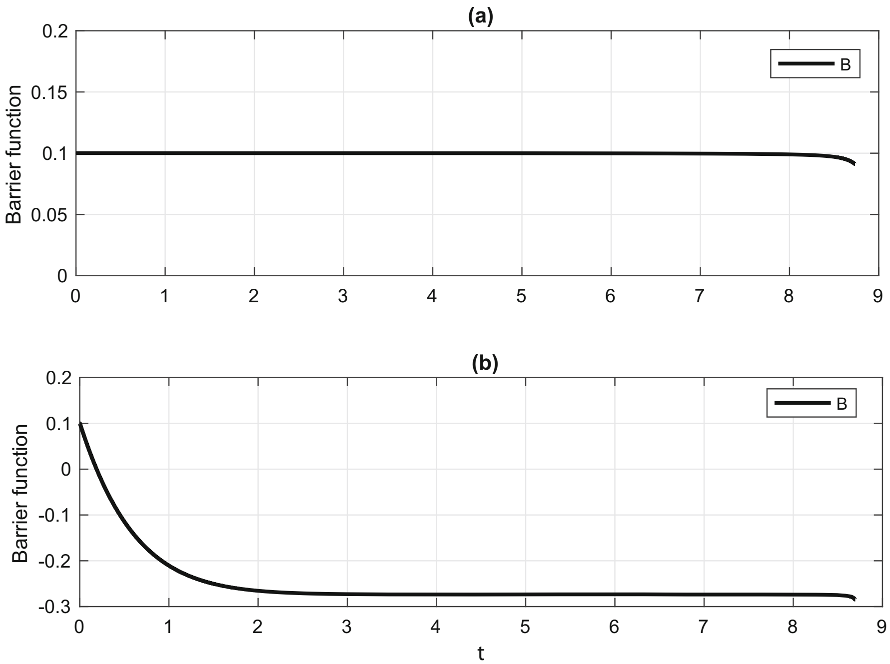

Fig. 9 Example 2. Barrier function. **a** Under optimal play, **b** when A implements PN

Figure 9b shows $\mathcal{B}(t)$ for the case when A uses the PN guidance law. Note that the agents, in this case only T & D (since A foolishly implemented a non-optimal strategy), monitor the sign of $\mathcal{B}(t)$ in order to decide which Game of Degree to play. If $\mathcal{B}(t) > 0$ then $\mathbf{x} \in \mathcal{R}_c$ and the T & D team implements the optimal strategies of the TAD game of capture derived

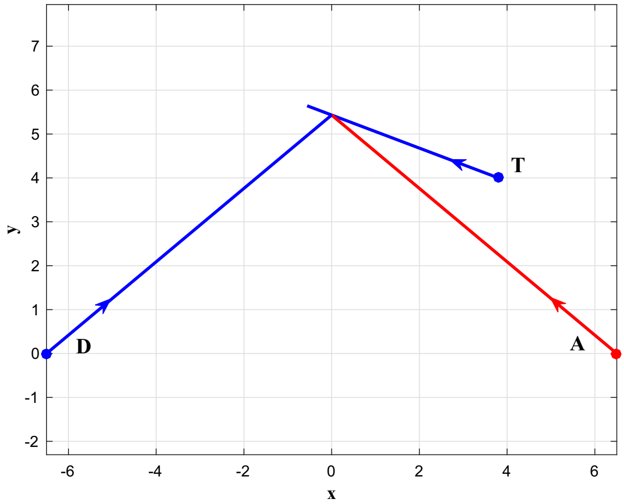

Fig. 10 Optimal trajectories in Example 3. $\mathbf{x}_0 \in \mathcal{R}_e$, under optimal play, D intercepts A

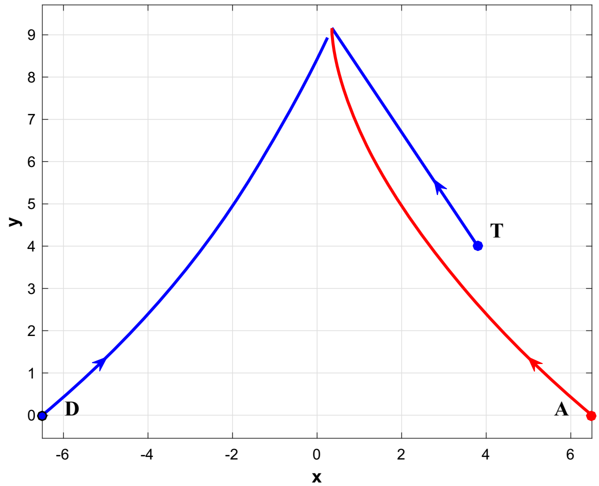

Fig. 11 Example 3. Target flees away from A along initial LOS. State of the system crosses from $\mathcal{R}_e$ into $\mathcal{R}_c$ and T is captured

in this paper. On the other hand, when $\mathcal{B}(t) < 0$ then the state of the game $\mathbf{x} \in \mathcal{R}_e$ and the T & D team switches to the optimal strategies of the ATDDG.

Example 3 Let us consider a scenario analogous to Example 2. The initial players' positions are $A_0 = (6.5, 0)$, $D_0 = (-6.5, 0)$, and $T_0 = (3.8, 4)$. The speed ratio parameter is $\alpha = 0.55$. It holds that $\mathbf{x}_0 \in \mathcal{R}_e$ and, under optimal play, the Attacker is intercepted by the Defender and the Target escapes. The optimal trajectories are shown in Fig. 10 where $V(\mathbf{x}) = 0.598$. In this example, the function $\mathcal{B}(t)$ is shown in Fig. 12a, where $\mathcal{B}(t) < 0$ during the engagement.

In Fig. 11, a non-optimal strategy of the Target is illustrated. In this case, the Target tries to evade the Attacker by fleeing outbound along the initial line-of-sight (LOS) to A—the Target runs for his life, as if D would not be there. The Attacker employs the TAD differential game saddle point strategy. (The Defender also implements his saddle point strategy but it will

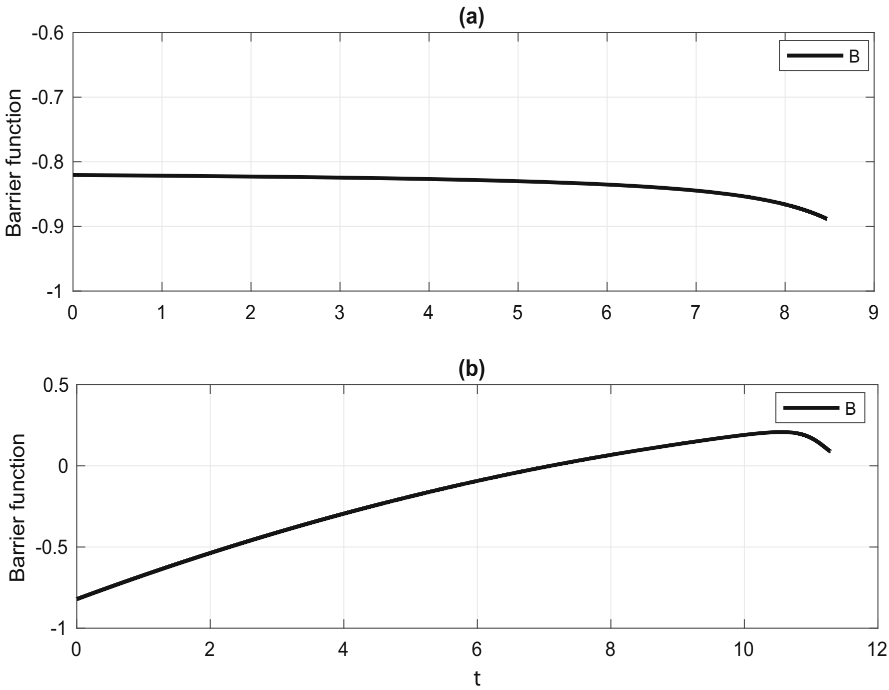

Fig. 12 Example 3. Barrier function. **a** Under optimal play, **b** when T flees away from A along initial LOS

be of little use since, in this case, the Target is not cooperating with D). This means that the Attacker checks the sign of the $\mathcal{B}(t)$ function, shown in Fig. 12b, and it implements its ATDDG optimal strategy when $\mathbf{x} \in \mathcal{R}_e$ and then it implements its TAD game of capture optimal strategy when the state of the system transitions into $\mathcal{R}_c$, courtesy of the non-optimal play of T. The result: A captures T before D can intercept A. The optimal trajectories in Figs. 5, 6, 7, 8, 9, 10, 11, and 12 are shown in the realistic plane.

8 Discussion

Isaacs' approach to the solution of differential games has very much the appearance of the Pontryagin maximum principle (PMP). The latter provides a necessary condition for optimality which, fortunately, is conducive to finding the optimal trajectory which connects points/states A and B. It requires the solution of a TPBVP.

But appearances can be misleading. Isaacs' method is revolutionary in that it entails the retrograde integration of the Euler–Lagrange equations "starting out" from the usable part of the terminal manifold. If by doing so one manages to cover the state space, with no terra incognita regions and without caustics/intersecting characteristics, the game has been solved—by construction. This is the case in our TAD game of capture: If the $A-T$ Apollonius circle does not intersect the orthogonal bisector of the segment AD, that is the state is in R_c, we have the players' constant controls which min max the terminal $T - D$ separation. The players' headings are constant because the co-states are constant.

As an historical aside concerning Isaacs', no pun intended, constructive approach to the solution of differential games, the following was recognized by Isaacs back in 1951. We reproduce verbatim footnote 13 from [9] where one could argue, not only the foundations of the theory of differential games, but also the foundations of Dynamic Programming and of the PMP were laid: "Our early statement that solutions of games of degree imply those of games of kind, may appear meretricious. It is not so, of course, when applied to the play as a whole."

9 Conclusions

The Target–Attacker–Defender (TAD) differential game of capture was introduced and analyzed in this paper using a geometric method. The TAD differential game belongs to the class of two termination set differential games where each team strives to end the game in its preferred terminal set. The Game of Degree in the region of win $\mathcal{R}_c$ of the Attacker provides the optimal strategy for the Attacker to capture the Target and maximize the terminal distance between Target and Defender at termination. It also provides the optimal strategy for the T & D team to minimize the same distance. The analysis is corroborated by the numerical results. When the ATDDG is played in $\mathcal{R}_e$ where the T & D team wins, the T & D team strives to maximize the $A - T$ separation at the time instant when A is intercepted by D whereas A works to minimize the same distance. In the ATDDG and as corroborated by the numerical results, also in the TAD game of capture, one gets away with a flow field consisting of primary optimal trajectories and the interesting surface is the Barrier surface $\mathcal{B}$ which separates the regions of win of the Attacker and the T & D team. Due to our judicious choice of performance functionals in both the ATDDG and the current TAD game of capture, the players' strategies have the desirable property important in two termination set differential games: the players' optimal strategies of the Game of Degree in each region coincide when the state of the system is on the Barrier surface that separates the winning regions of each team and the game terminates on the two-dimensional subspace which lies at the intersection of the players' target sets subspaces.

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