Finite-Time Connectivity-Preserving Consensus of Networked Nonlinear Agents With Unknown Lipschitz Terms

Yongcan Cao, Wei Ren, David W. Casbeer, and Corey Schumacher

***Abstract*—This technical note studies finite-time consensus problem for a team of networked nonlinear agents with unknown Lipschitz terms under communication constraints, where each agent has a limited sensing range. Because the induced interaction graph is typically state-dependent and dynamic, we propose a distributed nonlinear consensus algorithm that is capable of preserving the initial interaction patterns. By using tools from nonsmooth analysis, sufficient conditions are obtained such that finite-time consensus can be reached. An upper bound of the convergence time is derived via a two-step analysis. The validity of the theoretical result is shown by one simulation example.**

***Index Terms*—Consensus, cooperative control, limited sensing range, Lipschitz nonlinear dynamics.**

I. Introduction

Consensus of networked agents has been an active research topic in the systems and controls society. Emphasis is placed on the design and analysis of local control algorithms such that a team of networked agents can reach agreement on some common state. The local control algorithm, typically referred to as an *consensus algorithm*, is distributed in the sense that control input for each agent is based on information from its local (time-varying) neighbors. On one hand, the distributed nature is promising considering its high robustness, strong adaptivity, and flexible scalability. On the other hand, it is considerably more difficult to conduct stability analysis, especially when the interaction graph is dynamic.

A. Related Work

1) Consensus Under a State-Dependent Interaction Graph: One typical research direction in the study of consensus is to find the necessary and/or sufficient condition on the interaction graph such that consensus can be reached, where both *state-independent interaction graph* and *state-dependent interaction graph* have been considered. Roughly speaking, the state-independent interaction graph refers to the case when the interaction graph does not depend on the states of the agents. Research progress on consensus under a state-independent interaction graph can be found in survey papers [2], [3].

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A more relevant topic is on the study of consensus under a state-dependent interaction graph, where the interaction graph for a team of agents changes based on their states. A typical scenario is that any pair of agents can exchange their information if and only if their geometric distance is less than some threshold, often referred to as *communication/sensing range*. This is a rather practical model for the wireless communication among many physical vehicles. Reference [4] is an earlier work on the topic, where consensus of networked first-order agents is solved via a potential-based consensus algorithm. The main idea behind the algorithm is to guarantee that any existing interaction pattern is preserved afterwards. This connectivity preserving mechanism is due to the occurrence of an infinitely large gain associated with an edge, were that edge to break. When a special navigation function is used to replace the potential in [4], it is shown in [5] that consensus can still be reached when the control gains are always bounded. Similarly, a potential-based control algorithm is proposed in [6] to solve the consensus problem for networked second-order agents under a state-dependent interaction graph.

2) Consensus of Networked Nonlinear Agents: Consensus of networked linear agents has been considered extensively. Examples of the linear agents are first-order systems and second-order systems [7]. The study of consensus of networked linear agents serves as the basis for systems consisting of networked nonlinear agents. The consideration of nonlinear agents in the study of consensus (see [3] and references therein for some existing research) is mainly due to the nonlinearity of systems dynamics for many physical vehicles, such as UAVs. In many cases, system dynamics cannot even be completely identified due to uncertainties and the existence of some unidentifiable parameters. For agents with completely known (nonlinear) dynamics, the study of consensus can often be converted to the study of consensus of some linear agents by the feedback linearization technique. However, this does not necessarily apply to agents with partially known nonlinear dynamics. Research on consensus of agents with partially known nonlinear dynamics has been reported in [8]–[10], where consensus is studied for agents with first-order-like systems [10] and second-order-like systems [8], [9]. Generally speaking, the stability analysis for consensus of networked nonlinear agents is challenging when the nonlinear dynamics are only partially known. It is worthwhile to point out that both state-dependent interaction graph and agents with partially known nonlinear dynamics are considered in [9].

3) Finite-Time Consensus: The third related topic is on the study of consensus with finite convergence time, also referred to as *finite-time consensus*. Finite-time convergence is essentially a measure of the performance of a consensus algorithm. Finite-time consensus has potential benefits because it is often possible to decouple the consensus problem from other control objectives. Another benefit of finite-time consensus is its robustness to disturbance and noise. The study of finite-time consensus has been reported for both first-order systems [11]–[14] and second-order systems [15]. Because linear consensus algorithms can normally guarantee asymptotic convergence rather than finite-time convergence, the proposed consensus algorithms in [11]–[15] are nonlinear or even nonsmooth. It is worthwhile to point out that both state-dependent interaction graph and finite-time convergence are considered in [13] for first-order systems, whereas the interaction graphs considered in [11], [12], [14], [15] are state-independent.

B. Motivation of the Paper

Motivated by the study on consensus under a state-dependent interaction graph, consensus of networked agents with partially known dynamics, and finite-time consensus, this technical note aims at studying finite-time consensus of networked nonlinear agents under a state-dependent interaction graph by extending the preliminary result reported in [1]. As mentioned earlier, each topic has its own unique feature and motivation. A state-dependent interaction graph reflects practical communication capabilities for physical agents, such as UAVs, considering their power constraints among others. The consideration of agents with partially known dynamics admits the difficulty of system identification and aims to provide new insights of solving the well-known consensus problem even if agent dynamics are partially known. Finite-time convergence has the benefit of easy controller design by decoupling consensus from other control objectives and good system performance such as improved robustness. By considering the three features altogether, our objective is to study consensus problem from a practical point of view.

C. Contribution of the Paper

The contribution of the technical note is twofold. First, a distributed consensus algorithm is devised to address the aforementioned three features (i.e., a state-dependent interaction graph, agents with partially known nonlinear dynamics, and finite-time convergence) in a unified framework. Second, by employing a two-step analysis, an upper bound on the convergence time is derived explicitly. The analysis of the convergence time is challenging due to the partially known nonlinear dynamics and the state-dependent interaction graph.

D. Notation

$\mathbb{R}$ denotes the set of real numbers. $\mathbf{0}_n \in \mathbb{R}^n$ and $\mathbf{1}_n \in \mathbb{R}^n$ denote, respectively, the all-zero column vector and the all-one column vector. $I_n \in \mathbb{R}^{n\times n}$ denote the identity matrix. $\|\cdot\|$ denotes the 2-norm. For $x \in \mathbb{R}^n$, $\mathrm{sgn}(x)$ denotes the signum function of x defined entrywise. Define $\mathrm{sig}(x)^\alpha \triangleq \mathrm{sgn}(x)\|x\|^\alpha$. For a real symmetric positive semi-definite matrix A with only one eigenvalue equal to zero, $\lambda_2(A)$ and $\lambda_{\max}(A)$ denote, respectively, the minimum nonzero eigenvalue and the maximum eigenvalue of A.

II. Preliminaries

A. Graph Theory Notions

For a team of n agents, the interaction (communication or sensing) among all agents can be modeled by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the agent set and the edge set. An edge in an undirected graph denoted as (i, j) means that agents i and j can obtain information from each other. Accordingly, agents i and j are neighbors of each other. We assume that any agent is not a neighbor of itself. An undirected path is a sequence of undirected edges of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected graph is connected if there is an undirected path between every pair of distinct agents. An undirected graph is fully connected if any pair of agents can receive information from each other.

The adjacency matrix and Laplacian matrix are often used to describe an undirected graph. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that $a_{ij} > 0$ if $(j, i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that $\ell_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. By letting $a_{ij} = a_{ji}$ (i.e., $\mathcal{A}$ and $\mathcal{L}$ are symmetric), it is then well-known that $\mathcal{L}$ is symmetric positive semi-definite and $\mathcal{L}$ has at least one eigenvalue equal to 0 with a corresponding left eigenvector $\mathbf{1}_n^T$ and a corresponding right eigenvector $\mathbf{1}_n$.

B. Problem Statement

Consider a team of networked Lipschitz nonlinear agents

$$\dot{r}_i = f(r_i) + u_i, \quad i = 1, \ldots, n \tag{1}$$

where $r_i \in \mathbb{R}^m$ and $u_i \in \mathbb{R}^m$ are, respectively, the state and control input for the ith agent and $f(r_i) \in \mathbb{R}^m$ is an unknown nonlinear term characterizing the nonlinear dynamics for the ith agent. It is assumed that

$$\|f(r_i) - f(r_j)\| \leq \gamma \|r_i - r_j\|, \quad \forall\, r_i, r_j \in \mathbb{R}^m \tag{2}$$

where γ is a known positive constant. Here, any pair of agents, labeled as i and j, are neighbors of each other if $\|r_i - r_j\| < R$, where R is the sensing range of all agents. The objective is then to design local control laws u_i such that $\|r_i - r_j\| \to 0$ in finite time for all $i, j = 1, \ldots, n$. That is, all agents reach consensus in finite time.

C. Definitions and Lemmas

Definition 2.1—[16]: Consider a differential equation

$$\dot{x} = f(t, x) \tag{3}$$

with a piecewise continuous vector-valued function $f(t, x)$. Define the Filippov set-valued map $K[f](t, x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\mathrm{co}} f(t, B(x, \delta) - N)$, where $\bigcap_{\mu(N)=0}$ denotes the intersection over all sets N of Lebesgue measure zero, $B(x, r)$ denotes the open ball of radius r centered at x, and $\overline{\mathrm{co}}$ denotes the convex closure. A vector function $x(t)$ is called a Filippov solution of (3) if $x(t)$ is absolutely continuous and $\dot{x} \in K[f](t, x)$ almost everywhere.

Definition 2.2—[17]: For a locally Lipschitz function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$, define the *generalized gradient* of V at x by $\partial V(x) \triangleq \mathrm{Co}\{\lim(\partial V(x_i))/\partial x_i | x_i \to x, x_i \notin \Omega_V\}$, where Co denotes the convex hull and Ω_V is the set of measure zero where the gradient of V with respect to x is not defined. The *set-valued Lie derivative* of $V(x)$ with respect to (3) at x is defined as $\widetilde{L}_F V(x) \triangleq \bigcap_{\xi\in\partial V(x)} \xi^T K[f](t, x)$.

Definition 2.3 (see, e.g., [18]): The upper Dini derivative of a continuous function $\pi(t) : \mathbb{R} \mapsto \mathbb{R}$ is defined by $D^+\pi(t) \triangleq \limsup_{h\to 0^+}((\pi(t + h) - \pi(t))/h)$.

Lemma 2.1—[19]: Given (3), let $f(t, x)$ be measurable and locally essentially bounded, that is, bounded on a bounded neighborhood of every point excluding sets of measure zero. Then there exists a Filippov solution of (3) for any initial state.

Lemma 2.2—[18]: Consider the scalar differential equation $\dot{s} = g(t, s)$ where $s(t_0) = s_0$, $g(t, s)$ is continuous in t and locally Lipschitz in s for all $t \geq 0$, and all $s \in J \in \mathbb{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $s(t)$, and suppose $s(t) \in J$ for all $t \in [t_0, T)$. Let $\nu(t)$ be a continuous function whose upper Dini derivative $D^+\nu(t)$ satisfies $D^+\nu(t) \leq g(t, \nu(t))$ with $\nu(t_0) \leq s_0$ and $\nu(t) \in J$ for all $t \in [t_0, T)$. Then $\nu(t) \leq s(t)$ for all $t \in [t_0, T)$.

Lemma 2.3—[20]: Let $Q = [q_{ij}] \in \mathbb{R}^{n\times n}$ and $\phi : \mathbb{R} \mapsto \mathbb{R}$ be a function. If $Q = [q_{ij}] \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $\phi(-p_i) = -\phi(p_i)$, we have $(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}(\delta_i - \delta_j)\phi(\delta_i - \delta_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}\delta_i\phi(\delta_i - \delta_j)$.

Lemma 2.4—[21]: For $y \in \mathbb{R}^m$, $(d\|y\|^{\alpha+1})/dt = (\alpha+1)[\mathrm{sig}(y)^\alpha]^T \dot{y}$ and $d\|y\|^{\alpha+1}/dy = (\alpha+1)\mathrm{sig}(y)^\alpha$, where $\alpha \in \mathbb{R}$.

III. Main Result

The main result of this technical note is presented in this section. We first propose a distributed consensus algorithm and then show that this algorithm guarantees consensus in finite time by employing tools from nonsmooth analysis. An upper bound of the convergence time is derived by a two-step analysis.

The proposed consensus algorithm is given by

$$u_i = \sum_{j=1}^{n} \beta\left(\|r_i - r_j\|\right)\left[(r_j - r_i) + \mathrm{sig}(r_j - r_i)^\alpha\right] \tag{4}$$

where $\alpha \in (0,1)$ and $\beta(\|\cdot\|)$ is defined as

$$\beta\left(\|r_i - r_j\|\right) = \begin{cases} \beta_1\left(\|r_i - r_j\|\right), & \|r_i(0) - r_j(0)\| < R \\ \beta_2\left(\|r_i - r_j\|\right), & \text{otherwise} \end{cases}$$

where

$$\beta_1\left(\|r_i - r_j\|\right) = \begin{cases} \frac{\varpi R^2}{R^2 - \|r_i - r_j\|^2}, & \|r_i - r_j\| < R \\ 0, & \text{otherwise} \end{cases}$$

$$\beta_2\left(\|r_i - r_j\|\right) = \begin{cases} \varpi, & \|r_i - r_j\| < R \\ 0, & \text{otherwise} \end{cases}$$

where ϖ is a positive constant. Note that $\beta(\|r_i - r_j\|) = 0$ if $\|r_i - r_j\| \geq R$, which reflects the limited sensing range of the agents. In addition, if $\|r_i(0) - r_j(0)\| < R$ for some $i \neq j$, $\beta(\|r_i - r_j\|) \to \infty$ as $\|r_i - r_j\| \to R$. This property plays a crucial role in preserving the initial interaction patterns. In summary, the main idea behind (4) is that: (i) if ϖ is chosen properly, the initial interaction patterns are always preserved; (ii) consensus is reached asymptotically due to $r_j - r_i$; (iii) consensus is reached in finite time due to $\mathrm{sig}(r_j - r_i)^\alpha$. Here (i) is needed for the validity of (ii) while (i) and (ii) are needed for the validity of (iii). Thus both $r_j - r_i$ and $\mathrm{sig}(r_j - r_i)^\alpha$ are necessary. Note that the control algorithm (4) is discontinuous due to the discontinuity of $\beta(\|r_i - r_j\|)$. Accordingly, the solutions of (1) using (4) are considered in the Filippov sense (see Definition 2.1). Based on Lemma 2.1, there exists a Filippov solution of (1) using (4) because $f(r_i)$ and u_i are measurable and locally essentially bounded.

By letting $a_{ij}(t) = \beta(\|r_i(t) - r_j(t)\|)$, the associated Laplacian matrix $\mathcal{L}(t)$ is given by

$$\ell_{ij}(t) = \begin{cases} -\beta\left(\|r_i(t) - r_j(t)\|\right), & i \neq j \\ \sum_{j=1, j\neq i}^{n} \beta\left(\|r_i(t) - r_j(t)\|\right), & i = j. \end{cases}$$

Define a new matrix $C \triangleq [c_{ij}] \in \mathbb{R}^{n\times n}$, where

$$c_{ij} = \begin{cases} -\mathrm{sgn}\left[\beta\left(\|r_i(0) - r_j(0)\|\right)\right], & i \neq j \\ \sum_{j=1, j\neq i}^{n} \mathrm{sgn}\left[\beta\left(\|r_i(0) - r_j(0)\|\right)\right], & i = j. \end{cases} \tag{5}$$

C is a constant Laplacian matrix with each nonzero off-diagonal entry being either 0 or -1. Both $\mathcal{L}(t)$ and C are symmetric. Before presenting the main result, the following property regarding $\mathcal{L}(t)$ is needed.

Lemma 3.1: Let $\mathcal{G}(t)$ be a connected graph at $t = 0$. If $\|r_i(t) - r_j(t)\| < R$ holds for any $t \geq 0$ given that $\|r_i(0) - r_j(0)\| < R$, $i \neq j$, then the following two statements hold for any $t \geq 0$: (i) $\mathcal{L}(t) - \varpi C$ is symmetric positive semi-definite; (ii) $\lambda_2[\mathcal{L}(t)] \geq \varpi\lambda_2(C)$.

Proof—[Proof of the Statement (i)]: Based on the definitions of $\mathcal{L}(t)$ and C, each off-diagonal entry of $\mathcal{L}(t) - \varpi C$ is given by $-\beta(\|r_i(t) - r_j(t)\|) + \varpi$. Under the condition of the lemma and the definition of $\beta(\|r_i(t) - r_j(t)\|)$, $-\beta(\|r_i(t) - r_j(t)\|) + \varpi$ is always nonpositive. Therfore, the off-diagonal entries of $\mathcal{L}(t) - \varpi C$ are all nonpositive. Because $\mathcal{L}(t)$ and C are symmetric Laplacian matrices based on their definitions before this lemma, $\mathcal{L}(t) - \varpi C$ is a symmetric Laplacian matrix, which implies that $\mathcal{L}(t) - \varpi C$ is symmetric positive semi-definite.

Proof of the Statement (ii): From (i), $\mathcal{L}(t) - \varpi C$ is symmetric positive semi-definite. Let $y \in \mathbb{R}^n$ be the right nonzero eigenvector for $\mathcal{L}(t)$ associated with $\lambda_2[\mathcal{L}(t)]$. It follows that $\lambda_2[\mathcal{L}(t)]y^T y = y^T \mathcal{L}(t) y \geq \varpi y^T C y$. When $\mathcal{G}(t)$ is connected at $t = 0$, it follows from the definition of C that the interaction graph associated with C is also undirected and connected. This implies that C is a symmetric Laplacian matrix with a simple zero eigenvalue whose associated eigenvector is $\mathbf{1}_n$. Note that y is perpendicular to $\mathbf{1}_n$ because $\mathcal{L}(t)$ is a symmetric Laplacian matrix. Then, one has $y^T C y \geq \lambda_2(C) y^T y$. It then follows that $\lambda_2[\mathcal{L}(t)]y^T y \geq \varpi\lambda_2(C) y^T y$. Because $y^T y > 0$, it follows that $\lambda_2[\mathcal{L}(t)] \geq \varpi\lambda_2(C)$. ■

Define $r_0 \triangleq (1/n)\sum_{i=1}^{n} r_i$. By computation

$$\dot{r}_0 = \frac{1}{n}\sum_{i=1}^{n} f(r_i) \tag{6}$$

due to the fact that $\sum_{i=1}^{n} u_i = 0$ when the interaction graph is undirected. Define $\tilde{r}_i \triangleq r_i - r_0$ and $\tilde{r} \triangleq [\tilde{r}_1, \ldots, \tilde{r}_n]^T$. The main result of the technical note is given in the following theorem.

Theorem 3.1: Let $\mathcal{G}(t)$ be a connected graph at $t = 0$ and $V(\tilde{r}(0))$ be bounded, where

$$V(\tilde{r}) = \overline{V}(\tilde{r}) + \frac{\eta}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{0}^{\|\tilde{r}_i - \tilde{r}_j\|} \beta(s)(s + s^\alpha)ds \tag{7}$$

where $0 < \eta < (1/\gamma)$ with γ satisfying (2) and

$$\overline{V}(\tilde{r}) = \frac{1}{2}\tilde{r}^T\tilde{r}. \tag{8}$$

When the parameter ϖ is chosen such that

$$\varpi > \frac{\gamma}{(1 - \gamma\eta)\lambda_2(C)} \tag{9}$$

where C is defined in (5), the following three statements hold:

1) The initial interaction graph is always preserved;
2) For any $i \neq j$, $\|r_i(t) - r_j(t)\| < R$ when $t > (\ln\overline{V}(\tilde{r}(0)) + 3\ln 2 - 2\ln R)/(\varpi\lambda_2(C) - \gamma)$;
3) For any $i \neq j$, $\|r_i(t) - r_j(t)\| = 0$ when $t > (\ln\overline{V}(\tilde{r}(0)) + 3\ln 2 - 2\ln R)/(\varpi\lambda_2(C) - \gamma) + (2(n-1))/(n\varpi(1-\alpha))R^{1-\alpha}$.

We prove the previous three statements separately in the Appendix.

Remark 3.2: It is assumed in Theorem 3.1 that the interaction graph is connected initially. It is interesting to investigate if a relaxed condition on the interaction graph, such as connected interaction graphs in an integral sense (i.e., PE-like assumption in [22]), can be found to ensure consensus.

Remark 3.3: In Theorem 3.1, we analyzed the case when the interaction graph is connected initially without considering packet loss or dropout. It remains a challenging problem to address the issue of package loss or dropout.

IV. Simulation

This section presents a simulation example to illustrate the result in Theorem 3.1. We consider a team of 4 agents in 2-D space (i.e., $m = 2$). The parameters in (4) are chosen as $\varpi = 2$, $\alpha = 0.5$, and

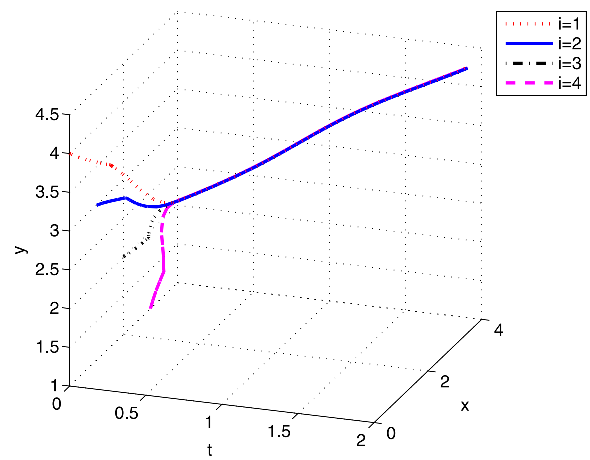

Fig. 1. Trajectories of the four agents using (4).

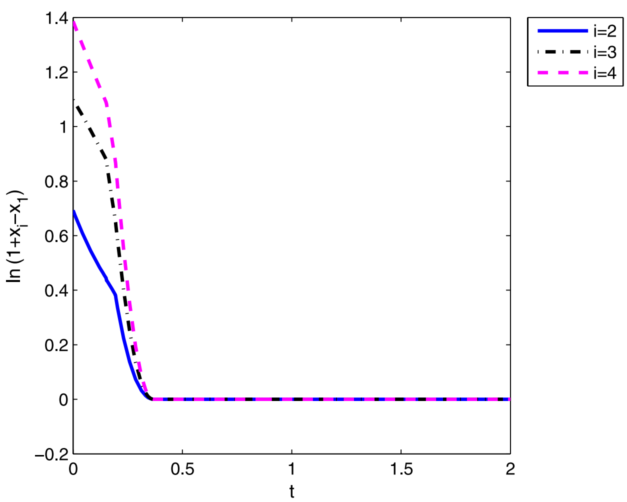

Fig. 2. Plot of $\ln(1+x_i-x_1)$, $i=2,3,4$.

the unknown nonlinear term in the agent dynamics (1) is defined as $f(r_i)=(1/4)\sin(4r_i)+1$. By computation, $\gamma=1$. The initial state is chosen as $r_1(0)=[0,4]^T$, $r_2(0)=[1,3]^T$, $r_3(0)=[2,2]^T$, and $r_4(0)=[3,1]^T$. The sensing range R is chosen as 2. Then the matrix C is $[1\ -1\ 0\ 0;\ -1\ 2\ -1\ 0;\ 0\ -1\ 2\ -1;\ 0\ 0\ -1\ 1]$ with $\lambda_2(C)=0.5858$. All conditions in Theorem 3.1 are thus satisfied. It can be computed from Theorem 3.1 that for any $i\neq j$, $\|r_i(t)-r_j(t)\|<R$ for $t>13.4183$ and $\|r_i(t)-r_j(t)\|=0$ for $t>15.5396$.

The trajectories of the four agents are shown in Fig. 1, where $r_i=[x_i,y_i]^T$ and t is the simulation time. It can be seen that consensus can be reached ultimately. To show that consensus can be reached in finite time, Fig. 2 and Fig. 3 respectively show how $\ln(1+x_i-x_1)$, $i=2,3,4$, and $\ln(1+y_i-y_4)$, $i=1,2,3$, evolve with t. Because consensus is reached if and only if $\ln(1+x_i-x_1)=0$, $i=2,3,4$, and $\ln(1+y_i-y_4)=0$, $i=1,2,3$, we can see from Figs. 2 and 3 that consensus is reached in less than 0.5 second. The upper bound of the convergence time in Theorem 3.1 is conservative because the conditions used in the proofs of Statements (ii) and (iii) to show that $\|r_i(t)-r_j(t)\|<R$ and $\|r_i(t)-r_j(t)\|=0$ are conservative.

V. Conclusion

In this technical note, we studied finite-time consensus of networked nonlinear agents with unknown Lipschitz terms under communication constraints. A nonlinear distributed consensus algorithm was proposed to solve this problem. By designing proper Lyapunov functions, sufficient conditions were derived to guarantee finite-time convergence and an upper bound of the convergence time was provided.

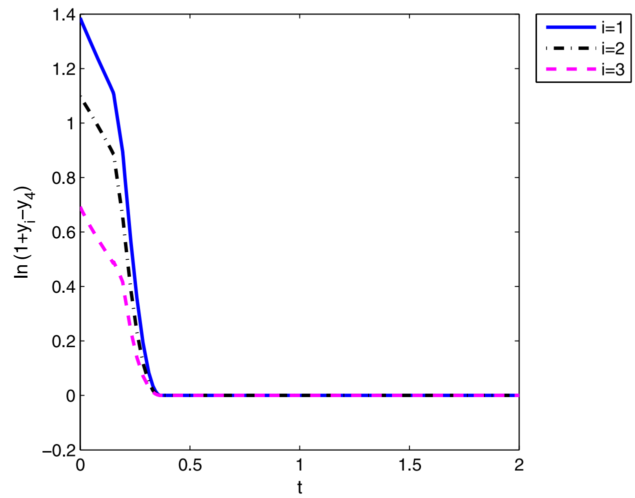

Fig. 3. Plot of $\ln(1+y_i-y_4)$, $i=1,2,3$.

Appendix

A. Proof of Statement (i)

The Lyapunov function candidate we are using here is given by (7). Because the gradient of $V(\tilde r)$ might not exist due to the possible discontinuity of $\beta(s)$, our focus is then to show that the set-valued Lie derivative of $V(t)$, denoted by $\widetilde{L}_F V(\tilde r)$, is nonpositive.[1] Since $V(\tilde r)$ itself is a function of $\tilde r$, it is thus necessary to compute the derivative of $\tilde r$, or equivalently $\tilde r_i$, in order to compute $\widetilde{L}_F V(\tilde r)$. Based on the definition of $\tilde r_i$, the derivative of $\tilde r_i$ is given by

$$\dot{\tilde r}_i=\dot r_i-\dot r_0=u_i+\frac{1}{n}\sum_{j=1}^{n}[f(r_i)-f(r_j)] \tag{10}$$

where u_i is given in (4).

Define $u\triangleq[u_1,\dots,u_n]^T$ and $f(r)\triangleq[f(r_1),\dots,f(r_n)]^T$. When $\tilde r\notin\Omega_V$, i.e., $V(\tilde r)$ is differentiable at $\tilde r$, the gradient of $V(\tilde r)$ with respect to $\tilde r_i$ is given by $(\partial V(\tilde r))/\partial\tilde r_i=\tilde r_i+\eta\sum_{j=1}^n\beta(\|\tilde r_i-\tilde r_j\|)(\|\tilde r_i-\tilde r_j\|+\|\tilde r_i-\tilde r_j\|^\alpha)(d/d\tilde r_i)\|\tilde r_i-\tilde r_j\|=\tilde r_i+\eta\sum_{j=1}^n\beta(\|\tilde r_i-\tilde r_j\|)(\|\tilde r_i-\tilde r_j\|+\|\tilde r_i-\tilde r_j\|^\alpha)\mathrm{sgn}(\tilde r_i-\tilde r_j)\ =\tilde r_i+\eta\sum_{j=1}^n\beta(\|\tilde r_i-\tilde r_j\|)[(\tilde r_i-\tilde r_j)+\mathrm{sig}(\tilde r_i-\tilde r_j)^\alpha]$, where Lemma 2.4 was used to derive the second equality. Let $\partial_i V(\tilde r)$ denote the ith component of $\partial V(\tilde r)$. According to Definition 2.2, the set-valued Lie derivative of $V(\tilde r)$ is given by $\widetilde{L}_F V(\tilde r)=\bigcap_{\xi\in\partial V(\tilde r)}\xi^T K[\dot{\tilde r}]=\sum_{i=1}^n\bigcap_{\xi_i\in\partial_i V(\tilde r)}\xi_i^T K[\dot{\tilde r}_i]$. For any $a\in\widetilde{L}_F V(\tilde r)$, it follows from Definition 2.2 that there exists $\psi_i\in K[\dot{\tilde r}_i]$ such that $a=\sum_{i=1}^n\xi_i^T\psi_i$ for all $\xi_i\in\partial_i V(\tilde r)$, $i=1,\dots,n$. Since $r_i, i=1,\dots,n$, and $f(r_i)$ are continuous, for any $\psi_i\in K[\dot{\tilde r}_i]$, there always exist $\hat\beta_{ij}=\hat\beta_{ji}\in\mathrm{Co}\{\lim\beta(\|\hat r_{\ell_1}-\hat r_{\ell_2}\|)\,|\,\hat r_{\ell_1}\to\tilde r_i,\hat r_{\ell_2}\to\tilde r_j\}$ such that $\psi_i=\sum_{j=1}^n\hat\beta_{ij}[(\tilde r_j-\tilde r_i)+\mathrm{sig}(\tilde r_j-\tilde r_i)^\alpha]\ +(1/n)\sum_{j=1}^n[f(r_i)-f(r_j)]$. Since $a=\sum_{i=1}^n\xi_i^T\psi_i$ for all

[1] $\widetilde{L}_F V(\tilde r)$ becomes $\dot V(\tilde r)$ if $V(\tilde r)$ is differentiable.

$\xi_i \in \partial_i V(\tilde{r}), i = 1, \ldots, n$, let's choose $\xi_i = \tilde{r}_i + \eta \sum_{j=1}^n \hat{\beta}_{ij}[(\tilde{r}_i - \tilde{r}_j) + \text{sig}(\tilde{r}_i - \tilde{r}_j)^\alpha] \in \partial_i V(\tilde{r})$. Therefore

$$\begin{aligned}
a =& \sum_{i=1}^n \tilde{r}_i^T \psi_i + \eta \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} \left[(\tilde{r}_i - \tilde{r}_j) + \text{sig}(\tilde{r}_i - \tilde{r}_j)^\alpha\right]^T \psi_i \\
=& \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} (\tilde{r}_i - \tilde{r}_j)^T \left[(\tilde{r}_j - \tilde{r}_i) + \text{sig}(\tilde{r}_j - \tilde{r}_i)^\alpha\right] \\
&+ \frac{1}{2n} \sum_{i=1}^n \sum_{j=1}^n (\tilde{r}_i - \tilde{r}_j)^T \left[f(r_i) - f(r_j)\right] \\
&+ \frac{\eta}{n} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \hat{\beta}_{ij} \left[(\tilde{r}_j - \tilde{r}_i) + \text{sig}(\tilde{r}_j - \tilde{r}_i)^\alpha\right]^T f(r_i) \\
&- \frac{\eta}{n} \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} \left[(\tilde{r}_j - \tilde{r}_i) + \text{sig}(\tilde{r}_j - \tilde{r}_i)^\alpha\right]^T \sum_{k=1}^n f(r_k) \\
\leq& \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} (\tilde{r}_i - \tilde{r}_j)^T \left[(\tilde{r}_j - \tilde{r}_i) + \text{sig}(\tilde{r}_j - \tilde{r}_i)^\alpha\right] \\
&+ \frac{\gamma}{2n} \sum_{i=1}^n \sum_{j=1}^n \|\tilde{r}_i - \tilde{r}_j\|^2 \\
&+ \frac{\eta}{2} \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} \left[(\tilde{r}_j - \tilde{r}_i) + \text{sig}(\tilde{r}_j - \tilde{r}_i)^\alpha\right]^T \left[f(r_i) - f(r_j)\right] \\
\leq& -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} \left(\|\tilde{r}_j - \tilde{r}_i\|^2 + \|\tilde{r}_j - \tilde{r}_i\|^{1+\alpha}\right) \\
&+ \frac{\gamma}{2n} \sum_{i=1}^n \sum_{j=1}^n \|\tilde{r}_i - \tilde{r}_j\|^2 \frac{\eta\gamma}{2} \sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij} \left(\|\tilde{r}_j - \tilde{r}_i\|^2 + \|\tilde{r}_j - \tilde{r}_i\|^{1+\alpha}\right)
\end{aligned}$$

where we used Lemma 2.3, (2), and the fact that $\sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij}(\tilde{r}_j - \tilde{r}_i) + \text{sig}(\tilde{r}_j - \tilde{r}_i)^\alpha = 0$ to derive the last two inequalities. Define $\Gamma \triangleq I_n - (1/n)\mathbf{1}_n\mathbf{1}_n^T$ and $Q = [q_{ij}]^{n \times n}$, where $q_{ij} = -\hat{\beta}_{ij}$ if $i \neq j$ and $q_{ii} = \sum_{j=1, j\neq i} \hat{\beta}_{ij}$. Then a can be written as $a \leq -(1/2)(1-\gamma\eta)\sum_{i=1}^n \sum_{j=1}^n \hat{\beta}_{ij}\|\tilde{r}_i - \tilde{r}_j\|^{\alpha+1} - (1/2)\tilde{r}^T\{[(1-\gamma\eta)Q - \gamma\Gamma] \otimes I_m\}\tilde{r}$.

Apparently, $a \leq 0$ if $(1-\gamma\eta)Q - \gamma\Gamma$ is symmetric positive semi-definite. Note that $\text{Co}\{\lim \beta(\|\hat{r}_i - \hat{r}_j\|) | \hat{r}_i \to \tilde{r}_i, \hat{r}_j \to \tilde{r}_j\} \in \{\{0\}, \{\beta(\|\tilde{r}_i - \tilde{r}_j\|)\}, [0, \beta(\|\tilde{r}_i - \tilde{r}_j\|)]\}$ and $\beta(\|\tilde{r}_i - \tilde{r}_j\|) \geq 0$. When $\mathcal{G}(t)$ is connected at $t = 0$ and preserved afterwards, by letting Q play the role of $\mathcal{L}(t)$ in Lemma 3.1, it follows from the Statement (i) in Lemma 3.1 that $Q - \varpi C$ is symmetric positive semi-definite. It then follows that $a \leq 0$ if $(1-\gamma\eta)\varpi C - \gamma\Gamma$ is symmetric positive semi-definite. Note that both C and Γ are symmetric Laplacian matrices and $\lambda_{\max}(\Gamma) = 1$. If C has only one eigenvalue equal to 0 and the smallest nonzero eigenvalue, denoted as $\lambda_2(C)$, satisfies that $\lambda_2(C) \geq \gamma/((1-\gamma\eta)\varpi)$, then $(1-\gamma\eta)\varpi C - \gamma\Gamma$ is symmetric positive semi-definite. This in turn implies that $\max \widetilde{L}_F V(\tilde{r}) \leq 0$ if $\lambda_2(C) \geq \gamma/((1-\gamma\eta)\varpi)$ and $\mathcal{G}(t)$ is connected at $t = 0$ and preserved afterwards. In other words, $V(\tilde{r})$ is a nonincreasing function if $\lambda_2(C) \geq \gamma/((1-\gamma\eta)\varpi)$ and $\mathcal{G}(t)$ is connected at $t = 0$ and preserved afterwards. When $\|r_i(0) - r_j(0)\| < R$ and $\|r_i(t) - r_j(t)\| \to R$ for some i, j, then the second term of the right-hand side of (7) is lower bounded by $(\eta/2)\lim_{\|r_i(t)-r_j(t)\|\to R} \int_0^{\|r_i - r_j\|} ((\varpi R^2)/(R^2 - s^2))(s + s^\alpha)ds > ((\eta\varpi R^2)/2)\lim_{\|r_i(t)-r_j(t)\|\to R} \int_0^{\|r_i - r_j\|} (s/(R^2 - s^2))ds = -(\eta\varpi R^2)/4 \ln(R^2 - s^2)|_0^R = \infty$. This means that if $\|r_i(0) - r_j(0)\| < R$, $\|r_i(t) - r_j(t)\| < R$ will always be true if $\lambda_2(C) \geq \gamma/((1-\gamma\eta)\varpi)$ because otherwise $V(\tilde{r}(t)) > V(\tilde{r}(0))$, which contradicts the fact that $V(\tilde{r})$ is a nonincreasing function before any initial interaction pattern is to be broken when $\lambda_2(C) \geq \gamma/((1-\gamma\eta)\varpi)$ and $\mathcal{G}(0)$ is connected. That is, the initial interaction graph will be preserved afterwards if (9) holds.

B. Proof of Statement (ii)

Based on the definition of $\tilde{r}_i$, one can verify that $\max_i \|\tilde{r}_i\| \geq \max_{i,j}(1/2)\|r_i - r_j\|$. Then $\overline{V}(\tilde{r})$, defined in (8), satisfies that $\overline{V}(\tilde{r}) \geq (1/8)\max_{i,j}\|\tilde{r}_i - \tilde{r}_j\|^2 = (1/8)\max_{i,j}\|r_i - r_j\|^2$. Then $\max_{i,j}\|r_i - r_j\| \leq R$ if $\overline{V}(\tilde{r}) \leq (1/8)R^2$. To prove the Statement (ii), it suffices to show that $\overline{V}(\tilde{r}) < (1/8)R^2$ for $t > (\ln \overline{V}(\tilde{r}(0)) + 3\ln 2 - 2\ln R)/(\varpi\lambda_2(C) - \gamma)$.

Let $\tilde{r}_{(k)} \in \mathbb{R}^n$ denote the vector consisting of the kth component of each $\tilde{r}_i$. Decompose $\tilde{r}_{(k)}$ in such a way that $\tilde{r}_{(k)} = \tilde{r}_{(k)}^{\|} + \tilde{r}_{(k)}^{\perp}$, where $\tilde{r}_{(k)}^{\|}$ denotes the projection of $\tilde{r}_{(k)}$ onto $\mathbf{1}_n$ while $\tilde{r}_{(k)}^{\perp}$ denotes the projection of $\tilde{r}_{(k)}$ onto the space that is perpendicular to $\mathbf{1}_n$. Based on the definition of projection, $\tilde{r}_{(k)}^{\|} = (\tilde{r}_{(k)}\mathbf{1}_n)/\|\mathbf{1}_n\|\mathbf{1}_n = \mathbf{0}_n$. That is, $\tilde{r}_{(k)} = \tilde{r}_{(k)}^{\perp}$. Then $\overline{V}(\tilde{r})$ can be further written as

$$\overline{V}(\tilde{r}) = \frac{1}{2}\sum_{k=1}^m \tilde{r}_{(k)}^T \tilde{r}_{(k)} = \frac{1}{2}\sum_{k=1}^m \left(\tilde{r}_{(k)}^{\perp}\right)^T \tilde{r}_{(k)}^{\perp}. \tag{11}$$

By following a similar analysis to that in the proof of Statement (i), it can be obtained that:

$$\begin{aligned}
\max \widetilde{L}_F \overline{V}(\tilde{r}) \leq& -\frac{1}{2}\tilde{r}^T \left[(\varpi C - \gamma\Gamma) \otimes I_m\right]\tilde{r} \\
=& -\frac{1}{2}\sum_{k=1}^m \tilde{r}_{(k)}^T (\varpi C - \gamma\Gamma)\tilde{r}_{(k)} \\
\leq& -\frac{1}{2}\left[\varpi\lambda_2(C) - \gamma\right]\sum_{k=1}^m \left(\tilde{r}_{(k)}^{\perp}\right)^T \tilde{r}_{(k)}^{\perp}
\end{aligned} \tag{12}$$

where we used the facts that C is a symmetric Laplacian matrix with only one eigenvalue equal to zero, all eigenvalues of Laplacian matrix Γ are not larger than one, and the smallest nonzero eigenvalue of C is $\lambda_2(C)$ to derive the last inequality. Combining (11) and (12), it follows that:

$$\max \widetilde{L}_F \overline{V}(\tilde{r}) \leq -\left[\varpi\lambda_2(C) - \gamma\right]\overline{V}(\tilde{r}). \tag{13}$$

Write $\overline{V}(\tilde{r}(t))$ as $\overline{V}(t)$ for simplicity. Although $\dot{\overline{V}}(t)$ is discontinuous at some time instant, it is Riemann integrable because $\overline{V}(t) \in [0, \overline{V}(0)]$ is bounded and the set of the points of discontinuity has measure zero [23]. Based on (13), we have $\overline{V}(t+h) - \overline{V}(t) = \int_t^{t+h} \dot{\overline{V}}(\sigma)d\sigma \leq -h[\varpi\lambda_2(C) - \gamma]\min_{\tau\in[t,t+h]}\overline{V}(\tau)$. It then follows from Definition 2.3 that $D^+\overline{V}(t) \leq -[\varpi\lambda_2(C) - \gamma]\overline{V}(t)$. By Lemma 2.2, $\overline{V}(t)$ is upper bounded by $s(t)$ satisfying $\dot{s} = -[\varpi\lambda_2(C) - \gamma]s, s(0) = \overline{V}(\tilde{r}(0))$. By computation, one has that $s(t) = e^{-t[\varpi\lambda_2(C)-\gamma]}\overline{V}(\tilde{r}(0))$. Therefore, $\overline{V}(t) \leq e^{-t[\varpi\lambda_2(C)-\gamma]}\overline{V}(\tilde{r}(0))$. When $t > (\ln \overline{V}(\tilde{r}(0)) + 3\ln 2 - 2\ln R)/(\varpi\lambda_2(C) - \gamma)$, it can be computed that $\overline{V}(t) \leq e^{-t[\varpi\lambda_2(C)-\gamma]}\overline{V}(\tilde{r}(0)) < e^{2\ln R - \ln \overline{V}(\tilde{r}(0)) - 3\ln 2}\overline{V}(\tilde{r}(0)) = (1/8)R^2$. This completes the proof of the second statement.

C. Proof of Statement (iii)

We finally prove the Statement (iii). From the Statement (ii), it is known that $\|r_i(t) - r_j(t)\| < R, i, j = 1, \ldots, n$, for any $t > T$, where $T \triangleq (\ln \overline{V}(\tilde{r}(0)) + 3\ln 2 - 2\ln R)/(\varpi\lambda_2(C) - \gamma)$. That is, the undirected graph for the n agents is fully connected when $t > T$. Consider again the Lyapunov function candidate $\overline{V}(t)$ given in (8). Based on the computation of $\widetilde{L}_F V(\tilde{r})$ in the proof of Statement (i), we can obtain that $\dot{\overline{V}}(\tilde{r}) \leq -(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}\varpi[\|\tilde{r}_i-\tilde{r}_j\|^2 + \|\tilde{r}_i-\tilde{r}_j\|^{1+\alpha}] + (1/(2n))\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma\|\tilde{r}_i-\tilde{r}_j\|^2$, where we used the fact that $\beta(\|\tilde{r}_i-\tilde{r}_j\|) \geq \varpi$, $\forall\, i \neq j$, for a fully connected graph. When (9) is satisfied, $\varpi \geq \gamma/(\lambda_2(C))$. By letting $nI_n - \mathbf{1}_n\mathbf{1}_n^T$ play the role of $\mathcal{L}(t)$ in Lemma 3.1, it follows from the Statement (ii) of Lemma 3.1 that $\lambda_2(C) \leq \lambda_2(nI_n - \mathbf{1}_n\mathbf{1}_n^T)$. Noting that $\lambda_2(nI_n - \mathbf{1}_n\mathbf{1}_n^T) \leq \lambda_{\max}(nI_n - \mathbf{1}_n\mathbf{1}_n^T) = n$, it then follows that $\varpi \geq \gamma/n$. Consequently, $-(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}\varpi\|\tilde{r}_i-\tilde{r}_j\|^2 + (1/(2n))\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma\|\tilde{r}_i-\tilde{r}_j\|^2 \leq 0$. This implies that $\dot{\overline{V}}(\tilde{r}) \leq -(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}\varpi\|r_i-r_j\|^{1+\alpha} \leq -(1/2)\varpi\max_{i,j}\|r_i-r_j\|^{1+\alpha}$, where we used Statement (ii) to derive the two inequalities. Because $\overline{V}(\tilde{r}) = \left(\sum_{i=1}^{n}\|nr_i - \sum_{j=1}^{n}r_j\|^2\right)/(2n^2) \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\|r_i - r_j\|^2\right)/(2n^2) \leq (n(n-1)\max_{i,j}\|r_i-r_j\|^2)/(2n^2) = ((n-1)\max_{i,j}\|r_i-r_j\|^2)/(2n)$, one can obtain that $\max_{i,j}\|r_i-r_j\|^{1+\alpha} \geq (2n)/(n-1)^{(1+\alpha)/2}\,[\overline{V}(\tilde{r})]^{(1+\alpha)/2}$. Therefore, we have that

$$\dot{\overline{V}}(\tilde{r}) \leq -\kappa\left[\overline{V}(\tilde{r})\right]^{\frac{1+\alpha}{2}} \tag{14}$$

where $\kappa = (\varpi/2)((2n)/(n-1))^{(1+\alpha)/2}$. Because $\alpha \in (0,1)$, it follows that $((1+\alpha)/2) \in (0,1)$. Write $\overline{V}(\tilde{r})$ as $\overline{V}(t)$. It then follows from (14) that $\dot{\overline{V}}(t)[\overline{V}(t)]^{-((1+\alpha)/2)} \leq -\kappa$. By computation, we then obtain that $((1-\alpha)/2)[\overline{V}(t)]^{(1-\alpha)/2} \leq ((1-\alpha)/2)\ [\widetilde{V}(T)]^{(1-\alpha)/2} - \kappa(t-T)$ for $t > T$. Because $\overline{V}(t) \geq 0$, when $t > T + (2/(\kappa(1-\alpha)))[\overline{V}(T)]^{(1-\alpha)/2}$, $\overline{V}(t) = 0$ must hold, which implies that $\tilde{r}(t) = 0$ must hold as well. That is, $\|r_i(t) - r_j(t)\| = 0$ for any $t > T + (2/(\kappa(1-\alpha)))[\overline{V}(T)]^{(1-\alpha)/2}$. Combining with the fact $\overline{V}(T) \leq ((n-1)/(2n))R^2$ completes the proof of this statement.

References

[1] Y. Cao, W. Ren, D. W. Casbeer, and C. Schumacher, "Finite-time consensus of networked lipschitz nonlinear agents under communication constraints," in *Proc. American Control Conf.*, Washington, DC, USA, Jun. 2013, pp. 1326–1331.

[2] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[3] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Inform.*, vol. 9, no. 1, pp. 427–438, 2013.

[4] M. Ji and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," *IEEE Trans. Robot.*, vol. 23, no. 4, pp. 693–703, Aug. 2007.

[5] D. V. Dimarogonas and K. H. Johansson, "Decentralized connectivity maintenance in mobile networks with bounded inputs," in *Proc. IEEE Int. Conf. Robot. Autom.*, Pasadena, CA, USA, May 2008, pp. 1507–1512.

[6] H. Su, X. Wang, and G. Chen, "Rendezvous of multiple mobile agents with preserved network connectivity," *Syst. Control Lett.*, vol. 59, no. 5, pp. 313–322, May 2010.

[7] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlin. Control*, vol. 17, no. 11, pp. 1002–1033, Jul. 2007.

[8] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multi-agent systems with directed topologies and nonlinear dynamics," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 3, pp. 881–891, 2010.

[9] H. Su, G. Chen, X. Wang, and Z. Lin, "Adaptive second-order consensus of networked mobile agents with nonlinear dynamics," *Automatica*, vol. 47, no. 2, pp. 368–375, 2011.

[10] W. Yu, G. Chen, and M. Cao, "Consensus in directed networks of agents with nonlinear dynamics," *IEEE Trans. Autom. Control*, vol. 56, no. 6, pp. 1436–1441, Jun. 2011.

[11] J. Cortes, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[12] Q. Hui, W. M. Haddad, and S. P. Bhat, "Finite-time semistability and consensus for nonlinear dynamical networks," *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1887–1890, Sep. 2008.

[13] Q. Hui, "Finite-time rendezvous algorithms for mobile autonomous agents," *IEEE Trans. Autom. Control*, vol. 56, no. 1, pp. 207–211, Jan. 2011.

[14] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[15] X. Wang and Y. Hong, "Finite-time consensus for multi-agent networks with second-order agent dynamics," in *Proc. IFAC World Congr.*, Seoul, Korea, Jul. 2008, pp. 15 185–15 190.

[16] B. E. Paden and S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Autom. Control*, vol. AC-34, no. 1, pp. 73–82, 1987.

[17] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Autom. Control*, vol. 39, no. 9, pp. 1910–1914, Sep. 1994.

[18] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[19] A. F. Filippov, *Differential Equations With Discontinuous Righthand Sides*. Norwell, MA, USA: Kluwer, 1988.

[20] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*, ser. Communications and Control Engineering, London, U.K.: Springer-Verlag, 2008.

[21] S. Yu, X. Yu, B. Shirinzadeh, and Z. Man, "Continuous finite-time control for robotic manipulators with terminal sliding mode," *Automatica*, vol. 41, no. 11, pp. 1957–1964, 2005.

[22] A. Loria, E. Panteley, D. Popovic, and A. R. Teel, "A nested matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 183–198, Feb. 2005.

[23] T. Apostol, *Mathematical Analysis*. Reading, MA, USA: Addison-Wesley, 1974.