Optimal Strategies of the Differential Game in a Circular Region

Eloy Garcia, David W. Casbeer, and Meir Pachter

***Abstract*—A two-player differential game is considered where a spy infiltrates the region of interest and a defender pursues and tries to capture the spy before it leaves such region. This letter provides the complete solution of this game by characterizing the Barrier surface which separates the state space into the regions of win of the defender and the spy. In each region the optimal strategies that guarantee the prescribed outcome are provided. This is in contrast to recent results addressing the same problem where the proposed strategies do not guarantee capture of the spy by the defender when initially this outcome was prescribed by the game of kind solution.**

***Index Terms*—Aerospace, autonomous systems, game theory.**

I. Introduction

DIFFERENTIAL game theory provides the right framework to analyze pursuit-evasion problems and combat games. Pursuit-evasion scenarios are representative of many important but challenging problems in aerospace, control, and robotics. Such interesting pursuit-evasion problems were formulated in the seminal work by Isaacs [1].

The problem discussed in this letter is related to the classical Lion and Man game in which a pursuer/lion is pitted against an evader/man where both players have the same speed and the game set is the interior of a circular region [2]–[6]. A novel interesting version of the Lion and Man game was presented by Yan *et al.* [7] where the evader wins by escaping from the region guarded by the pursuer; however, the pursuer is faster than the evader in this case. Reachability analysis can then be used to determine the winning regions of each player [8]–[10]. Similar reachability games where an evader strives to reach a goal set, while the pursuer tries to intercept the evader before reaching safe haven, include the capture the flag game [11]–[13], guarding a line [14], [15] and assisting and rescuing teammates [16]–[22].

This letter revisits the Differential Game in a Circular Region (DGCR) discussed in [7] where a Defender (D) protects the interior of the circle, the game set, and tries to capture a Spy (S) before it leaves the game set. The Spy tries to escape by reaching the boundary of the game set where the defender is ineffective. Two outcomes are possible: D wins if it captures S before the latter reaches the boundary and S wins if it is able to reach the boundary before being captured. These two outcomes give rise to the game of kind and its solution partitions the state space into two regions of win: $\mathcal{R}_c$, the capture region and $\mathcal{R}_e$, the escape region. Different Games of Degree are played in each of the two regions.

The game of kind and both games of degree are equally important and intrinsically connected. The game of kind solution is necessary for the players to determine which game of degree to play and the solutions to the games of degree need to sustain the Barrier surface obtained by solving the game of kind. The latter means, in technical terms, that optimal strategies should result in a semipermeable Barrier surface. This letter solves the complete game of defense of a circular region: the solution to the game of kind and the solutions to both games of degree are obtained in the sequel. Reference [7] recently addressed this problem; however, the proposed strategies do not return a semipermeable barrier surface. This is fundamental since the strategies designed in that reference do not guarantee that the Defender wins the game when the state of the system is in $\mathcal{R}_c$, where it is supposed to win.

The contributions of this letter are as follows: we present the correct formulation and solution of both Games of Degree which result in a semipermeable barrier surface; this was not the case in [7]. Also, we provide a more concise and general characterization of the Barrier surface than [7]. Finally, we identify the dispersal surface, where more than one optimal solution exists, which was not identified in [7] and where the Value function is not continuously differentiable (or C^1). Of great importance is the use of Isaacs' method, in particular, the Hamilton-Jacobi-Isaacs (HJI) equation, to solve both Games of Degree. Being able to carry through Isaacs' method is the ideal situation [8], if it is indeed attainable. We show that this differential game fits into Isaacs' paradigm, as opposed to the claim in [7]: we obtain the analytical solution of the DGCR, that is, we obtain the C^1 (except at the dispersal surface) Value function which is the solution of the HJI equation.

This letter is organized as follows. Section II introduces the Differential Game in a Circular Region (DGCR). The semipermeable Barrier surface is obtained in Section III. The capture and escape Games of Degree are respectively solved in Section IV and Section V. Illustrative examples are shown in Section VI and conclusions are drawn in Section VII.

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II. The Differential Game in a Circular Region

The DGCR is played by the Defender D and the Spy S; the game set Ω is a compact and convex set in the Cartesian plane. Similar to [7] we consider the game set Ω to be a disk of radius R and the speeds of D and S are constant; they are given by v_D and v_S, respectively. Without loss of generality we assume that the center of Ω is located at the origin. Thus, $\Omega := \{x \in \mathbb{R}, y \in \mathbb{R} | x^2 + y^2 \leq R^2\}$. In this scenario D pursues and tries to capture S before the latter reaches $\partial\Omega$, the boundary of Ω, which is considered the safe zone of S.

The states of D and S are respectively specified by their Cartesian coordinates $\mathbf{x}_D = (x_D, y_D)$, and $\mathbf{x}_S = (x_S, y_S)$. The complete state of the differential game is defined by $\mathbf{x} := (x_D, y_D, x_S, y_S) \in \mathbb{R}^4$. D is faster than S, which makes the game more tractable than the classical lion and man game, and the speed ratio parameter is $\alpha = v_S/v_D < 1$. The controls of D and S are the instantaneous heading angles, $\mathbf{u}_D = \{\psi\}$ and $\mathbf{u}_S = \{\phi\}$. The dynamics/kinematics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_D, \mathbf{u}_S)$ are specified by the system of ordinary differential equations

$$\begin{aligned} \dot{x}_D &= \cos\psi, \quad x_D(0) = x_{D_0} \\ \dot{y}_D &= \sin\psi, \quad y_D(0) = y_{D_0} \\ \dot{x}_S &= \alpha\cos\phi, \quad x_S(0) = x_{S_0} \\ \dot{y}_S &= \alpha\sin\phi, \quad y_S(0) = y_{S_0} \end{aligned} \tag{1}$$

where, without loss of generality, $v_D = 1$ and where the admissible controls are given by $\psi, \phi \in [-\pi, \pi)$. We consider non-anticipative strategies; each player has no knowledge of the instantaneous control input of the opponent. The initial state of the system is $\mathbf{x}_0 := (x_{D_0}, y_{D_0}, x_{D_0}, y_{D_0}) = \mathbf{x}(t_0)$, where it is assumed that the initial positions satisfy $x_{S_0}^2 + y_{S_0}^2 < R^2$ and $x_{D_0}^2 + y_{D_0}^2 < R^2$. The DGCR is a two-termination set differential game [23], [24]. One terminal condition is capture of S by D. The alternative is successful escape of S by reaching $\partial\Omega$ before being captured by D. Hence, the termination set of the DGCR is

$$\mathcal{T} := \mathcal{T}_c \bigcup \mathcal{T}_e \tag{2}$$

where

$$\mathcal{T}_c := \left\{\mathbf{x} | (x_S - x_D)^2 + (y_S - y_D)^2 = 0\right\} \tag{3}$$

represents the outcome where D captures S and

$$\mathcal{T}_e := \left\{\mathbf{x} \ | x_S^2 + y_S^2 = R^2\right\} \tag{4}$$

represents escape of S by reaching $\partial\Omega$.

The terminal time t_f is defined as the time instant when the state of the system satisfies (2), at which time the terminal state is $\mathbf{x}_f := (x_{D_f}, y_{D_f}, x_{S_f}, y_{S_f}) = \mathbf{x}(t_f)$.

Due to the two different outcomes specified in (2), the game of kind needs to be solved in order to partition the state space into two winning regions, one for each player. Since different games of degree are played in each region, it is essential for each player to determine which region the current state of the system lies in, and then the corresponding game of degree is solved in order to determine the appropriate optimal strategies. The state space $\mathbb{R}^4$ is partitioned into two sets: $\mathcal{R}_e$ and $\mathcal{R}_c$ which are defined as follows

$$\begin{aligned} \mathcal{R}_c &:= \{\mathbf{x} \mid B(\mathbf{x}; \alpha) > 0\} \\ \mathcal{R}_e &:= \{\mathbf{x} \mid B(\mathbf{x}; \alpha) < 0\}. \end{aligned} \tag{5}$$

The Barrier surface is defined as

$$\mathcal{B} := \{\mathbf{x} \mid B(\mathbf{x}; \alpha) = 0\} \tag{6}$$

where the Barrier function $B(\mathbf{x}; \alpha)$ is explicitly obtained in Section III. If $\mathbf{x} \in \mathcal{R}_c$ then the capture game of degree is played where the terminal performance functional is

$$J(\mathbf{u}_D(t), \mathbf{u}_S(t), \mathbf{x}_0) = \Phi_c(\mathbf{x}(t_f)) \tag{7}$$

where $\Phi_c(\mathbf{x}(t_f)) := x_{S_f}^2 + y_{S_f}^2$. The Value of the game is

$$V(\mathbf{x}_0) := \min_{\mathbf{u}_D(\cdot)} \max_{\mathbf{u}_S(\cdot)} J(\mathbf{u}_D(\cdot), \mathbf{u}_S(\cdot); \mathbf{x}_0) \tag{8}$$

subject to (1) and (3), where $\mathbf{u}_D(\cdot)$ and $\mathbf{u}_S(\cdot)$ are the teams' state feedback strategies. D aims at capturing S and maximize the terminal distance from the capture point to the boundary of Ω. S, knowing that it will be captured, strives to end up as close as possible to the safe zone $\partial\Omega$. Since Ω is a circular region with center at the origin, *S minimizes its distance with respect to $\partial\Omega$ by maximizing its distance with respect to the origin*. The analogous reasoning applies to D. This formulation provides a practical strategy: if D does not play optimally, S can further decrease its terminal distance with respect to $\partial\Omega$, possibly reach the safe zone, and win the game.

If $\mathbf{x} \in \mathcal{R}_e$ then the escape game of degree is played where the terminal performance functional is

$$J(\mathbf{u}_D(t), \mathbf{u}_S(t), \mathbf{x}_0) = \Phi_e(\mathbf{x}(t_f)) \tag{9}$$

where $\Phi_e(\mathbf{x}(t_f)) := \sqrt{(x_{S_f} - x_{D_f})^2 + (y_{S_f} - y_{D_f})^2}$. The Value of the game is given by

$$V(\mathbf{x}_0) := \min_{\mathbf{u}_D(\cdot)} \max_{\mathbf{u}_S(\cdot)} J(\mathbf{u}_D(\cdot), \mathbf{u}_S(\cdot); \mathbf{x}_0) \tag{10}$$

subject to (1) and (4), where $\mathbf{u}_D(\cdot)$ and $\mathbf{u}_S(\cdot)$ are the teams' state feedback strategies. In this case D, unable to capture S, strives to be as close as possible to it at the terminal time. S tries to escape while maximizing the terminal separation with respect to D. Similarly, this strategy would help D in case S does not play optimally by decreasing their terminal separation and possibly allowing capture of S.

Theorem 1: Consider the DGCR (1), (7), (9). The regular optimal headings of D and S are constant under optimal play and their trajectories are straight lines.

Proof: Consider (1) and (7) (the escape game of degree follows in the same way). The optimal control inputs (in terms of the co-state variables) can be immediately obtained from $\min_\psi \max_\phi \mathcal{H}$, where the Hamiltonian is

$$\mathcal{H} = \lambda_{x_D}\cos\psi + \lambda_{y_D}\sin\psi + \alpha\lambda_{x_S}\cos\phi + \alpha\lambda_{y_S}\sin\phi$$

and the co-state is $\lambda^T = (\lambda_{x_D}, \lambda_{y_D}, \lambda_{x_S}, \lambda_{y_S}) \in \mathbb{R}^4$.

Additionally, the co-state dynamics are: $\dot{\lambda}_{x_D} = \dot{\lambda}_{y_D} = \dot{\lambda}_{x_S} = \dot{\lambda}_{y_S} = 0$; hence, all co-states are constant and the optimal headings are constant as well. Consequently, the primary optimal trajectories are straight lines. ■

Note that in the Hamiltonian, the control inputs of the players are separable and Isaacs' condition holds, that is, $\min_\psi \max_\phi \mathcal{H} = \max_\phi \min_\psi \mathcal{H}$.

III. Game of Kind

The solution of the game of kind answers the question of which player wins the game under optimal play. We first solve the game of kind; we provide a closed form solution, that is, the Barrier surface function $B : \mathbb{R}^4 \to \mathbb{R}$ explicitly in terms of the state.

Theorem 2: For a given speed ratio parameter $0 < \alpha < 1$, the Barrier surface that separates the state space $\mathbb{R}^4$ into the two regions $\mathcal{R}_c$ and $\mathcal{R}_e$ is given by $B(\mathbf{x}; \alpha) = 0$, where

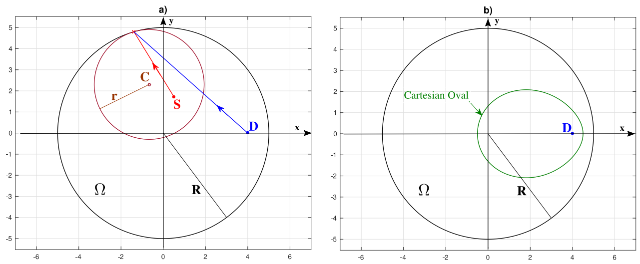

Fig. 1. DGCR with $\alpha = 0.5$. a) D-S Apollonius circle is tangent to $\partial\Omega$. b) Cross-section of Barrier Surface.

$$B(\mathbf{x}; \alpha) = R - \frac{\alpha}{1-\alpha^2}\sqrt{(x_S - x_D)^2 + (y_S - y_D)^2} - \frac{1}{1-\alpha^2}\sqrt{(x_S - \alpha^2 x_D)^2 + (y_S - \alpha^2 y_D)^2}. \quad (11)$$

Also, for fixed position of D the cross-section of $B(\mathbf{x}; \alpha)$ is a Cartesian Oval.

Proof: Since D is faster than S, their dominance region in the Cartesian plane is determined by an Apollonius circle.

The $D - S$ Apollonius circle is

$$(x - x_c)^2 + (y - y_c)^2 = r^2 \quad (12)$$

where

$$x_c = \frac{1}{1-\alpha^2}(x_S - \alpha^2 x_D), \quad y_c = \frac{1}{1-\alpha^2}(y_S - \alpha^2 y_D)$$
$$r = \frac{\alpha}{1-\alpha^2}\sqrt{(x_S - x_D)^2 + (y_S - y_D)^2}. \quad (13)$$

If the $D - S$ Apollonius circle does not intersect $\partial\Omega$ then D is able to capture S before the latter can escape. The limiting case is obtained when the Apollonius circle is tangent to $\partial\Omega$, as it is shown in Fig. 1.a. In such a case S can reach $\partial\Omega$ at the time instant when it is captured by D. When the Apollonius circle is tangent to $\partial\Omega$ it holds that

$$R - r - \sqrt{x_c^2 + y_c^2} = 0. \quad (14)$$

Substituting the variables in (13) into the left-hand-side of (14) we obtain $B(\mathbf{x}; \alpha) = 0$ where the Barrier function is explicitly given in terms of the state in (11). Also, if $B(\mathbf{x}; \alpha) > 0$ the Apollonius circle does not intersect $\partial\Omega$ and capture of S is guaranteed under optimal play, i.e., $\mathbf{x} \in \mathcal{R}_c$. On the other hand, if $B(\mathbf{x}; \alpha) < 0$ the Apollonius circle intersects $\partial\Omega$ and escape is guaranteed under optimal play, i.e., $\mathbf{x} \in \mathcal{R}_e$.

Finally, we notice that $\mathcal{B}$ is a surface in the state space $\mathbf{x} \in \mathbb{R}^4$. We can, for instance, fix the position of D anywhere in Ω (as it was done in [7]) and characterize the *cross-section* of the Barrier function. Let $x = x_S$ and $y = y_S$, from (11) we can write $B(\mathbf{x}; \alpha) = 0$ as follows

$$\sqrt{(x - \alpha^2 x_D)^2 + (y - \alpha^2 y_D)^2} + \alpha\sqrt{(x - x_D)^2 + (y - y_D)^2} = (1 - \alpha^2)R. \quad (15)$$

Therefore, for a fixed location of D:(x_D, y_D), the locus of points (x, y) that separate the winning regions of the players form a Cartesian Oval where the foci are (x_D, y_D) and $(\alpha^2 x_D, \alpha^2 y_D)$. The Cartesian Oval parameters are α and $(1 - \alpha^2)R$. An example of the cross-section of the Barrier surface is shown in Fig. 1.b. ■

IV. Capture Game of Degree

The Capture game of degree is played in the region $\mathcal{R}_c$ where there exists an strategy for D to capture S regardless of any control implemented by S. The performance functional is (7). When $\mathbf{x} \in \mathcal{R}_c$ the Apollonius circle is completely included in Ω as it is shown in Fig. 2.a. Player S is able to select its aimpoint in order to minimize its terminal distance with respect to the boundary of the circular region Ω. Thus, the optimal aimpoint is given by the point on the Apollonius circle closest to $\partial\Omega$.

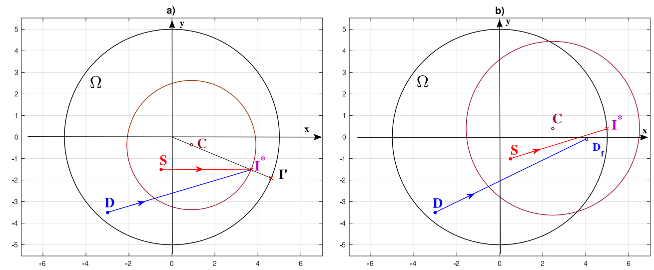

Fig. 2. Game of Degree with $\alpha = 0.6$ and different initial conditions. a) Capture Game. b) Escape Game.

Dispersal Surface: A dispersal surface exists within the capture game which we denote by $\mathcal{D} \in \mathcal{R}_c$ where

$$\mathcal{D} = \{\mathbf{x} | x_S = \alpha^2 x_D, y_S = \alpha^2 y_D\}. \quad (16)$$

Theorem 3: Consider the DGCR where the players are D and S. The speed ratio parameter is $\alpha = v_S/v_D < 1$ and assume that $\mathbf{x} \in \mathcal{R}_c$. The Value function is C^1, except at the dispersal surface $\mathcal{D}$, and it is the solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation. The Value function is given by

$$V(\mathbf{x}) = R - \frac{1}{1-\alpha^2}\Big(\alpha\sqrt{(x_S - x_D)^2 + (y_S - y_D)^2} + \sqrt{(x_S - \alpha^2 x_D)^2 + (y_S - \alpha^2 y_D)^2}\Big) \quad (17)$$

and the optimal state feedback strategies are

$$\cos\phi^* = \frac{x^* - x_S}{\sqrt{(x^* - x_S)^2 + (y^* - y_S)^2}}$$
$$\sin\phi^* = \frac{y^* - y_S}{\sqrt{(x^* - x_S)^2 + (y^* - y_S)^2}}$$
$$\cos\psi^* = \frac{x^* - x_D}{\sqrt{(x^* - x_D)^2 + (y^* - y_D)^2}}$$
$$\sin\psi^* = \frac{y^* - y_D}{\sqrt{(x^* - x_D)^2 + (y^* - y_D)^2}} \quad (18)$$

where

$$x^* = x_c\left(\frac{r}{\sqrt{x_c^2 + y_c^2}} + 1\right), \quad y^* = y_c\left(\frac{r}{\sqrt{x_c^2 + y_c^2}} + 1\right). \quad (19)$$

Proof: The closest point between $\partial\Omega$ and the Apollonius circle can be obtained by the transversality condition where the line joining the closest points between the two circles is perpendicular to the tangent of the circles at those points. This is illustrated in Fig. 2.a where the origin, the center of the Apollonius circle, and the points I^* and I' all lie on the perpendicular line. Hence, the slope of the line is $m = \frac{y_c}{x_c}$ and the coordinates of point I^* are given by (19). The coordinates of point I' are given by

$$x' = x_c\frac{R}{\sqrt{x_c^2 + y_c^2}}, \quad y' = y_c\frac{R}{\sqrt{x_c^2 + y_c^2}}. \quad (20)$$

The Value of the game is obtained by computing the distance between the optimal point I^* and point I'

$$V(\mathbf{x}) = \sqrt{(x' - x^*)^2 + (y' - y^*)^2}$$

$$= \sqrt{(1+m^2)(x'-x^*)^2}$$
$$= \frac{\sqrt{x_c^2+y_c^2}}{x_c}\left(x_c\frac{R}{\sqrt{x_c^2+y_c^2}} - x_c(\frac{r}{\sqrt{x_c^2+y_c^2}}+1)\right)$$
$$= R - r - \sqrt{x_c^2+y_c^2}.$$

The Value function is explicitly given in terms of the state in (17). The partial derivative of $V(\mathbf{x})$ with respect of x_S can be directly obtained

$$\frac{\partial V(\mathbf{x})}{\partial x_S} = -\frac{1}{1-\alpha^2}\left(\alpha\frac{x_s-x_D}{\sqrt{(x_S-x_D)^2+(y_S-y_D)^2}} + \frac{x_s-\alpha^2 x_D}{\sqrt{(x_S-\alpha^2 x_D)^2+(y_S-\alpha^2 y_D)^2}}\right). \quad (21)$$

Define the following $\lambda = \arctan(\frac{y_S-y_D}{x_S-x_D})$ and $\omega = \arctan(\frac{y_S-\alpha^2 y_D}{x_S-\alpha^2 x_D})$. Then, the gradient of $V(\mathbf{x})$ can be written in compact form

$$\frac{\partial V(\mathbf{x})}{\partial x_S} = -\frac{1}{1-\alpha^2}(\alpha\cos\lambda+\cos\omega)$$
$$\frac{\partial V(\mathbf{x})}{\partial y_S} = -\frac{1}{1-\alpha^2}(\alpha\sin\lambda+\sin\omega)$$
$$\frac{\partial V(\mathbf{x})}{\partial x_D} = \frac{\alpha}{1-\alpha^2}(\cos\lambda+\alpha\cos\omega)$$
$$\frac{\partial V(\mathbf{x})}{\partial y_D} = \frac{\alpha}{1-\alpha^2}(\sin\lambda+\alpha\sin\omega) \quad (22)$$

where $\sqrt{(x_S-x_D)^2+(y_S-y_D)^2} \neq 0$ for otherwise the evader has been captured and the game has ended. Therefore $V(\mathbf{x})$ is C^1 outside the dispersal surface $\mathcal{D}$. Let us now obtain

$$x^*-x_S = \alpha\frac{x_S-x_D}{1-\alpha^2}\Big(\frac{\cos\omega}{\cos\lambda}+\alpha\Big)$$
$$y^*-y_S = \alpha\frac{y_S-y_D}{1-\alpha^2}\Big(\frac{\sin\omega}{\sin\lambda}+\alpha\Big)$$
$$x^*-x_D = \frac{x_S-x_D}{1-\alpha^2}\Big(\alpha\frac{\cos\omega}{\cos\lambda}+1\Big)$$
$$y^*-y_D = \frac{y_S-y_D}{1-\alpha^2}\Big(\alpha\frac{\sin\omega}{\sin\lambda}+1\Big). \quad (23)$$

The HJI equation for regular solutions is given by $-\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \mathbf{x}}\cdot\mathbf{f}(\mathbf{x},\phi^*,\psi^*) + g(t,\mathbf{x},\phi^*,\psi^*)$. Note that in this problem $\frac{\partial V}{\partial t}=0$ and $g(t,\mathbf{x},\phi^*,\psi^*)=0$. Therefore, we compute

$$\frac{\partial V}{\partial \mathbf{x}}\cdot\mathbf{f}(\mathbf{x},\phi^*,\psi^*)$$
$$= \alpha\frac{\partial V}{\partial x_E}\cos\phi^* + \alpha\frac{\partial V}{\partial y_E}\sin\phi^* + \frac{\partial V}{\partial x_P}\cos\psi^* + \frac{\partial V}{\partial y_P}\sin\psi^*$$
$$= -\frac{\alpha(\alpha\cos\lambda+\cos\omega)(\frac{\cos\omega}{\cos\lambda}+\alpha)(x_S-x_D)}{(1-\alpha^2)\sqrt{(x_S-x_D)^2(\frac{\cos\omega}{\cos\lambda}+\alpha)^2+(y_S-y_D)^2(\frac{\sin\omega}{\sin\lambda}+\alpha)^2}}$$
$$- \frac{\alpha(\alpha\sin\lambda+\sin\omega)(\frac{\sin\omega}{\sin\lambda}+\alpha)(y_S-y_D)}{(1-\alpha^2)\sqrt{(x_S-x_D)^2(\frac{\cos\omega}{\cos\lambda}+\alpha)^2+(y_S-y_D)^2(\frac{\sin\omega}{\sin\lambda}+\alpha)^2}}$$
$$+ \frac{\alpha(\cos\lambda+\alpha\cos\omega)(\alpha\frac{\cos\omega}{\cos\lambda}+1)(x_S-x_D)}{(1-\alpha^2)\sqrt{(x_S-x_D)^2(\alpha\frac{\cos\omega}{\cos\lambda}+1)^2+(y_S-y_D)^2(\alpha\frac{\sin\omega}{\sin\lambda}+1)^2}}$$
$$+ \frac{\alpha(\sin\lambda+\alpha\sin\omega)(\alpha\frac{\sin\omega}{\sin\lambda}+1)(y_S-y_D)}{(1-\alpha^2)\sqrt{(x_S-x_D)^2(\alpha\frac{\cos\omega}{\cos\lambda}+1)^2+(y_S-y_D)^2(\alpha\frac{\sin\omega}{\sin\lambda}+1)^2}}.$$

Expanding the factors inside the square root of each denominator, it is seen that they are equivalent. Hence, the previous equation can be written as follows

$$\frac{\partial V}{\partial \mathbf{x}}\cdot\mathbf{f}(\mathbf{x},\phi^*,\psi^*)$$
$$= \frac{\alpha}{(1-\alpha^2)\sqrt{(1+\alpha^2+2\alpha\frac{\cos\omega}{\cos\lambda})(x_S-x_D)^2+(1+\alpha^2+2\alpha\frac{\sin\omega}{\sin\lambda})(y_S-y_D)^2}}$$
$$\times\left((\cos\lambda+\alpha^2\frac{\cos^2\omega}{\cos\lambda}-\alpha^2\cos\lambda-\frac{\cos^2\omega}{\cos\lambda})(x_S-x_D) + (\sin\lambda+\alpha^2\frac{\sin^2\omega}{\sin\lambda}-\alpha^2\sin\lambda-\frac{\sin^2\omega}{\sin\lambda})(y_S-y_D)\right)$$
$$= \frac{\alpha(1-\alpha^2)\Big((\cos\lambda-\frac{\cos^2\omega}{\cos\lambda})(x_S-x_D)+(\sin\lambda-\frac{\sin^2\omega}{\sin\lambda})(y_S-y_D)\Big)}{(1-\alpha^2)\sqrt{(1+\alpha^2+2\alpha\frac{\cos\omega}{\cos\lambda})(x_S-x_D)^2+(1+\alpha^2+2\alpha\frac{\sin\omega}{\sin\lambda})(y_S-y_D)^2}}$$
$$= \frac{\alpha(1-\cos^2\omega-\sin^2\omega)\sqrt{(x_S-x_D)^2+(y_S-y_D)^2}}{\sqrt{(1+\alpha^2+2\alpha\frac{\cos\omega}{\cos\lambda})(x_S-x_D)^2+(1+\alpha^2+2\alpha\frac{\sin\omega}{\sin\lambda})(y_S-y_D)^2}}$$
$$= 0.$$

Hence, $V(\mathbf{x})$ satisfies the HJI equation. ∎

Remark: At the dispersal surface $\mathcal{D}$ which is specified by (16), the gradient of $V(\mathbf{x})$ is discontinuous, as expected. The dispersal surface corresponds to the case where the Apollonius circle is concentric with the circular region Ω, that is, $x_c = y_c = 0$, and more than one optimal aimpoint exists; in fact, every point on the Apollonius circle is an optimal aimpoint. The state of the game leaves the dispersal surface and transitions into the regular surface once S makes a selection of aimpoint. Hence, there is no perpetual dilemma.

V. Escape Game of Degree

In this section we consider the case where $\mathbf{x} \in \mathcal{R}_e$ and S is guaranteed to escape under optimal play. The performance functional is (9). We will characterize points on $\partial\Omega$, which satisfy $x^2+y^2=R^2$, in terms of the angle θ, that is, $x = R\cos\theta$ and $y = R\sin\theta$. Also, let $\nu = e^{i\theta}$ denote the complex exponential form of θ. Define the following $b_1 = x_c R(x_D x_S - 2y_D y_S)$, $b_2 = iy_c R(y_D y_S - 2x_D x_S)$, $b_3 = R(\alpha^2 y_D^2 x_S - x_D y_S^2)$, $b_4 = iR(\alpha^2 x_D^2 y_S - y_D x_S^2)$, $R_S = R^2 + x_S^2 + y_S^2$, and $R_D = R^2 + x_D^2 + y_D^2$.

Theorem 4: Consider the DGCR where the players are D and S. The speed ratio parameter is $\alpha = v_S/v_D < 1$ and assume that $\mathbf{x} \in \mathcal{R}_e$. The Value function is C^1 and it is the solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation. The Value function is given by

$$V(\mathbf{x}) = \sqrt{(R\cos\theta^*-x_D)^2+(R\sin\theta^*-y_D)^2} - \frac{1}{\alpha}\sqrt{(R\cos\theta^*-x_S)^2+(R\sin\theta^*-y_S)^2} \quad (24)$$

and the optimal state feedback strategies are

$$\cos\phi^* = \frac{x^*-x_S}{\sqrt{(x^*-x_S)^2+(y^*-y_S)^2}}$$
$$\sin\phi^* = \frac{y^*-y_S}{\sqrt{(x^*-x_S)^2+(y^*-y_S)^2}}$$
$$\cos\psi^* = \frac{x^*-x_D}{\sqrt{(x^*-x_D)^2+(y^*-y_D)^2}}$$
$$\sin\psi^* = \frac{y^*-y_D}{\sqrt{(x^*-x_D)^2+(y^*-y_D)^2}} \quad (25)$$

where $x^* = R\cos\theta^*$, $y^* = R\sin\theta^*$, $\cos\theta^* = \frac{\nu^*+(\nu^*)^{-1}}{2}$, $\sin\theta^* = \frac{\nu^*-(\nu^*)^{-1}}{2i}$, and ν^* is the solution of the polynomial equation

$$[(1-\alpha^2)(b_1+b_2)+b_3+b_4]\nu^6 + [R_D(y_S+ix_S)^2+\alpha^2 R_S(x_D-iy_D)^2]\nu^5$$

$$
\begin{aligned}
&- [(1-\alpha^2)(b_1-b_2) - 3b_3 + 3b_4]\nu^4 \\
&+ 2[R_D(x_S^2+y_S^2) - \alpha^2 R_S(x_D^2+y_D^2)]\nu^3 \\
&- [(1-\alpha^2)(b_1+b_2) - 3b_3 - 3b_4]\nu^2 \\
&+ [R_D(y_S - ix_S)^2 + \alpha^2 R_S(x_D+iy_D)^2]\nu \\
&+ (1-\alpha^2)(b_1-b_2) + b_3 - b_4 = 0 \qquad (26)
\end{aligned}
$$

that maximizes (24).

Proof: Since the optimal trajectories are straight lines, S travels directly towards the optimal aimpoint (x^*, y^*) located in $\partial\Omega$ that maximizes its terminal separation with respect to D. In order to minimize the terminal distance with respect to S, D's unique choice is to aim at the same optimal point. The optimal play is illustrated in Fig. 2.b. Then, the terminal distance between D and S is

$$
\begin{aligned}
\overline{D_f S_f} &= \sqrt{(x-x_D)^2 + (y-y_D)^2} \\
&\quad - \frac{1}{\alpha}\sqrt{(x-x_S)^2 + (y-y_S)^2} \\
&= \sqrt{(R\cos\theta - x_D)^2 + (R\sin\theta - y_D)^2} \\
&\quad - \frac{1}{\alpha}\sqrt{(R\cos\theta - x_S)^2 + (R\sin\theta - y_S)^2}. \qquad (27)
\end{aligned}
$$

Setting the derivative of (27) with respect to θ equal to zero we obtain the polynomial equation (26) and the Value function can be written as in (24).

The gradient of $V(\mathbf{x})$ is $\frac{\partial V}{\partial \mathbf{x}} = [\frac{\partial V}{\partial x_i} + \frac{dV}{d\theta^*}\cdot\frac{d\theta^*}{dx_i}, \ \frac{\partial V}{\partial y_i} + \frac{dV}{d\theta^*}\cdot\frac{d\theta^*}{dy_i}]^T$ for $i = D, S$. The term $\frac{dV}{d\theta^*}$ is equivalent to the left-hand side of (26) where $\cos\theta^* = \frac{\nu^* + (\nu^*)^{-1}}{2}$, $\sin\theta^* = \frac{\nu^* - (\nu^*)^{-1}}{2i}$, and ν^* is a solution of the polynomial equation. Therefore, $\frac{dV}{d\theta^*} = 0$ and we have that $\frac{\partial V}{\partial \mathbf{x}} = [\frac{\partial V}{\partial x_D}\ \frac{\partial V}{\partial y_D}\ \frac{\partial V}{\partial x_S}\ \frac{\partial V}{\partial y_S}]^T$, where

$$
\begin{aligned}
\frac{\partial V}{\partial x_D} &= -\frac{x^* - x_D}{\sqrt{(x^*-x_D)^2 + (y^*-y_D)^2}} \\
\frac{\partial V}{\partial y_D} &= -\frac{y^* - y_D}{\sqrt{(x^*-x_D)^2 + (y^*-y_D)^2}} \\
\frac{\partial V}{\partial x_S} &= \frac{x^* - x_S}{\alpha\sqrt{(x^*-x_S)^2 + (y^*-y_S)^2}} \\
\frac{\partial V}{\partial y_S} &= \frac{y^* - y_S}{\alpha\sqrt{(x^*-x_S)^2 + (y^*-y_S)^2}}
\end{aligned}
$$

where $\sqrt{(x^*-x_D)^2 + (y^*-y_D)^2} \neq 0$ since D is unable to reach the optimal aimpoint before S does; also, $\sqrt{(x^*-x_S)^2 + (y^*-y_S)^2} \neq 0$ for otherwise S is already in safe haven and the game has ended.

Similar to the capture game, the HJI equation for regular solutions is given by $-\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \mathbf{x}}\cdot\mathbf{f}(\mathbf{x}, \phi^*, \psi^*) + g(t, \mathbf{x}, \phi^*, \psi^*)$, where $\frac{\partial V}{\partial t} = g(t, \mathbf{x}, \phi^*, \psi^*) = 0$. Then, we calculate

$$
\begin{aligned}
&\frac{\partial V}{\partial \mathbf{x}}\cdot\mathbf{f}(\mathbf{x}, \phi^*, \psi^*) \\
&= \alpha\frac{\partial V}{\partial x_E}\cos\phi^* + \alpha\frac{\partial V}{\partial y_E}\sin\phi^* + \frac{\partial V}{\partial x_P}\cos\psi^* + \frac{\partial V}{\partial y_P}\sin\psi^* \\
&= \frac{\alpha(x^*-x_S)^2}{\alpha[(x^*-x_S)^2 + (y^*-y_S)^2]} + \frac{\alpha(y^*-y_S)^2}{\alpha[(x^*-x_S)^2 + (y^*-y_S)^2]} \\
&\quad - \frac{(x^*-x_D)^2}{(x^*-x_D)^2 + (y^*-y_D)^2} - \frac{(y^*-y_D)^2}{(x^*-x_D)^2 + (y^*-y_D)^2} \\
&= \frac{(x^*-x_S)^2 + (y^*-y_S)^2}{(x^*-x_S)^2 + (y^*-y_S)^2} - \frac{(x^*-x_D)^2 + (y^*-y_D)^2}{(x^*-x_D)^2 + (y^*-y_D)^2} \\
&= 0.
\end{aligned}
$$

Hence, $V(\mathbf{x})$ satisfies the HJI equation. ■

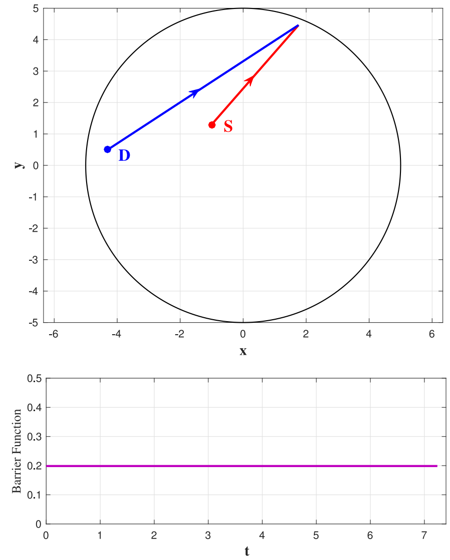

Fig. 3. Optimal play. Top: Optimal Trajectories. Bottom: Barrier function at each time instant.

A. Comparison to Previous Work

The same metric was used in the escape game in [7] which is the terminal distance between the players. It was stated in that reference that the solution was intractable. This, however, is not the case. The game of degree has been completely solved in this section and the solution is a closed-loop strategy that only requires the rooting of a polynomial. Importantly, it was shown that there are no singular surfaces in the escape game since $V(\mathbf{x})$, the solution of the HJI equation, was obtained and is C^1 for any $\mathbf{x} \in \mathcal{R}_e$.

The optimal strategies in (18) for the capture game are different than the corresponding strategies in [7]. This case represents a more significant issue which has been solved in this letter. The solution in that reference optimizes capture time, a different metric than the one used here. However, capture time is not the relevant metric in the DGCR. This aspect has important consequences: if, initially, $\mathbf{x} \in \mathcal{R}_c$ holds, then D might not be able to actually win the game if it implements the strategy in [7]. S will implement its strategy (18) and decrease the terminal distance with respect to $\partial\Omega$ and potentially win the game when it was initially doomed; this is illustrated in Example 2 below. Thus, the strategies of the capture game in [7] do not provide a semipermeable Barrier surface. In this letter we have used the correct performance functionals in both $\mathcal{R}_e$ and $\mathcal{R}_c$ which mesh naturally. Also, an elegant geometric solution using the Apollonius construct was presented. Finally, it was shown that Isaac's method applies for both Games of Degree.

VI. Examples

Example 1: Consider the initial conditions $x_{D_0} = -4.3$, $y_{D_0} = 0.5$, $x_{S_0} = -1$, $y_{S_0} = 1.3$ and the parameter $\alpha = 0.58$. The radius of Ω is $R = 5$. It holds that $\mathbf{x} \in \mathcal{R}_c$ and capture of S by D is guaranteed under optimal play.

The optimal trajectories are shown in the top plot of Fig. 3. Both players implement their optimal strategies in a closed-loop manner. At each time instant they compute the Barrier function (11) to determine if the state is in $\mathcal{R}_c$ or in $\mathcal{R}_e$ and apply the corresponding optimal heading. The Barrier function remains positive for all the duration of the engagement as it is expected since both agents played optimally. The Barrier

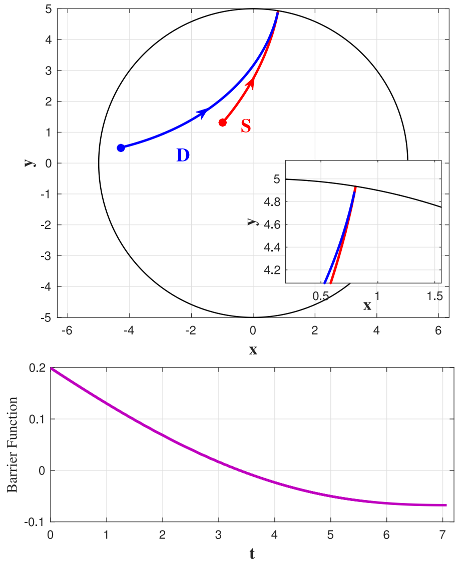

Fig. 4. Non-optimal play by D. Top: Trajectories. Bottom: Barrier function at each time instant.

function calculated at each time instant is shown in the bottom plot of Fig. 3.

Example 2: Let us now consider the same initial conditions and parameters but with the difference that D does not follow its optimal strategy. Instead, D tries to capture S in minimum time implementing the non-optimal strategy in [7]. S does not have any knowledge of the control applied by D and it implements the optimal strategies obtained in this letter in a closed-loop manner. At each time instant S checks the sign of the Barrier function and implements the corresponding strategy. At the beginning S plays the capture game (18)-(19). Since D does not play optimally, the Barrier function decreases as it is shown in the bottom plot of Fig. 4. The Barrier function becomes negative at about 3.3 *sec* and S is now able to escape by playing the escape game (25)-(26). The resulting trajectories are shown in the top plot of Fig. 4 where the heading of S is not constant anymore since it reacts to the non-optimal trajectory of D. A closer look at the terminal points of the players is included which shows that S is able to reach $\partial\Omega$ before being captured by D.

This is an illustrative example of how player S, who initially was prescribed to be captured and lose the game, is actually able to escape and defeat D. All this thanks to D who implemented a non-optimal strategy. This shows by counter-example that the strategy in [7] based on capture time is not optimal for the defense of a circular region. Moreover, the optimal strategies of the DGCR, which have been derived in this letter, allow S to take advantage of D's behavior in order to drive the state of the game into its preferred region, $\mathcal{R}_e$, where it can escape and win the game. In summary, the game of kind and the games of degree need to be correctly formulated and solved in order to obtain a semipermeable Barrier surface.

VII. Conclusion

The differential game of pursuit-escape in a circular region was considered. The complete solution of this differential game was presented which includes the solutions to the game of kind and the optimal strategies for each game of degree. The important relation between the game of kind and the games of degree as it relates to the semipermeability property of the Barrier surface was emphasized through this letter. Finally, examples were presented to illustrate this property and show how a player can react and take advantage of non-optimal behaviors by the adversary by implementing the optimal state-feedback strategies obtained in this letter.

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