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Engineering Notes

Cooperative Strategies for Optimal Aircraft Defense from an Attacking Missile

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I. Introduction

IN CURRENT military operations, a threat that is frequently faced by an air vehicle is that of a missile homing into the aircraft. An alternative aircraft protection method to common passive countermeasures, such as chaff and flares, entails firing a defending missile to intercept the attacking missile before it reaches the aircraft. This active self-defense system results in a special type of pursuit–evasion scenario involving three agents: the attacker missile, target aircraft, and defender missile.

Multi-agent pursuit–evasion scenarios represent important and challenging types of problems in control, aerospace, and robotics. In these problems, one or more agents, called pursuers, try to maneuver and reach a relatively small distance with respect to one or more evaders, which maneuver to escape the pursuers. Several types of pursuit–evasion problems involving many agents have been studied. This problem is usually posed as a dynamic game [1–3]. Approaches based on dynamic Voronoi diagrams have been used in problems with several pursuers to capture an evader within a bounded environment, as shown by Huang et al. [2] and by Bakolas and Tsiotras [4]. On the other hand, Sprinkle et al. [5] presented a receding-horizon approach that provides evasive maneuvers for a fixed-wing unmanned autonomous vehicle (UAV) assuming a known model of the pursuer's input, state, and constraints. In [6], a multi-agent scenario is presented in which a number of pursuers are assigned to intercept a group of evaders where the goals of the evaders are assumed to be known.

Cooperation between two agents with the goal of evading a single pursuer has been addressed in [7]. Scott and Leonard [8] investigated a bioinspired scenario where two evaders search for coordinated strategies to evade a single pursuer but also to keep them close to each other.

In the present Note, we consider a three-agent pursuit–evasion scenario. A two-agent team consists of a target and a defender who cooperate; the third agent, the attacker, is the opposition. The goal of the attacker is to capture the target while the target, assisted by the defender, tries to evade the attacker and avoid capture; the target cooperates with the defender, which tries to capture the attacker before the latter captures the target. Cooperation between the target and the defender is such that the defender will capture the attacker before the latter reaches the target.

This scenario has initially been analyzed in the context of cooperative missile operations [9,10]. In [11], Ratnoo and Shima presented the case when the defender chooses to implement line-of-sight (LOS) guidance to pursue the attacker, which requires the defender to have at least the same speed as the attacker. A different guidance law for the target–attacker–defender (TAD) scenario was given by Yamasaki et al. [12,13]. These authors propose an interception method, triangle guidance, where the defending missile is commanded to be on the line of sight between the attacking missile and the aircraft for all time, while the aircraft follows some predetermined trajectory. These previous approaches limit the level of cooperation between the target and the defender by implementing defender guidance laws without regard to the target's trajectory.

Rubinsky and Gutman [14,15] presented an analysis of the end-game TAD scenario focused on the attacker/target miss distance for a noncooperative target and defender. One is then solely interested in whether the missile wins (it captures the target) or it loses (the target evades capture). The authors provide linearization-based attacker maneuvers to evade the defender and continue pursuing the target.

In [16], optimal guidance laws, namely, the commanded lateral acceleration for each agent, including the attacker, are numerically obtained for the case of an aggressive defender; that is, the defender has a maneuverability advantage. A linear quadratic optimal control problem is posed where the defender's control effort weight is driven to zero to increase its aggressiveness. The work [17] provided a game theoretical analysis of the TAD problem using different guidance laws for both the attacker and the defender. The cooperative strategies in [18] allow for a maneuverability disadvantage for the defender with respect to the attacker and it is shown that the optimal target maneuver is either constant or arbitrary. Shaferman and Shima [19] implemented a multiple model adaptive estimator to identify the guidance law and parameters of the incoming missile and optimize a matched defender strategy to minimize its control effort. In a recent paper [20], Prokopov and Shima analyze different types of cooperation, assuming the attacker is unaware of the presence of the defender and its guidance law is known. In [21], the TAD differential game with a static target is analyzed. Other references discussing the TAD differential game are [22–24].

This work provides jointly optimal solutions, that is, the optimal headings of both the target and the defender, for pure pursuit (PP) and proportional navigation (PN) attacker guidance laws. These solutions are based on the actual nonlinear dynamics of the three-agent engagement, as opposed to linearization approaches and the method of linear quadratic control used in most of the papers discussed so far. In addition, the defender is not constrained to have similar or greater speed than the attacker. The optimal cooperation between target and defender allows the latter to capture the attacker even when it has a considerable speed disadvantage. The optimal control problem is solved as a two-point boundary value problem (TPBVP). Numerical solutions illustrate the efficiency of the cooperative behavior and show that, in general, little effort in terms of lateral acceleration is needed by both the target and defender, provided the defender is properly positioned from the start.

Our preliminary results were presented in [25,26]. In the present Note, the cooperative optimal guidance approach is extended to consider a differential game (DG) formulation where the attacker missile solves his optimal control problem to minimize the final separation between itself and the target. Assuming that the attacker knows the position of the defender, this strategy provides a better outcome for the attacker than using PP or PN in terms of final target–

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attacker separation. From the attacker's point of view, it is better to bring the (defender–attacker) interception collision point closer to the target's position (and hopefully produce some damage) even if the attacker is captured by the defender. The differential game provides target survival/capture guarantees given initial positions and assuming constant speeds of the agents. This means that, if the solution of the differential game provides a terminal separation greater than the attacker capture radius, then there exists an optimal policy for the target/defender team such that the target will avoid being captured regardless of the attacker's guidance law. On the other hand, if the terminal separation is smaller than the attacker capture radius then there exists an optimal policy for the attacker such that capture of the target is guaranteed, regardless of the target/defender team's maneuvers.

The Note is organized as follows. Section II describes the engagement scenario. In Sec. III, optimal policies to capture the attacker and maximize the final target–attacker separation are derived for the target and the defender when the attacker uses PP guidance. Section IV presents similar optimal policies for the target and the defender when the attacker uses PN guidance. Section V considers the differential game formulation of the active target defense scenario where the attacker computes its corresponding optimal policy to minimize the final target–attacker separation. In Sec. VI, details related to the numerical solution of the TPBVP are addressed. Examples are given in Secs. VII, and VIII concludes the Note.

II. Modeling

The target–attacker–defender engagement in the two-dimensional state space is illustrated in Fig. 1. The speeds of the target, attacker, and defender are V_T, V_A, and V_D, respectively, which are assumed to be constant.

The dynamics of the three vehicles in an inertial frame/realistic plane are given by

$$\dot{x}_T = V_T \cos(\lambda + \phi), \qquad \dot{y}_T = V_T \sin(\lambda + \phi) \tag{1}$$

$$\dot{x}_A = V_A \cos \hat{\chi}, \qquad \dot{y}_A = V_A \sin \hat{\chi} \tag{2}$$

$$\dot{x}_D = V_D \cos \hat{\psi}, \qquad \dot{y}_D = V_D \sin \hat{\psi} \tag{3}$$

where $\hat{\chi} = \lambda + \theta - \chi$ and $\hat{\psi} = \psi + \theta + \lambda - \pi$.

Define the speed ratio problem parameters $\alpha = V_T/V_A$ and $\beta = V_D/V_A$. In general, the attacker missile is faster than the target aircraft, so that $\alpha < 1$. In this work, the speed of the defender is not constrained to be equal to or greater than the speed of the attacker, and so the speed ratio β is unrestricted. The variables R and r represent the separation between the attacker and the target and between the attacker and the defender, respectively. The positive constant R_c represents the attacker's capture radius and the positive constant r_c represents the defender's capture radius.

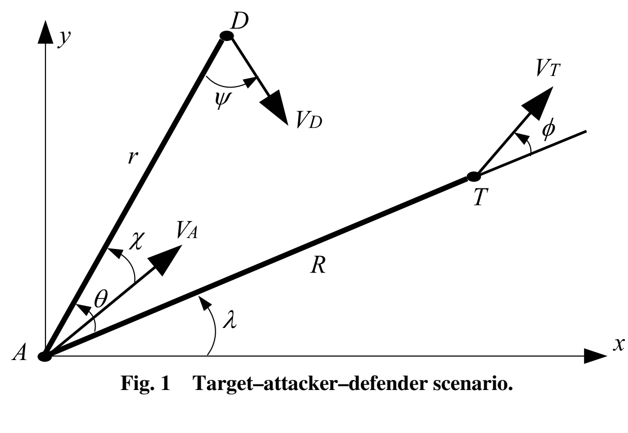


Fig. 1 Target–attacker–defender scenario.

The corresponding dynamics of the three-agent engagement will be expressed using the reduced state space R, r, and the angle between them, denoted by θ. The objective is to find the optimal heading angles of the target and the defender in this relative frame, which are denoted by ϕ and ψ, respectively. The distance $R(t_f)$ is maximized. Here, t_f is the time instant where the distance r becomes r_c and designates the capture of the attacker by the defender. Note that the relative heading angles can be easily transformed to heading angles with respect to the fixed coordinate axis x using the angle λ. The use of the reduced state space provides a three-state representation of the dynamics corresponding to this three-agent scenario.

III. Cooperative Defense Against an Attacker Using Pure Pursuit

Let us consider first the case where the attacker implements the PP guidance law (see Fig. 2). For pure pursuit guidance, we have that $\hat{\chi} = \lambda$ so that the vector V_A always points at target T. The dynamics in the reduced state space are given by

$$\dot{R} = \alpha \cos \phi - 1 \tag{4}$$

$$\dot{r} = -\cos \theta - \beta \cos \psi \tag{5}$$

$$\dot{\theta} = -\frac{\alpha}{R} \sin \phi - \frac{\beta}{r} \sin \psi + \frac{1}{r} \sin \theta \tag{6}$$

with

$$R(t_0) = R_0 \quad r(t_0) = r_0 \quad \theta(t_0) = \theta_0 \tag{7}$$

The objective of the target–defender team is to maximize the separation $R(t_f)$, that is, to solve the problem $\max_{\phi,\psi} J$, where

$$J = \int_{t_0}^{t_f} \dot{R} \, \mathrm{d}t \tag{8}$$

The selection of the maximum terminal separation $R(t_f)$ as the cost function represents an important measure to safeguard the target. The target will have a better shot at evading and deciding if additional countermeasures (such as releasing chaff and flares) are needed after the encounter of the defender and the attacker, if a higher separation from the attacker is provided.

To apply Pontryagin's maximum principle [27] to the optimal control problem (4–8), we introduce the Hamiltonian (where the target–defender team aims at minimizing $-J$ for convenience of notation of the solutions):

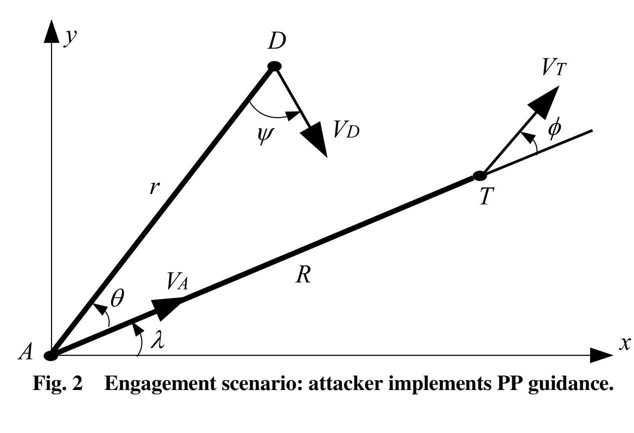


Fig. 2 Engagement scenario: attacker implements PP guidance.

$$H = -\alpha \cos \phi + 1 + (\alpha \cos \phi - 1)\lambda_R - (\cos \theta + \beta \cos \psi)\lambda_r + \left(-\frac{\alpha}{R}\sin \phi - \frac{\beta}{r}\sin \psi + \frac{1}{r}\sin \theta\right)\lambda_\theta \tag{9}$$

The costate dynamics are

$$\dot{\lambda}_R = -\frac{\partial H}{\partial R} = -\frac{\alpha \lambda_\theta}{R^2}\sin \phi \tag{10}$$

$$\dot{\lambda}_r = -\frac{\partial H}{\partial r} = \frac{\lambda_\theta}{r^2}(\sin \theta - \beta \sin \psi) \tag{11}$$

$$\dot{\lambda}_\theta = -\frac{\partial H}{\partial \theta} = -\lambda_r \sin \theta - \frac{\lambda_\theta}{r}\cos \theta \tag{12}$$

The terminal conditions are as follows:

$$\begin{aligned} r(t_f) &= r_c \\ \lambda_R(t_f) &= 0 \\ \lambda_\theta(t_f) &= 0 \\ 1 - \alpha - \lambda_r(t_f)[\cos \theta(t_f) + \beta] &= 0 \end{aligned} \tag{13}$$

Proposition 1: Assume that the attacker implements a PP guidance law to pursue the target. Then, the optimal headings that maximize the separation between the target and the attacker and achieve $r(t_f) = r_c$ are characterized by

$$\sin \psi^* = \frac{\lambda_\theta}{r\sqrt{\lambda_r^2 + \lambda_\theta^2/r^2}} \tag{14}$$

$$\cos \psi^* = \frac{\lambda_r}{\sqrt{\lambda_r^2 + \lambda_\theta^2/r^2}} \tag{15}$$

$$\sin \phi^* = \frac{\lambda_\theta}{R\sqrt{(1-\lambda_R)^2 + \lambda_\theta^2/R^2}} \tag{16}$$

$$\cos \phi^* = \frac{1-\lambda_R}{\sqrt{(1-\lambda_R)^2 + \lambda_\theta^2/R^2}} \tag{17}$$

The proof of this proposition is similar to the one for Proposition 3 in Sec. V and it is based on the optimality conditions $\partial H/\partial \psi = 0$ and $\partial H/\partial \phi = 0$.

The separation $R(t)$ is monotonically decreasing when the attacker implements the PP guidance law. This observation is corroborated by Eq. (4) and the fact that $\alpha < 1$. Then, the system of dynamic equations (4–6) can be reduced and the problem can be alternatively analyzed using a two-dimensional state space r and θ. For more details, see [25].

IV. Cooperative Defense Against an Attacker Using Proportional Navigation

The target–attacker–defender scenario, when the attacker implements PN guidance, is shown in Fig. 3. The PN guidance law is characterized by the rate of change of the attacker heading angle being proportional to the rate of change of the LOS between the attacker and the target. In this section, the three vehicles' dynamics are described by Eqs. (1) and (3) and the following equations:

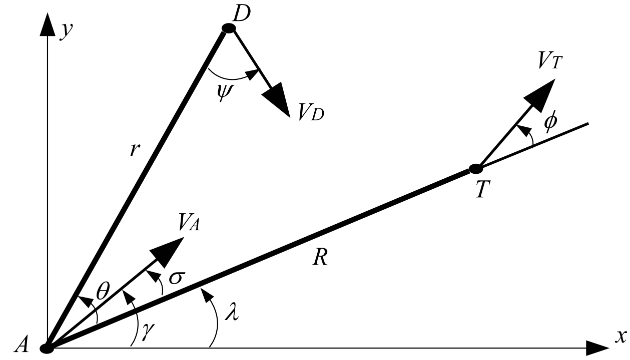


Fig. 3 Engagement scenario: attacker implements PN guidance.

$$\dot{x}_A = V_A \cos \gamma, \qquad \dot{y}_A = V_A \sin \gamma \tag{18}$$

where γ now represents the attacker's heading angle.

The dynamics of the three-agent engagement will be expressed using the reduced state space formed by R, r, θ, and by σ, which represents the difference between the attacker heading angle and the line-of-sight angle between the attacker and the target. Similar to Sec. III, the objective is to find the optimal heading angles of the target and the defender in this relative frame, which are denoted by ϕ and ψ, respectively, such that the distance $R(t_f)$ is maximized when the separation r becomes r_c (at some interception time t_f to be determined), which signals capture of the attacker by the defender.

For the derivation that follows, we use the angle σ, which represents the relative heading angle of the attacker with respect to the line of sight from the attacker to the target, that is,

$$\sigma = \gamma - \lambda \tag{19}$$

The rate of change of the attacker heading angle is given by

$$\dot{\gamma} = \frac{a_A}{V_A} \tag{20}$$

where a_A represents the lateral acceleration of the attacker.

The rate of change of the LOS angle λ is given by

$$\dot{\lambda} = \frac{V_T \sin \phi - V_A \sin \sigma}{R} \tag{21}$$

We consider the case where the attacker uses PN guidance with navigation constant N. Then the attacker's lateral acceleration is given by

$$a_A = N V_A \dot{\lambda} \tag{22}$$

and from Eqs. (19) and (20), we obtain the following relations:

$$\dot{\gamma} = N\dot{\lambda} \tag{23}$$

$$\dot{\sigma} = \dot{\gamma} - \dot{\lambda} = (N-1)\dot{\lambda} = (N-1)\frac{V_T \sin \phi - V_A \sin \sigma}{R} \tag{24}$$

Then, by using the speed ratio parameters α and β, the dynamics corresponding to the variables of interest in this problem can be expressed as follows:

$$\dot{R} = \alpha \cos \phi - \cos \sigma \tag{25}$$

$$\dot{r} = -\cos(\theta - \sigma) - \beta \cos \psi \tag{26}$$

$$\dot{\theta} = -\frac{\alpha}{R}\sin \phi + \frac{1}{R}\sin \sigma - \frac{\beta}{r}\sin \psi + \frac{1}{r}\sin(\theta - \sigma) \tag{27}$$

$$\dot{\sigma} = (N-1)\left(\frac{\alpha}{R}\sin \phi - \frac{1}{R}\sin \sigma\right) \tag{28}$$

with initial conditions given by Eq. (7) and $\sigma(t_0) = \sigma_0$. The objective of the target–defender team is to maximize the separation $R(t_f)$ between the target and the attacker at the interception time t_f, that is, to solve the problem $\max_{\phi,\psi} J$, where J was defined in Eq. (8). The Hamiltonian is given by (where the target–defender team aims at minimizing $-J$ for convenience of notation of the solutions)

$$\begin{aligned} H &= -\alpha \cos \phi + \cos \sigma + (\alpha \cos \phi - \cos \sigma)\lambda_R \\ &\quad - [\cos(\theta - \sigma) + \beta \cos \psi]\lambda_r \\ &\quad + \left(-\frac{\alpha}{R}\sin \phi + \frac{1}{R}\sin \sigma - \frac{\beta}{r}\sin \psi + \frac{1}{r}\sin(\theta - \sigma)\right)\lambda_\theta \\ &\quad + (N-1)\left(\frac{\alpha}{R}\sin \phi - \frac{1}{R}\sin \sigma\right)\lambda_\sigma \end{aligned} \tag{29}$$

and the costate dynamics are given by

$$\dot{\lambda}_R = -\frac{\partial H}{\partial R} = \frac{\lambda_\theta}{R^2}(\sin \sigma - \alpha \sin \phi) + (N-1)\frac{\lambda_\sigma}{R^2}(\alpha \sin \phi - \sin \sigma) \tag{30}$$

$$\dot{\lambda}_r = -\frac{\partial H}{\partial r} = \frac{\lambda_\theta}{r^2}[\sin(\theta - \sigma) - \beta \sin \psi] \tag{31}$$

$$\dot{\lambda}_\theta = -\frac{\partial H}{\partial \theta} = -\lambda_r \sin(\theta - \sigma) - \frac{\lambda_\theta}{r}\cos(\theta - \sigma) \tag{32}$$

$$\begin{aligned} \dot{\lambda}_\sigma &= -\frac{\partial H}{\partial \sigma} \\ &= (1 - \lambda_R)\sin \sigma + \left((N-1)\frac{\lambda_\sigma}{R} - \frac{\lambda_\theta}{R}\right)\cos \sigma \\ &\quad + \lambda_r \sin(\theta - \sigma) + \frac{\lambda_\theta}{r}\cos(\theta - \sigma) \end{aligned} \tag{33}$$

The terminal conditions for this free terminal time problem are as follows. The terminal state $r(t_f)$ is fixed and equal to r_c. The terminal states $R(t_f)$, $\theta(t_f)$, $\sigma(t_f)$ are free, and so $\lambda_R(t_f) = \lambda_\theta(t_f) = \lambda_\sigma(t_f) = 0$. The terminal condition for optimality requires that $H(x^*(t_f), u^*(t_f), \lambda^*(t_f), t_f) = 0$, which for this case takes the form

$$\cos \sigma(t_f) - \alpha - \{\cos[\theta(t_f) - \sigma(t_f)] + \beta\}\lambda_r(t_f) = 0 \tag{34}$$

Proposition 2: Assume that the attacker implements a PN guidance law with navigation constant N. Then, the optimal control headings that maximize the separation between the target and the attacker and achieve $r(t_f) = r_c$ are characterized by Eqs. (14) and (15) and

$$\sin \phi^* = \frac{\lambda_\theta - (N-1)\lambda_\sigma}{R\sqrt{(1-\lambda_R)^2 + (\lambda_\theta - (N-1)\lambda_\sigma)^2/R^2}} \tag{35}$$

$$\cos \phi^* = \frac{1 - \lambda_R}{\sqrt{(1-\lambda_R)^2 + (\lambda_\theta - (N-1)\lambda_\sigma)^2/R^2}} \tag{36}$$

The proof of this proposition is similar to the one for Proposition 3 in Sec. V and it is based on the optimality conditions $\partial H/\partial \psi = 0$ and $\partial H/\partial \phi = 0$.

V. Differential Game

We now consider the case when the attacker uses an optimal heading with a view to minimizing the final separation from the target, at the instant of his interception by the defender (see Fig. 1). For the differential game, we assume that attacker and defender have the same speed, that is, $\beta = 1$. The vehicle dynamics corresponding to the target, attacker, and defender can be represented by Eqs. (1–3), respectively. The attacker's heading in the reduced state space is denoted by χ.

The corresponding dynamics of the three-agent engagement will be expressed using the reduced state space formed by R, r, and θ. The main difference here is that there are now three control inputs: ϕ, ψ, and χ.

The dynamics corresponding to the variables of interest in this scenario can be expressed as follows:

$$\dot{R} = \alpha \cos \phi - \cos(\theta - \chi) \tag{37}$$

$$\dot{r} = -\cos \chi - \cos \psi \tag{38}$$

$$\dot{\theta} = -\frac{\alpha}{R}\sin \phi + \frac{1}{R}\sin(\theta - \chi) - \frac{1}{r}\sin \psi + \frac{1}{r}\sin \chi \tag{39}$$

with initial conditions given by Eq. (7). The objective of the target–defender team is to maximize the separation $R(t_f)$ between the target and the attacker at the interception time t_f, whereas the objective of the attacker is to minimize the same distance. The differential game can be formally expressed as

$$\min_{\chi} \max_{\phi,\psi} J \tag{40}$$

where the cost/payoff function J was defined in Eq. (8). Then, the Hamiltonian is given by [where the target–defender team aims at minimizing $-J$ and the attacker aims at maximizing $(-J)$ for convenience of notation of the solutions]

$$\begin{aligned} H &= \cos(\theta - \chi) - \alpha \cos \phi + [\alpha \cos \phi - \cos(\theta - \chi)]\lambda_R \\ &\quad - (\cos \chi + \cos \psi)\lambda_r \\ &\quad + \left(-\frac{\alpha}{R}\sin \phi + \frac{1}{R}\sin(\theta - \chi) - \frac{1}{r}\sin \psi + \frac{1}{r}\sin \chi\right)\lambda_\theta \end{aligned} \tag{41}$$

and the costate dynamics are given by

$$\dot{\lambda}_R = \frac{\lambda_\theta}{R^2}[\sin(\theta - \chi) - \alpha \sin \phi] \tag{42}$$

$$\dot{\lambda}_r = \frac{\lambda_\theta}{r^2}(\sin \chi - \sin \psi) \tag{43}$$

$$\dot{\lambda}_\theta = (1 - \lambda_R)\sin(\theta - \chi) - \frac{\lambda_\theta}{R}\cos(\theta - \chi) \tag{44}$$

The terminal conditions for this free terminal time problem are as follows. The terminal state $r(t_f)$ is fixed and equal to r_c. Because the terminal states $R(t_f)$ and $\theta(t_f)$ are free, we have $\lambda_R(t_f) = \lambda_\theta(t_f) = 0$. The final terminal condition for optimality for this problem requires that $H(x^*(t_f), u^*(t_f), \lambda^*(t_f), t_f) = 0$. For this differential game, the last condition is given by

$$\alpha^2 + 2[\alpha + \cos\,\theta(t_f)]\lambda_r(t_f) - 1 = 0 \tag{45}$$

Proposition 3: In the differential game, the target and defender optimal headings that maximize the separation between the target and the attacker and achieve $r(t_f) = r_c$ are characterized by Eqs. (14–17). The attacker's optimal heading that minimizes the separation between itself and the target is characterized by

$$\sin\,\chi^* = \frac{(1-\lambda_R)\sin\,\theta - (\lambda_\theta/R)\cos\,\theta + (\lambda_\theta/r)}{\sqrt{[(1-\lambda_R)\sin\,\theta - (\lambda_\theta/R)\cos\,\theta + (\lambda_\theta/r)]^2 + [(1-\lambda_R)\cos\,\theta + (\lambda_\theta/R)\sin\,\theta - \lambda_r]^2}} \tag{46}$$

$$\cos\,\chi^* = \frac{(1-\lambda_R)\ \cos\,\theta + (\lambda_\theta/R)\ \sin\,\theta - \lambda_r}{\sqrt{[(1-\lambda_R)\ \sin\,\theta - (\lambda_\theta/R)\ \cos\,\theta + (\lambda_\theta/r)]^2 + [(1-\lambda_R)\ \cos\,\theta + (\lambda_\theta/R)\ \sin\,\theta - \lambda_r]^2}} \tag{47}$$

Proof: To find the optimal heading angle equations involving ψ^*, we solve for this variable by differentiating the Hamiltonian (41) in ψ and setting the derivative to zero:

$$\frac{\partial H}{\partial \psi} = \lambda_r\,\sin\,\psi - \frac{\lambda_\theta}{r}\cos\,\psi = 0 \tag{48}$$

Using the trigonometric identity $\cos^2\psi = 1 - \sin^2\psi$, we can write Eq. (48) as

$$\sin^2\psi = \frac{\lambda_\theta^2}{r^2(\lambda_r^2 + \lambda_\theta^2/r^2)}$$

and Eq. (14) follows. The expression (15) is found in a similar way by letting $\sin^2\psi = 1 - \cos^2\psi$ in Eq. (48). We can compute the second partial derivative of the Hamiltonian with respect to ψ to show that this solution minimizes the cost $-J$. Doing so, we obtain

$$\frac{\partial^2 H}{\partial \psi^2} = \lambda_r\cos\psi + \frac{\lambda_\theta}{r}\sin\psi = \frac{\lambda_r^2}{\sqrt{\lambda_r^2 + \lambda_\theta^2/r^2}} + \frac{\lambda_\theta^2}{r^2\sqrt{\lambda_r^2 + \lambda_\theta^2/r^2}} > 0 \tag{49}$$

which means that ψ^* minimizes the cost $-J$; equivalently, it maximizes the final separation $R(t_f)$.

The optimal heading of the target can be found in a similar way. Let us evaluate

$$\frac{\partial H}{\partial \phi} = \alpha(1-\lambda_R)\sin\,\phi - \frac{\alpha}{R}\lambda_\theta\,\cos\,\phi = 0 \tag{50}$$

We use the trigonometric identity $\cos^2\phi = 1 - \sin^2\phi$ to write Eq. (50) as

$$\sin^2\phi^* = \frac{\lambda_\theta^2}{R^2[(1-\lambda_R)^2 + \lambda_\theta^2/R^2]}$$

and we obtain Eq. (16). The expression (17) is found in a similar way by setting $\sin^2\phi = 1 - \cos^2\phi$ in Eq. (50). Similarly, we compute

$$\frac{\partial^2 H}{\partial \phi^2} = \alpha(1-\lambda_R)\cos\,\phi + \frac{\alpha}{R}\lambda_\theta\,\sin\,\phi = \frac{\alpha(1-\lambda_R)^2}{\sqrt{(1-\lambda_R)^2 + \lambda_\theta^2/R^2}} + \frac{\alpha\lambda_\theta^2}{R^2\sqrt{(1-\lambda_R)^2 + \lambda_\theta^2/R^2}} > 0 \tag{51}$$

which means that ϕ^* minimizes the cost $-J$; equivalently, it maximizes the final separation $R(t_f)$.

The optimal heading χ^* is characterized in a similar way. We differentiate the Hamiltonian (41) in χ and set the derivative to zero:

$$\frac{\partial H}{\partial \chi} = (1-\lambda_R)\sin(\theta-\chi) + \lambda_r\sin\chi - \frac{\lambda_\theta}{R}\cos(\theta-\chi) + \frac{\lambda_\theta}{r}\cos\chi = 0 \tag{52}$$

Using the trigonometric identities

$$\sin(\theta-\chi) = \sin\,\theta\,\cos\,\chi - \cos\,\theta\,\sin\,\chi \tag{53}$$

$$\cos(\theta-\chi) = \cos\,\theta\,\cos\,\chi + \sin\,\theta\,\sin\,\chi \tag{54}$$

we can write Eq. (52) as follows:

$$\left((1-\lambda_R)\sin\,\theta - \frac{\lambda_\theta}{R}\cos\,\theta + \frac{\lambda\theta}{r}\right)\cos\,\chi = \left((1-\lambda_R)\cos\,\theta + \frac{\lambda_\theta}{R}\sin\,\theta - \lambda_r\right)\sin\,\chi \tag{55}$$

We now use the trigonometric identity $\cos^2\chi = 1 - \sin^2\chi$ to obtain

$$\sin^2\chi^* = \frac{[(1-\lambda_R)\sin\,\theta - (\lambda_\theta/R)\cos\,\theta + (\lambda_\theta/r)]^2}{[(1-\lambda_R)\sin\,\theta - (\lambda_\theta/R)\cos\,\theta + (\lambda_\theta/r)]^2 + [(1-\lambda_R)\cos\,\theta + (\lambda_\theta/R)\sin\,\theta - \lambda_r]^2}$$

and Eq. (46) follows. The expression (47) is found in a similar way by setting $\sin^2\chi = 1 - \cos^2\chi$ in Eq. (55).

To guarantee that the attacker optimal control maximizes the objective $-J$, we evaluate the second partial derivative of the Hamiltonian with respect to the attacker control input:

$$\frac{\partial^2 H}{\partial\chi^2} = -(1-\lambda_R)\cos(\theta-\chi) + \lambda_r \cos\chi - \frac{\lambda_\theta}{R}\sin(\theta-\chi) - \frac{\lambda_\theta}{r}\sin\chi \tag{56}$$

Inserting the expressions (53) and (54) into Eq. (56), we obtain the following:

$$\frac{\partial^2 H}{\partial\chi^2} = -\left((1-\lambda_R)\sin\theta - \frac{\lambda_\theta}{R}\cos\theta + \frac{\lambda_\theta}{r}\right)\sin\chi - \left((1-\lambda_R)\cos\theta + \frac{\lambda_\theta}{R}\sin\theta - \lambda_r\right)\cos\chi < 0 \tag{57}$$

Therefore, the solutions (46) and (47) maximize the objective $-J$, which is equivalent to minimizing the terminal separation $R(t_f)$. □

VI. Numerical Solution of Optimal Control Problems

In Secs. III and IV, we characterized the cooperative optimal heading angles for the target–defender team to guarantee interception of the attacker by the defender and to maximize the separation between target and attacker at the interception time t_f. Similarly, in Sec. V, we characterized the target and defender optimal heading angles and the attacker optimal heading angle in the differential game.

The derived characterizations of the optimal headings are used to numerically solve the TPBVP for the states and costates. In every case, this can be achieved by substituting the optimal control input expressions into the corresponding state and costate dynamic equations and using the corresponding boundary conditions.

Typical TPBVP solvers require an initial guess of the solution. In the special case when $\theta_0 = 0$ (or when $\theta_0 = \sigma_0 = 0$ in the PN case), the solution of the TPBVP is easy. For instance, the analytical solution for the differential game when $\theta_0 = 0$ is given by

$$\begin{aligned} R^*(t) &= (\alpha-1)t + R_0, \qquad \lambda_R^*(t) = 0 \\ r^*(t) &= -2t + r_0, \qquad \lambda_r^*(t) = \frac{1-\alpha^2}{2(\alpha+1)} \\ \theta^*(t) &= 0, \qquad \lambda_\theta^*(t) = 0 \end{aligned} \tag{58}$$

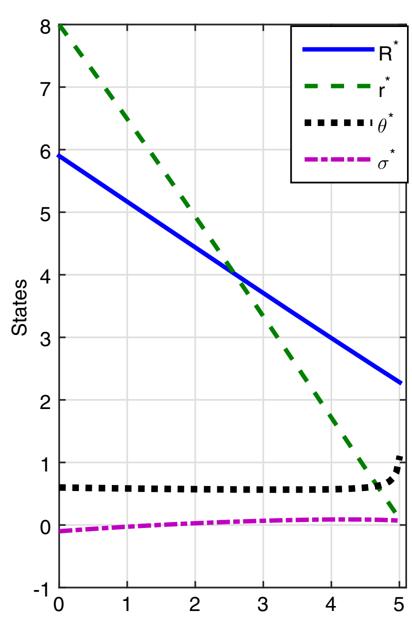
which can be used as an initial guess for problems with $\theta_0 \neq 0$. For $\theta_0 \neq 0$ and/or $\sigma_0 \neq 0$, the solution can be obtained after a few iterations of the solver starting with a solution for $\theta_0 \approx 0$, $\sigma_0 \approx 0$ and incrementing θ_0 and σ_0 to reach the required initial condition on θ and σ. The obtained solution at each iteration is used as an initial guess for the next iteration.

An additional issue is that the terminal value t_f is unknown. One approach to numerically find t_f is to relax one of the boundary conditions by introducing an unknown parameter and use Newton's method to update the initial guess for t_f and drive the unknown parameter to zero. For this particular problem, it has been found that faster convergence can be obtained by relaxing the initial condition r_0. For this choice, we want $r_0(k) - r_0 \to 0$, where $r_0(k)$ represents the relaxed initial condition when we provide $t_f(k)$ as the guess for the final time, and the index k denotes iterations of Newton's method to update t_f. The equations corresponding to one of these iterations are

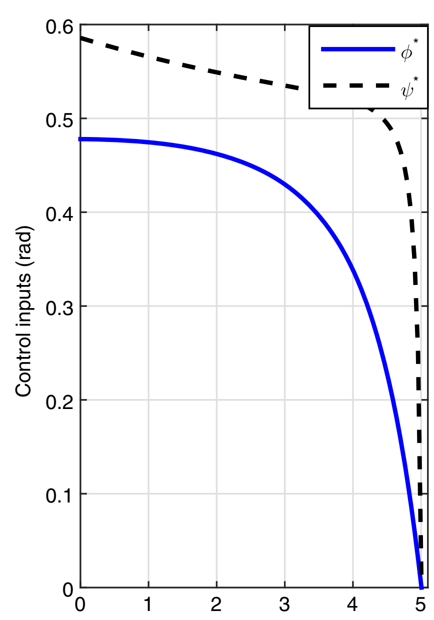
$$t_f'(k) = t_f'(k) - \frac{r_0(k) - r_0}{\dot r_0(k)} \qquad t_f(k) = t_f(0) - t_f'(k) \tag{59}$$

where the initial values of these iterations $t_f(0)$, $r_0(0)$, and $\dot r_0(0)$ can be obtained, for instance, from the analytical solution (58).

The TAD scenario has been analyzed as an open-loop optimal control problem or differential game in this Note. As such, the solutions are sensitive to changes in the optimal trajectories or unknown attacker guidance law. This can be solved by recomputing new solutions if new measurements do not agree with the expected solution. However, solving a TPBVP many times may not be suitable in this problem. Robust and computationally inexpensive solutions to the active target defense differential game are currently under study [28].



a) States



b) Control inputs

Fig. 4 Solution of Example 1: attacker implements PN guidance law.

VII. Examples

A. Example 1

Consider an attacker missile using PN guidance with navigation constant $N = 3$. Let the initial conditions be given by $R_0 = 5.9$ and $r_0 = 8$, and by $\theta_0 = 0.6$ and $\sigma_0 = -0.1$ rad. The initial LOS angle is given by $\lambda_0 = -1$ rad and the attacker is initially located at the coordinates (0,0). The capture radii are $R_c = 1$ and $r_c = 0.1$. The speed ratios are $\alpha = 0.3$ and $\beta = 0.85$. The defender missile is slower than the attacker missile. Without loss of generalization, we normalize with respect to the attacker's speed and we solve the problem using unitless separations (to obtain the real separations and time scale, we simply factor the attacker's speed as it is done in Example 3 in Sec. VII.C). Figure 4 shows the solution of the TPBVP. Figure 4a shows the optimal states: R^*, r^*, θ^*, and σ^*. Figure 4b shows the optimal control inputs for the target ϕ^* and for the defender ψ^*. The interception time is $t_f = 5.006$ and the final separation between target and attacker is $R(t_f) = 2.28$. Figure 5 shows the trajectories in the x-y plane, where the circles represent the starting points for each aircraft.

Remark 1: An important advantage of the cooperation between the target and the defender is that the defender missile's speed does not need to be equal to or greater than the attacker missile. As it was shown in the previous example, the defender is significantly slower than the attacker. Because the target and the defender cooperate for the defender to intercept the attacker, the target maneuvers to bring the attacker closer to the defender such that the terminal separation between target and attacker is maximized and, therefore, the target's chances of survival improve.

B. Example 2: Attacker Performance When $\beta = 1$

In this example, we consider an attacker missile and a defender missile with the same speed, and we investigate the performance of the attacker under the three different guidance laws discussed in this Note.

Let the initial conditions be given by $R_0 = 7$, $r_0 = 9$, and $\theta_0 = 0.8$ rad (for the PN guidance case, we consider $N = 3$ and $\sigma_0 = 0$ rad). The initial LOS angle is given by $\lambda_0 = 0$ rad and the attacker is initially located at the coordinates (0,0). The capture radii are $R_c = 1$ and $r_c = 0.1$. The relative target speed is given by $\alpha = 0.4$. The optimal states and optimal control inputs for the PP guidance law are shown in Fig. 6. Optimal trajectories in the inertial x-y frame are shown in Fig. 7.

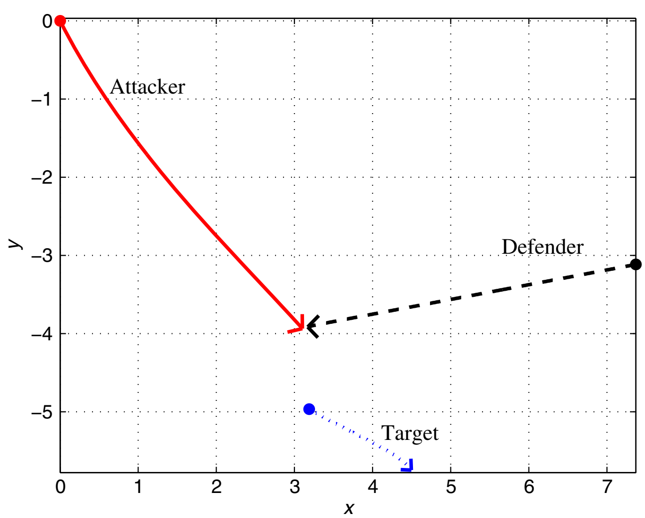


Fig. 5 Example 1: optimal trajectories when the attacker implements PN guidance.

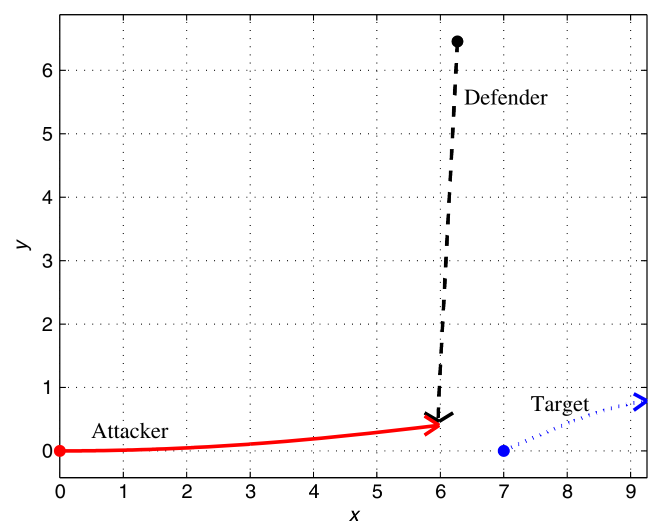
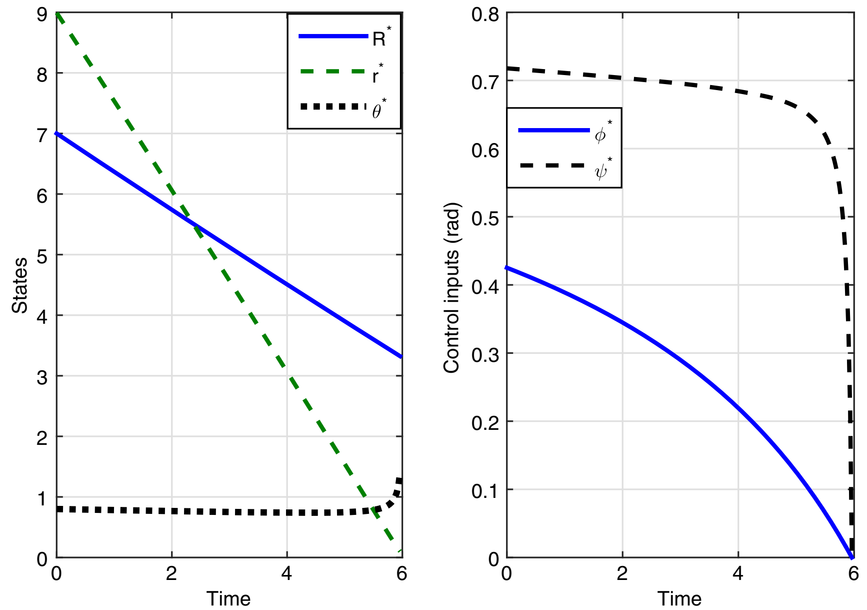
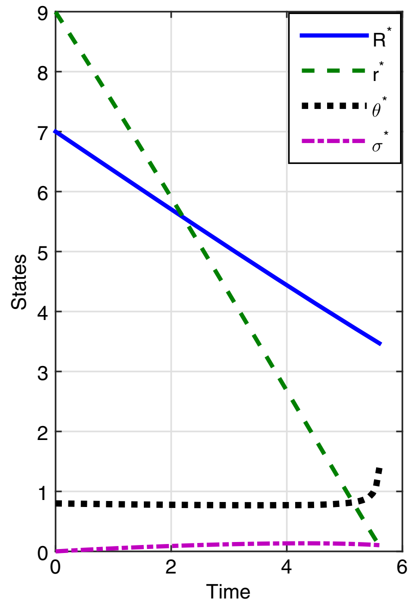


Fig. 7 Example 2: PP case; optimal trajectories in a fixed Cartesian frame.

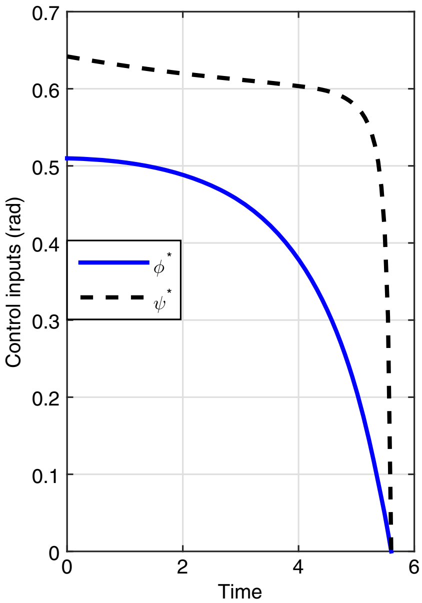


a) States **b) Control inputs**

Fig. 6 Solution of Example 2 for the PP case.



a) States



b) Control inputs

Fig. 8 Solution of Example 2 for the PN case.

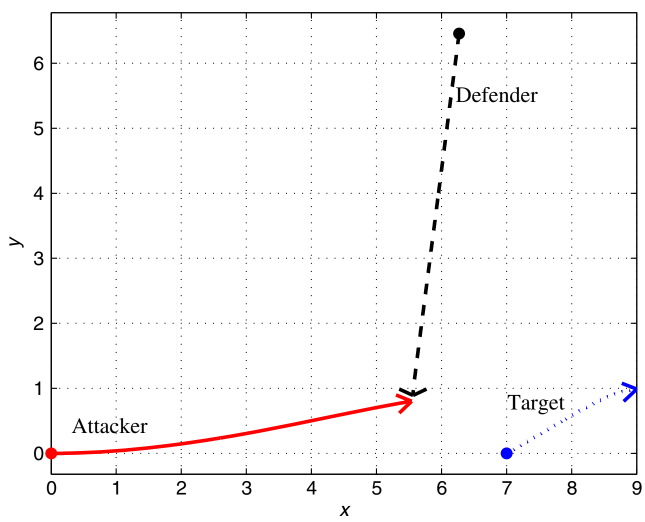


Fig. 9 Example 2: PN case; optimal trajectories in a fixed Cartesian frame.

Corresponding results when the attacker uses PN guidance are shown in Figs. 8 and 9, respectively. Finally, the optimal state response and the optimal control headings for the corresponding differential game are shown in Fig. 10, and the optimal trajectories are shown in Fig. 11. The interception times and, more important, the final separations between target and attacker are shown in Table 1.

It can be seen that, for the same problem parameters and initial conditions, the differential game strategy provides better outcomes for the attacker than using the PP or PN guidance laws. Although the attacker will be intercepted (if the defender is initially properly positioned), it can obtain the smallest final separation with respect to the target, and hopefully inflict some damage to it, if it implements its corresponding optimal strategy.

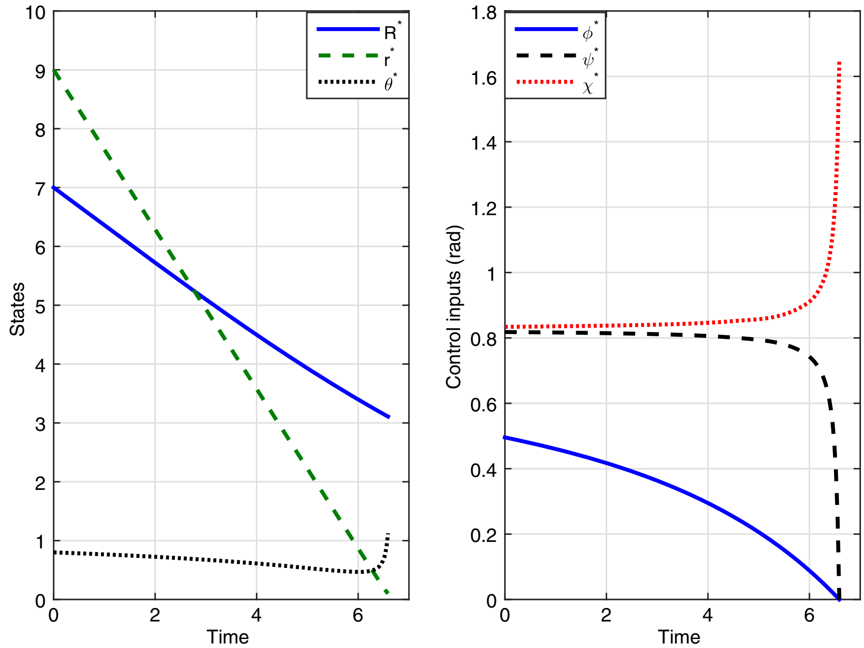
Remark 2: The scenarios addressed in this Note were formulated for applications involving beyond visual range (BVR) missiles. Thus, kinematic constraints associated with initial target and defender headings being different than the optimal headings provided within this document are considered as negligible. The numerical results show that optimal trajectories of the differential game are straight lines for every one of the three players. Thus, lateral accelerations are equal to zero. In the cases when the attacker implements PP or PN guidance laws, the defender lateral acceleration is equal to zero, whereas the target's optimal trajectory is not a straight line. However, lateral accelerations are low in general for the BVR scenario considered in this Note.

Here, we note that the angles ϕ^* and ψ^* are relative angles with respect to the reduced state space formed by the radials R and r and the angle θ. For instance, although in the end game, the angle ψ^* changes rapidly, in the inertial frame, the defender implements the heading angle $\hat{\psi}^*$. It can be seen that the change in ψ^* is compensated by the change in θ^* during the similar time interval and, consequently, the defender's actual heading $\hat{\psi}^*$ does not change rapidly, which is illustrated by the optimal defender trajectory in Fig. 5.

The study of optimal control laws for the target, defender, and attacker represents an important and challenging problem. One of the important aspects of this problem is that it provides existence of optimal trajectories for target evasion or capture. For instance, if the solution of the differential game provides a final separation $R(t_f) > R_c$, then there exists an optimal cooperative strategy for the target–defender team, such that the target will escape being captured by the attacker regardless of the attacker strategy. If, on the other hand, $R(t_f) < R_c$, then there exists an optimal strategy for the attacker that guarantees capture of the target regardless of the strategies implemented by the target–defender team. However, the approach described in this Note requires numerical solutions of a TPBVP. A closed-loop, analytical, robust (to nonoptimal maneuvers), optimal solution to the differential game will bring greater insight into the problem and better outcomes to each agent in this game. The differential game alternative is a topic of current research.

C. Example 3: Multi-Agents

Consider the operationally interesting scenario where several aircraft or UAVs can potentially fire a defender missile and help a target being pursued by an attacker missile. This scenario was described in [25,26]. In this problem, the target is already flying in a direction to evade the attacker, its potential fired defender will face the additional constraint to perform a 180 deg change of heading and being able to start following a coordinated trajectory that guarantees capture of the attacker. So far, air-to-air missiles that can turn around during a tail chase have not been operationally proven. Because the target is already running away from the attacker, launching its own



a) States b) Control inputs

Fig. 10 Solution of the differential game in Example 2.

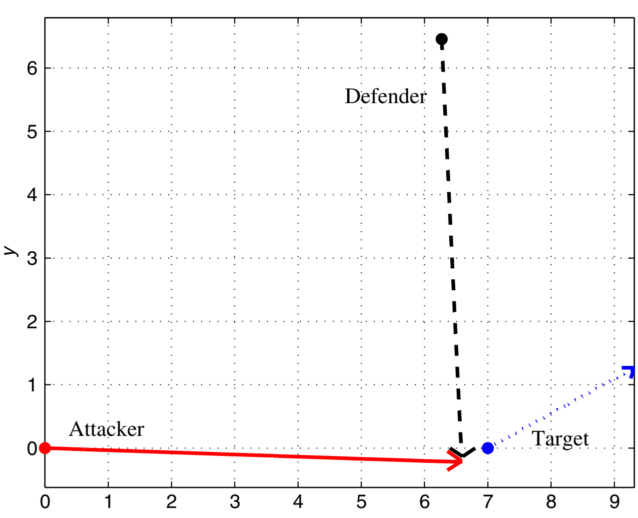


Fig. 11 Differential game in Example 2: optimal trajectories in a fixed Cartesian frame.

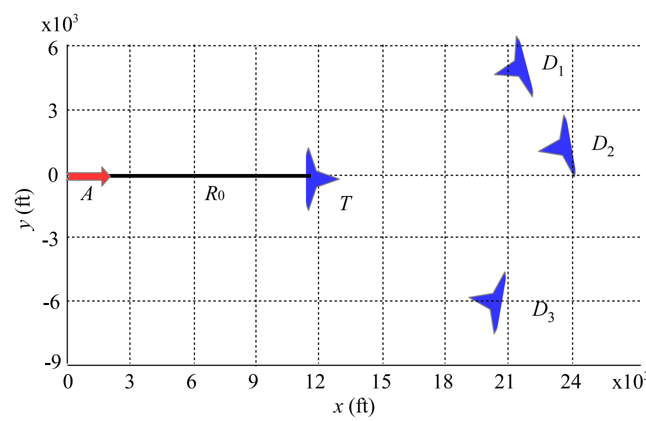


Fig. 12 Multi-UAV scenario.

defender would require an additional constraint on either the launched defender or on the target itself to turn back before firing. This maneuver could be problematic in practice. The other friendly UAVs may be aligned in such a way to have a better shot at the attacker even if they are significantly farther from the missile than the target. In addition, the other UAVs may be able to obtain better measurements of the attacker missile due to their particular headings. Thus, consider the scenario depicted in Fig. 12.

Table 1 Interception times and final separations in Example 2

Pursuit Method	t_f, s	$R(t_f)$
PP	5.962	3.323
PN	5.602	3.471
DG	6.589	3.108

The speed of the target is $V_T = 400$ ft/s. The speeds of the attacker and the defender are the same and are given by $V_A = V_D = 800$ ft/s. Then, the speed ratio is $\alpha = 0.5$. We consider the solutions of the differential game described in Sec. V. It is assumed that the initial position of the attacker is known. It is also assumed that the attacker knows the initial position of the selected defender as soon as it is fired and it computes its corresponding optimal solution based on that information. The attacker initial position is $A_0 = [0, 0]$ ft and the target initial position is $T_0 = [12,000; 0]$ ft. Three UAVs are within acceptable range and heading able to launch the defender missile; they are initially positioned as follows: $D_0^{(1)} = [21,500; 5000]$ ft, $D_0^{(2)} = [23,600; 1500]$ ft, and $D_0^{(3)} = [20,200; -6000]$ ft. Additionally, the capture radii are given by $R_c = 100$ and $r_c = 20$ ft. The initial conditions in the reduced state space can be obtained for every potential defender missile based on the previous data. Final potential separations are given by $R_1(t_f) = 6340$, $R_2(t_f) = 6082$, and $R_3(t_f) = 6505$ ft. It can be seen that UAV$_3$ provides the best outcome among the three choices because the corresponding final separation is the greatest of those three. The optimal state response and the optimal control headings for defender 3 (fired by UAV$_3$) are shown in Fig. 13, and the optimal trajectories are shown in Fig. 14.

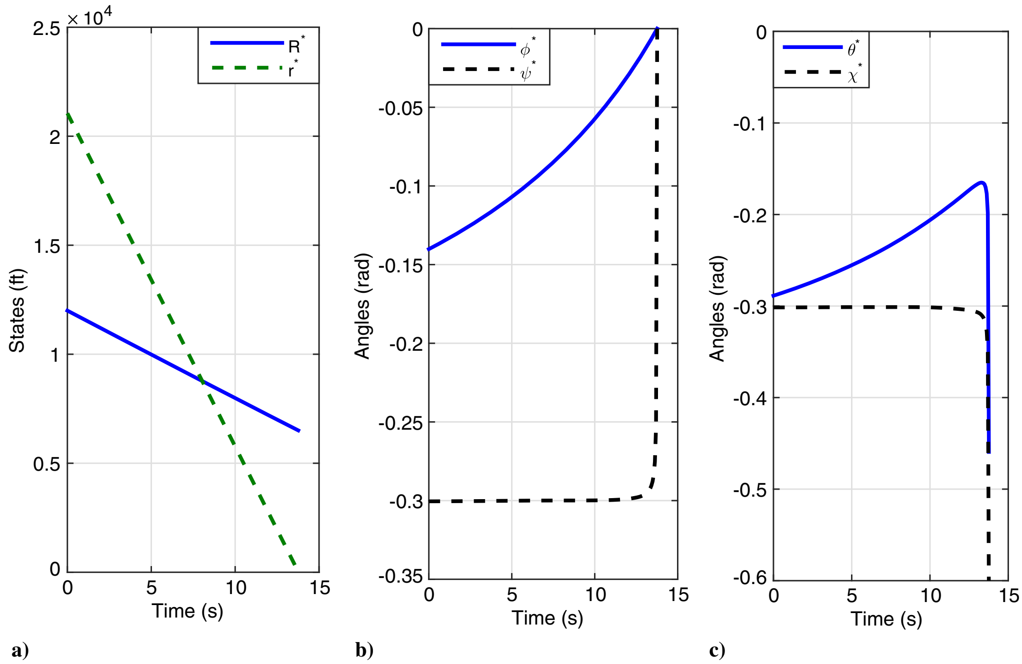


Fig. 13 Example 3, Optimal solutions: a) states R^* and r^*; b) control inputs ϕ^* and ψ^*; c) state θ^* and control input χ^*.

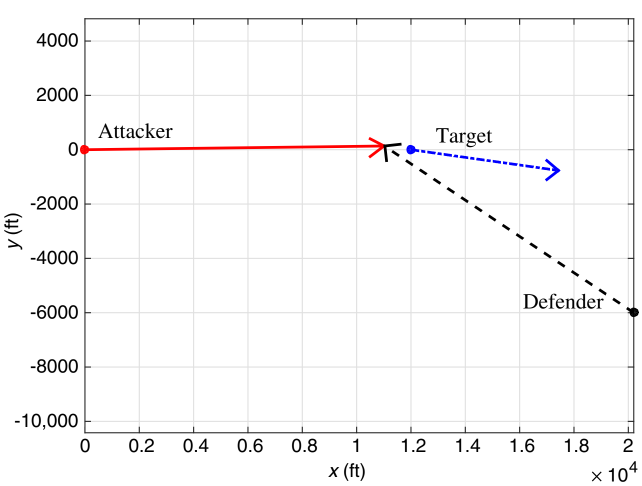


Fig. 14 Optimal trajectories of differential game in Example 3 for selected defender.

VIII. Conclusions

In this Note, optimal cooperative strategies for the target–attacker–defender scenario were derived. The first two scenarios investigated in this Note pertain to the case where the attacker uses pure pursuit and proportional navigation guidance. An optimal control problem was formulated and solved, which provides the joint optimal control inputs for the target aircraft and the defender missile team. The strategies presented in this Note do not restrict the defender to have greater speed than the attacker. This means that the interception of an attacking missile can be accomplished using a slower defender missile by implementing an optimal cooperative strategy between the target and the defender. A differential game formulation of the target defense scenario where the attacker is aware of the defender position was also analyzed. In this case, the attacker computes and implements the optimal heading angle that minimizes its final separation with respect to the target at his interception time instant. We showed that the solution to the differential game provides the best attacker strategy in terms of minimizing the separation between target and attacker to produce the greatest possible damage to the target.

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