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Brief paper

Decentralized event-triggered consensus with general linear dynamics[☆]

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ABSTRACT

The consensus problem with general linear dynamics and undirected graphs is studied in this paper by means of event-triggered control strategies. A novel consensus protocol is proposed, where each agent implements a model of the decoupled dynamics of its neighbors. Under this control strategy, transmission of information does not occur continuously but only at discrete points in time. The approach presented in this paper provides both a decentralized control law and a decentralized communication policy. We are able to design thresholds that only depend on local information and guarantee asymptotic consensus. Positive inter-event times are guaranteed for particular cases of the linear dynamics. In an extension, a positive constant is added to the thresholds in order to exclude Zeno behavior for general linear dynamics. The difference between agents trajectories can be bounded in this case and bounds on the state disagreement are derived.

1. Introduction

An increasing interest in controlling large scale dynamical systems composed of several to many autonomous mobile agents exists in different academic, commercial, and military areas. This thrust is related to the large number of applications in which a group of coordinated agents is potentially able to outperform a single or a number of systems operating independently (Ren, Beard, & Atkins, 2007). An important problem in Multi-Agent Systems (MAS) is to design and implement decentralized algorithms for control and communication of agents. It is well understood that each agent should be able to determine its own control laws independently and based only on local information (Ji & Egerstedt, 2007; Moreau, 2004; Ren et al., 2007). These papers consider agents with continuous-time dynamics and it is assumed that agents can have continuous access to the states of their neighbors. In many applications, continuous communication is not possible, and it becomes important to discern how frequently the agents should communicate in order to preserve the properties inherent in the corresponding control algorithms with continuous information exchange. The sampled-data approach is commonly used to estimate the sampling periods (Cao & Ren, 2010; Liu, Xie, & Wang, 2010; Qin & Gao, 2012). An important drawback of periodic transmission is that it requires synchronization between the agents, that is, all agents need to transmit their information at the same time instants and, in some cases, it requires a conservative sampling period for worst case situations. In the present paper, in lieu of periodic approaches, we use an asynchronous communication scheme based on event-triggered control strategies and we consider agents that are described by general linear models.

Consensus problems where all agents are described by general linear models have been considered by different authors (Li, Duan, & Chen, 2011; Ma & Zhang, 2010; Ren, 2008; Scardovi & Sepulchre, 2009; Su & Huang, 2012; Tuna, 2009). In Ren (2008) synchronization of oscillators described by second order dynamics and based on local interaction was studied. The authors of Scardovi and Sepulchre (2009) generalized the results in Ren (2008) to higher order systems described by linear models under the condition that all eigenvalues of the state matrix A lie in the imaginary axis. Similar work was presented in Tuna (2009). The work in Li et al. (2011) provided design methods for consensus of agents with linear dynamics and with a prescribed convergence speed. Consensus and leader-following tracking involving agents with linear dynamics and switching topology was studied in Su and Huang

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(2012). The authors of Ma and Zhang (2010) established conditions under which protocols will exist that achieve consensus. Such conditions are related to the communication network and to the stabilizability and detectability of the individual dynamics. All the previous references concerning linear systems assumed that continuous communication between agents is possible. The work in Wen, Duan, Ren, and Chen (2013) considers the consensus problem of agents with linear dynamics under communication constraints. Specifically, the authors consider the existence of continuous communication among agents for finite intervals of time and the total absence of communication among agents for other time intervals, and the minimum rate of continuous communication to no communication is given.

In this paper, apart from considering agents with general linear dynamics, we study consensus problems with limited communication and using event-triggered control strategies. Different to periodic (or time-triggered) implementations, in event-triggered control we have that information or measurements are not transmitted periodically in time but they are triggered by the occurrence of certain events. In event-triggered broadcasting (Anta & Tabuada, 2010; Astrom & Bernhardson, 2002; Donkers & Heemels, 2010; Garcia & Antsaklis, 2013; Tabuada, 2007; Wang & Lemmon, 2011), a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Event-triggered control strategies have been used for stabilization of multiple coupled subsystems as in Garcia and Antsaklis (2012) and Guinaldo, Dimarogonas, Johansson, Sanchez, and Dormido (2011). Consensus problems have also been studied using these techniques (Dimarogonas, Frazzoli, & Johansson, 2012; Garcia, Cao, Yu, Antsaklis, & Casbeer, 2013; Seyboth, Dimarogonas, & Johansson, 2013). Event-triggered control provides a more robust and efficient use of network bandwidth. Its implementation in MAS also provides a highly decentralized way to schedule transmission instants which does not require synchronization compared to periodic sampled-data approaches. Different authors have extended this approach, for instance, Chen and Hao (2012) studied event-triggered consensus for discrete time integrators. The authors of Yin and Yue (2013) used event-triggered techniques for consensus problems involving a combination of discrete time single and double integrators. The authors of Guo and Dimarogonas (2013) studied event-triggered consensus of single integrator systems using nonlinear consensus protocols.

The present paper represents one of the first attempts to extend the previous work on event-triggered control of MAS to the case of general linear dynamics. The contributions of this paper are three-fold. First, we present a novel method for consensus with limited communication in which each agent implements models of the decoupled dynamics of each one of their neighbors. This is in contrast to previous event-triggered control work where Zero-Order-Hold (ZOH) models are used, i.e., information from each neighbor is kept constant until a new update is received from the same neighbor. The consensus protocol proposed in the present paper is also different from the one used in Wen et al. (2013) where the states of neighbors are used to control the local input over the periods of continuous communication. However, during the time intervals when the agents cease to transmit, the input for each agent is chosen to be zero (no input is applied). In the limit case when only a single measurement is transmitted by each agent during a finite time interval instead of establishing continuous communication our proposed protocol offers a clear advantage. The results on this paper apply to multi-agent systems that can be represented as undirected graphs.

The second contribution of this paper consists on the design of decentralized event thresholds that guarantee asymptotic convergence. The threshold expressions for each agent are functions of only local information, i.e., of variables that can be obtained by the local agent without assuming continuous access to states of neighbors.

Since it is not possible, except for particular cases of the agent dynamics, to guarantee exclusion of Zeno phenomena, the third contribution of this paper extends the decentralized threshold design method in order to guarantee positive inter-event times for every agent. This is accomplished by including a positive constant term in the threshold expressions. Bounded consensus where the disagreement between any pair of states is bound is achieved in this case. Bounds on the state disagreement are provided.

The remainder of this paper is organized as follows. Section 2 provides a brief background on graph theory and describes the problem and the consensus protocol. Section 3 gives a result assuming continuous communication which will be used through the document. Decentralized thresholds are presented in Section 4. Section 5 offers modified decentralized event-triggered conditions in order to exclude Zeno behavior. Section 6 presents illustrative examples and Section 7 concludes the paper.

2. Preliminaries

2.1. Graph theory

Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consisting of a set of vertices or nodes $\mathcal{V} = \{1, \ldots, N\}$ and a set of edges $\mathcal{E}$. An edge between nodes i and j is represented by the pair $(i, j) \in \mathcal{E}$. A graph $\mathcal{G}$ is called undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ and the nodes are called adjacent. The adjacency matrix $\mathcal{A}$ is defined by $a_{ij} = 1$ if the nodes i and j are adjacent and $a_{ij} = 0$ otherwise. If $(j, i) \in \mathcal{E}$, then j is said to be a neighbor of i. The set $\mathcal{N}_i$ is called the set of neighbors of node i, and N_i is its cardinality. A node j is an element of $\mathcal{N}_i$ if $(j, i) \in \mathcal{E}$. A path from node i to node j is a sequence of distinct nodes that starts at i and ends at j, such that every pair of consecutive nodes is adjacent. An undirected graph is connected if there is a path between every pair of distinct nodes. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ where $\mathcal{D}$ represents the degree matrix which is a diagonal matrix with entries $d_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$. For undirected graphs, $\mathcal{L}$ is symmetric and positive semi-definite. $\mathcal{L}$ has zero row sums and, therefore, zero is an eigenvalue of $\mathcal{L}$ with associated eigenvector 1_N (a vector with all entries equal to one), that is, $\mathcal{L}1_N = 0$. If an undirected graph is connected then $\mathcal{L}$ has exactly one eigenvalue equal to zero and all its non-zero eigenvalues are positive; they can be set in increasing order $\lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \lambda_3(\mathcal{L}) \leq \cdots \leq \lambda_N(\mathcal{L})$, with $\lambda_1(\mathcal{L}) = 0$.

2.2. Problem statement

We consider the consensus problem with agents described by linear dynamics and with limited communication constraints where information from neighbors is not available continuously but only at some time instants. Event-triggered control implementations typically use a ZOH to compute the control input and the state error (Dimarogonas et al., 2012; Tabuada, 2007) in problems where continuous feedback is not available. Model-based approaches have been used more recently and it has been shown that they offer better performance by providing an estimate of the real state of a system between update intervals in Demir and Lunze (2012a) and Garcia and Antsaklis (2013). The model-based approach generalizes the traditional ZOH event-triggered control strategy. In ZOH strategies the agents that receive information from agent i maintain a piece-wise constant model of the state $x_i(t)$. The ZOH case is equivalent to implementing models when $A = 0$ in (3). However, the choice of ZOH is not suitable when considering general linear dynamics as it was in the case of single integrators (Dimarogonas et al., 2012; Garcia et al., 2013). Since trajectories can be unstable in general, a ZOH is not able to reduce communication as trajectories grow. In this case sensors need to generate events more frequently since the errors grow very quickly after each update. This situation increases communication and Zeno behavior may not be avoided. In contrast, the models are able to produce better estimates of real states than the ZOH and it is possible to show that Zeno behavior does not occur.

Consider a group of N agents with fixed communication graphs and fixed weights. Each agent can be described by the following:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1 \dots N, \tag{1}$$

with

$$u_i(t) = cF \sum_{j\in\mathcal{N}_i} (y_i(t) - y_j(t)), \quad i = 1 \dots N, \tag{2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$. The variables $y_i \in \mathbb{R}^n$ represent a model of the ith agent's state using the decoupled dynamics:

$$\begin{aligned} \dot{y}_i(t) &= Ay_i(t), \quad t \in [t_{k_i}, t_{k_i+1}) \\ y_i(t_{k_i}) &= x_i(t_{k_i}), \end{aligned} \tag{3}$$

for $i = 1 \dots N$. Every agent in the network implements a model of itself $y_i(t)$ and also models of its neighbors $y_j(t)$. Local events for agent i are defined as follows. When agent i triggers an event at time t_{k_i}, it will transmit its current state $x_i(t_{k_i})$ to its neighbors and agent i and its neighbors will update their local models $y_i(t)$. Since agent i and its neighbors use the same measurements to update the models and the model dynamics (3) represent the decoupled dynamics where all agents use the same state matrix, then the model states $y_i(t)$ implemented by agent i and by its neighbors are the same. The model update process is similar for all agents $i = 1 \dots N$. The local control input (2) is decentralized since it only depends on local information, that is, on the model states of the local agent and its neighbors. Note that the difference between the agent dynamics (1) and our proposed models (3) is given by the input term in (1) and this input decreases as the agents approach a consensus state. It can also be seen that in the particular case when systems (1) represent single integrator dynamics, then our models degenerate to ZOH models as in Dimarogonas et al. (2012) and Garcia et al. (2013).

Lemma 1. *Let $\mathcal{L}$ be the symmetric Laplacian of an undirected and connected graph. Then, consensus is achieved if and only if*

$$V = \xi^T \hat{\mathcal{L}} \xi = 0, \tag{4}$$

where $\hat{\mathcal{L}} = \mathcal{L} \otimes Q$, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\xi = \begin{bmatrix} \xi_1^T & \cdots & \xi_N^T \end{bmatrix}^T$, and $\xi_i \in \mathbb{R}^n$.

The proof is similar to the single dimension case in Olfati-Saber and Murray (2003).

3. Consensus with continuous measurements

This section provides an important result in Lemma 2 that will be useful in subsequent sections. Let us assume in this section that continuous communication between agents is possible; then (2) is given by:

$$u_i(t) = cF \sum_{j\in\mathcal{N}_i} (x_i(t) - x_j(t)), \quad i = 1 \dots N. \tag{5}$$

Assume that the pair (A, B) is controllable. Then, for $\alpha > 0$ there exists a symmetric and positive definite solution P to

$$PA + A^T P - 2PBB^T P + 2\alpha P < 0. \tag{6}$$

Let

$$F = -B^T P \tag{7}$$

$$c \geq 1/\lambda_2. \tag{8}$$

Theorem 1. *Assume that the pair (A, B) is controllable and the communication graph is connected and undirected. Define F and c as in (7) and (8). Then the following symmetric matrix*

$$\bar{\mathcal{L}} = \hat{\mathcal{L}} A_c + A_c^T \hat{\mathcal{L}} \tag{9}$$

has only n eigenvalues equal to zero and the rest of its eigenvalues are negative. In addition, the eigenvectors associated with its n zero eigenvalues belong to the subspace spanned by the eigenvectors associated with the n zero eigenvalues of $\hat{\mathcal{L}}$, where $\hat{\mathcal{L}} = \mathcal{L} \otimes P$, $A_c = \bar{A} + \bar{B}$, $\bar{A} = I_N \otimes A$, $\bar{B} = c\mathcal{L} \otimes BF$.

The proof of this theorem can be obtained following a similar approach to Li, Duan, Chen, and Huang (2010) and it is shown in Garcia, Cao, and Casbeer (2014).

Lemma 2. *Assume that the pair (A, B) is controllable and the communication graph is connected and undirected. Then, protocol (5), with F and c defined in (7) and (8), solves the consensus problem for agents described by (1). Furthermore, the Lyapunov function defined by $V = x^T \hat{\mathcal{L}} x$ has a time derivative along the trajectories of (1) with inputs (5) given by $\dot{V} = x^T \bar{\mathcal{L}} x$.*

From Theorem 1 it can be seen that $\dot{V}$ is negative when the overall system is in disagreement and is equal to zero only when the corresponding states are in total agreement. Different from consensus with single integrators, where the agents converge to a constant value, here it is only required that the difference between states of agents tends to zero, regardless of the particular response of the systems. As with many consensus algorithms, an estimate of the second smallest eigenvalue of the Laplacian matrix is required; this is the only global information needed by the agents. Algorithms for distributed estimation of the second eigenvalue of the Laplacian have been presented in Aragues, Shi, Dimarogonas, Sagues, and Johansson (2012) and Franceschelli, Gasparri, Giua, and Seatzu (2009). Readers are referred to these papers for details.

4. Decentralized event triggered consensus

In this section we derive decentralized thresholds that depend only on local information and can be measured and applied in a decentralized way.

Theorem 2. *Assume that the pair (A, B) is controllable and the communication graph is connected and undirected. Define F in (7) and $c_1 = c + c_2$ where $c \geq 1/\lambda_2$ and $c_2 > 0$. Then agents (1) with inputs (2) achieve consensus asymptotically if the events are triggered when*

$$\delta_i > \sigma z_i^T \Theta_i z_i, \tag{10}$$

where

$$\begin{aligned} \delta_i = {} & 2(c_2 - c)N_i z_i^T PBB^T Pe_i \\ & + \left[2cN_i^2(1 + b_i) + \frac{c_2 - c}{b_i} N_i \right. \\ & \left. + cN_i(N - 1)\left(b_i + \frac{3}{b_i}\right) \right] e_i^T PBB^T Pe_i, \end{aligned} \tag{11}$$

$0 < \sigma < 1$, $z_i = \sum_{j\in\mathcal{N}_i} (y_i - y_j)$, and

$$\Theta_i = (2c_2 - b_i N_i (c_2 - c))PBB^T P. \tag{12}$$

The parameter b_i is given by $0 < b_i < \frac{2c_2}{N_i(c_2 - c)}$ if $c_2 > c$, and $b_i > 0$ otherwise.

Proof. Let us start by defining the following: $\hat{\mathcal{L}}e = \left[\hat{\mathcal{L}}e_1^T \dots \hat{\mathcal{L}}e_N^T\right]^T$ and $\bar{B}e = \left[\bar{B}e_1^T \dots \bar{B}e_N^T\right]^T$ where $\hat{\mathcal{L}}e_i = P\sum_{j\in\mathcal{N}_i}\left(e_i - e_j\right)$, $\bar{B}e_i = cBF\sum_{j\in\mathcal{N}_i}\left(e_i - e_j\right)$, and $e_i(t) = y_i(t) - x_i(t)$. Furthermore,

$$e^T\hat{\mathcal{L}}\bar{B}e = -c\sum_{i=1}^{N}\left[\sum_{j\in\mathcal{N}_i}(e_i - e_j)^T PBB^T P\sum_{k\in\mathcal{N}_i}(e_i - e_k)\right] \tag{13}$$

$$\begin{aligned} y^T\hat{\mathcal{L}}\bar{B}y &= -c\sum_{i=1}^{N}\left[\sum_{j\in\mathcal{N}_i}(y_i - y_j)^T PBB^T P\sum_{k\in\mathcal{N}_i}(y_i - y_k)\right] \\ &= -c\sum_{i=1}^{N} z_i^T PBB^T Pz_i. \end{aligned} \tag{14}$$

Consider the candidate Lyapunov function $V = x^T\bar{\mathcal{L}}x$ and evaluate the derivative along the trajectories of systems (1) with inputs (2). Assume that $b_i = b$ for simplicity of notation (note that we can always select $b = \min(b_i)$). We can express $\dot{V}$ as follows:

$$\begin{aligned} \dot{V} &= x^T\hat{\mathcal{L}}\left(\bar{A}x + \bar{B}_1 y\right) + \left(\bar{A}x + \bar{B}_1 y\right)^T\hat{\mathcal{L}}x \\ &= (y-e)^T\hat{\mathcal{L}}\left(\bar{A}(y-e) + \bar{B}_1 y\right) \\ &\quad + \left(\bar{A}(y-e) + \bar{B}_1 y\right)^T\hat{\mathcal{L}}(y-e) \\ &= (y-e)^T\hat{\mathcal{L}}\left(A_{c_1}y - \bar{A}e\right) + \left(A_{c_1}y - \bar{A}e\right)^T\hat{\mathcal{L}}(y-e), \end{aligned} \tag{15}$$

where $A_{c_1} = \bar{A} + \bar{B}_1$, $\bar{B}_1 = c_1\mathcal{L}\otimes BF$. Also $x = \left[x_1^T \dots x_N^T\right]^T$, $y = \left[y_1^T \dots y_N^T\right]^T$, $e = \left[e_1^T \dots e_N^T\right]^T$.

Eq. (15) can be written as:

$$\begin{aligned} \dot{V} &= y^T\left(\hat{\mathcal{L}}\left(A_c + \bar{B}_2\right) + \left(A_c + \bar{B}_2\right)^T\hat{\mathcal{L}}\right)y \\ &\quad - y^T\hat{\mathcal{L}}\bar{A}e - e^T\hat{\mathcal{L}}A_{c_1}y + e^T\hat{\mathcal{L}}\bar{A}e \\ &\quad - y^T A_{c_1}^T\hat{\mathcal{L}}e - e^T\bar{A}^T\hat{\mathcal{L}}y + e^T\bar{A}^T\hat{\mathcal{L}}e \\ &= y^T\bar{\mathcal{L}}y + y^T\left(\hat{\mathcal{L}}\bar{B}_2 + \bar{B}_2^T\hat{\mathcal{L}}\right)y \\ &\quad - y^T\hat{\mathcal{L}}\bar{A}e - e^T\hat{\mathcal{L}}(A_c + \bar{B}_2)y + e^T\hat{\mathcal{L}}\bar{A}e \\ &\quad - y^T(A_c + \bar{B}_2)^T\hat{\mathcal{L}}e - e^T\bar{A}^T\hat{\mathcal{L}}y + e^T\bar{A}^T\hat{\mathcal{L}}e, \end{aligned} \tag{16}$$

where $A_c = \bar{A} + \bar{B}$, $\bar{B} = c\mathcal{L}\otimes BF$, $\bar{B}_2 = c_2\mathcal{L}\otimes BF$. Note that:

$$\begin{aligned} y^T\bar{\mathcal{L}}y &= (x+e)^T\bar{\mathcal{L}}(x+e) \\ &= x^T\bar{\mathcal{L}}x - e^T\bar{\mathcal{L}}e + e^T\bar{\mathcal{L}}y + y^T\bar{\mathcal{L}}e \\ &= x^T\bar{\mathcal{L}}x - e^T\left(\hat{\mathcal{L}}\bar{A} + \bar{A}^T\hat{\mathcal{L}} + \hat{\mathcal{L}}\bar{B} + \bar{B}^T\hat{\mathcal{L}}\right)e \\ &\quad + e^T\hat{\mathcal{L}}A_c y + e^T A_c^T\hat{\mathcal{L}}y + y^T\hat{\mathcal{L}}A_c e + y^T A_c^T\hat{\mathcal{L}}e. \end{aligned} \tag{17}$$

Substituting (17) in (16), canceling corresponding terms and by observing that $A_c = \bar{A} + \bar{B}$ we have:

$$\begin{aligned} \dot{V} &= x^T\bar{\mathcal{L}}x + y^T(\hat{\mathcal{L}}\bar{B}_2 + \bar{B}_2^T\hat{\mathcal{L}})y - e^T(\hat{\mathcal{L}}\bar{B} + \bar{B}^T\hat{\mathcal{L}})e \\ &\quad + y^T\hat{\mathcal{L}}\bar{B}e + e^T\bar{B}^T\hat{\mathcal{L}}y - y^T\bar{B}_2^T\hat{\mathcal{L}}e - e^T\hat{\mathcal{L}}\bar{B}_2 y. \end{aligned} \tag{18}$$

The expression at which we arrive at (18) after several steps is an important one. It contains the term $x^T\bar{\mathcal{L}}x \le 0$ for convergence and it also contains

$$y^T(\hat{\mathcal{L}}\bar{B}_2 + \bar{B}_2^T\hat{\mathcal{L}})y = \sum_{i=1}^{N} -2c_2 z_i^T PBB^T Pz_i.$$

Each term $z_i^T PBB^T Pz_i$ is only a function of model variables which are available locally and can be measured in order to design local thresholds that trigger measurement updates. In order to design decentralized threshold conditions we write (18) in the following form:

$$\begin{aligned} \dot{V} &= x^T\bar{\mathcal{L}}x + \sum_{i=1}^{N}\Bigg[-2c_2 z_i^T PBB^T Pz_i \\ &\quad + 2c\sum_{j\in\mathcal{N}_i}(e_i - e_j)^T PBB^T P\sum_{k\in\mathcal{N}_i}(e_i - e_k) \\ &\quad - 2cz_i^T PBB^T P\sum_{j\in\mathcal{N}_i}(e_i - e_j) \\ &\quad + 2c_2 z_i^T PBB^T P\sum_{j\in\mathcal{N}_i}(e_i - e_j)\Bigg] \\ &= x^T\bar{\mathcal{L}}x + \sum_{i=1}^{N}\Bigg[-2c_2 z_i^T PBB^T Pz_i \\ &\quad + 2c\sum_{j\in\mathcal{N}_i}(e_i - e_j)^T PBB^T P\sum_{k\in\mathcal{N}_i}(e_i - e_k) \\ &\quad + 2(c_2 - c)z_i^T PBB^T P\sum_{j\in\mathcal{N}_i}(e_i - e_j)\Bigg]. \end{aligned} \tag{19}$$

Note that:

$$\begin{aligned} &\sum_{j\in\mathcal{N}_i}(e_i - e_j)^T PBB^T P\sum_{k\in\mathcal{N}_i}(e_i - e_k) \\ &= \left(\sum_{j\in\mathcal{N}_i} e_i - \sum_{j\in\mathcal{N}_i} e_j\right)^T PBB^T P\left(\sum_{k\in\mathcal{N}_i} e_i - \sum_{k\in\mathcal{N}_i} e_k\right) \\ &= N_i^2 e_i^T PBB^T Pe_i - 2N_i e_i^T PBB^T P\sum_{j\in\mathcal{N}_i} e_j \\ &\quad + \sum_{j\in\mathcal{N}_i} e_j^T PBB^T P\sum_{k\in\mathcal{N}_i} e_k. \end{aligned} \tag{20}$$

The last term in (20) can be bounded as follows:

$$\begin{aligned} \sum_{j\in\mathcal{N}_i} e_j^T PBB^T P\sum_{k\in\mathcal{N}_i} e_k &\le \left|\sum_{j\in\mathcal{N}_i} e_j^T PBB^T P\sum_{k\in\mathcal{N}_i} e_k\right| \\ &\le \sum_{j\in\mathcal{N}_i}\sum_{k\in\mathcal{N}_i}\left|e_j^T PBB^T Pe_k\right| \\ &\le N_i\left(\frac{b}{2} + \frac{1}{2b}\right)\sum_{j\in\mathcal{N}_i} e_j^T PBB^T Pe_j \end{aligned} \tag{21}$$

where the inequality $\left|x^T y\right| \le \frac{b}{2}x^T x + \frac{1}{2b}y^T y$ for $b > 0$ was used. Since the communication graph is undirected we have the following properties

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} e_j^T PBB^T Pe_j = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} e_i^T PBB^T Pe_i \tag{22}$$

and

$$\begin{aligned} \sum_{i=1}^{N} N_i\sum_{j\in\mathcal{N}_i} e_j^T PBB^T Pe_j &= \sum_{i=1}^{N} N_j\sum_{j\in\mathcal{N}_i} e_i^T PBB^T Pe_i \\ &\le \sum_{i=1}^{N}(N-1)\sum_{j\in\mathcal{N}_i} e_i^T PBB^T Pe_i. \end{aligned} \tag{23}$$

Then (21) satisfies the following:

$$\sum_{i=1}^{N} N_i \left(\frac{b}{2}+\frac{1}{2b}\right)\sum_{j\in\mathcal{N}_i} e_j^T PBB^T Pe_j$$
$$= \sum_{i=1}^{N} N_j \left(\frac{b}{2}+\frac{1}{2b}\right)\sum_{j\in\mathcal{N}_i} e_i^T PBB^T Pe_i$$
$$\leq \sum_{i=1}^{N} (N-1) \left(\frac{b}{2}+\frac{1}{2b}\right)\sum_{j\in\mathcal{N}_i} e_i^T PBB^T Pe_i. \tag{24}$$

Applying (20) to (19) we obtain the following expression

$$\begin{aligned}\dot V =\ & x^T\bar{\mathcal{L}}x + \sum_{i=1}^{N}\Bigg[-2c_2 z_i^T PBB^T Pz_i \\ & + 2cN_i^2 e_i^T PBB^T Pe_i - 4cN_i \sum_{j\in\mathcal{N}_i} e_j^T PBB^T Pe_i \\ & + 2c\sum_{j\in\mathcal{N}_i} e_j^T PBB^T P\sum_{k\in\mathcal{N}_i} e_k + 2(c_2-c) z_i^T PBB^T P\sum_{j\in\mathcal{N}_i} e_i \\ & - 2(c_2-c) z_i^T PBB^T P\sum_{j\in\mathcal{N}_i} e_j\Bigg].\end{aligned}$$

Now we can use (24) and the inequality $|x^T y| \leq \frac{b}{2}x^T x + \frac{1}{2b}y^T y$ for $b > 0$ to obtain:

$$\begin{aligned}\dot V \leq\ & x^T\bar{\mathcal{L}}x + \sum_{i=1}^{N}\Bigg[-2c_2 z_i^T PBB^T Pz_i \\ & + 2cN_i^2 e_i^T PBB^T Pe_i + 2(c_2-c)N_i z_i^T PBB^T Pe_i \\ & + 4cN_i\sum_{j\in\mathcal{N}_i}\left(\frac{b}{2}e_i^T PBB^T Pe_i + \frac{1}{2b}e_j^T PBB^T Pe_j\right) \\ & + 2(c_2-c)\sum_{j\in\mathcal{N}_i}\frac{b}{2}z_i^T PBB^T Pz_i + 2(c_2-c)\sum_{j\in\mathcal{N}_i}\frac{1}{2b}e_j^T PBB^T Pe_j \\ & + 2c(N-1)\left(\frac{b}{2}+\frac{1}{2b}\right)\sum_{j\in\mathcal{N}_i} e_i^T PBB^T Pe_i\Bigg].\end{aligned}$$

Using the properties (22) and (23) and rearranging terms we have

$$\dot V \leq x^T\bar{\mathcal{L}}x + \sum_{i=1}^{N}\left[\left(bN_i(c_2-c) - 2c_2\right) z_i^T PBB^T Pz_i + \delta_i\right]. \tag{25}$$

When threshold (10) holds, say, at time t_{k_i} then the error resets to zero, that is, $e_i(t_{k_i}) = 0$. By definition of δ_i in (11) we have $e_i(t_{k_i}) = 0 \Rightarrow \delta_i(t_{k_i}) = 0$; then the following holds

$$\delta_i \leq \sigma z_i^T \Theta_i z_i. \tag{26}$$

Consequently,

$$\dot V \leq x^T\bar{\mathcal{L}}x + \sum_{i=1}^{N}(\sigma-1) z_i^T \Theta_i z_i. \tag{27}$$

Since $\sigma - 1 < 0$ and all $\Theta_i \geq 0$ we have that $(\sigma-1)z_i^T\Theta_i z_i \leq 0$ for $i = 1, \ldots, N$ and

$$\dot V \leq x^T\bar{\mathcal{L}}x \tag{28}$$

and the agents achieve consensus asymptotically. •

Remark. Note that the variables used to compute (10)–(12), which define the events at node i, are available locally. We have that the decentralized threshold (10) guarantees asymptotic convergence but it does not necessarily guarantee positive inter-event times. The matrix Θ_i used for computing the threshold is positive semi-definite in general and positive definite only for particular cases of B. In the less restrictive case, Θ_i being positive semi-definite implies that $z_i^T\Theta_i z_i = 0$ for some $z_i \neq 0$ and continuous communication may be unavoidable. A solution to this problem is proposed in the next section.

5. Decentralized event triggered consensus with guaranteed inter-event times

In this section we consider a threshold that incorporates a small positive constant η to guarantee positive inter-event times. Asymptotic convergence is not obtained but state disagreement can be bounded. The selection of η provides flexibility in the design of the event-triggered protocol by offering a tradeoff between reduction of communication and size of disagreement bounds.

Theorem 3. *Assume that the pair (A, B) is controllable and the communication graph is connected and undirected. Define F in (7) and $c_1 = c + c_2$ where $c \geq 1/\lambda_2$ and $c_2 > 0$. Then agents (1) with inputs (2) achieve bounded consensus where the difference between any two states is bounded by*

$$\lim_{t\to\infty}\|x_i(t) - x_j(t)\|^2 \leq \frac{N\eta}{\beta\lambda_{\min}(P)} \tag{29}$$

for $i, j = 1, \ldots, N$, if the events are triggered when

$$\delta_i > \sigma z_i^T\Theta_i z_i + \eta, \tag{30}$$

where δ_i and Θ_i are given by (11) and (12), respectively. Additionally $0 < \sigma < 1$, $\eta > 0$, $z_i = \sum_{j\in\mathcal{N}_i}(y_i - y_j)$, and $0 < b_i < \frac{2c_2}{N_i(c_2-c)}$ if $c_2 > c$, and $b_i > 0$ otherwise. Furthermore, the agents do not exhibit Zeno behavior and the inter-event times $t_{k_i+1} - t_{k_i}$ for every agent $i = 1, \ldots, N$ are bounded by the positive times τ_i, that is

$$\tau_i \leq t_{k_i+1} - t_{k_i} \tag{31}$$

where

$$\tau_i = \frac{\ln\left[\left(\frac{\eta}{k_2} + g^2\right)^{1/2} - g + 1\right]}{\|A\|} \tag{32}$$

and the parameters k_2 and g are given by:

$$\begin{aligned} k_2 &= \left|2cN_i^2(1+b_i) + \frac{c_2-c}{b_i}N_i + cN_i(N-1)\left(b_i + \frac{3}{b_i}\right)\right| \\ &\quad\times \left\|PBB^T P\right\|\left(\frac{z_{i,\max}\|cBF\|}{\|A\|}\right)^2 \\ g &= \frac{|2(c_2-c)N_i|\,\|A\|}{2\left|2cN_i^2(1+b_i) + \frac{c_2-c}{b_i}N_i + cN_i(N-1)\left(b_i+\frac{3}{b_i}\right)\right|\,\|cBF\|}\end{aligned} \tag{33}$$

and $z_{i,\max}$ represents a bound on $z_i(t)$, that is, $\|z_i(t)\| \leq z_{i,\max}$.

Proof. Let us start by deriving a bound on the difference between the states of any two agents. The difference between thresholds (10) and (30) is the positive constant term η. Then, by considering the Lyapunov function $V = x^T\hat{\mathcal{L}}x$, we observe that $\dot V$ satisfies (25). Now we use threshold (30) to limit the growth of the term δ_i. By applying this threshold condition we have that

$$\delta_i \leq \sigma z_i^T\Theta_i z_i + \eta \tag{34}$$
$$\dot V \leq x^T\bar{\mathcal{L}}x + N\eta. \tag{35}$$

We use the fact that $\hat{\mathcal{L}}$ is positive semi-definite and it has n zero eigenvalues with corresponding eigenvectors $v_1 \dots v_n$. Let $x = x_1 + x_2$ such that $\langle x_1^T, x_2 \rangle = 0$ and $\hat{\mathcal{L}} x_1 = 0$, that is, x_1 belongs to the subspace spanned by $v_1 \dots v_n$. Consider

$$x^T \hat{\mathcal{L}} x = (x_1 + x_2)^T \hat{\mathcal{L}} (x_1 + x_2) = x_2^T \hat{\mathcal{L}} x_2 \tag{36}$$

also

$$x_2^T \hat{\mathcal{L}} x_2 \le \lambda_{\max}(\hat{\mathcal{L}}) x_2^T x_2. \tag{37}$$

From Theorem 1 $-\bar{\mathcal{L}}$ is positive semi-definite with n zero eigenvalues and $\bar{\mathcal{L}} x_1 = 0$. We can see that

$$x^T \bar{\mathcal{L}} x = (x_1 + x_2)^T \bar{\mathcal{L}} (x_1 + x_2) = x_2^T \bar{\mathcal{L}} x_2 \tag{38}$$

and

$$x_2^T (-\bar{\mathcal{L}}) x_2 \ge \lambda_{\min \neq 0}(-\bar{\mathcal{L}}) x_2^T x_2 \tag{39}$$

where $\lambda_{\min \neq 0}(-\bar{\mathcal{L}}) > 0$ is the minimum eigenvalue of $-\bar{\mathcal{L}}$ other than zero. Combining the expressions above we have

$$x_2^T \bar{\mathcal{L}} x_2 \le -\frac{\lambda_{\min \neq 0}(-\bar{\mathcal{L}})}{\lambda_{\max}(\hat{\mathcal{L}})} x_2^T \hat{\mathcal{L}} x_2. \tag{40}$$

Then $\dot{V}$ can be bounded as follows:

$$\dot{V} \le -\beta x^T \hat{\mathcal{L}} x + N\eta = -\beta V + N\eta, \tag{41}$$

where $\beta = \frac{\lambda_{\min \neq 0}(-\bar{\mathcal{L}})}{\lambda_{\max}(\hat{\mathcal{L}})} > 0$. Solving (41) we have that

$$\begin{aligned} V(t) &\le e^{-\beta t} V(0) + N\eta \int_0^t e^{-\beta(t-\tau)} d\tau \\ &\le \left(V(0) - \frac{N\eta}{\beta} \right) e^{-\beta t} + \frac{N\eta}{\beta}. \end{aligned} \tag{42}$$

Expression (42) represents a bound on the consensus states as a function of the initial separation of the agents $V(0) = x(0)^T \hat{\mathcal{L}} x(0)$. We can express $V(t) = \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (x_i - x_j)^T P (x_i - x_j)$ and a bound on the difference between any two states i, j can be obtained as follows. Since the graph is undirected the term $(x_i - x_j)^T P (x_i - x_j)$ appears twice in the summation $V(t)$, then we can write

$$\begin{aligned} \lambda_{\min}(P) \left\| x_i - x_j \right\|^2 &\le (x_i - x_j)^T P (x_i - x_j) \\ &\le \left(V(0) - \frac{N\eta}{\beta} \right) e^{-\beta t} + \frac{N\eta}{\beta}. \end{aligned} \tag{43}$$

Finally, the difference between any two states can be bounded as in (29).

We now prove that the inter-event times are lower bounded by a positive constant. Consider the error dynamics of agent i for $t \in [t_{k_i}, t_{k_i+1})$ as follows:

$$\dot{e}_i = \dot{y}_i - \dot{x}_i = A y_i - A x_i - B u_i = A e_i - cBF z_i. \tag{44}$$

The error dynamics can be bounded as follows:

$$\begin{aligned} \frac{d}{dt} \|e_i\| &= (e_i^T e_i)^{-1/2} e_i^T \dot{e}_i \\ &\le \|A\| \|e_i\| + z_{i,\max} \|cBF\|. \end{aligned} \tag{45}$$

In the first part of this proof we showed a bound on the consensus dynamics, i.e., $z_{i,\max}$ exists and it is finite. Consider the following differential equation

$$\dot{q} = \|A\| q + z_{i,\max} \|cBF\|; \tag{46}$$

then we have that $\|e_i(t)\| \le q(t, q(0))$ where $q(t, q(0))$ is the solution of (46) with $q(0) = 0$. Such solution is given by:

$$q(t, 0) = \frac{z_{i,\max} \|cBF\|}{\|A\|} \left(e^{\|A\| t} - 1 \right). \tag{47}$$

We can bound the growth of δ_i as follows

$$\begin{aligned} \delta_i &\le z_{i,\max} \left| 2(c_2 - c) N_i \right| \left\| PBB^T P \right\| \|e_i\| \\ &\quad + \left| 2cN_i^2 (1 + b_i) + \frac{c_2 - c}{b_i} N_i \right. \\ &\quad \left. + cN_i (N-1) \left(b_i + \frac{3}{b_i} \right) \right| \left\| PBB^T P \right\| \|e_i\|^2 \\ &\le k_1 \left(e^{\|A\| t} - 1 \right) + k_2 \left(e^{\|A\| t} - 1 \right)^2, \end{aligned} \tag{48}$$

where

$$k_1 = z_{i,\max}^2 \left| 2(c_2 - c) N_i \right| \left\| PBB^T P \right\| \frac{\|cBF\|}{\|A\|}. \tag{49}$$

We can see that the time it takes δ_i to grow from zero to $\sigma z_i^T \Theta z_i + \eta$ is no smaller than the time it takes the last expression in (48) to reach η. Then we solve for the time τ_i in the following equation:

$$k_1 \left(e^{\|A\| \tau_i} - 1 \right) + k_2 \left(e^{\|A\| \tau_i} - 1 \right)^2 = \eta. \tag{50}$$

Eq. (50) can be written in the following form:

$$\left(e^{\|A\| \tau_i} - 1 + \frac{k_1}{2k_2} \right)^2 = \frac{\eta}{k_2} + \frac{k_1^2}{4k_2^2}. \tag{51}$$

Since the right hand side of (51) is positive we obtain:

$$e^{\|A\| \tau_i} = \left(\frac{\eta}{k_2} + \frac{k_1^2}{4k_2^2} \right)^{1/2} - \frac{k_1}{2k_2} + 1. \tag{52}$$

Note that $\left(\frac{\eta}{k_2} + \frac{k_1^2}{4k_2^2} \right)^{1/2} - \frac{k_1}{2k_2} > 0$ and therefore the right hand side of (52) is strictly greater than one. Solving for the time $\tau_i > 0$ we obtain (32) which shows that the inter-event times are strictly positive and we can guarantee that Zeno behavior does not occur at any node. •

Remark. The results provided in this paper hold for agents described by general linear dynamics. Single integrators and double integrators are particular cases covered by this framework. The single integrator is modeled as a ZOH and the double integrator is modeled similar to Seyboth et al. (2013), that is, velocity as a ZOH and position as a first-order-hold model. In Demir and Lunze (2012a,b) a similar approach to the one shown here was proposed where only constant thresholds were considered. Design and online computations are simplified by using only a constant threshold. The use of a time-varying threshold, as described in this paper, improves the overall performance by allowing large thresholds at the beginning to avoid frequent updates and then smaller thresholds are used to further reduce disagreement or to obtain asymptotic convergence when possible.

6. Example

Consider three third-order agents with unstable linear dynamics given by:

$$A = \begin{bmatrix} 0.48 & 0.29 & -0.3 \\ 0.13 & 0.23 & 0 \\ 0 & -1.2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ -1.5 & 1 \\ 0 & 1 \end{bmatrix}.$$

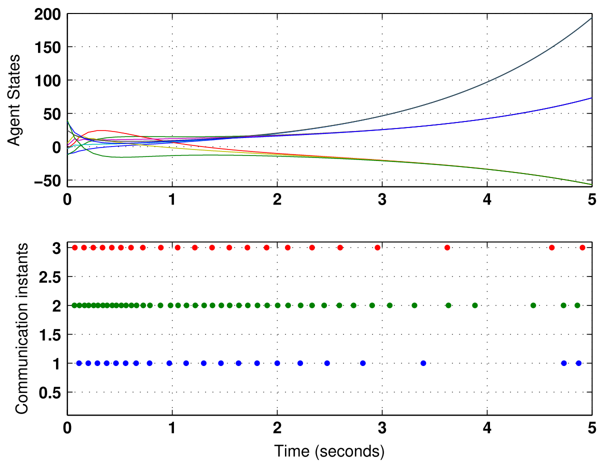

Fig. 1. Top: states of three agents. Bottom: transmission instants for each one of the three agents.

Matrix P is obtained by solving (6):

$$P = \begin{bmatrix} 4.8436 & 5.4783 & -1.1082 \\ 5.4783 & 7.0514 & -1.4299 \\ -1.1082 & -1.4299 & 0.3778 \end{bmatrix}$$

and $\eta = 0.3$. The nonzero elements of the undirected adjacency matrix are $a_{12} = a_{23} = 1$. Fig. 1 shows the response of the agents. It can be seen that each one of the three dimensions of the agents converge to a different trajectory. This figure also shows the time instants when each agent transmits a measurement update. It can be seen that agents reduce communication as the difference between their states is reduced and they converge to the same trajectories. This is expected since agents are able to obtain better estimates, as given by the models, of their neighbors states. Fig. 2 shows that the difference between states of agents in every dimension converges to zero and remains bounded. In order to draw a comparison we use the same example, parameters, and initial conditions, but the agents use ZOH devices which keep the last received update constant. Fig. 3 shows the response of the agents and their corresponding transmission instants. It is clear that in the ZOH case more communication is needed to obtain similar convergence. In the ZOH case agents increase communication with time since the associated errors grow very quickly after each transmission. The number of updates in the ZOH case shown in Fig. 3 for each agent were: 494, 560, and 477. In contrast, the number of updates when agents use models of neighbors shown in Fig. 1 (which covers the same simulation time) for each agent were: 21, 44, and 22. These numbers represent a very small fraction of the number of transmissions using ZOH.

7. Conclusions

A novel consensus protocol has been described in this paper. This protocol has been shown to be useful in problems where agents are not able to communicate continuously but they only transmit updates at some time instants. Decentralized events have been designed in order for each agent to communicate only when it is necessary, that is, when a function of discrepancy between real and model states is greater than a specified threshold. The decentralized event triggered technique allows each agent to transmit information based on its own decisions, since there is no requirement for synchronization of updates as in sampled-data-like approaches. Illustrative examples have been provided that show the performance of this framework in terms of convergence, reduction of communication, and freedom of every agent to determine its own broadcasting instants.

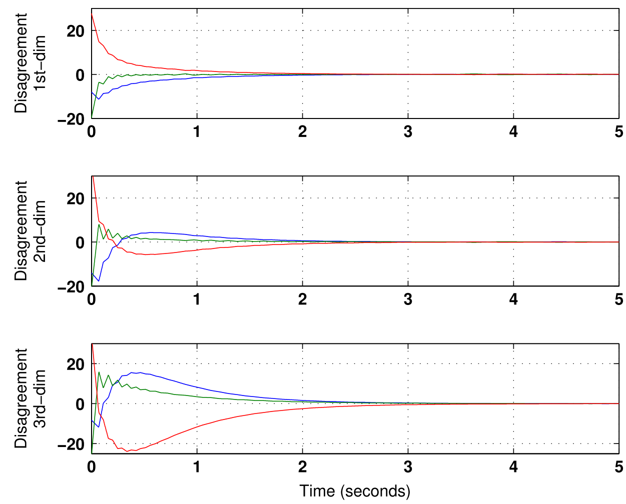

Fig. 2. Disagreement on the states of agents for each dimension.

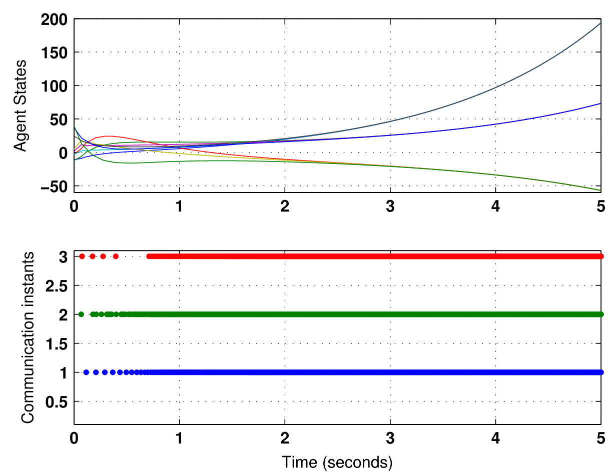

Fig. 3. States of three agents (top) and transmission instants (bottom) for each one of the three agents in the ZOH example.

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