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Decentralised event-triggered cooperative control with limited communication

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This note studies event-triggered control of Multi-Agent Systems (MAS) with first-order integrator dynamics. It extends previous work on event-triggered consensus by considering limited communication capabilities through strict peer-to-peer non-continuous information exchange. The approach provides both a decentralised control law and a decentralised communication policy. Communication events require no global information and are based only on local state errors; agents do not require a global sampling period or synchronous broadcasting as in sampled-data approaches. The proposed decentralised event-triggered control technique guarantees that the inter-event times for each agent are strictly positive. Finally, the ideas in this note are used to consider the practical scenario where agents are able to exchange only quantised measurements of their states.

Keywords: event-triggered control; consensus; quantisation; multi-agent systems

1. Introduction

An increasing interest in controlling large-scale dynamical systems composed of several to many autonomous mobile agents exists in different academic, commercial and military areas. This thrust is related to the large number of applications in which a group of coordinated agents is potentially able to outperform a single or a number of systems operating independently (Ren, Beard, & Atkins, 2007). An important problem in Multi-Agent Systems (MAS) is to design and implement decentralised algorithms for control and communication of agents. It is well understood that each agent should be able to determine its own control laws independently and based only on local information. This has been an important research topic (Ji & Egerstedt, 2007; Moreau, 2004; Ren et al., 2007; Tanner, Jadbabaie, & Pappas, 2003). These papers consider agents with continuous-time dynamics and it is assumed that agents can have continuous access to the states of their neighbours. In many applications the agents transmit their relevant variables such as position, velocity, heading, etc. to a subset of the agents not continuously but at discrete points in time. It is important to discern how frequently the agents should establish communication in order to preserve properties of similar control algorithms that assume continuous information exchange. The sample-data approach is commonly used to estimate the sampling periods (Cao & Ren, 2009; Cao & Ren, 2010; Hayakawa, Matsuzawa, & Hara, 2006; Liu, Xie, & Wang, 2010; Qin & Gao, 2012). An important drawback of periodic transmission is that it requires synchronisation between the agents, that is, all agents need to transmit their information at the same time instants and, in some cases, it requires a conservative sampling period for worst case situations.

In event-triggered broadcasting (Anta & Tabuada, 2010; Astrom, Astolfi, & Marconi, 2008; Astrom & Bernhardson, 2002; Donkers & Heemels, 2010; Garcia & Antsaklis, 2013; Tabuada, 2007; Wang & Lemmon, 2008; Wang & Lemmon, 2011) a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Event-triggered control schemes offer a new point of view, with respect to conventional time-driven strategies, on how information could be sampled for control purposes. The seminal work (Astrom & Bernhardson, 2002) provided an interesting comparison between conventional time-driven sampling and the new event-driven sampling, emphasising the practical advantages of the latter. Tabuada (2007) presented a triggering condition based on norms of the state and the state error $e = x(t_k) - x(t)$, that is, the last measured state minus the current state of the system, where the measurement received at the controller node is held constant until a new measurement arrives. When this happens, the error is set equal to zero and starts growing until it triggers a new measurement update.

The use of event-triggered control strategies in networked systems (Dimarogonas & Johansson, 2009; Dimarogonas & Frazzoli, 2009; Dimarogonas, Frazzoli, &

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Johansson, 2012; Garcia & Antsaklis, 2012; Seyboth, Dimarogonas, & Johansson, 2013; Sun & El-Farra, 2011; Yu & Antsaklis, 2012) provides a more robust and efficient use of network bandwidth. Its implementation in MAS also provides a highly decentralised way to schedule transmission instants that do not require synchronisation compared to periodic sampled-data approaches.

The work in the present paper is similar to (Dimarogonas & Frazzoli, 2009; Dimarogonas et al., 2012; Dimarogonas & Johansson, 2009) where the consensus problem with single integrator dynamics, event-based communication, and connected and undirected graphs was considered. The main advantage of our approach compared to these papers is that we consider both the reduction of actuation and communication updates while they only focus on reduction of update instants, i.e. they still assume that continuous communication exists among agents in order to calculate the error thresholds. Since continuous access to the states of neighbours is typically not possible we extend the work in (Dimarogonas et al., 2012) to consider the exchange of information among agents at discrete time instants that are, in general, non-periodic and based on local events. The present paper also provides an important extension to consider the case where the agents are able to transmit only a quantised version of its measured state. Similar work (Seyboth et al., 2013) uses a different threshold that does not require continuous access to the states of neighbours. The approach in this note preserves the decentralised nature of the event computations compared to (Seyboth et al., 2013) where an estimate of the second eigenvalue of the Laplacian matrix (L) is used to trigger communication events. The communication policy described in the present paper is decentralised in the sense that each agent computes its transmission instants based on local information. We provide asymptotic convergence to the initial average using the new threshold that considers only the last received states of the neighbours. The policy ensures strictly positive inter-event times. For the case when quantised measurements are used we are able to show convergence to a bounded region around the initial average; this bound is proportional to the quantisation parameter. An extended scheme is also proposed in order to guarantee strictly positive inter-event times in the presence of quantisation.

The remainder of this document is organised as follows: Section 2 addresses the event-triggered control strategy that considers limited knowledge of states of neighbours. Section 3 presents similar results using quantised measurements. Section 4 provides illustrative examples and conclusions are given in Section 5.

2. Decentralised consensus

We consider a set of n agents that are modelled as a single integrator:

$$\dot{x}_i = u_i, \quad i = 1 \dots n, \tag{1}$$

where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input associated with agent i. Since continuous measurements from neighbours are not available to each agent, then the control input is obtained using the last measurements received from each neighbour $j \in N_i$ as follows:

$$u_i(t) = u_i(t_{k_i}, t_{k_j}) = -\sum_{j \in N_i} (x_i(t_{k_i}) - x_j(t_{k_j})), \quad i = 1 \dots n, \tag{2}$$

where $x_i(t_{k_i})$ represents the last measurement transmitted by agent i at its update time t_{k_i} and N_i is the set of neighbours of agent i. Similarly, $x_j(t_{k_j})$ represents the last measurements received from neighbour j at corresponding time t_{k_j}. In general, the update intervals are non-periodic and the update instants for each agent are different from those of other agents, i.e. t_{k_i} and t_{k_j} are not necessarily equal.

The events are also computed based only on local information, that is, events are designed based on information that is available to each agent. We propose the following threshold:

$$e_i^2(t) > \frac{\sigma_i a(1 - a\,|N_i|)}{|N_i|} z_i^2(t_{k_i}, t_{k_j}), \tag{3}$$

where $e_i(t) = x_i(t_{k_i}) - x_i(t)$ represents the novel information with respect to the last transmitted measurement, $0 < a < (1/|N_i|)$, $0 < \sigma_i < 1$, $|N_i|$ is the cardinality of N_i, and

$$z_i(t_{k_i}, t_{k_j}) = \sum_{j \in N_i} (x_i(t_{k_i}) - x_j(t_{k_j})). \tag{4}$$

In this paper we use the notation (t_{k_i}, t_{k_j}) to represents piecewise constant variables that are updated at times t_{k_i}, when the local agent transmits an update, and also at all times t_{k_j} for $j \in N_i$, when the agent receives an update from any of its neighbours.

At each node the updates of the piecewise constant versions of the states $x_i(t_{k_i})$ and $x_j(t_{k_j})$ are as follows. When an event is triggered at time $t = t_{k_i}$ the local agent updates its local piecewise constant version of its state using the current measurement $x_i(t)$, i.e. $x_i(t_{k_i}) = x_i(t)$ and transmits this measurement to its neighbours.

On the other hand, when the local agent receives an update from any of its neighbours $j \in N_i$ at corresponding times $t = t_{k_j}$ containing a current measurement $x_j(t)$, the local agent uses this measurement to update its piecewise constant version of the state x_j, that is, $x_j(t_{k_j}) = x_j(t)$. Note that Equations (2) and (4) are functions of $x_i(t_{k_i})$ and all neighbour states $x_j(t_{k_j})$ for $j \in N_i$, therefore, they are updated at all corresponding time instants t_{k_i} and t_{k_j}.

Equation (3) is similar to the threshold used in Dimarogonas, Frazzoli, & Johansson (2012); however the threshold in this reference is based on the continuous variable $z_i(t)$

which is given by:

$$z_i(t) = \sum_{j \in N_i} (x_i(t) - x_j(t)). \tag{5}$$

It is clear that $z_i(t)$ is a function of the continuous measurements of local agent $x_i(t)$, and it is also a function of the continuous measurements of all neighbours $x_j(t)$. It is evident that the local agent is not able to design this threshold since continuous measurements from neighbours are not available. In this work we try to reduce both the actuation updates and the communication updates, while the authors (Dimarogonas et al., 2012) only considered the reduction of actuation updates assuming that the agents can have access to the continuous states of their neighbours. Therefore the threshold in (Dimarogonas et al., 2012) cannot be used in the present paper.

When an event is triggered by agent i we have $e_i(t_{k_i}) = x_i(t_{k_i}) - x_i(t) = x_i(t_{k_i}) - x_i(t_{k_i}) = 0$ because $t = t_{k_i}$ is an event time for agent i. We also have that

$$e_i^2(t) \le \frac{\sigma_i a(1 - a\,|N_i|)}{|N_i|} z_i^2(t_{k_i}, t_{k_j}) \tag{6}$$

holds for any value of $z_i(t_{k_i}, t_{k_j})$. Note that the triggering condition (3) guarantees that relation (6) is satisfied. Let $x(t_{k_1} \dots t_{k_n}) = [x_1(t_{k_1}) \ \dots \ x_n(t_{k_n})]^T$ represent the vector containing the latest broadcasted updates by each agent in the network, that is, this vector is a function of all update times t_{k_i} for $i = 1 \dots n$. Assume that input and communication delays are negligible. The next result shows convergence for a group of agents using the new threshold Equation (3) under control in Equation (2).

Theorem 1: *Consider a group of agents $\dot{x}_i = u_i$ for $i = 1 \dots n$, with control inputs given by Equation (2) and with event-based updates given by Equation (3). Assume that the communication graph is connected and undirected. Then all agents asymptotically stabilise to their initial average.*

Proof: Consider the ISS Lyapunov function $V = (1/2)x^T Lx$. We have that

$$\begin{aligned} \dot{V} &= x(t)^T L\dot{x}(t) = -x(t)^T LLx(t_{k_1} \dots t_{k_n}) \\ &= -\left(x(t_{k_1} \dots t_{k_n})^T - e(t)^T\right) LLx(t_{k_1} \dots t_{k_n}) \\ &= -\sum_i z_i^2(t_{k_i}, t_{k_j}) + \sum_i \sum_{j \in N_i} (e_i(t) - e_j(t)) z_i(t_{k_i}, t_{k_j}), \end{aligned}$$

where $x(t) = [x_1(t) \ \dots \ x_n(t)]^T$ and $e(t) = [e_1(t) \ \dots \ e_n(t)]^T$. By using the inequality $|xy| \le \frac{a}{2}x^2 + \frac{1}{2a}y^2$, for $a > 0$, we have:

$$\begin{aligned} \dot{V} \le& -\sum_i z_i^2(t_{k_i}, t_{k_j}) + \sum_i |N_i| \left(\frac{a}{2} z_i^2(t_{k_i}, t_{k_j}) + \frac{1}{2a} e_i^2(t)\right) \\ &+ \sum_i \sum_{j \in N_i} \left(\frac{a}{2} z_i^2(t_{k_i}, t_{k_j}) + \frac{1}{2a} e_j^2(t)\right). \end{aligned} \tag{7}$$

By symmetry of the undirected communication graph and using Equation (6), we have:

$$\begin{aligned} \dot{V} &\le -\sum_i (1 - a\,|N_i|)\, z_i^2(t_{k_i}, t_{k_j}) + \sum_i \frac{1}{a}\,|N_i|\, e_i^2(t) \\ &\le \sum_i (\sigma_i - 1)(1 - a\,|N_i|)\, z_i^2(t_{k_i}, t_{k_j}), \end{aligned} \tag{8}$$

which implies $\dot{V} \le 0$ for $0 < a < (1/|N_i|)$ and $0 < \sigma_i < 1$.

Because $V \ge 0$, $\dot{V} \le 0$ implies that V has a finite limit and $\dot{V} \to 0$ as $t \to \infty$. We have:

$$0 = \lim_{t \to \infty} \dot{V} \le \sum_i (\sigma_i - 1)(1 - a\,|N_i|)\, z_i^2(t_{k_i}, t_{k_j}) \le 0. \tag{9}$$

Since $(\sigma_i - 1)(1 - a\,|N_i|) < 0$ and $z_i^2(t_{k_i}, t_{k_j}) \ge 0$ then $(\sigma_i - 1)(1 - a\,|N_i|)\, z_i^2(t_{k_i}, t_{k_j}) \le 0$ for $i = 1 \dots n$. Thus, from Equation (9), we have $z_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i = 1 \dots n$. In view of Equations (6) and (2), when $z_i(t_{k_i}, t_{k_j}) = 0$ then all errors $e_i(t)$ are reset and remain equal to zero, that is, since $z_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i = 1 \dots n$, then we have that $\lim_{t \to \infty} e_i(t) = 0$ for $i = 1 \dots n$. We can also write

$$\begin{aligned} \dot{V} &= x(t)^T L\dot{x}(t) = -x(t)^T LL(x(t) + e(t)) \\ &= -z(t)^T z(t) - z(t)^T Le(t) \le 0. \end{aligned} \tag{10}$$

Similarly,

$$0 = \lim_{t \to \infty} \dot{V} = \lim_{t \to \infty} \left(-z(t)^T z(t) - z(t)^T Le(t)\right). \tag{11}$$

Because $\lim_{t \to \infty} e_i(t) = 0$ for $i = 1 \dots n$, it follows from Equations (10) and (11) that

$$\lim_{t \to \infty} \left(-z(t)^T z(t)\right) = -\lim_{t \to \infty} \sum_i z_i^2(t) = 0, \tag{12}$$

that is $\lim_{t \to \infty} z_i(t) = 0$ for $i = 1 \dots n$. Recall the definition of $z_i(t)$ in Equation (5), we have $\lim_{t \to \infty} \sum_{j \in N_i} (x_i(t) - x_j(t)) = 0$ for $i = 1 \dots n$ which can be written in vector form as

$$\lim_{t \to \infty} Lx(t) = 0_n. \tag{13}$$

When the interaction graph is connected the Laplacian L has a simple zero eigenvalue with the associated eigenvector 1_n. Therefore

$$\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t), \quad i, j = 1 \dots n. \tag{14}$$

For undirected graphs it can be shown that the initial average remains constant. Define the average $\bar{x}(t) = \frac{1}{N} \sum_i x_i(t)$, we have the following:

$$\dot{\bar{x}}(t) = \frac{1}{N}\sum_i \dot{x}_i(t) = -\frac{1}{N}\sum_i \sum_{j\in N_i}(x_i(t) - x_j(t)) \\ -\frac{1}{N}\sum_i \sum_{j\in N_i}(e_i(t) - e_j(t)) = 0 \tag{15}$$

and $\bar{x}(t) = \bar{x}(0) = \frac{1}{N}\sum_i x_i(0)$, then the initial average remains constant. □

The authors (Dimarogonas et al., 2012) were able to show that at any given time there exists at least one agent in the network for which its inter-event time is strictly positive. In this note we show that the inter-event times, not for at least one, but for all agents, are always strictly positive.

Corollary 2: *Consider a group of agents $\dot{x}_i = u_i$, $i = 1\ldots n$, with control inputs given by Equation (2) and with updates in Equation (3). Assume that the communication graph is connected. Then the inter-event times for each agent $i = 1\ldots n$, are strictly positive.*

Proof: Consider the evolution of the term $e_i^2(t)$ over the interval $t \in [t_{k_i}, t_{k_i+1})$ when $e_i(t)$ is continuous:

$$\frac{d}{dt}e_i^2(t) = 2e_i(t)\dot{e}_i(t) = 2e_i(t)z_i(t_{k_i}, t_{k_j}) \leq 2 \\ \times \left|e_i(t)z_i(t_{k_i}, t_{k_j})\right| \leq ae_i^2(t) + \frac{1}{a}z_i^2(t_{k_i}, t_{k_j}) \tag{16}$$

and consider the differential equation:

$$\dot{\phi}_i = a\phi_i + \frac{1}{a}z_i^2(t_{k_i}, t_{k_j}) \tag{17}$$

with initial condition $\phi_i(t_{k_i}) = e_i^2(t_{k_i}) = 0$. Then, we have:

$$e_i^2(t) \leq \phi_i(t) = \frac{1}{a}\int_{t_{k_i}}^{t} e^{a(t-\tau)}z_i^2(\tau)d\tau, \quad t \in [t_{k_i}, t_{k_i+1}). \tag{18}$$

A lower bound for the inter-event times of agent i is obtained by finding the minimum time t such as $\phi_i(t) > \chi_i z_i^2(t_{k_i}, t_{k_j})$, where $\chi_i = \frac{\sigma_i a(1-a|N_i|)}{|N_i|} > 0$.

We analyse two cases here, the first case is when $z_i(t_{k_i}, t_{k_j}) \neq 0$ at the last update instant t_{k_i}. In this case, from Equation (18), $\phi_i(t)$ takes a finite time $t > 0$ to grow from zero to $\chi_i z_i^2(t_{k_i}, t_{k_j})$ since $z_i^2(t_{k_i}, t_{k_j}) > 0$. The second case is when $z_i(t_{k_i}, t_{k_j}) = 0$. In this case $\phi_i(t) = 0$ for $t \in [t_{k_i}, t_{k_j})$, $t_{k_i} < t_{k_j}$ and we have that Equation (6) holds, therefore agent i does not generate any event during that time interval. When agent i receives an update from its neighbours then $z_i(t_{k_i}, t_{k_j}) \neq 0$ and the first case holds, i.e. the error takes a finite time $t > 0$ to grow from zero to $\chi_i z_i^2(t_{k_i}, t_{k_j})$. □

Remark 1: The selection of threshold (3) is intuitive because it really is a function of local information and it is also related to how fast the error will grow at any given time and trigger the next event. In fact, Equations (2), (3) and (18) tell us a clear picture of the communication pattern. Because $z_i(t_{k_i}, t_{k_j})$ is used in Equation (2), it determines how fast the corresponding agent moves with respect to its previous transmitted value and a proportional threshold is used for the same agent as seen in threshold (3). If $z_i(t_{k_i}, t_{k_j}) = 0$ for some agent i at some update instant t_{k_i} then $\dot{e}_i(t) = -\dot{x}_i(t) = z_i(t_{k_i}, t_{k_j}) = 0$, this means that the agent will not move, the error remains equal to zero, and the current $x_i(t)$ remains equal to the last update $x_i(t_{k_i})$. It is clear that there is no need to send additional updates if the current information has not changed and no events should be triggered. This is the main reason that the error is compared using 'strictly greater than' in threshold (3) instead of 'equal' (Dimarogonas et al., 2012). The main benefit is that we are able to lower bound the inter-event times, not for at least one agent, but for all of them.

Recent work (Seyboth et al., 2013) proposes a different threshold that does not require continuous access to the states of neighbours. The threshold is a function of time and other tuning parameters. The approach in this note preserves the decentralised nature of the solution compared to (Seyboth et al., 2013) since one of the tuning parameters depends on global information, i.e. on the second eigenvalue of L. Algorithms for estimation of the second eigenvalue of the Laplacian have been presented in (Aragues, Shi, Dimarogonas, Sagues, & Johansson, 2012; Franceschelli, Gasparri, Giua, & Seatzu, 2009; Yang et al., 2010). The algorithm (Aragues et al., 2012) is especially practical for implementation in the event-based approach (Seyboth et al., 2013) since the estimate of the second eigenvalue always remains smaller than the true second eigenvalue of L. This is a condition on the tuning parameter stated (Seyboth et al., 2013) for convergence of the consensus algorithm.

3. Decentralised consensus with quantisation

It was assumed in the last section that the sensor is able to measure the state of the system with infinite precision. In reality, however, the measured variables have to be quantised in order to be represented by a finite number of bits to be used in processor operations and to be transmitted over a digital communication channel. In this section we study the effects of signal quantisation on the convergence of the event-triggered control approach previously described in this paper.

We define a uniform quantiser as a function $q : \mathbb{R} \to \mathbf{V}$ such that:

$$q(\mu) = \nu_i \quad \text{if} \mu \in \left[\left(\nu_i - \frac{\delta}{2}\right), \left(\nu_i + \frac{\delta}{2}\right)\right), \\ |\mu - q(\mu)| \leq \frac{\delta}{2} \tag{19}$$

where δ represents the quantisation step and $\nu_i \in \mathbf{V} = \{\cdots - 2\delta, -\delta, 0, \delta, 2\delta, \ldots\}$. The above quantiser represents an infinite rate, uniform and passive quantiser with bounded quantisation error.

The only variables that are available to compute the control inputs and the state errors for each agent are the quantised states of the agents. The control inputs are now given by:

$$u_i(t) = -\sum_{j\in N_i} (q(x_i(t_{k_i})) - q(x_j(t_{k_j}))), \quad i = 1\ldots n \tag{20}$$

and the quantised state error is defined as follows:

$$e_i^q(t) = q(x_i(t_{k_i})) - q(x_i(t)), \quad i = 1\ldots n. \tag{21}$$

Theorem 3: *Consider a group of agents $\dot{x}_i = u_i$ for $i = 1\ldots n$, and each agent transmits its quantised output $q(x_i(t_{k_i}))$ to its neighbours at some time instants t_{k_i}. The control inputs are given by Equation (20) and the event-based updates are triggered when*

$$(e_i^q(t))^2 > \frac{\sigma_i a(1-a)}{|N_i|} M_i(t_{k_i}, t_{k_j}) \tag{22}$$

is satisfied, where $0 < a,\ \sigma_i < 1$, $M_i(t_{k_i}, t_{k_j}) = \sum_{j\in N_i} (q(x_i(t_{k_i})) - q(x_j(t_{k_j})))^2$. Assume that the communication graph is connected and undirected. Then all agents asymptotically stabilise to a bounded region around their initial average given by:

$$\lim_{t\to\infty} |x_i(t) - \bar{x}(0)| \le \delta, \quad i = 1\ldots n. \tag{23}$$

and the average remains constant, i.e. $\bar{x}(t) = \frac{1}{N}\sum_i x_i(t) = \frac{1}{N}\sum_i x_i(0)$.

Proof: Consider the candidate Lyapunov function $V = \sum_i V_i$ where $V_i = \int_0^{x_i} q(\upsilon)d\upsilon$ with $\dot{V}_i = q(x_i(t))\dot{x}_i(t) = q(x_i(t))u_i(t)$, for $i = 1,\ldots,n$. Note that $V_i \ge 0$ since the series interconnection of a single integrator and a passive memory less function is lossless (Khalil, 2002). We have

$$\begin{aligned}\dot{V} &= \sum_i u_i(t)q(x_i(t)) = \sum_i\sum_{j\in N_i} -(q(x_i(t_{k_i})) \\ &\quad - q(x_j(t_{k_j})))\left(q(x_i(t_{k_i})) - e_i^q(t)\right) \\ &= \sum_i\sum_{j\in N_i}\left[-q(x_i(t_{k_i}))^2 + q(x_i(t_{k_i}))q(x_j(t_{k_j}))\right. \\ &\quad \left. + \left(q(x_i(t_{k_i})) - q(x_j(t_{k_j}))\right)\left(e_i^q(t)\right)\right]\end{aligned} \tag{24}$$

and consider the following relation:

$$\sum_i\sum_{j\in N_i} q(x_i(t_{k_i}))^2 = \frac{1}{2}\sum_i\sum_{j\in N_i}\left[q(x_i(t_{k_i}))^2 + q(x_j(t_{k_j}))^2\right]. \tag{25}$$

Then we have:

$$\begin{aligned}\dot{V} &= \sum_i\sum_{j\in N_i} (q(x_i(t_{k_i})) - q(x_j(t_{k_j})))\left(e_i^q(t)\right) \\ &\quad - \frac{1}{2}\sum_i\sum_{j\in N_i}(q(x_i(t_{k_i})) - q(x_j(t_{k_j})))^2, \\ &\le \frac{a}{2}\sum_i M_i(t_{k_i}, t_{k_j}) + \frac{1}{2a}\sum_i\sum_{j\in N_i}(e_i^q(t))^2 \\ &\quad - \frac{1}{2}\sum_i M_i(t_{k_i}, t_{k_j}) \\ &\le -\frac{(1-a)}{2}\sum_i M_i(t_{k_i}, t_{k_j}) + \frac{1}{2a}\sum_i |N_i|\,(e_i^q(t))^2\end{aligned} \tag{26}$$

where the inequality $|xy| \le \frac{a}{2}x^2 + \frac{1}{2a}y^2$, for $a > 0$, has been used to obtain the second line in Equation (26). By using the threshold threshold (22) we can guarantee that

$$(e_i^q(t))^2 \le \frac{\sigma_i a(1-a)}{|N_i|} M_i(t_{k_i}, t_{k_j}) \tag{27}$$

holds. Then we obtain

$$\dot{V} \le \frac{(1-a)}{2}\sum_i(\sigma_i - 1)M_i(t_{k_i}, t_{k_j}) \tag{28}$$

which implies $\dot{V} \le 0$ for $0 < a < 1$ and $0 < \sigma_i < 1$.

Because $V \ge 0$, $\dot{V} \le 0$ implies that V has a finite limit and $\dot{V} \to 0$ as $t \to \infty$. We have the following:

$$0 = \lim_{t\to\infty}\dot{V} \le \frac{(1-a)}{2}\sum_i(\sigma_i - 1)M_i(t_{k_i}, t_{k_j}) \le 0. \tag{29}$$

We also have that $(\sigma_i - 1) < 0$ and $M_i(t_{k_i}, t_{k_j}) \ge 0$, for $i = 1\ldots n$; consequently $(\sigma_i - 1)\,M_i(t_{k_i}, t_{k_j}) \le 0$ for $i = 1\ldots n$. From Equation (29) we can see that $\sum_i(\sigma_i - 1)M_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ which means that $(\sigma_i - 1)\,M_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i = 1\ldots n$. Since $(\sigma_i - 1) \ne 0$ we have $M_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i = 1\ldots n$. By definition $M_i(t_{k_i}, t_{k_j}) = \sum_{j\in N_i}(q(x_i(t_{k_i})) - q(x_j(t_{k_j})))^2$, which consists of a summation of quadratic terms. Then we have that

$$\lim_{t\to\infty} q(x_i(t_{k_i})) = \lim_{t\to\infty} q(x_j(t_{k_j})), \quad i, j = 1\ldots n. \tag{30}$$

In view of Equations (20), (27) and (30) all errors $e_i^q(t)$ reset and remain equal to zero, that is, we have that $\lim_{t\to\infty} e_i^q(t) = 0$ for $i = 1\ldots n$, which is equivalent to

$$\lim_{t\to\infty} q(x_i(t_{k_i})) = \lim_{t\to\infty} q(x_i(t)), \quad i = 1\ldots n \tag{31}$$

therefore, it follows from Equations (30) and (31) that

$$\lim_{t\to\infty} q(x_i(t)) = \lim_{t\to\infty} q(x_j(t)), \quad i, j = 1\ldots n. \tag{32}$$

It can be shown that for undirected graphs, and using quantisation of the states in this case, the initial average remains constant. Define the average $\bar{x}(t) = \frac{1}{N}\sum_i x_i(t)$, we have the following:

$$\begin{aligned}\dot{\bar{x}}(t) &= \frac{1}{N}\sum_i \dot{x}_i(t) = -\frac{1}{N}\sum_i\sum_{j\in N_i}(q(x_i(t)) - q(x_j(t))) \\ &\quad -\frac{1}{N}\sum_i\sum_{j\in N_i}\left(e_i^q(t) - e_j^q(t)\right) = 0\end{aligned} \tag{33}$$

and $\bar{x}(t) = \bar{x}(0) = \frac{1}{N}\sum_i x_i(0)$.

Let $\bar{q} = \lim_{t\to\infty} q(x_i(t))$, from Equations (32) and (33) we have that

$$|\bar{q} - \bar{x}(0)| \le \frac{\delta}{2}. \tag{34}$$

The last statement can be shown by contradiction. Assume that $|\bar{q} - \bar{x}(0)| > \frac{\delta}{2}$ then, from relation (19), we have that either $x_i(t) > \bar{x}(0)$ for $i = 1\ldots n$, or $x_i(t) < \bar{x}(0)$ for $i = 1\ldots n$, and in both cases Equation (33) does not hold and we have a contradiction.

From Equations (34) and (19) we obtain Equation (23) and the proof is complete. □

It is important to note that inter-event times are not lower bounded when using quantisation. It is still possible to define solutions for this type of trajectories in the sense of Krasowskii, as it is shown in (Ceragioli, Persis, & Frasca, 2011), by introducing ideal sliding modes for trajectories that contain accumulation points. On the other hand, the computations associated with the event-triggered communication policy require continuous sensing, quantising and computing and comparing errors and time-varying thresholds. In practice, all these operations can be performed locally by each agent's processor unit frequently but not continuously. This implementation disassociates trajectories from ideal sliding modes and creates a chattering effect.

In order to prevent the undesired chattering effect that may be present when a system transmits updates very frequently in the boundary of a quantisation level and its associated Zeno behaviour we introduce a minimum update interval $\tau^* > 0$. The minimum update interval is useful not only for avoiding Zeno behaviour but also for a practical implementation of a non-continuous sensing and quantising scheme. In the following we relax the assumption that the errors need to be calculated continuously; instead, we introduce a sampling time T, $0 < T \le \tau^*$ which allows for a practical implementation of the event-triggered approach. Note that the sampling time T is only used to check the error periodically but communication between agents is still event-based since at every sampling time each agent decides if transmission of information is needed based on the size of the current error. In selecting τ^* we want to ensure that the error $e_i(t) = x_i(t_{k_i}) - x_i(t)$ remains bounded in a desired region $|e_i(t)| < \delta$ for the time interval $t \in [t_{k_i}, t_{k_i} + \tau^*]$. This means that if an update is triggered by the ith agent at time t_{k_i} then we have:

$$\left|e_i^q(t)\right| = \left|q(x_i(t_{k_i})) - q(x_i(t))\right| \le \delta, \ldots t \in [t_{k_i}, t_{k_i} + \tau^*]. \tag{35}$$

Let us consider in this case a fixed threshold. An event is triggered when:

$$\left|e_i^q(t)\right| \ge p\delta, \tag{36}$$

where $p \ge 1$ is an integer since $e_i^q(t)$ varies in increments of δ. In general, asymptotic convergence of the quantised outputs is not achieved in this case, but convergence to a bounded region around the initial average can be shown by evaluating the difference of the states of any one agent and the remaining agents in the network. Choose, without loss of generality, an agent and relabel it as x_c and the remaining agents as $x_1\ldots x_{n-1}$. Let A and L represent the adjacency and the Laplacian matrices associated with the communication graph corresponding to the remaining agents.

Corollary 4. *Consider a group of agents $\dot{x}_i = u_i$ for $i = 1\ldots n$, and each agent transmits its quantised output $q(x_i(t_{k_i}))$ to its neighbours at some time instants t_{k_i}. The control inputs are given by Equation (20) and the event-based updates are triggered according to threshold (36). Consider a sampled non-continuous event implementation. Assume that the original (before choosing an agent) communication graph is connected and undirected. Then all agents stabilise to a bounded region around their initial average and the following is satisfied:*

$$\begin{aligned}\lim_{\kappa\to\infty}\|x_i[\kappa] - x_c[\kappa]\|_\infty &\le \delta\left(1 + \left(p + \frac{1}{2}\right)\|G\|_\infty\right) \\ &\quad \left\|(\underline{L} + diag\{A_c\})^{-1}\right\|_\infty,\end{aligned} \tag{37}$$

when T is designed to satisfy

$$T < \min\left(\min_{i=1,\ldots,n} 1/|N_i|, \tau^*\right), \tag{38}$$

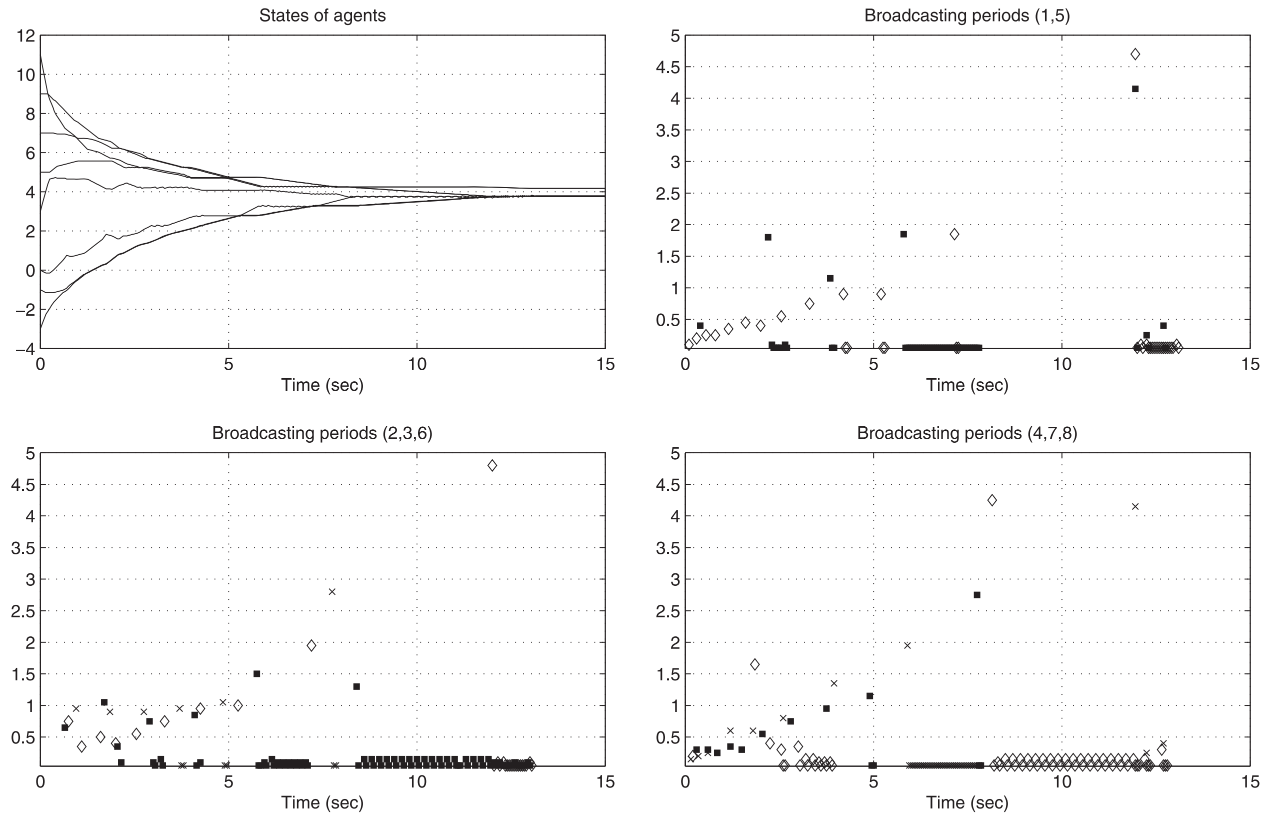

Figure 1. Quantised consensus. Top left: states of eight agents. Top right: broadcasting periods, agent 1 (◊) and agent 5 (■). Bottom left: broadcasting periods, agent 2 (◊), agent 3 (■) and agent 6 (x). Bottom right: broadcasting periods, agent 4 (◊), agent 7 (■) and agent 8 (x).

where A_c is a row vector containing the entries, other than the $a_{c,c}$ entry, in the cth row of the original adjacency matrix A and $G = [A_c^T \underline{L}]$.

Proof: First, given control inputs (20), triggering condition (36) and for any configuration with finite initial conditions and for fixed and connected topologies (such as the ones considered here) there always exists a finite and positive constant S such that $|\dot{x}_i| \leq S$. Consider the behaviour of the error $e_i(t) = x_i(t_{k_i}) - x_i(t)$ as follows:

$$\frac{d}{dt}|e_i(t)| \leq |\dot{e}_i(t)| = |\dot{x}_i(t)| \leq \dot{\phi}_i = S. \tag{39}$$

Solving Equation (39) with initial condition $\phi_i(t_{k_i}) = 0$ we obtain

$$|e_i(t_{k_i} + \tau^*)| \leq \phi_i(t_{k_i} + \tau^*) = S\tau^* = c \tag{40}$$

for $0 < c < \delta$. Then $\tau^* = \dfrac{c}{S} > 0$ is a lower bound on the inter-event times that the ith agent uses to broadcast its measurements. Additionally, using the minimum update interval τ^* estimated by Equation (40), we guarantee that $|q(x_i(t_{k_i} + \tau^*)) - q(x_i(t_{k_i}))| \leq \delta$. Also note that the initial average using quantisation remains constant as it was shown in Theorem 3.

Define $\xi_i[\kappa] = x_i[\kappa] - x_c[\kappa]$, which represents the difference between the chosen agent and any other remaining agent at the T-discretised time instants indexed by κ. We have the following:

$$\begin{aligned}\xi_i[\kappa+1] &= x_i[\kappa] - T\sum_{j\in N_i}\left(q(x_i[\kappa_{k_i}]) - q(x_j[\kappa_{k_j}])\right)\\ &\quad - x_c[\kappa+1]\\ &= \xi_i[\kappa] - T\sum_{j\in N_i}\left(\xi_i[\kappa] - \xi_j[\kappa] - \varepsilon_i[\kappa] + e_i^q[\kappa]\right.\\ &\quad \left. + \varepsilon_j[\kappa] - e_j^q[\kappa]\right) - x_c[\kappa+1] + x_c[\kappa] \end{aligned} \tag{41}$$

for $i = 1\ldots n$-1, where $T\kappa_{k_i} = t_{k_i}$ and $\varepsilon_i[\kappa] = x_i[\kappa] - q(x_i[\kappa])$. It is clear that in this case the event times t_{k_i} take place at some of the discrete time instants κ labelled as κ_{k_i}.

Equation (41) can be written in a compact form:

$$\xi[\kappa+1] = Q\xi[\kappa] + TG(\varepsilon[\kappa] - e^q[\kappa]) + X_c[\kappa], \tag{42}$$

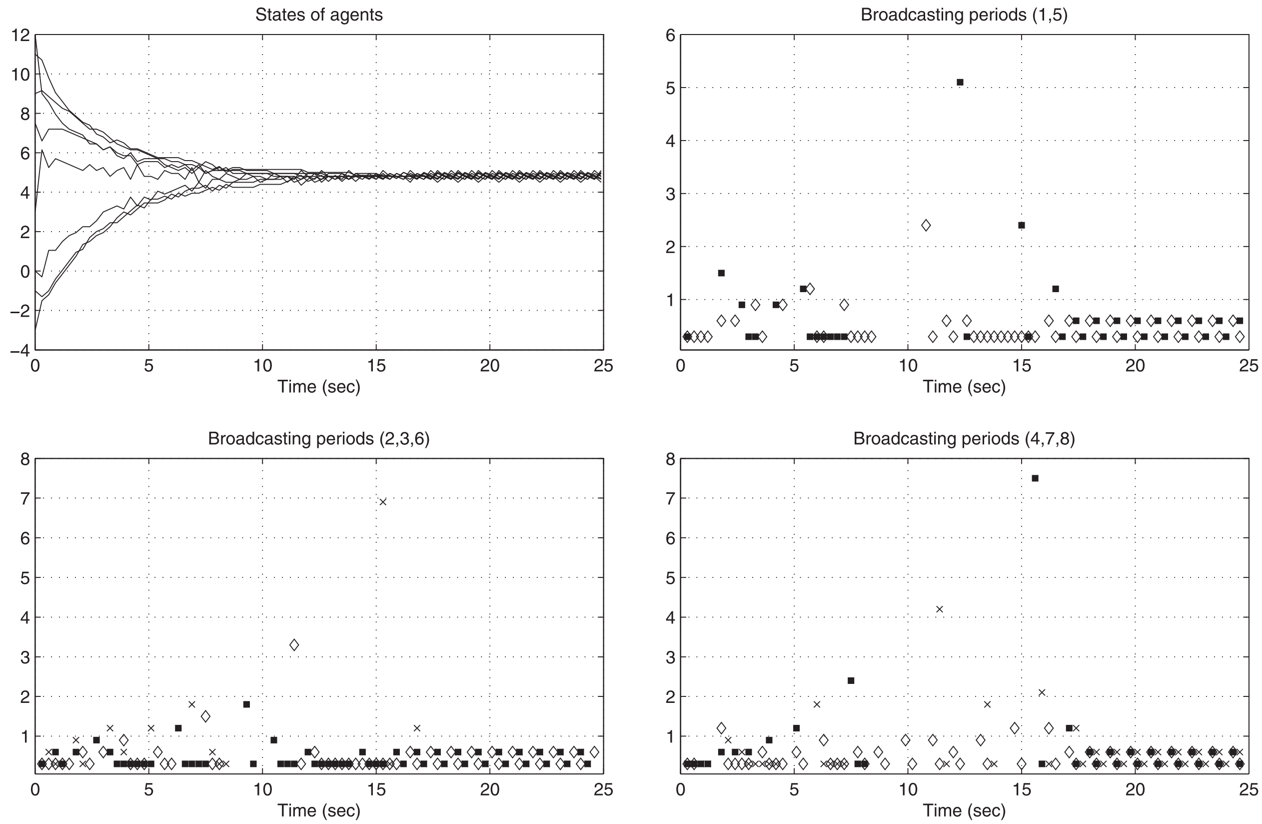

Figure 2. Quantised consensus. Top left: states of eight agents. Top right: broadcasting periods, agent 1 (◊) and agent 5 (■). Bottom left: broadcasting periods, agent 2 (◊), agent 3 (■) and agent 6 (x). Bottom right: broadcasting periods, agent 4 (◊), agent 7 (■) and agent 8 (x).

where $Q = I_n - T\underline{L} - T \bullet diag\{A_c\}$, $X_c[\kappa] = (-x_c[\kappa + 1] + x_c[\kappa])1_n$, $\xi[\kappa] = [\xi_1[\kappa] \ldots \xi_{n-1}[\kappa]]^T$, $\varepsilon[\kappa] = [\varepsilon_c[\kappa], \varepsilon_1[\kappa] \ldots \varepsilon_{n-1}[\kappa]]^T$, $e^q[\kappa] = [e_c^q[\kappa], e_1^q[\kappa] \ldots e_{n-1}^q[\kappa]]^T$. The response of Equation (42) to initial condition $\xi[0]$ is given by:

$$\xi[\kappa] = Q^\kappa \xi[0] - \sum_{l=1}^{\kappa} Q^{\kappa-l} T G(\varepsilon[l-1] - e^q[l-1]) + \sum_{l=1}^{\kappa} Q^{\kappa-l} X_c[l-1]. \tag{43}$$

The norm of $\xi[\kappa]$ satisfies:

$$\|\xi[\kappa]\|_\infty \leq \|Q^\kappa\|_\infty \|\xi[0]\|_\infty + \delta\left(p + \frac{1}{2}\right) T \|G\|_\infty \left\|\sum_{l=0}^{\kappa-1} Q^l\right\|_\infty + T\delta \left\|\sum_{l=0}^{\kappa-1} Q^l\right\|_\infty. \tag{44}$$

Since the original communication graph is connected then agent x_c has directed paths to all followers and $0 < T < \min_{i=1,\ldots,n} 1/|N_i|$ then, by Lemma 8.3 (Ren & Cao, 2011), Q has all its eigenvalues within the unit circle and $\lim_{\kappa\to\infty} Q^\kappa = 0$. Additionally, from Lemmas 1.26 and 1.28 (Ren & Cao, 2011), we have that $\lim_{\kappa\to\infty} \left\|\sum_{l=0}^{\kappa-1} Q^l\right\|_\infty = \left\|(I_n - Q)^{-1}\right\|_\infty$ and

$$\lim_{\kappa\to\infty} \|\xi[\kappa]\|_\infty \leq \delta\left(1 + \left(p + \frac{1}{2}\right)\|G\|_\infty\right) \left\|(\underline{L} + diag\{A_c\})^{-1}\right\|_\infty \tag{45}$$

which is equivalent to expression (37). □

Remark 2: Any agent in the network can be selected as x_c resulting in different expressions in Equation (37) according to the remaining agents' communication graph. The minimum of these expressions holds as a bound in Equation (37). Since the initial average is constant the agents converge around the initial average.

Remark 3: Threshold (36) is constant once we choose a quantisation parameter. This threshold choice makes sense because error (21) varies in increments of δ. Additionally, from Equation (37), the region of convergence can be reduced by choosing a smaller δ, by trading off the sampling inter-event time.

4. Examples

Example 1: Consider eight agents exchanging quantised measurements of positions using a negligible sampling time for computing the error. The quantisation parameter is $\delta = 0.5$. Simulation results are shown in Figure 1. This figure shows that the non-quantised states of the agents converge to a region around the initial average which is 3.875. All the quantised states reach a common value and the bound (23) is satisfied. In addition, since the quantised states reach a common value, the agents do not move and no additional events are generated after approximately 13 seconds as it can be seen in the remaining plots of Figure 1, where the broadcasting periods for each agent are shown.

Example 2: We consider the same system as in Example 1 but the difference is that we introduce a minimum inter-event time equal to 0.3 seconds which also serves as a sampling interval for calculating the error. We select $p = 1$ and $\delta = 0.5$. Simulation results are shown in Figure 2.

In this case the average of the non-quantised states also remains constant over time. The quantised values do not reach a common value although the difference of any pair of them remains bounded. The agents keep sending updates when they reach this region but using event times equal to or greater than the minimum update interval which is 0.3 seconds as it can be observed in the remaining plots of Figure 2. The agents converge to a bounded region around the initial average which is equal to 4.8125 and the bound (37) is satisfied. The minimum theoretical bound for this example is equal to 17 which is conservative since, from Figure 2 and after transient response, the difference between any two agents is less than 0.7.

5. Conclusions

Decentralised control and broadcasting laws for consensus were presented in this note. The main advantage of this formulation compared to similar work is that we were able to reduce both actuation and transmission updates; continuous monitoring of states of neighbours is no longer needed. Asymptotic convergence to initial average was shown. We offered an important extension to consider transmission of quantised measurements. In order to avoid Zeno behaviour in this case we introduced a strictly positive minimum update interval and convergence to a bounded region around the initial average was obtained.

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