Lecture notes: Trajectory tracking control for robot manipulators

Jie Fu

Department of Electrical and Computer Engineering
Robotics Engineering Program
Worcester Polytechnic Institute

RBE502, 2018

Outline

This lecture note is based on

- Chapter 8 in M. Spong **Robot modeling and control.**
- Chapter 2 of de Wit, Carlos Canudas, Bruno Siciliano, and Georges Bastin, eds. **Theory of robot control.** Springer Science & Business Media, 2012. Available online.
- Chapter 5: Position Control and Trajectory Tracking of Murray et. al. **A Mathematical Introduction to Robotic Manipulation**
- Refereces given herein.

Problem definition

Tracking control in the joint space: design a controller that allows the system to

- follow a given time-varying trajectory $q_d(t)$ and its successive derivatives $\dot{q}_d(t)$ and $\ddot{q}_d(t)$ — the desired velocity and acceleration.

Methods

Inverse dynamics control (or computed torque control)

- linearizing and decoupling robot manipulator dynamics;
- nonlinearities (Coriolis and centrifugal terms, gravity terms) compensated by adding these **forces** to the control input.
- CON: Compensational force can be difficult to compute. Not robust to modeling errors.

Lyapunov-based control:

- search for asymptotic stability —exponentially, if possible.

Passivity-based control:

- exploits the passivity properties of the robot manipulator dynamics.

Inverse dynamics control

Consider the dynamic model

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau.$$

Question: How to design trajectory tracking control?

- Recall: Trajectory tracking control for

$$\dot{x} = Ax + Bu \qquad \dot{x}_d = Ax_d + Bu_d$$

$$e = x - x_d$$

- Consider use it for nonlinear trajectory tracking control: What is the linear system now?

$$\dot{e} = \dot{x} - \dot{x}_d = Ax - Ax_d + Bu - Bu_d$$

$$= Ae + B\underbrace{(u-u_d)}_{v}$$

design v s.t. $e \to 0$: $v = -ke$

$u = u_d - ke$

$\dot{x} = Ax + Bu$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau.$$

Let $\tau = M(q)a_q + C(q,\dot{q})\dot{q} + N(q)$. Feedback linearization.

Since the inertia matrix is invertible, the system dynamical model reduces to

$M(q)\ddot{q} + \cancel{C(q,\dot{q})\dot{q}} + \cancel{N(q)} = M(q)a_q + \cancel{C(q,\dot{q})\dot{q}} + \cancel{N(q)}$

$\ddot{q} = a_q \longrightarrow \ddot{q}_i = a_{q_i}$

new input

$\vec{x} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

$\dot{\vec{x}} = \begin{bmatrix} 0 & I_{n\times n} \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} a_q$

$\dot{q} = \dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{q} = a_q$

$n: \; q \in \mathbb{R}^n$

Inverse dynamic control

Given

$$\ddot{q} = a_q$$

decoupled joint dynamics.

Write into the state space form:

Inverse dynamic control

Let $q^d(t)$ be the desired trajectory (cont. differentiable at least twice) the feedforward control input a_q^d should be

$$\vec{x} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\ddot{q}^d = a_q^d$$

$$\boxed{\dot{\vec{x}}^d = A\vec{x}^d + B a_q^d}$$

Traj tracking of linear system

$$e = \vec{x} - \vec{x}^d$$

$$a_q = a_q^d - Ke$$

$$\tau = M(q)a_q + C(q,\dot{q})\dot{q} + N(q)$$

$$= M(q)\left[\underbrace{a_q^d - Ke}_{\Downarrow} \right] + C(q,\dot{q})\dot{q} + N(q)$$

$$a_q^d - K_P(q - q_d) - K_D(\dot{q} - \dot{q}^d)$$

K_P, K_D positive. def.

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \qquad \overset{u}{B} = \begin{bmatrix} 0 \\ I \end{bmatrix} \left(\begin{bmatrix} K_P & K_D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$A - BK = \begin{bmatrix} 0 & I \\ -K_P & -K_D \end{bmatrix}$$

Inverse dynamic control with integral action

$$\tau = M(q)\,a_q + C(q,\dot q)\,\dot q + N(q)$$

$$a_q = \underbrace{\ddot q_d + K_D(\dot q_d - \dot q) + K_P(q_d - q)} + K_I \int_0^t (q_d - q)\,d\tau$$

lead to the error equation:

$e = q - q_d$

$$e^{(3)} + K_D\ddot e + K_P\dot e + K_I e = 0$$

$$K_x = \begin{bmatrix} K_{x_1} & & 0 \\ & K_{x_2} & \\ & & \ddots \\ 0 & & & K_{x_n} \end{bmatrix}, \quad x \in \{D, P, I\}$$

which is exponentially stable with a choice of K_P, K_D, K_I. — why?

$e_i = q_i - q_{d,i} \quad \in R$

$$\Rightarrow \boxed{e_i^{(3)} + k_{D_i}\ddot e_i + k_{P_i}\dot e_i + k_{I_i} e_i} = r(t)$$

Frequency Domain

$$S^3 E_i(S) + k_{D_i} S^2 E_i(S) + k_{P_i} S E_i(S) + k_{I_i} E_i(S) = R(S)$$

$$E_i(S) = \frac{1}{S^3 + k_{D_i} S^2 + k_{P_i} S + k_{I_i}} R(S) \Rightarrow \lim_{t \to ?} e_i(t)$$

$$e_i^{(3)} = -K_{D_i}\ddot{e}_i - K_{P_i}\dot{e}_i - K_I e_i$$

$$\Rightarrow \lim_{t\to\infty} e_i(t)$$

$$\lim_{s\to 0} E_i(s) = \lim_{s\to 0} \frac{R(s)}{s^3 + K_{D_i}s^2 + K_{P_i}s + K_{I_i}}$$

$$R(s) = C$$

$$\downarrow$$

$$\lim_{s\to 0} \frac{C}{s^3 + K_{D_i}s^2 + K_{P_i}s + K_{I_i}} = \lim_{s\to 0} \frac{C}{K_{I_i}}$$

$C = 0$ $\Rightarrow$ choose $K_{D_i}, K_{P_i}, K_{I_i}$ to make the system stable.

$\Rightarrow$ Time domain: $\vec{z} = \begin{bmatrix} e_i \\ \dot{e}_i \\ \ddot{e}_i \end{bmatrix}$

$$\dot{\vec{z}} = A\vec{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_{I_i} & -K_{P_i} & -K_{D_i} \end{bmatrix} \vec{z}$$

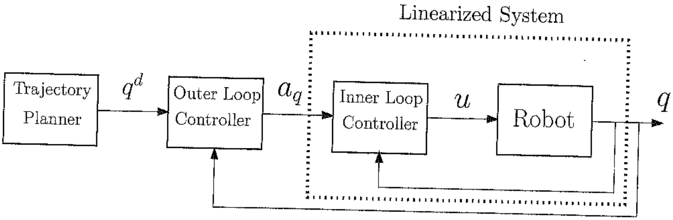
$$\dot{e}_i = \dot{z}_1 = z_2$$

$$\ddot{e}_i = \dot{z}_2 = z_3$$

$$e_i^{(3)} = \dot{z}_3 = -K_{D_i}z_3 - K_{P_i}z_2 - K_{I_i}z_1$$

Inverse dynamic control

the control architecture



- Outer: input $q^d, \dot{q}^d, q, \dot{q}$, output a_q
- Inner: input $q, \dot{q}, a_q$, and output u (or τ).

NOTE: This controller needs **exact knowledge** of the model parameters and requires an additional number of numerical computations.

Lyapunov-based control

introduce a change of coordinates:
$e = q - q_d$ — the tracking error.

$$\dot{\xi} = \dot{q}_d - \Lambda(q - q_d) = \dot{q}_d - \Lambda e;$$

$$\sigma = \dot{q} - \dot{\xi} = \dot{e} + \Lambda e; \quad = \dot{q} - \dot{q}_d + \Lambda e = \dot{e} + \Lambda e$$

$$= I\dot{e} + \Lambda e$$

where Λ— a constant, positive definite square matrix.
Consider the following controller:

$$u = M(q)\ddot{\xi} + C(q, \dot{q})\dot{\xi} + N(q) - K_D\sigma$$

where K_D — a constant, positive definite matrix.
The closed-loop system is

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \cancel{N(q)} = M(q)\ddot{\xi} + C(q,\dot{q})\dot{\xi} + \cancel{N(q)} - K_D\sigma$$

$$M(q)(\ddot{q} - \ddot{\xi}) + C(q,\dot{q})(\dot{q} - \dot{\xi}) + K_D\sigma = 0$$

$$M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K_D\sigma = 0$$

further, we simply the closed loop system with the definition of $\dot{\xi}$ and σ:

Control: Key idea: If $\sigma \to 0$ as $t \to \infty$, then $e \to 0$

$\sigma = e + \Lambda \dot{e}$

Proof of stability: The Lyapunov candidate :

$$V = \frac{1}{2}\sigma^T M(q)\sigma + e^T P e$$

where $P = P^T = \Lambda^T K_D > 0$. $\Rightarrow$ K_D is diagonal matrix. Λ : constant. pos. def matrix.

show this candidate is a valid Lyapunov function.

$\Lambda^T K_D$ symmetric matrix

$P = \Lambda^T K_D$

$P^T = K_D^T \Lambda \Rightarrow \Lambda^T K_D = K_D^T \Lambda$

$M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K_D\sigma = 0$

$V \geq 0.$ and $V = 0.$ if $\sigma = 0,\ e = 0$

$$\dot V = \frac{\partial}{\partial \sigma}\left(\tfrac{1}{2}\sigma^T M(q)\sigma\right)\cdot \frac{d\sigma}{dt} + \frac{\partial}{\partial e}\left(e^T P e\right)\cdot \dot e + \tfrac{1}{2}\sigma^T \dot M(q)\sigma$$

$$= \tfrac{1}{2}\left(\sigma^T M^T(q) + \sigma^T M(q)\right)\cdot \frac{d\sigma}{dt} + \left(e^T P^T + e^T P\right)\cdot \dot e + \tfrac{1}{2}\sigma^T \dot M(q)\sigma$$

$$= \sigma^T M(q)\dot\sigma + 2e^T P\dot e + \tfrac{1}{2}\sigma^T \dot M(q)\sigma$$

$$= \sigma^T\left(-K_D\sigma - C(q,\dot q)\sigma\right) + 2e^T P\dot e + \tfrac{1}{2}\sigma^T \dot M(q)\sigma$$

$$= \underbrace{\tfrac{1}{2}\sigma^T\left(\dot M(q) - 2C(q,\dot q)\right)\sigma}_{=0} - \sigma^T K_D\sigma + 2e^T P\dot e \quad \Leftarrow \sigma = \dot e + \Lambda e$$

$$= (\dot e + \Lambda e)^T(-K_D)(\dot e + \Lambda e) + 2e^T\Lambda^T K_D\dot e \quad \Leftarrow P = \Lambda^T K_D$$

$$= -\dot e^T K_D\dot e - e^T\Lambda^T K_D\Lambda e + (-2e^T\Lambda^T K_D\dot e) + 2e^T\Lambda^T K_D\dot e$$

$$= -\dot e^T K_D\dot e - e^T\Lambda^T K_D\Lambda e \leq 0$$

$$\dot V = 0 \Rightarrow \dot e = 0 \text{ and } e = 0$$

Passivity-based control: A general theorem

$\int_{t=0}^{T} -\dot{q}^T_{(s)}\tau(s)ds \geq -\beta.$

$M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K_D\sigma = 0.$ special case when $r = \sigma$ and $\phi = 0$

Consider the differential equation

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = \Phi$$

where $K_v = K_v^T > 0$ is a positive definite matrix.
Suppose $\int_{t=0}^{T} -r^{\mathsf{T}}(t)\Phi(t)dt \geq -\beta$ for all $T > 0$ and for some $\beta \geq 0$, then as $t \to \infty$, $r(t) \to 0$.

Proof

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = \phi \quad (*)$$

Consider a Lyapunov candidate V defined by

$$V = \underbrace{\frac{1}{2}r^T M(q) r}_{\geq 0} + \underbrace{\beta - \int_{t=0}^{T} r^\mathsf{T}(t)\Phi(t)dt}_{\geq 0}$$

passivity $\int_{t=0}^{T} -r^T(t)\phi(t)dt \geq -\beta$

Clearly, $V \geq 0$.
Differentiate V along the sytem traj.

$$\dot{V} = r^T M(q)\dot{r} + \frac{1}{2} r^T \dot{M}(q) r - r^T\phi$$

$$= r^T(\phi - C(q,\dot{q})r - K_v r) + \frac{1}{2} r^T \dot{M}(q) r - r^T\phi$$

$$= r^T(-C(q,\dot{q}))r - r^T K_v r + \frac{1}{2} r^T \dot{M}(q) r$$

$$= -r^T K_v r \leq 0.$$

and $\dot{V} = 0$ iff $r = 0 \Rightarrow \begin{cases} e = 0 \\ \dot{e} = 0 \end{cases}$

Insight for passivity-based control

Let $q^d(t)$ be a twice differentiable function, define

$$e(t) = q(t) - q^d(t).$$

$f(e, \dot{e})$

If the following condition holds: $\boxed{r(t) = f(\cancel{e(t)})}$ for some proper mapping $f(\cdot)$ such that $r(t) \to 0$ means $e(t) \to 0$ and $\dot{e}(t) \to 0$,
then select some control to make the system behaves like

$$M(q)\dot{r} + C(q, \dot{q})r + K_v r = \Phi$$

$t \to \infty$

$r \to 0$ $\xleftrightarrow{?}$ $\begin{cases} e \to 0 \\ \dot{e} \to 0 \end{cases}$

$r = f(e, \dot{e})$

Passivity based motion control

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau \quad \cdots \quad (1)$$

select τ and a definition of r to make the system dynamics as

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = 0 \qquad r = f(e, \dot{e}) \quad (2)$$

Note: A special case for $\Phi = 0$. Let $\tau = M(q)a + C(q,\dot{q})v + N(q) - K_v r$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \cancel{N(q)} = M(q)a + C(q,\dot{q})v + \cancel{N(q)} - K_v r$$

$$M(q)(\ddot{q} - a) + C(q,\dot{q})(\dot{q} - v) + K_v r = 0 \quad \cdots \quad (3)$$

$$\dot{r} = \ddot{q} - a$$

$$\boxed{r = \dot{q} - v} \Rightarrow \begin{cases} a = \ddot{q} - \dot{r} \\ v = \dot{q} - r \end{cases}$$

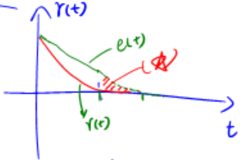
$r = f(e, \dot{e})$ such that $r \to 0$, $e \to 0$, $\dot{e} \to 0$.

$= \dot{e} + \Lambda e$

What is the mapping $r = f(e, \dot{e})$?

$\dot{e}(t)$
$e(t)$

Select $v = \dot{q}^d - \Lambda e$ where $\Lambda > 0$ is a positive definite matrix.



$r = \dot{e} + \Lambda e$:

$\boxed{r \to 0 \quad \text{and} \quad \Rightarrow \quad \dot{e} \to 0 \quad e \to 0}$ ✓

$r \to 0. \quad \dot{r} = 0$

$M(q)\dot{r} + C(q,\dot{q})r + K_r r = \phi \to$ Lyapunov. to show that

$\lim_{t \to \infty} r = 0$ and $\dot{r} = 0$

$$\begin{cases} \dot{r} = \ddot{e} + \Lambda \dot{e} = 0 \\ r = \dot{e} + \Lambda e = 0 \end{cases} \Rightarrow \begin{cases} \ddot{e} + \Lambda \dot{e} = 0 \\ \dot{e} + \Lambda e = 0 \end{cases} (*)$$

$$\vec{x} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \qquad \dot{\vec{x}} = \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} -\Lambda e \\ -\Lambda \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} -\Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}}_{A} [\vec{x}]$$

$\lim_{t \to \infty} \vec{x}(t) = 0$

What happens when $r \equiv 0$?

Conclusion

- Simplification of dynamic model (a simplified inertia matrix) for ease the computation with computed torque control is studied in [1].
- Lyapunov-based control [2].
- Passivity-based control [3].

References

A Aubin, C Canudas de Wit, and H Sidaoui.
Dynamic model simplification of industrial robot manipulators.
In *Robot Control 1991*, pages 15–20. Elsevier, 1992.

M. W. Spong, R. Ortega, and R. Kelly.
Comments on "adaptive manipulator control: a case study" by j. slotine and w. li.
IEEE Transactions on Automatic Control, 35(6):761–762, June 1990.

Romeo Ortega and Mark W Spong.
Adaptive motion control of rigid robots: A tutorial.
In *Decision and Control, 1988., Proceedings of the 27th IEEE Conference on*, pages 1575–1584. IEEE, 1988.