ECO2100 Part 2 Homework 1

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a)

Social welfare is given by:

$$S = \overbrace{f(k)}^{\text{Production of firm in match}}\overbrace{(1-u)}^{\text{Number of matches}} - \underbrace{vk}_{\text{total cost of vacancies}}$$

Note that we want to solve the steady state level of unemployment. First note that

$$\overbrace{\dot{u}}^{\text{change in unemployment across time, =0 in steady state}} = \underbrace{\delta(1-u)}_{\text{rate of job destruction times employed}} - \overbrace{\frac{m(\lambda)}{\lambda}u}^{\text{mathcing rate of unemployed workers times unemployed}}$$

Thus in steady state:

$$0 = \delta - (\delta + \frac{m(\lambda)}{\lambda})u$$

$$\iff u = \frac{\delta}{\delta + \frac{m(\lambda)}{\lambda}} = \frac{\lambda\delta}{\lambda\delta + m(\lambda)}$$

Furthermore, note that

$$\lambda = \frac{u}{v} \iff v = \frac{u}{\lambda} = \frac{\delta}{\lambda\delta + m(\lambda)}$$

Thus we can write the social planner's problem

$$\max_{\lambda \geq 0, k > 0} f(k)\frac{m(\lambda)}{\lambda\delta + m(\lambda)} - k\frac{\delta}{\lambda\delta + m(\lambda)}$$

The Lagrangian can be written as:

$$\mathcal{L}(\lambda, k, \mu_1, \mu_2) = \frac{f(k)m(\lambda) - k\delta}{\lambda\delta + m(\lambda)} + \mu_1(k - \epsilon) + \mu_2\lambda$$

where $\epsilon > 0$ is an arbitrary constant.

b)

First note, that ϵ is arbitrary, therefore we can always choose some ϵ such that $k > \epsilon > 0$, thus by complementary slackness $\mu_1 = 0$. Secondly, for finite The first order conditions are written as follows:

$$\frac{f'(k)m(\lambda) - \delta}{\lambda\delta + m(\lambda)} = 0 \tag{k}$$

$$\frac{f(k)m'(\lambda)\big(\lambda\delta + m(\lambda)\big) - \big(\delta + m'(\lambda)\big)\big(f(k)m(\lambda) - k\delta\big)}{\big(\lambda\delta + m(\lambda)\big)^2} + \mu_2 = 0 \tag{λ}$$

Inspecting condition (k) and taking the limit as $\lambda \to 0$ reveals that $\lambda \neq 0$. That is,

$$\lim_{\lambda \to 0} \frac{f'(k)m(\lambda) - \delta}{\lambda\delta + m(\lambda)} = \frac{-\delta}{\infty} < 0$$

By complementary slackness, it follows that $\mu_2 = 0$ Now inspect FOC (λ) we can derive:

$$f(k)m'(\lambda)\big(\lambda\delta + m(\lambda)\big) - \big(\delta + m'(\lambda)\big)\big(f(k)m(\lambda) - k\delta\big) = 0$$

by the denominator necessairily being nonzero. This can be simplified:

$$\lambda f(k)m'(\lambda) - f(k)m(\lambda) + k\big(\delta + m'(\lambda)\big) = 0$$

$$\iff \big(\lambda m'(\lambda) - m(\lambda)\big)f(k) + k\big(\delta + m'(\lambda)\big) = 0$$

$$\iff (1 - \frac{m(\lambda)}{\lambda m'(\lambda)})f(k) + k(\frac{\delta}{\lambda m'(\lambda)} + \frac{1}{\lambda}) = 0$$

Define

$$\eta(\lambda) = \frac{m(\lambda)}{\lambda m'(\lambda)}$$

Then re-write the above:

$$\big(1 - \eta(\lambda)\big) + k\big(\delta\eta(\lambda)m(\lambda) + \big)$$

c)

$$rJ(k) = f(k) - w - \delta\big(J(k) - V(k)\big) \tag{i}$$

$$rV(k) = -k + m(\lambda)\big(J(k) - V(k)\big) \tag{ii}$$

$$rW(k) = w - \delta\big(W(k) - U(k)\big) \tag{iii}$$

$$rU(k) = \frac{m(\lambda)}{\lambda}\big(W(k) - U(k)\big) \tag{iv}$$

d)

First, note that by $k = \bar{k}$ that the above Bellman equations are no longer functions of k. Furthermore, the equilibrium wage rate will be given by Nash Bargaining. That is

$$w = \arg\max_{\beta} (W - U)^{\beta}(J - V)^{(1-\beta)}$$

$$= \arg\max_{\beta} \beta \ln(W - U) + (1 - \beta)\ln(J - V)$$

The first order condition yields:

$$\beta\frac{\frac{\partial W}{\partial w}}{W-U}+(1-\beta)\frac{\frac{\partial J}{\partial w}}{J-V}=0$$

The partial derivatives come from the fact that W and J are functions of w, while V and U are not. Re-writing iii:

$$W=\frac{w-\delta U}{r+\delta}$$

Doing the same for J in terms of i yields:

$$J=\frac{f(k)-w-\delta V}{r+\delta}$$

It becomes obvious that $\frac{\partial W}{\partial w}=-\frac{\partial J}{\partial w}$ thus the first order condition can be re-written:

$$\beta\frac{1}{W-U}-(1-\beta)\frac{1}{J-V}=0$$

$$\iff \beta(J-V)=(1-\beta)(W-U)$$

Now we note the free entry condition which means that $V=0$ we can re-write the FOC to be:

$$\underbrace{\boxed{W-U}}_{=\frac{w-rU}{r+\delta}\text{ from iii}}=\beta\big(W-U+\underbrace{\boxed{J}}_{=\frac{f(k)-w}{r+\delta}\text{ from i}}\big)$$

$$\iff \frac{w-rU}{r+\delta}=\frac{\beta}{r+\delta}(f(k)-rU)$$

$$\iff rU(\beta-1)=\beta f(k)-w \tag{garbage}$$

From the FOC we can also derive:

$$W-U=\frac{\beta}{1-\beta}J$$

From ii we can obtain

$$J=\frac{k}{m(\lambda)}$$

Thus:

$$W-U=\frac{\beta}{1-\beta}\frac{k}{m(\lambda)}$$

Which can be subsitiuted into iv to obtain

$$rU=\frac{m(\lambda)}{\lambda}\frac{\beta}{1-\beta}\frac{k}{m(\lambda)}=\frac{\beta}{1-\beta}\frac{k}{\lambda}$$

Substituing into garbage and doing some simplifying:

$$-\beta\frac{k}{\lambda}=\beta f(k)-w$$

$$\iff w=\beta(f(k)+\frac{k}{\lambda})$$

Now inspect ii using J from i

$$0=-k+m(\lambda)\frac{f(k)-w}{r+\delta}$$

$$\iff w=f(k)-\frac{r+\delta}{m(\lambda)}k$$

Thus

$$\beta(f(k)+\frac{k}{\lambda})=f(k)-\frac{r+\delta}{m(\lambda)}k$$

$$\iff (1-\beta)f(k)-k\big(\frac{\beta}{\lambda}+\frac{r+\delta}{m(\lambda)}\big)=0$$