

OPTIMAL NEUMANN BOUNDARY AND DISTRIBUTED CONTROL OF THE WESTERVELT EQUATION WITH TIME-FRACTIONAL ATTENUATION

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ABSTRACT. Optimal control of nonlinear acoustic waves is relevant in many medical ultrasound technologies, ranging from cancer therapy to targeted drug delivery, where it can help guide the precise deposition of acoustic energy. In this work, we study Neumann boundary and distributed control problems for tracking a prescribed pressure field governed by the Westervelt equation with time-fractional dissipation. This model captures nonlinear ultrasonic wave propagation in biological media and accounts for the experimentally observed power-law attenuation. We begin by extending the existing well-posedness theory for time-fractional equations to include inhomogeneous Neumann boundary data used as control inputs, which requires constructing an appropriate data extension and regularization. Using these analytical results for the forward problem, we prove the existence of globally optimal controls and analyze the stability of the optimization problem with respect to perturbations in the target pressure field and to vanishing regularization parameters. Finally, we investigate the associated adjoint equation, which has state-dependent coefficients, and use it to derive first-order necessary optimality conditions.

1. INTRODUCTION

In medical acoustics, having precise control of ultrasonic waves in the region under treatment is often needed for both safety and efficiency; the applications include tissue ablation in cancer treatments [33], lithotripsy [22], and ultrasound-enhanced drug delivery [43]. This motivates the present study of an optimal control problem subject to Westervelt's wave model of nonlinear acoustic propagation through biological media. In such media, ultrasound attenuation exhibits a non-integer power-law dependence on frequency, which can be accurately captured through time-fractional dissipation terms; see [17, 19, 42, 50] for modeling details.

Control of the acoustic waves can be achieved through the boundary or by having a distributed acoustic source. We consider both settings and treat them in a largely unified theoretical framework. Given the desired acoustic pressure $p^d \in C([0, T]; L^2(\Omega))$, we consider a regularized objective

$$J(p, g, f) = \frac{\nu}{2} \|p - p^d\|_{L^2(0, T; L^2(\Omega_0))}^2 + \frac{1 - \nu}{2} \|p(T) - p^d(T)\|_{L^2(\Omega_0)}^2 + \mathcal{R}(g, f),$$

with $\nu \in \{0, 1\}$. We require that the pressure is close to the desired pressure locally on a region of interest $\Omega_0 \subset \Omega$ and possibly only at final time T (if $\nu = 0$). The regularizing

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functional is given by

$$(1.1) \quad \mathcal{R}(g, f) = \frac{\gamma}{2} \|g\|_{L^2(0,T;L^2(\partial\Omega))}^2 + \frac{\eta}{2} \|f\|_{L^2(0,T;L^2(\Omega))}^2, \quad \gamma, \eta \geq 0.$$

The acoustic pressure field is obtained by solving the Neumann initial boundary-value problem for the Westervelt wave equation [44, 51]:

$$(IBVP_{\text{West}}) \quad \begin{cases} p_{tt} - c^2 \Delta p - b \Delta \partial_t^\alpha p = k(x) (p^2)_{tt} + f, & \text{in } \Omega \times (0, T), \\ \frac{\partial p}{\partial n} = g & \text{on } \Gamma = \partial\Omega, \\ (p, p_t)|_{t=0} = (0, 0), \end{cases}$$

where the source term f and the Neumann boundary data g act as controls; the latter acts on the boundary $\Gamma = \partial\Omega$. In $(IBVP_{\text{West}})$, $p = p(x, t)$ denotes the acoustic pressure, $c > 0$ is the speed of sound in the medium, $b > 0$ the attenuation coefficient, and $k = k(x)$ is the nonlinearity coefficient. We allow for k to vary in space to model varying influences of nonlinearity closer and farther from the source; the size of k , which will later be assumed to be small in a suitable norm, also regulates the well-posedness of $(IBVP_{\text{West}})$, as seen in Theorem 3.1 below. The acoustic dissipation term $-b \Delta \partial_t^\alpha p$ in $(IBVP_{\text{West}})$ involves the Djrbashian–Caputo fractional derivative operator $\partial_t^\alpha(\cdot)$ of order $\alpha \in (0, 1)$ to accurately model experimentally observed power-law attenuation in tissue media; we provide its definition and further for this work relevant theoretical details in Section 2. Given the technical intricacies of studying optimal control problems involving nonlinear wave propagation, we focus on boundary and distributed excitation with homogeneous initial conditions.

Main contributions. The aim of this work is to deepen the theoretical understanding of optimal control of nonlinear acoustic waves in complex propagation media. We investigate the following optimal control problem:

$$(P) \quad \inf_{(p,g,f) \in M} J(p, g, f) \quad \text{with}$$

$$M = \left\{ (p, g, f) \in \mathcal{X}_p \times \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}} : p \text{ solves } (IBVP_{\text{West}}) \right\},$$

where the space \mathcal{X}_p to which the pressure field belongs will be made precise in the well-posedness analysis in Section 3. The admissible space $\mathcal{X}_g^{\text{adm}}$ of boundary controls is a subset of a closed ball in a suitable Banach space where the compatibility conditions with homogeneous initial data hold, whereas the admissible space X_f^{adm} of distributed controls is a closed ball in a Hilbert space; we refer to Section 4 for details. These choices are dictated by the needs of the well-posedness analysis of $(IBVP_{\text{West}})$. The main contributions of our work pertain to

- (i) showing the existence of globally optimal controls for (P),
- (ii) ensuring stability with respect to perturbations in the desired pressure p^d and the vanishing regularization limit, and
- (iii) deriving adjoint-based first-order optimality conditions for the distributed and Neumann control cases.

To the best of our knowledge, this is the first optimal control analysis for the Westervelt equation with time-fractional attenuation and Neumann boundary control. The principal difficulty in obtaining (i)–(iii), beyond adapting established optimal control frameworks [11,

[18, 38, 48], stems from the quasilinear nature of the state equation, which can be rewritten as

$$(1.2) \quad ((1 - 2kp)p_t)_t - c^2 \Delta p - b \Delta \partial_t^\alpha p = f.$$

A rigorous analysis of (1.2) must guarantee that $1 - 2kp$ stays strictly positive to avoid degeneracy, which requires a bound on $\|ku\|_{L^\infty(0,T;L^\infty(\Omega))}$. Because the fractional dissipation is weaker than classical strong damping, achieving such bounds necessitates higher-order energy estimates carried out on a linearized problem in combination with a fixed-point argument. The presence of a Neumann boundary control makes this analysis particularly intricate since the well-posedness theory requires a suitable extension of regularized boundary data into the interior of the domain. Furthermore, in the adjoint problem, which has state-dependent coefficients, the regularity of data is limited as a consequence of the localization of the tracking functional to Ω_0 , so the higher-order smoothness arguments used in the forward analysis cannot be transferred to it and we have to also devise well-posedness arguments in a lower-regularity regime.

Related work. Research on fractional evolution equations is very active, including in the neighboring area of inverse problems; we refer to the books [23, 25, 31] and the references contained therein. In closer relation to the present topic, we point out the work on the identification of the nonlinearity parameter in the fractionally attenuated Westervelt equation with Dirichlet data in [30]. Optimal control of fractional evolution equations is likewise an increasingly active area of investigations, although most works have focused on linear or non-wave problems; see, for example, [2, 15, 24, 41]. Analysis and numerical analysis of a distributed optimal control problem for a linear wave model with fractional attenuation in the form of $\partial_t^\alpha u$ and homogeneous Dirichlet data can be found in [49].

Boundary optimal control of the strongly damped Westervelt equation (that is, with the dissipation term $-b\Delta p_t$ in place of $-b\Delta \partial_t^\alpha p$ in (IBVP_{West})) has been rigorously investigated in [9]. In this strongly damped case, the equation is known to have parabolic-like character which can be exploited to ensure its global-in-time well-posedness in different settings in terms of boundary conditions; see, e.g., [26, 39]. With time-fractional attenuation as in (IBVP_{West}), this property is lost and the energy arguments do not carry over. The well-posedness of the fractionally damped Westervelt equation supplemented by Dirichlet boundary data has been established recently in [30]; see also [3, 28]. Well-posedness in the case of Neumann data as in (IBVP_{West}) is, to the best of our knowledge, still an open problem and it is thus the first step we take in this work toward understanding (P).

Organization of the paper. The remainder of the paper is structured as follows. Section 2 summarizes the necessary background on fractional calculus based on the Djrbashian–Caputo derivative, which we use in the analysis of the optimal control problem. Section 3 establishes well-posedness for a linearized wave equation in both low- and high-regularity settings. The low-regularity result is essential for the adjoint analysis and for proving differentiability of the control-to-state map. The high-regularity result, which requires the construction of a suitable extension of regularized Neumann data, is then combined with Banach’s fixed-point theorem to prove the well-posedness of the state problem when $k = k(x)$ is sufficiently small. This result is contained in Theorem 3.1. In Section 4 we prove the existence of optimal controls in Theorem 4.1 and then demonstrate stability of the problem with respect to perturbations in the target pressure, as well as in the limit of vanishing regularization parameters $\eta = \gamma \searrow 0$. Finally, building upon these results,

in Section 5 we analyze the adjoint problem, establish the differentiability of the control-to-state operator, and then derive the necessary optimality conditions for the Neumann ($J = J(p, g)$) and distributed ($J = J(p, f)$) optimal control problems separately. The optimality conditions are stated in Theorems 5.1 and 5.2, respectively.

Notation. We use $(\cdot, \cdot)_{L^2(\Omega)}$ to denote the $L^2(\Omega)$ inner product. When working with Bochner spaces $W^{m,p}(0, T; X)$, we often omit the time interval $(0, T)$ from the notation of norms. For example, $\|\cdot\|_{L^2(0, T; L^2(\Omega))}$ denotes the norm in $L^2(0, T; L^2(\Omega))$. A subscript t indicates that the temporal domain is $(0, t)$ for some $t \in [0, T]$. For example, $\|\cdot\|_{L_t^2(L^2(\Omega))}$ denotes the norm in $L^2(0, t; L^2(\Omega))$ for $t \in (0, T)$.

We frequently use $\text{lhs} \lesssim \text{rhs}$ in the estimates to denote $\text{lhs} \leq C \text{rhs}$, where $C > 0$ is a generic constant. When the hidden constant depends on the final time T , we use the notation $\text{lhs} \lesssim_T \text{rhs}$.

2. THEORETICAL PRELIMINARIES IN FRACTIONAL CALCULUS

In this section, we recall several results from fractional calculus involving the Djrbashian–Caputo fractional derivative, which will be needed for the subsequent analysis of the state and optimal control problems. Most results stated below can be found in [34, Ch. 2], where additional details are provided.

Fractional calculus. For $\alpha > 0$, the Riemann–Liouville fractional integral operator is given by

$$(2.1) \quad \mathcal{J}^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds \quad \text{with the domain } \mathcal{D}(\mathcal{J}^\alpha) = L^2(0, T),$$

where $\Gamma(\cdot)$ denotes the Gamma function; see [13, 34]. For $0 < \alpha < 1$, the fractional Sobolev space $H^\alpha(0, T)$ is endowed with the norm

$$\|v\|_{H^\alpha(0, T)} = \left(\|v\|_{L^2(0, T)}^2 + \int_0^T \int_0^T \frac{|v(t) - v(s)|^2}{|t-s|^{1+2\alpha}} dt ds \right)^{\frac{1}{2}};$$

cf. [10]. According to [34, Theorem 2.1], the fractional integral operator $\mathcal{J}^\alpha : L^2(0, T) \rightarrow H_\alpha(0, T)$ is bijective, where the space $H_\alpha(0, T)$ is defined as

$$H_\alpha(0, T) = \begin{cases} \{v \in H^\alpha(0, T) : v(0) = 0\} & \text{if } \frac{1}{2} < \alpha < 1, \\ \{v \in H^{1/2}(0, T) : \int_0^T \frac{|v(t)|^2}{t} dt < \infty\} & \text{if } \alpha = \frac{1}{2}, \\ H^\alpha(0, T) & \text{if } 0 < \alpha < \frac{1}{2}, \end{cases}$$

endowed with the norm

$$\|v\|_{H_\alpha(0, T)} = \begin{cases} \|v\|_{H^\alpha(0, T)}, & 0 < \alpha < 1, \alpha \neq \frac{1}{2}, \\ \left(\|v\|_{H^{1/2}(0, T)}^2 + \int_0^T \frac{|v(t)|^2}{t} dt \right)^{1/2}, & \alpha = \frac{1}{2}. \end{cases}$$

It is known that the space ${}_0C^1[0, T] = \{v \in C^1[0, T] : v(0) = 0\}$ is dense in $H_\alpha(0, T)$:

$$\overline{{}_0C^1[0, T]}^{H_\alpha(0, T)} = H_\alpha(0, T);$$

see [34, Lemma 2.2]. The (generalized) Djrbashian–Caputo fractional derivative is defined as the inverse of the fractional integral operator in (2.1), that is,

$$\partial_t^\alpha v = \mathcal{J}^{-\alpha} v, \quad \mathcal{D}(\partial_t^\alpha) = H_\alpha(0, T);$$

see [34, Definition 2.1]. On account of [34, Theorem 2.4, (2.25)], for $\alpha \in (0, 1)$, ∂_t^α is an isomorphism between $H_\alpha(0, T)$ and $L^2(0, T)$, and thus we have

$$\|\partial_t^\alpha v\|_{L^2(0, T)} \sim \|v\|_{H_\alpha(0, T)}.$$

If $v \in {}_0W^{1,1}(0, T) = \{v \in W^{1,1}(0, T) : v(0) = 0\}$, then the generalized Djrbashian–Caputo matches the usual definition

$$\partial_t^\alpha v = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v_t(s) \, ds, \quad \alpha \in (0, 1);$$

see [34, Theorem 2.4]. In this case, $\partial_t^\alpha v = \mathfrak{K}_{1-\alpha} * v_t$ with the weakly singular memory kernel

$$\mathfrak{K}_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}.$$

Moreover, $\mathfrak{K}_{1-\alpha} \in L^p(0, T)$ for $0 < \alpha < 1/p$ and $p \in (1, \infty)$.

Helpful inequalities. Given a Banach space X , we introduce

$$H_\alpha(0, T; X) = \mathcal{J}^\alpha L^2(0, T; X),$$

which is a Banach space endowed with the norm $\|u\|_{H_\alpha(0, T; X)} = \|\partial_t^\alpha u\|_{L^2(0, T; X)}$; see [20, Lemma 2]. For $u \in H_\alpha(0, t; L^2(\Omega))$, it holds that

$$(2.2) \quad \int_0^t \int_\Omega \partial_t^\alpha u \, \partial_t u \, dx ds \geq \frac{1}{2} (\mathfrak{K}_\alpha * \|\partial_t^\alpha u\|_{L^2(\Omega)}^2)(t) \geq C_\alpha(T) \|\partial_t^\alpha u\|_{L^2_t(L^2(\Omega))}^2,$$

where $C_\alpha(T) = \frac{T^\alpha}{2\Gamma(1-\alpha)}$, by [34, Theorem 3.1]. In the analysis, we use the α -uniform lower bound $C_\alpha(T) \geq \underline{C}(T)$ to guarantee that the constants in the estimates are α -independent. We also use the inequality

$$(2.3) \quad \int_0^t \int_\Omega u(s) (\mathcal{J}^\alpha u)(s) \, dx ds \geq 0, \quad u \in L^2(0, t; L^2(\Omega)),$$

see, e.g., [28, Sec. 5] and [13, Lemma 2]. Furthermore, for any $u \in H_\alpha(0, T; X)$, the identity $u = \mathcal{J}^\alpha \mathcal{J}^{-\alpha} u$, together with Young's convolution inequality, yields

$$\|u\|_{L^2(0, T; X)} \leq \|\mathfrak{K}_\alpha\|_{L^1(0, T)} \|\partial_t^\alpha u\|_{L^2(0, T; X)}.$$

Adjoint of the fractional derivative. In the derivation of first-order optimality conditions via the adjoint problem corresponding to $(\text{IBVP}_{\text{West}})$, we require the Banach space adjoint $\widetilde{\partial}_t^\alpha$ of the time-fractional derivative operator ∂_t^α :

$$\int_0^T (\partial_t^\alpha v)(t) \varphi(t) \, dt = \int_0^T v(t) (\widetilde{\partial}_t^\alpha \varphi)(t) \, dt, \quad v \in H_\alpha(0, T).$$

The adjoint operator is defined as

$$\widetilde{\partial}_t^\alpha = \widetilde{\mathcal{J}}^\alpha{}^{-1} \quad \text{with the domain } \mathcal{D}(\widetilde{\partial}_t^\alpha) = \tau H_\alpha(0, T), \text{ for } \alpha > 0,$$

where $(\tau v)(t) := v(T-t)$ denotes the time reversal operator and

$$(2.4) \quad \widetilde{\mathcal{J}}^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} v(s) \, ds = \tau \mathcal{J}^\alpha (\tau v)(t);$$

see [52, Theorem I-5]. To carry out the adjoint-based arguments in Section 4, we also require the following time-reversal identity:

$$(2.5) \quad \tau \left(\widetilde{\partial}_t^\alpha \varphi \right) = \partial_t^\alpha (\tau \varphi), \quad \varphi \in \tau H^\alpha(0, T).$$

Indeed, by (2.4), we have

$$\tau \mathcal{J}^\alpha \left(\tau \left((\widetilde{\mathcal{J}}^\alpha)^{-1} \varphi \right) \right) = \widetilde{\mathcal{J}}^\alpha \left(\widetilde{\mathcal{J}}^\alpha \right)^{-1} \varphi = \varphi.$$

Hence, since $(\tau(\tau v)(t)) = v(t)$, we have

$$\mathcal{J}^\alpha \left(\tau \left((\widetilde{\mathcal{J}}^\alpha)^{-1} \varphi \right) \right) = \tau \varphi.$$

Then because $\mathcal{J}^{-\alpha} = (\mathcal{J}^\alpha)^{-1}$ on $H_\alpha(0, T)$, it follows that

$$\tau \left((\widetilde{\mathcal{J}}^\alpha)^{-1} \varphi \right) = \mathcal{J}^{-\alpha}(\tau \varphi),$$

which is precisely (2.5).

For $v, \varphi \in H^1(0, T)$, with $v(0) = \varphi(T) = 0$, the adjoint operator can be expressed as

$$(\widetilde{\partial}_t^\alpha \varphi)(s) = -\frac{1}{\Gamma(1-\alpha)} \int_s^T (t-s)^{-\alpha} \varphi'(t) dt;$$

see [32, Sec. 3].

3. ANALYSIS OF WAVE EQUATIONS WITH TIME-FRACTIONAL ATTENUATION

In this section, we lay the groundwork for the analysis of the optimal control problem by studying the well-posedness of Neumann boundary value problems for both the linearized wave equation and the Westervelt equation in (IBVP_{West}). We consider a linearized wave equation of the form

$$(3.1) \quad \mathbf{a}(x, t) u_{tt} - c^2 \Delta u - b \Delta \partial_t^\alpha u + \mathbf{l}(x, t) u_t + \mathbf{n}(x, t) u = F(x, t)$$

The well-posedness result for (3.1) is relevant not only in the fixed-point argument for the forward problem, but also in the analysis of the adjoint problem and in establishing differentiability of the control-to-state mapping. We conduct the linear acoustic investigations under the following non-degeneracy assumption on the leading variable coefficient: we assume that there exist $\underline{\mathbf{a}}, \bar{\mathbf{a}}$, such that

$$(3.2) \quad 0 < \underline{\mathbf{a}} \leq \mathbf{a}(x, t) \leq \bar{\mathbf{a}} \quad \text{for all } (x, t) \in \Omega \times [0, T].$$

In terms of later studying the nonlinear pressure problem, this coefficient can be seen as a placeholder for $1 - 2kp$, whose non-degeneracy will be ensured through smallness of the nonlinearity coefficient $k = k(x)$ in a suitable norm in the fixed-point argument; cf. (1.2). The coefficient \mathbf{l} can be seen as a placeholder for $2ku_t$. The term \mathbf{n} is included to accommodate the analysis of differentiability of the control-to-state mapping Section 5; cf. Proposition 5.2.

Because the adjoint problem and the differentiability analysis will take place in a lower regularity setting than the state equation, we establish two well-posedness results: one in a lower-regularity regime and another in a higher-regularity regime.

3.1. Analysis of a linearized wave problem: Lower-regularity regime. The first well-posedness result is stated in a relatively low regularity setting for the data and variable coefficients, suitable for later analyzing the adjoint problem and the differentiability of the control-to-state mapping. As the adjoint problem contains a non-zero terminal condition

(see (5.1)), arising from the time-localized part of the objective function, we analyze here a linearized wave problem with $u_t|_{t=0} = u_1$:

$$(IBVP_{\text{lin}}) \quad \begin{cases} \mathbf{a}(x, t)u_{tt} - c^2\Delta u - b\Delta\partial_t^\alpha u + \mathbf{l}(x, t)u_t + \mathbf{n}(x, t)u = F(x, t), \\ \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma, \\ (u, u_t)|_{t=0} = (0, u_1). \end{cases}$$

We define weak solutions of (IBVP_{lin}) as functions satisfying

$$(3.3) \quad \begin{aligned} & - \int_0^T \int_\Omega (\mathbf{a}\varphi)_t u_t \, dx dt + \int_0^T \int_\Omega (c^2 \nabla u + b \nabla \partial_t^\alpha u) \cdot \nabla \varphi \, dx dt \\ & + \int_0^T \int_\Omega (\mathbf{l}u_t + \mathbf{n}u) \varphi \, dx dt - \int_0^T \int_\Gamma (c^2 g + b \partial_t^\alpha g) \varphi \, d\Gamma dt \\ & = \int_\Omega \mathbf{a}(0)u_1 \varphi(0) \, dx + \int_0^T \int_\Omega F \varphi \, dx dt \end{aligned}$$

for all test functions $\varphi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with $\varphi(T) = 0$.

Proposition 3.1. *Let $T > 0$ and let $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$ be a Lipschitz-regular bounded domain. Let $c > 0$, $b \in (0, \bar{b}]$ for some $\bar{b} > 0$, and $\alpha \in (0, 1)$. Assume that the variable coefficients in (IBVP_{lin}) satisfy*

$$\mathbf{a} \in W^{1,\infty}(0, T; L^\infty(\Omega)), \quad \mathbf{l} \in L^\infty(0, T; L^\infty(\Omega)), \quad \mathbf{n} \in L^2(0, T; L^3(\Omega)),$$

and that the non-degeneracy condition on \mathbf{a} in (3.46) holds. Furthermore, let $F \in L^2(0, T; L^2(\Omega))$, $g \in H^2(0, T; H^{-1/2}(\Gamma))$, and $u_1 \in L^2(\Omega)$. Then there exists a solution $u \in X_{\text{low}}$ of (3.3), where

$$(3.4) \quad X_{\text{low}} = \{u \in L^\infty(0, T; H^1(\Omega)) : u_t \in L^\infty(0, T; L^2(\Omega)), \partial_t^\alpha u \in (0, T; H^1(\Omega))\}.$$

The solution satisfies the estimate

$$(3.5) \quad \|u\|_{X_{\text{low}}}^2 \lesssim \Lambda_0(\mathbf{a}, \mathbf{l}, \mathbf{n}) \left(\|u_1\|_{L^2(\Omega)}^2 + \|F\|_{L^2(L^2(\Omega))}^2 + \|g\|_{H^2(H^{-1/2}(\Gamma))}^2 \right),$$

where

$$(3.6) \quad \Lambda_0(\mathbf{a}, \mathbf{l}, \mathbf{n}) = \exp \left\{ CT(1 + \|\mathbf{a}\|_{W^{1,\infty}(L^\infty(\Omega))} + \|\mathbf{l}\|_{L^\infty(L^\infty(\Omega))} + \|\mathbf{n}\|_{L^2(L^3(\Omega))}^2) \right\}.$$

The hidden constants are independent of b and α .

Proof. We establish well-posedness of (IBVP_{lin}) using a Faedo–Galerkin method, analogous to the approach used in the analysis of linear wave equations with time-fractional attenuation and homogeneous Dirichlet boundary conditions; see, e.g., [7, 28, 30]. The main difference lies in the testing procedure, which here is adapted to handle lower regularity. To discretize (IBVP_{lin}^{hom}) in space, we employ as the basis functions $\{\varphi^{(m)}\}_{m \geq 1}$ eigenfunctions of the Laplacian with homogeneous Neumann conditions

$$(3.7) \quad -\Delta \varphi^{(m)} = \lambda^{(m)} \varphi^{(m)} \quad \text{in } \Omega, \quad \frac{\partial \varphi^{(m)}}{\partial n} = 0 \quad \text{on } \Gamma.$$

The problem can then be posed in $V^{(m)} = \text{span}\{\varphi^{(1)}, \dots, \varphi^{(m)}\}$ and solved for

$$u^{(m)}(t) = \sum_{\ell=1}^m \xi_m^\ell(t) \varphi^{(m)},$$

by requiring that

$$(3.8) \quad (\mathbf{a}u_{tt}^{(m)}, v^{(m)})_{L^2(\Omega)} + (c^2 \nabla u^{(m)} + b \nabla \partial_t^\alpha u^{(m)}, \nabla v^{(m)})_{L^2(\Omega)} \\ - (c^2 g + b \partial_t^\alpha g, v^{(m)})_{L^2(\Gamma)} = (F - \mathbf{l}u_t^{(m)} - \mathbf{n}u^{(m)}, v^{(m)})_{L^2(\Omega)},$$

for all $v^{(m)} \in V^{(m)}$, where we impose $u^{(m)}(0) = 0$ and $(u_t^{(m)}(0), \varphi^{(m)})_{L^2(\Omega)} = (u_1, \varphi^{(m)})_{L^2(\Omega)}$, $k = 1, \dots, m$ (in other words we choose approximate initial data as L^2 projections of the exact data). Existence and uniqueness of $u^{(m)} \in H^2(0, T; V^{(m)})$ in this form follow in a directly analogous manner to the Dirichlet case by restating the problem in terms of $\boldsymbol{\mu} = \boldsymbol{\xi}_{tt}$ with $\boldsymbol{\xi} = [\xi_1^\ell \ \xi_2^\ell \ \dots \ \xi_m^\ell]^T$ and relying on the existence theory for Volterra integral equations of second kind; see [28, Appendix A] for details. We thus omit those arguments here and proceed directly to derive the energy estimate that is uniform with respect to the discretization parameter m .

• Testing with $u_t^{(m)}$: Testing (3.8) with $v^{(m)} = u_t^{(m)}$, and invoking the coercivity bound (2.2), we obtain the inequality

$$(3.9) \quad \frac{1}{2} \left\| \sqrt{\mathbf{a}(s)} u_t^{(m)}(s) \right\|_{L^2(\Omega)}^2 \Big|_0^t + \frac{c^2}{2} \|\nabla u^{(m)}(t)\|_{L^2(\Omega)}^2 + b C_\alpha(T) \|\nabla \partial_t^\alpha u^{(m)}\|_{L_t^2(L^2(\Omega))}^2 \\ \leq \frac{1}{2} \int_0^t (\mathbf{a}_t u_t^{(m)}, u_t^{(m)})_{L^2(\Omega)} ds + \int_0^t (\mathbf{l}u_t^{(m)} + \mathbf{n}u^{(m)}, u_t^{(m)})_{L^2(\Omega)} ds \\ + \int_0^t \int_\Gamma (c^2 g + b \partial_t^\alpha g) u_t^{(m)} d\Gamma ds + \int_0^t (F, u_t^{(m)})_{L^2(\Omega)} ds.$$

We estimate the first term on the right-hand side using Hölder's inequality:

$$(3.10) \quad \frac{1}{2} \int_0^t (\mathbf{a}_t u_t^{(m)}, u_t^{(m)})_{L^2(\Omega)} ds \lesssim \int_0^t \|\mathbf{a}_t\|_{L^\infty(\Omega)} \|u_t^{(m)}\|_{L^2(\Omega)}^2 ds.$$

Similarly, using also Young's ε -inequality, and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have, for any $\varepsilon > 0$,

$$(3.11) \quad \int_0^t (\mathbf{l}u_t^{(m)} + \mathbf{n}u^{(m)}, u_t^{(m)})_{L^2(\Omega)} ds \\ \lesssim \int_0^t \|\mathbf{l}\|_{L^\infty(\Omega)} \|u_t^{(m)}\|_{L^2(\Omega)}^2 ds + \int_0^t \|\mathbf{n}\|_{L^3(\Omega)} \|u^{(m)}\|_{L^6(\Omega)} \|u_t^{(m)}\|_{L^2(\Omega)} ds \\ \lesssim \int_0^t \left(\|\mathbf{l}\|_{L^\infty(\Omega)} + \|\mathbf{n}\|_{L^3(\Omega)}^2 \right) \|u_t^{(m)}\|_{L^2(\Omega)}^2 ds + \varepsilon \max_{s \in [0, t]} \|u^{(m)}(s)\|_{H^1(\Omega)}^2.$$

Concerning the boundary terms in (3.9), we integrate by parts in time:

$$\int_0^t \int_\Gamma (c^2 g + b \partial_t^\alpha g) u_t^{(m)} d\Gamma ds = \int_\Gamma (c^2 g + b \partial_t^\alpha g) u^{(m)} d\Gamma \Big|_0^t - \int_0^t \int_\Gamma (c^2 g + b \partial_t^\alpha g)_t u^{(m)} d\Gamma ds.$$

Since $u^{(m)}|_{t=0} = 0$ and $(\mathfrak{K}_{1-\alpha} * g_t)_t = \mathfrak{K}_{1-\alpha} * g_{tt} + \mathfrak{K}_{1-\alpha}(t)g_t(0)$, we obtain

$$\int_0^t \int_\Gamma (c^2 g + b \partial_t^\alpha g) u_t^{(m)} d\Gamma ds \\ = \int_\Gamma (c^2 g + b \partial_t^\alpha g)(t) u^{(m)}(t) d\Gamma - \int_0^t \int_\Gamma (c^2 g_t + b \partial_t^{1+\alpha} g + b \mathfrak{K}_{1-\alpha} g_t(0)) u^{(m)} d\Gamma ds.$$

Using Hölder's inequality and the trace theorem, we estimate

$$\begin{aligned} \int_0^t \int_{\Gamma} (c^2 g + b \partial_t^\alpha g) u_t^{(m)} d\Gamma ds &\lesssim \|g(t)\|_{H^{-1/2}(\Gamma)}^2 + \|\partial_t^\alpha g(t)\|_{H^{-1/2}(\Gamma)}^2 + \varepsilon \|u^{(m)}(t)\|_{H^1(\Omega)}^2 \\ &\quad + \|\mathfrak{K}_{1-\alpha}\|_{L^1(0,T)}^2 \|g_t(0)\|_{H^{-1/2}(\Gamma)}^2 + \varepsilon \max_{s \in [0,t]} \|u^{(m)}(s)\|_{H^1(\Omega)}^2 \\ &\quad + \|g_t\|_{L^2(H^{-1/2}(\Gamma))}^2 + \|\partial_t^{1+\alpha} g\|_{L^2(H^{-1/2}(\Gamma))}^2 \end{aligned}$$

for any $\varepsilon > 0$, where we have $\|\mathfrak{K}_{1-\alpha}\|_{L^1(0,T)} = \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \leq C$ for some $C > 0$ independent of α . By the assumed regularity of g , we have $g \in H^2(0,T; H^{-1/2}(\Gamma)) \hookrightarrow C^1([0,T]; H^{-1/2}(\Gamma))$. Additionally,

$$(3.12) \quad \int_0^t (F, u_t^{(m)})_{L^2(\Omega)} ds \lesssim \|F\|_{L^2(L^2(\Omega))}^2 + \int_0^t \|u_t^{(m)}\|_{L^2(\Omega)}^2 ds.$$

Substituting (3.10), (3.11), and (3.12) into (3.9) yields

$$\begin{aligned} (3.13) \quad &\frac{1}{2} \|\sqrt{\mathfrak{a}(t)} u_t^{(m)}(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla u^{(m)}(t)\|_{L^2(\Omega)}^2 + b C_\alpha(T) \|\nabla \partial_t^\alpha u^{(m)}\|_{L_t^2(L^2(\Omega))}^2 \\ &\lesssim \int_0^t \left\{ 1 + \|\mathfrak{l}\|_{L^\infty(\Omega)} + \|\mathfrak{n}\|_{L^3(\Omega)}^2 + \|\mathfrak{a}_t\|_{L^\infty(\Omega)} \right\} \|u_t^{(m)}\|_{L^2(\Omega)}^2 ds \\ &\quad + \varepsilon \max_{s \in [0,t]} \|u^{(m)}(s)\|_{H^1(\Omega)}^2 + \|F\|_{L^2(L^2(\Omega))}^2 + \int_0^t \int_{\Gamma} (c^2 g + b \partial_t^\alpha g) u_t^{(m)} d\Gamma ds. \end{aligned}$$

Adding the elementary estimate $\|u^{(m)}(t)\|_{L^2(\Omega)}^2 \lesssim T \|u_t^{(m)}\|_{L_t^2(L^2(\Omega))}^2 + \|u_0^{(m)}\|_{L^2(\Omega)}^2$ to (3.13), results in the energy inequality

$$\begin{aligned} (3.14) \quad &\|\sqrt{\mathfrak{a}} u_t^{(m)}(t)\|_{L^2(\Omega)}^2 + \|u^{(m)}(t)\|_{H^1(\Omega)}^2 + b C_\alpha(T) \|\nabla \partial_t^\alpha u^{(m)}\|_{L_t^2(L^2(\Omega))}^2 \\ &\lesssim_T (1 + \|\mathfrak{a}\|_{W^{1,\infty}(L^\infty(\Omega))} + \|\mathfrak{l}\|_{L^\infty(L^\infty(\Omega))} + \|\mathfrak{n}\|_{L^2(L^3(\Omega))}^2) \|u_t^{(m)}\|_{L_t^2(L^2(\Omega))}^2 \\ &\quad + \|u_1^{(m)}\|_{L^2(\Omega)}^2 + \|F\|_{L^2(L^2(\Omega))}^2 + \|g\|_{H^2(H^{-1/2}(\Gamma))}^2 + \varepsilon \max_{s \in [0,t]} \|u^{(m)}(s)\|_{H^1(\Omega)}^2. \end{aligned}$$

for $t \in [0, T]$. Thanks to the non-degeneracy assumption in (3.46), we have

$$\underline{\mathfrak{a}} \|u_t^{(m)}(t)\|_{L^2(\Omega)}^2 \leq \|\sqrt{\mathfrak{a}(t)} u_t^{(m)}(t)\|_{L^2(\Omega)}^2.$$

Employing this lower bound in (3.14) and taking the maximum over $t \in [0, \sigma]$ for $\sigma \in (0, T)$, we obtain, with sufficiently small ε ,

$$\begin{aligned} &\|u_t^{(m)}(\sigma)\|_{L^2(\Omega)}^2 + \|u^{(m)}(\sigma)\|_{H^1(\Omega)}^2 + b \underline{C}(T) \|\nabla \partial_t^\alpha u^{(m)}\|_{L^2(0,\sigma; L^2(\Omega))}^2 \\ &\lesssim \|u_1^{(m)}\|_{L^2(\Omega)}^2 + (1 + \|\mathfrak{a}\|_{W^{1,\infty}(L^\infty(\Omega))} + \|\mathfrak{l}\|_{L^\infty(L^\infty(\Omega))} + \|\mathfrak{n}\|_{L^2(L^3(\Omega))}^2) \|u_t^{(m)}\|_{L^2(0,\sigma; L^2(\Omega))}^2 \\ &\quad + \|F\|_{L^2(L^2(\Omega))}^2 + \|g\|_{H^2(H^{-1/2}(\Gamma))}^2. \end{aligned}$$

Applying Grönwall's inequality and using the estimate $\|u_1^{(m)}\|_{L^2(\Omega)} \lesssim \|u_1\|_{L^2(\Omega)}$, we obtain the semi-discrete version of the claimed estimate:

$$(3.15) \quad \|u^{(m)}\|_{X_{\text{low}}}^2 \lesssim \Lambda_0(\mathfrak{a}, \mathfrak{l}, \mathfrak{n}) \left(\|u_1\|_{L^2(\Omega)}^2 + \|F\|_{L^2(L^2(\Omega))}^2 + \|g\|_{H^2(H^{-1/2}(\Gamma))}^2 \right)$$

where Λ_0 is defined in (3.6).

• Passing to the limit as $m \rightarrow \infty$: Thanks to the m -uniform bound in (3.15), we may extract subsequences, that we do not relabel, such that

$$\begin{aligned} u^{(m)} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0, T; H^1(\Omega)), && u_t^{(m)} &\overset{*}{\rightharpoonup} u_t && \text{in } L^\infty(0, T; L^2(\Omega)), \\ \partial_t^\alpha u^{(m)} &\rightharpoonup \zeta && \text{in } L^2(0, T; H^1(\Omega)). \end{aligned}$$

We can identify $\zeta = \partial_t^\alpha u$ following the arguments in the proof of [23, Theorem 6.19, Sec. 6.4.2]. Passing to the limit $m \rightarrow \infty$ in

$$\begin{aligned} & - \int_0^T \int_\Omega (\mathbf{a}\varphi)_t u_t^{(m)} dx dt + \int_0^T \int_\Omega (c^2 \nabla u^{(m)} + b \nabla \partial_t^\alpha u^{(m)}) \cdot \nabla \varphi dx dt \\ & + \int_0^T \int_\Omega (\mathbf{l} u_t^{(m)} + \mathbf{n} u^{(m)}) \varphi dx dt - \int_0^T \int_\Gamma (c^2 g + b \partial_t^\alpha g) \varphi d\Gamma dt \\ & = \int_\Omega \mathbf{a}(0) u_1^{(m)} \varphi(0) dx + \int_0^T \int_\Omega F \varphi dx dt \end{aligned}$$

and arguing that $u|_{t=0} = 0$, we obtain a solution u of (3.3). \square

The previous result ensures existence but does not yet imply uniqueness. Uniqueness follows under higher regularity of the variable coefficients \mathbf{a} and \mathbf{l} .

Lemma 3.1. *Let the assumption of Proposition 3.1 hold. If, in addition,*

$$\mathbf{a} \in H^2(0, T; L^4(\Omega)) \quad \text{and} \quad \mathbf{l} \in H^1(0, T; L^3(\Omega)),$$

then the solution of (3.3) is unique.

Proof. To conclude uniqueness, we show that the homogeneous problem (3.3) with $F = 0$, $u_1 = 0$, and $g = 0$ admits only the trivial solution $u = 0$. We adapt the argument used in [12, Theorem 4, Ch. 7.2] and for a fixed $0 \leq s \leq T$, define

$$\Phi(t) = \begin{cases} \int_t^s u(\tau) d\tau, & 0 \leq t \leq s, \\ 0, & s \leq t \leq T. \end{cases}$$

Clearly,

$$\Phi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad \Phi(s) = 0,$$

and differentiation yields $\Phi_t(t) = -u(t)$ for $t \in (0, s)$. Using Φ as the test function in the homogeneous version of (3.3) and recalling that $u_t(0) = 0$, we obtain

$$\begin{aligned} & - \int_0^s \int_\Omega \mathbf{a}(x, t) u_t \Phi_t dx dt + c^2 \int_0^s \int_\Omega \nabla u \cdot \nabla \Phi dx dt \\ & = \int_0^s \int_\Omega \mathbf{a}_t u_t \Phi dt - b \int_0^s \int_\Omega \nabla \partial_t^\alpha u \cdot \nabla \Phi dx dt - \int_0^s \int_\Omega (\mathbf{l} u_t + \mathbf{n} u) \Phi dx dt. \end{aligned}$$

Substituting $\Phi_t = -u$ on $(0, s)$ yields

$$\begin{aligned} & \frac{1}{2} \int_0^s \frac{d}{dt} \left(\|\sqrt{\mathbf{a}} u\|_{L^2(\Omega)}^2 - c^2 \|\nabla \Phi\|_{L^2(\Omega)}^2 \right) dt \\ & = \frac{1}{2} \int_0^s (\mathbf{a}_t u, u)_{L^2(\Omega)} dt + \int_0^s \int_\Omega \mathbf{a}_t u_t \Phi dt - b \int_0^s \int_\Omega \nabla \partial_t^\alpha u \cdot \nabla \Phi dt - \int_0^s \int_\Omega (\mathbf{l} u_t + \mathbf{n} u) \Phi dx dt. \end{aligned}$$

Using $\Phi(s) = 0$ and the fact that $u(0) = 0$ yields

$$\begin{aligned}
 (3.16) \quad & \frac{1}{2}(\|\sqrt{\mathbf{a}}u(s)\|_{L^2(\Omega)}^2 + c^2\|\nabla\Phi(0)\|_{L^2(\Omega)}^2) \\
 &= \int_0^s \int_{\Omega} \mathbf{a}_t u_t \Phi \, dt - \int_0^s \int_{\Omega} b \nabla \partial_t^\alpha u \cdot \nabla \Phi \, dt + \frac{1}{2} \int_0^s (\mathbf{a}_t u, u)_{L^2(\Omega)} \, dt \\
 &\quad - \int_0^s \int_{\Omega} (\mathbf{l}u_t + \mathbf{n}u) \Phi \, dx dt.
 \end{aligned}$$

Moreover, by definition of Φ and the Cauchy–Schwarz inequality

$$(3.17) \quad \|\Phi(0)\|_{L^2(\Omega)}^2 \leq \left(\int_0^s \|u(t)\|_{L^2(\Omega)} \, dt \right)^2 \leq s \int_0^s \|u(t)\|_{L^2(\Omega)}^2 \, dt \leq T \int_0^s \|u(t)\|_{L^2(\Omega)}^2 \, dt.$$

Collecting (3.16) and (3.17), and recalling the non-degeneracy assumption (3.46), we obtain

$$\begin{aligned}
 (3.18) \quad & \frac{1}{2}(\|u(s)\|_{L^2(\Omega)}^2 + c^2\|\Phi(0)\|_{H^1(\Omega)}^2) + b \int_0^s \int_{\Omega} \nabla \partial_t^\alpha u \cdot \nabla \Phi \, dt \\
 & \lesssim \int_0^s (T^2 + \|\mathbf{a}_t\|_{L^\infty(\Omega)}) \|\sqrt{\mathbf{a}}u(t)\|_{L^2(\Omega)}^2 \, dt + \int_0^s \int_{\Omega} \mathbf{a}_t u_t \Phi \, dt - \int_0^s \int_{\Omega} (\mathbf{l}u_t + \mathbf{n}u) \Phi \, dx dt.
 \end{aligned}$$

We treat next the second integral on the right-hand side. Integrating by parts in time and using $\Phi(s) = 0$, $u(0) = 0$, and $\Phi_t = -u$ on $(0, s)$, we obtain

$$\int_0^s \int_{\Omega} \mathbf{a}_t u_t \Phi \, dt = \int_{\Omega} \mathbf{a}_t u \Phi \, dx \Big|_0^s - \int_0^s \int_{\Omega} (\mathbf{a}_{tt} \Phi + \mathbf{a}_t(-u)) u \, dx dt.$$

Using $\Phi(s) = 0 = u(0) = 0$, the first term on the right vanishes, and we arrive at

$$\int_0^s \int_{\Omega} \mathbf{a}_t u_t \Phi \, dt \lesssim \int_0^s \|\mathbf{a}_{tt}\|_{L^4(\Omega)} \|\Phi\|_{L^4(\Omega)} \|u\|_{L^2(\Omega)} \, dt + \int_0^s \|\mathbf{a}_t\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 \, dt.$$

Defining $U(t) = \int_0^t u(\tau) \, d\tau$, and noting that $\Phi(t) = U(s) - U(t)$, we use the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ to obtain

$$\begin{aligned}
 (3.19) \quad & \int_0^s \|\mathbf{a}_{tt}(t)\|_{L^4(\Omega)} \|\Phi(t)\|_{L^4(\Omega)} \|u(t)\|_{L^2(\Omega)} \, dt \\
 & \lesssim \int_0^s \left(\|U(s) - U(t)\|_{H^1(\Omega)}^2 + \|\mathbf{a}_{tt}\|_{L^4(\Omega)}^2 \|u\|_{L^2(\Omega)}^2 \right) \, dt \\
 & \lesssim s \|U(s)\|_{H^1(\Omega)}^2 + \int_0^s (\|U\|_{H^1(\Omega)}^2 + \|\mathbf{a}_{tt}\|_{L^4(\Omega)}^2 \|u\|_{L^2(\Omega)}^2) \, dt.
 \end{aligned}$$

Similarly, integrating by parts in time, we obtain

$$\begin{aligned}
 & - \int_0^s \int_{\Omega} (\mathbf{l}u_t + \mathbf{n}u) \Phi \, dx dt \\
 &= - \int_0^s \int_{\Omega} \mathbf{n}u \Phi \, dx dt + \int_0^s \int_{\Omega} (\mathbf{l}_t \Phi - \mathbf{l}u) u \, dx dt \\
 &= \int_0^s \|\mathbf{n}\|_{L^3(\Omega)} \|u\|_{L^2(\Omega)} \|\Phi\|_{L^6(\Omega)} \, dt + \int_0^s (\|\mathbf{l}_t\|_{L^3(\Omega)} \|\Phi\|_{L^6(\Omega)} + \|\mathbf{l}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}) \|u\|_{L^2(\Omega)} \, dt \\
 &\lesssim \int_0^s \left(\|\Phi\|_{H^1(\Omega)}^2 + \left(\|\mathbf{n}\|_{L^3(\Omega)}^2 + \|\mathbf{l}_t\|_{L^3(\Omega)}^2 + \|\mathbf{l}\|_{L^\infty(\Omega)}^2 \right) \|u\|_{L^2(\Omega)}^2 \right) \, dt.
 \end{aligned}$$

Using the representation $\Phi(t) = U(s) - U(t)$, we further obtain

$$(3.20) \quad \begin{aligned} & - \int_0^s \int_{\Omega} (\mathfrak{l}u_t + \mathfrak{n}u) \Phi \, dx dt \\ & \lesssim s \|U(s)\|_{H^1(\Omega)}^2 + \int_0^s \left\{ \|U\|_{H^1(\Omega)}^2 + \left(\|\mathfrak{n}\|_{L^3(\Omega)}^2 + \|\mathfrak{l}_t\|_{L^3(\Omega)}^2 + \|\mathfrak{l}\|_{L^\infty(\Omega)} \right) \|u\|_{L^2(\Omega)}^2 \right\} dt. \end{aligned}$$

Inserting the above estimates into (3.18), we obtain

$$(3.21) \quad \begin{aligned} & \frac{1}{2} (\|u(s)\|_{L^2(\Omega)}^2 + (c^2 - C_0 s) \|U(s)\|_{H^1(\Omega)}^2) + b \int_0^s \int_{\Omega} \nabla \partial_t^\alpha u \cdot \nabla \Phi \, dt \\ & \lesssim \int_0^s (T^2 + \|\mathfrak{a}_t\|_{L^\infty(\Omega)} + \|\mathfrak{l}\|_{L^\infty(\Omega)}) \|u\|_{L^2(\Omega)}^2 \, dt \\ & \quad + \int_0^s (\|W\|_{H^1(\Omega)}^2 + (\|\mathfrak{a}_{tt}\|_{L^4(\Omega)}^2 + \|\mathfrak{n}\|_{L^3(\Omega)}^2 + \|\mathfrak{l}_t\|_{L^3(\Omega)}^2) \|u\|_{L^2(\Omega)}^2) \, dt, \end{aligned}$$

where C_0 is the sum of the hidden constants in (3.19) and (3.20).

It remains to show that the time-fractional term is non-negative. Since we will integrate by parts to prove this, we first regularize. Let $u_n \in {}_0C^1([0, s]; H^1(\Omega))$ and define $\Phi_n(t) := \int_t^s u_n(\tau) d\tau$. Then

$$\begin{aligned} b \int_0^s \int_{\Omega} \nabla \partial_t^\alpha u_n \cdot \nabla \Phi_n \, dt &= b \int_0^s \int_{\Omega} (J^{-\alpha} \nabla u_n) \cdot \nabla \Phi_n \, dt \\ &= b \int_0^s \int_{\Omega} (J^{-1} J^{1-\alpha} \nabla u_n) \cdot \nabla \Phi_n \, dt. \end{aligned}$$

Since $J^{-1}v = \partial_t v$ for $v \in {}_0H^1(0, T)$ (see [34, Sec. 2.3]), integrating by parts in time yields

$$(3.22) \quad b \int_0^s \int_{\Omega} \nabla \partial_t^\alpha u_n \cdot \nabla \Phi_n \, dt = b \int_0^s \int_{\Omega} (J^{1-\alpha} \nabla u_n) \cdot \nabla u_n \, dt \geq 0,$$

because $\Phi_n(s) = 0$, $J^{1-\alpha} \nabla u_n(0) = 0$, and inequality (2.3) holds. By density, (3.22) extends to $u \in H_\alpha(0, s; H^1(\Omega))$. Thus from (3.21) we have

$$\begin{aligned} & \frac{1}{2} \|\sqrt{\mathfrak{a}}u(s)\|_{L^2(\Omega)}^2 + (c^2 - c_0 s) \|U(s)\|_{H^1(\Omega)}^2 \\ & \lesssim \int_0^s (T^2 + \|\mathfrak{a}_t\|_{L^\infty(\Omega)} + \|\mathfrak{l}\|_{L^\infty(\Omega)}) \|\sqrt{\mathfrak{a}}u(t)\|_{L^2(\Omega)}^2 \, dt \\ & \quad + \int_0^s (\|U(t)\|_{H^1(\Omega)}^2 + (\|\mathfrak{a}_{tt}\|_{L^4(\Omega)}^2 + \|\mathfrak{n}\|_{L^3(\Omega)}^2 + \|\mathfrak{l}_t\|_{L^3(\Omega)}^2) \|\sqrt{\mathfrak{a}}u\|_{L^2(\Omega)}^2) \, dt, \end{aligned}$$

from which we can conclude via Grönwall's inequality that $u \equiv 0$ on $(0, s)$ for small enough s . We can apply the same arguments starting from time s to extend the conclusion to $[0, T]$. \square

3.2. Analysis of a linearized wave problem: Higher-regularity regime. To prove the well-posedness of the nonlinear state problem, we need to employ a higher-order testing procedure for the linearized problem that mimics the one used in the analysis of a linearized Westervelt equation with time-fractional attenuation in the setting of homogeneous Dirichlet data in [3]. However, to be in a position to employ an analogous testing procedure to the Dirichlet case, we need to devise a suitable Neumann extension of the boundary data and at first work with smoother data $g^{(\nu)}$. We can then establish the required energy estimates

on the regularized solutions, and afterward pass to the limit $\nu \rightarrow \infty$. The following result will be used for that purpose. Throughout this and upcoming sections, we assume $\Omega \subset \mathbb{R}^d$, where $d \in \{1, 2, 3\}$ is a bounded domain with sufficient regularity so that the following elliptic regularity property holds:

$$(3.23) \quad \text{if } -\Delta z + z \in H^2(\Omega) \text{ and } \frac{\partial z}{\partial n} \in H^{5/2}(\partial\Omega), \text{ then } z \in H^4(\Omega).$$

We refer to, e.g., [14, Theorem 2.5.1.1] for assumptions on Ω under which (3.23) holds.

Lemma 3.2 (Compatible regularized boundary data). *Let $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, is a bounded domain that satisfies (3.23). Furthermore, let*

$$(3.24) \quad g \in \mathcal{X}_g^{\text{low}} = H_\alpha(0, T; H^{3/2}(\Gamma)) \cap H^1(0, T; H^{1/2}(\Gamma)) \cap H^2(0, T; H^{-1/2}(\Gamma)),$$

and assume the compatibility conditions

$$(3.25) \quad g(0) = 0 \text{ in } H^{1/2}(\Gamma), \quad g_t(0) = 0 \text{ in } H^{-1/2}(\Gamma).$$

Then there exists a family $\{g^{(\nu)}\}_\nu$ with

$$g^{(\nu)} \in C^2([0, T]; H^{5/2}(\Gamma)), \quad g^{(\nu)}(0) = g_t^{(\nu)}(0) = 0,$$

such that, as $\nu \rightarrow \infty$,

$$\begin{aligned} g^{(\nu)} &\rightarrow g \quad \text{in } H_\alpha(0, T; H^{3/2}(\Gamma)), & g_t^{(\nu)} &\rightarrow g_t \quad \text{in } L^2(0, T; H^{1/2}(\Gamma)), \\ g_{tt}^{(\nu)} &\rightarrow g_{tt} \quad \text{in } L^2(0, T; H^{-1/2}(\Gamma)). \end{aligned}$$

Proof. We conduct the proof by regularizing the function g and then employing a correction to guarantee homogeneous conditions at time zero, building upon the ideas in [40, Theorem 4].

- Regularization in time: We construct the family of mollifiers in time in the usual way; see [8, Ch. 1.4] and [1, Sec. 2.28]. Let $\eta \in C_0^\infty(\mathbb{R})$ be the non-negative function supported in $\{|t| \leq 1\}$:

$$\eta(t) = \begin{cases} C_0 \exp\left(-\frac{1}{1-|t|^2}\right) & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geq 1, \end{cases}$$

with C_0 chosen so that $\int_{-\infty}^{\infty} \eta(t) dt = 1$. For $\varepsilon > 0$, we set

$$\eta^\varepsilon(t) = \frac{1}{\varepsilon} \eta\left(\frac{t}{\varepsilon}\right).$$

Let \tilde{g} denote the extension of g by zero to a function in \mathbb{R} and introduce the time-mollification defined by

$$\tilde{g}^\varepsilon(t) = (\eta^\varepsilon * \tilde{g})(t) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \eta\left(\frac{t-\tau}{\varepsilon}\right) \tilde{g}(\tau) d\tau.$$

Then $\tilde{g}^\varepsilon \in C^\infty(\mathbb{R}; H^{3/2}(\Gamma))$ and, setting $g^\varepsilon = \tilde{g}^\varepsilon|_{[0, T]}$, we have

$$(3.26) \quad \begin{aligned} g^\varepsilon &\rightarrow g \quad \text{in } H_\alpha(0, T; H^{3/2}(\Gamma)), & g_t^\varepsilon &\rightarrow g_t \quad \text{in } L^2(0, T; H^{1/2}(\Gamma)), \\ g_{tt}^\varepsilon &\rightarrow g_{tt} \quad \text{in } L^2(0, T; H^{-1/2}(\Gamma)), & g_t^\varepsilon(0) &\rightarrow 0 \quad \text{in } H^{-1/2}(\Gamma), \end{aligned}$$

as $\varepsilon \searrow 0$.

- Regularization in space: By exploiting the fact that the space $C^2([0, T]; H^{5/2}(\Gamma))$ is dense

in $C^2([0, T]; H^{3/2}(\Gamma))$, we obtain a family $\{\bar{g}^\varepsilon\}_{\nu \geq 1} \subset C^2([0, T]; H^{5/2}(\Gamma))$ that converges to g in the sense of (3.26).

• Correction of regularized functions: We next introduce a correction to regularized functions in order to satisfy zero initial conditions:

$$(3.27) \quad g^{(\nu)}(x, t) = \bar{g}^\varepsilon(x, t) - \bar{g}^\varepsilon(0) - t\rho(R_\nu t)\bar{g}_t^\varepsilon(0),$$

where $\rho \in C_0^\infty(\mathbb{R})$ is equal to 1 in a neighborhood of zero and such that

$$\int_0^\infty (s\rho(s) + (1+s^2)\rho'(s) + s\rho''(s)) ds \leq C;$$

we include the construction of ρ in Lemma B.1 in Appendix B for completeness. Furthermore, we set

$$R_\nu = \nu \left(1 + \|\bar{g}_t^\varepsilon(0)\|_{H^{3/2}(\Gamma)}^2 \right),$$

and $\varepsilon = \frac{1}{\nu}$. The reason for including $1 + \|\bar{g}_t^\varepsilon(0)\|_{H^{3/2}(\Gamma)}^2$ in the definition of R_ν is that the resulting terms involving $\bar{g}_t^\varepsilon(0)$ below are then uniformly bounded; cf. (3.29). From (3.27), we find that

$$g_t^{(\nu)} = \bar{g}_t^\varepsilon - \chi_1(R_\nu t)\bar{g}_t^\varepsilon(0), \quad g_{tt}^{(\nu)} = \bar{g}_{tt}^\varepsilon - \chi_2(R_\nu t)\bar{g}_t^\varepsilon(0),$$

where we have introduced the functions

$$(3.28) \quad \chi_1(R_\nu t) = \rho(R_\nu t) + R_\nu t \rho'(R_\nu t), \quad \chi_2(R_\nu t) = 2R_\nu \rho'(R_\nu t) + R_\nu^2 t \rho''(R_\nu t),$$

which correspond to the first and second derivatives of $\rho(R_\nu t)$, respectively. We see that the regularized functions satisfy compatibility conditions, that is, $g^{(\nu)}(0) = 0$ and $g_t^{(\nu)}(0) = 0$.

It remains to establish the convergence of $g^{(\nu)} - \bar{g}^\varepsilon$ and its derivatives to zero in suitable norms. Below $C_1, \dots, C_4 > 0$ denote generic constants that do not depend on ν . To establish the desired convergence, it helps to note that

$$\begin{aligned} \int_0^T |\chi_1(R_\nu t)|^2 dt &\lesssim \int_0^T (t\rho(R_\nu t) + R_\nu t^2 \rho'(R_\nu t))^2 dt \\ &\leq \frac{1}{R_\nu^3} \int_0^\infty (s\rho(s) + s^2 \rho'(s))^2 ds \leq \frac{C_1}{R_\nu^3}. \end{aligned}$$

We then find that

$$(3.29) \quad \begin{aligned} \|\bar{g}_t^\varepsilon - g_t^{(\nu)}\|_{L^2(H^{1/2}(\Gamma))} &\leq \|\chi_1(R_\nu t)\|_{L^2(0,T)} \|\bar{g}_t^\varepsilon(0)\|_{H^{1/2}(\Gamma)} \\ &\leq C_2 \frac{1}{\nu^{3/2}} \frac{\|\bar{g}_t^\varepsilon(0)\|_{H^{3/2}(\Gamma)}}{\left(1 + \|\bar{g}_t^\varepsilon(0)\|_{H^{3/2}(\Gamma)}^2\right)^{3/2}} \leq C_3 \frac{1}{\nu^{3/2}} \rightarrow 0 \end{aligned}$$

as $\nu \rightarrow \infty$. Next, we have

$$\|\partial_t^\alpha (\bar{g}^\varepsilon - g^{(\nu)})\|_{L^2(H^{3/2}(\Gamma))} = \|\partial_t^\alpha (\chi_1(R_\nu t)) \bar{g}_t^\varepsilon(0)\|_{L^2(H^{3/2}(\Gamma))}.$$

Since we can write the fractional derivative term as the temporal convolution

$$\partial_t^\alpha (t\rho(R_\nu t)) = \mathfrak{K}_{1-\alpha} * \chi_1(R_\nu t),$$

an application of Young's convolution inequality yields

$$\|\partial_t^\alpha (\chi_1(R_\nu t)) \bar{g}_t^\varepsilon(0)\|_{L^2(H^{3/2}(\Gamma))} \leq \|\mathfrak{K}_{1-\alpha}\|_{L^1(0,T)} \|\chi(R_\nu t)\|_{L^2(0,T)} \|\bar{g}_t^\varepsilon(0)\|_{H^{3/2}(\Gamma)} \rightarrow 0$$

as $\nu \rightarrow \infty$, similarly to before. Therefore, we have $\|\partial_t^\alpha(\bar{g}^\varepsilon - g^{(\nu)})\|_{L^2(H^{3/2}(\Gamma))} \rightarrow 0$, which implies $\|\bar{g}^\varepsilon - g^{(\nu)}\|_{H_\alpha(H^{3/2}(\Gamma))} \rightarrow 0$ as $\nu \rightarrow \infty$.

Finally, we consider the convergence of $g_{tt}^{(\nu)}$. Recalling (3.28), we have

$$\int_0^T |\chi_2(R_\nu t)|^2 dt \lesssim \int_0^\infty (\rho'(s) + s\rho''(s))^2 ds \leq C_4.$$

Since $\|\bar{g}_t^\varepsilon(0)\|_{H^{-1/2}(\Gamma)} \rightarrow 0$ as $\nu = \frac{1}{\varepsilon} \rightarrow \infty$, we can conclude that

$$\int_0^T |\chi_2(R_\nu t)|^2 dt \cdot \|\bar{g}_t^\varepsilon(0)\|_{H^{-1/2}(\Gamma)}^2 \rightarrow 0$$

as $\nu \rightarrow \infty$. This in turn allows us to infer that

$$\|\bar{g}_{tt}^\varepsilon - g_{tt}^{(\nu)}\|_{L^2(H^{-1/2}(\Gamma))} \leq \|\chi_2(R_\nu t)\|_{L^2(0,T)} \|\bar{g}_t^\varepsilon(0)\|_{H^{-1/2}(\Gamma)} \rightarrow 0,$$

as $\nu \rightarrow \infty$, which completes the proof. \square

Equipped with the previous density property, we now analyze the linearized wave problem in a higher-regularity setting. This additional smoothness will later allow us to employ the result to show the well-posedness of the nonlinear state problem.

Proposition 3.2. *Let $T > 0$, $c > 0$, $b \in (0, \bar{b}]$ for some $\bar{b} > 0$, and $\alpha \in (0, 1)$. Assume that the variable coefficients in (IBVP_{lin}) satisfy*

$$\begin{aligned} \mathbf{a} &\in X_{\mathbf{a}} = L^\infty(0, T; W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega)) \cap W^{1,\infty}(0, T; L^\infty(\Omega)), \\ \mathbf{l} &\in X_{\mathbf{l}} = L^\infty(0, T; H^2(\Omega)), \quad \mathbf{n} \in X_{\mathbf{n}} = L^\infty(0, T; H^2(\Omega)), \end{aligned}$$

and that the non-degeneracy condition (3.46) holds. Furthermore, assume that $u_1 = 0$, $F \in L^2(0, T; H^2(\Omega))$, and

$$(3.30) \quad \begin{aligned} g &\in \mathcal{X}_g = L^\infty(0, T; H^{3/2}(\Gamma)) \cap H_\alpha(0, T; H^{3/2}(\Gamma)) \cap W^{1,\infty}(0, T; H^{1/2}(\Gamma)) \\ &\quad \cap H^2(0, T; H^{-1/2}(\Gamma)) \end{aligned}$$

together with the compatibility conditions (3.25). Then there exists a unique solution u of (IBVP_{lin}), such that

$$(3.31) \quad u \in \mathcal{X}_p = \{p \in L^\infty(0, T; H^3(\Omega)) : p_t \in L^\infty(0, T; H^2(\Omega)), \partial_t^\alpha p \in L^2(0, T; H^3(\Omega)), p_{tt} \in L^2(0, T; H^1(\Omega))\}.$$

Moreover, the solution satisfies the estimate

$$(3.32) \quad \|u\|_{\mathcal{X}_p}^2 \lesssim \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) \left(\|F\|_{L^2(H^2(\Omega))}^2 + \|g\|_{\mathcal{X}_g}^2 \right),$$

where

$$(3.33) \quad \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) = \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \exp \{CT\mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^6\}, \quad \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} = 1 + \|\mathbf{a}\|_{X_{\mathbf{a}}} + \|\mathbf{n}\|_{X_{\mathbf{n}}} + \|\mathbf{l}\|_{X_{\mathbf{l}}},$$

where the hidden constants are independent of b and α .

Proof. We conduct the proof through several steps. In the first step, we construct a Neumann extension of the boundary data, which allows us to reformulate the problem with homogeneous boundary and initial conditions.

Step I: Neumann extension. In the spirit of, e.g., [6, 27, 36], where integer-order wave problems are analyzed, we introduce the following Neumann extension of g

$$-\Delta G + G = 0, \quad \frac{\partial G}{\partial n} = g$$

pointwise in time. It is known by elliptic theory that $\mathcal{N}^\Delta : g \mapsto G$ is a linear bounded mapping:

$$\mathcal{N}^\Delta : H^s(\Gamma) \rightarrow H^{s+\frac{3}{2}}(\Omega), \quad s \in \mathbb{R}$$

at each $t \in [0, T]$; see, e.g., [37, 47]. Note that since \mathcal{N}^Δ is linear, it holds that $\partial_t \mathcal{N}^\Delta(g^{(\nu)}) = \mathcal{N}^\Delta(\partial_t g^{(\nu)})$. By the compatibility conditions in (3.25), we conclude that $G(0) = G_t(0) = 0$.

We can then consider the following initial boundary-value problem for

$$w := u - G$$

with homogeneous boundary and initial data:

$$(\text{IBVP}_{\text{lin}}^{\text{hom}}) \quad \begin{cases} \mathfrak{a}(x, t)w_{tt} - c^2 \Delta w - b \Delta \partial_t^\alpha w + \mathfrak{l}(x, t)w_t + \mathfrak{n}(x, t)w = F_G(x, t), \\ \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma, \\ (w, w_t)|_{t=0} = (0, 0), \end{cases}$$

where the right-hand side is given by

$$F_G = F - \mathfrak{a}(x, t)G_{tt} + c^2 \Delta G + b \Delta \partial_t^\alpha G - \mathfrak{l}(x, t)G_t - \mathfrak{n}(x, t)G.$$

Step II: Regularized Neumann data. Thanks to Lemma 3.2, we can introduce the regularized Neumann data

$$\{g^{(\nu)}\}_\nu \subset C^2([0, T]; H^{3/2}(\Gamma)), \quad g^{(\nu)}(0) = g_t^{(\nu)}(0) = 0,$$

and instead of $(\text{IBVP}_{\text{lin}}^{\text{hom}})$ consider the following problem:

$$(\text{IBVP}_{\text{lin}}^{\text{hom}, \nu}) \quad \begin{cases} \mathfrak{a}(x, t)w_{tt}^{(\nu)} - c^2 \Delta w^{(\nu)} - b \Delta \partial_t^\alpha w^{(\nu)} + \mathfrak{l}(x, t)w_t^{(\nu)} + \mathfrak{n}(x, t)w^{(\nu)} = F_{G^{(\nu)}}, \\ \frac{\partial w^{(\nu)}}{\partial n} = 0 \quad \text{on } \Gamma, \\ (w^{(\nu)}, w_t^{(\nu)})|_{t=0} = (0, 0). \end{cases}$$

where the right-hand side is now

$$(3.34) \quad F_{G^{(\nu)}} = F - \mathfrak{a}(x, t)G_{tt}^{(\nu)} + c^2 \Delta G^{(\nu)} + b \Delta \partial_t^\alpha G^{(\nu)} - \mathfrak{l}(x, t)G_t^{(\nu)} - \mathfrak{n}(x, t)G^{(\nu)},$$

with $G^{(\nu)} = \mathcal{N}^\Delta g^{(\nu)}$. By the regularity of $g^{(\nu)}$ and properties of \mathcal{N}^Δ , we conclude that

$$G^{(\nu)} \in H^2(0, T; H^4(\Omega)).$$

Due to compatibility of regularized data, we also know that $G^{(\nu)}(0) = 0$ and $G_t^{(\nu)}(0) = 0$.

The right-hand side of $(\text{IBVP}_{\text{lin}}^{\text{hom}, \nu})$ can be bounded using the inequality

$$\begin{aligned} & \|v_1 v_2\|_{L^2(H^1(\Omega))} \\ & \lesssim \|v_1\|_{L^\infty(L^\infty(\Omega))} \|v_2\|_{L^2(L^2(\Omega))} + \|\nabla v_1\|_{L^\infty(L^4(\Omega))} \|v_2\|_{L^2(L^4(\Omega))} + \|v_1\|_{L^\infty(L^\infty(\Omega))} \|\nabla v_2\|_{L^2(L^2(\Omega))} \\ & \lesssim \|v_1\|_{L^\infty(H^2(\Omega))} \|v_2\|_{L^2(H^1(\Omega))} \end{aligned}$$

as follows:

$$\begin{aligned} \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))} &\lesssim \|F\|_{L^2(H^1(\Omega))} + \|\mathfrak{a}\|_{X_{\mathfrak{a}}}\|G_{tt}^{(\nu)}\|_{L^2(H^1(\Omega))} + \|\Delta G^{(\nu)}\|_{L^2(H^1(\Omega))} \\ &\quad + \|\bar{b}\|\|\Delta\partial_t^\alpha G^{(\nu)}\|_{L^2(H^1(\Omega))} + \|\mathfrak{l}\|_{X_{\mathfrak{l}}}\|G_t^{(\nu)}\|_{L^2(H^1(\Omega))} + \|\mathfrak{n}\|_{X_{\mathfrak{n}}}\|G^{(\nu)}\|_{L^2(H^1(\Omega))}. \end{aligned}$$

Then by the properties of the extension mapping \mathcal{N}^Δ , we have

$$(3.35) \quad \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))} \lesssim \|F\|_{L^2(H^1(\Omega))} + \mathcal{L}_{\mathfrak{a},\mathfrak{n},\mathfrak{l}}\|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}},$$

where $\mathcal{X}_g^{\text{low}}$ is defined in (3.24).

Step III: Galerkin approximation. We next show the well-posedness of $(\text{IBVP}_{\text{lin}}^{\text{hom},\nu})$ by employing the Galerkin procedure as in Proposition 3.1. We apply a testing procedure with test functions used in the Dirichlet case in [3, Lemma 3.2]; we focus here on pointing out the main differences that arise due to the present Neumann setting.

- Testing with $w_t^{(\nu,m)}$: We test the semi-discrete version of $(\text{IBVP}_{\text{lin}}^{\text{hom},\nu})$ with $w_t^{(\nu,m)}$. Analogously to Proposition 3.1, this yields the energy estimate

$$(3.36) \quad \begin{aligned} &\|\sqrt{\mathfrak{a}(t)} w_t^{(\nu,m)}(t)\|_{L^2(\Omega)}^2 + \|w^{(\nu,m)}(t)\|_{H^1(\Omega)}^2 + b\underline{C}(T)\|\nabla\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 \\ &\lesssim_T \mathcal{L}_{\mathfrak{a},\mathfrak{n},\mathfrak{l}}^2\|w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 + \|F_{G^{(\nu)}}\|_{L^2(L^2(\Omega))}^2 \end{aligned}$$

for $t \in [0, T]$.

- Testing with $-\Delta w_t^{(\nu,m)}$: Testing the semi-discrete equation with $-\Delta w_t^{(\nu,m)}$ and applying energy arguments analogous to [3] (see Appendix A.1 for details) yields

$$(3.37) \quad \begin{aligned} &\|\sqrt{\mathfrak{a}(t)}\nabla w_t^{(\nu,m)}(t)\|_{L^2(\Omega)}^2 + \|\Delta w^{(\nu,m)}(t)\|_{L^2(\Omega)}^2 + b(\underline{C}(T) - \varepsilon C_0)\|\Delta\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 \\ &\lesssim \mathcal{L}_{\mathfrak{a},\mathfrak{n},\mathfrak{l}}^4(\|\Delta w^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 + \|w_t^{(\nu,m)}\|_{L_t^2(H^1(\Omega))}^2 + \|w^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2) \\ &\quad + \|F_{G^{(\nu)}}\|_{L^2(L^2(\Omega))}^2 + \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 \end{aligned}$$

for some $C_0 > 0$, independent of b and α . We emphasize that we cannot yet control the term $\|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2$ by the left-hand side. This becomes possible after the third testing step.

- Applying Δ and testing with $\Delta w_t^{(\nu,m)}$: In the third testing step, we apply the operator Δ to the semi-discrete equation associated to $(\text{IBVP}_{\text{lin}}^{\text{hom}})$ and then test the resulting equation with $\Delta w_t^{(\nu,m)}$. This yields

$$(3.38) \quad \begin{aligned} &(\Delta(\mathfrak{a}w_{tt}^{(\nu,m)}) - c^2\Delta^2w^{(\nu,m)} - b\Delta^2\partial_t^\alpha w^{(\nu,m)}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)} \\ &= (\Delta F_{G^{(\nu)}} - \Delta(\mathfrak{l}w_t^{(\nu,m)}) - \Delta(\mathfrak{n}w^{(\nu,m)}), \Delta w_t^{(\nu,m)})_{L^2(\Omega)}. \end{aligned}$$

The terms involving variable coefficients \mathfrak{a} , \mathfrak{l} and \mathfrak{n} are treated analogously to the Dirichlet case in [3, 28]. The estimate of the $F_{G^{(\nu)}}$ term is new and we thus focus on it here. Recalling how $F_{G^{(\nu)}}$ is defined in (3.34) and using the fact that $\Delta G^{(\nu)} = G^{(\nu)}$ and $\Delta\partial_t^\alpha G^{(\nu)} = \partial_t^\alpha G^{(\nu)}$,

we obtain

$$\begin{aligned} & \int_0^t (\Delta F_{G^{(\nu)}}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)} ds \\ &= \int_0^t \left(\Delta F - \Delta \left(\mathbf{a} G_{tt}^{(\nu)} \right) + c^2 G^{(\nu)} + b \partial_t^\alpha G^{(\nu)} - \Delta \left(\mathbf{l} G_t^{(\nu)} \right) - \Delta \left(\mathbf{n} G^{(\nu)} \right), \Delta w_t^{(\nu,m)} \right)_{L^2(\Omega)} ds. \end{aligned}$$

For $\Delta(\mathbf{a} G_{tt}^{(\nu)})$, we use the product rule and $\Delta G_{tt}^{(\nu)} = G_{tt}^{(\nu)}$:

$$\int_0^t (\Delta(\mathbf{a} G_{tt}^{(\nu)}), \Delta w_t^{(\nu,m)})_{L^2(\Omega)} ds = \int_0^t ((\Delta \mathbf{a}) G_{tt}^{(\nu)} + 2 \nabla \mathbf{a} \cdot \nabla G_{tt}^{(\nu)} + \mathbf{a} G_{tt}^{(\nu)}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)} ds.$$

Analogously, we have

$$\begin{aligned} \int_0^t \left(\Delta \left(\mathbf{l} G_t^{(\nu)} + \mathbf{n} G^{(\nu)} \right), \Delta w_t^{(\nu,m)} \right)_{L^2(\Omega)} &= \int_0^t (\Delta \mathbf{l} G_t^{(\nu)} + 2 \nabla \mathbf{l} \cdot \nabla G_t^{(\nu)} + \mathbf{l} G_t^{(\nu)}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)} \\ &\quad + \int_0^t (\Delta \mathbf{n} G^{(\nu)} + 2 \nabla \mathbf{n} \cdot \nabla G^{(\nu)} + \mathbf{n} G^{(\nu)}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)}. \end{aligned}$$

By Hölder's inequality, we then obtain the estimate

$$\begin{aligned} & \int_0^t (\Delta F_{G^{(\nu)}}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)} ds \\ & \lesssim \int_0^t \|\Delta F\|_{L^2(\Omega)} \|\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)} ds + \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}} \int_0^t \left(\|G^{(\nu)}\|_{H^1(\Omega)} + \|G_t^{(\nu)}\|_{H^2(\Omega)} + \|\partial_t^\alpha G^{(\nu)}\|_{L^2(\Omega)} \right. \\ & \quad \left. + \|G_{tt}^{(\nu)}\|_{H^1(\Omega)} \right) \|\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)} ds. \end{aligned}$$

Above, we have relied on the bound

$$\begin{aligned} \int_\Omega \Delta \mathbf{a} G_{tt}^{(\nu)} \Delta w_t^{(\nu,m)} dx &\lesssim \|\Delta \mathbf{a}\|_{L^3(\Omega)} \|G_{tt}^{(\nu)}\|_{L^6(\Omega)} \|\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)} \\ &\lesssim \|\mathbf{a}\|_{W^{2,3}(\Omega)} \|G_{tt}^{(\nu)}\|_{H^1(\Omega)} \|\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)}, \end{aligned}$$

and we have treated the \mathbf{l} and \mathbf{n} terms analogously. Then by elliptic regularity for the Neumann extension operator, we infer

$$\begin{aligned} & \int_0^t (\Delta F_{G^{(\nu)}}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)} ds \\ & \lesssim \int_0^t \left\{ \|\Delta F\|_{L^2(\Omega)} + \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}} \|g^{(\nu)}\|_{\mathcal{X}_{g^{\text{low}}}^1} \right\} \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))} ds. \end{aligned}$$

By testing the semi-discrete problem with $w_{tt}^{(\nu,m)}$ and its space-differentiated version with $\nabla w_{tt}^{(\nu,m)}$, it can be seen, analogously to the Dirichlet case in [28], that

$$\begin{aligned} & \|w_{tt}^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} \\ (3.39) \quad & \lesssim \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}}^2 (\|\Delta \partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} + \|w^{(\nu,m)}\|_{L_t^2(H^2(\Omega))} + \|w_t^{(\nu,m)}\|_{L^2(H^1(\Omega))} + \\ & \quad \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))}). \end{aligned}$$

By otherwise also proceeding analogously to the Dirichlet case, we arrive at the outcome of the third testing step in the form of

$$\begin{aligned}
(3.40) \quad & \frac{1}{2} \|\sqrt{\mathbf{a}(t)} \Delta w_t^{(\nu, m)}(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla \Delta w^{(\nu, m)}(t)\|_{L^2(\Omega)}^2 \\
& + b \underline{C}(T) \|\nabla \Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 \\
& \lesssim \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^6 \left(\|w_t^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 + \|w^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 \right) + \varepsilon b \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(H^1(\Omega))}^2 \\
& + \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))}^2 + \|\Delta F\|_{L^2(L^2(\Omega))}^2 + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}}^2
\end{aligned}$$

for any $\varepsilon > 0$ and all $t \in [0, T]$. The skipped details are provided in Appendix A.2 for completeness.

• Combining the estimates: Adding (3.36), (3.37), and (3.40) and using the lower bound $\underline{\mathbf{a}} \leq \mathbf{a}$ together with a suitable reduction of $\varepsilon > 0$, we obtain

$$\begin{aligned}
& \|w_t^{(\nu, m)}(t)\|_{H^2(\Omega)}^2 + \|w^{(\nu, m)}(t)\|_{H^3(\Omega)}^2 + \|\nabla \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 \\
& \lesssim_T \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^6 \left(\|w_t^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 + \|w^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 \right) + \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))}^2 + \|F\|_{L^2(H^2(\Omega))}^2 \\
& + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}}^2.
\end{aligned}$$

Using the estimate for $F_{G^{(\nu)}}$ in (3.35) and combining with (3.39), we further obtain

$$\begin{aligned}
(3.41) \quad & \|w_{tt}^{(\nu, m)}\|_{L_t^2(H^1(\Omega))}^2 + \|w_t^{(\nu, m)}(t)\|_{H^2(\Omega)}^2 + \|w^{(\nu, m)}(t)\|_{H^3(\Omega)}^2 + \|\nabla \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 \\
& \lesssim_T \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^6 \left(\|w_t^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 + \|w^{(\nu, m)}\|_{L_t^2(H^2(\Omega))}^2 \right) + \|F\|_{L^2(H^2(\Omega))}^2 + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}}^2.
\end{aligned}$$

Applying Grönwall's inequality to (3.41) yields the uniform estimate

$$\begin{aligned}
(3.42) \quad & \|w_{tt}^{(\nu, m)}\|_{L^2(H^1(\Omega))}^2 + \|w_t^{(\nu, m)}\|_{L^\infty(H^2(\Omega))}^2 + \|w^{(\nu, m)}\|_{L^\infty(H^3(\Omega))}^2 + \|\partial_t^\alpha w^{(\nu, m)}\|_{L^2(H^3(\Omega))}^2 \\
& \lesssim_T \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) \left(\|F\|_{L^2(H^2(\Omega))}^2 + \|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}}^2 \right),
\end{aligned}$$

where Λ is defined in (3.33).

• Passing to the limit as $m \rightarrow \infty$: The estimates derived above imply that the sequence $\{w^{(\nu, m)}\}_m$ is bounded uniformly in m in the space \mathcal{X}_p . Hence, by standard weak compactness arguments, we can extract a (not relabeled) subsequence and pass to the limit in the semi-discrete formulation shows that $w^{(\nu)}$ satisfies $(\text{IBVP}_{\text{lin}}^{\text{hom}, \nu})$. By taking the limit $m \rightarrow \infty$ in (3.42) and using the lower semi-continuity of norms, together with the fact that

$$\|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}} \lesssim \|g\|_{\mathcal{X}_g^{\text{low}}},$$

we obtain the ν -uniform estimate

$$\begin{aligned}
(3.43) \quad & \|w_{tt}^{(\nu)}\|_{L^2(H^1(\Omega))}^2 + \|w_t^{(\nu)}\|_{L^\infty(H^2(\Omega))}^2 + \|w^{(\nu)}\|_{L^\infty(H^3(\Omega))}^2 + \|\partial_t^\alpha w^{(\nu)}\|_{L^2(H^3(\Omega))}^2 \\
& \lesssim_T \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) \left(\|F\|_{L^2(H^2(\Omega))}^2 + \|g\|_{\mathcal{X}_g^{\text{low}}}^2 \right).
\end{aligned}$$

Step IV: Passing to the limit as $\nu \rightarrow \infty$. From the uniform estimate (3.43), $\{w^{(\nu)}\}_\nu$ is bounded in \mathcal{X}_p . Hence, we may extract a subsequence (that we do not relabel), such that

$$\begin{aligned} w^{(\nu)} &\overset{*}{\rightharpoonup} w \quad \text{in } L^\infty(0, T; H^3(\Omega)), & w_t^{(\nu)} &\overset{*}{\rightharpoonup} w_t \quad \text{in } L^\infty(0, T; H^2(\Omega)), \\ \partial_t^\alpha w^{(\nu)} &\rightharpoonup \partial_t^\alpha w \quad \text{in } L^2(0, T; H^3(\Omega)), & w_{tt}^{(\nu)} &\rightharpoonup w_{tt} \quad \text{in } L^2(0, T; H^1(\Omega)). \end{aligned}$$

By the Aubin–Lions compactness theorem (see [46, (6.5)]), we also have strong convergence along a subsequence

$$w^{(\nu)} \rightarrow w \quad \text{in } C([0, T]; H^2(\Omega)), \quad w_t^{(\nu)} \rightarrow w_t \quad \text{in } C([0, T]; H^1(\Omega)).$$

Passing to the limit $\nu \rightarrow \infty$ in the regularized problem (IBVP_{lin}^{hom, ν}) shows that w solves (IBVP_{lin}^{hom}). Uniqueness follows by testing the homogeneous problem with w_t , following an analogous estimating strategy as in the first testing step of the Galerkin procedure.

By passing to the $\nu \rightarrow \infty$ limit in (3.43), we see that w satisfies

$$\begin{aligned} (3.44) \quad & \|w_{tt}\|_{L^2(H^1(\Omega))}^2 + \|w_t\|_{L^\infty(H^2(\Omega))}^2 + \|w\|_{L^\infty(H^3(\Omega))}^2 + \|\partial_t^\alpha w\|_{L^2(H^3(\Omega))}^2 \\ & \lesssim_T \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) \left(\|F\|_{L^2(H^2(\Omega))}^2 + \|g\|_{\mathcal{X}_g^{\text{low}}}^2 \right). \end{aligned}$$

Step V: The result for the inhomogeneous problem. In the final step of the proof, we return to the function

$$u = w + G$$

which solves (IBVP_{lin}). From estimate (3.44), we have

$$\|w\|_{\mathcal{X}_p}^2 \lesssim \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) \left(\|F\|_{L^2(H^2(\Omega))}^2 + \|g\|_{\mathcal{X}_g^{\text{low}}}^2 \right),$$

and by the properties of the elliptic operator \mathcal{N}^Δ , we have

$$\|G\|_{\mathcal{X}_p} = \|\mathcal{N}^\Delta g\|_{\mathcal{X}_p} \lesssim \|g\|_{\mathcal{X}_g}.$$

Combining these bounds yields

$$\|u\|_{\mathcal{X}_p}^2 \lesssim_T \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) (\|F\|_{L^2(0, T; H^2)}^2 + \|g\|_{\mathcal{X}_g}^2),$$

which is exactly (3.32). This completes the proof. \square

3.3. Well-posedness of the state problem. Equipped with the well-posedness result for the linearized problem in the higher-regularity setting (Proposition 3.2), we now apply Banach’s fixed-point theorem to obtain existence and uniqueness for the full nonlinear state problem.

Theorem 3.1. *Let $\alpha \in (0, 1)$, $c > 0$, $b \in (0, \bar{b})$ for some $\bar{b} > 0$, and fix a final time $T > 0$. Assume that the coefficient satisfies $k \in \mathcal{X}_k = W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega)$, that the distributed source satisfies $f \in X_f = L^2(0, T; H^2(\Omega))$, and that the boundary data $g \in \mathcal{X}_g$ satisfies the compatibility conditions (3.25), where \mathcal{X}_g is defined in (3.30). Then there exists a constant $\delta = \delta(T) > 0$ such that if*

$$(3.45) \quad \|k\|_{\mathcal{X}_k} \leq \delta,$$

the nonlinear problem (IBVP_{West}) admits a unique solution $p \in \mathcal{X}_p$, where \mathcal{X}_p is defined in (3.31). The solution satisfies

$$\|p\|_{\mathcal{X}_p} \lesssim_T \|f\|_{X_f} + \|g\|_{\mathcal{X}_g},$$

where the hidden constant is independent of the dissipation parameter b and the fractional order α . Moreover, there exist $\underline{\mathfrak{a}}, \bar{\mathfrak{a}}$, independent of b and α , such that

$$(3.46) \quad 0 < \underline{\mathfrak{a}} \leq 1 - 2k(x)p(x, t) \leq \bar{\mathfrak{a}} \quad \text{for all } (x, t) \in \Omega \times [0, T].$$

Remark 1 (On the smallness assumption). *Theorem 3.1 requires smallness of the nonlinearity coefficient k . This coefficient can be expressed as*

$$k = \frac{1 + B/(2A)}{\rho c^2},$$

where ρ denotes the background density and the ratio B/A is the acoustic nonlinearity parameter, which is proportional to the ratio of the quadratic and linear coefficients in the Taylor expansion of pressure as a function of density perturbations; see [5, 44]. In biological tissues, $B/A \in [5, 12]$; see [16, Ch. 2, Table III]. To assess the validity of the smallness assumption (3.45), we consider a nondimensional formulation obtained by introducing the scaling

$$x = L\tilde{x}, \quad t = \frac{L}{c}\tilde{t}, \quad p = p^{\text{ref}}\tilde{p},$$

where L is a characteristic length scale (e.g., the diameter of Ω) and p^{ref} is a reference pressure amplitude. After the change of variables and division by $\frac{c^2 p^{\text{ref}}}{L^2}$, we obtain the dimensionless equation

$$\tilde{p}_{tt} - \Delta \tilde{p} - \tilde{b} \Delta \partial_t^\alpha \tilde{p} = \tilde{k}(\tilde{p}^2)_{tt},$$

where

$$\tilde{b} = \frac{b}{c^2} \left(\frac{L}{c} \right)^{-\alpha} \quad \text{and} \quad \tilde{k} = \frac{(1 + B/(2A)) p^{\text{ref}}}{\rho c^2}.$$

The parameter \tilde{k} is therefore small whenever $p^{\text{ref}} \ll \rho c^2$. Typical ultrasound pressure amplitudes used in therapeutic acoustics are several orders of magnitude smaller than $\rho c^2 \sim 10^9 - 10^{10}$ Pa; see, for example, [4, Fig. 21.72]. Hence the smallness condition (3.45) is meaningful in all practically relevant regimes.

Proof. The proof of Theorem 3.1 follows by applying Banach's fixed-point theorem to the mapping

$$\mathcal{T} : p^* \mapsto p,$$

where p^* is taken from a closed ball of radius R in \mathcal{X}_p :

$$\mathbb{B}_R = \{p^* \in \mathcal{X}_p : \|p^*\|_{\mathcal{X}_p} \leq R\},$$

and p is defined as the solution of the linearized problem (IBVP_{lin}) with

$$\mathfrak{a} = 1 - 2kp^*, \quad \mathfrak{l} = -2kp_t^*, \quad \mathfrak{n} = 0, \quad \text{and} \quad F = f.$$

We first verify that the assumptions of Proposition 3.2 are satisfied. The non-degeneracy condition on \mathfrak{a} in (3.46) follows by employing the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ (see [1, Theorem 4.12]):

$$\|2kp^*\|_{L^\infty(L^\infty(\Omega))} \lesssim \|k\|_{L^\infty(\Omega)} \|p^*\|_{L^\infty(H^2(\Omega))} \lesssim \|k\|_{L^\infty(\Omega)} R.$$

By choosing $\|k\|_{L^\infty(\Omega)}$ sufficiently small, we ensure that $\mathfrak{a} = 1 - 2kp^*$ remains positive. The additionally required regularity

$$\mathfrak{a} \in X_{\mathfrak{a}} = L^\infty(0, T; W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega)) \cap W^{1,\infty}(0, T; L^\infty(\Omega))$$

follows from the fact that $W^{2,3}(\Omega)$ and $W^{1,\infty}(\Omega)$ are algebras in our at most three-dimensional setting $d \leq 3$ (see [1, Theorem 4.39]), so that

$$\begin{aligned} \|\mathbf{a}\|_{X_{\mathbf{a}}} &\lesssim 1 + \|kp^*\|_{L^\infty(W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega))} + \|k\|_{L^\infty(\Omega)} \|p^*\|_{W^{1,\infty}(L^\infty(\Omega))} \\ &\lesssim 1 + \|k\|_{\mathcal{X}_k} \|p^*\|_{L^\infty(W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega))} + \|k\|_{L^\infty(\Omega)} \|p^*\|_{W^{1,\infty}(L^\infty(\Omega))} \\ &\lesssim 1 + \|k\|_{\mathcal{X}_k} R, \end{aligned}$$

where in the last step we have used the Sobolev embedding $H^3(\Omega) \hookrightarrow W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega)$. Similarly,

$$\|\mathbf{l}\|_{L^\infty(H^2(\Omega))} \lesssim \|k\|_{H^2(\Omega)} R.$$

Therefore the assumptions of Proposition 3.2 are fulfilled, and \mathcal{T} is well-defined.

To show that $\mathcal{T}(\mathbb{B}_R) \subset \mathbb{B}_R$, let $p^* \in \mathbb{B}_R$ and $p = \mathcal{T}(p^*)$. From Proposition 3.2, we have

$$\|p\|_{\mathcal{X}_p}^2 \lesssim_T \Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) \left(\|g\|_{\mathcal{X}_g}^2 + \|f\|_{\mathcal{X}_f}^2 \right),$$

where

$$\mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} = 1 + \|\mathbf{a}\|_{X_{\mathbf{a}}} + \|\mathbf{l}\|_{X_{\mathbf{l}}} \lesssim 1 + \|k\|_{\mathcal{X}_k} R,$$

and

$$\Lambda(\mathbf{a}, \mathbf{n}, \mathbf{l}) = \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \exp \{CT \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^6\} \lesssim (1 + \|k\|_{\mathcal{X}_k} R)^2 \exp \{CT(1 + \|k\|_{\mathcal{X}_k} R)^6\}.$$

Hence,

$$\|p\|_{\mathcal{X}_p}^2 \lesssim (1 + \|k\|_{\mathcal{X}_k} R)^2 \exp \{CT(1 + \|k\|_{\mathcal{X}_k} R)^6\} (L_1^2 + L_2^2) \leq R^2,$$

where the last inequality holds if the radius $R = R(L_1, L_2)$ is sufficiently large and $\|k\|_{\mathcal{X}_k}$ sufficiently small relative to R . Thus, in that setting, $\mathcal{T}(\mathbb{B}_R) \subset \mathbb{B}_R$.

To verify that \mathcal{T} is a strict contraction, take $p^{*,(1)}, p^{*,(2)} \in \mathbb{B}_R$ and set $p^{(1)} = \mathcal{T}(p^{*,(1)})$, $p^{(2)} = \mathcal{T}(p^{*,(2)})$. Let $\bar{p}^* = p^{*,(1)} - p^{*,(2)}$. The difference $\bar{p} = p^{(1)} - p^{(2)}$ solves

$$((1 - 2kp^{*,(1)})\bar{p}_t)_t - c^2 \Delta \bar{p} - b \Delta \partial_t^\alpha \bar{p} - 2k(\bar{p}^* p_t^{(2)})_t = 0$$

with homogeneous initial and Neumann data. Using the uniform bounds

$$\|p^{*,(1)}\|_{\mathcal{X}_p}, \|p^{*,(2)}\|_{\mathcal{X}_p} \lesssim R,$$

and exploiting the estimate (3.5) from Proposition 3.1 with

$$\mathbf{a} = 1 - 2kp^{*,(1)}, \mathbf{l} = 1 - 2kp_t^{*,(1)}, \mathbf{n} = 0, \text{ and } F = -2k(\bar{p}^* p_t^{(2)})_t,$$

we obtain strict contractivity of \mathcal{T} in the norm of $W^{1,\infty}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ for $\|k\|_{L^\infty(\Omega)}$ sufficiently small. We can argue that \mathbb{B}_R is closed in this norm by exploiting the Banach–Alaoglu theorem (see [21, Theorem B.1.7]), analogously to [29, Theorem 4.1]. Banach’s fixed-point theorem now yields the existence of a unique fixed-point $p \in \mathbb{B}_R \subset \mathcal{X}_p$, which is the unique solution of the problem. \square

4. EXISTENCE OF OPTIMAL CONTROLS

In this section, we establish the existence of a minimizer for the optimal control under study. To this end, we first introduce the control-to-state operator. Let $L_1, L_2 > 0$ be given and fixed. We define the sets of admissible boundary controls as follows:

$$(4.1) \quad \mathcal{X}_g^{\text{adm}} = \left\{ g \in \mathcal{X}_g = L^\infty(0, T; H^{3/2}(\Gamma)) \cap H_\alpha(0, T; H^{3/2}(\Gamma)) \cap W^{1,\infty}(0, T; H^{1/2}(\Gamma)) \right. \\ \left. \cap H^2(0, T; H^{-1/2}(\Gamma)) : \|g\|_{\mathcal{X}_g} \leq L_1, g(\cdot, 0) = g_t(\cdot, 0) = 0 \right\},$$

The set $\mathcal{X}_g^{\text{adm}}$ represents a set of smooth controls in the Banach space \mathcal{X}_g subject to appropriate compatibility conditions with respect to the initial data, as dictated by Theorem 3.1. The admissible set of distributed controls is defined as the closed ball in the Hilbert space X_f

$$(4.2) \quad X_f^{\text{adm}} = \left\{ f \in X_f = L^2(0, T; H^2(\Omega)) : \|f\|_{X_f} \leq L_2 \right\}.$$

The control-to-state mapping is then defined by

$$(4.3) \quad \mathcal{S} : \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}} \rightarrow \mathcal{X}_p, \quad (g, f) \mapsto \mathcal{S}(g, f) = p \text{ solving (IBVP}_{\text{West}}).$$

Under the assumptions of Theorem 3.1 with the nonlinearity coefficient $\|k\|_{\mathcal{X}_k} \leq \delta$, the control-to-state mapping is well-defined. We can then write the optimal control problem (P) in the reduced form as

$$(4.4) \quad \boxed{\text{Find } (\tilde{g}, \tilde{f}) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}, \text{ such that } j(\tilde{g}, \tilde{f}) = \inf_{(g,f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}} j(g, f)}$$

with

$$(4.5) \quad j(g, f) = J(\mathcal{S}(g, f), g, f).$$

We next address the question of existence of optimal controls.

Theorem 4.1 (Existence of optimal controls). *Let the assumptions of Theorem 3.1 hold and let \mathcal{S} be defined in (4.3). Then there exists at least one optimal control $(\tilde{g}, \tilde{f}) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$, which minimizes the cost functional $J(\mathcal{S}(g, f), g, f)$ over $(g, f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$.*

Proof. The proof follows via the direct method of calculus of variations; see, for example, [38, Ch. 9.2]. We first note that $\mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}} \neq \emptyset$. Since J is non-negative, the infimum

$$\tilde{j} = \inf_{(g,f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}} J(\mathcal{S}(g, f), g, f)$$

exists. Let $\{(g^m, f^m)\}_{m \geq 1} \subset \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$ be a minimizing sequence, such that

$$\tilde{j} = \lim_{m \rightarrow \infty} J(\mathcal{S}(g^m, f^m), g^m, f^m).$$

Then $p^{(m)} = \mathcal{S}(g^m, f^m)$ is the solution of

$$(4.6) \quad \begin{cases} ((1 - 2kp^{(m)})p_t^{(m)})_t - c^2 \Delta p^{(m)} - b \Delta \partial_t^\alpha p^{(m)} = f^m & \text{in } \Omega \times (0, T), \\ \frac{\partial p^{(m)}}{\partial n} = g^m & \text{on } \Gamma \times (0, T), \\ (p^{(m)}, p_t^{(m)}) = (0, 0) & \text{on } \Gamma \times \{0\}, \end{cases}$$

in the sense of Theorem 3.1. On account of $g^m \in \mathcal{X}_g^{\text{adm}}$, $f^m \in X_f^{\text{adm}}$, and Theorem 3.1, we have the uniform bounds

$$\|g^m\|_{\mathcal{X}_g} \leq L_1, \quad \|f^m\|_{X_f} \leq L_2, \quad \|p^{(m)}\|_{\mathcal{X}_p} \lesssim_T \|g^m\|_{\mathcal{X}_g} + \|f^m\|_{X_f} \lesssim_T L_1 + L_2.$$

Therefore, using the Banach–Alaoglu theorem, we may extract weakly(-star) convergent subsequences, that we do not relabel, such that

$$(4.7) \quad \begin{aligned} p^{(m)} &\overset{*}{\rightharpoonup} p^* && \text{in } L^\infty(0, T; H^3(\Omega)), && p_t^{(m)} &\overset{*}{\rightharpoonup} p_t^* && \text{in } L^\infty(0, T; H^2(\Omega)), \\ \partial_t^\alpha p^{(m)} &\rightharpoonup \partial_t^\alpha p^* && \text{in } L^2(0, T; H^3(\Omega)), && p_{tt}^{(m)} &\rightharpoonup p_{tt}^* && \text{in } L^2(0, T; H^1(\Omega)), \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} g^m &\overset{*}{\rightharpoonup} \tilde{g} && \text{in } L^\infty(0, T; H^{3/2}(\Gamma)), && g_t^m &\overset{*}{\rightharpoonup} \tilde{g}_t && \text{in } L^\infty(0, T; H^{1/2}(\Gamma)), \\ \partial_t^\alpha g^m &\rightharpoonup \partial_t^\alpha \tilde{g} && \text{in } L^2(0, T; H^{3/2}(\Gamma)), && g_{tt}^m &\rightharpoonup \tilde{g}_{tt} && \text{in } L^2(0, T; H^{-1/2}(\Gamma)), \end{aligned}$$

as well as

$$f^m \rightharpoonup \tilde{f} \quad \text{in } L^2(0, T; H^2(\Omega))$$

as $m \rightarrow \infty$. Further, (4.7) and the Aubin–Lions compactness theorem implies the following strong convergence:

$$p^{(m)} \rightarrow p^* \quad \text{in } C([0, T]; H^2(\Omega)), \quad p_t^{(m)} \rightarrow p_t^* \quad \text{in } C([0, T]; H^1(\Omega)).$$

The weak lower semi-continuity of norms guarantees that

$$\|\tilde{g}\| \leq L_1, \quad \|\tilde{f}\| \leq L_2.$$

Further, (4.8) and [53, Lemma 3.1.7] imply

$$g^m(0) \overset{*}{\rightharpoonup} \tilde{g}(0) \quad \text{in } H^{1/2}(\Gamma), \quad g_t^m(0) \overset{*}{\rightharpoonup} \tilde{g}_t(0) \quad \text{in } H^{-1/2}(\Gamma).$$

By the uniqueness of limits, we have $\tilde{g}(0) = \tilde{g}_t(0) = 0$. Thus, we conclude that $(\tilde{g}, \tilde{f}) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$.

Now, to show that $p^* = \mathcal{S}(\tilde{g}, \tilde{f})$, we will pass to the limit $m \rightarrow \infty$ in the weak form of (4.6). Since $p^{(m)} \rightarrow p^*$ in $C([0, T]; H^2(\Omega))$ strongly and $p_{tt}^{(m)} \rightharpoonup p_{tt}^*$ in $L^2(0, T; H^1(\Omega))$ weakly, we have

$$(4.9) \quad p^{(m)} p_{tt}^{(m)} \rightharpoonup p^* p_{tt}^* \quad \text{in } L^2(0, T; H^1(\Omega)).$$

The remaining terms can be handled analogously or in a simpler manner as $m \rightarrow \infty$, to conclude that p^* solves (4.6).

Since $p^{(m)} \rightarrow p^*$ in $C([0, T]; H^1(\Omega))$ strongly, we also have

$$p^{(m)}(T) \rightarrow p^*(T) \quad \text{in } L^2(\Omega)$$

as $m \rightarrow \infty$. For the statement to follow, we then need the lower semi-continuity of J :

$$J(p^*, \tilde{g}, \tilde{f}) \leq \liminf_{m \rightarrow \infty} J(p^{(m)}, g^m, f^m) \leq \tilde{j},$$

which holds due to the weak lower semi-continuity property of norms. By the definition of infimum, we conclude that

$$J(p^*, \tilde{g}, \tilde{f}) = \tilde{j},$$

which shows optimality. \square

Because the state problem is nonlinear, and the reduced cost functional thus non-convex, we cannot expect the uniqueness of minimizers without imposing additional assumptions; we refer to the discussion in [48, Sec. 4.4] for further details on this topic.

4.1. Stability with respect to perturbations. We next discuss the stability of minimizers with respect to perturbations in the targeted pressure distribution. Subsequently, motivated also by numerical simulations, we consider a setting with approximate optimal controls (g_γ, f_γ) , such that

$$J_\gamma^{p^\text{d}}(\mathcal{S}(g_\gamma, f_\gamma), g_\gamma, f_\gamma) \leq J_\gamma^{p^\text{d}}(\mathcal{S}(g, f), g, f) + o(\gamma)$$

and discuss the limiting behavior as $\gamma \searrow 0$.

Proposition 4.1 (Stability with respect to perturbations in p^d). *Let the assumptions of Theorem 4.1 hold. Let $\eta > 0$ in the regularizing functional (1.1). If $p^{\text{d},m} \rightarrow p^\text{d}$ in $C([0, T]; L^2(\Omega))$ as $m \rightarrow \infty$, then the corresponding minimizers $\{(g^m, f^m)\}_{m \geq 1}$ of*

$$\begin{aligned} & J^{p^{\text{d},m}}(\mathcal{S}(g, f), g, f) \\ &= \frac{\nu}{2} \|\mathcal{S}(g, f) - p^{\text{d},m}\|_{L^2(L^2(\Omega_0))}^2 + \frac{1-\nu}{2} \|\mathcal{S}(g, f)(T) - p^{\text{d},m}(T)\|_{L^2(\Omega_0)}^2 + \mathcal{R}(g, f) \end{aligned}$$

over $\mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$ have a strongly convergent subsequence in $C([0, T]; H^{1/2}(\Gamma)) \times L^2(0, T; L^2(\Omega))$ and the limit of each convergent subsequence as $m \rightarrow \infty$ is a minimizer of $J^{p^\text{d}}(\mathcal{S}(g, f), g, f)$.

Proof. The proof follows by adapting ideas from [11, Theorem 2.1]. Since $\{(g^m, f^m)\}_{m \geq 1}$ are minimizers, we have

$$(4.10) \quad J^{p^{\text{d},m}}(\mathcal{S}(g^m, f^m), g^m, f^m) \leq J^{p^{\text{d},m}}(\mathcal{S}(g, f), g, f)$$

for any $(g, f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$. Let $p^{(m)} = \mathcal{S}(g^m, f^m)$. Thanks to the uniform boundedness of $\{(g^m, f^m)\}_{m \geq 1}$ and $\{p^{(m)}\}_{m \geq 1}$ (the latter, similarly to before, follows by Theorem 3.1), there exist subsequences, not relabeled, such that

$$\begin{aligned} p^{(m)} &\overset{*}{\rightharpoonup} p^* & \text{in } L^\infty(0, T; H^3(\Omega)), & \quad p_t^{(m)} &\overset{*}{\rightharpoonup} p_t^* & \text{in } L^\infty(0, T; H^2(\Omega)), \\ \partial_t^\alpha p^{(m)} &\rightharpoonup \partial_t^\alpha p^* & \text{in } L^2(0, T; H^3(\Omega)), & \quad p_{tt}^{(m)} &\rightharpoonup p_{tt}^* & \text{in } L^2(0, T; H^1(\Omega)). \end{aligned}$$

and

$$\begin{aligned} g^m &\overset{*}{\rightharpoonup} \tilde{g} & \text{in } L^\infty(0, T; H^{3/2}(\Gamma)), & \quad g_t^m &\overset{*}{\rightharpoonup} \tilde{g}_t & \text{in } L^\infty(0, T; H^{1/2}(\Gamma)), \\ \partial_t^\alpha g^m &\rightharpoonup \partial_t^\alpha \tilde{g} & \text{in } L^2(0, T; H^{3/2}(\Gamma)) & \quad g_{tt}^m &\rightharpoonup \tilde{g}_{tt} & \text{in } L^2(0, T; H^{-1/2}(\Gamma)), \end{aligned}$$

as well as $f^m \rightharpoonup \tilde{f}$ in $L^2(0, T; H^2(\Omega))$ as $m \rightarrow \infty$. By the Aubin–Lions theorem, we have the strong convergence along a subsequence

$$g^m \rightarrow \tilde{g} \text{ in } C([0, T]; H^{1/2}(\Gamma)).$$

We can then pass to the limit in the weak form of the problem solved by $p^{(m)}$ as $m \rightarrow \infty$ to obtain $p^* = \mathcal{S}(\tilde{g}, \tilde{f})$, analogously to the proof of Theorem 4.1.

By the weak lower semi-continuity of norms and (4.10),

$$\begin{aligned}
 J^{p^d}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}) &\leq \liminf_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m) \\
 (4.11) \quad &\leq \limsup_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m) \\
 &\leq \lim_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g, f), g, f) = J^{p^d}(\mathcal{S}(g, f), g, f),
 \end{aligned}$$

for all $(g, f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$, where we have used the strong convergence of $p^{d,m}$ in the last step. This inequality implies that (\tilde{g}, \tilde{f}) is a minimizer and, combined with (4.10), that

$$(4.12) \quad \lim_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m) = J^{p^d}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}).$$

Indeed, from (4.10) we have, because $(\tilde{g}, \tilde{f}) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$,

$$J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m) \leq J^{p^{d,m}}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}).$$

From here,

$$\begin{aligned}
 \limsup_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m) &\leq \limsup_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}) \\
 &= \lim_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}) = J^{p^d}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}).
 \end{aligned}$$

On the other hand, due to (4.11),

$$J^{p^d}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}) \leq \liminf_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m).$$

These two inequalities combined yield

$$\limsup_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m) \leq J^{p^d}(\mathcal{S}(\tilde{g}, \tilde{f}), \tilde{g}, \tilde{f}) \leq \liminf_{m \rightarrow \infty} J^{p^{d,m}}(\mathcal{S}(g^m, f^m), g^m, f^m),$$

which allows us to conclude that (4.12) holds.

In the last step, we prove strong convergence of f^m in $L^2(0, T; L^2(\Omega))$. Assume that the opposite is true, that is, $f^m \rightharpoonup \tilde{f}$ in $L^2(0, T; L^2(\Omega))$. Since $\|f^m\|_{L^2(L^2(\Omega))}$ is bounded, then

$$C := \limsup_{m \rightarrow \infty} \|f^m\|_{L^2(L^2(\Omega))} > \|\tilde{f}\|_{L^2(L^2(\Omega))}$$

and there exists a subsequence of $\{(g^m, f^m)\}_{m \geq 1}$, denoted $\{(g^n, f^n)\}_{n \geq 1}$, for which

$$\|f^n\|_{L^2(L^2(\Omega))} \rightarrow C$$

as $n \rightarrow \infty$. Then (4.12) implies

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \left(J^{p^{d,n}}(\mathcal{S}(g^n, f^n), g^n, f^n) - \frac{\eta}{2} \|f^n\|_{L^2(L^2(\Omega))}^2 \right) \\
 &= \frac{\nu}{2} \int_0^T \int_{\Omega} (\mathcal{S}(\tilde{g}, \tilde{f}) - p^d)^2 \chi_{\Omega_0} \, dx dt + \frac{1-\nu}{2} \int_{\Omega} (\mathcal{S}(\tilde{g}, \tilde{f})(T) - p^d(T))^2 \chi_{\Omega_0} \, dx dt \\
 &\quad + \frac{\gamma}{2} \|\tilde{g}\|_{L^2(L^2(\Gamma))}^2 + \frac{\eta}{2} \left(\|\tilde{f}\|_{L^2(L^2(\Omega))}^2 - C^2 \right) \\
 &< \frac{\nu}{2} \int_0^T \int_{\Omega} (\mathcal{S}(\tilde{g}, \tilde{f}) - p^d)^2 \chi_{\Omega_0} \, dx dt + \frac{1-\nu}{2} \int_{\Omega} (\mathcal{S}(\tilde{g}, \tilde{f})(T) - p^d(T))^2 \chi_{\Omega_0} \, dx dt \\
 &\quad + \frac{\gamma}{2} \|\tilde{g}\|_{L^2(L^2(\Gamma))}^2,
 \end{aligned}$$

which is in contradiction with the lower semi-continuity property of norms. Therefore, $f^m \rightarrow \tilde{f}$ in $L^2(0, T; L^2(\Omega))$. \square

We next discuss the vanishing regularization limit. To simplify the exposition, we set $\eta = \gamma$.

Proposition 4.2. *Let the assumptions of Theorem 4.1 hold and let $\eta = \gamma > 0$ in the regularization functional (1.1). Assume that p^d is attainable, that is, there exists (g^\dagger, f^\dagger) , such that $\mathcal{S}(g^\dagger, f^\dagger) = p^d$. Let $(g_\gamma, f_\gamma) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$ be defined as an approximate minimizer in the sense that*

$$(4.13) \quad J_\gamma^{p^d}(\mathcal{S}(g_\gamma, f_\gamma), g_\gamma, f_\gamma) \leq J_\gamma^{p^d}(\mathcal{S}(g, f), g, f) + \zeta(\gamma)$$

holds for every $(g, f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$. Then for any sequence of regularizing parameters $\{\gamma_m\}_{m \geq 1}$ with

$$(4.14) \quad \begin{aligned} \gamma_m &\rightarrow 0 \text{ as } m \rightarrow \infty, & \zeta(\gamma_m) &= o(\gamma_m), \\ \|p_{\gamma_m}^d - p^d\|_{L^2(L^2(\Omega))} &= o(\sqrt{\gamma_m}), & \|p_{\gamma_m}^d(T) - p^d(T)\|_{L^2(\Omega)} &= o(\sqrt{\gamma_m}), \end{aligned}$$

the sequence $\{(g_{\gamma_m}, f_{\gamma_m})\}_{m \geq 1}$ of approximate minimizers has a strongly convergent subsequence in $L^2(0, T; L^2(\Gamma)) \times L^2(0, T; L^2(\Omega))$ and the limit (\tilde{g}, \tilde{f}) of every convergent subsequence satisfies $\mathcal{S}(\tilde{g}, \tilde{f}) = p^d$.

Proof. The proof is based on [11, Theorem 2.3], which, in turn, employs techniques from [45, Sec. 3]. Analogously to our previous arguments, we can leverage uniform boundedness of $\{(g_{\gamma_m}, f_{\gamma_m})\}_{m \geq 1} \subset \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$ and $\{\mathcal{S}(g_{\gamma_m}, f_{\gamma_m})\}_{m \geq 1}$, together with suitable compact embeddings, to argue that there is a subsequence and $(\tilde{g}, \tilde{f}) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$, not relabeled, such that

$$(4.15) \quad \begin{aligned} g_{\gamma_m} &\rightarrow \tilde{g} && \text{strongly in } L^2(0, T; L^2(\Gamma)), \\ f_{\gamma_m} &\rightharpoonup \tilde{f} && \text{weakly in } L^2(0, T; H^2(\Omega)), \\ \mathcal{S}(g_{\gamma_m}, f_{\gamma_m}) &\rightarrow \mathcal{S}(\tilde{g}, \tilde{f}) && \text{strongly in } L^2(0, T; L^2(\Omega)), \\ \mathcal{S}(g_{\gamma_m}, f_{\gamma_m})(T) &\rightarrow \mathcal{S}(\tilde{g}, \tilde{f})(T) && \text{strongly in } L^2(\Omega), \end{aligned}$$

as $m \rightarrow \infty$.

Next, thanks to the approximate minimization property in (4.13), by choosing $(g, f) = (g^\dagger, f^\dagger)$, we obtain

$$(4.16) \quad J_{\gamma_m}^{p^d}(\mathcal{S}(g_{\gamma_m}, f_{\gamma_m}), g_{\gamma_m}, f_{\gamma_m}) \leq J_{\gamma_m}^{p^d}(\mathcal{S}(g^\dagger, f^\dagger), g^\dagger, f^\dagger) + \zeta(\gamma_m).$$

The right-hand side of (4.16) tends to zero as $m \rightarrow \infty$ due to (4.14) and the fact that $\mathcal{S}(g^\dagger, f^\dagger) = p^d$. Hence,

$$\lim_{m \rightarrow \infty} J_{\gamma_m}^{p^d}(\mathcal{S}(g_{\gamma_m}, f_{\gamma_m}), g_{\gamma_m}, f_{\gamma_m}) = 0.$$

We thus conclude that $\{\mathcal{S}(g_{\gamma_m}, f_{\gamma_m})\}_{m \geq 1}$ converges to p^d strongly in $L^2(0, T; L^2(\Omega))$. By the uniqueness of limits, it must hold that $\mathcal{S}(\tilde{g}, \tilde{f}) = p^d$.

It remains to show strong convergence of (a subsequence of) $\{f_{\gamma_m}\}_{m \geq 1}$ in $L^2(0, T; L^2(\Omega))$.

Taking $(g, f) = (\tilde{g}, \tilde{f})$ in (4.13) and dividing by $\gamma_m/2$ yields

$$\begin{aligned} & \|g_{\gamma_m}\|_{L^2(L^2(\Gamma))}^2 + \|f_{\gamma_m}\|_{L^2(L^2(\Omega))}^2 \\ & \leq \frac{\nu}{\gamma_m} \|\mathcal{S}(\tilde{g}, \tilde{f}) - p_{\gamma_m}^d\|_{L^2(L^2(\Omega_0))}^2 + \frac{1-\nu}{\gamma_m} \|\mathcal{S}(\tilde{g}, \tilde{f})(T) - p_{\gamma_m}^d(T)\|_{L^2(\Omega_0)}^2 \\ & \quad + \|\tilde{g}\|_{L^2(L^2(\Gamma))}^2 + \|\tilde{f}\|_{L^2(L^2(\Omega))}^2 + \frac{2}{\gamma_m} \zeta(\gamma_m). \end{aligned}$$

Thanks to (4.14), the right-hand side is uniformly bounded and, because $\mathcal{S}(\tilde{g}, \tilde{f}) = p^d$, we have

$$\limsup_{m \rightarrow \infty} (\|g_{\gamma_m}\|_{L^2(L^2(\Gamma))}^2 + \|f_{\gamma_m}\|_{L^2(L^2(\Omega))}^2) \leq \|\tilde{g}\|_{L^2(L^2(\Gamma))}^2 + \|\tilde{f}\|_{L^2(L^2(\Omega))}^2.$$

We wish to show that $\|f_{\gamma_m}\|_{L^2(L^2(\Omega))} \rightarrow \|\tilde{f}\|_{L^2(L^2(\Omega))}$ as $m \rightarrow \infty$, which combined with the weak convergence in (4.15) will yield the claimed strong convergence. Suppose that

$$\liminf_{m \rightarrow \infty} (\|g_{\gamma_m}\|_{L^2(L^2(\Gamma))}^2 + \|f_{\gamma_m}\|_{L^2(L^2(\Omega))}^2) < \|\tilde{g}\|_{L^2(L^2(\Gamma))}^2 + \|\tilde{f}\|_{L^2(L^2(\Omega))}^2.$$

By (4.15) and the lower semi-continuity of norms, we would then have

$$\|\tilde{g}\|_{L^2(L^2(\Gamma))}^2 + \|\tilde{f}\|_{L^2(L^2(\Omega))}^2 < \|\tilde{g}\|_{L^2(L^2(\Gamma))}^2 + \|\tilde{f}\|_{L^2(L^2(\Omega))}^2.$$

which is a contradiction. This concludes the proof. \square

5. FIRST-ORDER NECESSARY OPTIMALITY CONDITIONS

In this final section, we derive the necessary optimality conditions for the distributed and Neumann boundary control problems, treating each case separately. We begin with the analysis of the adjoint problem.

5.1. The adjoint problem. Under the assumptions of Theorem 3.1, let p denote the solution of the forward Westervelt problem ($\text{IBVP}_{\text{West}}$). The adjoint pressure problem in strong form is formally given by

$$(5.1) \quad \begin{cases} (1 - 2kp)p_{tt}^{\text{adj}} - c^2 \Delta p^{\text{adj}} - b \Delta \widetilde{\partial_t^\alpha} p^{\text{adj}} = \nu(p - p^d)\chi_{\Omega_0} & \text{in } \Omega \times (0, T), \\ \frac{\partial}{\partial n} (c^2 p^{\text{adj}} + b \widetilde{\partial_t^\alpha} p^{\text{adj}}) = 0 & \text{on } \Gamma, \\ p^{\text{adj}}(T) = 0, \quad p_t^{\text{adj}}(T) = -(1 - \nu) \frac{p(T) - p^d(T)}{1 - 2kp(T)} \chi_{\Omega_0} & \text{on } \Omega. \end{cases}$$

Due to the space-localization of the source and final data via χ_{Ω_0} , the right-hand side and final data of the adjoint problem (5.1) lack sufficient spatial regularity to apply analogous arguments to those of Proposition 3.2. Instead we interpret (5.1) in a weak sense where, after time reversal, we can apply Proposition 3.1.

Proposition 5.1. *Assume the hypotheses of Theorem 3.1 hold. Then there exists a unique*

$$p^{\text{adj}} \in \left\{ p^{\text{adj}} \in L^\infty(0, T; H^1(\Omega)) \cap \tau H_\alpha(0, T; H^1(\Omega)) : p_t^{\text{adj}} \in L^\infty(0, T; L^2(\Omega)) \right\},$$

which solves

$$(5.2) \quad \begin{aligned} & - \int_0^T \int_\Omega ((1 - 2kp)\varphi)_t p_t^{\text{adj}} \, dx dt + \int_0^T \int_\Omega (c^2 \nabla p^{\text{adj}} + b \nabla \widetilde{\partial_t^\alpha} p^{\text{adj}}) \cdot \nabla \varphi \, dx dt \\ & = (1 - \nu) \int_\Omega (p(T) - p^d(T)) \chi_{\Omega_0} \varphi(T) \, dx + \nu \int_0^T \int_\Omega (p - p^d) \chi_{\Omega_0} \varphi \, dx dt \end{aligned}$$

with $p^{\text{adj}}(T) = 0$, for all test functions $\varphi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ satisfying $\varphi(0) = 0$. Furthermore, the following estimate holds:

$$\|p^{\text{adj}}\|_{\mathcal{X}_{p^{\text{adj}}}} \leq C(T, p) \left((1 - \nu) \|p(T) - p^{\text{d}}(T)\|_{L^2(\Omega_0)} + \nu \|p - p^{\text{d}}\|_{L^2(L^2(\Omega_0))} \right),$$

where

$$C(T, p) = C_1 \exp\{C_2 T(1 + \|p\|_{W^{1,\infty}(L^\infty(\Omega))})\}, \quad C_1, C_2 > 0,$$

and the involved constants do not depend on b or α .

Proof. We analyze first the time reversed adjoint problem, given for $\tilde{p}^{\text{adj}} = \tau p^{\text{adj}} = p^{\text{adj}}(T - t)$ by

$$\begin{aligned} (5.3) \quad & - \int_0^T \int_{\Omega} ((1 - 2k\tilde{p})\varphi)_t \tilde{p}_t^{\text{adj}} \, dx dt + \int_0^T \int_{\Omega} (c^2 \nabla \tilde{p}^{\text{adj}} + b \nabla \partial_t^\alpha \tilde{p}^{\text{adj}}) \cdot \nabla \varphi \, dx dt \\ & = (1 - \nu) \int_{\Omega} (\tilde{p}(0) - \tilde{p}^{\text{d}}(0)) \chi_{\Omega_0} \varphi(0) \, dx + \nu \int_0^T \int_{\Omega} (\tilde{p} - \tilde{p}^{\text{d}}) \chi_{\Omega_0} \varphi \, dx dt \end{aligned}$$

with $\tilde{p}^{\text{adj}}(0) = 0$, for all test functions $\varphi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ satisfying $\varphi(T) = 0$. The statement follows from Proposition 3.1 together with Lemma 3.1 by choosing

$$\mathbf{a} = 1 - 2k\tilde{p}, \quad \mathbf{l} = \mathbf{n} = 0, \quad F = \nu(\tilde{p} - \tilde{p}^{\text{d}}) \chi_{\Omega_0}, \quad g = 0, \quad u_1 = -(1 - \nu) \frac{\tilde{p}(0) - \tilde{p}^{\text{d}}(0)}{\mathbf{a}(0)} \chi_{\Omega_0},$$

$$\Lambda_0(\mathbf{a}, \mathbf{l}, \mathbf{n}) = \exp\{CT(1 + \|\mathbf{a}\|_{W^{1,\infty}(L^\infty(\Omega))})\}.$$

Indeed, by Theorem 3.1, we have

$$1 - 2k\tilde{p} \in W^{1,\infty}(0, T; L^\infty(\Omega)) \cap H^2(0, T; L^4(\Omega)), \quad 0 < \underline{\mathbf{a}} \leq 1 - 2k\tilde{p} \leq \bar{\mathbf{a}} \text{ in } \Omega \times [0, T].$$

Similarly, $u_1 \in L^2(\Omega)$ because $\tilde{p}(0) - \tilde{p}^{\text{d}}(0) = p(T) - p^{\text{d}}(T) \in L^2(\Omega)$ and $0 < \underline{\mathbf{a}} \leq \mathbf{a}(0) \leq \bar{\mathbf{a}}$ in Ω . We then obtain the solution \tilde{p}^{adj} of (5.3) by Proposition 3.1 and of (5.2) by setting $p^{\text{adj}} = \tau \tilde{p}^{\text{adj}}$. \square

5.2. Differentiability of the control-to-state mapping. Using the control-to-state operator (4.3), consider again the reduced optimization problem

$$(4.4) \quad \text{Find } (g^*, f^*) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}, \text{ such that } j(g^*, f^*) = \inf_{(g,f) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}} j(g, f)$$

where $j = J(S(g, f), g, f)$, and $\mathcal{X}_g^{\text{adm}}$ and X_f^{adm} are defined in (4.1) and (4.2), respectively. To establish differentiability of j , we first show that differentiability of the control-to-state mapping \mathcal{S} .

Proposition 5.2. *Let the assumptions of Theorem 3.1 hold. Then the control-to-state mapping \mathcal{S} defined in (4.3) is directionally differentiable in the following sense. Let $(h, \phi) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$ be such that $(g + \varepsilon h, f + \varepsilon \phi) \in \mathcal{X}_g^{\text{adm}} \times X_f^{\text{adm}}$ for $\varepsilon \in [0, \bar{\varepsilon})$. Define*

$$z^\varepsilon = \frac{p^\varepsilon - p}{\varepsilon}, \quad p = \mathcal{S}(g, f), \quad \text{and} \quad p^\varepsilon = \mathcal{S}(g + \varepsilon h, f + \varepsilon \phi).$$

Then there exists a subsequence $\{z^{\varepsilon_n}\}_{n \in \mathbb{N}}$ of $\{z^\varepsilon\}_{\varepsilon \in (0, \bar{\varepsilon})}$, such that

$$(5.4) \quad \begin{aligned} z^{\varepsilon_n} &\xrightarrow{*} z && \text{in } L^\infty(0, T; H^1(\Omega)), && z_t^{\varepsilon_n} &\xrightarrow{*} z_t && \text{in } L^\infty(0, T; L^2(\Omega)), \\ \partial_t^\alpha z^{\varepsilon_n} &\rightharpoonup \partial_t^\alpha z && \text{in } L^2(0, T; H^1(\Omega)), \end{aligned}$$

as $n \rightarrow \infty$. The limit z satisfies

$$(5.5) \quad - \int_0^T \int_{\Omega} ((1 - 2kp)z)_t \varphi_t \, dxdt + \int_0^T \int_{\Omega} (c^2 \nabla z + b \nabla \partial_t^\alpha z) \cdot \nabla \varphi \, dxdt \\ - \int_0^T \int_{\Gamma} (c^2 h + b \partial_t^\alpha h) \phi \, d\Gamma dt = \int_0^T \int_{\Omega} \phi \varphi \, dxdt$$

with $z(0) = 0$, for all $\varphi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with $\varphi(T) = 0$.

Proof. Thanks to Theorem 3.1, the family $\{p^\varepsilon\}_{\varepsilon \geq 0}$ is uniformly bounded:

$$(5.6) \quad \|p^\varepsilon\|_{\mathcal{X}_p} \lesssim_T \|g + \varepsilon h\|_{\mathcal{X}_g} + \|f + \varepsilon \phi\|_{\mathcal{X}_f} \lesssim_T (1 + \bar{\varepsilon})(L_1 + L_2).$$

The quotient $z^\varepsilon = \frac{p^\varepsilon - p}{\varepsilon}$ satisfies

$$(5.7) \quad - \int_0^T \int_{\Omega} (1 - 2kp)z_t^\varepsilon \varphi_t \, dxdt + \int_0^T \int_{\Omega} 2kz^\varepsilon p_t^\varepsilon \varphi_t \, dxdt \\ + \int_0^T \int_{\Omega} (c^2 \nabla z^\varepsilon + b \nabla \partial_t^\alpha z^\varepsilon) \cdot \nabla \varphi \, dxdt - \int_0^T \int_{\Gamma} (c^2 h + b \partial_t^\alpha h) \varphi \, d\Gamma dt \\ = \int_0^T \int_{\Omega} \phi \varphi \, dxdt$$

for all $\varphi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with $\varphi(T) = 0$. We now apply estimate (3.5) from Proposition 3.1 by choosing

$$\mathbf{a} = 1 - 2kp, \quad \mathbf{l} = -2kp_t^\varepsilon, \quad \mathbf{n} = -2kp_{tt}^\varepsilon, \quad F = \phi, \quad g = h, \quad \text{and } u_1 = 0.$$

This yields

$$(5.8) \quad \|z^\varepsilon\|_{X_{\text{low}}} \lesssim \Lambda_0(\mathbf{a}, \mathbf{l}, \mathbf{n}) \left(\|h\|_{H^2(H^{-1/2}(\Gamma))} + \|\phi\|_{L^2(L^2(\Omega))} \right),$$

where Λ_0 is defined in (3.6). Thanks to Theorem 3.1 with the assumption $\|k\|_{\mathcal{X}_k} \leq \delta$, we obtain

$$\Lambda_0(\mathbf{a}, \mathbf{l}, \mathbf{n}) = \exp \left\{ CT(1 + \|\mathbf{a}\|_{W^{1,\infty}(L^\infty(\Omega))} + \|\mathbf{l}\|_{L^\infty(L^\infty(\Omega))} + \|\mathbf{n}\|_{L^2(L^3(\Omega))}^2) \right\} \\ \lesssim \exp \left\{ CT \left(1 + \delta \left(\|p\|_{W^{1,\infty}(L^\infty(\Omega))} + \|p_t^\varepsilon\|_{L^\infty(L^\infty(\Omega))} + \|p_{tt}^\varepsilon\|_{L^2(H^1(\Omega))}^2 \right) \right) \right\} \\ \lesssim \exp \left\{ CT \left(1 + \delta \|p^\varepsilon\|_{\mathcal{X}_p} + \delta \|p^\varepsilon\|_{\mathcal{X}_p}^2 \right) \right\}.$$

Together with (5.6), this shows that Λ_0 is uniformly bounded. Therefore, estimate (5.8) implies that $\{z^\varepsilon\}_\varepsilon$ is uniformly bounded in X_{low} , defined in (3.4). Consequently, there exists a subsequence $\{z^{\varepsilon_n}\}_{n \in \mathbb{N}}$, such that the convergences in (5.4) hold. To pass to the limit in (5.7) as $\varepsilon_n \rightarrow 0$, we use the weak convergence

$$-(1 - 2kp)z_t^{\varepsilon_n} + 2kz^{\varepsilon_n}p_t^{\varepsilon_n} \rightharpoonup -(1 - 2kp)z_t + 2kzp_t = -((1 - 2kp)z)_t$$

in $L^2(0, T; L^2(\Omega))$, which follows analogously to (4.9) and to [9, p. 749]. Passing to the limit in (5.7) as $\varepsilon_n \rightarrow 0$ therefore yields (5.5), as claimed. \square

5.3. Derivative of the reduced objective. Next, we compute the derivative of the reduced objective. Using the chain rule and recalling (4.5), together with the fact that $p = \mathcal{S}(g, f)$, we obtain

$$j'(g, f; h, \phi) = D_f J(p, g, f; \phi) + D_g J(p, g, f; h) + D_p J(p, g, f; \mathcal{S}'(g, f; h, \phi)),$$

where

$$(5.9) \quad D_f J(p, g, f; \phi) = \eta \int_0^T \int_{\Omega} f \phi \, dx dt, \quad D_g J(p, g, f; h) = \gamma \int_0^T \int_{\Gamma} gh \, d\Gamma dt,$$

and

$$\begin{aligned} & D_p J(p, g, f; \mathcal{S}'(g, f; h, \phi)) \\ &= \nu \int_0^T \int_{\Omega} (p - p^d) \chi_{\Omega_0} \mathcal{S}'(g, f; h, \phi) \, dx dt + (1 - \nu) \int_{\Omega} (p(T) - p^d(T)) \chi_{\Omega_0} \mathcal{S}'(g, f; h, \phi)(T) \, dx. \end{aligned}$$

We now use the adjoint problem to express these terms without the explicit appearance of \mathcal{S}' . Testing (5.2) with $z = \mathcal{S}'(g, f; h, \phi)$ yields

$$\begin{aligned} & (1 - \nu) \int_{\Omega} (p(T) - p^d(T)) \chi_{\Omega_0} z(T) \, dx + \nu \int_0^T \int_{\Omega} (p - p^d) \chi_{\Omega_0} z \, dx dt \\ &= - \int_0^T \int_{\Omega} ((1 - 2kp)z)_t p_t^{\text{adj}} \, dx dt + \int_0^T \int_{\Omega} (c^2 \nabla p^{\text{adj}} + b \nabla \widetilde{\partial}_t^{\alpha} p^{\text{adj}}) \cdot \nabla z \, dx dt \\ &= - \int_0^T \int_{\Omega} ((1 - 2kp)z)_t p_t^{\text{adj}} \, dx dt + \int_0^T \int_{\Omega} (c^2 \nabla z + b \nabla \partial_t^{\alpha} z) \cdot \nabla p^{\text{adj}} \, dx dt. \end{aligned}$$

Testing the weak form (5.5) solved by z with p^{adj} gives an equivalent form:

$$(5.10) \quad D_p J(p, g, f; \mathcal{S}'(g, f; h, \phi)) = \int_0^T \int_{\Gamma} (c^2 h + b \partial_t^{\alpha} h) p^{\text{adj}} \, d\Gamma dt + \int_0^T \int_{\Omega} \phi p^{\text{adj}} \, dx dt.$$

Combining (5.10) with (5.9), we finally obtain

$$(5.11) \quad \begin{aligned} j'(g, f; h, \phi) &= \int_0^T \int_{\Gamma} (c^2 h + b \partial_t^{\alpha} h) p^{\text{adj}} \, d\Gamma dt + \int_0^T \int_{\Omega} \phi p^{\text{adj}} \, dx dt \\ &\quad + \gamma \int_0^T \int_{\Gamma} gh \, d\Gamma dt + \eta \int_0^T \int_{\Omega} f \phi \, dx dt. \end{aligned}$$

We now state the necessary optimality conditions. From this point onward, we treat the distributed and boundary control cases separately.

5.4. Optimality conditions for the Numann boundary control problem. We focus now on the reduced boundary control problem

Find $g^* \in \mathcal{X}_g^{\text{adm}}$, such that $j(g^*) = \inf_{g \in \mathcal{X}_g^{\text{adm}}} j(g)$

with the objective $j(g) = J(\mathcal{S}(g), g)$, where

$$J(p, f) = \frac{\nu}{2} \|p - p^d\|_{L^2(L^2(\Omega_0))}^2 + \frac{1 - \nu}{2} \|p(T) - p^d(T)\|_{L^2(\Omega_0)}^2 + \frac{\gamma}{2} \|g\|_{L^2(L^2(\Gamma))}^2.$$

For $h \in \mathcal{X}_g^{\text{adm}}$ (cf. (4.1)), we have the identity

$$b \int_0^T \int_{\Gamma} \partial_t^{\alpha} h p^{\text{adj}} \, d\Gamma dt = -b \int_0^T \int_{\Gamma} h \widetilde{\partial}_t^{\alpha} p^{\text{adj}} \, d\Gamma dt.$$

Thus in the setting of exerting control only via the boundary excitation, we find that

$$j'(g; h) = \int_0^T \int_{\Gamma} (c^2 p^{\text{adj}} - b \widetilde{\partial}_t^\alpha p^{\text{adj}} + \gamma g) h \, d\Gamma dt.$$

The first-order necessary optimality conditions are then immediately obtained by employing [48, Lemma 2.21]; see also [38, Theorem 9.2].

Theorem 5.1 (Necessary optimality conditions for boundary control). *Let the assumptions of Theorem 3.1 hold. Let \tilde{g} be a local solution of the reduced minimization problem*

$$\min_{g \in \mathcal{X}_g} j(g) \quad \text{s.t.} \quad g \in \mathcal{X}_g^{\text{adm}},$$

where $\mathcal{X}_g^{\text{adm}}$ is defined in (4.1). Then \tilde{g} satisfies the following variational inequality:

$$\int_0^T \int_{\Gamma} (c^2 p^{\text{adj}} - b \widetilde{\partial}_t^\alpha p^{\text{adj}} + \gamma \tilde{g})(\tilde{g} - g) \, d\Gamma dt \geq 0,$$

where p solves (IBVP_{West}) and p^{adj} the adjoint problem in the weak sense (5.2).

5.5. Optimality conditions for the distributed control problem. Secondly, we focus on the reduced distributed control problem

Find $f^* \in X_f^{\text{adm}}$, such that $j(f^*) = \inf_{f \in X_f^{\text{adm}}} j(f)$

with $j(f) = J(\mathcal{S}(f), f)$, where here

$$J(p, f) = \frac{\nu}{2} \|p - p^{\text{d}}\|_{L^2(L^2(\Omega_0))}^2 + \frac{1-\nu}{2} \|p(T) - p^{\text{d}}(T)\|_{L^2(\Omega_0)}^2 + \frac{\eta}{2} \|f\|_{L^2(L^2(\Omega))}^2.$$

In this setting, analogously to (5.11), we have

$$j'(f; \phi) = \int_0^T \int_{\Omega} (p^{\text{adj}} + \eta f) \phi \, dx dt.$$

Similarly to Theorem 5.1, we have the following first-order necessary optimality conditions.

Theorem 5.2 (Necessary optimality conditions for distributed control). *Let the assumptions of Theorem 3.1 hold. Let $\tilde{f} \in X_f^{\text{adm}}$ be a solution of the reduced minimization problem*

$$\text{Find } \tilde{f} \in X_f^{\text{adm}}, \text{ such that } j(\tilde{f}) = \inf_{f \in X_f^{\text{adm}}} j(f),$$

where X_f is defined in (4.2). Then \tilde{f} satisfies the following variational inequality:

$$\int_0^T \int_{\Omega} (p^{\text{adj}} + \eta f)(\tilde{f} - f) \, dx dt \geq 0 \quad \forall f \in X_f^{\text{adm}},$$

where p solves the state problem (IBVP_{West}) and p^{adj} the adjoint problem in the weak sense (5.2).

CONCLUSION

The aim of this work was to deepen the understanding of how nonlinear acoustic waves can be controlled in complex propagation media. To this end, we analyzed distributed and boundary optimal control problems subject to the Westervelt equation with time-fractional attenuation, a model relevant ultrasound propagation through biological media and, consequently, for various medical applications. First, we extended the existing well-posedness theory for the Westervelt equation with nonlocal damping to the setting of inhomogeneous Neumann data. This required constructing a suitable extension of regularized boundary data first. Second, we established the existence of globally optimal controls and analyzed the stability of the optimization problem with respect to perturbations in the targeted pressure distribution and the vanishing regularization parameters. Third, we showed that the adjoint problem, which has state-dependent coefficients, admits a solution even though it does not share the regularity level of the state equation. This allowed us to prove differentiability of the control-to-state mapping in a reduced regularity setting. Finally, these results justified the derivation of adjoint-based first-order optimality conditions for the control problem.

Natural directions for future research include developing and analyzing efficient numerical optimization algorithms in this context, as well as building upon the present theoretical framework to incorporate Westervelt-based systems as state constraints, such as the wave-heat systems motivated by ultrasound heating phenomena [7]. Additionally, as the analysis of the state problem in Theorem 3.1 is uniform in b and α , this could be used as a basis for investigating the limiting behavior of minimizers as $b \searrow 0$ and $\alpha \nearrow 1$ and establishing a connection to problems constrained by the inviscid and strongly damped Westervelt equations, respectively.

A. DETAILS ON THE DERIVATION OF ENERGY ESTIMATES

We provide here the missing details of the second and third testing steps within the Faedo–Galerkin procedure in the proof of Theorem 3.1.

A.1. Derivation of estimate (3.37). Testing the semi-discrete equation with $-\Delta w_t^{(\nu,m)}$ yields

$$(A.1) \quad \begin{aligned} & (\mathbf{a}w_{tt}^{(\nu,m)} - c^2\Delta w^{(\nu,m)} - b\Delta\partial_t^\alpha w^{(\nu,m)}, -\Delta w_t^{(\nu,m)})_{L^2(\Omega)} \\ &= (F_{G^{(\nu)}} - \mathbf{l}w_t^{(\nu,m)} - \mathbf{n}w^{(\nu,m)}, -\Delta w_t^{(\nu,m)})_{L^2(\Omega)}. \end{aligned}$$

Since $w_t^{(\nu,m)}$ satisfies homogeneous Neumann data, integration by parts yields the identity

$$\begin{aligned} & (\mathbf{a}w_{tt}^{(\nu,m)}, -\Delta w_t^{(\nu,m)})_{L^2(\Omega)} \\ &= (\nabla(\mathbf{a}w_{tt}^{(\nu,m)}), \nabla w_t^{(\nu,m)})_{L^2(\Omega)} - \int_{\Gamma} \mathbf{a}w_{tt}^{(\nu,m)} \frac{\partial w_t^{(\nu,m)}}{\partial n} \, d\Gamma \\ &= \frac{1}{2} \frac{d}{dt} (\mathbf{a}\nabla w_t^{(\nu,m)}, \nabla w_t^{(\nu,m)})_{L^2(\Omega)} - \frac{1}{2} (\mathbf{a}_t \nabla w_t^{(\nu,m)}, \nabla w_t^{(\nu,m)})_{L^2(\Omega)} + (w_{tt}^{(\nu,m)} \nabla \mathbf{a}, \nabla w_t^{(\nu,m)})_{L^2(\Omega)}. \end{aligned}$$

The last two terms above are placed on the right-hand side and, after integrating in time, estimated by

$$\begin{aligned} & \frac{1}{2} \int_0^t (\mathbf{a}_t \nabla w_t^{(\nu, m)}, \nabla w_t^{(\nu, m)})_{L^2(\Omega)} - (w_{tt}^{(\nu, m)} \nabla \mathbf{a}, \nabla w_t^{(\nu, m)})_{L^2(\Omega)} \, ds \\ & \lesssim \int_0^t \|\mathbf{a}_t\|_{L^\infty(\Omega)} \|\nabla w_t^{(\nu, m)}\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \mathbf{a}\|_{L^\infty(\Omega)} \|w_{tt}^{(\nu, m)}\|_{L^2(\Omega)} \|\nabla w_t^{(\nu, m)}\|_{L^2(\Omega)}. \end{aligned}$$

The terms involving \mathbf{l} and \mathbf{n} are controlled via Hölder's and Young's inequalities:

$$\begin{aligned} & \int_0^t (-\mathbf{l} w_t^{(\nu, m)} - \mathbf{n} w^{(\nu, m)}, -\Delta w_t^{(\nu, m)})_{L^2(\Omega)} \, ds \\ & \leq \frac{1}{2} \| -\mathbf{l} w_t^{(\nu, m)} - \mathbf{n} w^{(\nu, m)} \|_{L_t^2(L^2(\Omega))}^2 + \frac{1}{2} \|\Delta w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2. \end{aligned}$$

Similarly,

$$\int_0^t (F_{G^{(\nu)}}, -\Delta w_t^{(\nu, m)})_{L^2(\Omega)} \, ds \lesssim \int_0^t \|F_G\|_{L^2(\Omega)} \|\Delta w_t^{(\nu, m)}\|_{L^2(\Omega)} \, ds.$$

Integrating (A.1) over $(0, t)$, inserting these estimates, and invoking the coercivity estimate (2.2), we obtain

$$\begin{aligned} & \frac{1}{2} \|\sqrt{\mathbf{a}(t)} \nabla w_t^{(\nu, m)}(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} (T) \|\Delta w^{(\nu, m)}(t)\|_{L^2(\Omega)}^2 + b C_\alpha(T) \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 \\ & \lesssim \|\mathbf{a}_t\|_{L^\infty(L^\infty(\Omega))} \|\nabla w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 + \|\nabla \mathbf{a}\|_{L^\infty(L^\infty(\Omega))} \|w_{tt}^{(\nu, m)}\|_{L^2(\Omega)} \|\nabla w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \\ & \quad + \|F_{G^{(\nu)}}\|_{L^2(L^2(\Omega))}^2 + \|\Delta w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 \\ & \quad + \|\mathbf{l}\|_{L^\infty(L^\infty(\Omega))}^2 \|w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 + \|\mathbf{n}\|_{L^\infty(L^\infty(\Omega))}^2 \|w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2. \end{aligned}$$

To estimate $\|w_{tt}^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}$, we can rely on the identity

$$(\mathbf{a} w_{tt}^{(\nu, m)}, w_{tt}^{(\nu, m)})_{L^2(\Omega)} = (c^2 \Delta w^{(\nu, m)} + b \Delta \partial_t^\alpha w^{(\nu, m)} - \mathbf{l} w_t^{(\nu, m)} - \mathbf{n} w^{(\nu, m)} + F_{G^{(\nu)}} w_{tt}^{(\nu, m)})_{L^2(\Omega)},$$

which can be seen as an additional testing by $w_{tt}^{(\nu, m)}$, from which we obtain by recalling (3.46)

$$\begin{aligned} & \|\mathbf{a} w_{tt}^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \lesssim \|c^2 \Delta w + b \Delta \partial_t^\alpha w^{(\nu, m)} - \mathbf{l} w_t^{(\nu, m)} - \mathbf{n} w^{(\nu, m)} + F_{G^{(\nu)}}\|_{L_t^2(L^2(\Omega))} \\ & \lesssim \|\Delta w\|_{L_t^2(L^2(\Omega))} + b \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} + \|\mathbf{l}\|_{L^\infty(L^\infty(\Omega))} \|w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \\ & \quad + \|\mathbf{n}\|_{L^\infty(L^\infty(\Omega))} \|w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} + \|F_{G^{(\nu)}}\|_{L_t^2(L^2(\Omega))}. \end{aligned}$$

Thus (A.2) becomes

$$\begin{aligned}
& \frac{1}{2} \|\sqrt{\mathbf{a}(t)} \nabla w_t^{(\nu, m)}(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\Delta w^{(\nu, m)}(t)\|_{L^2(\Omega)}^2 + b \underline{C}(T) \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 \\
& \lesssim \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} \|\nabla w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \left\{ \|\Delta w\|_{L_t^2(L^2(\Omega))} + b \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \right. \\
& \quad + \|w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} + \|w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} + \|F_{G^{(\nu)}}\|_{L_t^2(L^2(\Omega))} \left. \right\} \|\nabla w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \\
& \quad + \|F_{G^{(\nu)}}\|_{L^2(L^2(\Omega))}^2 + \|\Delta w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \|w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 \\
& \quad + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 \|w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2.
\end{aligned} \tag{A.4}$$

Young's ε -inequality with $\varepsilon > 0$ sufficiently small allows the resulting $\varepsilon \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2$ term on the right to be absorbed into the left-hand side of (A.4):

$$\begin{aligned}
& \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^2 b \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \|\nabla w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))} \\
& \leq \varepsilon b \|\Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 + \frac{1}{4\varepsilon} \bar{b} \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}}^4 \|\nabla w_t^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2,
\end{aligned}$$

which leads to (3.37).

A.2. Derivation of estimate (3.40). We provide here the missing details for testing step 3 in the proof of Theorem 3.1. To treat the \mathbf{a} term, we use the identity

$$\begin{aligned}
& (\Delta(\mathbf{a} w_{tt}^{(\nu, m)}), \Delta w_t^{(\nu, m)})_{L^2(\Omega)} \\
& = (\mathbf{a} \Delta w_{tt}^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} + (w_{tt}^{(\nu, m)} \Delta \mathbf{a} + 2 \nabla w_{tt}^{(\nu, m)} \cdot \nabla \mathbf{a}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} \\
& = \frac{1}{2} \frac{d}{dt} (\mathbf{a} \Delta w_t^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} - \frac{1}{2} (\mathbf{a}_t \Delta w_t^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} \\
& \quad + (w_{tt}^{(\nu, m)} \Delta \mathbf{a} + 2 \nabla w_{tt}^{(\nu, m)} \cdot \nabla \mathbf{a}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)}.
\end{aligned}$$

From (3.7), we have $\frac{\partial}{\partial n} \Delta w^{(\nu, m)} = 0$ on Γ . Hence, integration by parts yields

$$-(c^2 \Delta^2 w^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2} = c^2 (\nabla \Delta w^{(\nu, m)}, \nabla \Delta w_t^{(\nu, m)})_{L^2(\Omega)} = \frac{1}{2} c^2 \frac{d}{dt} \|\nabla \Delta w^{(\nu, m)}\|_{L^2(\Omega)}^2.$$

Integrating by parts, using the fact that $\frac{\partial}{\partial n} \Delta \partial_t^\alpha w^{(\nu, m)} = 0$, and employing the coercivity estimate (2.2), we obtain

$$\begin{aligned}
& - \int_0^t (b \Delta^2 \partial_t^\alpha w^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} ds = b \int_0^t (\nabla \partial_t^\alpha \Delta w^{(\nu, m)}, \nabla \Delta w_t^{(\nu, m)})_{L^2(\Omega)} ds \\
& \geq b C_\alpha \|\nabla \Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2.
\end{aligned}$$

Substituting these identities and lower bounds in (3.38) yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\mathbf{a} \Delta w_t^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} + \frac{1}{2} c^2 \frac{d}{dt} \|\nabla \Delta w^{(\nu, m)}\|_{L^2(\Omega)}^2 + b C_\alpha \|\nabla \Delta \partial_t^\alpha w^{(\nu, m)}\|_{L_t^2(L^2(\Omega))}^2 \\
& \leq \frac{1}{2} (\mathbf{a}_t \Delta w_t^{(\nu, m)}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} - (w_{tt}^{(\nu, m)} \Delta \mathbf{a} + 2 \nabla w_{tt}^{(\nu, m)} \cdot \nabla \mathbf{a}, \Delta w_t^{(\nu, m)})_{L^2(\Omega)} \\
& \quad + (\Delta F_{G^{(\nu)}} - \Delta(\mathbf{l} w_t^{(\nu, m)}) - \Delta(\mathbf{n} w^{(\nu, m)}), \Delta w_t^{(\nu, m)})_{L^2}.
\end{aligned}$$

We treat the \mathbf{l} and \mathbf{n} terms on the right-hand side as follows:

$$\begin{aligned} & -(\Delta(\mathbf{l}w_t^{(\nu,m)}), \Delta w_t^{(\nu,m)})_{L^2} - (\Delta(\mathbf{n}w_t^{(\nu,m)}), \Delta w_t^{(\nu,m)})_{L^2(\Omega)} \\ = & -(\mathbf{l}\Delta w_t^{(\nu,m)} + 2\nabla\mathbf{l} \cdot \nabla w_t^{(\nu,m)} + w_t^{(\nu,m)}\Delta\mathbf{l} + \mathbf{n}\Delta w_t^{(\nu,m)} + 2\nabla\mathbf{n} \cdot \nabla w_t^{(\nu,m)} + w_t^{(\nu,m)}\Delta\mathbf{n}, \Delta w_t^{(\nu,m)})_{L^2(\Omega)}. \end{aligned}$$

We estimate the \mathbf{l} -dependent terms by

$$\begin{aligned} & \int_0^t (\mathbf{l}\Delta w_t^{(\nu,m)} + 2\nabla\mathbf{l} \cdot \nabla w_t^{(\nu,m)} + w_t^{(\nu,m)}\Delta\mathbf{l}, \Delta w_t^{(\nu,m)})_{L^2} ds \\ \lesssim & \|\mathbf{l}\|_{L^\infty(L^\infty(\Omega))} \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 + \|\nabla\mathbf{l}\|_{L^\infty(L^3(\Omega))} \|\nabla w_t^{(\nu,m)}\|_{L_t^2(L^6(\Omega))} \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))} \\ & + \|\Delta\mathbf{l}\|_{L^\infty(L^2(\Omega))} \|w_t^{(\nu,m)}\|_{L_t^2(L^\infty(\Omega))} \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))} \\ \lesssim & \|\mathbf{l}\|_{X_1} \|w_t^{(\nu,m)}\|_{L_t^2(H^2(\Omega))}^2, \end{aligned}$$

where we have used the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ in the last step. Similarly,

$$\begin{aligned} & \int_0^t (\mathbf{n}\Delta w_t^{(\nu,m)} + 2\nabla\mathbf{n} \cdot \nabla w_t^{(\nu,m)} + w_t^{(\nu,m)}\Delta\mathbf{n}, \Delta w_t^{(\nu,m)})_{L^2} ds \\ \lesssim & \|\mathbf{n}\|_{X_n} \|w_t^{(\nu,m)}\|_{L_t^2(H^2(\Omega))} \|w_t^{(\nu,m)}\|_{L_t^2(H^2(\Omega))}. \end{aligned}$$

After integrating (3.38) over $(0, t)$ for $t \in (0, T)$ and inserting the derived estimates, we obtain

$$\begin{aligned} & \frac{1}{2} \|\sqrt{\mathbf{a}}\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)}^2 \Big|_0^t + \frac{c^2}{2} \|\nabla\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)}^2 \Big|_0^t + b\underline{C}(T) \|\nabla\Delta\partial_t^\alpha w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 \\ (A.5) \leq & \frac{1}{2} \int_0^t (\mathbf{a}_t\Delta w_t^{(\nu,m)}, \Delta w_t^{(\nu,m)})_{L^2} ds + \int_0^t (w_{tt}^{(\nu,m)}\Delta\mathbf{a} + 2\nabla w_{tt}^{(\nu,m)} \cdot \nabla\mathbf{a}, \Delta w_t^{(\nu,m)})_{L^2} ds \\ & + \|\Delta F\|_{L^2(L^2(\Omega))}^2 + \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2 + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} \|g^{(\nu)}\|_{\mathcal{X}_g^{\text{low}}} \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))} \\ & + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} \|w_t^{(\nu,m)}\|_{L_t^2(H^2(\Omega))}^2 + \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} \|w_t^{(\nu,m)}\|_{L_t^2(H^2(\Omega))} \|w_t^{(\nu,m)}\|_{L_t^2(H^2(\Omega))}. \end{aligned}$$

The first term on the right-hand side is bounded using

$$\frac{1}{2} \int_0^t (\mathbf{a}_t\Delta w_t^{(\nu,m)}, \Delta w_t^{(\nu,m)})_{L^2} ds \leq \frac{1}{2} \int_0^t \|\mathbf{a}_t\|_{L^\infty(\Omega)} \|\Delta w_t^{(\nu,m)}\|_{L^2(\Omega)}^2 ds.$$

To estimate the $w_{tt}^{(\nu,m)}$ terms on the right-hand side of (A.5), we first use Hölder's and Young's inequalities:

$$\begin{aligned} & \int_0^t (w_{tt}^{(\nu,m)}\Delta\mathbf{a} + 2\nabla w_{tt}^{(\nu,m)} \cdot \nabla\mathbf{a}, \Delta w_t^{(\nu,m)})_{L^2} ds \\ \lesssim & \left(\|w_{tt}^{(\nu,m)}\|_{L^2(L^6(\Omega))} \|\Delta\mathbf{a}\|_{L^\infty(L^3(\Omega))} + \|\nabla w_{tt}^{(\nu,m)}\|_{L^2(L^2(\Omega))} \|\nabla\mathbf{a}\|_{L^\infty(L^\infty(\Omega))} \right) \|\Delta w_t^{(\nu,m)}\|_{L^2(L^2(\Omega))}. \end{aligned}$$

By the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^3(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we obtain

$$\|\Delta\mathbf{a}\|_{L^\infty(L^3(\Omega))} + \|\nabla\mathbf{a}\|_{L^\infty(L^\infty(\Omega))} \lesssim \|\Delta\mathbf{a}\|_{L^\infty(H^1(\Omega))} + \|\nabla\mathbf{a}\|_{L^\infty(H^2(\Omega))} \lesssim \|\mathbf{a}\|_{X_a}.$$

Hence,

$$\begin{aligned} (A.6) \quad & \int_0^t (w_{tt}^{(\nu,m)}\Delta\mathbf{a} + 2\nabla w_{tt}^{(\nu,m)} \cdot \nabla\mathbf{a}, \Delta w_t^{(\nu,m)})_{L^2} ds \\ & \leq \mathcal{L}_{\mathbf{a}, \mathbf{n}, \mathbf{l}} \|w_{tt}^{(\nu,m)}\|_{L^2(H^1(\Omega))} \|\Delta w_t^{(\nu,m)}\|_{L^2(L^2(\Omega))}. \end{aligned}$$

To bound $\|w_{tt}^{(\nu,m)}\|_{L^2(H^1(\Omega))}$, we note that

$$\|w_{tt}^{(\nu,m)}\|_{L^2(H^1(\Omega))} \lesssim \|w_{tt}^{(\nu,m)}\|_{L^2(L^2(\Omega))} + \|\nabla w_{tt}^{(\nu,m)}\|_{L^2(L^2(\Omega))}.$$

We estimated the first term on the right in (A.3). For the second one, we use the identity

$$\begin{aligned} & (\mathbf{a}\nabla w_{tt}^{(\nu,m)} + \nabla \mathbf{a}w_{tt}^{(\nu,m)}, \nabla w_{tt}^{(\nu,m)})_{L^2(\Omega)} \\ &= (c^2\nabla\Delta w^{(\nu,m)} + b\nabla\Delta\partial_t^\alpha w^{(\nu,m)} - \nabla[\mathbf{l}w_t^{(\nu,m)} + \mathbf{n}w^{(\nu,m)}] + \nabla F_{G^{(\nu)}}, \nabla w_{tt}^{(\nu,m)})_{L^2(\Omega)}, \end{aligned}$$

obtained by applying ∇ to the semi-discrete PDE and testing with $\nabla w_{tt}^{(\nu,m)}$. We then have

$$\begin{aligned} & \|\nabla w_{tt}^{(\nu,m)}\|_{L_t^2(L^2(\Omega))} \\ (A.7) \quad & \lesssim \|\nabla \mathbf{a}\|_{L^\infty(L^\infty(\Omega))} \|c^2\Delta w^{(\nu,m)} + b\Delta\partial_t^\alpha w^{(\nu,m)} - \mathbf{l}w_t^{(\nu,m)} - \mathbf{n}w^{(\nu,m)} + F_{G^{(\nu)}}\|_{L_t^2(L^2(\Omega))} \\ & \quad + \|b\nabla\Delta\partial_t^\alpha w^{(\nu,m)} + c^2\nabla\Delta w^{(\nu,m)} - \nabla[\mathbf{l}w_t^{(\nu,m)}] - \nabla[\mathbf{n}w^{(\nu,m)}] + \nabla F_{G^{(\nu)}}\|_{L_t^2(L^2(\Omega))}. \end{aligned}$$

Estimating the right-hand side terms in (A.7) and incorporating (A.3) yields

$$\begin{aligned} (A.8) \quad & \|w_{tt}^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} \\ & \lesssim \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}} \left(b\|\Delta\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} + \|\Delta w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} + \|\mathbf{l}\|_{X_1} \|w_t^{(\nu,m)}\|_{L^2(H^1(\Omega))} \right. \\ & \quad \left. + \|\mathbf{n}\|_{X_n} \|w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} + \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))} \right) \\ & \lesssim \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}}^2 \left(b\|\Delta\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} + \|w^{(\nu,m)}\|_{L_t^2(H^2(\Omega))} + \|w_t^{(\nu,m)}\|_{L^2(H^1(\Omega))} + \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))} \right). \end{aligned}$$

Returning to (A.6) and applying (A.8) gives

$$\begin{aligned} & \int_0^t (w_{tt}^{(\nu,m)} \Delta \mathbf{a} + 2\nabla w_{tt}^{(\nu,m)} \cdot \nabla \mathbf{a}, \Delta w_t^{(\nu,m)})_{L^2} ds \\ & \lesssim \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}}^3 \left\{ \|w^{(\nu,m)}\|_{L_t^2(H^2(\Omega))} + \|w_t^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} + b\|\Delta\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} \right. \\ & \quad \left. + \|F_{G^{(\nu)}}\|_{L^2(H^1(\Omega))} \right\} \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}. \end{aligned}$$

Analogously to before, Young's inequality applied for any $\varepsilon > 0$ yields:

$$\begin{aligned} & \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}}^3 b \|\Delta\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))} (\|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))} + \|\nabla w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}) \\ & \leq \varepsilon b \|\Delta\partial_t^\alpha w^{(\nu,m)}\|_{L_t^2(H^1(\Omega))}^2 + \frac{1}{4\varepsilon} \bar{b} \mathcal{L}_{\mathbf{a},\mathbf{n},\mathbf{l}}^6 \|\Delta w_t^{(\nu,m)}\|_{L_t^2(L^2(\Omega))}^2, \end{aligned}$$

allowing to absorb the first term on the right by reducing ε . Employing the derived estimates in (A.5) leads to (3.40).

B. CONSTRUCTION OF THE FUNCTION ρ

We present here the construction of the function ρ used in the proof of Lemma 3.2.

Lemma B.1. *Let $0 < r < R$ for $R > 0$. There exists $\rho \in C_0^\infty(\mathbb{R})$, such that*

$$(B.1) \quad \rho(\tau) = 1 \quad \text{for } |\tau| \leq r, \quad \rho(\tau) \in [0, 1] \quad \text{for } r \leq |\tau| \leq R, \quad \rho(\tau) = 0 \quad \text{for } |\tau| \geq R.$$

Proof. Such a function can be constructed as follows (see [35]). We first consider the smooth $C^\infty(\mathbb{R})$ transition function

$$f(\tau) = \begin{cases} e^{-1/\tau}, & \text{for } \tau > 0 \\ 0, & \text{for } \tau \leq 0. \end{cases}$$

Then by induction

$$\partial_\tau^\ell f(\tau) = \begin{cases} p_\ell(1/\tau)e^{-1/\tau}, & \text{for } \tau > 0 \\ 0, & \text{for } \tau \leq 0 \end{cases}$$

for a certain polynomial p_ℓ , $\ell = 0, 1, 2, \dots$ of degree at most $\ell + 1$. Since for any polynomial we have

$$\lim_{\tau \rightarrow 0} p(\tau)e^{-1/\tau} = 0,$$

we conclude that $\partial_\tau^\ell f$ is differentiable at 0. We next set

$$f_1(\tau) = f(\tau - r)f(R - \tau).$$

It is clear that $f_1 \in C_0^\infty(\mathbb{R})$ since $\text{supp}(f_1) \subset [r, R]$. Now, let

$$f_2(\tau) = \int_\tau^\infty f_1(s) \, ds.$$

We then have

$$f_2(\tau) = \begin{cases} 0, & \text{for } \tau \geq R \\ \int_\tau^R f_1(s) \, ds & \text{for } \tau \leq r. \end{cases}$$

Hence, the function

$$\rho(\tau) = \frac{f_2(|\tau|)}{\int_r^R f_1(s) \, ds}$$

belongs to $C_0^\infty(\mathbb{R})$ and satisfies (B.1). Integrating by parts, we obtain

$$\int_0^\infty (s\rho(s) + (1 + s^2)\rho'(s) + s\rho''(s)) \, ds = - \int_0^\infty s\rho(s) \, ds \leq 0.$$

□

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