Straight-Edge and Compass: Constructing the Heptadecagon

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Western Washington University

May 15, 2013

Outline

1. Background
   - Straightedge-And-Compass Construction
   - History
2. The Construction.
   - Basic Tools and Techniques.
   - Constructing Polygons.
   - How To Draw a Heptadecagon

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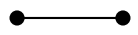

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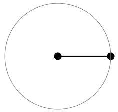

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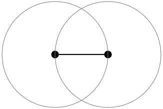

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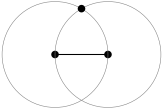

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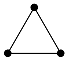

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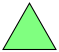

Applications.
Why bother?

We can use geometry to prove that a construction is exactly correct. This is significant in many fields. In particular, consider…

- Drafting, both by hand and with CAD software.
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And most importantly, it's fun to do!

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- Bisection of angles.
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The known Fermat primes are 3, 5, 17, 257, and 65537.

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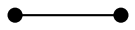

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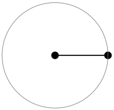

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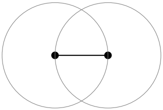

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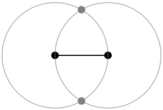

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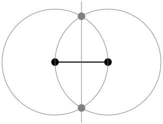

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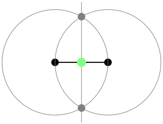

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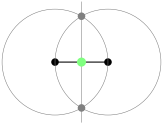

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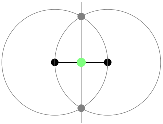

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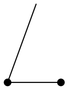

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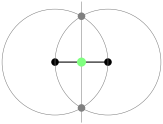

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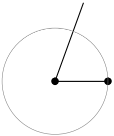

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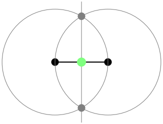

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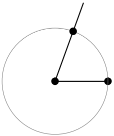

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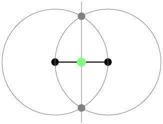

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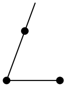

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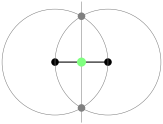

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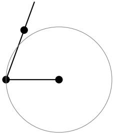

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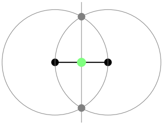

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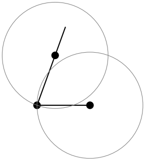

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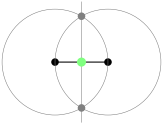

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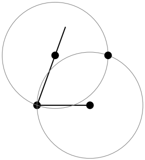

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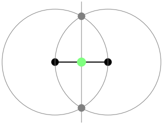

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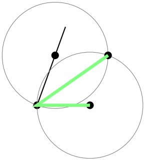
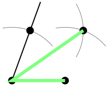


A Time-Saving Tip.

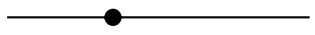
It is usually only necessary to draw short arcs of your circles:



But perhaps it is not as visually pleasing.

Perpendicular lines.

Given a line and a point, we can construct a perpendicular line:



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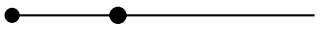

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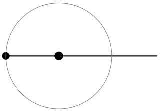

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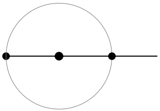

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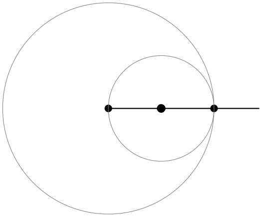

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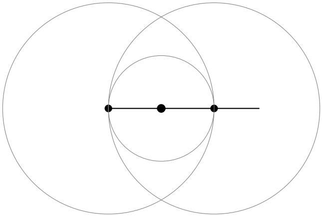

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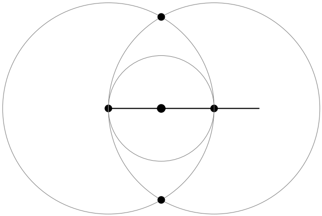

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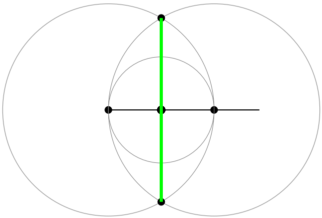

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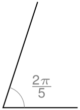
Polygons and Angles.

Suppose we wish to construct the regular pentagon. It turns out that this is equivalent to constructing the angle $2\pi/5$.



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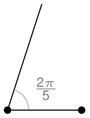

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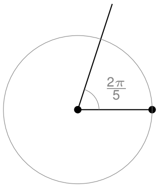

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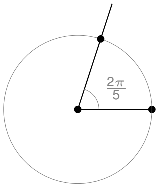

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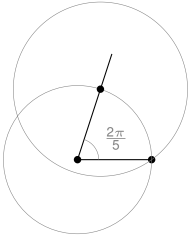

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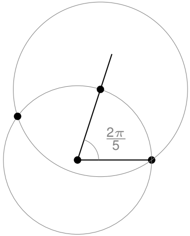

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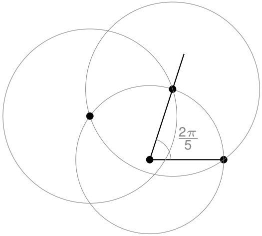

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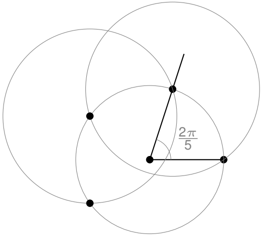

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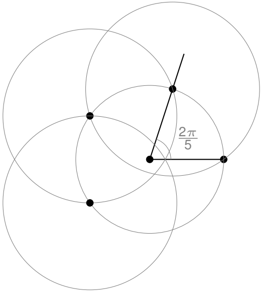

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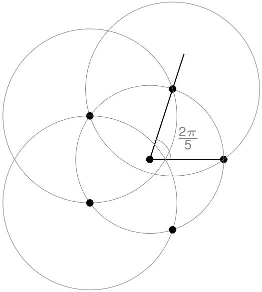

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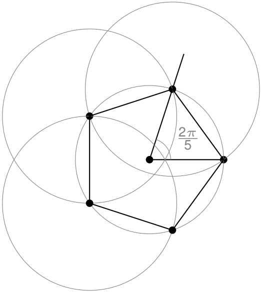

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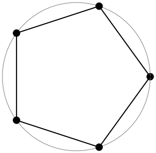

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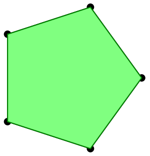

Primitive Roots of Primes.

In fact, if n is prime, we can draw the n-gon given any angle $\frac{2k\pi}{n}$, with k an integer and $0 < k < n$. We can find each of the n vertices of the polygon by copying $\frac{2k\pi}{n}$ around a circle as we did on the previous slide with $\frac{2\pi}{5}$.

Primitive Roots of Primes.

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We can understand this in group theory terms, noting that any nonzero element k of the group $\mathbb{Z}_n$ of integers mod n generates the whole group, so the sets

$$\{0, k, 2k, \ldots, (n-1)k\} = \{0, 1, 2, \ldots, n-1\}$$

are equal under mod n arithmetic.

Primitive Roots of Primes.

Careful! This is only guaranteed if n is prime.
In general, it works if $\gcd(k, n) = 1$.

Sines and Cosines.

Constructing the cosine of an angle (i.e., constructing a line segment of that length) is just as good as constructing the angle itself:

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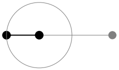

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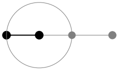

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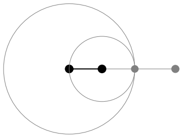

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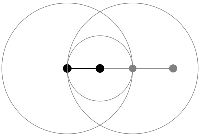

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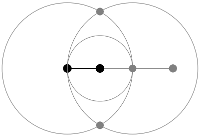

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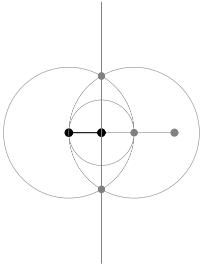

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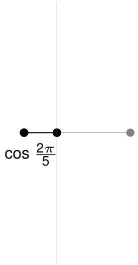

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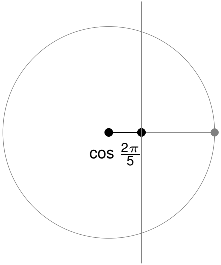

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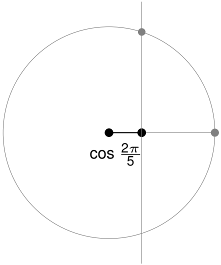

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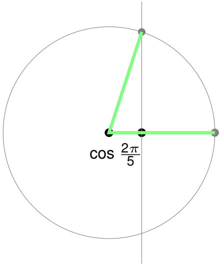

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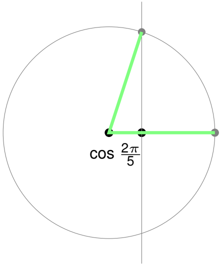

Of course, the sine is also sufficient.

Outline

1. Background
   - Straightedge-And-Compass Construction
   - History
2. The Construction.
   - Basic Tools and Techniques.
   - Constructing Polygons.
   - How To Draw a Heptadecagon

A Brief Acknowledgment.

The construction method I will demonstrate today is due to Herbert William Richmond, who published it in 1893. I have only seen a few techniques, but I find his to be very elegant and I am proud to share it with you today.

Getting Started.

As we have seen, constructing the 17-gon is equivalent to constructing the length $\cos\frac{2k\pi}{17}$ for any $0 < k < 17$.
In particular, today we will directly construct

- $\cos\frac{6\pi}{17}$
- and $\cos\frac{10\pi}{17}$.

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Let's see what they look like ...

The Big, Complicated Numbers.

We have $\cos 3\alpha = \frac{1}{4096}(\sqrt{-2\sqrt{17}+34}$

$$+\sqrt{2\sqrt{-2\sqrt{17}+34}(\sqrt{17}-1)-16\sqrt{2\sqrt{17}+34}+12\sqrt{17}+68}$$

$$+\sqrt{17}-1)^3+\frac{3}{4096}((\sqrt{-2\sqrt{17}+34}$$

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The Big, Complicated Numbers.

And $\cos 5\alpha = \frac{1}{1048576}(\sqrt{-2\sqrt{17}+34}$

$+\sqrt{2\sqrt{-2\sqrt{17}+34}(\sqrt{17}-1) - 16\sqrt{2\sqrt{17}+34} + 12\sqrt{17}+68}$

$+\sqrt{17}-1)^5 + \frac{5}{524288}((\sqrt{-2\sqrt{17}+34}$

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The Easy Way

Consider the acute angle ϕ such that $\tan 4\phi = 4$ (so ϕ is about 18°).

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It can be shown that

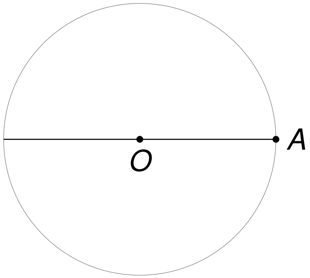
$$\tan\phi = 2(\cos 3\alpha + \cos 5\alpha)$$

and

$$\tan(\phi - \frac{\pi}{4}) = 4\,\cos 3\alpha\,\cos 5\alpha.$$

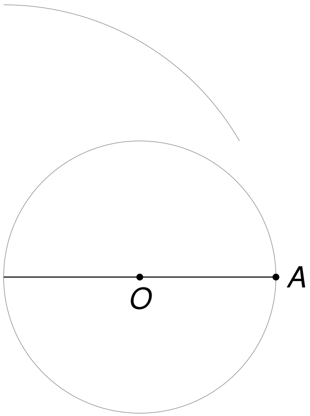
We will now construct these two lengths.

First Steps.



Start with a unit circle and arbitrary diameter. The point A is your 0^{th} vertex.
Next, find a perpendicular radius.

First Steps.



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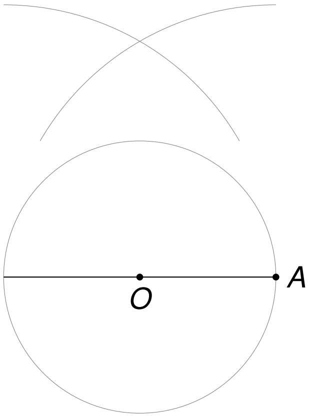

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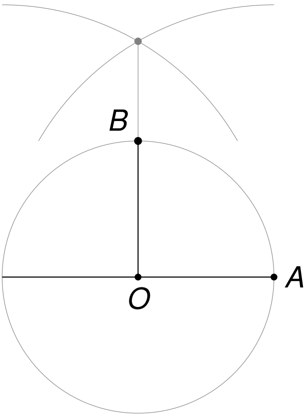

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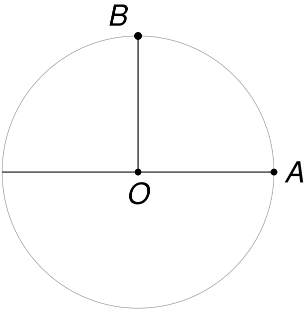

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Bisect $\overline{OB}$ twice to find I at $(0, \frac{1}{4})$.

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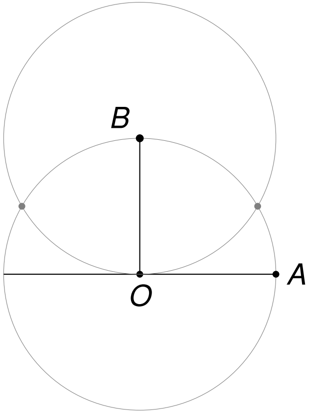

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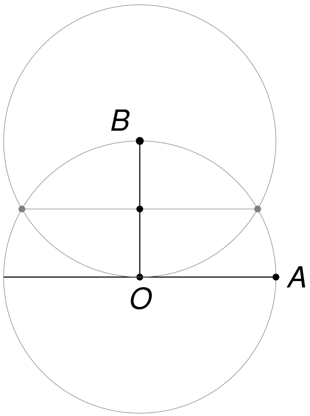

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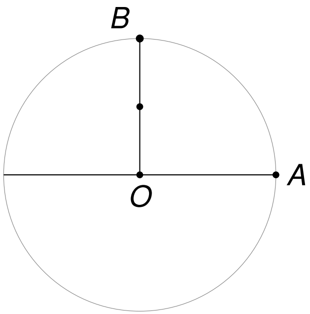

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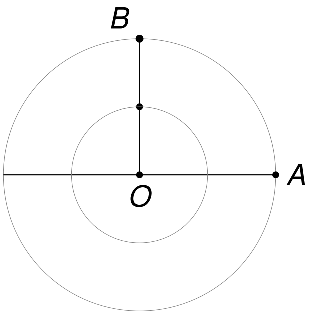

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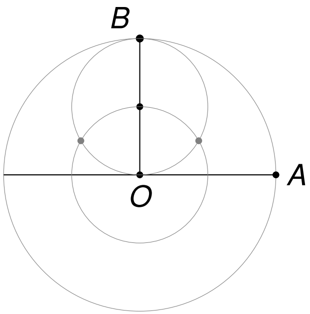

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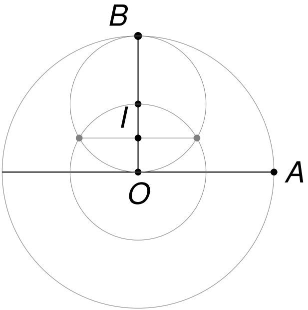

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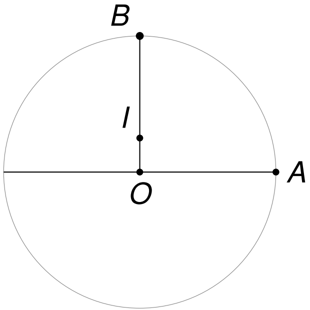

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Using your straightedge, draw the line segment $\overline{AI}$.

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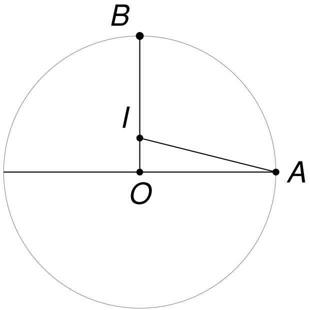

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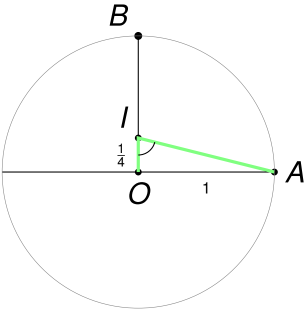
The Angle 4ϕ.

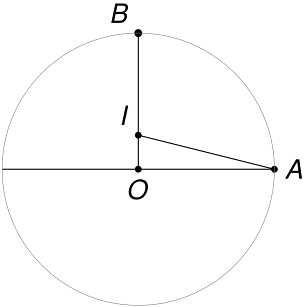


Recall that ϕ is acute and that $\tan 4\phi = 4$. Since

$$\begin{aligned}\tan \angle OIA &= \overline{OA}/\overline{OI} \\ &= 4,\end{aligned}$$

we have already constructed the angle 4ϕ.

The Angle ϕ and Its Tangent.



Next, bisect $\angle OIA$ twice to construct ϕ.

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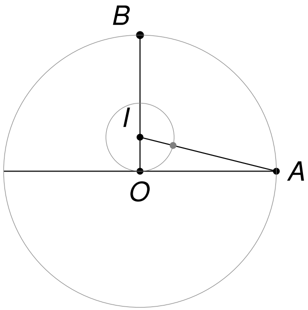

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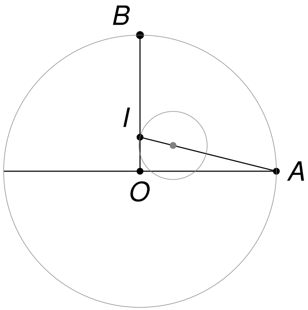

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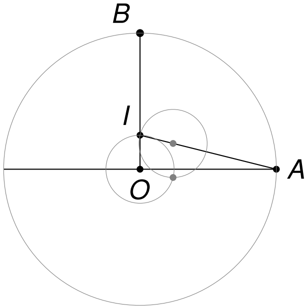

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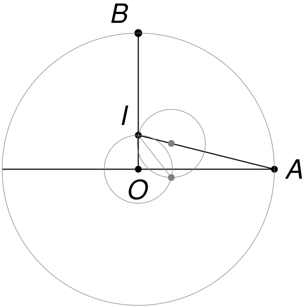

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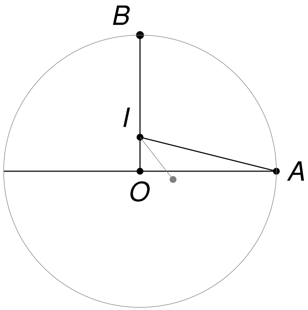

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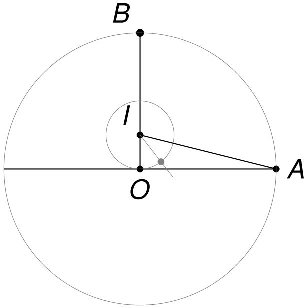

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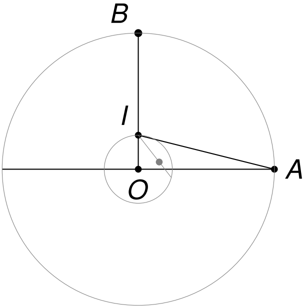

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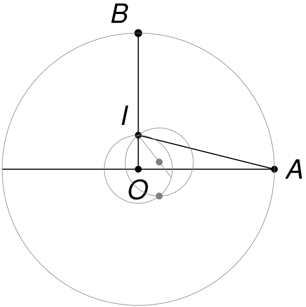

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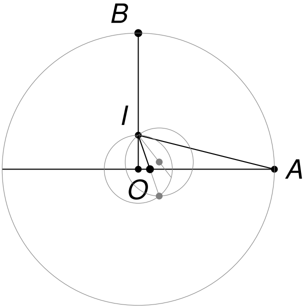

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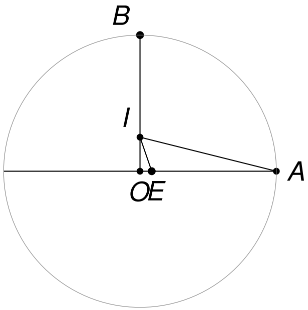

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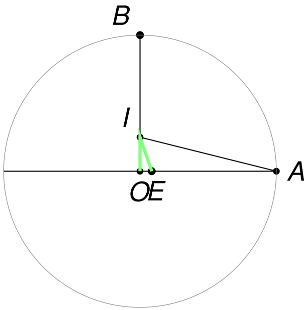
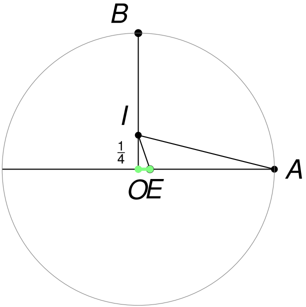


Next, bisect $\angle OIA$ twice to construct ϕ.
There it is, $\phi = \angle OIE$.

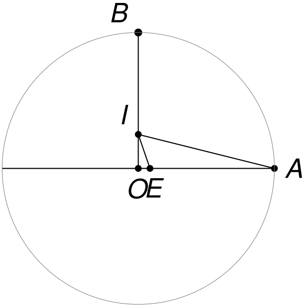
The Angle ϕ and Its Tangent.



Next, bisect $\angle OIA$ twice to construct ϕ.
There it is, $\phi = \angle OIE$.
And in fact,

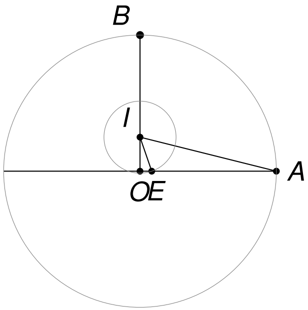
$$\begin{aligned} \tan\phi &= \tan\angle OIE \\ &= \overline{OE}/\overline{OI} \\ &= 4\,\overline{OE}. \end{aligned}$$

The Angle $\phi - \frac{\pi}{4}$.



To construct $\phi - \frac{\pi}{4}$, begin by finding a perpendicular line segment to $\overline{EI}$ through I.

The Angle $\phi - \frac{\pi}{4}$.



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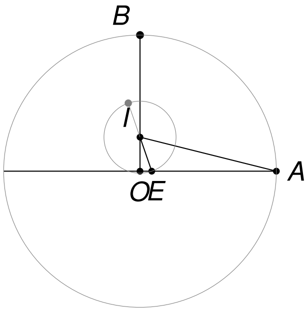

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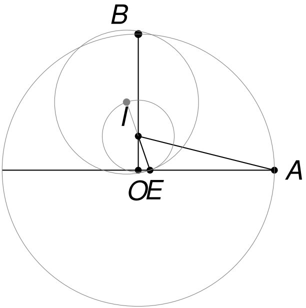

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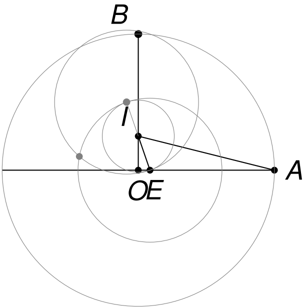

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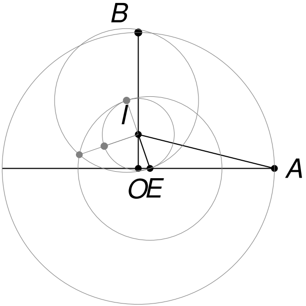
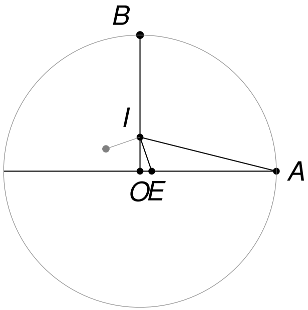


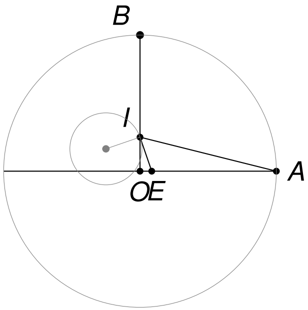
To construct $\phi - \frac{\pi}{4}$, begin by finding a perpendicular line segment to $\overline{EI}$ through I.

The Angle $\phi - \frac{\pi}{4}$.



To construct $\phi - \frac{\pi}{4}$, begin by finding a perpendicular line segment to $\overline{EI}$ through I.
Next, bisect that right angle ...

The Angle $\phi - \frac{\pi}{4}$.



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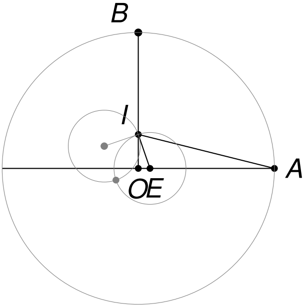

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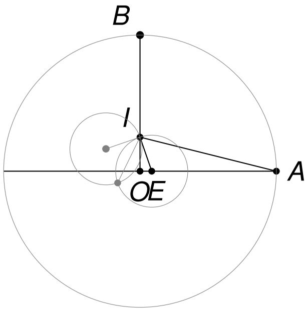

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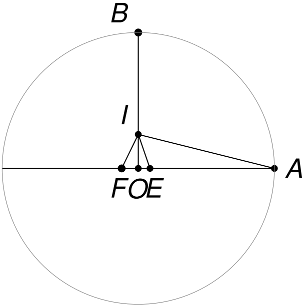

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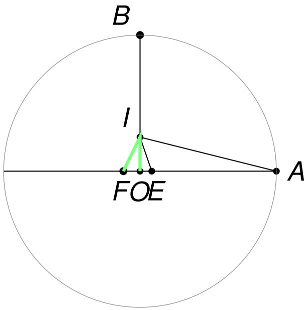
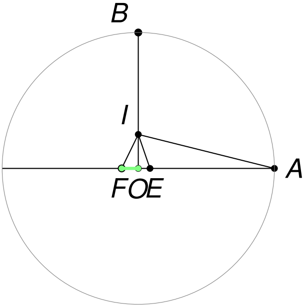


To construct $\phi - \frac{\pi}{4}$, begin by finding a perpendicular line segment to $\overline{EI}$ through I.

Next, bisect that right angle ...

And you have it.

The Angle $\phi - \frac{\pi}{4}$.



To construct $\phi - \frac{\pi}{4}$, begin by finding a perpendicular line segment to $\overline{EI}$ through I.

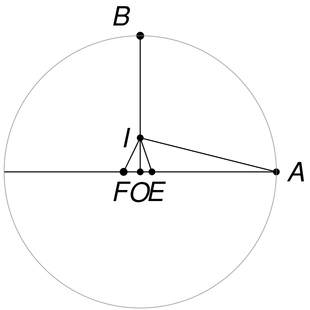
Next, bisect that right angle ...

And you have it.

Further, notice that

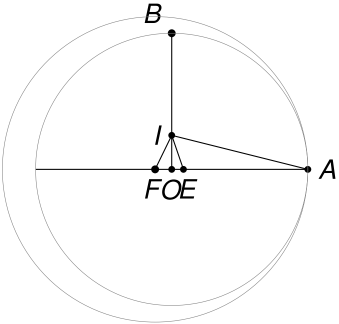
$$\tan(\phi - \frac{\pi}{4}) = 4\overline{OF}.$$

Nearly There.



Now we use what we know about the relationships among $\cos 3\alpha$, $\cos 5\alpha$, $\tan\phi$, and $\tan(\phi - \frac{\pi}{4})$ to construct the two lengths we want.
Draw a circle with diameter $\overline{AF}$.

Nearly There.



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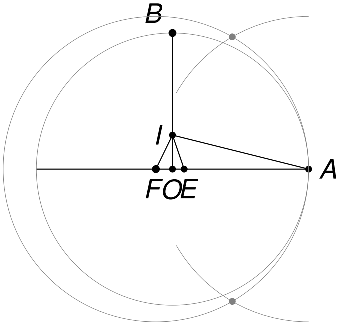

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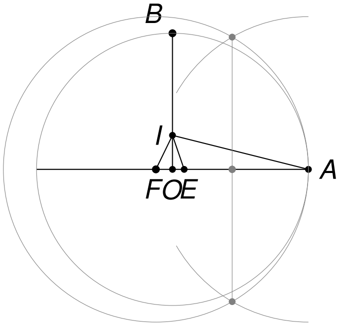

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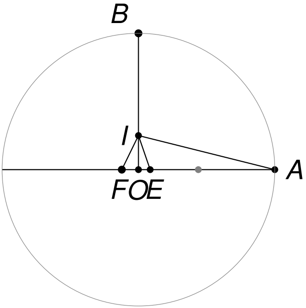

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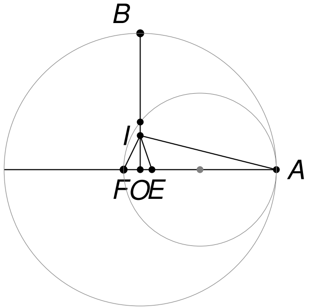

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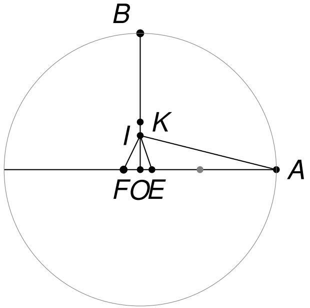

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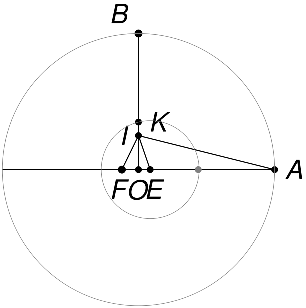

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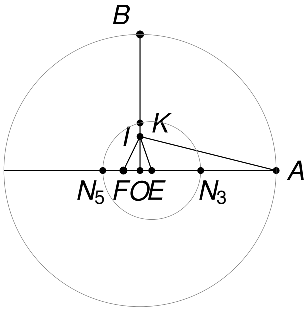
Drumroll . . .



This last circle intersects our original diameter at N_3 and N_5. I now claim that

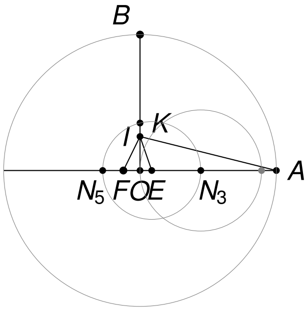
$$\overline{ON_3} = \cos 3\alpha$$

and

$$\overline{ON_5} = -\cos 5\alpha.$$

There is only one thing left to do!

Drumroll . . .



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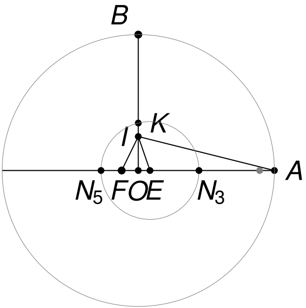

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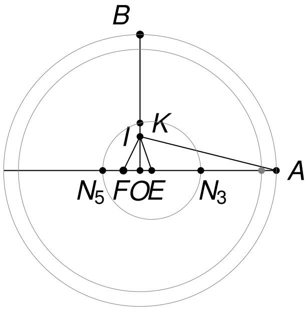

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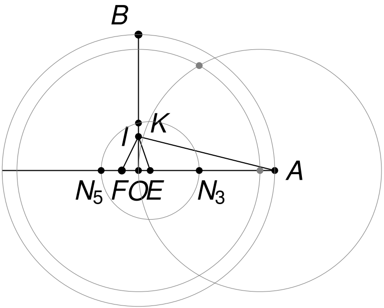

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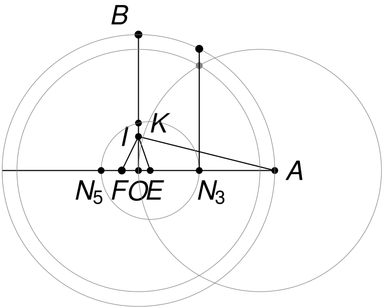

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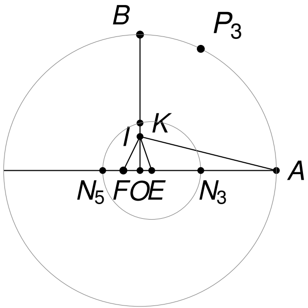

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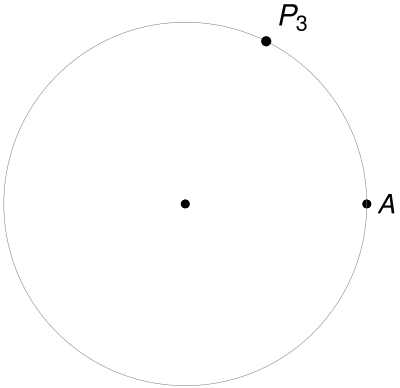
The Big Finish.



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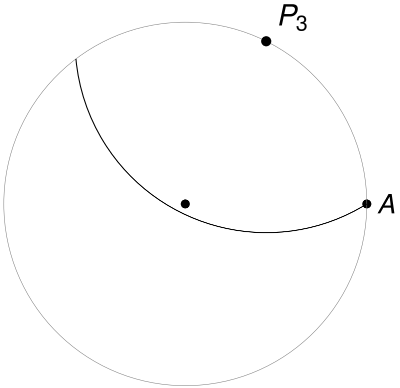

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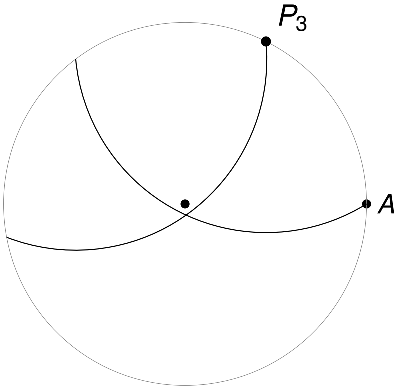

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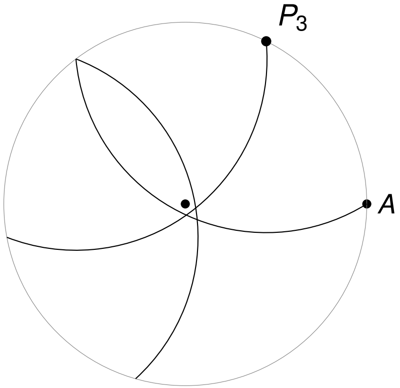

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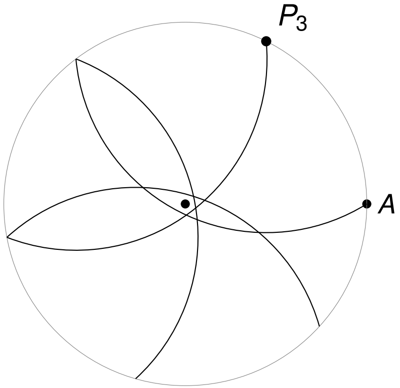

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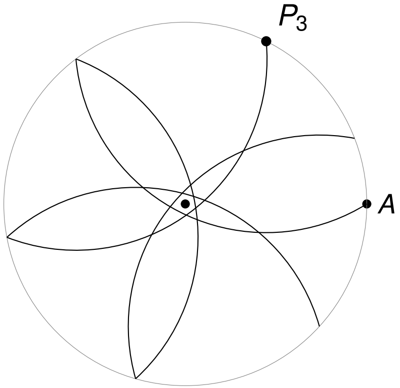

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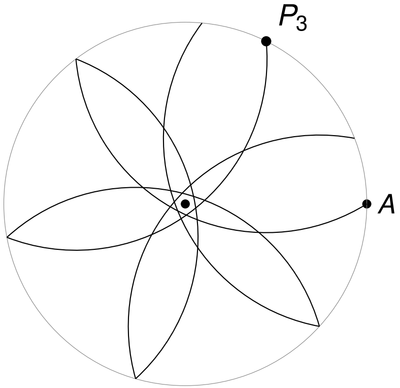

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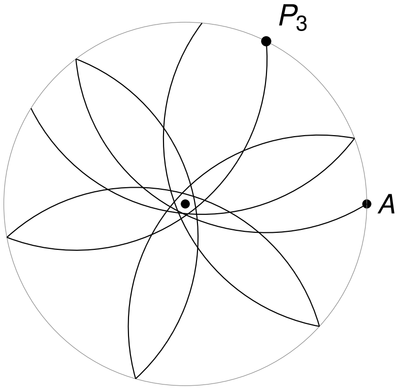

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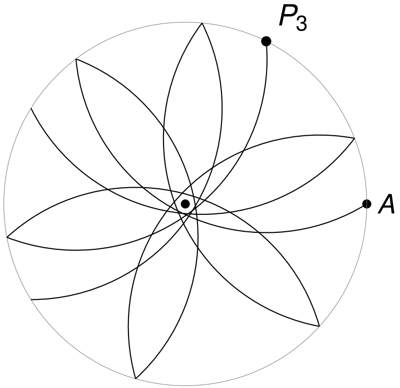

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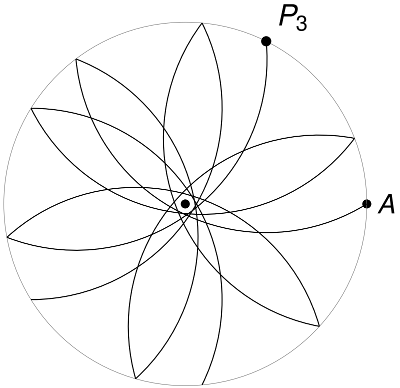

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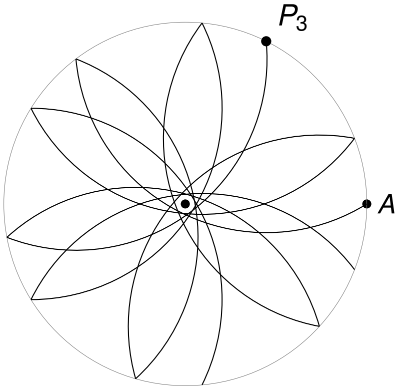

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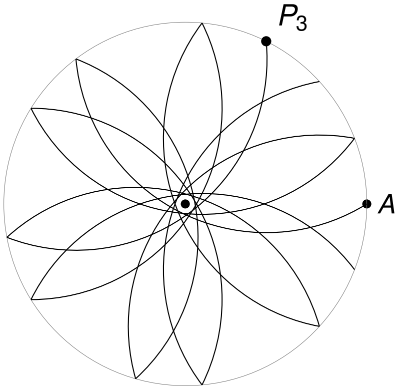

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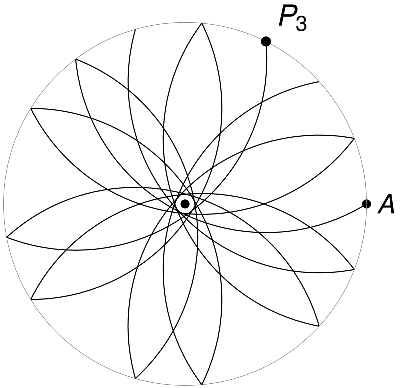

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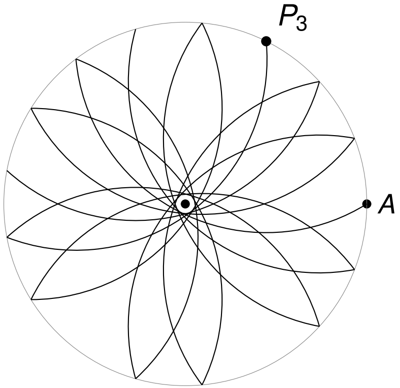

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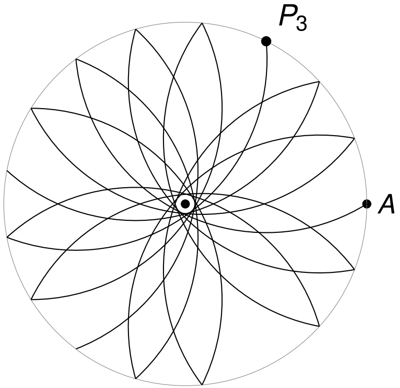

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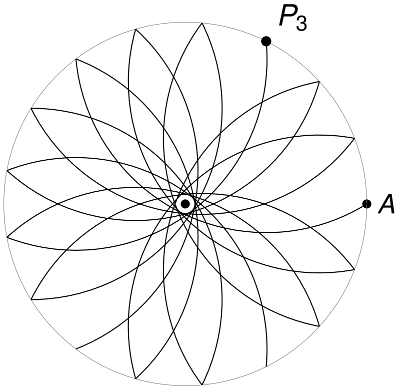

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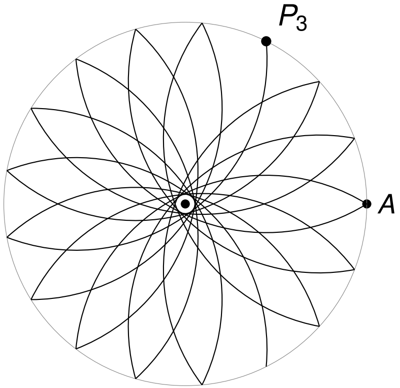

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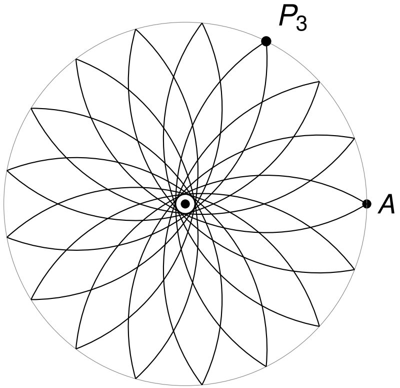

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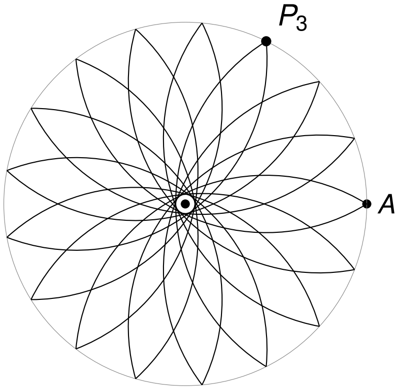

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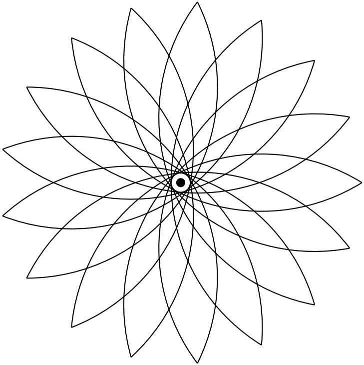

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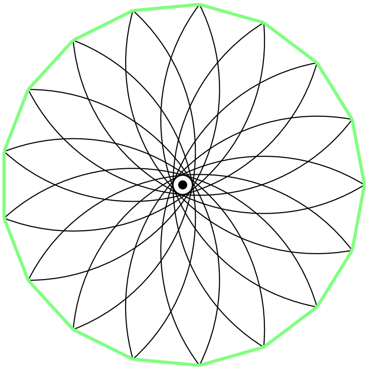

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