Macro Impacts

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The linearized elastic wave equations are (Hemmann)

$$0 = (\lambda + 2\mu)\nabla^2\phi - \rho\ddot{\phi} + f\,, \tag{1}$$

$$0 = \mu\nabla(\nabla\cdot A) - \mu\nabla\times(\nabla\times A) - \rho\ddot{A} + G\,, \tag{2}$$

where $u = \nabla\phi + \nabla\times A$ is the displacement and $F = \nabla f + \nabla\times G$ is the body force. Hence, we see to first order that shear waves and compressional waves decouple in the linear regime. If we assume that displacements due to the impact are very small (as we would expect for a very dense macro impact), then we may assume hydrostatic equilibrium, in which case the stress tensor takes the form

$$-p\delta_{ij} = \tau_{ij} = \lambda g_{ij} u^s_{,s} + \mu(u_{i,j} + u_{j,i})\,. \tag{3}$$

Note, the commas denote covariant differentiation. We now transform into cylindrical coordinates. In the case that we are interested in, the only important component is the radial one, for which we have

$$-p = \tau_{rr}\,, \tag{4}$$

$$= \lambda u^s_{,s} + 2\mu u_{r,r}\,, \tag{5}$$

$$= \lambda\nabla u + 2\mu u_{r,r}\,. \tag{6}$$

The metric of cylindrical coordinates is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2\,, \tag{7}$$

leading to the Christoffel symbols

$$\Gamma^r_{\theta\theta} = -r\,, \tag{8}$$

$$\Gamma^r_{r\theta} = \frac{1}{r}\,. \tag{9}$$

Thus, assuming only the radial component has non-zero derivatives (which will be the case for a cylindrically symmetric source),

$$\nabla u = \partial_r u_r + \frac{1}{r}u_r\,, \tag{10}$$

$$= \frac{1}{r}\partial_r(ru_r)\,. \tag{11}$$

Hence

$$-p = \tau_{rr} = \lambda\frac{1}{r}\partial_r(ru_r) + 2\mu\partial_r u_r\,, \tag{12}$$

$$= \lambda\frac{1}{r}\partial_r(ru_r) + 2\mu\frac{1}{r}(\partial_r r u_r) - 2\mu\frac{1}{r}(u_r)\,, \tag{13}$$

$$= (\lambda + 2\mu)\frac{1}{r}\partial_r(ru_r) - 2\mu\frac{1}{r}u_r\,, \tag{14}$$

$$= (\lambda + 2\mu)\frac{1}{r}\partial_r(r\partial_r\phi) - 2\mu\frac{1}{r}\partial_r\phi \tag{15}$$

With $f = 0$, this and the scalar equation of motion yield

$$0 = (\lambda + 2\mu)\nabla^2\phi - \rho\ddot{\phi}\,, \tag{16}$$

$$= (\lambda + 2\mu)\left(\frac{1}{r}\partial_r\left(r\partial_r\phi\right)\right) - \rho\ddot{\phi}\,, \tag{17}$$

$$= -p + 2\mu\frac{1}{r}\partial_r\phi - \rho\ddot{\phi}\,, \tag{18}$$

Thus, observe that all the solutions from Hemmann apply with small modification (compare 19 with his equation 9).

Assuming only outgoing modes are generated, we may write $\phi = F(t - r/\alpha)$ where $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ is the compressional wave velocity. We can, for the sake of reusing Hemmann's work, assume that the cavity produced by the macro existed prior to the generation of compressional modes. This is actually somewhat convenient, since it is the production of the cavity that generates shear modes. Denote differentiation with respect to the retarded time $t' = t - \frac{r}{\alpha}$ by a prime. I can't figure out how to get the following from equation 11 in Hemmann, but the evolution equation is

$$-r^3\alpha p = \alpha\rho r^2 F'' + 2\mu r F' + 2\mu\alpha F\,. \tag{19}$$

After some guessing and checking, it seems that one can arrive at this equation by the substitution $\phi = \frac{1}{r}F(t - r/\alpha)$. Either way, this simply confirms that we can use Hemmann directly with only slight modification. In particular, replace μ in Hemmann's paper with $\mu/2$.

The equation between 43 and 44 provides that

$$S_r = -\lambda\dot{u}_r\nabla u - 2\mu u_r\partial_r u_r\,, \tag{20}$$

$$= -\lambda\dot{u}_r\partial_r u_r - \frac{\lambda}{r}\dot{u}_r u_r - 2\mu u_r\partial_r u_r\,, \tag{21}$$

$$= -(\lambda + 2\mu)\dot{u}_r\partial_r u_r - \frac{\lambda}{r}\dot{u}_r u_r\,. \tag{22}$$

Again, this only differs from his work by a factor of two on the right-most term. However, in this case, it is not so simple as substituting $\mu/2$ for μ. In this case, we may need to follow through some calculations with the modification unless there is another simple change we can make. However, he appears to drop the problem term in the following steps...

Now, the obvious modification to calculate Q, the total energy radiated from the source, is that instead of integrating over a spherical shell, we must integrate over the length of the explosion and the circle of radius r. This yields

$$Q = 2\pi r L\int_{-\infty}^{\infty} S_r\,\mathrm{d}t\,. \tag{23}$$

Dropping the terms as in Hemmann, we find that

$$Q = -2\pi r L(\lambda + 2\mu)\int_{-\infty}^{\infty} u_r\partial_r u_r\,\mathrm{d}t\,. \tag{24}$$

Observe, our version of equation 34 will be different by factors of r since he had to undo the spherical gradient while we have to undo only the polar gradient. I will go through this tomorrow.