Homework

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1 Scalar Modes

In the following, lower case denotes a quantity in position space while capital letters denote their components in Fourier space.

Denote the displacement field $u(\vec{x},t) = \nabla\phi(\vec{x},t) + \nabla \times a(\vec{x},t)$. The linearized (acoustic) wave equation for ϕ is then

$$\alpha^2 \nabla^2 \phi = \partial_t^2 \phi \,, \tag{1}$$

where $\alpha^2 = \frac{\lambda + 2\mu}{\rho}$. We may write

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\vec{k},\omega) e^{\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} \,, \tag{2}$$

from which we obtain the dispersion relation $\alpha k = \omega$, and hence the solution

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\vec{k},\omega) e^{\mathrm{i}(\vec{k}\cdot\vec{x} - \alpha k t)} \,. \tag{3}$$

We now impose the constraint equation

$$\sigma_{ij} = \delta_{ij} \lambda \nabla \cdot u + \mu(u_{i,j} + u_{j,i}) \tag{4}$$

which, for the scalar modes becomes

$$\sigma_{ij} = \delta_{ij} \lambda \nabla^2 \phi + 2\mu \partial_i \partial_j \phi \tag{5}$$

and whose trace is

$$-p = K \nabla^2 \phi \,, \tag{6}$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus and $p = -\frac{1}{3} \operatorname{tr} \sigma_{ij}$. We take this constraint as an initial condition at $t = 0$. In Fourier space

$$P(\vec{k}) = K k^2 \Phi(\vec{K}) \,, \tag{7}$$

from which we obtain the displacement field components

$$U(\vec{k}) = \mathrm{i} \frac{P(\vec{k})}{K} \frac{\vec{k}}{k^2} \,, \tag{8}$$

and hence

$$u(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2}e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)}\,, \tag{9}$$

The equation for the energy of the compressional modes is

$$E = \frac{1}{2}\int_V \mathrm{d}^3x\left(\rho|\partial_t u|^2 + (\lambda+2\mu)|\nabla\cdot u|^2\right)\,, \tag{10}$$

$$= \frac{\lambda+2\mu}{K^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|P(\vec{k})\right|^2 \tag{11}$$

In the case that P depends only on the frequency

$$E = 4\pi\frac{\lambda+2\mu}{K^2}\int\frac{\mathrm{d}\tilde{\lambda}}{\tilde{\lambda}^4}\left|P(\tilde{\lambda})\right|^2. \tag{12}$$

Now consider the case of a step function type pressure source - a cylinder of height h and radius r_X

$$p(\vec{x},0) = p_0\theta(r_X - r)\left[\theta(z+h/2)-\theta(z-h/2)\right]\,, \tag{13}$$

whose Fourier components are

$$P(\vec{k}) = \frac{4\pi r_X p_0}{\sqrt{k_x^2+k_y^2}k_z}J_1\left(\sqrt{k_x^2+k_y^2}r_X\right)\sin\left(\frac{h}{2}k_z\right)\,, \tag{14}$$

which, in polar k-space is

$$P(k,\varphi,\theta) = \frac{4\pi r_X p_0}{k^2\sin\theta\cos\theta}J_1\left(r_X k\sin\theta\right)\sin\left(\frac{h}{2}k\cos\theta\right)\,, \tag{15}$$

Directly integrating in Mathematica (over θ first) yields

$$E_{\mathrm{total}} = \frac{\lambda+2\mu}{K^2}4\pi^2p_0^2\sigma_X h \tag{16}$$

where $\sigma_X = \pi r_X^2$. To calculate the energy deposition into the low frequency spectrum, we integrate k from 0 to k_0. Observe, we can make the following approximation

$$P(\vec{k}) \approx_{k\ll r_X} \frac{2\pi r_X p_0}{k^2\cos\theta\sin\theta}r_X k\sin\theta\sin\left(\frac{h}{2}k\cos\theta\right)\,, \tag{17}$$

$$= \frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right)\,. \tag{18}$$

From this we obtain the portion of the energy relegated to the long wavelength spectrum

$$E_{\mathrm{propagated}} = \frac{\lambda+2\mu}{K^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|\frac{2\pi r_X^2p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right)\right|^2, \tag{19}$$

$$= \left[\frac{1}{2h}(r_X^2p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[hk_0\cos(hk_0)+\sin(hk_0)+hk_0\left(-2+hk_0\mathrm{Si}(hk_0)\right)\right]\,, \tag{20}$$

$$= \left[(r_X^2p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi}{\lambda_0}\cos\left(\frac{2\pi h}{\lambda_0}\right)+\sin\left(\frac{2\pi h}{\lambda_0}\right)+\left(\frac{\pi}{\lambda_0}\right)\left(-2+\left(\frac{2\pi h}{\lambda_0}\right)\mathrm{Si}\left(\frac{2\pi h}{\lambda_0}\right)\right)\right]\,, \tag{21}$$

$$\approx_{h\gg\lambda_0\gg 1}\left[(r_X^2p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi^3h}{\lambda_0^2}\right]\,, \tag{22}$$

$$= \left[(\sigma_Xp_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi h}{\lambda_0^2}\right]\,. \tag{23}$$

From this we obtain the fractional energy deposition into the unattenuated wavelengths

$$\Xi = \frac{\sigma_X}{4\pi\lambda_0^2} \, . \tag{24}$$

This approximation holds for $\lambda_0^2 \gg \sigma_X$, which is appropriate for the case that λ_0 is on the order of kilometers and σ_X is on the order of centimeters squared.

2 Vector Modes

The following is the equation of motion for the vector potential a without body force:

$$\mu\nabla^2 a - \rho\ddot{a} = \mu\nabla(\nabla\cdot a) - \mu\nabla\times(\nabla\times a) - \rho\ddot{a} = 0\,, \tag{25}$$

where the leftmost term is the vector laplacian. This gives us three wave equations for the three components of a. The constraint equation takes the form

$$\begin{aligned}
\sigma_{ij} &= \lambda\delta_{ij}\nabla\cdot\nabla\times a + \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j)\,, && (26)\\
&= \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j)\,, && (27)\\
&= \mu\begin{pmatrix} 2\partial_x(\partial_y a_z - \partial_z a_y) & (\partial_y^2 - \partial_x^2)a_z + \partial_z(\partial_x a_x - \partial_y a_y) & (\partial_x^2 - \partial_z^2)a_y + \partial_y(\partial_z a_z - \partial_x a_x) \\ & 2\partial_y(\partial_z a_x - \partial_x a_z) & (\partial_z^2 - \partial_y^2)a_x + \partial_x(\partial_y a_y - \partial_z a_z) \\ & & 2\partial_z(\partial_x a_y - \partial_y a_x) \end{pmatrix}\,. && (28)
\end{aligned}$$