Realistic Macro Impacts

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1 The Green's Function

This should be correct. We have the differential equation

$$v(r)^2\nabla^2\phi - \partial_t^2\phi = -\frac{f}{\rho} \tag{1}$$

We want to write ϕ as eigenfunctions of the linear differential operator on the LHS. Fourier transforming in time $\phi \to \tilde{\phi}$, we may write an eigenvalue problem

$$v(r)^2\nabla^2\tilde{\phi} + \lambda\tilde{\phi} = 0\,. \tag{2}$$

Multiplying by $\frac{r^2}{v(r)^2\tilde{\phi}}$ and supposing $\tilde{\phi} = R(r)Y(\Omega)$, we have

$$\frac{\partial_r(r^2\partial_r R)}{R} + \lambda\frac{r^2}{v(r)^2} + \frac{r^2\nabla_\Omega Y}{Y} = 0 \tag{3}$$

It is clear that $\frac{r^2\nabla_\Omega Y}{Y}$ depends only on Ω and that the remaining terms only depend on r, hence the equation is separable, and our ansatz is justified. The solutions Y are just the spherical harmonics with eigenvalues $r^2\nabla_\Omega Y = -l(l+1)Y$, $l \in \mathbb{N} \cup \{0\}$. R obeys a more complicated equation

$$\partial_r(r^2\partial_r R) + \lambda\frac{r^2}{v(r)^2}R - l(l+1)R = 0 \tag{4}$$

which is just a Sturm–Liouville problem for the appropriate boundary conditions at $r = 0$ and $r = \frac{a}{b}$. The weight function is $\rho = \frac{r^2}{v(r)^2}$. Since this equation is in terms of a Sturm–Liouville operator, its eigenfunctions form an orthogonal set with respect to ρ, i.e.

$$\delta(\lambda - \lambda') = \int \mathrm{d}r\rho(r)\bar{R}_\lambda(r)R_{\lambda'}(r)\,. \tag{5}$$

Assuming completeness of the eigenfunctions, we may write

$$\delta(r - r') = \int \mathrm{d}\lambda\rho(r)\bar{R}_\lambda(r)R_\lambda(r')\,. \tag{6}$$

We may decompose any function in terms of R, Y, and $e^{\mathrm{i}\omega t}$ as

$$f(r,\Omega,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int \mathrm{d}\lambda\,\mathrm{d}\omega F_{lm}(\lambda,\omega)R_{\lambda l}(r)Y_{lm}(\Omega)e^{\mathrm{i}\omega t}\,. \tag{7}$$

Notice

$$\left[v(r)^2\nabla^2 - \partial_t^2\right]\left[R_\lambda(r)Y_{lm}(\Omega)e^{\mathrm{i}\omega t}\right] = R_\lambda(r)Y_{lm}(\Omega)e^{\mathrm{i}\omega t}\left[\frac{v(r)^2\nabla_r^2 R}{R} + \frac{v(r)^2\nabla_\Omega^2 Y}{Y} + \omega^2\right], \tag{8}$$

$$= R_\lambda(r)Y_{lm}(\Omega)e^{\mathrm{i}\omega t}\left[-\lambda + \frac{v(r)^2}{r^2}l(l+1) - \frac{v(r)^2}{r^2}l(l+1) + \omega^2\right], \tag{9}$$

$$= R_\lambda(r)Y_{lm}(\Omega)e^{\mathrm{i}\omega t}\left[-\lambda + \omega^2\right]. \tag{10}$$

Thus, we see that

$$G(\vec{r},t;\vec{r}',t') = \rho(r)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{a^2b^2}^{\infty}\mathrm{d}\lambda\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\bar{R}_{\lambda l}(r)R_{\lambda l}(r')\bar{Y}_{lm}(\Omega)Y_{lm}(\Omega')e^{\mathrm{i}\omega(t-t')}}{\omega^2-\lambda-\mathrm{i}\epsilon}. \tag{11}$$

This may be incorrect because of the dependence of R on l... However, we can probably formulate (4) as a S-L problem with eigenvalue $-l(l+1)$ instead, and we'll obtain a second orthogonality relation for R. This should mean that (11) is correct up to a weight function.

2 The Source

We must find an appropriate body force potential f. As a practice problem, let's consider $f = \nabla g = -4\pi G\left[M\delta(\vec{r}-\vec{v}t) + \delta\rho\right]$ as in the detection of a small black hole problem, where $\delta\rho = -\rho\nabla u$, where u is the displacement field. Then the source term of the wave equation is

$$-\frac{f}{\rho} = \frac{4\pi MG}{\rho}\delta(\vec{r}-\vec{v}t) - 4\pi G\phi \tag{12}$$

Since f has a dependent variable in it, we need to slightly modify our Green's function. The eigenvectors $\tilde{u}$ of $\tilde{D}$ will now satisfy