Macro Impacts

July 16, 2016

1 Scalar Modes

In the following, lower case denotes a quantity in position space while capital letters denote their components in Fourier space.

Denote the displacement field $u(\vec{x},t) = \nabla\phi(\vec{x},t) + \nabla \times a(\vec{x},t)$. The linearized (acoustic) wave equation for ϕ is then

$$\alpha^2\nabla^2\phi = \partial_t^2\phi\,, \tag{1}$$

where $\alpha^2 = \frac{\lambda+2\mu}{\rho}$. We may write

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}\,, \tag{2}$$

from which we obtain the dispersion relation $\alpha k = \omega$, and hence the solution

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)}\,. \tag{3}$$

We now impose the constraint equation

$$\sigma_{ij} = \delta_{ij}\lambda\nabla\cdot u + \mu(u_{i,j}+u_{j,i}) \tag{4}$$

which, for the scalar modes becomes

$$\sigma_{ij} = \delta_{ij}\lambda\nabla^2\phi + 2\mu\partial_i\partial_j\phi \tag{5}$$

and whose trace is

$$-p = K\nabla^2\phi = K\nabla\cdot u\,, \tag{6}$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus and $p = -\frac{1}{3}\,\mathrm{tr}\,\sigma_{ij}$. We take this constraint as an initial condition at $t=0$. In Fourier space

$$P(\vec{k}) = Kk^2\Phi(\vec{K})\,, \tag{7}$$

from which we obtain the displacement field components

$$U(\vec{k}) = \mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2}\,, \tag{8}$$

and hence

$$u(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2}e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)}\,, \tag{9}$$

The equation for the energy of the compressional modes is

$$E = \frac{1}{2}\int_V \mathrm{d}^3x \left(\rho|\partial_t u|^2 + (\lambda+2\mu)|\nabla\cdot u|^2\right) , \tag{10}$$

$$= \frac{\lambda+2\mu}{K^2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left|P(\vec{k})\right|^2 \tag{11}$$

In the case that P depends only on the frequency

$$E = 4\pi\frac{\lambda+2\mu}{K^2}\int \frac{\mathrm{d}\tilde{\lambda}}{\tilde{\lambda}^4}\left|P(\tilde{\lambda})\right|^2 . \tag{12}$$

Observe, the total energy can be calculated directly from p as it is always true that half the energy is spring potential and half is kinetic.

$$E = \frac{\lambda+2\mu}{K^2}\int_V \mathrm{d}^3x(p^2) \tag{13}$$

Now consider the case of a step function type pressure source - a cylinder of height h and radius r_X

$$p(\vec{x},0) = p_0\theta(r_X - r)\left[\theta(z+h/2) - \theta(z-h/2)\right] , \tag{14}$$

whose Fourier components are

$$P(\vec{k}) = \frac{4\pi r_X p_0}{\sqrt{k_x^2+k_y^2}k_z}J_1\left(\sqrt{k_x^2+k_y^2}r_X\right)\sin\left(\frac{h}{2}k_z\right) , \tag{15}$$

which, in polar k-space is

$$P(k,\varphi,\theta) = \frac{4\pi r_X p_0}{k^2\sin\theta\cos\theta}J_1\left(r_X k\sin\theta\right)\sin\left(\frac{h}{2}k\cos\theta\right) , \tag{16}$$

The total energy is clearly

$$E_{\mathrm{total}} = \frac{\lambda+2\mu}{K^2}p_0^2\sigma_X h \tag{17}$$

where $\sigma_X = \pi r_X^2$. To calculate the energy deposition into the low frequency spectrum, we integrate k from 0 to k_0. Observe, we can make the following approximation

$$P(\vec{k}) \approx_{k\ll r_X} \frac{2\pi r_X p_0}{k^2\cos\theta\sin\theta}r_X k\sin\theta\sin\left(\frac{h}{2}k\cos\theta\right) , \tag{18}$$

$$= \frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right) . \tag{19}$$

From this we obtain the portion of the energy relegated to the long wavelength spectrum

$$E_{\mathrm{propagated}} = \frac{\lambda+2\mu}{K^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|\frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right)\right|^2 , \tag{20}$$

$$= \left[\frac{1}{2h}(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[hk_0\cos(hk_0) + \sin(hk_0) + hk_0\left(-2 + hk_0\mathrm{Si}(hk_0)\right)\right] , \tag{21}$$

$$\approx_{h\gg\lambda_0\gg 1}\left[(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi^3 h}{\lambda_0^2}\right] , \tag{22}$$

$$= \left[(\sigma_X p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi h}{\lambda_0^2}\right] . \tag{23}$$

From this we obtain the fractional energy deposition into the unattenuated wavelengths

$$\Xi = \frac{\sigma_X}{\lambda_0^2} \, . \tag{24}$$

This approximation holds for $\lambda_0^2 \gg \sigma_X$, which is appropriate for the case that λ_0 is on the order of kilometers and σ_X is on the order of centimeters squared.

It is wrong, however, to assume that the wave evolves linearly close to the source. Modes will be coupled to one another, and our expression (24) will only hold in a small neighborhood of the event. Effects such as heating and rock fracturing will cause the high frequency modes in the shock to rapidly attenuate. They are, however, difficult to quantify. It is not controversial to say that the energy of the shock after its non-linear evolution (when the overpressure exceeds the elastic limit of the Earth) will be less than its initial energy. It is also well known that the behavior of shockwaves for long time tends towards a sharp wave-front with a linearly decreasing tail. To produce an over-estimate of the detectible energy, we hypothesize an approximate pressure waveform and endow it with energy equivalent to that of the initial blast. Then, we calculate the energy per mode, and sum over only those modes whose frequency is in the regime that will not rapidly attenuate.

We describe the long-time waveform of the shock by

$$p = \bar{p}\frac{r - r_0}{\Delta r}\left[\theta(r_0 + \Delta r - r) - \theta(r + \Delta r)\right]\left[\theta(z + h/2) - \theta(z - h/2)\right] . \tag{25}$$

Δr is the length of the tail, and r_0 is its base. $\bar{p}$ is the peak pressure of the shock, and we can take it to be the stress corresponding to the elastic limit of rock. The total energy of this pressure wave is

$$E_{\text{total}} = \frac{\lambda + 2\mu}{K^2}\frac{1}{6}\bar{p}^2\pi\Delta r h(4r_0 + 3\Delta r) . \tag{26}$$

We require that this be equal to the initial total energy. From this, we can obtain an expression for r_0 in terms of Δr.

$$r_0 = \frac{3(2p_0^2 r_X^2 - \bar{p}^2\Delta r^2)}{4\bar{p}^2\Delta r} . \tag{27}$$

A lower bound for r_0 is clearly 0, which sets an upper bound for Δr, i.e.

$$\Delta r \leq \frac{\sqrt{2}p_0 r_X}{\bar{p}} , \tag{28}$$

$$= \frac{1}{\bar{p}}\sqrt{\frac{2}{\pi}\frac{K^2}{\lambda + 2\mu}\left|\frac{dE}{dx}\right|} . \tag{29}$$

Plugging in some numbers (from below) yields

$$\Delta r \leq 1.2\text{m} . \tag{30}$$

The energy in the longest modes is, for $h \gg \lambda_0 \gg r_0 + \Delta r$

$$E_{\text{propagated}} \approx \frac{\lambda + 2\mu}{K^2}\frac{\bar{p}^2\pi^3 h\Delta r^2(3r_0 + 2\Delta r)^2}{9\lambda_0^2} . \tag{31}$$

Then we have that the fraction of energy that will reach seismometers is

$$\Xi = \frac{2\pi^2\Delta r(3r_0 + 2\Delta r)^2}{3(4r_0 + 3\Delta r)\lambda_0^2} , \tag{32}$$

$$= \pi^2\left(\frac{18p_0^2 r_X^2 - \bar{p}^2\Delta r^2}{12p_0\bar{p}r_X\lambda_0}\right)^2 \tag{33}$$

Since we know that the total energy is (assuming that the velocity of the macro remains approximately constant throughout its journey through Earth) we can calculate the initial overpressure.

$$\frac{\lambda + 2\mu}{K^2}p_0^2\sigma_X h = E_{\text{total}} = h\left|\frac{dE}{dx}\right| = h\rho_\oplus\sigma_X v_X^2 . \tag{34}$$

Thus we obtain the following expression for p_0

$$p_0 = \sqrt{\frac{\rho}{\lambda + 2\mu}} K v_X \,. \tag{35}$$

Thus, the suppression factor is

$$\Xi = \pi \frac{(\bar{p}^2 \pi \Delta r^2 (\lambda + 2\mu) - 18 K^2 \sigma_X v_X^2 \rho_\oplus)^2}{144 K^2 (\lambda + 2\mu) \rho_\oplus \bar{p}^2 \sigma_X v_X^2 \lambda_0^2} \,. \tag{36}$$

Using the upper bound for Δr from earlier

$$\Xi \geq \frac{16}{9} \left(\frac{K^2}{\lambda + 2\mu} \right) \left(\frac{\pi}{\bar{p}\lambda_0} \right)^2 \left| \frac{dE}{dx} \right| . \tag{37}$$

Taking Δr to be 0, we find

$$\Xi < \frac{9}{4} \left(\frac{K^2}{\lambda + 2\mu} \right) \left(\frac{\pi}{\bar{p}\lambda_0} \right)^2 \left| \frac{dE}{dx} \right| . \tag{38}$$

Note the strict inequality: $\Delta r = 0$ corresponds to $r_0 = \infty$.

The following are the approximate quantities used

$$h = 1.2 \times 10^7 \text{m} \,, \tag{39}$$

$$v_X = 2.5 \times 10^6 \text{m s}^{-1} \,, \tag{40}$$

$$\rho_\oplus = 6 \times 10^3 \text{kg m}^{-3} \,, \tag{41}$$

$$\sigma_X = 10^{-11} \text{m}^2 \,, \tag{42}$$

$$\left| \frac{dE}{dx} \right| = \rho_\oplus \sigma_X v_X^2 = 3.75 \times 10^5 \text{J m}^{-1} \,, \tag{43}$$

$$E_{\text{total}} = 4.5 \times 10^{12} \text{J} \,, \tag{44}$$

$$r_X = 1.8 \times 10^{-6} \text{m} \,, \tag{45}$$

$$\alpha = 6 \times 10^3 \text{m s}^{-1} \,, \tag{46}$$

$$\beta = 3.5 \times 10^3 \text{m s}^{-1} \,, \tag{47}$$

$$\mu = 7.2 \times 10^{10} \text{Pa} \,, \tag{48}$$

$$\lambda = 7.2 \times 10^{10} \text{Pa} \,, \tag{49}$$

$$K = 1.2 \times 10^{11} \text{Pa} \,, \tag{50}$$

$$\bar{p} = 10^8 \text{Pa} \,. \tag{51}$$

These lead to the results, (using my estimation scheme $\Delta r = \frac{v_X}{\alpha} r_X$ from earlier, which may be incorrect)

$$\Delta r = 7.4 \times 10^{-4} \text{m} \,, \tag{52}$$

$$r_0 = 1.6 \times 10^3 \text{m} \,, \tag{53}$$

$$E_{\text{propagated}} = \frac{8 \times 10^{13} \text{J m}^2}{\lambda_0^2} \,, \tag{54}$$

$$\Xi = \frac{17.6 \text{m}^2}{\lambda_0^2} \,. \tag{55}$$

Conservatively we set λ_0 at about 6 kilometers. This yields a total suppression factor of $\Xi = 4.9 \times 10^{-7}$. Keep in mind that this estimate is incredibly conservative, and does not take into account any non-conservative processes in the nonlinear regime $r < r_0$. Note, even though both λ_0 and r_0 are on the same order of magnitude, the approximations used are still good since they are overestimates of the Bessel J and Struve H functions.

Our result shows that the suppression factor only depends on the stopping factor, which in turn only depends on the velocity and cross-section of the macro. This is sensible since the velocity and cross-section of the macro are the defining characteristics of the impact. The only factor missing is the density of the macro. However, we are assuming that the density is large enough that the average velocity of the macro on its journey through the earth is close to its initial velocity.

It is significant that Ξ is proportional to σ_X. For the very largest macros, our very conservative suppression factor is near unity. However, it may still be enough to rule out detection.

Note, it is incorrect to consider the expression for Ξ in the small and large parameter limits: we required (in some approximations) that $r_0 \ll \lambda_0$, which places a lower bound on $\bar{p}$. Moreover $\bar{p}$ is supposed to be less than p_0.

2 Vector Modes

The following is the equation of motion for the vector potential a without body force:

$$\mu\nabla^2 a - \rho\ddot{a} = \mu\nabla(\nabla\cdot a) - \mu\nabla\times(\nabla\times a) - \rho\ddot{a} = 0\,, \tag{56}$$

where the leftmost term is the vector laplacian. This gives us three wave equations for the three components of a. The constraint equation takes the form

$$\begin{aligned}
\sigma_{ij} &= \lambda\delta_{ij}\nabla\cdot\nabla\times a + \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j)\,, &(57)\\
&= \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j)\,, &(58)\\
&= \mu\begin{pmatrix} 2\partial_x(\partial_y a_z - \partial_z a_y) & (\partial_y^2 - \partial_x^2)a_z + \partial_z(\partial_x a_x - \partial_y a_y) & (\partial_x^2 - \partial_z^2)a_y + \partial_y(\partial_z a_z - \partial_x a_x) \\ & 2\partial_y(\partial_z a_x - \partial_x a_z) & (\partial_z^2 - \partial_y^2)a_x + \partial_x(\partial_y a_y - \partial_z a_z) \\ & & 2\partial_z(\partial_x a_y - \partial_y a_x) \end{pmatrix}. &(59)
\end{aligned}$$

We now write a_i in terms of its Fourier components

$$a_i(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} A_i(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}\,. \tag{60}$$

Observe the dispersion relation $\beta k = \omega$, which holds for each component a_i, and that $\beta^2 = \frac{\mu}{\rho}$. Thus, the solutions to the wave equation are

$$a_i(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} A_i(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\beta k t)}\,. \tag{61}$$

We now use the constraint equation to write down the initial vector potential in terms of the initial stress. Note, we will obtain three equations constraining the possible initial stresses (i.e., we will be able to eliminate three components of σ from the equations), and three equations writing A_i in terms of Σ_{ij}.

In Fourier space we have

$$\begin{aligned}
-\Sigma_{xx} &= \mu 2k_x(k_y A_z - k_z A_y)\,, &(62)\\
-\Sigma_{yy} &= \mu 2k_y(k_z A_x - k_x A_z)\,, &(63)\\
-\Sigma_{zz} &= \mu 2k_z(k_x A_y - k_y A_x)\,, &(64)\\
-\Sigma_{xy} &= \mu\left[(k_y^2 - k_x^2)A_z + k_z(k_x A_x - k_y A_y)\right]\,, &(65)\\
-\Sigma_{yz} &= \mu\left[(k_z^2 - k_y^2)A_x + k_x(k_y A_y - k_z A_z)\right]\,, &(66)\\
-\Sigma_{zx} &= \mu\left[(k_x^2 - k_z^2)A_y + k_y(k_z A_z - k_x A_x)\right]\,. &(67)
\end{aligned}$$

First observe that any three of these equations together forms a singular linear system in A_i. Thus we cannot invert to solve for A_i. However, we can get some information about the Σ's.

Taking the sum of 31-33 yields

$$0 = \Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz}\,. \tag{68}$$

Observe that

$$\begin{aligned}
\frac{-\Sigma_{xx}k_y}{2\mu k_x k_z} &= \frac{k_y^2}{k_z}A_z - k_y A_y\,, &(69)\\
\frac{-\Sigma_{yy}k_x}{2\mu k_y k_z} &= k_x A_x - \frac{k_x^2}{k_z}A_z\,, &(70)\\
\frac{-\Sigma_{xy}}{\mu k_z} &= \left[\frac{k_y^2 - k_z^2}{k_z}\right]A_z + k_x A_x - k_y A_y\,. &(71)
\end{aligned}$$

Taking the sum of the first two reveals that‘

$$2k_xk_y\Sigma_{xy} = k_y^2\Sigma_{xx} + k_x^2\Sigma_{yy}\,. \tag{72}$$

If you want to be silly, we find

$$0 = (\Sigma_x k_y - \Sigma_y k_x)^2\,. \tag{73}$$

Anyway, we then have the system of three equations

$$2k_xk_y\Sigma_{xy} = k_y^2\Sigma_{xx} + k_x^2\Sigma_{yy}\,, \tag{74}$$
$$2k_yk_z\Sigma_{yz} = k_z^2\Sigma_{yy} + k_y^2\Sigma_{zz}\,, \tag{75}$$
$$2k_zk_x\Sigma_{zx} = k_x^2\Sigma_{zz} + k_z^2\Sigma_{xx}\,. \tag{76}$$

which is non-singular. They yield

$$\Sigma_{xx} = \frac{k_x}{k_yk_z}\left(k_z\Sigma_{xy} - k_x\Sigma_{yz} + k_y\Sigma_{zx}\right)\,, \tag{77}$$
$$\Sigma_{yy} = \frac{k_y}{k_zk_x}\left(k_z\Sigma_{xy} + k_x\Sigma_{yz} - k_y\Sigma_{zx}\right)\,, \tag{78}$$
$$\Sigma_{zz} = \frac{k_z}{k_xk_y}\left(-k_z\Sigma_{xy} + k_x\Sigma_{yz} + k_y\Sigma_{zx}\right)\,. \tag{79}$$

This, along with being traceless, shows that the shear only has two independent degrees of freedom: the polarizations.

It is impossible to solve for A_i in terms of the stress as is, hence we must fix a gauge. To do this, we fix $\nabla\cdot a = 0$, whence

$$0 = k_xA_x + k_yA_y + k_zA_z\,. \tag{80}$$

From earlier

$$-\Sigma_{yz} = \mu\left[(k_z^2 - k_y^2)A_x + k_x(k_yA_y - k_zA_z)\right]\,, \tag{81}$$
$$-\Sigma_{zx} = \mu\left[(k_x^2 - k_z^2)A_y + k_y(k_zA_z - k_xA_x)\right]\,. \tag{82}$$

Imposing the Gauge condition,

$$-\Sigma_{yz} = \mu\left[(k_x^2 - k_y^2 + k_z^2)A_x + 2k_xk_yA_y\right]\,, \tag{83}$$
$$-\Sigma_{zx} = \mu\left[(k_x^2 - k_y^2 - k_z^2)A_y - 2k_xk_yA_x\right]\,. \tag{84}$$

Inverting:

$$A_z = \frac{k_x\Sigma_{yz} - k_y\Sigma_{zx}}{\mu k_z k^2} \tag{85}$$

Hence, by symmetry

$$A_x = \frac{k_y\Sigma_{zx} - k_z\Sigma_{xy}}{\mu k_x k^2}\,, \tag{86}$$
$$A_y = \frac{k_z\Sigma_{xy} - k_x\Sigma_{yz}}{\mu k_y k^2}\,, \tag{87}$$
$$A_z = \frac{k_x\Sigma_{yz} - k_y\Sigma_{zx}}{\mu k_z k^2}\,. \tag{88}$$

Now we take the curl of A to find the shear mode displacement and use the relations (73) through (75) to obtain

$$U_x = -\mathrm{i}\frac{\Sigma_{xx}}{2k_x\mu}\,, \tag{89}$$

$$U_y = -\mathrm{i}\frac{\Sigma_{yy}}{2k_y\mu}\,, \tag{90}$$

$$U_z = -\mathrm{i}\frac{\Sigma_{zz}}{2k_z\mu}\,, \tag{91}$$

The energy of these shear waves is given

$$\begin{aligned}
E &= \frac{1}{2}\int_V \left(\rho|u_t|^2 + \mu|\nabla\cdot u|^2\right)\,, &(92)\\
&= \rho\int_V |u_t|^2\,, &(93)\\
&= \rho\beta^2\int_V\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\left(kk'U(\vec{k})\bar{U}(\vec{k}')\right)e^{-\mathrm{i}(\vec{k}\cdot\vec{x}-i\beta kt)}e^{\mathrm{i}(\vec{k}'\cdot\vec{x}-i\beta k't)}\,, &(94)\\
&= \mu\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(k^2\left|U(\vec{k})\right|^2\right)\,, &(95)\\
&= \mu\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\frac{\Sigma_{zz}^2\sec^2\theta + \csc^2\theta(\Sigma_{yy}^2\csc^2\phi + \Sigma_{xx}^2\sec^2\phi)}{4\mu^2}\right)\,. &(96)
\end{aligned}$$

Assuming axial symmetry, we have $\Sigma_{xx} = \Sigma\cos^2\phi$, $\Sigma_{yy} = \Sigma\sin^2\phi$, and $\Sigma_{zz} = -\Sigma$. Note that this choice is justified since the full stress tensor is invariant under rotations about the z-axis. Hence,

$$\begin{aligned}
E &= \mu\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Sigma^2\left(\frac{\sec^2\theta + \csc^2\theta}{4\mu^2}\right)\,, &(97)\\
&= \frac{1}{\mu}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Sigma^2\csc^2(2\theta)\,, &(98)\\
&= \frac{1}{\mu}\int\frac{\mathrm{d}k\,\mathrm{d}\theta}{(2\pi)^2}\Sigma^2k^2\csc^2(2\theta)\sin\theta\,, &(99)\\
&= \frac{1}{4\mu}\int\frac{\mathrm{d}k\,\mathrm{d}\theta}{(2\pi)^2}\Sigma^2k^2\csc(\theta)\sec^2(\theta)\,. &(100)
\end{aligned}$$

In cartesian coordinates:

$$E = \frac{1}{4\mu}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Sigma^2\frac{k^4}{(k_x^2+k_y^2)k_z^2}\,. \tag{101}$$

Something makes me think that $\csc^2(2\theta)$ should be absorbed into the definition of Σ, but I don't know how to show this.

3

For a given macro of given cross-section and velocity, how much energy gets deposited into seismic waves that propagated unattenuated? A better question: as a function of the sensitivity of a seismometer, will the seismometers have measured the macro? Look at the lunar limits.

4 P-Wave Vector Propagation

Observe, by Snell's law, we have that

$$p = \frac{r \sin\theta}{v}\,, \tag{102}$$

is constant along any fixed ray, and is given by the above form in a spherically symmetric velocity field. θ is the angle of incidence, r is the distance from the origin, and v is the velocity at radius r. Let $\sigma(s)$ be the parametrized curve followed by a p-wave. Then

$$\cos\theta = \frac{\hat{r}\cdot\dot{\sigma}(s)}{\|\sigma(s)\|} = \frac{\sigma(s)\cdot\dot{\sigma}(s)}{\|\sigma(s)\|\|\dot{\sigma}(s)\|}\,. \tag{103}$$

From this we obtain

$$\left(\frac{pv}{r}\right)^2 = 1 - \frac{(\sigma\cdot\dot{\sigma})^2}{(\|\sigma(s)\|\|\dot{\sigma}(s)\|)^2}\,, \tag{104}$$

If we denote the radial component as $r(s)$ and the angular component $\theta(s)$ we obtain the differential equation

$$\left(\frac{pv}{r}\right)^2 = \frac{r^2\dot{\theta}^2}{(\dot{r}^2+\dot{\theta}^2 r^2)}\,. \tag{105}$$

But observe, the velocity of the curve σ is just the velocity in the medium, hence

$$\left(\frac{pv}{r}\right)^2 = r^2\frac{\dot{\theta}^2}{v^2}\,, \tag{106}$$

from which we obtain the equation for θ,

$$\dot{\theta} = \pm\frac{pv^2}{r^2} \tag{107}$$

We now use the velocity definition to obtain an implicit equation for r. Observe

$$v^2 = \dot{r}^2 + \dot{\theta}^2 r^2\,, \tag{108}$$

$$= \dot{r}^2 + \frac{p^2 v^4}{r^2}\,, \tag{109}$$

hence

$$1 = \pm\frac{r\dot{r}}{\sqrt{v^2r^2 - p^2v^4}} \tag{110}$$

This can be integrated directly, yielding

$$s + C = \pm\int\frac{r\,\mathrm{d}r}{\sqrt{v^2r^2-p^2v^4}}\,. \tag{111}$$

In the case that v is a power law, i.e.

$$v(r) = v_0\left(\frac{r_0}{r}\right)^{\alpha} \tag{112}$$

we can directly integrate for integer α, yielding

$$s + C = \frac{r^{2\alpha}r_0^{-2\alpha}\sqrt{r^{-4\alpha}r_0^{2\alpha}v_0^2(r^{2+2\alpha}-p^2r_0^{2\alpha}v_0^2)}}{v_0^2(1+\alpha)}\,. \tag{113}$$

It is possible to invert this equation for any particular α. One solution which holds for all α is

$$r(s,\alpha) = r_0^{\frac{\alpha}{1+\alpha}} v_0^{\frac{1}{1+\alpha}} (p^2 + (C+s)^2(1+\alpha)^2)^{\frac{1}{2(1+\alpha)}} \, . \tag{114}$$

We thus obtain the following expression for θ:

$$\theta(s,\alpha) = \frac{1}{1+\alpha} \arctan\left[\frac{1+\alpha}{p}(C+s)\right] + \theta_0 \tag{115}$$

WOAH! We expect straight lines when $\alpha = 0$. Let's check:

$$r(s,0) = v_0\sqrt{p^2 + (C+s)^2}\,, \tag{116}$$

$$\theta(s,0) = \arctan\left[\frac{C+s}{p}\right] + \theta_0 \tag{117}$$

Setting θ_0 to zero reveals the equation for a vertical line. Thus, adding θ_0 rotates the line, and p sets the initial distance from the origin.

Now we consider the realistic velocity distribution $v(r) = a^2 - b^2r^2$. Proper application of mathematica to the above formulas reveals

$$r(s) = \frac{a\sqrt{b^2 - 2be^{2ab(s+C)} + e^{4ab(s+C)} + 4a^2b^4p^2}}{b\sqrt{b^2 + 2be^{2ab(s+C)} + e^{4ab(s+C)} + 4a^2b^4p^2}}\,, \tag{118}$$

$$\theta(s) = \arctan\left[\frac{-b^2 + e^{4ab(s+C)} + 4a^2b^4p^2}{4ab^3p}\right] + \theta_0 \tag{119}$$

These are arcs of circles which touch the boundary $r = \frac{a}{b}$ at right angles. That is, the trajectory of rays is given by geodesics in the Poincare disk. This can be seen easily by inspection of the metric a modified Poincare Sphere:

$$\mathrm{d}s^2 = \frac{\sum_i \mathrm{d}x_i^2}{(a^2 - b^2\sum_i x_i^2)^2}\,, \tag{120}$$

$$= \frac{\sum_i \mathrm{d}x_i^2}{(a^2 - b^2\sum_i x_i^2)^2}\,, \tag{121}$$

$$= \frac{1}{v^2}\sum_i \mathrm{d}x_i^2 \tag{122}$$

It is then obvious that $b = 0$ corresponds to straight lines. The importance of this observation is that $b^2 \approx 10^{-7}\mathrm{s}^{-1}$ and $a^2 \approx 10\mathrm{km\ s}^{-1}$, so the radius of the Poincare sphere corresponding to the trajectories of p-waves in earth is approximately $7 \times 10^3 - 1 \times 10^4$km, while the earth is of radius 10^3km. In particular, for the different layers of the earth, the ratio radius of the layer's outer boundary and the radius of its corresponding Poincare sphere is 7.0 for the inner core, 2.0 for the outer core, and 1.8 for the mantle. Thus, it is reasonable to approximate the trajectories of P-waves in the inner core as straight lines, but necessary to treat those in the other layers as bending.

It is clear that numerical methods will be necessary to get an answer. We will now consider a sequence of increasingly realistic models. I will begin typing up the first few tomorrow.

5 Melt Zone

As derived in "A time dependent model for long-rod penetration" the plastic zone (i.e. the melt zone) around an impacting projectile is given by $\alpha * R$, where α is given by

$$(1 + \frac{\rho_t u^2}{Y_t})\sqrt{K_t - \rho_t \alpha^2 u^2} = (1 + \frac{\rho_t \alpha^2 u^2}{2G_t})\sqrt{K_t - \rho_t u^2}. \tag{123}$$

where u is the velocity of the projectile (i.e. the macro), ρ_t, K_t, G_t and Y_t are the density, bulk modulus, sheer modulus, and flow stress of the target (i.e. earth or the moon) respectively.

The connection between the energy of the projectile and the value $\alpha * R$ is given in the paper posted "Hypervelocity penetration modeling- momentum vs. energy and energy transfer mechanisms". A derivation is given for the decomposition of the energy of a projectile into kinetic, elastic and plastic terms in, for example, equations 32, 33, and 38.

Note that the results in the section "Penetration" in this paper "Hypervelocity penetration..." are for eroding rods – but macro doesn't erode. If we decided to use equation 39 and the resulting equations which partition the energy of the impact into kinetic, elastic and plastic energy terms of the target (i.e. earth or moon), we'd set $u = v$, and $s = 0$ to eliminate the "eroding" effect.

6 Realistic Model

Since I have realized that the curving of rays is non-negligible, Professor Tolley has suggested doing a realistic treatment of the problem. The displacement potential satisfies the inhomogeneous wave equation

$$-\frac{f}{\rho} = v(r)^2 \nabla^2 \phi - \partial_t^2 \phi \,, \tag{124}$$

where $v(r) = a^2 - b^2 r^2$. Note, b is about 8 orders of magnitude smaller than a, so that the background varies slowly with respect to the frequency of the waves so the WKB approximation is appropriate. Denote

$$\hat{L} = v(r)^2 \frac{1}{r^2} \partial_r (r^2 \partial_r) - \frac{1}{r^2}(l(l+1)) + \omega^2 \,, \tag{125}$$

It is easy to see that $\hat{L}$ is Hermitian since it can be put in Sturm–Liouville form when stated as an eigenvalue problem

$$-\frac{v(r)^2}{r^2} \left[\partial_r (r^2 \partial_r) + \frac{r^2 \omega^2 - l(l+1)}{v(r)^2} \right] \phi = \lambda \phi \,. \tag{126}$$

Call the expression in the bracket on the left hand side $\hat{L}'$. We now impose finite boundary conditions, i.e. that the 0th and 1st derivatives of a solution vanish at R_E. Denote the eigenfunction corresponding to λ as ϕ_λ. Define the L^2 inner product

$$\langle f, g \rangle = \int_0^{R_E} \mathrm{d}r \frac{v(r)^2}{r^2} \bar{f}(r) g(r) \,, \tag{127}$$

with respect to which $\hat{L}'$ is self-adjoint. We now employ the WKB ansatz (where $\delta \ll \frac{a^2}{b^2 R_E^2}$ *this is a guess - think about it!)

$$\phi_\lambda(r) = \exp\left(\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_{\lambda n}(r) \right) \,, \tag{128}$$

and hence (note that δ is just an auxiliary variable that we will absorb into S at the end of the calculation)

$$\partial_r \phi_\lambda(r) = \left(\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S'_{\lambda n}(r) \right) \phi_\lambda(r) \,, \tag{129}$$

$$\sim \frac{1}{\delta} S'_{\lambda 0}(r) \phi_\lambda(r) \,, \tag{130}$$

$$\partial_r^2 \phi_\lambda(r) = \left(\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S''_{\lambda n}(r) \right) \phi_\lambda(r) + \left(\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S'_{\lambda n}(r) \right)^2 \phi_\lambda(r) \,, \tag{131}$$

$$\sim \frac{1}{\delta} S''_{\lambda 0}(r) \phi_\lambda(r) + \frac{1}{\delta^2} S'^2_{\lambda 0}(r) \phi_\lambda(r) + \frac{2}{\delta} S'_{\lambda 0}(r) S'_{\lambda 1}(r) \phi_\lambda(r) \,. \tag{132}$$

Thus

$$r^2 \left(\frac{1}{\delta} S''_{\lambda 0}(r) + \frac{1}{\delta^2} S'^2_{\lambda 0}(r) + \frac{2}{\delta} S'_{\lambda 0}(r) S'_{\lambda 1}(r) \right) + 2r \frac{1}{\delta} S'_{\lambda 0}(r) = -\frac{r^2(\lambda - \omega^2) + l(l+1)}{v(r)^2} \,. \tag{133}$$

As we take δ to zero, the dominant balance is

$$r^2 S'^2_{\lambda 0}(r) = -\frac{r^2(\lambda - \omega^2) + l(l+1)}{v(r)^2} \,, \tag{134}$$

from which we obtain

$$S_{\lambda 0}(r) = C_0 \pm \mathrm{i}\frac{\sqrt{(b^2 l(l+1) + a^2(\lambda - \omega^2))v(r)}}{a^2 b}\mathrm{AppellF1}\left[\frac{1}{2}, -\frac{1}{2}, 1, \frac{3}{2}, \frac{(\lambda-\omega^2)v(r)}{b^2 l(l+1) + a^2(\lambda - \omega^2)}, \frac{v(r)}{a^2}\right], \tag{135}$$

The first order contribution is then determined by

$$S'_{\lambda 1}(r) = -\frac{1}{r} - \frac{S''_{\lambda 0}(r)}{2S'_{\lambda 0}(r)}\,. \tag{136}$$

The solution is quite simple

$$S_{\lambda 1}(r) = C_1 - \frac{1}{4}\ln\left[\frac{r^2\left[l(l+1) + r^2(\lambda - \omega^2)\right]}{v(r)}\right]. \tag{137}$$

Hence, the eigenfunctions are, semiclassically

$$\begin{aligned}\phi_\lambda = {}& K_1\left[\frac{r^2\left[l(l+1) + r^2(\lambda-\omega^2)\right]}{v(r)}\right]^{-1/4} e^{+\mathrm{i}\frac{\sqrt{(b^2 l(l+1)+a^2(\lambda-\omega^2))v(r)}}{a^2 b}\mathrm{AppellF1}\left[\frac{1}{2},-\frac{1}{2},1,\frac{3}{2},\frac{(\lambda-\omega^2)v(r)}{b^2 l(l+1)+a^2(\lambda-\omega^2)},\frac{v(r)}{a^2}\right]} \\ &+ K_2\left[\frac{r^2\left[l(l+1) + r^2(\lambda-\omega^2)\right]}{v(r)}\right]^{-1/4} e^{-\mathrm{i}\frac{\sqrt{(b^2 l(l+1)+a^2(\lambda-\omega^2))v(r)}}{a^2 b}\mathrm{AppellF1}\left[\frac{1}{2},-\frac{1}{2},1,\frac{3}{2},\frac{(\lambda-\omega^2)v(r)}{b^2 l(l+1)+a^2(\lambda-\omega^2)},\frac{v(r)}{a^2}\right]}.\end{aligned} \tag{138}$$

Note that K_1 and K_2 will depend on λ. Because R_E has no physical relation to $\frac{a}{b}$, it is likely that imposing the boundary conditions will be ugly.