Homework

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1 Scalar Modes

In the following, lower case denotes a quantity in position space while capital letters denote their components in Fourier space.

Denote the displacement field $u(\vec{x},t) = \nabla\phi(\vec{x},t) + \nabla\times a(\vec{x},t)$. The linearized (acoustic) wave equation for ϕ is then

$$\alpha^2\nabla^2\phi = \partial_t^2\phi\,, \tag{1}$$

where $\alpha^2 = \frac{\lambda+2\mu}{\rho}$. We may write

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}\,, \tag{2}$$

from which we obtain the dispersion relation $\alpha k = \omega$, and hence the solution

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)}\,. \tag{3}$$

We now impose the constraint equation

$$\sigma_{ij} = \delta_{ij}\lambda\nabla\cdot u + \mu(u_{i,j} + u_{j,i}) \tag{4}$$

which, for the scalar modes becomes

$$\sigma_{ij} = \delta_{ij}\lambda\nabla^2\phi + 2\mu\partial_i\partial_j\phi \tag{5}$$

and whose trace is

$$-p = K\nabla^2\phi\,, \tag{6}$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus and $p = -\frac{1}{3}\,\mathrm{tr}\,\sigma_{ij}$. We take this constraint as an initial condition at $t = 0$. In Fourier space

$$P(\vec{k}) = Kk^2\Phi(\vec{K})\,, \tag{7}$$

from which we obtain the displacement field components

$$U(\vec{k}) = \mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2}\,, \tag{8}$$

and hence

$$u(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2} e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)} , \tag{9}$$

The equation for the energy of the compressional modes is

$$E = \frac{1}{2}\int_V \mathrm{d}^3x \left(\rho|\partial_t u|^2 + (\lambda+2\mu)|\nabla\cdot u|^2\right) , \tag{10}$$

$$= \frac{\lambda+2\mu}{K^2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left|P(\vec{k})\right|^2 \tag{11}$$

In the case that P depends only on the frequency

$$E = 4\pi\frac{\lambda+2\mu}{K^2}\int \frac{\mathrm{d}\tilde{\lambda}}{\tilde{\lambda}^4}\left|P(\tilde{\lambda})\right|^2 . \tag{12}$$

Now consider the case of a step function type pressure source - a cylinder of height h and radius r_X

$$p(\vec{x},0) = p_0\theta(r_X - r)\left[\theta(z+h/2) - \theta(z-h/2)\right] , \tag{13}$$

whose Fourier components are

$$P(\vec{k}) = \frac{4\pi r_X p_0}{\sqrt{k_x^2+k_y^2}k_z} J_1\left(\sqrt{k_x^2+k_y^2}r_X\right)\sin\left(\frac{h}{2}k_z\right) , \tag{14}$$

which, in polar k-space is

$$P(k,\varphi,\theta) = \frac{4\pi r_X p_0}{k^2\sin\theta\cos\theta} J_1\left(r_X k\sin\theta\right)\sin\left(\frac{h}{2}k\cos\theta\right) , \tag{15}$$

Directly integrating in Mathematica (over θ first) yields

$$E_{\mathrm{total}} = \frac{\lambda+2\mu}{K^2}4\pi^2 p_0^2\sigma_X h \tag{16}$$

where $\sigma_X = \pi r_X^2$. To calculate the energy deposition into the low frequency spectrum, we integrate k from 0 to k_0. Observe, we can make the following approximation

$$P(\vec{k}) \approx_{k\ll r_X} \frac{2\pi r_X p_0}{k^2\cos\theta\sin\theta} r_X k\sin\theta\sin\left(\frac{h}{2}k\cos\theta\right) , \tag{17}$$

$$= \frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right) . \tag{18}$$

From this we obtain the portion of the energy relegated to the long wavelength spectrum

$$E_{\mathrm{propagated}} = \frac{\lambda+2\mu}{K^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|\frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right)\right|^2 , \tag{19}$$

$$= \left[\frac{1}{2h}(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[hk_0\cos(hk_0) + \sin(hk_0) + hk_0\left(-2 + hk_0\mathrm{Si}(hk_0)\right)\right] , \tag{20}$$

$$= \left[(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi}{\lambda_0}\cos\left(\frac{2\pi h}{\lambda_0}\right) + \sin\left(\frac{2\pi h}{\lambda_0}\right) + \left(\frac{\pi}{\lambda_0}\right)\left(-2 + \left(\frac{2\pi h}{\lambda_0}\right)\mathrm{Si}\left(\frac{2\pi h}{\lambda_0}\right)\right)\right] , \tag{21}$$

$$\approx_{h\gg\lambda_0\gg 1}\left[(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi^3 h}{\lambda_0^2}\right] , \tag{22}$$

$$= \left[(\sigma_X p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi h}{\lambda_0^2}\right] . \tag{23}$$

From this we obtain the fractional energy deposition into the unattenuated wavelengths

$$\Xi = \frac{\sigma_X}{4\pi\lambda_0^2} \,. \tag{24}$$

This approximation holds for $\lambda_0^2 \gg \sigma_X$, which is appropriate for the case that λ_0 is on the order of kilometers and σ_X is on the order of centimeters squared.

2 Vector Modes

The following is the equation of motion for the vector potential a without body force:

$$\mu\nabla^2 a - \rho\ddot{a} = \mu\nabla(\nabla\cdot a) - \mu\nabla\times(\nabla\times a) - \rho\ddot{a} = 0\,, \tag{25}$$

where the leftmost term is the vector laplacian. This gives us three wave equations for the three components of a. The constraint equation takes the form

$$\begin{aligned}
\sigma_{ij} &= \lambda\delta_{ij}\nabla\cdot\nabla\times a + \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j)\,, &(26)\\
&= \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j)\,, &(27)\\
&= \mu\begin{pmatrix} 2\partial_x(\partial_y a_z - \partial_z a_y) & (\partial_y^2 - \partial_x^2)a_z + \partial_z(\partial_x a_x - \partial_y a_y) & (\partial_x^2 - \partial_z^2)a_y + \partial_y(\partial_z a_z - \partial_x a_x) \\ & 2\partial_y(\partial_z a_x - \partial_x a_z) & (\partial_z^2 - \partial_y^2)a_x + \partial_x(\partial_y a_y - \partial_z a_z) \\ & & 2\partial_z(\partial_x a_y - \partial_y a_x)\end{pmatrix}. &(28)
\end{aligned}$$

We now write a_i in terms of its Fourier components

$$a_i(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} A_i(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}\,. \tag{29}$$

Observe the dispersion relation $\beta k = \omega$, which holds for each component a_i, and that $\beta = \frac{\mu}{\rho}$. Thus, the solutions to the wave equation are

$$a_i(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} A_i(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\beta k t)}\,. \tag{30}$$

We now use the constraint equation to write down the initial vector potential in terms of the initial stress. Note, we will obtain three equations constraining the possible initial stresses (i.e., we will be able to eliminate three components of σ from the equations), and three equations writing A_i in terms of Σ_{ij}.

In Fourier space we have

$$\begin{aligned}
-\Sigma_{xx} &= 2k_x(k_yA_z - k_zA_y)\,, &(31)\\
-\Sigma_{yy} &= 2k_y(k_zA_x - k_xA_z)\,, &(32)\\
-\Sigma_{zz} &= 2k_z(k_xA_y - k_yA_x)\,, &(33)\\
-\Sigma_{xy} &= (k_y^2 - k_x^2)A_z + k_z(k_xA_x - k_yA_y)\,, &(34)\\
-\Sigma_{yz} &= (k_z^2 - k_y^2)A_x + k_x(k_yA_y - k_zA_z)\,, &(35)\\
-\Sigma_{zx} &= (k_x^2 - k_z^2)A_y + k_y(k_zA_z - k_xA_x)\,. &(36)
\end{aligned}$$

First observe that any three of these equations together forms a singular linear system.

Taking the sum of 31-33 yields

$$0 = \Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz}\,. \tag{37}$$

Observe that

$$\begin{aligned}
\frac{-\Sigma_{xx}k_y}{2k_xk_z} &= \frac{k_y^2}{k_z}A_z - k_yA_y\,, &(38)\\
\frac{-\Sigma_{yy}k_x}{2k_yk_z} &= k_xA_x - \frac{k_x^2}{k_z}A_z\,, &(39)\\
\frac{-\Sigma_{xy}}{k_z} &= \left[\frac{k_y^2 - k_z^2}{k_z}\right]A_z + k_xA_x - k_yA_y\,. &(40)
\end{aligned}$$

Taking the sum of the first two reveals that

$$2k_x k_y \Sigma_{xy} = k_y^2 \Sigma_{xx} + k_x^2 \Sigma_{yy} \, . \tag{41}$$

If you want to be silly, we find

$$0 = (\Sigma_x k_y - \Sigma_y k_x)^2 \, . \tag{42}$$

Anyway, we then have the system of three equations

$$2k_x k_y \Sigma_{xy} = k_y^2 \Sigma_{xx} + k_x^2 \Sigma_{yy} \, , \tag{43}$$
$$2k_y k_z \Sigma_{yz} = k_z^2 \Sigma_{yy} + k_y^2 \Sigma_{zz} \, , \tag{44}$$
$$2k_z k_x \Sigma_{zx} = k_x^2 \Sigma_{zz} + k_z^2 \Sigma_{xx} \, . \tag{45}$$

which are non-singular. They yield

$$\Sigma_{xx} = \frac{k_x}{k_y k_z} \left(k_z \Sigma_{xy} - k_x \Sigma_{yz} + k_y \Sigma_{zx} \right) \, , \tag{46}$$
$$\Sigma_{yy} = \frac{k_y}{k_z k_x} \left(k_z \Sigma_{xy} + k_x \Sigma_{yz} - k_y \Sigma_{zx} \right) \, , \tag{47}$$
$$\Sigma_{zz} = \frac{k_z}{k_x k_y} \left(-k_z \Sigma_{xy} + k_x \Sigma_{yz} + k_y \Sigma_{zx} \right) \, . \tag{48}$$

This reveals two facts. The first fact is that the only relevant equations are 34-36. The second is that 37 implies that

$$0 = k^2 \left[\frac{\Sigma_{yz}}{k_y k_z} + \frac{\Sigma_{zx}}{k_z k_x} + \frac{\Sigma_{xy}}{k_x k_y} \right] \, , \tag{49}$$

i.e.

$$0 = k_x \Sigma_{yz} + k_y \Sigma_{zx} + k_z \Sigma_{xy} \, . \tag{50}$$

That is to say $(\Sigma_{yz}, \Sigma_{zx}, \Sigma_{xy})$ is transverse. Because the only three relevant components are those of this vector, denote $\Sigma = (\Sigma_{yz}, \Sigma_{zx}, \Sigma_{xy}) := (\Sigma_x, \Sigma_y, \Sigma_z)$.