Homework

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1 Scalar Modes

In the following, lower case denotes a quantity in position space while capital letters denote their components in Fourier space.

Denote the displacement field $u(\vec{x}, t) = \nabla \phi(\vec{x}, t) + \nabla \times a(\vec{x}, t)$. The linearized (acoustic) wave equation for ϕ is then

$$\alpha^2 \nabla^2 \phi = \partial_t^2 \phi \,, \tag{1}$$

where $\alpha^2 = \frac{\lambda + 2\mu}{\rho}$. We may write

$$\phi(\vec{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{x} - \omega t)} \,, \tag{2}$$

from which we obtain the dispersion relation $\alpha k = \omega$, and hence the solution

$$\phi(\vec{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{x} - \alpha k t)} \,. \tag{3}$$

We now impose the constraint equation

$$\sigma_{ij} = \delta_{ij} \lambda \nabla \cdot u + \mu (u_{i,j} + u_{j,i}) \tag{4}$$

which, for the scalar modes becomes

$$\sigma_{ij} = \delta_{ij} \lambda \nabla^2 \phi + 2\mu \partial_i \partial_j \phi \tag{5}$$

and whose trace is

$$-p = K \nabla^2 \phi \,, \tag{6}$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus and $p = -\frac{1}{3} \operatorname{tr} \sigma_{ij}$. We take this constraint as an initial condition at $t = 0$. In Fourier space

$$P(\vec{k}) = K k^2 \Phi(\vec{K}) \,, \tag{7}$$

from which we obtain the displacement field components

$$U(\vec{k}) = i \frac{P(\vec{k})}{K} \frac{\vec{k}}{k^2} \,, \tag{8}$$

and hence

$$u(\vec{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{i} \frac{P(\vec{k})}{K} \frac{\vec{k}}{k^2} e^{\mathrm{i}(\vec{k}\cdot\vec{x} - \alpha k t)} , \tag{9}$$

The equation for the energy of the compressional modes is

$$E = \frac{1}{2} \int_V \mathrm{d}^3 x \left(\rho |\partial_t u|^2 + (\lambda + 2\mu)|\nabla \cdot u|^2 \right) , \tag{10}$$

$$= \frac{\lambda + 2\mu}{K^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left| P(\vec{k}) \right|^2 \tag{11}$$

In the case that P depends only on the frequency

$$E = 4\pi \frac{\lambda + 2\mu}{K^2} \int \frac{\mathrm{d}\tilde{\lambda}}{\tilde{\lambda}^4} \left| P(\tilde{\lambda}) \right|^2 . \tag{12}$$

Now consider the case of a step function type pressure source - a cylinder of height h and radius r_X

$$p(\vec{x}, 0) = p_0 \theta(r_X - r) \left[\theta(z + h/2) - \theta(z - h/2) \right] , \tag{13}$$

whose Fourier components are

$$P(\vec{k}) = \frac{4\pi r_X p_0}{\sqrt{k_x^2 + k_y^2} k_z} J_1 \left(\sqrt{k_x^2 + k_y^2} r_X \right) \sin \left(\frac{h}{2} k_z \right) , \tag{14}$$

which, in polar k-space is

$$P(k, \varphi, \theta) = \frac{4\pi r_X p_0}{k^2 \sin\theta \cos\theta} J_1 \left(r_X k \sin\theta \right) \sin \left(\frac{h}{2} k \cos\theta \right) , \tag{15}$$

Directly integrating in Mathematica (over θ first) yields

$$E_{\text{total}} = \frac{\lambda + 2\mu}{K^2} 4\pi^2 p_0^2 \sigma_X h \tag{16}$$

where $\sigma_X = \pi r_X^2$. To calculate the energy deposition into the low frequency spectrum, we integrate k from 0 to k_0. Observe, we can make the following approximation

$$P(\vec{k}) \approx_{k \ll r_X} \frac{2\pi r_X p_0}{k^2 \cos\theta \sin\theta} r_X k \sin\theta \sin \left(\frac{h}{2} k \cos\theta \right) , \tag{17}$$

$$= \frac{2\pi r_X^2 p_0}{k \cos\theta} \sin \left(\frac{h}{2} k \cos\theta \right) . \tag{18}$$

From this we obtain the portion of the energy relegated to the long wavelength spectrum

$$E_{\text{propagated}} = \frac{\lambda + 2\mu}{K^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left| \frac{2\pi r_X^2 p_0}{k \cos\theta} \sin \left(\frac{h}{2} k \cos\theta \right) \right|^2 , \tag{19}$$

$$= \left[\frac{1}{2h} (r_X^2 p_0)^2 \frac{\lambda + 2\mu}{K^2} \right] \left[h k_0 \cos(h k_0) + \sin(h k_0) + h k_0 \left(-2 + h k_0 \mathrm{Si}(h k_0) \right) \right] , \tag{20}$$

$$= \left[(r_X^2 p_0)^2 \frac{\lambda + 2\mu}{K^2} \right] \left[\frac{\pi}{\lambda_0} \cos \left(\frac{2\pi h}{\lambda_0} \right) + \sin \left(\frac{2\pi h}{\lambda_0} \right) + \left(\frac{\pi}{\lambda_0} \right) \left(-2 + \left(\frac{2\pi h}{\lambda_0} \right) \mathrm{Si} \left(\frac{2\pi h}{\lambda_0} \right) \right) \right] , \tag{21}$$

$$\approx_{h \gg \lambda_0 \gg 1} \left[(r_X^2 p_0)^2 \frac{\lambda + 2\mu}{K^2} \right] \left[\frac{\pi^3 h}{\lambda_0^2} \right] , \tag{22}$$

$$= \left[(\sigma_X p_0)^2 \frac{\lambda + 2\mu}{K^2} \right] \left[\frac{\pi h}{\lambda_0^2} \right] . \tag{23}$$

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From this we obtain the fractional energy deposition into the unattenuated wavelengths

$$\Xi = \frac{\sigma_X}{4\pi\lambda_0^2} \, . \tag{24}$$

This approximation holds for $\lambda_0^2 \gg \sigma_X$, which is appropriate for the case that λ_0 is on the order of kilometers and σ_X is on the order of centimeters squared.

2 Vector Modes

The following is the equation of motion for the vector potential a without body force:

$$\mu\nabla^2 a - \rho\ddot{a} = \mu\nabla(\nabla\cdot a) - \mu\nabla\times(\nabla\times a) - \rho\ddot{a} = 0 \,, \tag{25}$$

where the leftmost term is the vector laplacian. This gives us three wave equations for the three components of a. The constraint equation takes the form

$$\sigma_{ij} = \lambda\delta_{ij}\nabla\cdot\nabla\times a + \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j) \,, \tag{26}$$

$$= \mu(\partial_j(\nabla\times a)_i + \partial_i(\nabla\times a)_j) \,, \tag{27}$$

$$= \mu\begin{pmatrix} 2\partial_x(\partial_y a_z - \partial_z a_y) & (\partial_y^2 - \partial_x^2)a_z + \partial_z(\partial_x a_x - \partial_y a_y) & (\partial_x^2 - \partial_z^2)a_y + \partial_y(\partial_z a_z - \partial_x a_x) \\ & 2\partial_y(\partial_z a_x - \partial_x a_z) & (\partial_z^2 - \partial_y^2)a_x + \partial_x(\partial_y a_y - \partial_z a_z) \\ & & 2\partial_z(\partial_x a_y - \partial_y a_x) \end{pmatrix} \,. \tag{28}$$

We now write a_i in terms of its Fourier components

$$a_i(\vec{x}, t) = \int\frac{\mathrm{d}^3 k}{(2\pi)^3} A_i(\vec{k}, \omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} \,. \tag{29}$$

Observe the dispersion relation $\beta k = \omega$, which holds for each component a_i, and that $\beta = \frac{\mu}{\rho}$. Thus, the solutions to the wave equation are

$$a_i(\vec{x}, t) = \int\frac{\mathrm{d}^3 k}{(2\pi)^3} A_i(\vec{k}, \omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x} - \beta k t)} \,. \tag{30}$$

We now use the constraint equation to write down the initial vector potential in terms of the initial stress. Note, we will obtain three equations constraining the possible initial stresses (i.e., we will be able to eliminate three components of σ from the equations), and three equations writing A_i in terms of Σ_{ij}.

In Fourier space we have

$$-\Sigma_{xx} = 2k_x(k_y A_z - k_z A_y) \,, \tag{31}$$

$$-\Sigma_{yy} = 2k_y(k_z A_x - k_x A_z) \,, \tag{32}$$

$$-\Sigma_{zz} = 2k_z(k_x A_y - k_y A_x) \,, \tag{33}$$

$$-\Sigma_{xy} = (k_y^2 - k_x^2)A_z + k_z(k_x A_x - k_y A_y) \,, \tag{34}$$

$$-\Sigma_{yz} = (k_z^2 - k_y^2)A_x + k_x(k_y A_y - k_z A_z) \,, \tag{35}$$

$$-\Sigma_{zx} = (k_x^2 - k_z^2)A_y + k_y(k_z A_z - k_x A_x) \,. \tag{36}$$

First observe that any three of these equations together forms a singular linear system in A_i. Thus we cannot invert to solve for A_i. However, we can get some information about the Σ's.

Taking the sum of 31-33 yields

$$0 = \Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz} \,. \tag{37}$$

Observe that

$$\frac{-\Sigma_{xx}k_y}{2k_x k_z} = \frac{k_y^2}{k_z}A_z - k_y A_y \,, \tag{38}$$

$$\frac{-\Sigma_{yy}k_x}{2k_y k_z} = k_x A_x - \frac{k_x^2}{k_z}A_z \,, \tag{39}$$

$$\frac{-\Sigma_{xy}}{k_z} = \left[\frac{k_y^2 - k_z^2}{k_z}\right]A_z + k_x A_x - k_y A_y \,. \tag{40}$$

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Taking the sum of the first two reveals that'

$$2k_x k_y \Sigma_{xy} = k_y^2 \Sigma_{xx} + k_x^2 \Sigma_{yy} \,. \tag{41}$$

If you want to be silly, we find

$$0 = (\Sigma_x k_y - \Sigma_y k_x)^2 \,. \tag{42}$$

Anyway, we then have the system of three equations

$$2k_x k_y \Sigma_{xy} = k_y^2 \Sigma_{xx} + k_x^2 \Sigma_{yy} \,, \tag{43}$$

$$2k_y k_z \Sigma_{yz} = k_z^2 \Sigma_{yy} + k_y^2 \Sigma_{zz} \,, \tag{44}$$

$$2k_z k_x \Sigma_{zx} = k_x^2 \Sigma_{zz} + k_z^2 \Sigma_{xx} \,. \tag{45}$$

which is non-singular. They yield

$$\Sigma_{xx} = \frac{k_x}{k_y k_z} \left(k_z \Sigma_{xy} - k_x \Sigma_{yz} + k_y \Sigma_{zx} \right) \,, \tag{46}$$

$$\Sigma_{yy} = \frac{k_y}{k_z k_x} \left(k_z \Sigma_{xy} + k_x \Sigma_{yz} - k_y \Sigma_{zx} \right) \,, \tag{47}$$

$$\Sigma_{zz} = \frac{k_z}{k_x k_y} \left(-k_z \Sigma_{xy} + k_x \Sigma_{yz} + k_y \Sigma_{zx} \right) \,. \tag{48}$$

This, along with being traceless, shows that the shear only has two independent degrees of freedom: the polarizations.

It is impossible to solve for A_i in terms of the stress as is, hence we must fix a gauge. To do this, we fix $\nabla \cdot a = 0$, whence

$$0 = k_x A_x + k_y A_y + k_z A_z \,. \tag{49}$$

From earlier, we have

$$-\Sigma_{yy} = 2k_y (k_z A_x - k_x A_z) \,, \tag{50}$$

$$-\Sigma_{zz} = 2k_z (k_x A_y - k_y A_x) \,, \tag{51}$$

which becomes

$$-\Sigma_{yy} = 2k_y \left[-\frac{k_z}{k_x} \left(k_y A_y + k_z A_z \right) - k_x A_z \right] \,, \tag{52}$$

$$-\Sigma_{zz} = 2k_z \left[k_x A_y + \frac{k_y}{k_x} \left(k_y A_y + k_z A_z \right) \right] \,, \tag{53}$$

which simplifies to

$$-\Sigma_{yy} = -\frac{2k_y}{k_x} \left[k_y k_z A_y + \left(k_z^2 + k_x^2 \right) A_z \right] \,, \tag{54}$$

$$-\Sigma_{zz} = \frac{2k_z}{k_x} \left[\left(k_x^2 + k_y^2 \right) A_y + k_y k_z A_z \right] \,, \tag{55}$$

Hence

$$A_x = \frac{(k_y^2 - k_z^2)(\Sigma_{yy} - \Sigma_{zz}) + k_x^2 (\Sigma_{yy} + \Sigma_{zz})}{2k_x^2 k^2} \,, \tag{56}$$

$$= \frac{(k_y^2 - k_z^2)(\Sigma_{yy} - \Sigma_{zz}) - k_x^2 \Sigma_{xx}}{2k_x^2 k^2} \,, \tag{57}$$

$$= \frac{(k_y^2 - k_z^2)}{k_x^2} \frac{(\Sigma_{yy} - \Sigma_{zz})}{2k^2} - \frac{\Sigma_{xx}}{2k^2} \,, \tag{58}$$

$$\tag{59}$$

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By symmetry, we have

$$A_x = \frac{(k_y^2 - k_z^2)}{k_x^2} \frac{(\Sigma_{yy} - \Sigma_{zz})}{2k^2} - \frac{\Sigma_{xx}}{2k^2}, \tag{60}$$

$$A_y = \frac{(k_z^2 - k_x^2)}{k_y^2} \frac{(\Sigma_{zz} - \Sigma_{xx})}{2k^2} - \frac{\Sigma_{yy}}{2k^2}, \tag{61}$$

$$A_z = \frac{(k_x^2 - k_y^2)}{k_z^2} \frac{(\Sigma_{xx} - \Sigma_{yy})}{2k^2} - \frac{\Sigma_{zz}}{2k^2}. \tag{62}$$

Observe, in the case that all stresses are equal, we obtain that $A_x = A_y = A_z = 0$, as we should. We now solve for A_i in terms of Σ_i.

Now we take the curl of A to find the shear mode displacement

$$U_x = kyA_z - kzA_y, \tag{63}$$

$$= \frac{1}{2k^2 k_y^2 k_z^2} \left[-k_z^3 \left(-k_y^2 \Sigma_{yy} + (k_x^2 - k_z^2)(\Sigma_{xx} - \Sigma_{zz}) \right) + k_y^3 \left((k_x^2 - k_y^2)(\Sigma_{xx} - \Sigma_{yy}) - k_z^2 \Sigma_{zz} \right) \right] \tag{64}$$

Just cyclically permute the indices to obtain the expressions for U_y and U_z.

For shocks, check the laser physics book.

3 Comparison to "Passage of Small BH..."

Their considerations in the acoustic limit account for a body force generated by the gravitational pull of the miniature black hole. They discover that only supersonic blackholes would generate this type of radiation. Our calculations, on the other hand, seem independent of these considerations. In our case, the gravitational pull of something as dense as a Macro is not negligible, but should not generate as significant an effect this way. The principal mechanism for energy deposition is the scattering of matter off the macro. However, their methods do bring into question whether the free wave equation is appropriate. They also consider the transfer of energy due to pressure from the black hole to matter along its path. This is a different calculation from ours since black hole radiation is very weak, as opposed to collisions between baryonic matter.

However, one key point is that they do neglect shear by using a scalar displacement potential ψ below their equation (3). Moreover, they neglect the subtlety of the behavior of the collision near the boundary of the Earth.

4   Our Argument

As a Macro passes through the earth, there are several phases to its journey. The first is its passage through the atmosphere. The second is its crossing the boundary from air to earth, which we suspect will be relatively uneventful, since the air is $1000\times$ lower density than earth, and thus would displace relatively little of it. The third is the transit of the macro through the earth. The shockwave it produces should be generated primarily by elastic and inelastic scattering of baryonic matter off the Macro. If we estimate that the macro produces an initial overpressure in a tube of its own radius, we can calculate the initial overpressure from the total energy deposition and its cross section. Given that the macro is dense enough, its loss of momentum will be approximately linear, and we can approximate its energy loss as being approximately constant. Estimates have already been made of the stopping power, so the total energy deposition will just be the

$$\left|\frac{dE}{dx}\right|h = E_{\text{total}} = \frac{\lambda + 2\mu}{K^2} 4\pi^2 p_0^2 \sigma_X h \tag{65}$$

whence the initial overpressure is

$$p_0 = \frac{K}{2\pi}\sqrt{\left|\frac{dE}{dx}\right|\frac{1}{\sigma_X(\lambda + 2\mu)}} \tag{66}$$

It is not appropriate to use the linear regime until the overpressure is close to 0 compared to some pressure, which we suspect may be $p_{\text{non-linear}} \approx 60\text{GPa}$. It is reasonable to assume that the elastic limit holds when the pressures are an order of magnitude smaller than $p_{\text{non-linear}}$, denoted $p_{\text{linear}} \approx 6\text{GPa}$. (This stuff needs to be checked). Note that energy losses due to heating, i.e. anelastic frequency attenuation, will dilute the energy density of the shock. However, for a conservative estimate, we only consider the geometric attenuation, which will dilute the energy density as r^{-1}, where r is the radial distance from the Macro's trajectory to the shock front. The shape of the shockwave will rapidly achieve a right-triangular profile. Geometric considerations show that the overpressure will go as r^{-1}. Thus, suppose that the average pressure in the shock is $\bar{p}$. Then the average pressure at radius r is given

$$\bar{p}(r) = p_0\frac{r_0 + l}{r} . \tag{67}$$

The overpressure is then given by

$$p = \bar{p}_{\text{linear}}\frac{r - r_0}{\ell}\left[\theta(r_0 + \ell - r) - \theta(r + \ell)\right]\left[\theta(z + h/2) - \theta(z - h/2)\right] \tag{68}$$

where ℓ is the radial length of the shock and r_0 is its base. We now calculate the total energy contained in this shock. Fourier transforming reveals

$$P = \frac{\bar{p}_{\text{linear}}}{k_r^2 k_z \ell}\pi \sin\left(\frac{h}{2}k_z\right)\left[(\ell + r_0)(2\sqrt{2}k_r\ell - \pi H_0(\sqrt{2}k_r(\ell + r_0))J_1(\sqrt{2}k_r(\ell + r_0))\right.$$

$$\left. + \pi(\ell + r_0)H_1(\sqrt{2}k_r(\ell + r_0))J_0(\sqrt{2}k_r(\ell + r_0)) + \pi r_0 H_0(\sqrt{2}k_r r_0)J_1(\sqrt{2}k_r r_0) - \pi r_0 H_1(\sqrt{2}k_r r_0)J_0(\sqrt{2}k_r r_0)\right] \tag{69}$$

This reduces to, for the limit that $r_0 \ll \lambda_{\text{attenuation}}$

5 What we still need to do

What we've calculated is the the pressure from the total energy deposition. We need to figure out the total energy, pressure, and shape of the shock after it becomes sufficiently acoustic. It would also be good to know the velocity of the shock. From this, we can calculate the energy deposited in the measurable range.

It appears as though *nobody* considers the shear waves... Should we?

The gravitational effects can be taken into account as in the Black Hole paper.