Homework

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In the following, lower case denotes a quantity in position space while capital letters denote their components in Fourier space.

Denote the displacement field $u(\vec{x},t) = \nabla\phi(\vec{x},t) + \nabla \times a(\vec{x},t)$. The linearized (acoustic) wave equation for ϕ is then

$$\alpha^2\nabla^2\phi = \partial_t^2\phi\,, \tag{1}$$

where $\alpha^2 = \frac{\lambda+2\mu}{\rho}$. We may write

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}\,, \tag{2}$$

from which we obtain the dispersion relation $\alpha k = \omega$, and hence the solution

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(\vec{k},\omega)e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)}\,. \tag{3}$$

We now impose the constraint equation

$$\sigma_{ij} = \delta_{ij}\lambda\nabla\cdot u + \mu(u_{i,j} + u_{j,i}) \tag{4}$$

which, for the scalar modes becomes

$$\sigma_{ij} = \delta_{ij}\lambda\nabla^2\phi + 2\mu\partial_i\partial_j\phi \tag{5}$$

and whose trace is

$$-p = K\nabla^2\phi\,, \tag{6}$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus and $p = -\frac{1}{3}\operatorname{tr}\sigma_{ij}$. We take this constraint as an initial condition at $t = 0$. In Fourier space

$$P(\vec{k}) = Kk^2\Phi(\vec{K})\,, \tag{7}$$

from which we obtain the displacement field components

$$U(\vec{k}) = \mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2}\,, \tag{8}$$

and hence

$$u(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{i}\frac{P(\vec{k})}{K}\frac{\vec{k}}{k^2}e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)}\,, \tag{9}$$

The equation for the energy of the compressional modes is

$$E = \frac{1}{2}\int_V \mathrm{d}^3x \left(\rho|\partial_t u|^2 + (\lambda+2\mu)|\nabla\cdot u|^2\right), \tag{10}$$

$$= \frac{\lambda+2\mu}{K^2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left|P(\vec{k})\right|^2 \tag{11}$$

In the case that P depends only on the frequency

$$E = 4\pi\frac{\lambda+2\mu}{K^2}\int \frac{\mathrm{d}\tilde{\lambda}}{\tilde{\lambda}^4}\left|P(\tilde{\lambda})\right|^2. \tag{12}$$

Now consider the case of a step function type pressure source - a cylinder of height h and radius r_X

$$p(\vec{x},0) = p_0\theta(r_X - r)\left[\theta(z+h/2) - \theta(z-h/2)\right], \tag{13}$$

whose Fourier components are

$$P(\vec{k}) = \frac{4\pi r_X p_0}{\sqrt{k_x^2+k_y^2}k_z}J_1\left(\sqrt{k_x^2+k_y^2}r_X\right)\sin\left(\frac{h}{2}k_z\right), \tag{14}$$

which, in polar k-space is

$$P(k,\varphi,\theta) = \frac{4\pi r_X p_0}{k^2\sin\theta\cos\theta}J_1\left(r_X k\sin\theta\right)\sin\left(\frac{h}{2}k\cos\theta\right), \tag{15}$$

Directly integrating in Mathematica (over θ first) yields

$$E_{\mathrm{total}} = \frac{\lambda+2\mu}{K^2}4\pi^2 p_0^2\sigma_X h \tag{16}$$

where $\sigma_X = \pi r_X^2$. To calculate the energy deposition into the low frequency spectrum, we integrate k from 0 to k_0. Observe, we can make the following approximation

$$P(\vec{k}) \approx_{k\ll r_X} \frac{2\pi r_X p_0}{k^2\cos\theta\sin\theta}r_X k\sin\theta\sin\left(\frac{h}{2}k\cos\theta\right), \tag{17}$$

$$= \frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right). \tag{18}$$

From this we obtain the portion of the energy relegated to the long wavelength spectrum

$$E_{\mathrm{propagated}} = \frac{\lambda+2\mu}{K^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|\frac{2\pi r_X^2 p_0}{k\cos\theta}\sin\left(\frac{h}{2}k\cos\theta\right)\right|^2, \tag{19}$$

$$= \left[\frac{1}{2h}(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[hk_0\cos(hk_0) + \sin(hk_0) + hk_0\left(-2 + hk_0\mathrm{Si}(hk_0)\right)\right], \tag{20}$$

$$= \left[(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi}{\lambda_0}\cos\left(\frac{2\pi h}{\lambda_0}\right) + \sin\left(\frac{2\pi h}{\lambda_0}\right) + \left(\frac{\pi}{\lambda_0}\right)\left(-2 + \left(\frac{2\pi h}{\lambda_0}\right)\mathrm{Si}\left(\frac{2\pi h}{\lambda_0}\right)\right)\right], \tag{21}$$

$$\approx_{h\gg\lambda_0\gg 1}\left[(r_X^2 p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi^3 h}{\lambda_0^2}\right], \tag{22}$$

$$= \left[(\sigma_X p_0)^2\frac{\lambda+2\mu}{K^2}\right]\left[\frac{\pi h}{\lambda_0^2}\right]. \tag{23}$$

From this we obtain the fractional energy deposition into the unattenuated wavelengths

$$\Xi = \frac{\sigma_X}{4\pi\lambda_0^2}. \tag{24}$$

This approximation holds for $\lambda_0^2 \gg \sigma_X$, which is appropriate for the case that λ_0 is on the order of kilometers and σ_X is on the order of centimeters squared.