

**Beyond modelocking: High repetition-rate frequency  
combs derived from a continuous-wave laser**

by

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Reader Two

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Beyond modelocking: High repetition-rate frequency combs derived from a continuous-wave laser

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Optical frequency combs based on modelocked lasers have revolutionized precision metrology by facilitating measurements of optical frequencies, with implications both for fundamental scientific questions and for applications such as fast, broadband spectroscopy. In this thesis, I describe advances in the generation of frequency combs without modelocking in platforms with smaller footprints and higher repetition rates, with the ultimate goal of bringing frequency combs to new applications in a chip-integrated package. I discuss two approaches for comb generation: parametric frequency conversion in Kerr microresonators and active electro-optic modulation of a continuous-wave laser. After introducing microresonator-based frequency combs (microcombs), I discuss two specific developments in microcomb technology: First, I describe a new, extremely reliable method for generation of soliton pulses through the use of a phase-modulated pump laser. This technique eliminates the dependence on initial conditions that was formerly a universal feature of these experiments, presenting a solution to a significant technical barrier to the practical application of microcombs. Second, I present observations of *soliton crystal* states with highly structured ‘fingerprint’ optical spectra that correspond to ordered pulse trains exhibiting crystallographic defects. These pulse trains arise through interaction of solitons with avoided mode crossings in the resonator spectrum. I also discuss generation of Kerr soliton combs in the Fabry-Perot (FP) geometry, with a focus on the differences between the FP geometry and the ring geometry that has been the choice of most experimenters to date. Next, I discuss combs based on electro-optic modulation. I introduce the operational principle, and then describe the first self-referencing of a frequency comb of this kind and a proof-of-principle application experiment. Finally, I discuss a technique for reducing the repetition rate of a high repetition-rate frequency comb, which will be a necessary post-processing step for some applications.

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