Beyond modelocking: High repetition-rate frequency combs derived from a continuous-wave laser

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Daniel C. Cole

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M.S., University of Colorado, 2015

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Optical frequency combs based on modelocked lasers have revolutionized precision metrology by facilitating measurements of optical frequencies, with implications both for fundamental scientific questions and for applications such as fast, broadband spectroscopy. In this thesis, I describe advances in the generation of frequency combs without modelocking in platforms with smaller footprints and higher repetition rates, with the ultimate goal of bringing frequency combs to new applications in a chip-integrated package. I discuss two approaches for comb generation: parametric frequency conversion in Kerr microresonators and active electro-optic modulation of a continuous-wave laser. After introducing microresonator-based frequency combs (microcombs), I discuss two specific developments in microcomb technology: First, I describe a new, extremely reliable method for generation of soliton pulses through the use of a phase-modulated pump laser. This technique eliminates the dependence on initial conditions that was formerly a universal feature of these experiments, presenting a solution to a significant technical barrier to the practical application of microcombs. Second, I present observations of soliton crystal states with highly structured fingerprint optical spectra that correspond to ordered pulse trains exhibiting crystallographic defects. These pulse trains arise through interaction of solitons with avoided mode-crossings in the resonator spectrum. I also discuss generation of Kerr soliton combs in the Fabry-Perot (FP) geometry, with a focus on the differences between the FP geometry and the ring geometry that has been the choice of most experimenters to date. Next, I discuss combs based on electro-optic modulation. I introduce the operational principle, and then describe the first self-referencing of a frequency comb of this kind and a proof-of-principle metrology experiment. Finally, I discuss a technique for reducing the repetition rate of a high-repetition-rate frequency comb, which will be a necessary post-processing step for some applications. I conclude with a discussion of avenues for future research.

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Chapter 1

Summary and outlook

This thesis has described advances in systems for generation of optical frequency combs derived from a continuous wave laser. A major focus of the thesis has been microresonator-based frequency combs. I also described generation of a frequency comb by active-modulation of a CW laser, and I presented results on the downsampling of frequency combs for repetition-rate reduction.

In the context of microresonator-based frequency combs, I described three results: 1. The investigation and implementation of a technique for spontaneous soliton generation in Kerr resonators using a phase-modulated pump laser, 2. The observation and explanation of soliton crystals in Kerr resonators, and 3. A theoretical investigation of Kerr-comb generation in Fabry-Perot cavities, with an emphasis on the properties of solitons and soliton generation. These results all help to more clearly define what is possible with these systems, and suggest avenues for further research.

Soliton generation with a phase-modulated pump laser is a promising candidate for the mechanism by which Kerr-soliton combs can be generated deterministically on chip. Two directions for continued work are additional theoretical investigations of the full LLE with a phase-modulated pump, which could provide insight into the dynamics beyond what is possible using the approximations described in Chapter **??**; and implementation of the technique with resonators that have electronically-inaccessible free-spectral ranges, using the subharmonic-modulation approach that we proposed. Incorporation of the technique into a chip-integrated Kerr-soliton comb may also require modification of the technique that we demonstrated to overcome thermal instabilities associated with the increasing-frequency pump-laser scan.

The investigation of soliton crystals presented here serves several important purposes. First, it represented an important step towards full explanation of observed Kerr-comb phenomena in terms of the LLE model. Second, soliton crystals have the attractive properties of single-soliton Kerr combs, with the additional property that a soliton crystal of N pulses has conversion efficiency of pump-laser power into the comb that is roughly N times higher than a comparable single-soliton comb. With careful preparation of a particular crystal state, this could make them attractive for applications like optical arbitrary waveform generation and nonlinear spectroscopy. Additionally, soliton crystals present a hugely degenerate configuration space that could be useful in implementations, for example, of an on-chip optical buffer or in communications applications [**Leo2010**]. Finally, experimental generation of soliton crystals is significantly simpler than generation of single solitons, where the change in the duty cycle of the optical waveform from extended pattern to single soliton leads to thermal instabilities that are alleviated only with precise control of the pump-laser power and frequency. Thus, it is possible to propose a scheme for deterministic on-chip soliton crystal generation that makes use of two resonators, each constructed of looped single-mode optical waveguides. One resonator is pumped by a laser and hosts the soliton crystal. The second resonator need not be pumped, and exists to provide a specific perturbation to the mode structure of the first resonator to enable soliton crystallization; this could be achieved through careful engineering of the coupling between the resonators. If the free-spectral range of the second resonator is considerably higher than the free-spectral range of the first, and not near one of its harmonics, then realization of single-mode perturbation to the mode structure of the first resonator could be achieved. Implementing deterministic soliton crystal generation on a chip in this way could greatly simplify requirements on the other components in a system for full-integration of Kerr solitons, as soliton generation could be achieved through slow tuning of the pump laser.

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The theoretical investigation of Kerr-comb generation in the Fabry-Perot geometry will provide useful guidance for future experimental work. An obvious direction for continued work is the generation of solitons in Fabry-Perot cavities that make use of the additional degree of freedom provided by the dispersion applied by reflection at the ends of the cavity. This would build on previous experiments [13, 14]. In fact, soliton generation in Fabry-Perot cavities constructed of potted fiber ferrules with high-reflectivity end-coatings has already been realized at NIST Boulder [**Zhang2018**], but there remains work to be done to achieve control the total cavity dispersion with chirped mirror-coatings. Unresolved questions include the effect of uncontrolled expansion of the mode in the coating on both the mirror reflectivity and its group-velocity disperson. Looking to the chip scale, integrated Fabry-Perot cavities construcued of single-mode waveguides with photonic-crystal mirrors is a promising route for development that would further reduce the footprint of Kerr-comb systems. This work is ongoing at NIST Boulder, and primary comb has been observed in such a cavity [**Yu2018**]. Finally, I note that the proposal for deterministic chip-scale generation of soliton crystals presented above could be realized with two co-linear on-chip Fabry-Perot cavities, where the first cavity hosts the crystal, which is out-coupled in reflection, and the second cavity provides a perturbation to the first cavity's mode structure.

Appendix A

Numerical simulations of nonlinear optics

This appendix describes the algorithm used for numerical simulation of the generalized nonlinear Schrodinger equation (GNLSE) and Lugiato-Lefever equation (LLE) to obtain the results presented in the preceding chapters in this thesis. These equations are simulated with Matlab using a fourth-order Runge-Kutta interaction picture (RK4IP) method [89] with adaptive step size [88]. The RK4IP method is a particular algorithm in the broader class of split-step Fourier algorithms, in which nonlinearity is implemented in the time domain and dispersion is implemented in the frequency domain. An illustrative example of this split-step Fourier approach is a far simpler algorithm carried out with a single line of Matlab code to simulate the LLE:

```
psi=ifft(exp(delta*L).*fft(exp(delta*(1i*abs(psi).^2+F./psi)).*psi));
```

where `delta` is the step size and `L` is a linear frequency-domain dispersion operator ($\hat{L}$, see below) that has been defined in the preceding code. The RK4IP algorithm with adaptive step size is advantageous over this simple algorithm in calculation time and in the scaling of error with the step size.

A.1 RK4IP algorithm

The LLE (NLSE) describes the evolution of the field ψ (A), a function of a fast variable θ (T), over a timescale parametrized by a slow variable τ (z). In what immediately follows we use the variable names corresponding to the LLE for simplicity. Each of these equations can be written as the sum of a nonlinear operator $\hat{N}$ and a linear operator $\hat{L}$ acting on ψ, so that the field ψ evolves as

$$\frac{\partial\psi}{\partial\tau} = (\hat{N} + \hat{L})\psi, \tag{A.1}$$

which can be implemented with the split-step Fourier approach.

The RK4IP algorithm specifies a recipe for advancing the field a single step δ in the slow variable τ to obtain $\psi(\theta, \tau + \delta)$ from $\psi(\theta, \tau)$. This specific algorithm has the attractive feature that it reduces the number of Fourier transformations that must be performed to achieve a given calculation accuracy relative to other common algorithms. The RK4IP algorithm is[89]:

$$\psi_I = \exp\left(\frac{\delta}{2}\hat{L}\right)\psi(\theta, \tau) \tag{A.2}$$

$$k_1 = \exp\left(\frac{\delta}{2}\hat{L}\right)\left[\delta\tau\hat{N}(\psi(\theta, \tau))\right]\psi(\theta, \tau) \tag{A.3}$$

$$k_2 = \delta\hat{N}(\psi_I + k_1/2)\left[\psi_I + k_1/2\right] \tag{A.4}$$

$$k_3 = \delta\hat{N}(\psi_I + k_2/2)\left[\psi_I + k_2/2\right] \tag{A.5}$$

$$k_4 = \delta\hat{N}\left(\exp\left(\frac{\delta}{2}\hat{L}\right)(\psi_I + k_3)\right) \tag{A.6}$$

$$\times \exp\left(\frac{\delta}{2}\hat{L}\right)(\psi_I + k_3) \tag{A.7}$$

$$\psi(\theta, \tau + \delta) = \exp\left(\frac{\delta}{2}\hat{L}\right)\left[\psi_I + k_1/6 + k_2/3 + k_3/3\right] + k_4/6. \tag{A.8}$$

In the above it is understood that $\hat{L}$ is applied in the frequency domain and $\hat{N}$ is applied in the time domain. Calculation of $\psi(\theta, \tau+\delta)$ from $\psi(\theta, \tau)$ therefore requires eight Fourier transformations.

A.2 Adaptive step-size algorithm

An adaptive step-size algorithm is a strategy for adjusting the magnitude of the steps δ that are taken to optimize the simulation speed while maintaining a desired degree of accuracy. The RK4IP algorithm exhibits error that scales locally as $O(\delta^5)$. Since reducing the step size naturally requires more steps and therefore increases the number of small errors that accumulate, the resulting global accuracy of the algorithm is $O(\delta^4)$. One appropriate step-size adjustment algorithm for this scaling is described by Heidt [88]. For a given goal error e_G, the algorithm goes as follows:

- Calculate a field ψ_{coarse} by advancing the field $\psi(\theta, \tau)$ according to RK4IP by a step of size δ.

- Calculate a field ψ_{fine} by advancing the field $\psi(\theta, \tau)$ according to RK4IP by two steps of size $\delta/2$.

- Calculate the measured error $e = \sqrt{\sum_j |\psi_{coarse,j} - \psi_{fine,j}|^2 / \sum_j |\psi_{fine,j}|^2}$, where j indexes over the discrete points parametrizing the fast variable θ.

  * If $e > 2e_G$, discard the solution and repeat the process with coarse step size $\delta' = \delta/2$.

  * If $e_G < e < 2e_G$, the evolution continues and the step size is reduced to $\delta' = \delta/2^{1/5} \approx 0.87\delta$.

  * If $e_G/2 < e < e_G$, the evolution continues and the step size is not changed.

  * If $e < e_G/2$, the evolution continues and the step size is increased to $\delta' = 2^{1/5}\delta \approx 1.15\delta$.

When the simulation continues, the new field $\psi(\theta, \tau+\delta)$ is taken to be $\psi(\theta, \tau+\delta) = 16\psi_{fine}/15 - \psi_{coarse}/15$. In the calculations described in this thesis, the goal error e_G is typically 10^{-6}.

A.3 Pseudocode for numerical simulation with the RK4IP algorithm and adaptive step size

The pseudocode shown in Algorithm 1 shows how the RK4IP algorithm with adaptive step size is implemented. This pseudocode neglects the specific details of the RK4IP algorithm.

Two notes:

- The current field $\psi(\theta, \tau)$ is stored until the approximation to the new field $\psi(\theta, \tau + \delta)$ is found to be acceptable.

- This implementation makes use of an extra efficiency that is possible when the solution is discarded and the step size is halved: the first step of the fine solution $\psi_{fine,1}$ for the previous attempt becomes the coarse solution ψ_{coarse} for the current attempt.

Algorithm 1 Pseudocode showing the implementation of RK4IP with adaptive step size.

```
procedure
    while τ < τ_end do
        e = 1                                   ▷ Initialize the error to a large value
        firsttry = TRUE                         ▷ For more efficiency if this is not the first attempt (see below)
        δ = 2δ                                  ▷ To account for halving on the first iteration

        while e > 2e_G do
            if firsttry then
                ψ_coarse = RK4IP(ψ, δ)
            else
                ψ_coarse = ψ_fine,1             ▷ We get to re-use the first step of the previous attempt's fine solution

            δ = δ/2
            ψ_fine = ψ

            for j_step = 1 : 2 do
                ψ_fine = RK4IP(ψ_fine, δ)
                if j_step = 1 then
                    ψ_fine,1 = ψ_fine

            e = sqrt( Σ|ψ_coarse − ψ_fine|^2 / Σ|ψ_fine|^2 )
            firsttry = FALSE

        ψ = 16ψ_fine/15 − ψ_coarse/15
        τ = τ + 2δ                              ▷ We took two fine steps of size δ

        if e > e_G then
            δ = δ/2^(1/5)
        if e < e_G/2 then
            δ = 2^(1/5)δ
```

A.3.1 Simulation of the LLE

For simulation of the LLE, the operators are:

$$\hat{N} = i|\psi|^2 + F/\psi, \tag{A.9}$$

$$\hat{L} = -(1 + i\alpha_\mu), \text{where} \tag{A.10}$$

$$\alpha_\mu = \alpha - \sum_{n=1}^{N} \beta_n \mu^n / n!. \tag{A.11}$$

The subscript μ indicates the pump-referenced mode number upon which the operator acts. Note, in particular, that the pump term F has been incorporated into the nonlinear operator, so that it is implemented in the time domain. The quantity $\hat{N}\psi$ then becomes $i|\psi|^2\psi + F$, as required for computation of $\partial\psi/\partial\tau$.

A.3.2 Simulation of the GNLSE

In addition to self-phase modulation, the GNLSE used in the simulations conducted for Chapter **??** contains nonlinear terms that describe the medium's Raman response and self-steepening. The equation employed can be written as [17, 89]:

$$\frac{\partial A}{\partial z} = -\left(\sum_n \beta_n \frac{i^{n-1}}{n!}\frac{\partial^n}{\partial T^n}\right) A + i\gamma\left(1+\frac{1}{\omega_0}\frac{\partial}{\partial T}\right) \times \left((1-f_R)A|A|^2 + f_R A\int_0^{\infty} h_R(\tau)|A(z,T-\tau)|^2 d\tau\right). \quad \text{(A.12)}$$

For Chapter **??**, second- and third-order dispersion is used with β_2 =-7.7 ps^2/km and β_3 =0.055 ps^3/km, where β_n is the n^{th} frequency-derivative of the propagation constant. The nonlinear coefficient $\gamma = \frac{2\pi}{\lambda}\frac{n_2}{A_{eff}}$ used is 11 W/km [82], coming from an effective mode-field diameter of ~3.5 μm for the HNLF used in the experiment and the nonlinear index $n_2 = 2.7\times10^{-16}$ cm^2/W of silica. The quantity $\omega_0 = 2\pi c/\lambda_0$ is the (angular) carrier frequency of the pulse, and the parameter $f_R = 0.18$ and function

$$h_R(\tau > 0) = (\tau_1^2 + \tau_2^2)/(\tau_1\tau_2^2) \times e^{-\tau/\tau_2}\sin\tau/\tau_1 \quad \text{(A.13)}$$

describe the medium's Raman response, with $\tau_1 = 12.2$ fs and $\tau_2 = 32$ fs used here [17, 89, 93].

The linear frequency-domain operator applied in the RK4IP algorithm is

$$\hat{L} = i\frac{\beta_2}{2}(\omega_\mu - \omega_0)^2 - \frac{\beta_3}{6}(\omega_\mu - \omega_0)^3 \quad \text{(A.14)}$$

Here ω_μ is defined by the discretization of the frequency domain due to Fourier-transformation of a finite temporal window of length T_{comp} via $\omega_\mu = \omega_0 + 2\pi\mu/T_{comp}$; where T_{comp} is the size of the domain for the fast time variable T.

The nonlinear operator $\hat{N}$ for the GNLSE implements the convolution as a product in the frequency domain. That is,

$$\hat{N} = i\gamma\frac{1}{A}\left(1+\frac{1}{\omega_0}\frac{\partial}{\partial T}\right) \times \left[(1-f_R)A|A|^2 + f_R A\,\mathcal{F}^{-1}\left\{\chi_R\cdot\mathcal{F}(|A|^2)\right\}\right], \quad \text{(A.15)}$$

where $\chi_R = \mathcal{F}\{h_R(\tau)\}$ and $\mathcal{F}$ denotes Fourier transformation. Procedurally, the quantity in the square brackets is calculated first, and then the fast-time derivative is implemented and the sum in the curved brackets is calculated.

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