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Optical frequency combs have revolutionized precision metrology by enabling measurements of optical frequencies, with implications both for fundamental scientific questions and for applications such as fast, broadband spectroscopy. In this thesis, I describe the development of comb generation platforms with smaller footprints and higher repetition rates, with the ultimate goal of bringing frequency combs to new applications in a chip-integrated package. I present two new types of frequency combs: electro-optic modulation (EOM) combs and Kerr-microresonator-based frequency combs (microcombs). First I describe the EOM comb scheme and, in particular, techniques for mitigating noise in the comb generation process, and I present the results of a proof-of-principle metrology experiment and some possible applications. Then I discuss developments in microcomb technology. I present novel soliton crystal states, which have highly structured fingerprint optical spectra that correspond to ordered pulse trains exhibiting crystallographic defects. These pulse trains arise through interaction of the solitons with avoided mode-crossings in the resonator spectrum. Next, I describe the direct and deterministic generation of single microresonator solitons using a phase-modulated pump laser. This technique removes the dependence on initial conditions that was formerly a universal feature of these experiments, presenting a solution to a significant technical barrier to the practical application of microcombs. I also discuss generation of Kerr combs in the Fabry-Perot (FP) geometry. I introduce a nonlinear partial differential equation describing dynamics in an FP cavity and discuss the differences between the FP geometry and the ring cavity, which is the geometry used in previous Kerr-comb experiments. Finally, I discuss a technique for reducing the repetition rate of a high-repetition-rate frequency comb, which will be a necessary post-processing step for some applications. I conclude with a discussion of avenues for future research, including the chip-integration of Fabry-Perot Kerr resonators and the use of band-engineered photonic crystal cavities to further simplify soliton generation.

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The work in this thesis would not have been possible...

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- Acknowledgement line 2

# Contents

# Figures

# Chapter 1

# Introduction

The invention of the optical frequency comb two decades ago provided a revolution in precision measurement by dramatically improving the resolution with which we can measure time. This revolution came about through the development of a simple scheme (that required markedly *less* simple advancements in capabilities in nonlinear optics) by which the hundreds-of-terahertz-scale optical frequencies of a mode-locked laser could be effectively measured by electronics with bandwidth limitations on the gigahertz scale. The first frequency combs immediately permitted measurement of fundamental properties of matter, for example the electronic transition frequency in hydrogen, with unprecedented levels of precision. Since those first demonstrations, optical frequency combs have played an integral part in experiments and applications in contexts ranging from record-setting optical clocks, systems for ultra-low-noise microwave synthesis, broadband spectroscopy applications, and stable long-term calibration of astronomical spectrographs for exoplanet detection. Further development of the technology beyond the first stabilization of the Ti:sapphire laser that heralded the frequency comb's arrival has enabled frequency combs to reach applications across many wavelength bands. The technology is reaching maturity, and frequency combs have been commercially available for some time.

In the last decade, methods for generating optical frequency combs without a mode-locked laser have suggested the possibility of bringing their capabilities to a wide set of applications outside the controlled environment of the research laboratory. These new frequency combs come with higher repetition rates, which makes them particularly attractive for applications where high power per comb mode, individual accessibility of comb modes, and fast acquisition times are desired; these applications include arbitrary microwave and optical waveform generation, telecommunications, and broadband, fast-acquisition-time spectroscopy. Moreover, these combs come with lower size, weight, and power (SWAP) requirements, which will enable them to bring the features that make mode-locked laser-based combs attractive into the field, enabling e.g. direct optical frequency synthesis on a chip [1].

This thesis focuses on this second generation of optical frequency combs. The bulk of the thesis covers microresonator-based frequency combs, and especially the nonlinear dynamics involved in the generation of these frequency combs via the Kerr nonlinearity. The penultimate chapter presents a second method for generating a high-repetition-rate frequency comb without modelocking that is based on active modulation of a seed CW laser and subsequent nonlinear spectral broadening. In the final chapter, I present experimental and theoretical investigations of a technique for repetition-rate reduction of frequency combs, which may prove useful for adapting low-SWAP combs and their intrinsically high repetition rates to some applications as the technology continues to develop.

In the remainder of this chapter, I discuss the basic properties of frequency combs and explain how the optical frequencies making up a comb can be fully determined by electronics operating with gigahertz-scale bandwidths.

## 1.1 Optical frequency combs

An optical frequency comb is obtained by fully stabilizing the spectrum of an optical pulse train. The first frequency combs came about through full frequency-stabilization of modelocked lasers; this thesis focuses on frequency combs with pulse trains generated through other means.

### 1.1.1 Optical pulse trains and their spectra

In the time domain, a frequency comb consists of a train of uniformly spaced optical pulses arriving at the pulse train's repetition rate $f_r$. These pulses are typically very short compared to their repetition period $T = 1/f_r$. In the frequency domain, the comb consists of a set of modes that are spaced by $f_r$ in frequency and that have amplitudes determined by an overall spectral envelope centered at the optical carrier frequency, with bandwidth inversely related to the temporal duration of the pulses. The usual description of a frequency comb, which is natural for modelocked-laser-based combs that are not derived from a CW laser, gives the frequencies of these modes as

$$\nu_n = nf_r + f_0, \tag{1.1}$$

where $n \sim f_{carrier}/f_r$ for the optical modes that make up the comb and $f_0$ is the carrier-envelope offset frequency, which may be defined to be between 0 and $f_r$. The offset frequency results from the pulse-to-pulse evolution of the carrier wave underneath the temporal intensity envelope of the pulses due to a difference in group and phase velocities. An equivalent representation of the frequencies of the comb that is more natural for frequency combs directly derived from a CW laser, as described in this thesis, is

$$\nu_\mu = \nu_c + \mu f_r, \tag{1.2}$$

where $\nu_c$ is the frequency of the CW laser, the 'pump' or 'seed' laser, from which the frequency comb is derived and $\mu$ is a pump-referenced mode number, in contrast with the zero-referenced mode number $n$ of Eq. 1.1. Fig. **??** depicts the properties of a frequency comb in the time domain and the frequency domain.

It is useful to consider a mathematical treatment of an optical pulse train to understand the relationships presented above. In the time domain, the electric field $E(t)$ of the pulse train consists of optical pulses that arrive periodically and have baseband (centered at zero frequency) field envelope $A(t)$ multiplying the carrier wave of angular frequency $\omega_c$:

$$E(t) = \sum_{k=-\infty}^{\infty} A(t - kT)e^{i\omega_c t}. \tag{1.3}$$

Here, $T$ is the repetition period of the pulse train. Eq. 1.3 can be viewed as describing a laser of angular frequency $\omega_c$ with a time-varying amplitude. This temporal modulation leads to the distribution of the power across a spectrum whose width scales inversely with the temporal duration of $A$. Intuitively, the spectrum of the comb is the spectrum of the periodic baseband field envelope

$\Sigma_k A(t-kT)$, shifted by the multiplication with $e^{i\omega_c t}$ so that it is centered around the optical carrier. More formally, we can calculate the spectrum $|\mathcal{F}\{E\}|^2$ by calculating

$$\mathcal{F}\{E\}(\omega) \sim \left( \sum_{k=-\infty}^{\infty} \mathcal{F}\{A(t-kT)\} \right) * \delta(\omega - \omega_c), \tag{1.4}$$

which results from the convolution (denoted by $*$) theorem for Fourier transforms. We use the Fourier transform's property that a temporal translation results in a linear spectral phase shift to obtain:

$$\mathcal{F}\{E\} \sim \left( \mathcal{F}\{A\} \times \sum_{k=-\infty}^{\infty} e^{-i\omega kT} \right) * \delta(\omega - \omega_c). \tag{1.5}$$

The quantity $\Sigma_k e^{-i\omega kT}$ is the Fourier-series representation of the series of $\delta$-functions $\Sigma_\mu \delta(\omega - 2\pi\mu/T)$, so we get

$$\mathcal{F}\{E\}(\omega) \sim \left( \mathcal{F}\{A\} \times \sum_{\mu=-\infty}^{\infty} \delta\left(\omega - 2\pi\mu/T\right) \right) * \delta(\omega - \omega_c), \tag{1.6}$$

and performing the convolution leads to the replacement of $\omega$ with $\omega - \omega_c$, leading to:

$$\mathcal{F}\{E\} \sim \sum_{\mu=-\infty}^{\infty} \delta\left(\omega - \omega_c - \mu\omega_r\right) \mathcal{F}\{A\}(\omega - \omega_c). \tag{1.7}$$

This expression indicates that the spectrum of the comb has frequency content at modes $\nu_\mu = \nu_c + \mu f_r$, and that their amplitudes are determined by the spectrum of the baseband field envelope, shifted up to the optical carrier frequency $\nu_c$. This is the natural formulation in the case of a comb derived from a CW laser, but it obscures the carrier-envelope offset frequency in the difference between $\nu_c$ and the nearest multiple of the repetition rate, so that $f_0$ is the remainder of $\nu_c \div f_r$. In practice, if $f_r$ is known, then a measurement of $f_0$ is equivalent to a measurement of the frequency of the input CW laser.

### 1.1.2 Frequency stabilization of optical pulse trains

The scientific need for a method to measure optical frequencies motivated the development of optical frequency combs. While the measurement bandwidth of electronic frequency counters has improved since 1999, it remains limited to frequencies roughly one *million* times lower than the frequency of, e.g., visible red light. Frequency combs present a method for measurement of the unknown frequency $f_{opt}$ of an optical signal through heterodyne with a frequency comb—if $f_{opt}$ falls within the bandwidth of the frequency comb, then the frequency of the heterodyne between the comb and the signal is guaranteed to be less than $f_r/2$, which is typically a frequency that can be measured electronically, at least for modelocked-laser-based combs. Therefore, if the frequencies of the comb are known, measurement of the heterodyne of the comb with the signal reveals the frequency of the signal, provided that the comb mode number $n$, as defined by Eq. 1.1, can be determined. This can be done via a wavelength measurement if sufficient precision is available, or by measuring the change $\partial f_b/\partial f_r = \pm n$, where $f_b$ is the measured frequency of the beat.

The unique utility of the optical frequency comb lies in the fact that measurement of the two frequencies $f_r$ and $f_0$, along with a measurement of the spectral envelope, is sufficient to determine the optical frequencies of all of the modes of the comb, thereby enabling frequency measurement of optical signals. Measurement of the repetition rates of optical pulse trains was possible for many years before the realization of optical frequency comb technology, as this can be done by simply impinging the pulse train on a photodetector. It was the confluence of several technological developments around the turn of the twenty-first century that allowed detection and measurement of the carrier-envelope offset frequency , thereby enabling creation of fully-stabilized modelocked-laser pulse trains: optical frequency combs. cite

The carrier-envelope offset frequency of a pulse train is challenging to measure because it describes evolution of the optical carrier wave underneath the intensity envelope, and therefore cannot be measured through straightforward detection of the intensity of the pulse train. Presently, the most straightforward way to measure $f_0$ is $f-2f$ *self-referencing*, which is illustrated in Fig.**??**. make fig This can be performed only with a pulse train whose spectrum spans an octave—a factor of two in frequency. Given such an octave-spanning supercontinuum spectrum, a group of modes near mode number $N$ is frequency-doubled in a medium with the $\chi^{(2)}$nonlinearity[7]. This frequency-doubled light is heterodyned with the native light in the supercontinuum with mode number near $2N$. The frequency of the resulting beat $f_b$ is:

$$f_b = f_{doubled} - f_{native} \tag{1.8}$$

$$= 2(Nf_r + f_0) - (2Nf_r + f_0) \tag{1.9}$$

$$= f_0. \tag{1.10}$$

Generating the necessary octave-spanning supercontinuum spectrum typically requires nonlinear spectral broadening of the pulse train after its initial generation, except for in specific, carefully engineered cases. Achieving the required degree of spectral broadening while preserving the coherence properties of the pulse train is a significant challenge—typically this requires launching a train of high energy (∼1 nJ), temporally short (≤ 100 fs) pulses into the spectral-broadening stage, and meeting these requirements is one of the important engineering considerations in designing optical frequency comb systems, as discussed in Chapters **??** and **??**.

cite Tara, others

# Chapter 2

# Introduction to microresonator-based frequency combs

This chapter introduces the basic physics of optical frequency-comb generation in Kerr-nonlinear microresonators and discusses some preliminary considerations for the applications of these combs. The two subsequent chapters describe results obtained in these systems. correct?

For simplicity, we will refer to broadband optical spectra generated through frequency conversion in Kerr-nonlinear resonators as 'Kerr combs,' even when the output is not strictly a frequency comb. So far researchers have focused on Kerr-comb generation using microresonators with a ring geometry—so-called microring resonators. It is also possible to generate Kerr combs in a Kerr-nonlinear Fabry-Perot (FP) cavity, as has been recently demonstrated by Obrzud et al [**Obrzud2018**]. Theoretical investigations of Kerr-comb generation with the FP geometry are presented in Chapter **??**.

## 2.1 Optical microring resonators

An optical microring resonator, shown schematically in Fig. **??**, guides light around a closed path in a dielectric medium by total internal reflection. The principle is the same as the guiding of light in an optical fiber, and indeed a 'macroring' resonator can be constructed from a loop of fiber, using a fiber-optic coupler with a small coupling ratio as an input/output port. Ring resonators are sometimes referred to as whispering-gallery mode resonators due to the similarity between their guided modes and the acoustic 'whispering-gallery' waves that permit a listener on one side of St. Paul's cathedral to hear whispers uttered by a speaker on the other side of the cathedral, as explained by Lord Rayleigh beginning in 1910. Optical microring resonators have a host of characteristics that make them useful for nonlinear optics and photonics applications; these include the ease with which they can be integrated, the ultra-high quality ($Q$) factors that have been demonstrated, and the ability to tailor the spectral distribution of guided modes through careful resonator design.

A microring resonator supports propagating guided optical modes of electromagnetic radiation with (vacuum) wavelengths that evenly divide the optical round-trip path length: $\lambda_m = n_{eff}(\lambda_m)L/m$, with associated resonance frequencies $\nu_m = c/\lambda_m = mc/n_{eff}(\nu_m)L$. This leads to constructive interference from round-trip to round-trip. Here $L$ is the physical circumference of the resonator, $m$ is the azimuthal mode number, and $n_{eff}(\lambda_m)$ is an effective index of refraction that depends on the resonator geometry and the transverse intensity profile of the mode (see e.g. [**REFHERE**] for more information). The free-spectral range $f_{FSR}$ of a resonator is the *local* frequency spacing between modes, calculated via:

$$
\begin{aligned}
f_{FSR} &\approx \nu_{m+1} - \nu_m \approx \nu_m - \nu_{m-1}, && (2.1)\\
&= \frac{\partial \nu_m}{\partial m}, && (2.2)\\
&= \frac{c}{n_{eff}(\nu)L} - \frac{mc}{n_{eff}^2(\nu)L}\frac{\partial n_{eff}}{\partial \nu}\frac{\partial \nu}{\partial m}, && (2.3)\\
\Rightarrow f_{FSR} &= \frac{c/L}{\left(n_{eff} + \nu\frac{\partial n_{eff}}{\partial \nu}\right)} = \frac{c}{n_g L} = 1/T_{RT}, && (2.4)
\end{aligned}
$$

where $n_g = n_{eff} + \nu\frac{\partial n_{eff}}{\partial \nu}$ is the group velocity of the mode and $T_{RT}$ is the mode's round-trip time. The effective index $n_{eff}$ is frequency dependent due to both intrinsic material dispersion and geometric dispersion, where the latter results for example from different sampling of core versus cladding material properties for different wavelength-dependent mode areas. A frequency-dependent $n_{eff}$ leads to a frequency dependence of $n_g$ and $f_{FSR}$, and a resulting non-uniform spacing in the cavity modes in frequency despite the linearity of $\nu_m$ in $m$.

Depending on the design, microring resonators can support a single propagating transverse mode profile or may be multi-mode, meaning that many different transverse mode profiles are supported. The former can be readily achieved using chip-integrated photonic waveguides that provide index contrast and transverse confinement on four sides; the latter is typical of resonators that lack an inner radius dimension and therefore exhibit less spatial confinement, such as free-standing silica microrod resonators [2]. For a given resonator geometry, to calculate the frequency-dependent effective index $n_{eff}(\nu)$, thereby enabling calculation of the resonance frequencies and wavelengths, one must solve Maxwell's equations for the resonator geometry. Except in special cases of high symmetry [3], this is typically done numerically using finite-element modeling tools like COMSOL. The modes of an optical resonator, both within a mode family defined by a transverse mode profile and between mode families, must be orthogonal[4], meaning that there is no linear coupling between them.

### 2.1.1 Resonant enhancement in a microring resonator

The lifetime $\tau_\gamma$ of circulating photons in a resonator is fundamental to its fitness for applications. Generally, two processes lead to the loss of circulating photons: intrinsic dissipation that occurs at a rate $1/\tau_{int}$ and outcoupling to an external waveguide that occurs at a rate $1/\tau_{ext}$, leading to a total loss rate of $\tau_\gamma^{-1} = \tau_{ext}^{-1} + \tau_{int}^{-1}$. To understand the quantitative role of these parameters, we consider a cavity mode of frequency $\omega_0$ and amplitude $a$ (normalized such that $|a|^2 = N$, the number of circulating photons) driven by a field with frequency $\Omega_0$ and rotating amplitude $s \propto \exp(i\Omega_0 t)$ (normalized such that $|s|^2 = S$, the rate at which photons in the coupling waveguide pass the coupling port) that is in-coupled with strength $\kappa$. The equation of motion for such a system is[4]:

$$\frac{da}{dt} = i\omega_0 a - \left(\frac{1}{2\tau_{int}} + \frac{1}{2\tau_{ext}}\right) a + \kappa s. \tag{2.5}$$

We can immediately solve this equation by assuming that $a \propto \exp(i\Omega_0 t)$, and we obtain:

check this and all the math here

$$a = \frac{\kappa s}{\left(\frac{1}{2\tau_{int}} + \frac{1}{2\tau_{ext}}\right) + i(\Omega_0 - \omega_0)}. \tag{2.6}$$

To extract anything further from this equation, we must derive a relationship between $\tau_{ext}$ and $\kappa$, which so far are not related. To do this, we exploit the time-reversal symmetry that is inherent in this system when there is no dissipation, that is, when $1/\tau_{int} = 0$. In the case of only an initial excitation $N_0$ decaying into the waveguide with the driving term $s$ set to zero, we have $N = N_0 e^{-t/\tau_{ext}}$. In this case, energy conservation guarantees that the rate $S_{out}$ at which photons propagate away from the resonator in the waveguide is $S_{out} = -dN/dt = N_0 e^{-t/\tau_{ext}}/\tau_{ext}$; we therefore have $S_{out} = N/\tau_{ext}$. In the time-reversed system with $t \to -t$, this amplitude $S_{out}$ describes the rate of pumping: the cavity is resonantly driven with increasing power $S = S_{out}(-t) = N_0 e^{t/\tau_{ext}}/\tau_{ext}$. In this case the frequency of the driving field $s$ can be written $\omega_0 - i/2\tau_{ext}$. Inserting this frequency into Eq. 2.6 gives the equations

$$a = \kappa s \tau_{ext} \tag{2.7}$$

and

$$N = |\kappa|^2 S \tau_{ext}^2. \tag{2.8}$$

By comparing the relationships between $S_{out}$ and $N$ for the forward-evolving system and $S$ and $N$ for the backward-evolving system, we arrive at the relationship $|\kappa|^2 = 1/\tau_{ext}$. We can return to the case including dissipation and insert this expression for $\kappa$ into Eq. 2.6, which can then be squared to obtain:

$$N = \frac{\Delta\omega_{ext} S}{\Delta\omega_{tot}^2/4 + (\Omega_0 - \omega_0)^2} \tag{2.9}$$

Here we define the linewidths $\Delta\omega_{ext} = 1/\tau_{ext}$, $\Delta\omega_{int} = 1/\tau_{int}$, and $\Delta\omega_{tot} = \Delta\omega_{ext} + \Delta\omega_{int}$. Two important observations can be drawn from Eq. 2.9: First, the cavity response is Lorentzian with a full-width at half-maximum (FWHM) linewidth that is related to the photon lifetime via $\tau_\gamma = 1/\Delta\omega_{tot}$, and second, on resonance the number of circulating photons is related to the input rate by the factor $\Delta\omega_{ext}/\Delta\omega_{tot}^2$. The combination of this resonant enhancement and a small cavity mode volume enables very large circulating optical intensities, which is important for the application of microresonators in nonlinear optics.

Two commonly used practical quantities are linked to the photon lifetime: the resonator finesse $\mathcal{F} = 2\pi\tau_\gamma/T_{RT}$, which for a ring resonator can be interpreted literally as the azimuthal resonator angle traced out by a typical photon over its lifetime; and the resonator quality factor $Q = \omega_c \tau_\gamma$, the phase over which the optical field evolves during the photon lifetime. Using the relationship $\tau_\gamma = 1/\Delta\omega_{tot}$, the finesse and quality factor can be rewritten as simple ratios of the relevant frequencies:

$\mathcal{F} = f_{FSR}/\Delta\nu$; $Q = \nu_c/\Delta\nu$, where $\Delta\nu = \Delta\omega_{tot}/2\pi$.

### 2.1.2 Thermal effects in microresonators

In a typical microresonator frequency-comb experiment, a frequency-tunable pump laser is coupled evanescently into and out of the resonator using a tapered optical fiber[**Spillane2003**] (for e.g. free-standing silica disc resonators) or a bus waveguide (for chip-integrated resonators, e.g. in silicon nitride rings). When spatial overlap between the evanescent mode of the fiber and a whispering-gallery mode of the resonator is achieved, with the frequency of the pump laser close to the resonant frequency of that mode, light will build up in the resonator and the transmission of the pump laser past the resonator will decrease.

In any experiment in which a significant amount of pump light is coupled into a resonator, one immediately observes that the cavity resonance lineshape in a scan of the pump-laser frequency is not Lorentzian as expected from Eq. 2.9. This is due to heating of the resonator as it absorbs circulating optical power. Since the volume of a pumped mode and the physical volume of the microresonator are both small, thermal effects have significant practical implications in microresonator experiments. As the volume of the mode heats (over a 'fast thermal timescale') and this energy is conducted to and heats the rest of the resonator (over the 'slow thermal timescale') [5], the resonance frequency of a given cavity mode shifts due to the thermo-optic coefficient $\partial n/\partial T$ and the coefficient of thermal expansion of the mode volume $\partial V/\partial T$. For typical microresonator materials the thermo-optic effect dominates, and $\partial n/\partial T > 0$ leads to a decrease in the resonance frequency with increased circulating power in steady state.

A calculation of the thermal dynamics of the system [6] composed of the pump laser and the resonator reveals that there is a range of pump-laser frequencies $\Omega_0$ (that depends on the pump laser power) near and below the 'cold-cavity' resonance frequency of a given cavity mode over which the system has three possible thermally-shifted resonance frequencies $\omega_{0,shifted}$ at which thermal steady state is achieved. Generally, these points are:

(1) $\Omega_o > \omega_{0,shifted}$, blue detuning with significant coupled power and thermal shift

(2) $\Omega_o < \omega_{0,shifted}$, red detuning with significant coupled power and thermal shift

(3) $\Omega_o \ll \omega_{0,shifted}$, red detuning with insignificant coupled power and no thermal shift

Steady-state point (1) is experimentally important, because in the presence of pump-laser frequency and power fluctuations it leads to so-called thermal 'self-locking.' Specifically for steady-state point (1), this can be seen as follows:

- If the pump-laser power increases the cavity heats, the resonance frequency decreases, the detuning increases, and the change in coupled power is minimized.
- If the pump-laser power decreases the cavity cools, the resonance frequency increases, the detuning decreases, and the change in coupled power is minimized.
- If the pump-laser frequency increases the cavity cools, the resonance frequency increases, and the change in coupled power is minimized.
- If the pump-laesr frequency decreases the cavity heats, the resonance frequency decreases, and the change in coupled power is minimized.

This is in contrast with steady-state point (2), where each of the four pump-laser fluctuations considered above generates a positive feedback loop, with the result that any fluctuation will push the system towards point (1) or point (3). This preference of the system to occupy point (1) or point (3) over a range of pump-laser detuning is referred to as thermal bistability. One consequence of this bistability is that the transmission profile of the pump laser takes on hysteretic behavior in a scan over a cavity resonance with significant pump power: in a decreasing frequency scan, the lineshape takes on a broad sawtooth shape, while in an increasing frequency scan, the resonance takes on a narrow pseudo-Lorentzian profile whose exact shape depends on the scan parameters relative to the thermal timescale. This is shown in Fig. **??**. A second consequence is that operation at red detuning with significant coupled power in a microresonator experiment requires special efforts to mitigate the effects of thermal instability.

## 2.2 Microring resonator Kerr frequency combs

The high circulating optical intensities accessible in resonators with long photon lifetimes find immediate application in the use of microresonators for nonlinear optics. The experiments described in this thesis are conducted in silica microresonators. Silica falls into a broader class of materials that exhibit both centro-symmetry, which dictates that the second-order nonlinear susceptibility $\chi^{(2)}$ must vanish, and a significant third-order susceptibility $\chi^{(3)}$. The $n^{\text{th}}$-order susceptibility is a term in the Taylor expansion describing the response of the medium's polarization to an external electric field[7]: $P = P_0 + \epsilon_0\chi^{(1)}E + \epsilon_0\chi^{(2)}E^2 + \epsilon_0\chi^{(3)}E^3 + ....$ The effect of $\chi^{(3)}$ can be described in a straightforward way as a dependence of the refractive index on the local intensity[8],

$$n = n_0 + n_2 I \tag{2.10}$$

where $n_2 = \frac{3\chi^{(3)}}{4n_0^2\epsilon_0 c}$[8, 9]. The intensity-dependence of the refractive index resulting from the third-order susceptibility $\chi^{(3)}$ is referred to as the optical Kerr effect.

The combination of the Kerr effect and the high circulating intensities that are accessible in high-finesse cavities provides a powerful platform for nonlinear optics. Specifically, the Kerr effect enables self-phase modulation, cross-phase modulation, and four-wave mixing, the last of which is depicted schematically in Fig. **??**.

In 2007, a remarkable result brought the beginning of a new era for frequency comb research. Del'Haye et al reported *cascaded four-wave mixing* (CFWM) in toroidal silica microcavities on silicon chips, the result of which was a set of many co-circulating optical fields that were uniformly spaced by $f_{rep}$ ranging from 375 GHz to ∼750 GHz (depending on the platform)[10]. Measurements indicated that the frequency spacing was uniform to a precision of $7.3 \times 10^{-18}$, thereby establishing that the output of the system was a frequency comb. This result built on previous demonstrations of few-mode parametric oscillations in microresonators [11–13], and showed that the non-uniform distribution of cavity resonance frequencies due to dispersion could be overcome to generate an output with equidistant frequency modes. Demonstrations of Kerr-comb generation in other platforms followed shortly, with realizations in ...A second important development occurred in 2012, when Herr et al

brief history

reported the generation of frequency combs corresponding in the time domain to single circulating optical 'soliton' pulses. This observation followed the observation of solitons in formally-equivalent passive fiber-ring resonators in 2010[14]. Due to unique properties that make them particularly well-suited for applications, as discussed in Sec. 2.6, the generation and manipulation of soliton combs has become a significant priority in microcomb research.

## 2.3 A model for Kerr-comb nonlinear optics: The Lugiato-Lefever equation

Kerr-comb generation can be motivated and partially understood through the CFWM picture [15], but the phase and amplitude degrees of freedom for each comb line mean that CFWM gives rise to a rich space of comb phenomena—it is now known that Kerr combs can exhibit several fundamentally distinct outputs. A useful model for understanding this rich space is the Lugiato-Lefever equation (LLE), which was shown to describe microcomb dynamics by Chembo and Menyuk [16] through Fourier-transformation of a set of coupled-mode equations describing CFWM and by Coen et al [17] through time-averaging of an Ikeda map for a low-loss resonator (as first performed by Haelterman, Trillo, and Wabnitz [18]). The LLE is a nonlinear partial-differential equation that describes evolution of the normalized cavity field envelope $\psi$ over a slow time $\tau = t/2\tau_\gamma$ in a frame check parametrized by the ring's azimuthal angle $\theta$ (running from $-\pi$ to $\pi$) co-moving at the group velocity. The equation as formulated by Chembo and Menyuk, as it will be used throughout this thesis, reads:

$$\frac{\partial \psi}{\partial \tau} = -(1 + i\alpha)\psi + i|\psi|^2\psi - i\frac{\beta}{2}\frac{\partial^2 \psi}{\partial \theta^2} + F. \tag{2.11}$$

This equation describes $\psi$ over the domain $-\pi \leq \theta \leq +\pi$ with periodic boundary conditions $\psi(-\pi, \tau) = \psi(\pi, \tau)$. Here $F$ is the normalized strength of the pump laser, with $F$ and $\psi$ both normalized so that they take the value 1 at the absolute threshold for cascaded four-wave mixing: $F = \sqrt{\frac{8 g_0 \Delta\omega_{ext}}{\Delta\omega_{tot}^3}\frac{P_{in}}{\hbar\Omega_0}}$, $|\psi|^2 = \frac{2 g_0 T_{RT}}{\hbar\omega\Delta\omega_{tot}} P_{circ}(\theta, \tau)$, so that $|\psi(\theta, \tau)|^2$ is the instantaneous normalized power at the co-moving azimuthal angle $\theta$. Here $g_0 = n_2 c\hbar\Omega_0^2/n_g^2 V_0$ is a parameter describing the four-wave mixing gain, $\Delta\omega_{ext}$ is the rate of coupling at the input/output port, $\Delta\omega_{tot} = 1/\tau_\gamma$ is the FWHM resonance linewidth, $P_{in}$ is the pump-laser power, $P_{circ}$ is the circulating power in the cavity, $\hbar$ is Planck's constant, and $\Omega_0$ is the pump-laser frequency. The parameters $n_2$, $n_g$, and $V_0$ describe the nonlinear (Kerr) index (see Eqn. 2.10), the group index of the mode, and the effective nonlinear mode volume at the pump frequency; $L$ is the physical round-trip length of the ring cavity.

The parameters $\alpha$ and $\beta$ describe the normalized frequency detuning of the pump laser and second-order dispersion of the resonator mode family into which the pump laser is coupled: $\alpha = -\frac{2(\Omega_0-\omega_0)}{\Delta\omega_{tot}}$, $\beta = -\frac{2D_2}{\Delta\omega_{tot}}$; here $D_2 = \frac{\partial^2\omega_\mu}{\partial\mu^2}\Big|_{\mu=0}$ is the second-order modal dispersion parameter, where $\mu$ is the pump-referenced mode number of Eq. 1.2. The parameters $D_1 = \frac{\partial\omega_\mu}{\partial\mu}\Big|_{\mu=0} = 2\pi f_{FSR}$ and $D_2$ are related to the derivatives of the propagation constant $\beta_{prop} = n_{eff}(\omega)\omega/c$ via $D_1 = 2\pi/L\beta_1$ and $D_2 = -\frac{D_1^2}{\beta_{prop,1}}\beta_{prop,2}$, where $\beta_{prop,n} = \partial^n\beta_{prop}/\partial\omega^n$. The subscript *prop* is used here to distinguish the propagation constant from the LLE dispersion coefficients $\beta_n = -2D_n/\Delta\omega_{tot}$, as unfortunately the use of the symbol $\beta$ for both of these quantities is standard. Expressions for higher-order modal dispersion parameters $D_n$ in terms of the expansion of the propagation constant can be obtained by evaluating the equation $D_n = (D_1\frac{\partial}{\partial\mu})^n\omega_\mu$. check

The formulation of the LLE in terms of dimensionless normalized parameters helps to elucidate the fundamental properties of the system and facilitates comparison of results obtained in platforms with widely different experimental conditions. In words, the LLE relates the time-evolution of the intracavity field (normalized to its threshold value for cascaded four-wave mixing) to the power of the pump laser (normalized to its value at the threshold for cascaded four-wave mixing), the pump-laser detuning (normalized to half the cavity linewidth), and the cavity second-order disperison quantified by the change in the FSR per mode (normalized to half the cavity linewidth). One example of the utility of this formulation is that it makes apparent the significance of the cavity linewidth in determining the output comb, and underscores the fact that optimization of the dispersion, for example, without paying heed to the effect of this optimization on the cavity linewidth, may not yield the desired results.

The LLE is, of course, a simplified description of the dynamics occurring in the microresonator. It abstracts the nonlinear dynamics and generally successfully describes the various outputs that can be generated in a microresonator frequency comb experiment. The LLE is a good description of these nonlinear dynamics when the resonator photon lifetime, mode-field overlap, and nonlinear index $n_2$ are roughly constant over the bandwidth of the generated comb, and when the dominant contribution to nonlinear dynamics is simply the self-phase modulation term $i|\psi|^2\psi$ arising from the Kerr nonlinearity. The LLE neglects polarization effects, thermal effects, and the Raman scattering and self-steepening nonlinearities, although in principle each of these can be included. It is also worth emphasizing that the LLE can be derived from a more formally-accurate Ikeda map (as is done by Coen et al [17]), in which the effect of localized input- and output-coupling is included in the model. This is achieved by 'delocalizing' the pump field and the output-coupling over the round trip, including only their averaged effects. This is an approximation that is valid in the limit of high finesse due to the fact that the cavity field cannot change on the timescale of a single round trip, but as a result the LLE necessarily neglects all dynamics that might have some periodicity at the round-trip time; the fundamental timescale of LLE dynamics is the photon lifetime.

The LLE provides a useful framework for the prediction and interpretation of experimental results. Basically, it predicts the existence of two fundamentally distinct types of Kerr-combs: extended temporal patterns and localized soliton pulses. These predictions are born out by experiments, the interpretation of which is facilitated by insight gained from the LLE. In the remainder of this chapter I briefly present some simple analytical results that can be obtained from the LLE, and then discuss these two types of comb outputs. This discussion provides context for the results presented in the next two chapters. Fig. **??** summarizes the results that will be presented in the remainder of this chapter.

## 2.4 Analytical investigation of the resonator's CW response

Some insight into comb dynamics can be obtained via analytical investigations of the LLE, Eq. 2.11. This section largely follows the analysis of Ref. [19], with similar analysis having been performed elsewhere, for example in Refs. [17] and [**Barashenkov1996**]. When the derivative term $\partial^2\psi/\partial\theta^2$ inthe LLE is non-zero, $\psi$ is necessarily broadband, and a Kerr comb has been formed. There are no known exact analytical solutions to the LLE to describe Kerr-comb outputs. However, flat solutions to the LLE $\psi_s$ may be calculated by setting all derivatives to zero—when these solutions can be realized physically (discussed below), they describe the behavior of the CW field that exists in the resonator in the absence of Kerr-comb formation. Upon setting the derivatives to zero, one finds:

$$F = (1 + i\alpha)\psi_s - i|\psi_s|^2\psi_s. \tag{2.12}$$

The circulating intensity $\rho = |\psi_s|^2$ is obtained by taking the modulus-square of Eq. 2.12 to get:

$$F^2 = \left(1 + (\alpha - \rho)^2\right)\rho, \tag{2.13}$$

whereupon this equation can be solved for $\rho$. With $\alpha$ held constant, the function $F^2_\alpha(\rho)$ defined by this equation uniquely determines $F^2$ for a given value of $\rho$. By noting that $F^2(\rho = 0) = 0$ and $\partial F^2/\partial\rho\big|_{\rho=0} > 0$, one can conclude that three real solutions for the inverted function $\rho_\alpha(F^2)$ exist between the values $\rho_\pm$ that extremize $F^2_\alpha(\rho)$:

$$\rho_\pm = \frac{2\alpha \pm \sqrt{\alpha^2 - 3}}{3}, \tag{2.14}$$

while outside of this interval there is only one real solution $\rho_\alpha(F^2)$ exists.

Physically, the coexistence of multiple flat solutions $\rho$ at a given point $(\alpha, F^2)$ corresponds to a 'tilting' of the Lorentzian transmission profile of the cavity and leads to bistability, even before taking into account thermal effects. This is illustrated in Fig. **??**. Generally speaking, extended patterns exist on the upper branch of this curve, highlighted in blue, and solitons exist on the lower branch, highlighted in red. For flat solutions $\rho$, an effective Kerr-shifted detuning can be defined as $\alpha_{eff} = \alpha - \rho$. The discussion of thermal effects in Sec. 2.1.2 then applies to the effective detuning $\alpha_{eff}$; that is, operating with significant coupled power at *effective* red-detuning is thermally unstable, while thermal locking occurs with significant coupled power at effective blue detuning. The effective detuning simply incorporates the Kerr nonlinearity into the round-trip phase shift that describes the constructive or destructive interference of the circulating field with the pump at the coupling port. By noting that $\alpha = F^2 = \rho$ solves Eq. 2.13, we can conclude that the position of the effective Kerr-shifted resonance is on the line $\alpha = F^2$, where $\alpha_{eff} = 0$. For fixed $F^2$, an effectively red-detuned branch of the tilted resonance exists above the value of $\alpha$ where $\rho$ becomes multivalued. This value of $\alpha$ can be determined by inserting $\rho_-$ (Eq. 2.14) into Eq. 2.13 and solving for $\alpha$.

Once the circulating intensity $\rho$ is known, the corresponding flat solution $\psi_s$ can be determined from Eq. 2.12 by inserting the known value of $\rho$ and solving for $\psi_s$, with the result:

$$\psi_s = \frac{F}{1 + i(\alpha - \rho)}. \tag{2.15}$$

This expression reveals that the flat solution acquires a phase $\phi_{CW} = \tan^{-1}(\rho - \alpha)$ relative to the pump.

If the flat solution(s) at a point $(\alpha, F^2)$ is (are) unstable, a Kerr comb will form spontaneously. Stability analysis of the flat solutions can be performed, and the results are [19]:

- In the region of multi-stability, if the flat solutions are ordered with increasing magnitude as $\rho_1$, $\rho_2$, and $\rho_3$, the middle solution $\rho_2$ is always unstable.

- A flat solution $\rho$ that is not the middle solution is stable if $\rho < 1$; otherwise it is unstable. When the flat solution is unstable, the mode that experiences the greatest instability has mode number $\mu_{max} = \sqrt{\frac{2}{\beta}(\alpha - 2\rho)}$.

Therefore, the pump-laser threshold curve for Kerr-comb generation can be determined in the $\alpha - F^2$ plane by setting $\rho = 1$ in Eq. 2.12:

$$F^2_{thresh} = 1 + (\alpha - 1)^2, \tag{2.16}$$

$$\alpha_{thresh} = 1 \pm \sqrt{F^2 - 1}, \tag{2.17}$$

for an experiment in which the pump power or detuning is tuned while the other is held fixed.

## 2.5 Kerr comb outputs: extended modulation-instability patterns

Extended temporal patterns arise spontaneously as a result of the instability of the flat solution to the LLE when the pump laser is tuned above the threshold curve. These patterns can be stationary, in which case they are typically referred to as ‘Turing patterns’ or ‘primary comb,’ or can evolve in time, in which case they are typically referred to as 'noisy comb' or 'spatiotemporal chaos.' In general, the former occurs for lower values of the detuning $\alpha$ and smaller pump strengths $F^2$; although some studies of the transition from Turing patterns to chaos have been conducted [**others**, 20], a well-defined boundary between the two has not been established, and may not exist.

In the spatial domain parametrized by $\theta$, a Turing pattern consists of a pulse train with (typically) $n \gg 1$ pulses in the domain $-\pi \leq \theta \leq \pi$—the pulse train's repetition rate is a multiple of the cavity FSR: $f_{rep} = n \times f_{FSR}$. Corresponding to the $n$-fold decreased period (relative to the round-trip time) of an $n$-pulse Turing pattern's modulated waveform in the time domain, the optical spectrum of a Turing pattern consists of modes spaced by $n$ resonator FSR—it is this widely-spaced spectrum that is referred to as 'primary comb.' Analytical approximations for Turing patterns are possible near threshold [21, 22] and in the small damping limit [23]. The stability analysis results from the last section can be used to predict the spacing $n$ of a primary comb (equivalently the number of Turing-pattern pulses) generated in a decreasing-frequency scan across the resonance with fixed normalized pump power $F^2$: $n = \mu_{max,thresh} = \sqrt{\Delta\omega_0(1+\sqrt{F^2-1})/D_2}$.

Spatiotemporal chaos can be understood as a Turing pattern whose pulses oscillate in height, with adjacent pulses oscillating out of phase. From such an oscillating Turing pattern, if $\alpha$ and/or $F^2$ is increased, one moves deeper into the chaotic regime and pulses begin to exhibit lateral motion and collisions; the number of pulses present in the cavity is no longer constant in time. Depending on the severity of the chaos (higher for larger $\alpha$ and $F^2$), a chaotic comb may correspond to a primary-comb-type spectrum with each primary-comb mode exhibiting sidebands at the resonator FSR, so-called 'subcombs,' or it may correspond to a densely-populated spectrum with light in each cavity mode.

Relative to generation of solitons, experimental generation of an extended pattern is straightforward. As shown in Fig. **??**, these patterns are generated with blue effective pump-laser detuning $\alpha_{eff} < 0$, where thermal locking occurs. Because they arise spontaneously from noise, their generation is (comparatively) straightforward: simply decrease the pump-laser frequency until a pattern is generated. Unfortunately, operation of a Kerr-comb in the extended pattern regime is disadvantageous for applications: the $n$-FSR spacing of primary comb presents a challenge for measurement of the repetition rate of the frequency comb due to the bandwidth of measurement electronics, and the aperiodic time-evolution of chaotic comb corresponds to modulation sidebands on the comb modes within the linewidth of the cavity that preclude the use of the comb as a set of stable optical reference frequencies.

An important property of these extended patterns is that they fill the resonator—the characteristic size of temporal features scales roughly as $1/\sqrt{-\beta}$, but these features are distributed densely and uniformly throughout the resonator. This means that the total circulating power of an extended pattern $\int d\theta\, |\psi|^2$ is large relative to the localized pulses discussed in the next section, and therefore that extended patterns come with a comparatively large thermal shift of the resonance.

## 2.6 Kerr comb outputs: solitons

The term 'soliton' generally refers to a localized excitation that can propagate without changing its shape due to a delicate balance between dispersion (or diffraction) and nonlinearity. Solitons are found in several contexts within the field of nonlinear optics, and temporal Kerr-soliton pulses in optical fibers are particularly well known. Microresonators support so-called dissipative cavity solitons, which are localized pulses circulating the resonator that are out-coupled once per round trip. In the case of a single circulating soliton, this leads to a train of pulses propagating away from the resonator with repetition rate $1/T_{RT}$. Thus the mode spacing of the comb matches the FSR of the resonator, in contrast with widely-spaced primary comb spectra, and the soliton can, in principle, propagate indefinitely as a stationary solution to the LLE. This makes Kerr combs based on solitons particularly attractive for applications.

Solitons in optical fibers are solutions of the nonlinear Schrodinger equation (NLSE) that describes pulse-propagation in optical fiber [8]:

$$\frac{\partial A}{\partial z} = i\gamma|A|^2 A - i\frac{\beta}{2}\frac{\partial^2 A}{\partial T^2}. \tag{2.18}$$

This equation describes the evolution of the pulse envelope $A$ in the 'fast-time' reference frame parametrized by $T$ as it propagates down the length of the fiber, parametrized by the distance variable $z$. Here $\gamma$ is the nonlinear coefficient of the fiber and $\beta prop, 2$ is the GVD parameter. The LLE can be viewed as an NLSE with additional loss and detuning terms $-(1 + i\alpha)\psi$ and a driving term $F$.

reference to the above, also do I need a 2 here

The fundamental soliton solution to the NLSE is:

$$A_{sol} = \sqrt{P_0}\,\text{sech}\,(T/\tau)\; e^{i\gamma P_0 z/2 + i\phi_0}, \tag{2.19}$$

where $P_0$ is the peak power of the pulse and is related to the duration of the pulse $\tau$ via $\tau = \sqrt{-\beta/\gamma P_0}$, and $\phi_0$ is an arbitrary phase. Thus, this equation admits a *continuum* of pulsed fundamental 'soliton' solutions, with one existing for each value of the peak power. Each of these solutions propagates down the fiber without changing shape; only the phase evolves with distance as $\phi(z) = \gamma P_0 z/2 + \phi_0$.

The introduction of the loss, detuning, and driving terms into the NLSE to obtain the LLE has several important consequences for solitons. First, exact analytical expressions for the soliton solution to the LLE in terms of elementary functions are not known, in contrast with the situation for the NLSE. However, the soliton solutions to the LLE, Eq. 2.11, can be approximated well as:

$$\psi_{sol} = \psi_{s,min} + e^{i\phi_0}\sqrt{2\alpha}\,\text{sech}\sqrt{\frac{2\alpha}{-\beta}}\theta. \tag{2.20}$$

Here $\psi_{s,min}$ is the flat solution to the LLE from Eq. 2.15 at the point where the soliton solution is desired; when multiple flat solutions exist, $\psi_{s,min}$ is the one corresponding to the smallest intensity $\rho_1$. The phase $\phi_0 = \cos^{-1}(\sqrt{8\alpha}/\pi F)$ arises from the intensity-dependent phase shift in the cavity due to the Kerr effect, mathematically described by the term $i|\psi|^2\psi$.

This approximation $\psi_{sol}$ from Eq. 2.20 for the soliton solution of the LLE illustrates a second important consequence of the differences between the NLSE and the LLE: while the NLSE admits a continuum of fundamental soliton solutions parametrized by their peak power $P_0$ and arbitrary phase $\phi_0$, the LLE supports only one shape for the envelope of a soliton for fixed experimental parameters $\alpha$ and $F^2$. Intuitively, this can be understood as arising from the need for a balance between dispersion and nonlinearity, as in the NLSE, *and* between loss (described by $\partial\psi/\partial\tau = -\psi + ...$) and the pump (described by $\partial\psi/\partial\tau = ... + F$) for stable evolution of an LLE soliton—the driving term is not scaled by $\psi$, which instead would represent linear gain, and therefore provides an absolute reference that fixes the amplitude of the soliton.

The amplitude of the LLE soliton depends only on the detuning $\alpha$, and the width of the soliton increases with larger detuning $\alpha$ and smaller dispersion $\beta$. These characteristics are apparent from the analytical approximation in Eq. **??**, but are also evident in numerical calculations of the exact soliton solution to the LLE[24].

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Solitons exist only where there is a flat solution $\psi_s$ that is effectively red detuned that can form the background for the pulse[**Barashenkov1996**, **Coen2013**]. . Consistent with the phase $\phi_0$ in the approximation $\psi_{sol}$ in Eq. 2.20, solitons can exist up to a maximum detuning of $\alpha_{max} = \pi^2 F^2/8$ [25].

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Solitons are strongly localized: the deviation of the background intensity from $\rho_1$ near a soliton at $\theta_0$ is proportional to $e^{-(\theta-\theta_0)/\delta\theta}$, where $\delta\theta = \sqrt{-\beta/2\alpha}$. If $\delta\theta$ is sufficiently small, multiple solitons can be supported in the resonator domain $-\pi \leq \theta \leq \pi$ with negligible interactions between solitons. Simulations reveal that if $(\theta - \theta_0)/\delta\theta$ is too small, solitons exhibit attractive interactions; the result of this attraction can be pair-wise annihilation or pair-wise merger, with the ultimate result being a stable soliton ensemble with fewer solitons. The maximum number of solitons that can coexist in a resonator in the absence of higher-order stabilizing effects (see Chapter **??**) can be approximated as $N_{max} \approx \sqrt{-2/\beta}$ [25].

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The spectrum of a single-soliton comb has a $\mathrm{sech}^2(\Omega_0/\Delta\Omega)$ envelope, where $\Delta\Omega \approx \sqrt{32\alpha/|\beta|T_{RT}^2}$ is the bandwidth of the pulse in angular frequency. Equivalently, the bandwidth of the soliton in (linear) optical frequency is $\sqrt{\frac{16\Delta\nu f_{rep}^2}{D_2}\alpha}$. For a soliton at the maximum detuning $\alpha_{max} = \pi^2 F^2/8$ for fixed normalized pump power $F^2$, the bandwidth is then $\sqrt{\frac{\pi^2\Delta\nu f_{rep}^2}{2D_2}F^2}$. Because solitons have single-FSR spacing, have the output localized into a high peak-power pulse, and are stationary (in contrast with chaos, which has single-FSR spacing but is not-stationary), they are useful for applications. Many of the proposals for and demonstrations of applications with Kerr-combs have used single-soliton operation.

### 2.6.1 Experimental generation of solitons

Relative to the generation of extended modulation-instability patterns, experimental generation of solitons in microring resonators is challenging. Solitons are localized excitations below threshold, which means that their existence is degenerate with their absence—a resonator can host $N = 0, 1, 2, ...$ up to $N_{max}$ solitons for a given set of parameters $\alpha$ and $F^2$; this degeneracy is illustrated in Fig. **??**. If $\alpha$ and $F^2$ are experimentally tuned to a point at which solitons may exist, $\psi$ will evolve to a form determined by the initial conditions of the field $\psi$. To provide appropriate initial conditions, most experimental demonstrations of soliton generation have involved first generating an extended pattern in the resonator, and then tuning to an appropriate point $(\alpha, F^2)$ so that 'condensation' of solitons from the extended pattern occurs.

Condensation of solitons from an extended pattern presents additional challenges. First, it is difficult to control the number of solitons that emerge, due to the high degree of soliton-number degeneracy. This typically leads to a success rate somewhat lower than 100 % in the generation of single solitons. Second, the transition from a high duty-cycle extended pattern to a lower duty-cycle ensemble of one or several solitons comes with a dramatic drop in intracavity power that occurs on the timescale of the photon lifetime. If the resonator is in thermal steady-state before this drop occurs, the resonator will cool and the resonance frequency will increase. If this increase is large enough that the final detuning $\alpha$ exceeds $\alpha_{max} = \pi^2 F^2/8$, the soliton is lost. This challenge can be addressed by preparing initial conditions for soliton generation and then tuning to an appropriate point $(\alpha, F^2)$ before the cavity can come into thermal steady-state at the temperature determined by the larger power of the extended pattern; this is possible because the timescale over which an extended pattern can be generated is related to the photon lifetime, which is typically much faster than the thermal timescale.

The first report of soliton generation in microresonators came in a paper by Herr et al published in 2014[25]. These authors described optimizing the speed of a decreasing-frequency scan of the pump laser across the cavity resonance so that a soliton could be condensed from an extended pattern and the scan could then be halted at a laser frequency where the soliton could be maintained, with the system in thermal steady-state at the temperature determined by the circulating power of the soliton. Other approaches for dealing with the challenges described above have been developed since this first demonstration; these include fast manipulation of the pump power [**Brash2015**, 24] and periodic modulation of the pump laser's phase or power at $f_{FSR}$[**Lobanov2015**, **Obrzud2017**]. These methods continue to make use of extended patterns to provide initial conditions for soliton generation. In formally-equivalent fiber-ring resonators, direct generation of solitons without condensation from an extended pattern has been demonstrated using transient phase and/or amplitude modulation of the pump laser [**Jang1**, **Jang2**, **AMpaper**]. Chapter **??** of this thesis presents a new variation on these schemes that enables direct generation of solitons using only phase modulation at $f_{FSR}$ without transient manipulation of the system parameters; this approach is based on a proposal by Taheri et al [**Taheri2016**].

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A variety of applications of soliton-based Kerr frequency combs have already been demonstrated.

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