

**Beyond modelocking: High repetition-rate frequency
combs derived from a continuous-wave laser**

by

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Beyond modelocking: High repetition-rate frequency combs derived from a continuous-wave laser

Thesis directed by Dr. Scott A. Diddams

Optical frequency combs based on modelocked lasers have revolutionized precision metrology by facilitating measurements of optical frequencies, with implications both for fundamental scientific questions and for applications such as fast, broadband spectroscopy. In this thesis, I describe advances in the generation of frequency combs without modelocking in platforms with smaller footprints and higher repetition rates, with the ultimate goal of bringing frequency combs to new applications in a chip-integrated package. I discuss two approaches for comb generation: parametric frequency conversion in Kerr microresonators and active electro-optic modulation of a continuous-wave laser. After introducing microresonator-based frequency combs (microcombs), I discuss two specific developments in microcomb technology: First, I describe a new, extremely reliable method for generation of soliton pulses through the use of a phase-modulated pump laser. This technique eliminates the dependence on initial conditions that was formerly a universal feature of these experiments, presenting a solution to a significant technical barrier to the practical application of microcombs. Second, I present observations of *soliton crystal* states with highly structured ‘fingerprint’ optical spectra that correspond to ordered pulse trains exhibiting crystallographic defects. These pulse trains arise through interaction of solitons with avoided mode-crossings in the resonator spectrum. I also discuss generation of Kerr soliton combs in the Fabry-Perot (FP) geometry, with a focus on the differences between the FP geometry and the ring geometry that has been the choice of most experimenters to date. Next, I discuss combs based on electro-optic modulation. I introduce the operational principle, and then describe the first self-referencing of a frequency comb of this kind and a proof-of-principle application experiment. Finally, I discuss a technique for reducing the repetition rate of a high repetition-rate frequency comb, which will be a necessary post-processing step for some applications.

Contents

1	Introduction	1
1.1	Optical frequency combs	3
1.1.1	Optical pulse trains and their spectra	3
1.1.2	Frequency stabilization of optical pulse trains	6
2	Introduction to microresonator-based frequency combs	10
2.1	Optical microring resonators	11
2.1.1	Resonant enhancement in a microring resonator	13
2.1.2	Thermal effects in microresonators	15
2.2	Microring resonator Kerr frequency combs	17
2.2.1	A model for Kerr-comb nonlinear optics: The Lugiato-Lefever equation . . .	19
2.3	Description of Kerr-comb outputs using the Lugiato-Lefever equation	22
2.3.1	Analytical investigation of the resonator's CW response	23
2.3.2	Kerr comb outputs: extended modulation-instability patterns	26
2.3.3	Kerr comb outputs: solitons	29
	References	36

Figures

1.1	Optical frequency combs in the time and frequency domains	5
1.2	Measurement of the carrier-envelope offset frequency via $f - 2f$ self-referencing . . .	9
2.1	Optical microdisk resonator	12
2.2	Thermal bistability in microresonators	17
2.3	An illustration of four-wave mixing and frequency-comb generation.	19
2.4	Solution space for the Lugiato-Lefever equation	23
2.5	Investigation of the circulating CW power in a Kerr resonator	25
2.6	Extended-pattern solutions to the LLE	28
2.7	Soliton solutions to the LLE	31
2.8	Kerr-soliton energy-level diagram	33

Chapter 1

Introduction

The optical frequency comb is an array of uniformly-spaced, phase-coherently oscillating optical fields [1, 2]. This frequency-domain ‘ruler’ is a versatile, critically important tool for science and technology—its invention two decades ago initiated a revolution in precision metrology by dramatically improving the resolution with which we can conveniently measure time and frequency [3–6]. This revolution was brought about by the development of the femtosecond modelocked Ti:sapphire laser [7] and the introduction of new methods for tailoring the fundamental nonlinear interaction between light and matter [8, 9]. The powerful combination of these two advances enabled the generation of laser pulse trains with coherent, octave-spanning spectra and allowed implementation of a simple scheme (see Sec. 1.1.2) by which their hundreds-of-terahertz-scale optical frequencies could be measured electronically.

The connection between the optical and electrical domains afforded by the frequency comb has found an important role in many contexts. These range from basic scientific inquiry, such as measurements of the size of the proton [10], searches for time-variation of the fundamental constants of nature [11, 12], and calibration of astronomical spectrographs for exoplanet detection [13], to applications such as ultra-low-noise microwave synthesis [14, 15], fast and broadband spectroscopy [16, 17], and optical arbitrary waveform generation [18]. Further development of the technology beyond the first stabilization of the Ti:sapphire laser that heralded the frequency comb’s arrival has enabled combs to reach applications across many wavelength bands [19–22], and in particular modelocked fiber-laser frequency combs have proven to be versatile, compact systems that are useful

for many applications [23–25]. The technology is reaching maturity—frequency combs have been deployed outside the laboratory for spectroscopy applications [26, 27] and have been deployed in space [28]. Packaged frequency combs have been commercially available as laboratory tools for some time.

In the last decade, methods for generating optical frequency combs without a modelocked laser have emerged. These new frequency combs come with higher repetition rates and lower fundamental size, weight, and power (SWAP) requirements. Higher repetition rates make them particularly attractive for applications where high power per comb mode, individual accessibility of comb modes, and fast acquisition times are desired; these applications include arbitrary microwave and optical waveform generation, telecommunications, and broadband, temporally-resolved spectroscopy. With their low SWAP requirements, these combs present a promising route towards planar integration of frequency combs, which could enable their seamless inclusion in compact devices. This will continue to drive forward the revolution that was initiated some twenty years ago. There remains much work to be done, however, to develop these combs based on continuous-wave (CW) lasers to the level of technological maturity that has been reached by modelocked-laser-based combs.

This thesis discusses these new frequency combs and has several related, discrete areas of focus. The bulk of the thesis covers microresonator-based frequency combs (microcombs), and especially the nonlinear dynamics involved in the generation of these frequency combs via the Kerr nonlinearity. An introduction to this field is provided in Chapter 2. Chapters ??-?? describe advancements in the field, and Chapter ?? provides a brief summary and a discussion of avenues for further research on this topic. Then, Chapter ?? presents a second method for generating a high repetition-rate frequency comb without modelocking that is based on active modulation of a CW seed laser and subsequent nonlinear spectral broadening. The first self-referencing of a comb of this type is described, and a proof-of-principle application to the generation of low-noise microwaves is discussed. An outlook for further development of this type of comb is presented. Finally, in Chapter ??, I present experimental and theoretical investigations of repetition-rate reduction of frequency combs via pulse gating. This technique may prove useful for adapting high repetition-rate combs to some applications as the

technology continues to develop.

In the remainder of this chapter I discuss the basic properties of frequency combs and explain how the optical frequencies making up a comb can be fully determined by electronics operating with gigahertz-scale bandwidths.

1.1 Optical frequency combs

An optical frequency comb is obtained by fully stabilizing the spectrum of an optical pulse train so that each mode in the spectrum serves as a stable optical reference frequency. The first frequency combs came about through full frequency-stabilization of modelocked lasers, in which many co-lasing modes in a cavity with broadband gain are made to synchronize and generate a train of pulses. This is achieved through the introduction of a modelocking mechanism that favors pulsed operation by lowering loss at higher peak power; such mechanisms include Kerr self-focusing [29, 30], nonlinear polarization rotation [31, 32], and saturable absorption [33]. This thesis focuses on frequency comb pulse trains that are generated through other means—in particular, the combs presented here are derived from a CW laser that functions as the central mode of the comb, and there is no laser cavity with broadband, active gain and multi-mode lasing. The modes of the comb are generated from this input CW laser through passive nonlinear (Chapters 2-??) and active linear (Chapter ??) frequency conversion.

1.1.1 Optical pulse trains and their spectra

In the time domain, a frequency comb consists of a train of uniformly spaced optical pulses arriving at the pulse train's repetition rate f_{rep} , which within the growing space of frequency comb technology is between ~ 10 MHz and ~ 1 THz; the combs discussed in this thesis have repetition rates between 10 GHz and 30 GHz. In most implementations the pulses are very short compared to the repetition period $T = 1/f_{rep}$, with durations on the order of 100 fs. In the frequency domain, the comb consists of a set of modes that are spaced by f_{rep} in frequency and that have amplitudes determined by an overall spectral envelope centered at the optical carrier frequency ν_c (~ 193 THz

in this thesis), with bandwidth inversely related to the temporal duration of the pulses. The usual description of a frequency comb, which is natural for modelocked-laser-based combs that are not derived from a CW laser, gives the frequencies of the comb modes as

$$\nu_n = n f_{rep} + f_0, \quad (1.1)$$

where $n \sim \nu_c / f_{rep}$ for the optical modes that make up the comb and f_0 is the carrier-envelope offset frequency, which may be defined to be between 0 and f_{rep} . The offset frequency results from the pulse-to-pulse evolution of the carrier wave underneath the temporal intensity envelope of the pulses due to a difference in group and phase velocities. An equivalent representation of the frequencies of the comb that is more natural for frequency combs directly derived from a CW laser, as described in this thesis, is

$$\nu_\mu = \nu_c + \mu f_{rep}, \quad (1.2)$$

where ν_c is the frequency of the CW laser, the ‘pump’ or ‘seed’ laser, from which the frequency comb is derived and μ is a pump-referenced mode number, in contrast with the zero-referenced mode number n of Eq. 1.1. Now the carrier-envelope offset frequency f_0 is found in the difference between ν_c and the closest harmonic of f_{rep} : $f_0 = \nu_c - N f_{rep}$, where N is the largest integer such that $f_0 > 0$. Fig. 1.1 depicts the properties of a frequency comb in the time domain and the frequency domain.

It is useful to consider a mathematical treatment of an optical pulse train to understand the relationships presented above. In the time domain, the electric field $E(t)$ of the pulse train consists of optical pulses that arrive periodically and have baseband (centered at zero frequency) field envelope $A(t)$ multiplying the carrier wave of angular frequency $\omega_c = 2\pi\nu_c$:

$$E(t) = \sum_{k=-\infty}^{\infty} A(t - kT) e^{i\omega_c t}. \quad (1.3)$$

Here, T is the repetition period of the pulse train. Eq. 1.3 can be viewed as describing a laser of angular frequency ω_c with a time-varying amplitude. This temporal modulation leads to the distribution of the power across a spectrum whose width scales inversely with the temporal duration

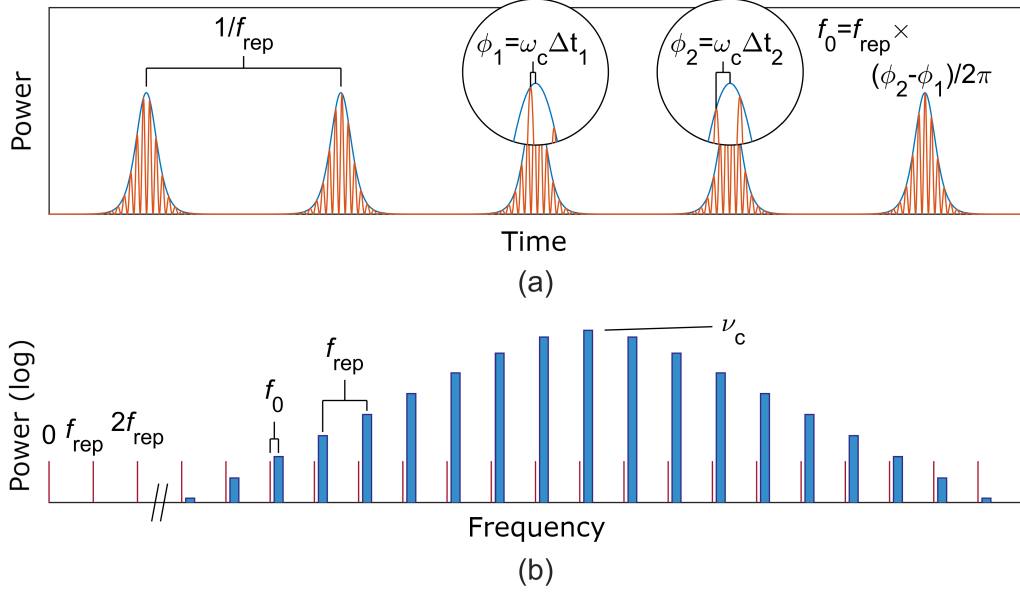


Figure 1.1: **Optical frequency combs in the time and frequency domains.** (a) Time-domain depiction of a frequency comb as a train of pulses spaced by $1/f_{rep}$. The intensity envelope is shown in blue, and the carrier wave is shown in orange. The carrier-envelope offset frequency f_0 arises from a phase-slip of the carrier with respect to the intensity envelope from pulse to pulse. Specifically, if phases $\phi_j = \omega_c \Delta t_j$ are traced out by the carrier wave between its maximum and the j^{th} peak of the pulse train, then $f_0 = \frac{\phi_{j+1} - \phi_j}{2\pi} f_{rep}$. (b) Frequency-domain depiction of the same frequency comb. The comb modes (shown in blue) are centered around an optical frequency ν_c and offset from harmonics of the repetition rate f_{rep} (shown in red) by a frequency shift f_0 . Note that the x-axis has been broken, and the zero-referenced mode numbers of the comb modes shown are large, e.g. $n \sim 19340$ for a 10 GHz repetition-rate comb centered at 1550 nm wavelength (see Chapter ??).

of A . Intuitively, the spectrum of the comb is the spectrum of the periodic baseband field envelope¹ $\Sigma_k A(t - kT)$, shifted in frequency by the multiplication with $e^{i\omega_c t}$ so that it is centered around the optical carrier. More formally, we can calculate the frequency content of the comb by calculating

$$\mathcal{F}\{E\}(\omega) \sim \left(\sum_{k=-\infty}^{\infty} \mathcal{F}\{A(t - kT)\} \right) * \delta(\omega - \omega_c). \quad (1.4)$$

Here \mathcal{F} denotes Fourier transformation and $*$ denotes convolution; this expression results from the Fourier transform's property that the transform of a product is the convolution of the transforms: $\mathcal{F}(A \cdot B) = \mathcal{F}(A) * \mathcal{F}(B)$. Now we use the Fourier transform's property that a temporal translation results in a linear spectral phase shift to obtain:

$$\mathcal{F}\{E\} \sim \left(\mathcal{F}\{A\} \times \sum_{k=-\infty}^{\infty} e^{-i\omega kT} \right) * \delta(\omega - \omega_c). \quad (1.5)$$

¹ which, as the spectrum of a periodic function, is already a comb.

The quantity $\sum_k e^{-i\omega_k T}$ is the Fourier-series representation of the series of δ -functions $\sum_\mu \delta(\omega - 2\pi\mu/T)$ (the *Dirac comb*), so we have

$$\mathcal{F}\{E\}(\omega) \sim \left(\mathcal{F}\{A\} \times \sum_{\mu=-\infty}^{\infty} \delta(\omega - 2\pi\mu/T) \right) * \delta(\omega - \omega_c), \quad (1.6)$$

and performing the convolution leads to the replacement of ω with $\omega - \omega_c$, leading to:

$$\mathcal{F}\{E\} \sim \sum_{\mu=-\infty}^{\infty} \delta(\omega - \omega_c - \mu\omega_r) \mathcal{F}\{A\}(\omega - \omega_c), \quad (1.7)$$

where $\omega_{rep} = 2\pi f_{rep} = 2\pi/T$. This expression indicates that the spectrum of the comb has frequency content at modes $\nu_\mu = \nu_c + \mu f_{rep}$, and that their amplitudes are determined by the spectrum of the baseband field envelope, shifted up to the optical carrier frequency ν_c . This is the natural formulation in the case of a comb derived from a CW laser, but it obscures the carrier-envelope offset frequency in the difference between ν_c and the nearest multiple of the repetition rate, as discussed above. In practice, if f_{rep} is known, then a measurement of f_0 is equivalent to a measurement of the frequency of the input CW laser.

1.1.2 Frequency stabilization of optical pulse trains

The scientific need for a method to measure optical frequencies motivated the development of optical frequency combs. While the measurement bandwidth of electronic frequency counters has improved since 1999, it remains limited to frequencies roughly *ten thousand* times lower than the frequency of, e.g., visible red light. Frequency combs present a method for measurement of the unknown frequency f_{opt} of an optical signal through heterodyne with a frequency comb—if f_{opt} falls within the bandwidth of the frequency comb, then the frequency of the heterodyne between the comb and the signal is guaranteed to be less than $f_{rep}/2$. If the frequencies of the comb are known, measurement of the heterodyne with the signal reveals its frequency f_{opt} , provided that the comb mode number and sign of the beat can be determined. This can be done via a wavelength measurement if sufficient precision is available, or by measuring the change $\partial f_b / \partial f_{rep}$, where f_b is the measured frequency of the beat.

The utility of the optical frequency comb lies in the fact that measurement of the two frequencies f_{rep} and f_0 is sufficient to determine the optical frequencies of all of the modes of the comb, thereby enabling frequency measurement of optical signals. Measurement of the repetition rates of optical pulse trains was possible before the realization of optical frequency comb technology, as this can be done by simply impinging the pulse train on a photodetector. Some pulse trains generated in new platforms have repetition rates too high for direct measurement in this way, but this challenge can be addressed by e.g. spectrally interleaving a lower repetition-rate comb [34, 35]. In general, measurement of f_0 presents the more difficult challenge. It was the confluence of several technological developments around the turn of the twenty-first century that allowed detection and measurement of this frequency, thereby enabling creation of fully-stabilized modelocked-laser pulse trains: optical frequency combs.

The carrier-envelope offset frequency of a pulse train is challenging to measure because it describes evolution of the optical carrier wave underneath the intensity envelope, and therefore cannot be measured through straightforward detection of the intensity of the pulse train. Presently, the most straightforward way to measure f_0 is $f - 2f$ *self-referencing*. This can be performed only with a pulse train whose spectrum spans an octave—a factor of two in frequency. Given such an octave-spanning supercontinuum spectrum, a group of modes near mode number N is frequency-doubled in a medium with the $\chi^{(2)}$ nonlinearity [36]. This frequency-doubled light is heterodyned with the native light in the supercontinuum with mode number near $2N$. The frequency of the resulting beat f_b is:

$$f_b = f_{doubled} - f_{native} \quad (1.8)$$

$$= 2(Nf_{rep} + f_0) - (2Nf_{rep} + f_0) \quad (1.9)$$

$$= f_0. \quad (1.10)$$

Such a scheme is implemented in an $f - 2f$ *interferometer*, which is depicted in Fig. 1.2. Generating the necessary octave-spanning supercontinuum spectrum typically requires nonlinear spectral broadening of the pulse train after its initial generation except for in specific, carefully engineered cases

(e.g. [35, 37]). Achieving the required degree of spectral broadening while preserving the coherence properties of the pulse train is a significant challenge—in the past this has typically required launching a train of high energy (~ 1 nJ), temporally short (≤ 100 fs) pulses into the spectral-broadening stage. Recent developments in nonlinear fiber and waveguide technology have relaxed these requirements slightly (e.g. Ref. 38, also Chapter ??), but maintaining the coherence of the pulse train during spectral broadening remains an important consideration in designing optical frequency comb systems.

The application of $f - 2f$ self-referencing for full frequency-comb stabilization is discussed in Chapters ?? and ?. Self-referencing of microresonator-based frequency combs is not a result presented explicitly in this thesis, but it is nonetheless a key step in the preparation of microcombs for applications and is a motivation for the investigations into microcomb nonlinear dynamics that are presented in Chapters ??-??.

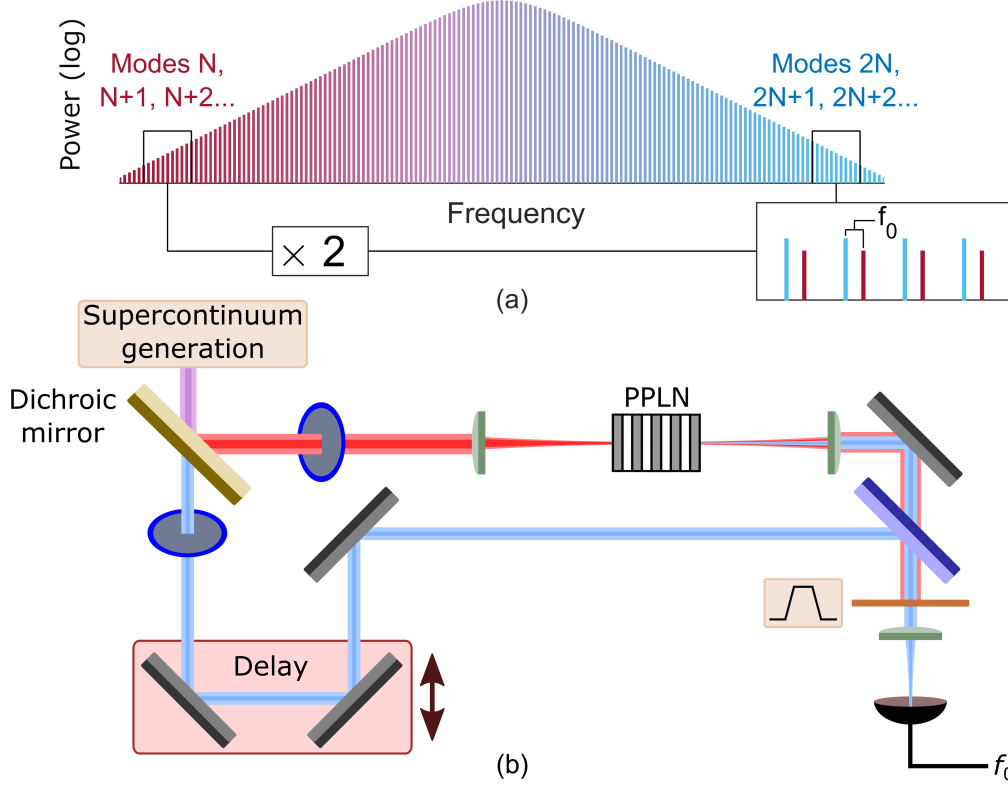


Figure 1.2: **Measurement of the carrier-envelope offset frequency f_0 via $f - 2f$ self-referencing.** (a) Frequency-domain depiction of $f - 2f$ self-referencing: Light on the low frequency end of an octave-spanning supercontinuum is frequency-doubled, and then heterodyned with light on the high frequency end near twice its frequency, enabling measurement of the carrier-envelope offset frequency. (b) Schematic depiction of an $f - 2f$ interferometer: After supercontinuum generation, a dichroic mirror splits the light by wavelength, and the low-frequency end of the supercontinuum (red) is sent through a nonlinear crystal for frequency-doubling. Here the crystal is periodically-poled lithium niobate (PPLN), where quasi-phasematching is employed for efficient doubling of the target modes [39]. The high-frequency end (blue) is sent through a delay stage, which can be adjusted to compensate for temporal walk-off between the spectral components (modes $\sim N$ and modes $\sim 2N$) required for self-referencing during the supercontinuum generation process. The two beams are then recombined by a beamsplitter and sent through a narrow optical band-pass filter centered around the doubled modes, which filters out light not necessary for f_0 measurement to increase the signal-to-noise ratio of the detection. Photodetection of the band-passed beam then reveals f_0 . Waveplates in each path are used to optimize the polarization of the long-wavelength light for frequency-doubling and to ensure co-polarization of the two beams on the detector.

Chapter 2

Introduction to microresonator-based frequency combs

This chapter introduces the basic physics of optical frequency-comb generation in Kerr-nonlinear microring resonators, with a particular emphasis on providing context for the results described in the subsequent chapters. This field emerged in 2007 with the first report of comb generation in silica microtoroids [40], and has evolved rapidly. There are facets to the field that are not discussed here; we note that a number of papers that review this topic have been published, each of which provides a unique perspective [41–44]. The combs generated in Kerr-nonlinear ring resonators, excluding those generated in definitively ‘macro’ fiber loops, have generally been called microcombs, despite the fact that some of the resonators used to generate them have dimensions on the scale of several millimeters. Microcombs are an attractive technology because of their high repetition rates and small footprints, especially relative to modelocked-laser-based combs, which make them promising candidates for inclusion in integrated photonics systems. Microcomb generation has been reported in a variety of platforms, including the aforementioned silica microtoroids, silica wedge [45, 46] and rod [47] resonators, crystalline magnesium-fluoride [48] and calcium-fluoride [49] resonators, and silicon-nitride waveguide resonators [50, 51], which have the advantage of being immediately amenable to photonic integration.

For simplicity, and following the terminology of the field, we will refer to broadband optical spectra generated through frequency conversion in Kerr-nonlinear microring resonators as ‘Kerr combs,’ even when the output is not strictly a coherent frequency comb. Finally, we note that although researchers have so far focused on Kerr-comb generation with the ring geometry, is also

possible to generate Kerr combs in a Kerr-nonlinear Fabry-Perot (FP) cavity, as has been demonstrated in several experiments [52, 53]. Theoretical investigations of Kerr-comb generation with the FP geometry are presented in Chapter ??.

2.1 Optical microring resonators

An optical microring resonator guides light for many round trips around a closed path in a dielectric medium by total internal reflection. The principle is the same as the guiding of light in an optical fiber, and indeed a ‘macroring’ resonator can be constructed from a loop of fiber, using a fiber-optic coupler with a small coupling ratio as an input/output port. Microring resonators can be constructed by looping an optical waveguide back on itself, in which case the resonator provides index contrast and light confinement over a full 360° of the modal cross-section. Alternatively, resonators can be realized with geometries that lack an inner radius dimension and therefore provide less spatial confinement. In this case they can host ‘whispering-gallery modes,’¹ so-called due to their similarity with the acoustic ‘whispering-gallery’ waves that permit a listener on one side of St. Paul’s cathedral (for example) to hear whispers uttered by a speaker on the other side of the cathedral. A schematic depiction of the basic components of a typical microring-resonator experiment is shown in Fig. 2.1. Optical microring resonators have a host of characteristics that make them useful for photonics applications in general and for nonlinear optics in particular; these include the ease with which they can be integrated and the ability to tailor the spectral distribution of guided modes through careful resonator design, as well as the ultra-high quality factors that have been demonstrated (\geq several hundred million). The resonator quality factor Q is defined as $Q = \omega_0 \tau_{ph} = \nu_0 / \Delta\nu$, where $\omega_0 = 2\pi\nu_0$ is the optical angular frequency, τ_{ph} is the photon lifetime, and $\Delta\nu$ is the resonance linewidth. The Q can be interpreted literally as the optical phase that is traversed by the carrier wave during the photon lifetime and is a useful figure of merit for nonlinear optics.

¹ In some sources the terminology ‘whispering-gallery mode resonator’ has been applied more generally, but the analogy to the acoustic case seems most appropriate for resonators in which index contrast is not provided over a full 360° of the modal cross-section. Otherwise it is unclear what makes a WGM resonator different from a fiber loop, which in the limit of large radius obviously does not host whispering-gallery modes. This issue of terminology is discussed in Ref. 54.

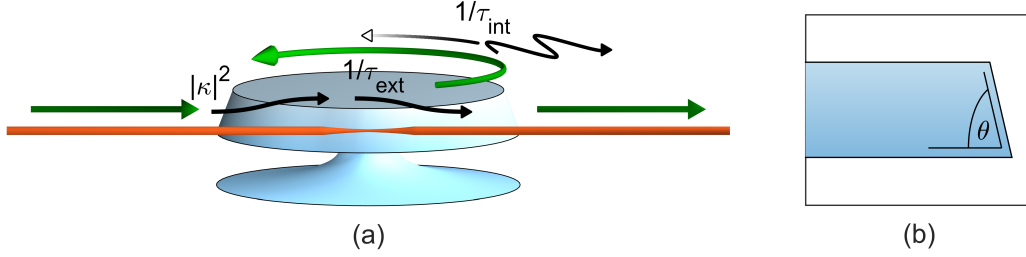


Figure 2.1: **Optical microdisk resonator.** (a) An optical microring resonator with the disk geometry as described in Ref. 45, operated in a through-coupled configuration. Light (green) is evanescently coupled into and out of the resonator through a tapered optical fiber, shown in orange, which contacts the resonator near the fiber's point of smallest diameter. Light circulates in whispering-gallery modes concentric to the resonator's circumference. The black labels indicate the coupling and loss rates discussed in Sec. 2.1.1: $|\kappa|^2$ is the rate at which incoming photons are coupled into the resonator, $1/\tau_{ext}$ is the rate at which circulating photons are coupling into the waveguide, and $1/\tau_{int}$ is the intrinsic loss rate. Here contributions to $1/\tau_{int}$ from absorption and radiative losses are depicted. (b) The wedge angle θ can be adjusted to control the geometric dispersion of the propagating whispering-gallery modes as described in Ref. 55, as θ dictates, for example, the extent to which larger (longer-wavelength) modes are confined further from the circumference of the wedge.

A microring resonator supports propagating guided modes of electromagnetic radiation with (vacuum) wavelengths that evenly divide the optical round-trip path length: $\lambda_m = n_{eff}(\lambda_m)L/m$, with associated resonance frequencies $\nu_m = c/\lambda_m = mc/n_{eff}(\nu_m)L$. This leads to constructive interference from round trip to round trip. Here m is the azimuthal mode number and the quantity $Ln_{eff}(\lambda_m)$ is the optical round-trip path length of the mode, where $n_{eff}(\lambda_m)$ defines an effective index of refraction related to the mode's propagation constant $k(\omega)$ via $k(\omega) = n_{eff}(\omega)\omega/c$ (see e.g. Refs. [56, 57]; we use the symbol k here and reserve the standard symbol β for another quantity). The free-spectral range f_{FSR} of a resonator is the *local* frequency spacing between modes, calculated via:

$$f_{FSR} \approx \frac{\nu_{m+1} - \nu_{m-1}}{2} \quad (2.1)$$

$$= \frac{\partial \nu_m}{\partial m} \quad (2.2)$$

$$= \frac{c}{n_{eff}(\nu)L} - \frac{mc}{n_{eff}^2(\nu)L} \frac{\partial n_{eff}}{\partial \nu} \frac{\partial \nu}{\partial m}, \quad (2.3)$$

so that, rearranging, we obtain:

$$f_{FSR} = \frac{c/L}{\left(n_{eff} + \nu \frac{\partial n_{eff}}{\partial \nu}\right)} = \frac{c}{n_g L} = 1/T_{RT}, \quad (2.4)$$

where $n_g = n_{eff} + \nu \frac{\partial n_{eff}}{\partial \nu}$ is the group velocity of the mode and T_{RT} is the mode's round-trip time. The effective index n_{eff} is frequency dependent due to both intrinsic material dispersion and geometric dispersion, where the latter results for example from different sampling of material properties for different wavelength-dependent mode areas. A frequency-dependent n_{eff} leads to a non-uniform spacing in the cavity modes in frequency despite the linearity of ν_m in m ; equivalently this results in a frequency dependence of n_g and f_{FSR} .

Depending on the design, microring resonators can support many transverse mode profiles, or just one. The former is typical of whispering-gallery-mode resonators that lack an inner radius, such as the wedge resonator shown in Fig. 2.1 or free-standing silica microrod resonators [47]; the latter can be readily achieved using chip-integrated single-mode photonic waveguides. For a given resonator geometry, to calculate the frequency-dependent effective index $n_{eff}(\nu)$, thereby enabling calculation of the resonance frequencies and wavelengths, one must solve Maxwell's equations for the resonator geometry. Except in special cases of high symmetry (e.g. a dielectric sphere [58]), this is typically done numerically using finite-element modeling tools like COMSOL. The modes of an optical resonator, both within a mode family defined by a transverse mode profile (such that they differ only by azimuthal mode number m) and between mode families, must be orthogonal [59], with no linear coupling between them.

2.1.1 Resonant enhancement in a microring resonator

The lifetime τ_{ph} of circulating photons in a resonator is fundamental to its fitness for applications. Generally, two processes lead to the loss of circulating photons: intrinsic dissipation that occurs at a rate $1/\tau_{int}$ and out-coupling to an external waveguide that occurs at a rate $1/\tau_{ext}$, leading to a total loss rate of $\tau_{ph}^{-1} = \tau_{ext}^{-1} + \tau_{int}^{-1}$. To understand the quantitative role of these parameters, we consider a cavity mode of frequency ω_0 and described by instantaneous amplitude $a(t)$ (normalized

such that $|a|^2 = N$, the number of circulating photons) driven by a pump field with frequency ω_p and rotating amplitude $s \propto \exp(i\omega_p t)$ (normalized such that $|s|^2 = S$, the rate at which photons in the coupling waveguide pass the coupling port) that is in-coupled with strength κ . The equation of motion for such a system is [59]:

$$\frac{da}{dt} = i\omega_0 a - \left(\frac{1}{2\tau_{int}} + \frac{1}{2\tau_{ext}} \right) a + \kappa s, \quad (2.5)$$

and the rates that determine the evolution of a are shown schematically in Fig. 2.1. We can immediately solve this equation by assuming that $a \propto \exp(i\omega_p t)$, and we obtain:

$$a = \frac{\kappa s}{\left(\frac{1}{2\tau_{int}} + \frac{1}{2\tau_{ext}} \right) + i(\omega_p - \omega_0)}. \quad (2.6)$$

The coupling strength κ into the waveguide and the out-coupling rate $1/\tau_{ext}$ are related by $|\kappa|^2 = 1/\tau_{ext}$; one can arrive at this conclusion by considering the special case $1/\tau_{int} = 0$ and exploiting the time-reversal symmetry of the system under this condition [59]. By squaring Eq. 2.6 and inserting this relationship between κ and τ_{ext} , we find:

$$N = \frac{\Delta\omega_{ext} S}{\Delta\omega^2/4 + (\omega_p - \omega_0)^2}, \quad (2.7)$$

where we have defined the rates $\Delta\omega_{ext} = 1/\tau_{ext}$, $\Delta\omega_{int} = 1/\tau_{int}$, and $\Delta\omega = \Delta\omega_{ext} + \Delta\omega_{int}$. Two important observations can be drawn from Eq. 2.7: First, the cavity response is Lorentzian with a full-width at half-maximum (FWHM) linewidth that is related to the photon lifetime via $\tau_{ph} = 1/\Delta\omega$, and second, on resonance the number of circulating photons is related to the input rate by the factor $\Delta\omega_{ext}/\Delta\omega^2 \ll 1$. This factor is not yet the resonant enhancement, which we now calculate by considering the circulating power $P = N\hbar\omega_p/T_{RT}$ on resonance (when $\omega_p = \omega_0$):

$$P = \frac{4\Delta\omega_{ext} P_{in}/T_{RT}}{\Delta\omega^2} \quad (2.8)$$

$$= \frac{2}{\pi} P_{in} \eta \mathcal{F}, \quad (2.9)$$

where $\mathcal{F} = 2\pi\tau_{ph}/T_{RT} = f_{FSR}/\Delta\nu$ is the resonator finesse, $\eta = \Delta\omega_{ext}/\Delta\omega$ is the coupling ratio, typically of order $\sim \frac{1}{2}$, and $P_{in} = \hbar\omega_p S$ is the power in the waveguide. Thus, the circulating power is approximately a factor \mathcal{F} greater than the input power. The combination of this resonant

enhancement and a small cavity mode volume enables very large circulating optical intensities in high finesse resonators, which is important for the application of microresonators in nonlinear optics.

2.1.2 Thermal effects in microresonators

In a typical microresonator frequency-comb experiment, a frequency-tunable pump laser is coupled evanescently into and out of the resonator using a tapered optical fiber [60, 61] (for e.g. free-standing silica disc resonators) or a bus waveguide (for chip-integrated resonators, e.g. in silicon nitride rings). When spatial overlap and phase-matching ($n_{eff,res} \sim n_{eff,coupler}$ [62]) between the evanescent mode of the coupler and a whispering-gallery mode of the resonator is achieved, with the frequency of the pump laser close to the resonant frequency of that mode, light will build up in the resonator and the transmission of the pump laser past the resonator will decrease.

In any experiment in which a significant amount of pump light is coupled into a resonator, one immediately observes that the cavity resonance lineshape in a scan of the pump-laser frequency is not Lorentzian as expected from Eq. 2.7; plots of measured resonance lineshapes are shown in Fig. 2.2a. This is because the resonator heats as it absorbs circulating optical power. Associated with this change in temperature are changes in the mode volume and the refractive index, described respectively by the coefficient of thermal expansion $\partial V/\partial T$ and the thermo-optic coefficient $\partial n/\partial T$. For typical microresonator materials the thermo-optic effect dominates, and $\partial n/\partial T > 0$ leads to a decrease in the resonance frequency with increased circulating power in thermal steady state. Thus, for an adiabatic scan across the cavity resonance with decreasing laser frequency, as the laser approaches the resonance in frequency space and power is coupled into the resonator, the resonance frequency will begin to shift with the laser frequency, and a sawtooth-shaped resonance emerges.

The thermal dynamics related to $\partial n/\partial T$ and $\partial V/\partial T$ dictate the signs and values of detuning $\omega_0 - \omega_p$ that are readily accessible in experiment. Specifically, a calculation of the thermal dynamics of the system composed of the pump laser and the resonator reveals that when the pump laser with frequency ω_p is near the ‘cold-cavity’ resonance frequency of a given cavity mode $\omega_{0,cold}$ the resonance has three possible thermally-shifted resonance frequencies $\omega_{0,shifted}$ at which thermal steady state

is achieved [63]. Generally, these points are:

- (1) $\omega_p > \omega_{0,shifted}$, blue detuning² with significant coupled power and thermal shift
- (2) $\omega_p < \omega_{0,shifted}$, red detuning with significant coupled power and thermal shift
- (3) $\omega_p \ll \omega_0$, red detuning with insignificant coupled power and insignificant thermal shift

These points are depicted schematically in Fig. 2.2b. Steady-state point (1) is experimentally important, because in the presence of pump-laser frequency and power fluctuations it leads to so-called thermal ‘self-locking.’ Specifically for steady-state point (1), this can be seen as follows:

- If the pump-laser power increases, the cavity heats, the resonance frequency decreases, the detuning increases, and the change in coupled power is minimized.
- If the pump-laser power decreases, the cavity cools, the resonance frequency increases, the detuning decreases, and the change in coupled power is minimized.
- If the pump-laser frequency increases, the cavity cools, the resonance frequency increases, and the change in coupled power is minimized.
- If the pump-laser frequency decreases, the cavity heats, the resonance frequency decreases, and the change in coupled power is minimized.

This is in contrast with steady-state point (2), where each of the four pump-laser fluctuations considered above generates a positive feedback loop, with the result that any fluctuation will push the system towards point (1) or point (3) and so point (2) is unstable. This preference of the system to occupy point (1) or point (3) over a range of pump-laser detuning is referred to as thermal bistability. As a result of this bistability, point (2) (i.e. red detuning with significant coupled power) cannot be observed in an experimental scan of the pump laser across the resonance in either direction. As explained above, when the pump-laser frequency is decreased the resonance takes on a broad sawtooth shape, while in an increasing-frequency scan the resonance takes on a narrow

² Here we use the convention that the ‘color’ of the detuning specifies the position of the laser with respect to the resonance—‘blue’ detuning means that the laser is more blue, or higher in frequency.

pseudo-Lorentzian profile whose exact shape depends on the scan parameters relative to the thermal timescale. A second consequence is that, in the absence of other stabilizing effects, operation at red detuning with significant coupled power in a microresonator experiment requires special efforts to mitigate the effects of thermal instability.

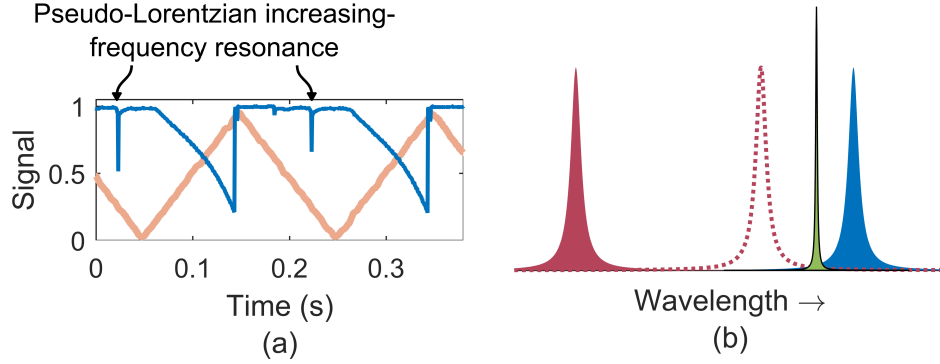


Figure 2.2: **Thermal bistability in microresonators.** (a) Measurement of power transmitted past the microresonator (blue) in an experiment using a ~ 16.5 GHz-FSR microdisk resonator and a tapered fiber. The wavelength of the pump laser is controlled by a piezo-electric crystal that adjusts the length of the laser cavity. Here, larger control signal (orange) corresponds to longer laser wavelength. As the laser wavelength is increased, the resonator heats and a sawtooth-shaped resonance is observed. Ultimately the resonator reaches a maximum temperature that depends on the pump power, and the laser then becomes red-detuned as the wavelength continues to increase; then the resonator rapidly cools and the resonance is lost. Shortly thereafter, the direction of the scan is reversed. As the resonator wavelength is decreased, the system will ‘flip’ from steady-state point (3) to steady-state point (1), leading to observation of a narrow pseudo-Lorentzian resonance, with the exact shape depending on the thermal and scanning timescales. (b) Depiction of the three steady-state points for the laser detuning. For fixed laser wavelength (green), stable steady-state points exist with relatively small blue detuning and significant coupled power (solid blue), and relatively large red detuning and little coupled power (solid red). An unstable steady-state point also exists with red detuning and significant coupled power (dashed red). Note in this terminology that the color of the detuning (red or blue) refers to the position of the laser relative to the position of the resonance in wavelength space.

2.2 Microring resonator Kerr frequency combs

The high circulating optical intensities accessible in resonators with long photon lifetimes find immediate application in the use of microresonators for nonlinear optics. The experiments described in this thesis are conducted in silica microresonators. Silica falls into a broader class of materials that exhibit both centro-symmetry, which dictates that the second-order nonlinear susceptibility

$\chi^{(2)}$ must vanish, and a significant third-order susceptibility $\chi^{(3)}$. The n^{th} -order susceptibility is a term in the Taylor expansion describing the response of the medium's polarization to an external electric field [36]: $P = P_0 + \epsilon_0\chi^{(1)}E + \epsilon_0\chi^{(2)}E^2 + \epsilon_0\chi^{(3)}E^3 + \dots$. The effect of $\chi^{(3)}$ can be described in a straightforward way as a dependence of the refractive index on the local intensity [56],

$$n = n_0 + n_2 I \quad (2.10)$$

where $n_2 = \frac{3\chi^{(3)}}{4n_0^2\epsilon_0 c}$ is called the Kerr index [56, 64]. The intensity-dependence of the refractive index resulting from the third-order susceptibility $\chi^{(3)}$ is referred to as the optical Kerr effect and enables the self-phase modulation (SPM), cross-phase modulation (XPM), and four-wave mixing (FWM) nonlinear processes [36]. Four-wave mixing is a general frequency-domain description of an energy-conserving interaction between fields of up to four different frequencies, as depicted in Figs. 2.3a and b; self-phase modulation and cross-phase modulation can be thought of as time-domain descriptions of particular cases of FWM. In SPM and XPM the nonlinear interaction leads to intensity-dependent phase shifts of the field. If the intensity varies in time, for example in the case of an optical pulse, then the resulting time-varying nonlinear phase shift applies a chirp to the pulse. This can lead to modification of the pulse's spectrum, including the generation of new frequency components.³

For FWM to efficiently lead to the generation of new frequencies in some medium or waveguide it must be effectively phasematched, meaning that the quantity [65]

$$\Delta k = k(\omega_1) + k(\omega_2) - k(\omega') - k(\omega''), \quad (2.11)$$

where $k(\omega) = n_{\text{eff}}(\omega)\omega/c$ is the propagation constant, should be made as small as possible; in the above the four frequencies correspond to those depicted in Figs. 2.3a and b. In a ring resonator, FWM is intrinsically phasematched in interactions between fields propagating in modes with azimuthal mode numbers m_1 , m_2 , m' , and m'' such that $m_1 + m_2 = m' + m''$ [66]. Reports of few-mode parametric oscillation in microresonators based on FWM preceded the first observations of Kerr-comb generation [66–68].

³ SPM doesn't *always* lead to the generation of new frequency components. For example, solitons can propagate without becoming chirped through a balance between SPM and dispersion, and SPM can even lead to narrowing of the bandwidth of a pulse when e.g. the chirp generated by SPM has sign opposite to an existing chirp on the pulse.

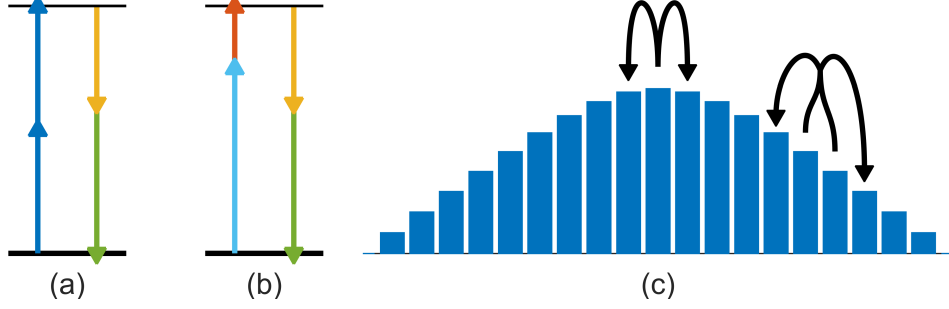


Figure 2.3: **An illustration of four-wave mixing and frequency-comb generation.** (a) Degenerate four-wave mixing, in which two fields of the same frequency ω_1 (blue) mix and generate fields at two new frequencies ω' and ω'' (yellow and green). The schematic indicates the energy-conservation requirements of the process, which can be written as $2\omega_1 = \omega' + \omega''$. (b) Non-degenerate four-wave mixing, in which two fields of different frequencies ω_2 and ω_3 (light blue and orange) mix to generate fields at frequencies ω' and ω'' (yellow and green). Energy conservation is now expressed as $\omega_2 + \omega_3 = \omega' + \omega''$. (c) Schematic depiction of one degenerate FWM step and one non-degenerate FWM step in a cascaded four-wave mixing process that generates a frequency comb. Figure after Ref. 41.

In 2007, the remarkable observation by Del’Haye et al. of *cascaded four-wave mixing* (CFWM, shown in Fig. 2.3c) in anomalously-dispersive ($k'' = \frac{\partial^2 n_{eff}(\omega)\omega}{\partial\omega^2} < 0$) toroidal silica microcavities on silicon chips brought about a new era for frequency comb research. They observed the generation many co-circulating optical fields that were uniformly spaced by f_{rep} ranging from 375 GHz to ~ 750 GHz (depending on the platform) [40]. This result showed that the non-uniform distribution of cavity resonance frequencies due to dispersion could be overcome to generate an output with many equidistant frequency modes. A second important development occurred in 2012, when Herr et al. reported the generation of frequency combs corresponding in the time domain to single circulating optical ‘soliton’ pulses [69, 70]. This observation followed the observation of solitons in formally-equivalent passive fiber-ring resonators in 2010 [71]. Due to unique properties that make them particularly well-suited for applications, as discussed in Sec. 2.3.3, the generation and manipulation of soliton combs has become a significant priority in microcomb research.

2.2.1 A model for Kerr-comb nonlinear optics: The Lugiato-Lefever equation

Kerr-comb generation can be motivated and partially understood through the CFWM picture [72], but the phase and amplitude degrees of freedom for each comb line mean that CFWM gives

rise to a rich space of comb phenomena—it is now known that Kerr combs can exhibit several fundamentally distinct outputs. A useful model for understanding this rich space is the Lugiato-Lefever equation (LLE), which was shown to describe microcomb dynamics by Chembo and Menyuk [73] through Fourier-transformation of a set of coupled-mode equations describing CFWM and by Coen, Randle, Sylvestre, and Erkintalo [74] through time-averaging of a more formally-accurate model for a low-loss resonator (as first performed by Haelterman, Trillo, and Wabnitz [75]). The LLE is a nonlinear partial-differential equation that describes evolution of the normalized cavity field envelope ψ over a slow time $\tau = t/2\tau_{ph}$ in a frame parametrized by the ring’s azimuthal angle θ (running from $-\pi$ to π) co-moving at the group velocity.⁴ A derivation of the LLE is provided in Appendix ???. The equation in the notation of Chembo and Menyuk, as it will be used throughout this thesis, reads:

$$\frac{\partial \psi}{\partial \tau} = -(1 + i\alpha)\psi + i|\psi|^2\psi - i\frac{\beta_2}{2}\frac{\partial^2 \psi}{\partial \theta^2} + F. \quad (2.12)$$

This equation describes ψ over the domain $-\pi \leq \theta \leq +\pi$ with periodic boundary conditions $\psi(-\pi, \tau) = \psi(\pi, \tau)$. Here F is the field strength of the pump laser, with F and ψ both normalized so that they take the value 1 at the absolute threshold for parametric oscillation: $F = \sqrt{\frac{8g_0\Delta\omega_{ext}}{\Delta\omega^3} \frac{P_{in}}{\hbar\omega_p}}$, $|\psi|^2 = \frac{2g_0T_{RT}}{\hbar\omega_p\Delta\omega} P_{circ}(\theta, \tau)$, so that $|\psi(\theta, \tau)|^2$ is the instantaneous normalized power at the co-moving azimuthal angle θ . Here $g_0 = n_2c\hbar\omega_p^2/n_g^2V_0$ is a parameter describing the four-wave mixing gain, $\Delta\omega_{ext}$ is the rate of coupling at the input/output port, $\Delta\omega = 1/\tau_{ph}$ is the FWHM resonance linewidth, P_{in} is the pump-laser power, $P_{circ}(\theta, \tau)$ is the local circulating power in the cavity, \hbar is Planck’s constant, and ω_p is the pump-laser frequency. The parameters n_2 , n_g , and V_0 describe the nonlinear (Kerr) index (see Eqn. 2.10), the group index of the mode, and the effective nonlinear mode volume at the pump frequency; L is the physical round-trip length of the ring cavity.

The parameters α and β_2 describe the frequency detuning of the pump laser and second-order dispersion of the resonator mode family into which the pump laser is coupled, both normalized to

⁴ The co-moving azimuthal angle θ is analogous to the ‘fast time’ variable that appears in, for example, the nonlinear Schrodinger equation for fiber-optic pulse propagation [56], and it can be transformed explicitly to a fast time t via $t = T_{RT} \times \frac{\theta}{2\pi}$.

half the cavity linewidth:

$$\alpha = -\frac{2(\omega_p - \omega_0)}{\Delta\omega}, \quad (2.13)$$

$$\beta_2 = -\frac{2D_2}{\Delta\omega}; \quad (2.14)$$

here $D_2 = \left. \frac{\partial^2 \omega_\mu}{\partial \mu^2} \right|_{\mu=0}$ is the second-order modal dispersion parameter, where μ is the pump-referenced mode number of Eq. 1.2. The parameters $D_1 = \left. \frac{\partial \omega_\mu}{\partial \mu} \right|_{\mu=0} = 2\pi f_{FSR}$ and D_2 are related to the derivatives of the propagation constant $k(\omega) = \frac{n_{eff}(\omega)\omega}{c}$ via $D_1 = 2\pi/Lk'$ and $D_2 = -D_1^2 \frac{k''}{k'}$. It is useful to note that $k' = 1/v_g$, where v_g is the group velocity in the medium, and k'' is often referred to as the GVD parameter and denoted by β_2 , which here is reserved for the dispersion parameter in the LLE. Expressions for higher-order modal dispersion parameters D_n in terms of the expansion of the propagation constant can be obtained by evaluating the equation $D_{n>1} = (D_1 \frac{\partial}{\partial \omega})^{n-1} D_1$, and may be incorporated into the LLE up to desired order N through the replacement:

$$-i \frac{\beta_2}{2} \frac{\partial^2 \psi}{\partial \theta^2} \rightarrow + \sum_{n=1}^N i^{n+1} \frac{\beta_n}{n!} \frac{\partial^n \psi}{\partial \theta^n}, \quad (2.15)$$

where $\beta_n = -2D_n/\Delta\omega$. This thesis describes frequency-comb generation in anomalously-dispersive resonators, and so $\beta_2 < 0$ throughout.

The formulation of the LLE in terms of dimensionless normalized parameters helps to elucidate the fundamental properties of the system and facilitates comparison of results obtained in platforms with widely different experimental conditions. The LLE relates the time-evolution of the intracavity field (normalized to its threshold value for cascaded four-wave mixing) to the power of the pump laser (normalized to its value at the threshold for cascaded four-wave mixing), the pump-laser detuning (normalized to half the cavity linewidth), and the cavity second-order dispersion quantified by the change in the FSR per mode (normalized to half the cavity linewidth). One example of the utility of this formulation is that it makes apparent the significance of the cavity linewidth in determining the output comb, and underscores the fact that optimization of the dispersion, for example, without paying heed to the effect of this optimization on the cavity linewidth, may not yield the desired results. This adds an additional layer of complexity to dispersion engineering relative to straight waveguides.

The LLE is, of course, a simplified description of the dynamics occurring in the microresonator. It abstracts the nonlinear dynamics and generally successfully describes the various outputs that can be generated in a microresonator frequency comb experiment. The LLE is a good description of these nonlinear dynamics when the resonator photon lifetime, mode overlap, and nonlinear index n_2 are roughly constant over the bandwidth of the generated comb, and when the dominant contribution to nonlinear dynamics is simply the self-phase modulation term $i|\psi|^2\psi$ arising from the Kerr nonlinearity. The LLE neglects the polarization of the electric field (ψ is a scalar), as well as thermal effects and the Raman scattering and self-steepening nonlinearities, although in principle each of these can be included [56, 70, 76, 77]. It is also worth emphasizing that the LLE can be derived from a more formally-accurate Ikeda map (as explained by Coen et al. [74]), in which the effect of localized input- and output-coupling is included in the model. This derivation is accomplished by ‘delocalizing’ the pump field and the output-coupling over the round trip, including only their averaged effects. This is an approximation that is valid in the limit of high finesse due to the fact that the cavity field cannot change on the timescale of a single round trip, but as a result the LLE necessarily neglects all dynamics that might have some periodicity at the round-trip time; the fundamental timescale of LLE dynamics is the photon lifetime.

2.3 Description of Kerr-comb outputs using the Lugiato-Lefever equation

The LLE provides a useful framework for the prediction and interpretation of experimental results. Basically, it predicts the existence of two fundamentally distinct types of Kerr-combs: extended temporal patterns and localized soliton pulses. These predictions are born out by experiments, the interpretation of which is facilitated by insight gained from the LLE. In the remainder of this chapter I briefly present some analytical results that can be obtained from the LLE about the behavior of the continuous-wave (CW) field that exists in the resonator in the absence of Kerr-comb formation, and then discuss these two types of comb outputs. This discussion provides context for the results presented in the next two chapters. Fig. 2.4 summarizes the results that will be presented

in the remainder of this chapter, and in particular shows the values of the parameters α and F^2 at which solitons and extended patterns can be obtained.

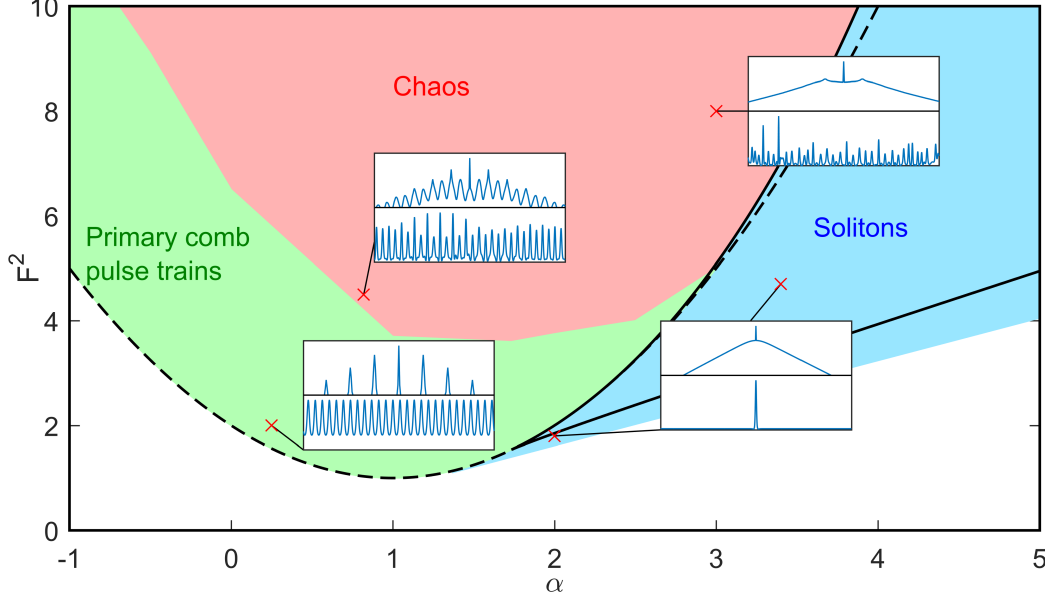


Figure 2.4: **Solution space for the Lugiato-Lefever equation.** Depiction of the various behaviors exhibited by ψ as a function of its position in the $\alpha - F^2$ plane; this predicts the type of Kerr-comb output as a function of the pump-laser detuning and power, the parameters that are most readily adjusted in experiment. Curves plotted in black are obtained through analytical investigation of the LLE; these include the threshold curve for parametric oscillation (dashed black, Eq. 2.22) and the lines obtained via $\rho(\alpha, F^2) = \rho_{\pm}(\alpha)$ (solid black, Eq. 2.19), which define the region where the LLE exhibits multiple flat solutions (i.e. solutions such that $\partial\psi/\partial\theta = 0$, Eq. 2.16). Extended patterns arise above the threshold curve through modulation instability. Solitons exist outside of the threshold curve at higher red detuning, up to an approximate maximum $\alpha_{max} = \pi^2 F^2/8$. The lines bounding the existence of chaos are not known precisely, and in fact chaos can be observed in simulation outside of the threshold curve at values $\alpha > \alpha_{thresh,+}$ (Eq. 2.23). Insets show representative simulation results for the various types of comb outputs in the frequency (top) and time (bottom) domains. Fig. after Ref. 78.

2.3.1 Analytical investigation of the resonator's CW response

Some insight into comb dynamics can be obtained via analytical investigations of the LLE, Eq. 2.12. This section largely follows the analysis of Ref. 78, with similar analysis having been performed elsewhere, for example in Refs. [74] and [79]. When the derivative term $\partial^2\psi/\partial\theta^2$ in the LLE is non-zero, ψ is necessarily broadband, and a Kerr comb has been formed. There are no known exact analytical solutions to the LLE to describe Kerr-comb outputs, which must instead be numerically simulated (see Appendix ??). However, flat solutions ψ_{CW} to the LLE may be calculated

by setting all derivatives to zero—when these solutions can be realized physically (discussed below), they describe a CW field in the resonator. Upon setting the derivatives in the LLE to zero, one finds:

$$F = (1 + i\alpha)\psi_{CW} - i|\psi_{CW}|^2\psi_{CW}. \quad (2.16)$$

The circulating intensity $\rho = |\psi_{CW}|^2$ is obtained by taking the modulus-square of Eq. 2.16 to obtain:

$$F^2 = (1 + (\alpha - \rho)^2) \rho, \quad (2.17)$$

$$= \rho^3 - 2\alpha\rho^2 + (\alpha^2 + 1)\rho, \quad (2.18)$$

whereupon this equation can be numerically solved for ρ . As a third-order polynomial in ρ this equation has three solutions, one or three of which may be real; the complex solutions are unphysical. The function $F^2(\alpha, \rho)$ defined by this equation uniquely determines F^2 given α and ρ . We now consider plotting a graph of $F^2(\alpha, \rho)$ with α held constant; examples are given in Fig. 2.5. By noting that $F^2(\alpha, \rho = 0) = 0$ and $\partial F^2 / \partial \rho|_{\rho=0} > 0$, we can conclude that a graph of $F^2(\alpha, \rho)$ will cross the same value F^2 three times if F^2 is between the extremal values $F_{\pm}^2(\alpha)$ at which $\partial F^2 / \partial \rho = 0$. This means that three real solutions ρ_1 , ρ_2 , and ρ_3 for the inverted function $\rho(\alpha, F^2)$ exist for each value of F^2 between $F_-^2(\alpha)$ and $F_+^2(\alpha)$. The values $F_{\pm}^2(\alpha)$ bounding this region of degeneracy in ρ are found by inserting the values ρ_{\pm} at which $\partial F^2 / \partial \rho = 0$ into Eq. 2.17. That is, $F_{\pm}^2(\alpha) = F^2(\alpha, \rho_{\mp})$, where:

$$\rho_{\pm} = \frac{2\alpha \pm \sqrt{\alpha^2 - 3}}{3}. \quad (2.19)$$

For pump powers outside of the interval $[F_-^2(\alpha), F_+^2(\alpha)]$, which varies with α , there is only one real solution ρ ; within this interval there are three. This is illustrated in Fig. 2.5. The smallest value of F^2 at which the stationary curve ρ becomes multivalued is found to be $F^2 = 8\sqrt{3}/9$ by solving for $\rho_- = \rho_+$ and inserting the corresponding values into Eq. 2.17.

Physically, the coexistence of multiple flat solutions ρ at a given point (α, F^2) corresponds to a ‘tilting’ of the Lorentzian transmission profile of the cavity and leads to bistability, even before taking into account thermal effects. This is illustrated in Fig. 2.5. For flat solutions ρ , an effective Kerr-shifted detuning can be defined as $\alpha_{eff} = \alpha - \rho$. The effective detuning simply incorporates

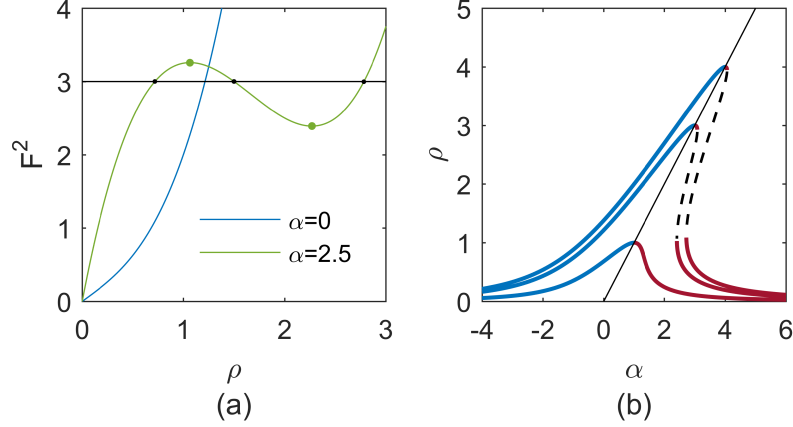


Figure 2.5: **Investigation of the circulating CW power in a Kerr resonator.** (a) Plots of F^2 as a function of ρ for $\alpha = 0$ (blue) and $\alpha = 2.5$ (green), according to Eq. 2.17. When real values of ρ exist that extremize F^2 according to this equation, multiple real solutions for the circulating power ρ exist between these extremal values of F^2 . For $\alpha = 2.5$ we indicate the extremal values of F^2 as green dots. For an example value $F^2 = 3$, the corresponding allowed values ρ_1 , ρ_2 , and ρ_3 are the intersections of the green curve and the black line (black dots); such a line would have three intersections with the green curve for any value of F^2 between $F_\alpha^2(\rho_-)$ and $F_\alpha^2(\rho_+)$. (b) Kerr-tilted resonances curves $\rho(\alpha)$ for $F^2 = 1$ (smallest), $F^2 = 3$, and $F^2 = 4$ (largest). The line $\rho = \alpha = F^2$ (solid black) marks the highest circulating power for a given input power F^2 and separates the effectively blue-detuned and effectively red-detuned branches. When $F^2 > 8\sqrt{3}/9$ (obtained by solving for $\rho_+ = \rho_-$, Eq. 2.19), the resonance becomes tilted steeply enough that an unstable middle branch (dashed black) exists.

the Kerr nonlinearity into the round-trip phase shift that describes the constructive or destructive interference of the circulating field with the pump at the coupling port. By noting that $\alpha = F^2 = \rho$ solves Eq. 2.17, we can conclude that the position of the effective Kerr-shifted resonance is on the line $\alpha = F^2$, where $\alpha_{eff} = 0$.

Once the circulating intensity ρ is known, the corresponding flat solution ψ_{CW} can be determined from Eq. 2.16 by inserting the known value of ρ and solving for ψ_{CW} , with the result:

$$\psi_{CW} = \frac{F}{1 + i(\alpha - \rho)}. \quad (2.20)$$

This expression reveals that the flat solution acquires a phase $\phi_s = \tan^{-1}(\rho - \alpha)$ relative to the pump.

If the flat solution(s) at a point (α, F^2) is (are) unstable, a Kerr comb will form spontaneously. Stability analysis of the flat solutions can be performed, and for the case of second-order dispersion

alone the results are [78]:

- In the region of multi-stability, if the flat solutions are ordered with increasing magnitude as ρ_1 , ρ_2 , and ρ_3 , the middle solution ρ_2 is always unstable.
- When $\alpha < 2$, a flat solution ρ that is not the middle solution is stable if $\rho < 1$; otherwise it is unstable. When the flat solution is unstable, the mode that experiences the greatest instability has mode number given by:

$$\mu_{max} = \sqrt{\frac{2}{\beta_2}(\alpha - 2\rho)} \quad (2.21)$$

Therefore, the pump-laser threshold curve for Kerr-comb generation can be determined in the region $\alpha < 2$ of the $\alpha - F^2$ plane by setting $\rho = 1$ in Eq. 2.16:

$$F_{thresh}^2 = 1 + (\alpha - 1)^2, \quad (2.22)$$

$$\alpha_{thresh,\pm} = 1 \pm \sqrt{F^2 - 1}. \quad (2.23)$$

These equations explicitly describe the point at which comb is generated in an experiment in which the pump power or detuning is varied while the other is held fixed.

2.3.2 Kerr comb outputs: extended modulation-instability patterns

Extended temporal patterns arise spontaneously as a result of the instability of the flat solution to the LLE when the pump laser is tuned above the threshold curve. Two types of extended patterns are shown in Fig. 2.6. These patterns can be stationary, in which case they are typically referred to as ‘Turing patterns’ or ‘primary comb,’ or can evolve in time, in which case they are typically referred to as ‘noisy comb’ or ‘spatiotemporal chaos.’ In general, the former occurs for lower values of the detuning α and smaller pump strengths F^2 ; although some studies of the transition from Turing patterns to chaos have been conducted (e.g. Ref. 80), a well-defined boundary between the two has not been established, and may not exist.

In the spatial domain parametrized by θ , a Turing pattern consists of a pulse train with (typically) $n \gg 1$ pulses in the domain $-\pi \leq \theta \leq \pi$ —the pulse train’s repetition rate is a multiple of

the cavity FSR: $f_{rep} = n \times f_{FSR}$. Corresponding to the n -fold decreased period (relative to the round-trip time) of an n -pulse Turing pattern's modulated waveform in the time domain, the optical spectrum of a Turing pattern consists of modes spaced by n resonator FSR—it is this widely-spaced spectrum that is referred to as ‘primary comb.’ Analytical approximations for Turing patterns are possible near threshold [81, 82] and in the small damping limit [83]. The stability analysis results from the last section can be used to predict the spacing n of a primary comb (equivalently the number of Turing-pattern pulses) generated in a decreasing-frequency scan across the resonance with fixed normalized pump power F^2 :

$$n = \mu_{max,thresh} = \sqrt{\Delta\omega_0(1 + \sqrt{F^2 - 1})/D_2}, \quad (2.24)$$

which is obtained by inserting $\alpha_{thresh,-}$ from Eq. 2.23 and $\rho = 1$ into the expression for μ_{max} in Eq. 2.21 above and moving to the dimensionful dispersion parameter D_2 . Fig. 2.6a shows measured and simulated primary comb spectra and Fig. 2.6b shows the corresponding simulated time-domain waveform.

Spatiotemporal chaos can be understood as a Turing pattern whose pulses oscillate in height, with adjacent pulses oscillating out of phase. From such an oscillating Turing pattern, if α and/or F^2 is increased, one moves deeper into the chaotic regime and pulses begin to exhibit lateral motion and collisions; the number of pulses present in the cavity is no longer constant in time. Depending on the severity of the chaos (greater for larger α and F^2), a chaotic comb may correspond to a primary-comb-type spectrum with each primary-comb mode exhibiting sidebands at the resonator FSR, so-called ‘subcombs,’ or it may correspond to a spectrum with light in each cavity mode. Fig. 2.6c shows measured and simulated time-averaged spectra of chaotic combs and Fig. 2.6d shows a corresponding simulated time-domain waveform.

Relative to generation of solitons, discussed below, experimental generation of an extended pattern is straightforward. These patterns are generated with blue effective pump-laser detuning $\alpha_{eff} < 0$, where thermal locking can occur. Because they arise spontaneously from noise, their generation is (comparatively) straightforward: simply decrease the pump-laser frequency until a

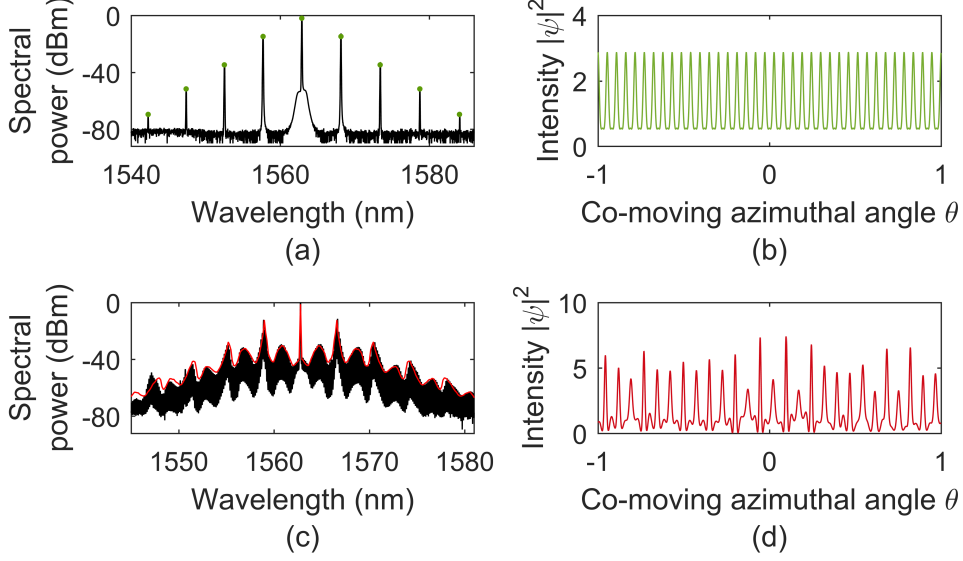


Figure 2.6: **Extended-pattern solutions to the LLE.** (a,b) Primary-comb pulse train in the frequency (a) and time (b) domains. The primary-comb spectrum corresponds to 39 time-domain pulses. The experimental optical spectrum (black) was obtained in a microdisk resonator with 17.32 GHz free-spectral range, and the simulation (green) is conducted with parameters near typical experimental values: $F^2=6$, $\alpha = -0.6$, and $\beta_2 = -0.0044$. (c,d) Spatiotemporal chaos obtained in the same resonator. The experimental measurement (black) yields a time-averaged optical spectrum, with a simulation of qualitatively similar dynamics shown in red. Simulation parameters are $F^2 = 4.2$, $\alpha = 1.2$, and $\beta_2 = -0.0054$. A snapshot of the evolving time-domain waveform is shown in (d).

pattern is generated. Unfortunately, operation of a Kerr-comb in the extended pattern regime is disadvantageous for applications: the n -FSR spacing of primary comb presents a challenge for measurement of the repetition rate of the frequency comb due to the bandwidth of measurement electronics and is also an inefficient use of physical space (i.e. for an n -pulse primary comb pulse train, an equivalent pulse train can always be obtained using the single-soliton output of a resonator with area that is smaller by a factor of $1/n^2$), and the aperiodic time-evolution of chaotic comb corresponds to modulation sidebands on the comb modes within the linewidth of the cavity that preclude the use of the comb as a set of stable optical reference frequencies.

An important property of these extended patterns is that they fill the resonator—the characteristic size of temporal features scales roughly as $1/\sqrt{-\beta_2}$, but these features are distributed densely and uniformly throughout the resonator. This means that the total circulating power of an

extended pattern $\int d\theta |\psi|^2$ is large relative to the localized pulses discussed in the next section, and therefore that extended patterns come with a comparatively large thermal shift of the resonance. As explained below, this contributes to the experimental challenges in soliton generation.

2.3.3 Kerr comb outputs: solitons

The term ‘soliton’ generally refers to a localized excitation that can propagate without changing its shape due to a delicate balance between dispersion (or diffraction) and nonlinearity; sometimes known as ‘solitary waves,’ solitons entered the scientific consciousness in the nineteenth century with their observation by John Scott Russell [84]. They are fundamental solutions to nonlinear partial-differential equations that describe a host of physical phenomena, and are found in several contexts within the field of nonlinear optics: spatial[81, 85] and spatiotemporal solitons (light bullets) [86] have been studied, and soliton modelocking [87, 88] is an important method of femtosecond pulse generation. Temporal Kerr-soliton pulses in optical fibers are particularly well known [56, 89], and have been considered as a candidate for fiber-optic communications protocols [90, 91]. Microresonators support so-called dissipative cavity solitons, which are localized pulses circulating the resonator that are out-coupled once per round trip. In the case of a single circulating soliton, this leads to a train of pulses propagating away from the resonator with repetition rate $1/T_{RT}$. Thus the mode spacing of the comb matches the FSR of the resonator, in contrast with widely-spaced primary comb spectra, and the soliton can, in principle, remain stable and propagate indefinitely as a stationary solution to the LLE. This makes Kerr combs based on solitons particularly attractive for applications.

2.3.3.1 Mathematical description of solitons

Solitons in optical fibers are solutions of the nonlinear Schrodinger equation (NLSE) that describes pulse-propagation in optical fiber [56]:

$$\frac{\partial A}{\partial z} = i\gamma|A|^2 A - i\frac{k''}{2} \frac{\partial^2 A}{\partial T^2}. \quad (2.25)$$

This equation describes the evolution of the pulse envelope A in the ‘fast-time’ reference frame parametrized by T as it propagates down the length of the fiber, parametrized by the distance variable z . Here $\gamma = \frac{2\pi}{\lambda} \frac{n_2}{A_{eff}}$ is the nonlinear coefficient of the fiber, where n_2 is the Kerr index, A_{eff} is the effective nonlinear mode area and λ is the carrier wavelength, and $k'' < 0$ is the GVD parameter. The LLE can be viewed as an NLSE with additional loss and detuning terms $-(1+i\alpha)\psi$ and a driving term F .

The fundamental soliton solution to the NLSE is:

$$A_{sol} = \sqrt{P_0} \operatorname{sech}(T/\tau) e^{i\gamma P_0 z/2 + i\phi_0}, \quad (2.26)$$

where P_0 is the peak power of the pulse and is related to the duration of the pulse τ via $\tau = \sqrt{-l''/\gamma P_0}$, and ϕ_0 is an arbitrary phase. Thus, this equation admits a *continuum* of pulsed fundamental ‘soliton’ solutions, with one existing for each value of the peak power. Each of these solutions propagates down the fiber without changing shape; only the phase evolves with distance as $\phi(z) = \gamma P_0 z/2 + \phi_0$.

The introduction of the loss, detuning, and driving terms into the NLSE to obtain the LLE has several important consequences for solitons. First, exact analytical expressions for the soliton solution to the LLE in terms of elementary functions are not known, in contrast with the situation for the NLSE. However, the soliton solutions to the LLE, Eq. 2.12, can be approximated well as:

$$\psi_{sol} = \psi_{CW,min} + e^{i\phi_0} \sqrt{2\alpha} \operatorname{sech} \sqrt{\frac{2\alpha}{-\beta_2}} \theta. \quad (2.27)$$

Here $\psi_{CW,min}$ is the flat solution to the LLE from Eq. 2.20 at the point where the soliton solution is desired; when multiple flat solutions exist, $\psi_{CW,min}$ is the one corresponding to the smallest intensity ρ_1 . The phase $\phi_0 = \cos^{-1}(\sqrt{8\alpha}/\pi F)$ arises from the intensity-dependent phase shift in the cavity due to the Kerr effect, mathematically described by the term $i|\psi|^2\psi$. We depict this approximation, alongside numerical calculations of exact soliton solutions to the LLE, in Fig. 2.7.

This approximation ψ_{sol} from Eq. 2.27 for the soliton solution of the LLE illustrates a second important consequence of the differences between the NLSE and the LLE: while the NLSE admits a continuum of fundamental soliton solutions parametrized by their peak power P_0 and arbitrary phase

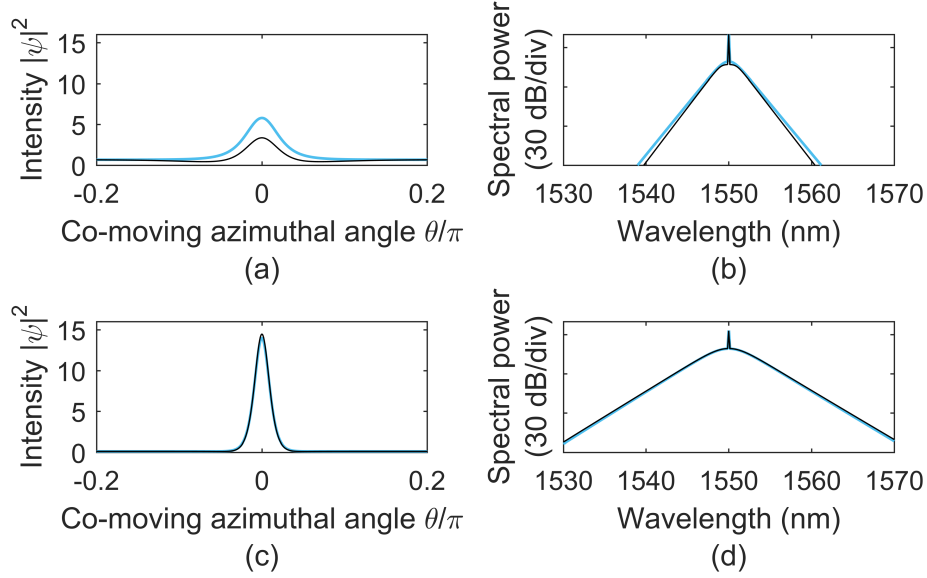


Figure 2.7: **Soliton solutions to the LLE.** Analytical approximations (color) and numerically-calculated exact solutions (black) to the LLE in the time (a,c) and frequency (b,d) domains. The solitons are calculated at $\alpha = 0.95 \alpha_{max} = 0.95 \pi^2 F^2 / 8$ for $F^2 = 8\sqrt{3}/9$ (a,b) and $F^2 = 6$ (c,d) with $\beta_2 = -0.02$ in both cases. The isolated spectral spike is at the pump frequency and corresponds to the CW background $\psi_{CW,min}$. Spectra are calculated using $f_{rep} = 16.5$ GHz with pump wavelength of $\lambda_p = 1550$ nm. For experimental measurements of solitons in microring resonators, see Chapters ?? and ??.

ϕ_0 , the LLE supports only one shape for the envelope of a soliton for fixed experimental parameters. Intuitively, this can be understood as arising from the need for the round-trip phase shift for all points on the soliton to be zero in steady-state; the introduction of the detuning parameter α breaks the degeneracy that exists for the NLSE within the continuum of soliton solutions.

The analytical approximation in Eq. 2.27 indicates the scaling of the amplitude and width of the LLE soliton with the experimental parameters: the amplitude of the LLE soliton, prior to its summation with the CW background, depends only on the detuning α , and the width of the soliton increases with larger detuning α and smaller dispersion β_2 . Importantly, if one is concerned with maximizing the bandwidth of the soliton, it is important to minimize β_2 and maximize α , due to the inverse relationship between temporal duration and spectral bandwidth. The spectrum of a single-soliton Kerr comb has a $\text{sech}^2((\omega - \omega_p)/\Delta\omega_{sol})$ envelope, where ω is the optical angular frequency and $\Delta\omega_{sol} \approx \sqrt{32\alpha/|\beta_2|T_{RT}^2}$ is the bandwidth of the pulse in angular frequency. Equivalently,

the bandwidth of the soliton in (linear) optical frequency is $\sqrt{\frac{16\Delta\nu f_{rep}^2}{D_2}}\alpha$, where $\Delta\nu$ is the resonance linewidth in linear frequency; the spectral width in mode number is $\Delta\mu_{sol} \approx 4\sqrt{\alpha\Delta\nu/D_2}$. Consistent with the phase ϕ_0 in the approximation ψ_{sol} in Eq. 2.27, solitons can exist up to a maximum detuning of $\alpha_{max} \sim \pi^2 F^2/8$ [70]. For a soliton at the maximum detuning for fixed normalized pump power F^2 , the bandwidth is then $\sqrt{\frac{\pi^2\Delta\nu f_{rep}^2}{2D_2}}F^2$.

Solitons exist only where there is a stable flat solution ψ_{CW} that is effectively red detuned that can form the background for the pulse [79, 92]. This effectively red-detuned background is itself thermally unstable (see Sec. 2.1.2), but the existence of the soliton acts to stabilize the pump detuning. As explained by Herr et al., the soliton provides a local modulation of the refractive index through the Kerr effect, which changes the round-trip phase shift of pump light that arrives coincidentally with the soliton at the coupling port [70]. This leads to a *local* increase in the resonant wavelength for this pump light. Thus there are effectively two resonant wavelengths, a smaller one determined by the round-trip phase shift including the Kerr shift from the CW background, and a larger one determined by the round-trip phase shift including the Kerr shift from the soliton [93]. The pump laser can be effectively blue-detuned with respect to the latter resonance, which can lead to thermally stable operation in the soliton regime.

Solitons are strongly localized: as can be seen from Eq. 2.27, the deviation of the background intensity from ρ_1 near a soliton at θ_0 is proportional to $e^{-(\theta-\theta_0)/\delta\theta}$, where $\delta\theta = \sqrt{-\beta_2/2\alpha}$. If $\delta\theta$ is sufficiently small, multiple solitons can be supported in the resonator domain $-\pi \leq \theta \leq \pi$ with very weak interactions between solitons. If the separation between solitons i and j at θ_i and θ_j is small relative to $\delta\theta$, the solitons will interact. The topic of soliton interactions is complicated in general, with different types of interactions in different systems (see e.g. Refs. [94–97]). Simulations reveal that if $(\theta_i - \theta_j)/\delta\theta$ is too small, LLE solitons exhibit attractive interactions as a result of the monotonic (as opposed to oscillatory) decay of the localized pulse to ψ_{CW} [98], which precludes the existence of stable equilibrium separations. The result of this attraction can be pair-wise annihilation or merger, with the ultimate result being an ensemble with fewer solitons. The maximum number of solitons that can coexist in a resonator in the absence of higher-order stabilizing effects (see Chapter

?? and Refs. [98, 99]) can be approximated as $N_{max} \approx \sqrt{-2/\beta_2}$ [70]. An approximation to the form of a soliton ensemble is possible as:

$$\psi_{ens} = \psi_{CW,min} + e^{i\phi_0} \sqrt{2\alpha} \sum_j \text{sech} \left(\sqrt{\frac{2\alpha}{-\beta_2}} (\theta - \theta_j) \right), \quad (2.28)$$

where $\{\theta_j\}$ define the positions of the solitons in the ensemble and $\phi_0 = \cos^{-1}(\sqrt{8\alpha}/\pi F)$ as above.

Fig. 2.8 provides an example illustrating the degeneracy in soliton number of Kerr-combs operating in the soliton regime.

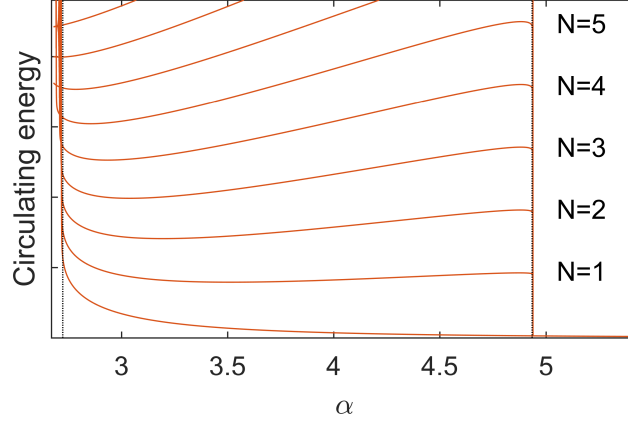


Figure 2.8: **Kerr-soliton energy-level diagram.** Some of the possible values of the circulating energy (proportional to $\int d\theta |\psi|^2$) in the soliton regime as a function of the detuning parameter α . Level curves correspond to the number of circulating solitons. This diagram is obtained from numerical solutions using $F^2 = 4$, $\beta_2 = -0.0187$, and is quantitatively dependent on both of these parameters. Dotted vertical lines indicate approximations to the minimum and maximum detunings for solitons. The approximation for the minimum detuning is the value of α at which the effectively red-detuned branch vanishes, obtained by inserting ρ_- (Eq. 2.19) into Eq. 2.17 for $F^2 = 4$ and solving for α , and the approximate maximum detuning is $\alpha_{max} = \pi^2 F^2 / 8$.

2.3.3.2 Microresonator solitons in experiments

Relative to the generation of extended modulation-instability patterns, experimental generation of solitons in microring resonators is challenging. Solitons are localized excitations below threshold, which means that their existence is degenerate with their absence—a resonator can host $N = 0, 1, 2, \dots$ up to N_{max} solitons for a given set of parameters α and F^2 ; as discussed above and illustrated in Fig. 2.8. If α and F^2 are experimentally tuned to a point at which solitons may

exist, ψ will evolve to a form determined by the initial conditions of the field ψ_0 . To provide initial conditions that evolve to $N > 0$ solitons, most experimental demonstrations of soliton generation have involved first generating an extended pattern in the resonator, and then tuning to an appropriate point (α, F^2) so that ‘condensation’ of solitons from the extended pattern occurs.

Condensation of solitons from an extended pattern presents additional challenges. First, it is difficult to control the number of solitons that emerge, due to the high degree of soliton-number degeneracy as shown in Fig. 2.8. This typically leads to a success rate somewhat lower than 100 % in the generation of single solitons. Second, the transition from a high duty-cycle extended pattern to a lower duty-cycle ensemble of one or several solitons comes with a dramatic drop in intracavity power that occurs on the timescale of the photon lifetime. If the resonator is in thermal steady-state before this drop occurs, the resonator will cool and the resonance frequency will increase. If this increase is large enough that the final detuning α exceeds $\alpha_{max} = \pi^2 F^2 / 8$, the soliton is lost. This challenge can be addressed by preparing initial conditions for soliton generation and then tuning to an appropriate point (α, F^2) faster than the cavity can come into thermal steady-state at the temperature determined by the larger power of the extended pattern; this is possible because the timescale over which an extended pattern can be generated is related to the photon lifetime, which is typically much faster than the thermal timescale.

The first report of soliton generation in microresonators came in 2012 in a paper by Herr et al. [70] (2012 pre-print [69]). These authors described optimizing the speed of a decreasing-frequency scan of the pump laser across the cavity resonance so that solitons could be condensed from an extended pattern and the scan could then be halted at a laser frequency where the solitons could be maintained with the system in thermal steady-state at the temperature determined by the circulating power of the solitons. Stochastic reduction in the number of solitons in the resonator after condensation from an extended pattern was identified in these experiments. This corresponds to transitions between levels in the diagram in Fig. 2.8, and is associated with discrete steps in a measurement of the ‘comb power,’ the output power of the resonator with the pump frequency ν_p filtered out. The resulting staircase-like nature of a comb power measurement is a useful experimental

signature of soliton generation in microresonators, and is important for comparison with the results described in Chapter ??.

Other approaches for dealing with the challenges described above have been developed since this first demonstration; these include fast manipulation of the pump power [46, 100] or frequency [101], periodic modulation of the pump laser’s phase or power at f_{FSR} [53, 102], tuning of the cavity resonance frequency using chip-integrated heaters instead of tuning the pump-laser frequency [103, 104], and soliton-ensemble preparation and subsequent population reduction through manipulation of the pump laser [93]. These methods continue to make use of extended patterns to provide initial conditions for soliton generation. In formally-equivalent fiber-ring resonators, direct generation of solitons without condensation from an extended pattern has been demonstrated using transient phase and/or amplitude modulation of the pump laser [105–107].

2.3.3.3 Microresonator solitons in applications

Because solitons have single-FSR spacing, have the output localized into a high peak-power pulse, and are stationary (in contrast with chaos, which has single-FSR spacing but is not stationary), they are promising for applications. Many of the proposals for and demonstrations of applications with Kerr-combs have used single-soliton operation. Some of the applications already demonstrated include an optical clock [108], dual-comb spectroscopy [109], coherent communications [110], and direct on-chip optical frequency synthesis [34]. Additionally, soliton combs have been self-referenced both with [111, 112] and without [35, 113] external spectral broadening. Nevertheless, there remains work to be done to bring microresonator-soliton technology to the level of maturity that will be required for deployment in the field. Chapters ?? and ?? describe two recent advancements: the development of a method for direct on-demand generation of single solitons by use of a phase-modulated pump laser, and the observation and explanation of a soliton-interaction mechanism that imparts rigid structure on the allowed configurations of multi-soliton ensembles.

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