Natural Language Processing
Probabilistic Language Models

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Overview

- The language modeling problem
- Trigram models
- Evaluating language models: perplexity
- Estimation techniques:
    1. Linear interpolation
    2. Discounting methods
- This slides are based on the course material by Michael Collins: `http://www.cs.columbia.edu/~mcollins/cs4705-spring2019/slides/lmslides.pdf`

The Language Modeling Problem

- We have some (finite) vocabulary, say $\mathcal{V} = \{$the, a, man, telescope, Beckham, two, . . .$\}$
- We have an (infinite) set of strings, $\mathcal{V}^*$.
- For example:
    - the STOP
    - a STOP
    - the fan STOP
    - the fan saw Beckham STOP
    - the fan saw saw STOP
    - the fan saw Beckham play for Real Madrid STOP
- Where STOP is a special symbol indicating the end of a sentence.

The Language Modeling Problem (Continued)

- We have a training sample of example sentences in English.
- We need to ”learn” a probability distribution p.
- p is a function that satisfies:

$$\sum_{x \in V^*} p(x) = 1$$

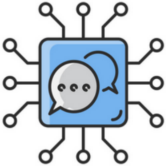
$$p(x) \geq 0 \quad \text{for all } x \in V^*$$

- Examples of probabilities assigned to sentences:

$$\begin{aligned}
p(\text{the STOP}) &= 10^{-12} \\
p(\text{the fan STOP}) &= 10^{-8} \\
p(\text{the fan saw Beckham STOP}) &= 2 \times 10^{-8} \\
p(\text{the fan saw saw STOP}) &= 10^{-15} \\
&\ldots \\
p(\text{the fan saw Beckham play for Real Madrid STOP}) &= 2 \times 10^{-9}
\end{aligned}$$

The Language Modeling Problem (Continued)

- Idea 1: The model assigns a higher probability to fluent sentences (those that make sense and are grammatically correct).
- Idea 2: Estimating this probability function from text (corpus).
- The language model helps text generation models distinguish between good and bad sentences.



Why would we want to do this?

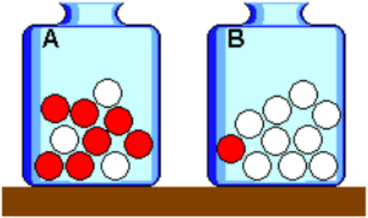
- Speech recognition was the original motivation.
- Consider the sentences: 1) recognize speech and 2) wreck a nice beach.
- These two sentences sound very similar when pronounced, making it challenging for automatic speech recognition systems to accurately transcribe them.
- When the speech recognition system analyzes the audio input and tries to transcribe it, it takes into account the language model probabilities to determine the most likely interpretation.
- The language model would favor p(recognize speech) over p(wreck a nice beach).
- This is because the former is a more common sentence and should occur more frequently in the training corpus.

Why on earth would we want to do this?

- By incorporating language models, speech recognition systems can improve accuracy by selecting the sentence that aligns better with linguistic patterns and context, even when faced with similar-sounding alternatives.
- Related problems are optical character recognition, handwriting recognition.
- Actually, Language Models are useful in any NLP tasks involving the generation of language (e.g., machine translation, summarization, chatbots).
- The estimation techniques developed for this problem will be VERY useful for other problems in NLP.

Language Models are Generative

- Language models can generate sentences by sequentially sampling from probabilities.
- This is analogous to drawing balls (words) from an urn where their sizes are proportional to their relative frequencies.
- Alternatively, one could always draw the most probable word, which is equivalent to predicting the next word.



A Naive Method

- A very naive method for estimating the probability of a sentence is to count the occurrences of the sentence in the training data and divide it by the total number of training sentences (N) to estimate the probability.
- We have N training sentences.
- For any sentence $x_1, x_2, \ldots, x_n$, $c(x_1, x_2, \ldots, x_n)$ is the number of times the sentence is seen in our training data.
- A naive estimate:

$$p(x1, x2, \ldots, xn) = \frac{c(x_1, x_2 \ldots, x_n)}{N}$$

- Problem: As the number of possible sentences grows exponentially with sentence length and vocabulary size, it becomes increasingly unlikely for a specific sentence to appear in the training data.
- Consequently, many sentences will have a probability of zero according to the naive model, leading to poor generalization.

Markov Processes

- Consider a sequence of random variables $X_1, X_2, \ldots, X_n$.
- Each random variable can take any value in a finite set V.
- For now, we assume the length n is fixed (e.g., $n = 100$).
- Our goal: model $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$

First-Order Markov Processes

$$
\begin{aligned}
P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) &= P(X_1 = x_1) \prod_{i=2}^{n} P(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \\
&= P(X_1 = x_1) \prod_{i=2}^{n} P(X_i = x_i | X_{i-1} = x_{i-1})
\end{aligned}
$$

The first-order Markov assumption: For any $i \in \{2, \ldots, n\}$ and any $x_1, \ldots, x_i$,

$$
P(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) = P(X_i = x_i | X_{i-1} = x_{i-1})
$$

Second-Order Markov Processes

$$
\begin{aligned}
& P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \\
P(X_1 = x_1) \cdot P(X_2 = x_2 | X_1 = x_1) \cdot & \prod_{i=3}^{n} P(X_i = x_i | X_{i-2} = x_{i-2}, X_{i-1} = x_{i-1}) \\
& = \prod_{i=1}^{n} P(X_i = x_i | X_{i-2} = x_{i-2}, X_{i-1} = x_{i-1})
\end{aligned}
$$

(For convenience, we assume $x_0 = x_{-1} = *$, where $*$ is a special ”start” symbol.)

Modeling Variable Length Sequences

- We would like the length of the sequence, n, to also be a random variable.
- A simple solution: always define $X_n = \text{STOP}$, where STOP is a special symbol.
- Then use a Markov process as before:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i | X_{i-2} = x_{i-2}, X_{i-1} = x_{i-1})$$

- (For convenience, we assume $x_0 = x_{-1} = *$, where $*$ is a special "start" symbol.)

Trigram Language Models

- A trigram language model consists of:
  1. A finite set $\mathcal{V}$
  2. A parameter $q(w|u, v)$ for each trigram u, v, w such that $w \in \mathcal{V} \cup \{\text{STOP}\}$, and $u, v \in \mathcal{V} \cup \{*\}$
- For any sentence $x_1 \ldots x_n$ where $x_i \in \mathcal{V}$ for $i = 1 \ldots (n-1)$, and $x_n = \text{STOP}$, the probability of the sentence under the trigram language model is:

$$p(x_1 \ldots x_n) = \prod_{i=1}^{n} q(x_i|x_{i-2}, x_{i-1})$$

- We define $x_0 = x_{-1} = *$ for convenience.

An Example

For the sentence `the dog barks STOP`, we would have:

$$p(\text{the dog barks STOP}) = q(\text{the}|*, *) \times q(\text{dog}|*, \text{the}) \times q(\text{barks}|\text{the, dog}) \times q(\text{STOP}|\text{dog, barks})$$

The Trigram Estimation Problem

Remaining estimation problem:

$$q(w_i | w_{i-2}, w_{i-1})$$

For example:

$$q(\text{laughs}|\text{the, dog})$$

A natural estimate (the ”maximum likelihood estimate”):

$$q(w_i | w_{i-2}, w_{i-1}) = \frac{\text{Count}(w_{i-2}, w_{i-1}, w_i)}{\text{Count}(w_{i-2}, w_{i-1})}$$

For instance,

$$q(\text{laughs}|\text{the, dog}) = \frac{\text{Count(the, dog, laughs)}}{\text{Count(the, dog)}}$$

Sparse Data Problems

A natural estimate (the ”maximum likelihood estimate”):

$$q(w_i | w_{i-2}, w_{i-1}) = \frac{\text{Count}(w_{i-2}, w_{i-1}, w_i)}{\text{Count}(w_{i-2}, w_{i-1})}$$

$$q(\text{laughs}|\text{the, dog}) = \frac{\text{Count(the, dog, laughs)}}{\text{Count(the, dog)}}$$

- Say our vocabulary size is $N = |V|$, then there are N^3 parameters in the model.
- For example, $N = 20,000 \Rightarrow 20,000^3 = 8 \times 10^{12}$ parameters.

Evaluating a Language Model: Perplexity

- We have some test data, m sentences: $s_1, s_2, s_3, \ldots, s_m$
- We could look at the probability under our model $\prod_{i=1}^{m} p(s_i)$. Or more conveniently, the log probability:

$$\log\left(\prod_{i=1}^{m} p(s_i)\right) = \sum_{i=1}^{m} \log p(s_i)$$

- In fact, the usual evaluation measure is perplexity:

$$\text{Perplexity} = 2^{-l} \quad \text{where} \quad l = \frac{1}{M}\sum_{i=1}^{m} \log p(s_i)$$

- M is the total number of words in the test data

Some Intuition about Perplexity

- Say we have a vocabulary V, and $N = |V| + 1$, and a model that predicts:

$$q(w|u,v) = \frac{1}{N} \quad \text{for all } w \in V \cup \{\text{STOP}\}, \text{ for all } u,v \in V \cup \{*\}$$

- It's easy to calculate the perplexity in this case:

$$\text{Perplexity} = 2^{-l} \quad \text{where} \quad l = \log \frac{1}{N} \Rightarrow \text{Perplexity} = N$$

- Perplexity can be seen as a measure of the effective "branching factor"

Some Intuition about Perplexity

- **Proof:** Let's asume we have m sentences of length n in the corpus, and M the amount of tokens in the corpus, $M = m \cdot n$.
- Let's consider the log (base 2) probability of a sentence $s = w_1 w_2 \dots w_n$ under the model:

$$\log p(s) = \log \prod_{i=1}^{n} q(w_i | w_{i-2}, w_{i-1}) = \sum_{i=1}^{n} \log q(w_i | w_{i-2}, w_{i-1})$$

- Since each $q(w_i | w_{i-2}, w_{i-1})$ is equal to $\frac{1}{N}$, we have:

$$\log p(s) = \sum_{i=1}^{n} \log \frac{1}{N} = n \cdot \log \frac{1}{N} = -n \cdot \log N$$

$$l = \frac{1}{M} \sum_{i=1}^{m} \log p(s_i) = \frac{1}{M} \sum_{i=1}^{m} -n \cdot \log N = \frac{1}{M} \cdot -m \cdot n \cdot \log N = -\log N$$

- Therefore, the perplexity is given by:

$$\text{Perplexity} = 2^{-l} = 2^{-(-\log N)} = N$$

Typical Values of Perplexity

- Results from Goodman ("A bit of progress in language modeling"), where $|V| = 50,000$ [Goodman, 2001].
- A trigram model: $p(x_1, \ldots, x_n) = \prod_{i=1}^{n} q(x_i|x_{i-2}, x_{i-1})$
  Perplexity = 74
- A bigram model: $p(x_1, \ldots, x_n) = \prod_{i=1}^{n} q(x_i|x_{i-1})$
  Perplexity = 137
- A unigram model: $p(x_1, \ldots, x_n) = \prod_{i=1}^{n} q(x_i)$
  Perplexity = 955

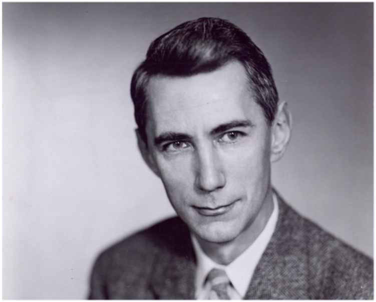
Some History

- Shannon conducted experiments on the entropy of English, specifically investigating how well people perform in the perplexity game.
- Reference: C. Shannon. “Prediction and entropy of printed English.” *Bell Systems Technical Journal*, 30:50–64, 1951. [Shannon, 1951]

Prediction and Entropy of Printed English

By C. E. SHANNON

(*Manuscript Received Sept. 15, 1950*)

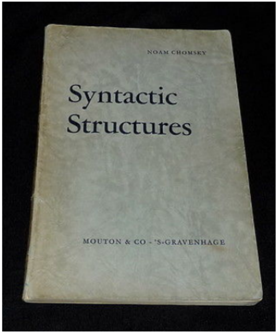


Some History

- Chomsky, in his book *Syntactic Structures* (1957), made several important points regarding grammar. [Chomsky, 2009]
- According to Chomsky, the notion of ”grammatical” cannot be equated with ”meaningful” or ”significant” in a semantic sense.
- He illustrated this with two nonsensical sentences:
    - (1) Colorless green ideas sleep furiously.
    - (2) Furiously sleep ideas green colorless.
- While both sentences lack meaning, Chomsky argued that only the first one is considered grammatical by English speakers.

Some History

- Chomsky also emphasized that grammaticality in English cannot be determined solely based on statistical approximations.
- Even though neither sentence (1) nor (2) has likely occurred in English discourse, a statistical model would consider them equally ”remote” from English.
- However, sentence (1) is grammatical, while sentence (2) is not, highlighting the limitations of statistical approaches in capturing grammaticality.



The Bias-Variance Trade-Off

- Trigram maximum-likelihood estimate:

$$q_{\mathrm{ML}}(w_i|w_{i-2}, w_{i-1}) = \frac{\mathrm{Count}(w_{i-2}, w_{i-1}, w_i)}{\mathrm{Count}(w_{i-2}, w_{i-1})}$$

- Bigram maximum-likelihood estimate:

$$q_{\mathrm{ML}}(w_i|w_{i-1}) = \frac{\mathrm{Count}(w_{i-1}, w_i)}{\mathrm{Count}(w_{i-1})}$$

- Unigram maximum-likelihood estimate:

$$q_{\mathrm{ML}}(w_i) = \frac{\mathrm{Count}(w_i)}{\mathrm{Count}()}$$

Linear Interpolation

- Take our estimate $q(w_i|w_{i-2}, w_{i-1})$ to be

$$q(w_i|w_{i-2}, w_{i-1}) = \lambda_1 \cdot q_{\mathrm{ML}}(w_i|w_{i-2}, w_{i-1}) + \lambda_2 \cdot q_{\mathrm{ML}}(w_i|w_{i-1}) + \lambda_3 \cdot q_{\mathrm{ML}}(w_i)$$

  where $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_i \geq 0$ for all i.
- Our estimate correctly defines a distribution (define $V' = V \cup \{\mathrm{STOP}\}$):

$$\begin{aligned}
& \sum_{w \in V'} q(w|u, v) \\
= & \sum_{w \in V'} [\lambda_1 \cdot q_{\mathrm{ML}}(w|u, v) + \lambda_2 \cdot q_{\mathrm{ML}}(w|v) + \lambda_3 \cdot q_{\mathrm{ML}}(w)] \\
= & \lambda_1 \sum_w q_{\mathrm{ML}}(w|u, v) + \lambda_2 \sum_w q_{\mathrm{ML}}(w|v) + \lambda_3 \sum_w q_{\mathrm{ML}}(w) \\
= & \lambda_1 + \lambda_2 + \lambda_3 = 1
\end{aligned}$$

- We can also show that $q(w|u, v) \geq 0$ for all $w \in V'$.

Estimating λ Values

- Hold out part of the training set as *validation* data.
- Define $c'(w_1, w_2, w_3)$ to be the number of times the trigram (w_1, w_2, w_3) is seen in the validation set.
- Choose λ_1, λ_2, λ_3 to maximize:

$$L(\lambda_1, \lambda_2, \lambda_3) = \sum_{w_1, w_2, w_3} c'(w_1, w_2, w_3) \log q(w_3 | w_1, w_2)$$

such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_i \geq 0$ for all i, and where

$$q(w_i | w_{i-2}, w_{i-1}) = \lambda_1 \cdot q_{\text{ML}}(w_i | w_{i-2}, w_{i-1}) + \lambda_2 \cdot q_{\text{ML}}(w_i | w_{i-1}) + \lambda_3 \cdot q_{\text{ML}}(w_i)$$

Discounting Methods

- Consider the following counts and maximum-likelihood estimates:

Sentence	Count	$q_{\mathbf{ML}}(w_i \mid w_{i-1})$
the	48	
the, dog	15	$15/48$
the, woman	11	$11/48$
the, man	10	$10/48$
the, park	5	$5/48$
the, job	2	$2/48$
the, telescope	1	$1/48$
the, manual	1	$1/48$
the, afternoon	1	$1/48$
the, country	1	$1/48$
the, street	1	$1/48$

- The maximum-likelihood estimates are high, particularly for low count items.

Discounting Methods

- Define "discounted" counts as follows:

$$\text{Count}^*(x) = \text{Count}(x) - 0.5$$

Sentence	Count	$\mathbf{Count^*(x)}$	$\mathbf{q_{ML}(w_i \mid w_{i-1})}$
the	48		
the, dog	15	14.5	$14.5/48$
the, woman	11	10.5	$10.5/48$
the, man	10	9.5	$9.5/48$
the, park	5	4.5	$4.5/48$
the, job	2	1.5	$1.5/48$
the, telescope	1	0.5	$0.5/48$
the, manual	1	0.5	$0.5/48$
the, afternoon	1	0.5	$0.5/48$
the, country	1	0.5	$0.5/48$
the, street	1	0.5	$0.5/48$

- The new estimates are based on the discounted counts.

Discounting Methods (Continued)

- We now have some ”missing probability mass”:

$$\alpha(w_{i-1}) = 1 - \sum_{w} \frac{\text{Count}^*(w_{i-1}, w)}{\text{Count}(w_{i-1})}$$

- For example, in our case:

$$\alpha(\text{the}) = \frac{10 \times 0.5}{48} = \frac{5}{48}$$

Katz Back-Off Models (Bigrams)

- For a bigram model, define two sets:

$$A(w_{i-1}) = \{w : \text{Count}(w_{i-1}, w) > 0\}$$

$$B(w_{i-1}) = \{w : \text{Count}(w_{i-1}, w) = 0\}$$

- A bigram model:

$$q_{\text{BO}}(w_i|w_{i-1}) = \begin{cases} \frac{\text{Count}^*(w_{i-1}, w_i)}{\text{Count}(w_{i-1})} & \text{if } w_i \in A(w_{i-1}) \\ \frac{\alpha(w_{i-1}) q_{\text{ML}}(w_i)}{\sum_{w \in B(w_{i-1})} q_{\text{ML}}(w)} & \text{if } w_i \in B(w_{i-1}) \end{cases}$$

- Where:

$$\alpha(w_{i-1}) = 1 - \sum_{w \in A(w_{i-1})} \frac{\text{Count}^*(w_{i-1}, w)}{\text{Count}(w_{i-1})}$$

Katz Back-Off Models (Trigrams)

- For a trigram model, first define two sets:

$$A(w_{i-2}, w_{i-1}) = \{w : \text{Count}(w_{i-2}, w_{i-1}, w) > 0\}$$

$$B(w_{i-2}, w_{i-1}) = \{w : \text{Count}(w_{i-2}, w_{i-1}, w) = 0\}$$

- A trigram model is defined in terms of the bigram model:

$$q_{\text{BO}}(w_i | w_{i-2}, w_{i-1}) = \begin{cases} \frac{\text{Count}^*(w_{i-2}, w_{i-1}, w_i)}{\text{Count}(w_{i-2}, w_{i-1})} & \text{if } w_i \in A(w_{i-2}, w_{i-1}) \\ \frac{\alpha(w_{i-2}, w_{i-1}) q_{\text{BO}}(w_i | w_{i-1})}{\sum_{w \in B(w_{i-2}, w_{i-1})} q_{\text{BO}}(w | w_{i-1})} & \text{if } w_i \in B(w_{i-2}, w_{i-1}) \end{cases}$$

- Where:

$$\alpha(w_{i-2}, w_{i-1}) = 1 - \sum_{w \in A(w_{i-2}, w_{i-1})} \frac{\text{Count}^*(w_{i-2}, w_{i-1}, w)}{\text{Count}(w_{i-2}, w_{i-1})}$$

Summary

- Deriving probabilities in probabilistic language models involves three steps:
    1. Expand $p(w_1, w_2, \ldots, w_n)$ using the Chain rule.
    2. Apply Markov Independence Assumptions $p(w_i|w_1, w_2, \ldots, w_{i-2}, w_{i-1}) = p(w_i|w_{i-2}, w_{i-1})$.
    3. Smooth the estimates using low order counts.
- Other methods for improving language models include:
    - Introducing latent variables to represent topics, known as topic models. [Blei et al., 2003]
    - Replacing $p(w_i|w_1, w_2, \ldots, w_{i-2}, w_{i-1})$ with a predictive neural network and an "embedding layer" to better represent larger contexts and leverage similarities between words in the context. [Bengio et al., 2000]
- Modern language models utilize deep neural networks in their backbone and have a vast parameter space.

Questions?

Thanks for your Attention!

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