

Non-Local Boxes for Networks

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Nonlocal boxes are conceptual tools that capture the essence of the phenomenon of quantum non-locality, central to modern quantum theory and quantum technologies. We introduce network nonlocal boxes tailored for quantum networks under the natural assumption that these networks connect independent sources and do not allow signaling. Hence, these boxes satisfy the No-Signaling and Independence (NSI) principle. For the case of boxes without inputs, connecting pairs of bipartite sources and producing binary outputs, we prove that the sources and boxes producing local random outputs and maximal 2-box correlations, i.e. $E_2 = \sqrt{2} - 1$, $E_2^o = 1$, are essentially unique.

I. INTRODUCTION

Non-locality is a key feature of quantum physics and one of the major discovery - arguably the major discovery - of last century physics. Modern quantum technology promises, in addition to quantum computers, quantum networks that will connect these quantum processors and offer proven confidentiality of all communications. It is thus natural and timely to study quantum non-locality in networks, a field that has been burgeoning for about a decade under the names of bilocality and n -locality, for 2 and n independent sources, respectively [1–21]. Let us stress that in networks the characteristic feature of quantum physics, namely entanglement, enters twice. First, entanglement of the subsystems emitted by the sources. Second, entanglement produced by the joint measurements that connect independent subsystems emitted by different sources. This second form of entanglement is much less studied and understood than the first one [22].

In this work we go beyond quantum physics and study n -locality for arbitrary boxes only limited by no-signaling and independence, a principle we name NSI [23]. This is in the spirit of the non-local boxes introduced by Popescu and Rochlich - the so-called PR-boxes [24] - as a conceptual tool to study Bell non-locality. Here, however, we go beyond Bell non-locality, our boxes connect independent sources and have no inputs. A priori, such network non-local boxes could have any number of outputs and connect any number of sources, with sources connected to any number of boxes. In this paper we restrict ourselves to boxes connecting two sources, producing binary outputs and sources connected to two boxes, see Fig 1. We name the networks obtained with these resources binary networks. Similarly to the PR-boxes the aim is the set the limits of non-locality in networks imposed by the NSI principle and to offer conceptual tools to study this new form of non-locality.

Assuming that locally each output of a box is totally random and that the outputs of two neighboring boxes are maximally correlated when using a particular type of source, we find that the statistics produced by this box in presence of this source in a binary network are unique

(up to flipping all outputs).

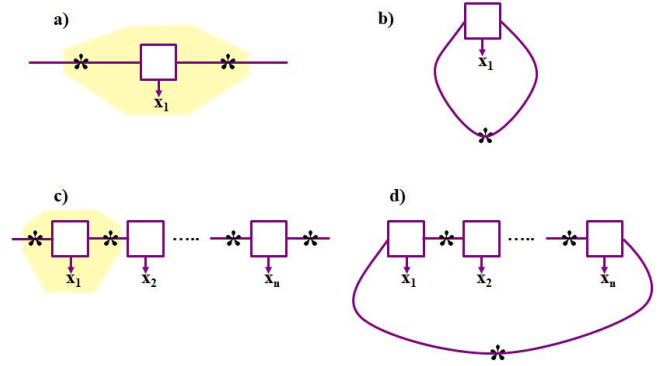


FIG. 1. Various networks. Boxes are indicated with a square that outputs x_j . Sources are marks as *. A priori, the sources and boxes could be asymmetrical (i.e. non invariant under exchange of their arms). Since this turns out to be irrelevant here, we simply depict them in a symmetric fashion.

- a) One box in a line. b) One box in a loop. c) n boxes in a line. d) n boxes in a loop.

The NSI principle states that the statistics produced in situations which are indistinguishable to the parties must be identical, such as the statistics of x_1 in situations a) and c), by identification of the yellow regions (see [23] for more details). Note that this principle does not impose the statistics of x_1 to be identical in b), however, because the change from two sources in a) to just one source in b) can be detected by the party when producing x_1 , i.e. no equivalent of the yellow region in a) can be found in b).

II. BINARY NETWORK NON-LOCAL BOXES

First, consider two identical but independent sources, sending out subsystems in two opposite directions, coupled by one box as illustrated in Fig. 1.a. Each box outputs a bit $x_j = \pm 1$, which we often label merely \pm . We assume that this bit is random, hence its expectation value is zero: $E_1 = \langle x_j \rangle = 0$. Figure 1.b illustrates the case of a box connected to the two parts of a single

source. Here also we assume $E_1^o = \langle x_j \rangle = 0$, where the superscript “o” indicates that this correlator corresponds to a closed loop.

Next, let us add more independent sources and boxes. We focus on the case in which all boxes are identical, i.e. copies of our box of interest, and similarly all sources are also copies of our source of interest. Only two fully connected configurations are possible. Either all sources and boxes are on a line, Fig. 1c, or they form a loop, Fig. 1d. It is important to realize that no other configurations is possible. Hence, proving the existence of a source and box compatible with these two configurations suffices to prove that they are conceptually possible.

In both configurations, the probability of the outputs x_j can conveniently be expressed in terms of all correlators. Since all sources and boxes are identical, the correlators between k adjacent boxes are all equal, denoted $E_k = \langle x_{m+1} \cdot x_{m+2} \cdot \dots \cdot x_{m+k} \rangle$. In case of a polygon configuration with n boxes at the vertices, i.e. a loop, there is one additional correlator: $E_n^o = \langle x_1 \cdot x_2 \cdot \dots \cdot x_n \rangle$.

Let us consider some examples. First, the joint probability of outputs for 3 boxes on a line reads:

$$p_3(x_1, x_2, x_3) = \frac{1}{2^3} (1 + (x_1 + x_2 + x_3)E_1 + (x_1x_2 + x_2x_3)E_2 + x_1x_2x_3E_3 + x_1x_3E_1^2) \quad (1)$$

where the bipartite correlator associated to x_1x_3 is E_1^2 due to the independence assumption. Since by assumption $E_1 = 0$, we obtain

$$p_3(x_1, x_2, x_3) = \frac{1}{2^3} (1 + (x_1x_2 + x_2x_3)E_2 + x_1x_2x_3E_3) \quad (2)$$

where the first term guarantees normalization. Next, for 4 boxes in a square we have:

$$p_4^o(x_1x_2x_3x_4) = \frac{1}{2^4} (1 + (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1)E_2 + (x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2)E_3 + x_1x_2x_3x_4E_4^o) \quad (3)$$

and so on.

The correlators corresponding to disconnected sets factorize, because of the assumed independence of the sources, as in the last term in (1). It is important to realize that for hexagons and 5 boxes in a line and larger networks, independence implies some non-linear correlators. For example, in a line with 5 boxes the last correlator equals the square of the 2-box correlator:

$$p_5(x_1 \dots x_5) = \frac{1}{2^5} (1 + \sum_{k=1}^4 x_k x_{k+1} E_2 + \sum_{k=1}^3 x_k x_{k+1} x_{k+2} E_3 + \sum_{k=1}^2 x_k x_{k+1} x_{k+2} x_{k+3} E_4 + \prod_{k=1}^5 x_k E_5 + x_1 x_2 x_4 x_5 E_2^2) \quad (4)$$

Obviously, the correlators have to be such that $p(x_j) \geq 0$ for all outputs strings $\{x_j\}$. In [23], using 6 sources and 6 boxes in an hexagonal loop, we proved the following bound on the 2-box correlator:

$$E_2 \leq \sqrt{2} - 1 \approx 0.4142 \quad (5)$$

Here we prove that this upper bound can be saturated for all possible binary networks and that the saturation of (5) is achieved in an essentially unique way (up to flipping all outputs).

Our main results are the correlators for boxes in a line (13), section IV, and in a loop (22), section V. These correlators are unique given $E_1 = 0$, $E_2 = \sqrt{2} - 1$ and $E_2^o = 1$, and we prove that they guarantee that all probabilities $p(\vec{x})$ and $p^o(\vec{x})$, where $\vec{x} = (x_1 \dots x_n)$, are non-negative. But first, in the next section, we consider 3 boxes in a line and prove that the value of the corresponding correlator E_3 is unique (up to its sign).

III. 3-BOX CORRELATOR

Consider 3 boxes in a line as in eq. (2). We have $p(+-) = \frac{1}{8} (1 - 2E_2 - E_3) \geq 0$. Hence, $E_2 = \sqrt{2} - 1$ imposes $E_3 \leq 1 - E_2 = 3 - 2\sqrt{2}$. Interestingly, when considering lines with more copies of the box, this inequality becomes an equality (up to a sign flip), namely

$$E_3 = \pm(3 - 2\sqrt{2}), \quad (6)$$

as implied by the following lemma proven in the Appendix.

Lemma 1. *In a line configuration with 7 boxes, whenever $E_1 = 0$ and $E_2 = \sqrt{2} - 1$, the following inequality holds:*

$$E_3^2 \geq (3 - 2\sqrt{2})^2. \quad (7)$$

Henceforth we assume the positive sign for E_3 . We'll see that this choice implies that all correlators are non-negative. Note that by flipping all outputs, all odd correlators, in particular E_3 , merely change sign.

The relation (6), inserted in (2), has the following important consequence: $p(+-) = 0$. Accordingly, there are never 3 adjacent boxes with outputs $+-$:

$$p_n(x_1 \dots x_{j-1} + - + x_{j+3} \dots x_n) = p(+ - +) \cdot \quad (8)$$

$$p(x_1 \dots x_{j-1} x_{j+3} \dots x_n | x_j = +, x_{j+1} = -, x_{j+2} = +) = 0$$

This in turn implies that whenever two connected boxes output $+-$, then the next box necessarily outputs $-$. This is key in proving existence and uniqueness of our network non-local box, as we show in the two next sections.

IV. RECURSION FORMULA FOR THE E_k

Let us define $q_n(\vec{x}) = 2^n p_n(\vec{x})$, so that the normalization factors drop out. We still have the non-negativity conditions $q(\vec{x}) \geq 0$ for all \vec{x} . The case of $n+1$ boxes in a line can be reduced to n boxes as follows:

$$\begin{aligned} q_{n+1}(x_1 \dots x_{n+1}) &= q_n(x_1 \dots x_n) + (\Pi_{j=1}^{n+1} x_j) E_{n+1} \\ &\quad + \sum_{k=0}^{n-1} (\Pi_{j=k+2}^{n+1} x_j) q_k E_{n-k} \quad (9) \\ &= q(x_1 \dots x_n) \\ &\quad + (\Pi_{j=1}^{n+1} x_j) (E_{n+1} + Q_n) \quad (10) \end{aligned}$$

where $Q_n = \sum_{k=0}^{n-1} (\Pi_{j=1}^{k+1} x_j) q_k E_{n-k}$, $q_k = q_k(x_1 \dots x_k)$ and we used $x_j^2 = 1$ for all j . The first term on the left of (10) takes into account the first n boxes, the second term all $n+1$ boxes and the last term all cases that involve the last box but not all boxes.

Considering successively the cases $(\Pi_{j=1}^{n+1} x_j) = \pm 1$ one gets:

$$q_n(x_1 \dots x_n) - Q_n \geq E_{n+1} \geq -q_n(x_1 \dots x_n) - Q_n \quad (11)$$

Apply this to the string of alternating outputs $q_n(+ - + - + \dots)$, the result of the previous section implies $q_k = 0$ for all $k \geq 3$. Hence, from (10) and (11) one obtains the recursion formula:

$$E_{n+1} = -E_n + E_{n-1} + (1 - E_2) E_{n-2} \quad (12)$$

This leads to the following closed form for all correlators of boxes in a line:

$$E_n = \frac{1}{\mu} \left[\left(\frac{2}{\mu-1} \right)^{n-1} - \left(\frac{-2}{\mu+1} \right)^{n-1} \right] \quad (13)$$

where $\mu = \sqrt{5+4\sqrt{2}} \approx 3.2645$. Notice that $E_n > 0$ for all $n \geq 2$. Obviously, flipping all outputs would change the sign of E_n for all odd n 's. Table I lists the 12 first values of E_n , remarkably they are all of the form $n+m\sqrt{2}$, with n and m integers.

It remains to prove that these correlators guarantee $q_n(\vec{x}) \geq 0$ for all \vec{x} .

First, notice that by inserting the recursion formula (12) into (10) one gets, after some algebra:

$$q_n(\vec{1}_n) = (\sqrt{2})^{n-1} \quad (14)$$

where $\vec{1}_n = (+1, +1, \dots, +1)$, with $n+1$'s.

Second, using $q_n(x_1 \dots x_{j-1} + - + x_{j+3} \dots x_n) = 0$ for all $2 \leq j \leq n-3$ one sees that the line of boxes can be split in parts each time the output string changes sign. Without loss of generality we may assume $x_1 = +1$. Then, either all outputs are $+1$, i.e. $\vec{x} = \vec{1}_n$ in which case $q_n(\vec{1}_n) = \sqrt{2}^{n+1} > 0$, or the first output -1 happen at position j_0 , i.e. $x_k = +1$ for all $k < j_0$ and $x_{j_0} = -1$. In such a case, the box at position $j_0 + 1$ can be removed, as

anyway the output -1 is certain. But if the $j_0 + 1$ box is removed, then the independence assumption implies that the probability factorizes: $p_n(x_1 x_2 \dots x_n) = p_{j_0}(+ + \dots + -) \cdot p_{n-j_0-1}(x_{j_0+2} \dots x_n)$.

Eventually, all probabilities are products of the following terms: $p_n(\vec{1}_n)$, $p_n(\vec{1}_{n-1}, -)$, $p_n(-, \vec{1}_{n-1})$ and $p_n(-, \vec{1}_{n-2}, -)$. Remains to prove that these 4 terms are all non-negative for all n . Relation (14) proves the positivity of the first term and will be heavily used to prove the positivity of the 3 other term:

$$\begin{aligned} p_{n+1}(\vec{1}_n, -) &= p_{n+1}(-, \vec{1}_n) = p_n(\vec{1}_n) - p_{n+1}(\vec{1}_n, +) \\ &= \frac{1}{\sqrt{2}^{n+1}} \left(1 - \frac{1}{\sqrt{2}} \right) > 0 \quad (15) \end{aligned}$$

$$\begin{aligned} p_{n+2}(-, \vec{1}_n, -) &= p_n(\vec{1}_n) - p_{n+2}(-, \vec{1}_n, +) \\ &\quad - p_{n+2}(+, \vec{1}_n, -) - p_{n+2}(+, \vec{1}_n, +) \quad (16) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}^{n+1}} \left(1 - \frac{2}{\sqrt{2}} \left(1 - \frac{1}{\sqrt{2}} \right) - \frac{1}{2} \right) \\ &> 0 \quad (17) \end{aligned}$$

In summary, all the correlators (13) corresponding to n boxes in a line are fixed by the assumptions $E_1 = 0$ and $E_2 = \sqrt{2} - 1$ (and $E_3 \geq 0$) and they guarantee that all probabilities $p(\vec{x}) \geq 0$, for all output strings \vec{x} . In the next section we prove that these correlators are also compatible with n boxes in any polygon configuration.

V. POLYGONS

The smallest closed loop has a single box fed by the two links produced by a single source, see Fig. 1a. In this case we assume, similarly to a single box in a line, $E_1^o = 0$, where the upper index o indicates a closed loop. Accordingly, single boxes always produce fully random outputs. The second shortest loop has 2 boxes and 2 sources. Since E_2^o is not limited by the NSI principle (see Fig. 1), we assume it takes the largest possible value¹: $E_2^o = 1$.

Let's now consider polygons with $n+1$ vertices and edges, for $n \geq 2$. Using a similar technique as for the line we have:

$$\begin{aligned} q_{n+1}^o(x_1 \dots x_{n+1}) &= q_n(x_1 \dots x_n) + \Pi_{j=1}^{n+1} x_j \cdot E_{n+1}^o \\ &\quad + \sum_{k=1}^n E_k \cdot \sum_{\ell=1}^k (\Pi_{j=n+1-k+\ell}^{n+\ell} x_j) \\ &\quad \cdot q_{n-1-k}(x_{n-1-k+\ell} \dots x_{n+\ell+2}) \quad (18) \end{aligned}$$

¹ At first sight, one may fear that combining the two end sources of the 2-box configuration in a line into a single source, thus changing the line into a loop configuration, leads to signaling, since $E_2 \neq E_2^o$. However, the change in the source takes time to influence the boxes, the time of flight of the subsystems emitted by the sources. Hence, $E_2 \neq E_2^o$ does not lead to signaling.

where all indices of x_j 's are assumed modulo $n+1$ (e.g. $x_{n+\ell+2} = x_{\ell+1}$) and we set $q_{-1} = q_0 = 1$.

Using $q_3(+ - +) = 0$ we deduce that for all $n \geq 3$ $q_{n+1}(\vec{1}_n, -) = 0$. Let's apply this to the above formula:

$$\begin{aligned} q_{n+1}^o(\vec{1}_n, -) &= q_n(\vec{1}_n) - E_{n+1}^o \\ &+ \sum_{k=1}^n E_k \cdot \sum_{\ell=1}^k x_1 \dots \hat{x}_\ell \dots \hat{x}_{\ell+n-k} \dots x_{n+1} \\ &\cdot q_{n-1-k}(x_\ell \dots x_{\ell+n-k-1}) \end{aligned} \quad (19)$$

$$\begin{aligned} &= \sqrt{2}^{n-1} - E_{n+1}^o - \sum_{k=1}^{n-2} E_k \cdot k \cdot \sqrt{2}^{n-k-2} \\ &- (n-1)E_{n-1} - nE_n = 0 \end{aligned} \quad (20)$$

Consequently:

$$\begin{aligned} E_{n+1}^o &= \sqrt{2}^{n-1} - \sum_{k=1}^{n-2} E_k \cdot k \cdot \sqrt{2}^{n-k-2} \\ &- n \cdot E_n - (n-1) \cdot E_{n-1} \end{aligned} \quad (21)$$

Using eq. (13), with the same $\mu = \sqrt{5+4\sqrt{2}}$, one gets, for all $n \geq 2$:

$$E_n^o = \left(\frac{2}{\mu-1} \right)^n + \left(\frac{-2}{\mu+1} \right)^n \quad (22)$$

Note that $E_n^o > 0$ for all n . Table I lists the 12 first values of E_n^o , remarkably they are all of the form $n+m\sqrt{2}$, with n and m integers, as we found for the correlators in a line.

Remains to prove that $q_{n+1}^o(\vec{x}) \geq 0$ for all \vec{x} . First, assume that not all outputs are equal. Then, there are 2 adjacent boxes with outputs $+1-1$. Since $p_3(+ - +) = 0$, we can remove the next box as it necessarily outputs -1 . In this way one opens the loop and reduces it to a line for which we already proved non-negativity of all probabilities. Next, assume all outputs are $+1$. Since $E_n^o \geq 0$ for all n , $p_n^o(\vec{1}_n) \geq 0$. Finally, assume all outputs are -1 :

$$p_n^o(\vec{-1}_n) = \frac{1}{2^n} \left(1 + n \sum_{k=2}^{n-1} (-1)^k E_k + (-1)^n E_n^o \right) \quad (23)$$

From (13) and (22) straightforward computations shows that $E_{2n} - E_{2n+1} > 0$ and $1 + E_{n-1} - E_n^o > 0$. Hence, $p_n^o(\vec{-1}_n) \geq 0$ for all n .

n	1	2	3	4	5	6	7	8	9	10	11	12
E_n	0	$\sqrt{2}-1$	$3-2\sqrt{2}$	$3\sqrt{2}-4$	$3-2\sqrt{2}$	$3-2\sqrt{2}$	$10\sqrt{2}-14$	$27-19\sqrt{2}$	$22\sqrt{2}-31$	$10-7\sqrt{2}$	$51-36\sqrt{2}$	$104\sqrt{2}-147$
E_n^o	0	1	$2-\sqrt{2}$	$4\sqrt{2}-5$	$9-6\sqrt{2}$	$6\sqrt{2}-8$	$\sqrt{2}-1$	$23-16\sqrt{2}$	$37\sqrt{2}-52$	$71-50\sqrt{2}$	$32\sqrt{2}-45$	$44-62\sqrt{2}$

TABLE I. Table of the first 12 correlators in line, E_n , and in a loop, E_n^o . (The indicated values of E_2^o and E_3^o are the maximal values compatible with $p_2^o(x_1 x_2) \geq 0$ and $p_3^o(x_1 x_2 x_3) \geq 0$, respectively.)

VI. CONCLUSION

We proved that under the natural assumption $E_1 = E_1^o = 0$ there is a single pair of source and box in binary networks that maximizes the 2-box correlators E_2 and E_2^o , in the sense that the statistics produced by these resources in any binary network is uniquely defined. The existence and uniqueness of such a source and box is highly non-trivial. Admittedly, the deep reason why such a source and box exist and - especially - are unique is left open. Another question is whether this binary non-local resource can be realized within quantum theory and, if not, how close quantum can come?

A natural challenge is whether a similar result holds also for boxes with more outputs, boxes connecting more than 2 sources, and/or sources connecting more than 2 boxes. We investigated numerically the case of boxes with 4 outputs under the natural assumption of output permutation symmetry, i.e. for all permutations π of $\{1, 2, 3, 4\}$, $p_n(\vec{x}) = p_n(\pi(\vec{x}))$ (same permutations for all outputs). However, we found no sign of the uniqueness for 4-output network boxes, though the question remains open and should be investigated in future work. Also, other types of networks should be studied, in particular configurations beyond lines and loops, like, e.g. star networks.

Network non-locality is a new and fascinating form of non-locality. It combines non-locality and "entangling" joint measurements. It deserves to be analyzed within quantum theory and, as we do here, from outside quantum theory under the very general NSI principle: no-signaling and independence. The presented source and box for binary networks is a conceptual tool, similar in spirit to the PR-boxes introduced by Popescu and Rohrlich [24]; such conceptual tool are useful to understand and simulate quantum correlations, and for applications in the spirit of device independent quantum information processing.

Among the many fascinating research directions to which this one contributes are all questions on the limits of quantum non-locality (e.g. information causality [25]). In particular, macroscopic locality [27] and classical limits of non-local boxes [28, 29] are especially interesting and should be applied to network boxes.

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Appendix A: Proof of Lemma 1

In this appendix we prove the lemma used in section III:

Lemma 1. *In a line configuration with 7 boxes, whenever $E_1 = 0$ and $E_2 = \sqrt{2} - 1$, the following inequality holds:*

$$E_3^2 \geq (3 - 2\sqrt{2})^2. \quad (\text{A1})$$

Proof. Let us identify 17 probabilities:

$$P_1 = P(- - - - -) \quad (\text{A2})$$

$$P_2 = P(- - - + - - -) \quad (\text{A3})$$

$$P_3 = P(- - + - + - -) \quad (\text{A4})$$

$$P_4 = P(+ - + - + - +) \quad (\text{A5})$$

$$P_5 = P(+ + - + - + +) \quad (\text{A6})$$

$$P_6 = P(+ + + - + + +) \quad (\text{A7})$$

$$P_7 = P(+ + + + + + +) \quad (\text{A8})$$

$$P_8 = P(- + + - + - +) \quad (\text{A9})$$

$$P_9 = P(- + - + - - +) \quad (\text{A10})$$

$$Q_1 = P(- - - - + - +) \quad (\text{A11})$$

$$Q_2 = P(- - - - + + +) \quad (\text{A12})$$

$$Q_3 = P(- - - + - + +) \quad (\text{A13})$$

$$Q_4 = P(- - - + + + +) \quad (\text{A14})$$

$$Q_5 = P(- - + - + + +) \quad (\text{A15})$$

$$Q_6 = P(- - + + - + -) \quad (\text{A16})$$

$$Q_7 = P(- + - + + + +) \quad (\text{A17})$$

$$Q_8 = P(+ - + - - + +). \quad (\text{A18})$$

We define three vectors in \mathbb{R}^{24} :

$$\begin{aligned} u &= (8, 9, 6, 1, 9, 4, 5, 3, 7, 1, 6, 2, 7, 1, 7, 5, 3, 7, 1, 6, 2, 6, 7, 1) \\ v &= (4, 2, 7, 6, 4, 2, 2, 4, 1, 7, 6, 8, 6, 3, 5, 4, 2, 7, 1, 1, 3, 5, 3, 5) \end{aligned} \quad (\text{A19})$$

and

$$\begin{aligned} c = & \left(\frac{11146 + 16545\sqrt{2}}{1160712}, \frac{2581 + 4686\sqrt{2}}{773808}, \frac{3 + 34\sqrt{2}}{18424}, \right. \\ & \frac{109 + 2003\sqrt{2}}{515872}, \frac{10253 + 16404\sqrt{2}}{1160712}, \frac{9529 + 14340\sqrt{2}}{1547616}, \\ & \frac{29063 + 10799\sqrt{2}}{3095232}, \frac{50719 - 4773\sqrt{2}}{9285696}, \frac{1517 + 304\sqrt{2}}{147392}, \\ & \frac{9(139 + 40\sqrt{2})}{147392}, \frac{3(-38 + 337\sqrt{2})}{257936}, \frac{6 + 19\sqrt{2}}{5488}, \\ & \frac{227 + 1805\sqrt{2}}{515872}, \frac{16154 + 2677\sqrt{2}}{3095232}, \frac{107(80 + 139\sqrt{2})}{3095232}, \\ & \frac{127277 - 41427\sqrt{2}}{9285696}, \frac{41879 + 4049\sqrt{2}}{3095232}, \frac{1387 + 366\sqrt{2}}{147392}, \\ & \frac{1381 + 298\sqrt{2}}{147392}, \frac{3 + 34\sqrt{2}}{18424}, \frac{44 + 25\sqrt{2}}{5488}, \\ & \left. \frac{3(636 + 299\sqrt{2})}{257936}, \frac{2066 - 383\sqrt{2}}{442176}, \frac{8536 + 9995\sqrt{2}}{3095232} \right) \end{aligned} \quad (\text{A20})$$

A lengthy but direct computation shows that

$$2^{14} \cdot \sum_{i=1}^{24} c_i P_{u_i} Q_{v_i} = 12\sqrt{2} - 17 + E_3^2. \quad (\text{A21})$$

In other words, this expression does not involve any correlator with more than 3 parties. Since the probabilities and all components of c are positive, this expression also is positive, i.e.

$$12\sqrt{2} - 17 + E_3^2 \geq 0, \quad (\text{A22})$$

which is equivalent to Eq. (7). \square

The proof above was derived by considering probabilities for seven parties in a line in the space of correlator monomials. According to (10) and (13), every probability in this scenario can be expressed as a linear combination of some products of correlators. The task of finding a linear combination of probabilities with positive coefficients for which all monomial except E_3^2 vanish then amounts to solving a linear program. The proof constitutes a solution of this program.