THE MEANING OF LIFE

By Tony Stark

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Approved:

Évariste Galois, Ph.D., Chair

John von Neumann, Ph.D.

Srinivasa Ramanujan, Ph.D.

ABSTRACT

THE MEANING OF LIFE

Tony Stark

Everything you always wanted to know will be discussed.

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Chapter 1

Preliminaries

1.1 Introduction

To be written once we know where everything is going to go.

1.2 Coxeter System

A *Coxeter System* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter Group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [4]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *short braid relation* and s and t commute. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *long braid relation.* Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *braid move.*

We can represent a Coxeter System, (W, S), with a unique *Coxeter graph*, Γ, having

1. vertex set $S = \{s_1, s_2, \ldots, s_n\}$ and

2. edges $\{s_i, s_j\}$ for each $m(s,t) \geq 3$.

Each edge $\{s_i, s_j\}$ is labeled with its corresponding bond strength $m(s,t)$. Since $m(s,t) = 3$ occurs most frequently, it is customary to leave the edge unlabeled. If (W,S) is a Coxeter group with corresponding Coxeter system Γ, we may denote the group as $W(\Gamma)$ for emphasis. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. Given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. Note that s and t are not connected in the graph if and only if $m(s,t) = 2$. Also, the Coxeter group $W(\Gamma)$ is said to be *irreducible* if and only if Γ is connected. Otherwise, $W(\Gamma)$ is said to be *reducible*. Furthermore, if the graph is disconnected, the connected components correspond to factors in a direct product of irreducible Coxeter groups [4].

Example 1.2.1.

(a) The Coxeter graph of type A_n is seen in Figure reference the proper figure. Given A_n, we can construct the corresponding Coxeter system, $W(A_n)$, with generating set $S = \{s_1, s_2, \dots s_n\}$ and defining relations

1. $s_i^2 = e$ for all i
2. $s_i s_j = s_j s_i$ when $|i - j| > 1$
3. $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} elements sending $s_i \to (i, i+1)$. This thesis will briefly touch on Coxeter systems of type A_n.

(b) The Coxeter graph of type B_n is seen in Figure reference the proper figure. From B_n, we can construct the Coxeter group $W(B_n)$ with generating set $S = \{s_0, s_1, \dots s_{n-1}\}$ and defining relations

1. $s_i^2 = e$ for all i
2. $s_i s_j = s_j s_i$ when $|i - j| > 1$
3. $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \dots, n-1\}$
4. $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to S_n^B, where S_n^B is the group of all signed permutations. This thesis will also touch on Coxeter systems of type B_n.

(c) The Coxeter graph of Type $\widetilde{C}_n$ is seen in Figure reference the proper figure. From $\widetilde{C}_n$ we can construct the Coxeter group $W(\widetilde{C}_n)$ with generating set $S = \{s_0, s_1, \dots s_n\}$ and defining relations

1. $s_i^2 = e$ for all i
2. $s_i s_j = s_j s_i$ when $|i-j| > 1$ for $i \in \{1, 2, \dots, n-1\}$
3. $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$ for $i \in \{1, 2, \dots, n-1\}$
4. $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$
5. $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

The Coxeter group $W(\widetilde{C}_n)$ will be touched upon in this thesis as well.

Given a Coxeter system, (W, S), a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S, is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal over all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions to represent it. If we wish to emphasize a specific possibly reduced expression for $w \in W$ we will represent it as $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$. The following theorem tells us more about how reduced expressions relate.

Theorem 1.2.2 (Matsumoto, [3]). *If $w \in W$, then every reduced expression for w can be obtained by a sequence of braid moves of the form*

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. □

It follows from Matsumoto's Theorem that any reduced expression for $w \in W$ contains the same number of generators in the expression. Let $w \in W$ and fix a reduced expression $\overline{w}$ for w. Then the *support* of w, denoted $\text{supp}(\overline{w})$, is the set of all generators of that appear in $\overline{w}$. It follows from Matsumoto's Theorem that s appears in $\text{supp}(\overline{w})$ if and only if s appears in the support of all reduced expressions for w. If $\text{supp}(\overline{w}) = S$, we say that s has *full support*. Given $w \in W$ and a fixed reduced expression $\overline{w}$ for w, any subsequence of $\overline{w}$ is called a *subexpression* of $\overline{w}$.

Example 1.2.3. Let $w \in W(A_7)$ and let $\overline{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be a fixed expression for w. Then we have

$$\begin{aligned} s_7s_2s_4s_5s_3s_2s_3s_6 &= s_7s_4s_2s_5s_3s_2s_3s_6 \\ &= s_7s_4s_5s_2s_3s_2s_3s_6 \\ &= s_7s_4s_5s_3s_2s_3s_3s_6 \\ &= s_7s_4s_5s_3s_2s_6. \end{aligned}$$

This shows that $\overline{w}$ was not reduced. However, $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\text{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

It should be noted that $s \in \mathcal{L}(w)$ if and only if there is a reduced expression for w that begins with s and $s \in \mathcal{R}(w)$ if and only if there is a reduced expression for w that ends with s.

Example 1.2.4. Let $w \in W(B_4)$ and let $\overline{w} = s_0s_1s_2s_1s_3$ be a reduced expression for w. Note that all reduced expressions for w are as follows

$$\begin{matrix} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3. \end{matrix}$$

We see that $l(w) = 5$, and w has full support. Note that $\mathcal{L}(w) = \{s_0, s_2\}$ and $\mathcal{R}(w) = \{s_1, s_3\}$.

1.3 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W$. Following [5], we define a relation $\sim$ on the set of reduced expressions for w. Let $\overline{w}_1$ and $\overline{w}_2$ be two reduced expressions for w. We define $\overline{w}_1 \sim \overline{w}_2$ if we can obtain $\overline{w}_2$ from $\overline{w}_1$ by applying a single braid move of the form $s_is_j \mapsto s_js_i$ where $m(s_i, s_j) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative*, or just FC.

The set of fully commutative elements of $W(\Gamma)$ is denoted by $\text{FC}(\Gamma)$. We say that a reduced expression $\overline{w}$ is FC if it is a reduced expression for $w \in \text{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$, the definition of fully commutative tells us that to obtain all the reduced expressions for w, one must only perform short braid moves. The following theorem tells us that the we can not obtain a reduced expression for w using long braid relations.

Theorem 1.3.1 (Stembridge, [5]). *An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all $s_i \neq s_j$ when $m(s,t) \geq 3$.* □

Example 1.3.2. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0s_1s_2s_0s_3s_1$ be a reduced expression for w. We see that

$$s_0s_1s_2s_0s_3s_1 = s_0s_1s_0s_2s_3s_1 = s_0s_1s_0s_2s_1s_3,$$

where the purple indicates applying a short braid relation. Note that there is no way possible to perform a long braid relation. Hence w is FC.

Example 1.3.3. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0s_1s_2s_0s_1s_2$ be a reduced expression for w. We see that

$$s_0s_1s_3s_0s_1s_2 = s_0s_1s_0s_3s_1s_2 = s_0s_1s_0s_1s_3s_2,$$

where the purple indicates applying a short braid relation and the blue indicates applying a long braid relation. Thus w is not FC since a long braid relation can be applied.

Stembridge classified the irreducible Coxeter groups that contain a finite number of fully commutative elements, the so-called *FC-finite Coxeter groups*. This thesis is mainly concerned with $W(A_n)$, $W(B_n)$, $W(\widetilde{C}_n)$. Both $W(A_n)$, $W(B_n)$ are finite Coxeter groups, and thus are FC finite. On the other hand $W(\widetilde{C}_n)$ is infinite and has infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, E_n for $n \geq 9$ are infinite cite coxeter graph figure, but contain only finitely many fully commutative elements.

Theorem 1.3.4 (Stembridge, [5]). *The FC-finite irreducible Coxeter groups are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. The corresponding Coxeter graphs are shown in reference coxeter graph figure.* □

1.4 Heaps

We can now discuss another representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap.

Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps of straight line Coxeter groups in [1], [2] and [5].

Let (W, S) be a Coxeter system. Suppose $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ is a fixed reduced expression for $w \in W$. As in [5], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$

Chapter 2

My Cool Stuff

2.1 Classification of T-Avoiding Elements in Type B

An introduction should go here regarding the awesome sauce that is to follow.

And we probably need some other stuff to go here but Sarah S. told me to work on typing page 1 today.

Proposition 2.1.1 (Björner, cite when in Mendalay)**.** *Let $w \in W(B_n)$. Then*

$$\mathcal{R}(w) = \{s_i \in S : w(i) > w(i+1)\}$$

where w(0)=0 by definition.

Proof. This is, cite when in Mendelay Proposition 8.1.2. □

Lemma 2.1.2. *Let $s, t \in S$ such that $m(s,t) = 3$. Then w has a reduced expression ending in sts if and only if w has the consecutive pattern* 321*.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in cite BB when in Mendelay. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

($\Rightarrow$) Suppose w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+1)$. By 2.1.1 $s_i, s_{i+1} \in \mathcal{R}(w)$. By Tyson's reference to simply laced coxeter group stuff 1.2.1 w ends in $s_i s_{i+1} s_{i+2}$. ($\Leftarrow$) Suppose w ends in $s_i s_i + 1 s_i$. This implies that either $w_I = s_i s_{i+1}$ or $w_I = s_{i+1} s_i$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by 2.1.1. Thus w has the consecutive pattern 321. Therefore, w has a reduced expression ending in sts if and only if w has the consecutive pattern 321. □

Lemma 2.1.3. *Let $s, t \in S$ such that $m(s,t) = 3$. Then w has a reduced expression ending in st if and only if w has the consecutive pattern* 231*.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in cite BB when in Mendelay. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

($\Rightarrow$) Suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$. By 2.1.1 $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by s_{i+1} we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, this implies that $s_i \in \mathcal{R}(ws_{i+1})$. This implies w has a reduced expression that ends in $s_i s_{i+1}$. ($\Leftarrow$) Suppose that w has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence w has the consecutive pattern 231. Therefore, w has a reduced expression ending in st if and only if w has the consecutive pattern 231. $\square$

Lemma 2.1.4. *Let $s, t \in S$ such that $m(s,t) = 3$. Then w has a reduced expression ending in ts if and only if w has the consecutive pattern* 312.

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in cite BB when in Mendelay. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

($\Rightarrow$) Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. By 2.1.1 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by s_i we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$. By above $w(i) > w(i+2)$, and by 2.1.1 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that w has a reduced expression ending in $s_{i+1}s_i$. ($\Leftarrow$) Conversely suppose w ends in a reduced expression with $s_{i+1}s_i$ Then $w_I = s_{i+1}s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, w has the consecutive pattern 312. Therefore, w has a reduced expression ending in ts if and only if w has the consecutive pattern 312. $\square$

Chapter 3

Title of Chapter 3

Slightly boring, but still intelligent. I think.

Chapter 4

Title of Chapter 4

In this chapter, we'll say the most intelligent stuff anyone has ever said.

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