

A Study of T-Avoiding Elements in Coxeter Groups

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Definition

A **Coxeter system** consists of a group W (called a **Coxeter group**) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

where $m(s, t) \geq 2$ for all $s \neq t$.

Comment

Since s and t are involutions, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$m(s, t) = 2 \implies st = ts \quad \} \quad \text{commutations}$$

$$\left. \begin{array}{l} m(s, t) = 3 \implies sts = tst \\ m(s, t) = 4 \implies stst = tsts \\ \vdots \end{array} \right\} \quad \text{braid relations}$$

Definition

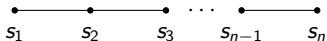
We can encode (W, S) with a unique Coxeter graph Γ having:

- vertex set S ;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$;

Comments

- if $m(s, t) = 3$, we omit label.
- If s and t are not connected in Γ , then s and t commute.
- Given Γ , we can uniquely reconstruct the corresponding (W, S) .

Coxeter groups of type A_n ($n \geq 1$) are defined by:



Then $W(A_n)$ is generated by $\{s_1, s_2, \dots, s_n\}$ and is subject to defining relations

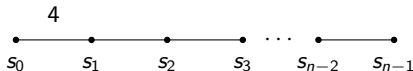
1. $s_i^2 = e$ for all i ,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

$W(A_n)$ is isomorphic to the symmetric group, Sym_{n+1} , under the correspondence

$$s_i \mapsto (i, i + 1),$$

where $(i, i + 1)$ is the adjacent transposition exchanging i and $i + 1$.

Coxeter groups of type B_n ($n \geq 2$) are defined by:

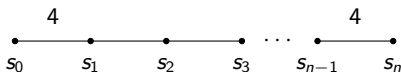


Then $W(B_n)$ is generated by $\{s_0, s_2, \dots, s_{n-1}\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i ,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j \leq n - 1$,
4. $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

$W(B_n)$ is a finite group of order $n!2^n$ (wreath product of \mathbb{Z}_2 and the symmetric group).

Coxeter groups of type \tilde{C}_n ($n \geq 2$) are defined by:



Here, we see that $W(\tilde{C}_n)$ is generated by $\{s_0, \dots, s_n\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i ,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j < n$,
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$ if $\{i, j\} = \{0, 1\}$ or $\{n - 1, n\}$.

$W(\tilde{C}_n)$ is an infinite group.

Comment

We can obtain $W(A_n)$ and $W(B_n)$ from $W(\tilde{C}_n)$ by removing the appropriate generators and corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m}$ is called an **expression** for $w \in W$ if it is equal to w when considered as a group element.

We define the length of w , $\ell(w)$, to be the smallest m for which w is a product of m generators, such an expression is called reduced.

Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w , we represent it as

$$\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}.$$

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of commutations and braid moves.

Definition

We define $\text{supp}(w)$ to be the set of generators appearing in any reduced expression for w . This is well defined by Matsumoto's theorem.

Definition

We define the **left descent set** w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

Example

Let $\overline{w} = s_2 s_1 s_2 s_3 s_1$ be a fixed expression for $w \in W(A_3)$. We see that

$$s_2 s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 s_1 = s_1 s_2 s_1 s_1 s_3 = s_1 s_2 s_3$$

Definition

Let (W, S) be a Coxeter system of type Γ . We say that $w \in W(\Gamma)$ is **fully commutative** (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements is denoted $FC(\Gamma)$.

Theorem (Stembridge)

$w \in FC(\Gamma)$ if and only if no reduced expression for w contains a braid.

Comment

It follows from Stembridge that $W(\tilde{C}_n)$ contains an infinite number of FC elements, while $W(A_n)$ and $W(B_n)$ do not.

Comment

The elements of $\text{FC}(\tilde{C}_n)$ are precisely those whose reduced expressions avoid the consecutive subwords $s_i s_j s_i$ for $m(s_i, s_j) = 3$, $s_0 s_1 s_0 s_1$, and $s_{n-1} s_n s_{n-1} s_n$.

Example

Let $\overline{w} = s_0 s_2 s_4 s_3 s_2 s_1$ be a reduced expression for $w \in W(\tilde{C}_4)$. We see that

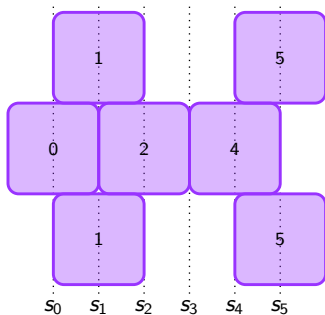
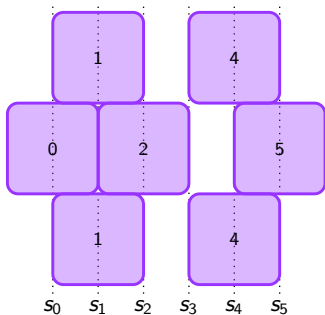
$$s_0 \textcolor{red}{s_2} \textcolor{red}{s_4} s_3 s_2 s_1 = s_0 s_4 \textcolor{red}{s_2} \textcolor{red}{s_3} \textcolor{red}{s_2} s_1.$$

Since w has one of the forbidden consecutive subwords, w is **not** FC.

Every reduced expression \overline{w} can be represented with a labeled partially ordered set (poset) called a heap, denoted $H(\overline{w})$. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators.

Example

Let $\overline{w} = s_4 s_5 s_1 s_0 s_2 s_4 s_1$ be a reduced expression for $w \in W(B_6)$.

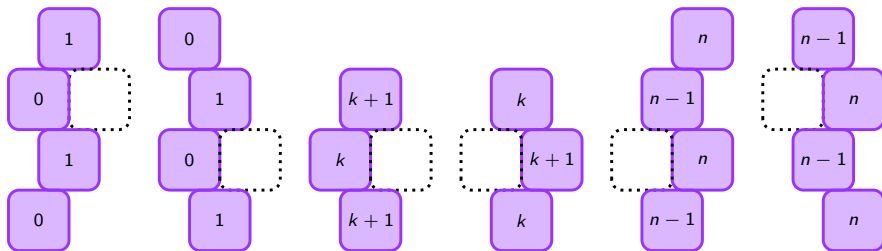


Theorem (Stembridge)

There is a unique heap for w if and only if w is FC.

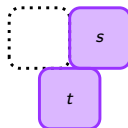
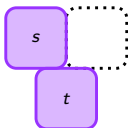
Lemma

Let $w \in \text{FC}(\tilde{C}_n)$. Then $H(w)$ can not contain any of the following convex subheaps



Definition

We define w to be **left star reducible by s with respect to t** if $m(s, t) \geq 3$, $s \in \mathcal{L}(w)$ and $t \in \mathcal{L}(sw)$.



Definition

We define $W(\Gamma)$ to be **star reducible** if every element of $FC(\Gamma)$ is star reducible to a product of commuting generators.

Theorem (Green)

Coxeter systems of type A_n ($n \geq 1$), type B_n ($n \geq 2$), type D_n ($n \geq 4$), type F_n ($n \geq 4$), type H_n ($n \geq 2$), type $I_2(m)$ ($m \geq 3$), type \tilde{A}_n ($n \geq 3$ and n even), type \tilde{C}_n ($n \geq 3$ and n odd), type \tilde{E}_6 , or type \tilde{F}_5 , are star reducible.

Definition

We define w to have **Property T** if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s, t) \geq 3$.

We say w is **T-avoiding** if w does not have Property T.

Proposition

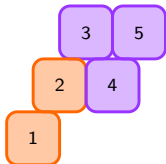
A product of commuting generators is T-avoiding.

Definition

We define w to be **trivially T-avoiding** if w is a product of commuting generators. Otherwise, we say w is **non-trivially T-avoiding**.

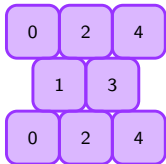
Example

Let $\overline{w}_1 = s_5 s_3 s_2 s_4 s_1$ be a reduced expression for $w \in W(A_5)$.



Example

Let $\overline{w}_2 = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$ be a reduced expression for $w \in W(\tilde{C}_4)$.



Classification of T-Avoiding Elements: Already Known \tilde{A}_n

Theorem (Fan)

If n is odd, and $n \geq 2$ there are no non-trivial T-avoiding elements in $W(\tilde{A}_n)$. If n is even, and $n \geq 2$ then $W(\tilde{A}_n)$ contains non-trivial T-avoiding elements.

Conjecture

The only non-trivial T-avoiding elements of $W(\tilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

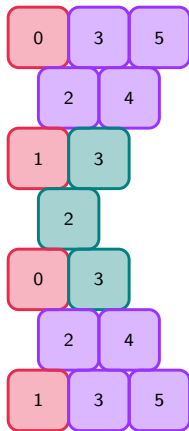
Theorem

There are no non-trivial T-avoiding elements in $W(A_n)$.

Classification of T-Avoiding Elements : Already Known D_n

Theorem (Gern)

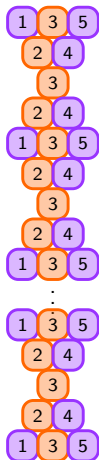
There are non-trivial T-avoiding elements in $W(D_n)$.



Classification of T-Avoiding Elements: Already Known F_n

Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

The only non-trivial T-avoiding elements in F_5 are stacks of bowties.

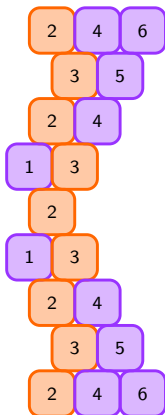


Classification of T-Avoiding Elements: Already Known F_n

Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no non-trivial T-avoiding elements in F_4 .

Classifying non-trivial T-avoiding elements in F_n for $n \geq 6$ gets very difficult.



Classification of T-Avoiding Elements: $I_2(m)$

Theorem

There are no non-trivial T-avoiding elements in $W(I_2(m))$.

Since $W(B_n) \cong \text{Sym}_n^B$, we can write $w \in W(B_n)$ as a signed permutation

$$[w(1), w(2), \dots, w(n)]$$

where we write a bar underneath a number in place of a negative sign.

Definition

We

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(B_n)$.