THE MEANING OF LIFE

By Tony Stark

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Approved:

Évariste Galois, Ph.D., Chair

John von Neumann, Ph.D.

Srinivasa Ramanujan, Ph.D.

Contents

List of Figures

Chapter 1

Preliminaries

1.1 Introduction

To be written once we know where everything is going to go.

1.2 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [8]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

(1) vertex set S and

(2) labeled edges $\{s, t\}$ for each $m(s, t) \geq 3$ with labeled with its corresponding bond strength $m(s, t)$.

Since $m(s, t) = 3$ occurs most frequently, it is customary to leave the edge unlabeled. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. If (W, S) is a Coxeter system with corresponding Coxeter graph Γ, we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. That is, given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. Note that s and t are not connected in the graph if and only if $m(s, t) = 2$. Also, the Coxeter system (W, S) is said to be *irreducible* if and only if Γ is connected. Further, if the graph Γ is disconnected, the connected components correspond to factors in a direct product of irreducible Coxeter systems [8].

Example 1.2.1.

(a) The Coxeter system of type A_n is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group with generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} under the correspondence $s_i \mapsto (i, i + 1)$, where $(i, i + 1)$ is the adjacent transposition that swaps i and $i + 1$.

(b) The Coxeter system of type B_n is given by the graph in Figure 1.1(b). We can construct the corresponding Coxeter group with generating set $S(B_n) = \{s_0, s_1, \ldots, s_{n-1}\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \ldots, n - 1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to Sym_n^B, where Sym_n^B is the group of all signed permutations on the set $\{1, 2, \ldots, n\}$.

(c) The Coxeter system of type $\widetilde{C}_n$ is seen in Figure 1.2(d). We can construct the corresponding Coxeter group $W(\widetilde{C}_n)$ with generating set $S(\widetilde{C}_n) = \{s_0, s_1, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i-j| > 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

The Coxeter graphs given in Figure 1.1 correspond to the collection of irreducible finite Coxeter groups, while the Coxeter graphs given in Figure 1.2 are the irreducible affine Coxeter groups, which are infinite [8]. These affine Coxeter groups are unique in that if a vertex is removed along with its edges from the Coxeter graph, the newly created graph will result in a finite Coxeter group. This thesis will mainly focus upon Coxeter groups of types A_n, B_n and $\widetilde{C}_n$ and will also briefly touch upon types $\widetilde{A}_n$ and F_n.

Given a Coxeter system (W, S), a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent it as $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$. The following theorem tells us more about how reduced expressions for a given group element are related.

Theorem 1.2.2 (Matsumoto, [6]). *Let (W, S) be a Coxeter system. If $w \in W$, then given a reduced expression for w we can obtain every other reduced expression for w by a sequence of braid moves and commutations of the form*

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. □

It follows from Matsumoto's Theorem that if a generator s appears in a reduced expression for $w \in W$, then s appears in all reduced expressions for w. Let $w \in W$ and fix a reduced expression $\overline{w}$ for w. Then the *support* of w, denoted $\mathrm{supp}(w)$, is the set of all generators of that appear in any reduced expression for w. If $\mathrm{supp}(w) = S$, we say that w has *full support*.

Given $w \in W$ and a fixed reduced expression $\overline{w}$ for w, any subsequence of $\overline{w}$ is called a *subexpression* of $\overline{w}$. We will refer to a subexpression consisting of a consecutive subsequence of $\overline{w}$ as a *subword* of $\overline{w}$.

Example 1.2.3. Let $w \in W(A_7)$ and let $\overline{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be a fixed expression for w. Then we have

$$\begin{aligned} s_7 s_2 s_4 s_5 s_3 s_2 s_3 s_6 &= s_7 s_4 s_2 s_5 s_3 s_2 s_3 s_6 \\ &= s_7 s_4 s_5 s_2 s_3 s_2 s_3 s_6 \\ &= s_7 s_4 s_5 s_3 s_2 s_3 s_3 s_6 \\ &= s_7 s_4 s_5 s_3 s_2 s_6, \end{aligned}$$

where the blue highlighted text corresponds to a commutation, the teal highlighted text corresponds to a braid move, and the red highlighted text corresponds to cancellation. This shows that $\overline{w}$ is not reduced. However it turns out that, $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\text{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $s \in \mathcal{R}(w)$) if and only if there is a reduced expression for w that begins (respectively, ends) with s.

Example 1.2.4. Let $w \in W(B_4)$ such that all reduced expressions for w are as follows

$$\begin{matrix} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{matrix}$$

We see that $l(w) = 5$ and w has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

1.3 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W$. Following [11], we define a relation $\sim$ on the set of reduced expressions for w. Let $\overline{w}_1$ and $\overline{w}_2$ be two reduced expressions for w. We define $\overline{w}_1 \sim \overline{w}_2$ if we can obtain $\overline{w}_2$ from $\overline{w}_1$ by applying a single commutation move of the form $st \mapsto ts$ where $m(s, t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative* (FC).

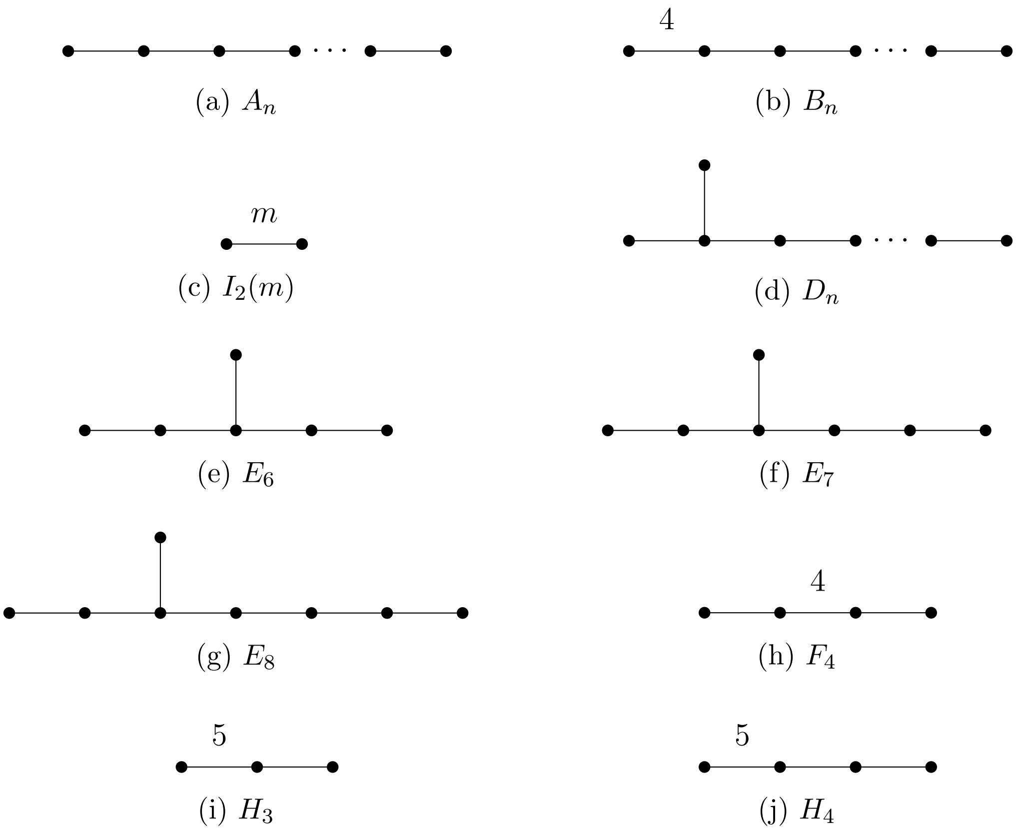


Figure 1.1: Coxeter graphs corresponding to the finite Coxeter groups.

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$, and a starting reduced expression for w, observe the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for w, but the following result due to Stembridge [11] states that when w is FC performing commutations is the only possible way to obtain another reduced expression for w.

Theorem 1.3.1 (Stembridge, [11])**.** *An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all when $m(s,t) \geq 3$.* □

In other words, w is FC if and only if we never have the opportunity to apply a braid move.

Example 1.3.2. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0s_1s_2s_0s_3s_1$ be a reduced expression for w. We see that

$$s_0s_1s_2s_0s_3s_1 = s_0s_1s_0s_2s_3s_1 = s_0s_1s_0s_2s_1s_3,$$

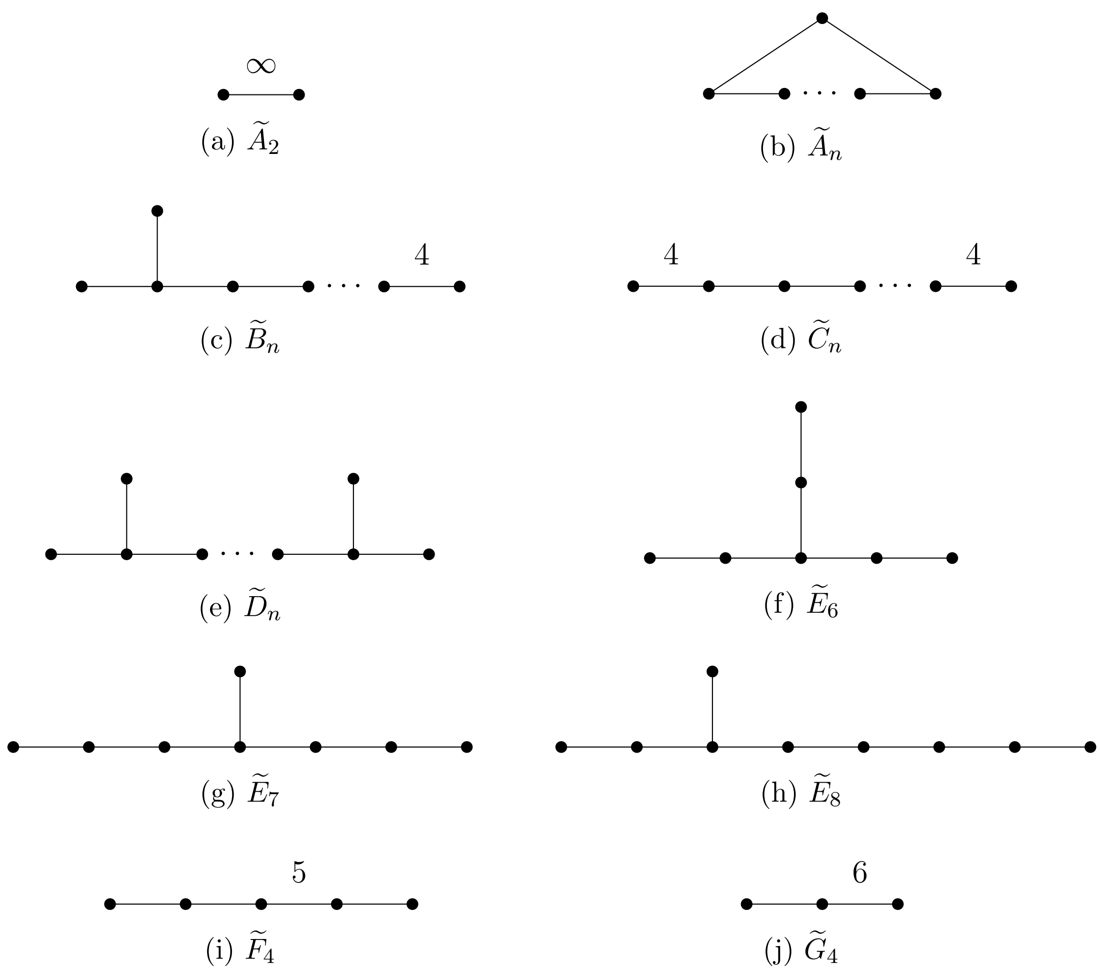


Figure 1.2: Coxeter graphs corresponding to the infinite Coxeter groups

where the purple indicates applying a commutation. Note that there is no possible way to perform a braid move. Hence w is FC.

Example 1.3.3. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0s_1s_2s_0s_1s_2$ be a reduced expression for w. We see that

$$s_0s_1s_3s_0s_1s_2 = s_0s_1s_0s_3s_1s_2 = s_0s_1s_0s_1s_3s_2,$$

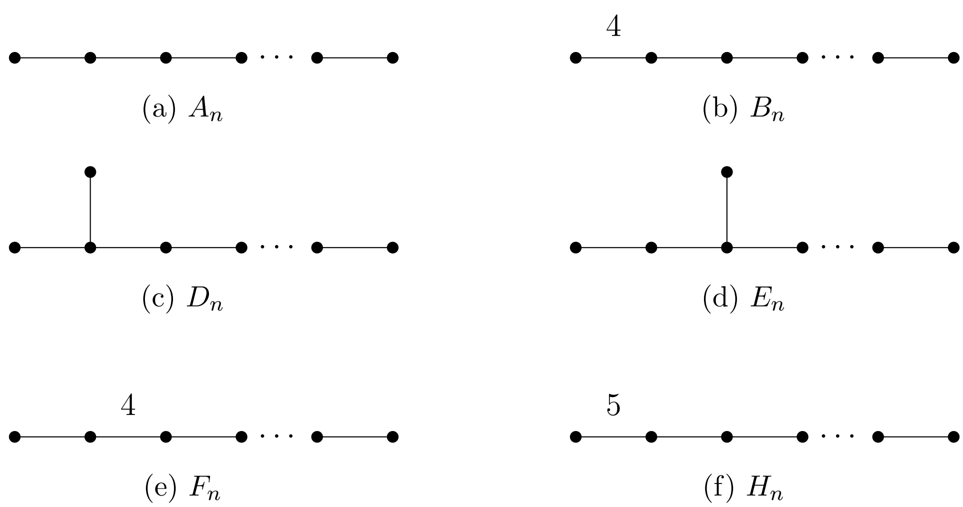
where the purple indicates applying a commutation and the orange indicates applying a braid move. Thus w is not FC since a braid move can be applied.

Example 1.3.4. Let $\overline{w} = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. Applying the commutation $s_4s_2 = s_2s_4$, we can obtain another reduced expression for w, namely $\overline{w}_2 = s_1s_0s_4s_1s_3s_5s_4s_2s_6$ which is in the same commutation class as $\overline{w_1}$. However, applying the braid move $s_2s_3s_2 = s_3s_2s_3$, we obtain another reduced expression $\overline{w_3} = s_1s_3s_2s_3s_4s_0$. Note that since $\overline{w}_3$ was obtained by applying a braid move, $\overline{w_3}$ is in a different commutation class than $\overline{w_1}$ and $\overline{w_2}$. It turns out w has exactly two commutation classes, one containing $\overline{w}_1$ and $\overline{w}_2$ and another containing $\overline{w}_3$.

Stembridge classified the irreducible Coxeter Systems that contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. This thesis is mainly concerned with $W(A_n)$, $W(B_n)$, $W(\widetilde{C}_n)$. Both $W(A_n)$, and $W(B_n)$ are finite Coxeter groups, and thus are FC-finite. On the other hand, $W(\widetilde{C}_n)$ is infinite and has infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, E_n for $n \geq 9$ (see Figure **??**) is infinite, but contains only finitely many FC elements.

Theorem 1.3.5 (Stembridge, [11])**.** *The FC-finite irreducible Coxeter groups are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$.* □

The FC-finite Coxeter graphs are given in Figure 1.3. Note that we have already encountered the FC-finite Coxeter groups in Figure 1.1. Since these are finite Coxeter groups it is clear that they will have a finite number of FC elements. However, we haven't yet encountered the coxeter groups seen in Figures 1.3(d), 1.3(e), 1.3(f). All of these Coxeter systems are infinite for large n, yet contain only finitely many FC elements.



(a) A_n (b) B_n

(c) D_n (d) E_n

(e) F_n (f) H_n

Figure 1.3: Coxeter graphs corresponding to the FC-finite Coxeter groups.

Chapter 2

Heaps

2.1 Heaps

We can now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps of straight line Coxeter groups in [1], [4], and [11].

Let (W, S) be a Coxeter system of type Γ. Suppose $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ is a fixed reduced expression for $w \in W$. As in [11], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, since $\overline{w}$ is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of $\overline{w}$, where i is labeled by s_{x_i}. Note that for simplicity we are omitting the labels of the underlying poset but retaining the labels of the corresponding generators.

It follows from [11] that heaps are well-defined up to commutation class. That is, given two reduced expressions $\overline{w}$ and $\overline{w'}$ for $w \in W$ that are in the same commutation class, then the heaps for $\overline{w}$ and $\overline{w'}$ will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then w has a one commutation class, and thus w has a unique heap.

When w is FC, we wish to make a canonical choice for the representation of $H(w)$ by assembling the entries in a particular way. To do so, we position all of the entries corresponding to elements in $\mathcal{L}(w)$ in the same vertical position, and all of the remaining elements should be positioned as high as possible in the lattice point representation. For example, the representation in Figure 2.2 is the canonical representation for w. Note that our canonical representation of heaps corresponds to Cartier-Foata normal form for monomials [3, 7]. There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap for each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

Example 2.1.1. Let $\overline{w} = s_1 s_0 s_4 s_1 s_3 s_5 s_2 s_4 s_6$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that $\overline{w}$ is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $1 \lessdot 0$ since $0 < 1$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 2.1.

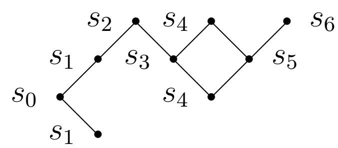


Figure 2.1: Labeled hasse diagram for the heap of an element in $\mathrm{FC}(\widetilde{C}_n)$

Let $\overline{w}$ be a reduced expression for an element in $w \in W(\widetilde{C}_n)$. As in [1] and [4] we can represent a heap for $\overline{w}$ as a set of lattice points embedded in $\{0, 1, 2, \ldots, n\} \times \mathbb{N}$. To do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \ldots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of $\overline{w}$ in such a way that:

(1) An entry with coordinates (x, y) is labeled s_i (or i) in the heap if and only if $x = i$;

(2) If an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight line Coxeter graph with the appropriate adjustments made to the label set and assignments of coordinates. Specifically for type A_n our label set is $\{1, 2, \ldots, n\} \times \mathbb{N}$ and for type B_n our label set is $\{0, 1, \ldots, n-1\} \times \mathbb{N}$.

In the case of any straight line Coxeter graph it follows from the definition that (x, y) covers (x', y') in the heap if and only if $x = x' \pm 1$, $y > y'$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries in the top correspond to the generators occurring in the right descent set of the corresponding reduced expression.

Let $\overline{w}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice representation of the heap poset in $\{0, 1, 2, \ldots n\} \times n$ described in the preceding paragraphs via $H(\overline{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ (note the absence of sans serif font) and we refer to $H(w)$ as the heap of w. As above the necessary adjustments to the lattice representation will result in a general result for heaps of all straight line coxeter graphs.

Given a heap, there are many possible coordinate assignments, yet the x-coordinates will be fixed for each entry will be fixed for all of them. In particular, two entries labeled by the same generator will only differ by the amount of vertical space between them while they will maintain the same horizontal position to adjacent entries in the heap.

Let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_j} at a level that is *above* the level of the point labeled by s_{x_i}. Because generators in a Coxeter graph that are not adjacent do commute, points whose x-coordinates differ more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners in such a way that if one entry covers another the squares overlap halfway. In addition, we will also label each square with i representing the generator s_i.

Example 2.1.2. Let $\overline{w} = s_1 s_0 s_4 s_1 s_3 s_5 s_2 s_4 s_6$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.4.1. Figure 2.2 shows a possible lattice point representation for $H(w)$.

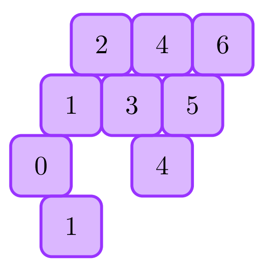
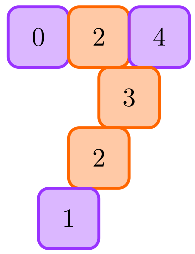
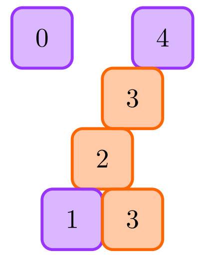


Figure 2.2: A possible lattice point representation of for the heap of an FC element in $W(\widetilde{C}_6)$.

Example 2.1.3. Let $\overline{w}_1 = s_1 s_2 s_3 s_4 s_2 s_0$ be a reduced expression for $w \in W(\widetilde{C}_4)$. Applying the commutation $s_4 s_2 = s_2 s_4$, we can obtain another reduced expression for w, namely $\overline{w}_2$ which is in the same commutation class as $\overline{w_1}$ and hence has the same heap. However, applying the braid move $s_2 s_3 s_2 = s_3 s_2 s_3$, we obtain another reduced expression $\overline{w_3} = s_1 s_3 s_2 s_3 s_4 s_0$. Note that since $\overline{w}_3$ was obtained by applying a braid move, $\overline{w_3}$ is in a different commutation class than $\overline{w_1}$ and $\overline{w_2}$. Representations of $H(\overline{w}_1), H(\overline{w}_2)$ and $H(\overline{w_3})$ are seen in Figure 2.3 where the braid relation is colored in orange.



(a) $H(\overline{w}_1)$ and $H(\overline{w}_2)$ in $W(\widetilde{C}_4$.



(b) $H(\overline{w}_3)$

Figure 2.3: Two heaps of a non-FC element

Chapter 3

Star Operations

3.1 Star Operations

The notion of star operations was originally introduced by Kazhdan and Lusztig in [9] for simply laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s,t \in S$), and was later generalized to all Coxeter systems in [10]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to define only the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Dana will you make sure that this definition is correct. Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of noncommuting generators whose product has order m. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We define w to be *left star reducible by s with respect to t* if there exists $t \in \mathcal{L}(sw)$. We analogously define w to be *right star reducible by s with respect to t*. Observe that if $m(s,t) \geq 3$, then w is left (respectively, right) star reducible if and only if there is a reduced expression for w such that $\overline{w} = stv$ (respectively, $\overline{w} = vts$). We say that w is *star reducible* if it is either left or right star reducible.

Example 3.1.1. Let $w \in W(B_4)$ and let $\overline{w} = s_0s_1s_0s_2s_3$ be a reduced expression for w. We see that w is left star reducible by s_0 with respect to s_1 to $s_1s_0s_2s_3$, since $m(s_0,s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0w)$. Also w is right star reducible by s_3 with respect to s_2 to $s_0s_1s_0s_2$, since $m(s_2,s_3) = 3$ and $s_3 \in \mathcal{R}(w)$ and $s_2 \in \mathcal{R}(ws_3)$.

It may be helpful to visualize star reducible in terms of heaps. Figure 3.1(a) represents $\overline{w}$. Note that, we can see s_0 is in the left descent set of w since s_0 is in the bottom row of the heap. Furthermore, multiplying on the left by s_0 we get the heap in Figure 3.1(b). Again, since s_1 is in the bottom row of the heap, $s_1 \in \mathcal{L}(s_1w)$.

In figure 3.1(a) we also see that s_3 is in the right descent set of w since $s_3 \in \mathcal{R}(w)$. Multiplying on the left by s_3 we can see that s_2 would be in the top level of the heap so $s_2 \in \mathcal{R}(ws_2)$. From this we can interpret visually an element $w \in W(\Gamma)$ is left star reducible (respectively, right star reducible) if there exists a heap that we can pull a block off the bottom row of the heap (respectively, top of the heap) and a new block that wasn't previously in the bottom row (respectively, top row) is now in the bottom row (respectively, top row) of the heap.

(a) I don't know what to call this (b) I don't know what to call this

Figure 3.1: Visualization of Example 3.1.1

Using the notion of star reducible we are now able to introduce the concept of a star reducible Coxeter group. We say that a Coxeter group $W(\Gamma)$, or it's Coxeter graph Γ, is *star reducible* if every element of $\mathrm{FC}(\Gamma)$ is star reducible to a product of commuting generators. That is, $W(\Gamma)$ is star reducible if when we apply star operations repeatedly to $w \in \mathrm{FC}(\Gamma)$ and eventually w is a product of commuting generators. In[7], Green classified all star reducible Coxeter groups.

Theorem 3.1.2. *Let $W(\Gamma)$ be a Coxeter group with (finite) generating set S. Then $W(\Gamma)$ is star reducible if an only if each component of Γ is either a complete graph with labels $m(s,t) \geq 3$, or is one of the following types: type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type H_n $(n \geq 2)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_{n-1}$ $(n \geq 3$ and n odd $)$, type $\widetilde{C}_{n-1}$ $(n \geq 4$ and n even $)$, type $\widetilde{E}_6$ or type $\widetilde{F}_5$.* □

3.2 Non-Cancellable Elements

We now introduce the concept of weak star reducible which is related to the notion of cancellable in [5]. Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of noncommuting generators of the Coxeter group $W(\Gamma)$. If $w \in \mathrm{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t to sw* if

(1) w is left star reducible by s with respect to t;

(2) and $tw \notin \mathrm{FC}(W)$.

Notice that (2) implies that $l(tw) > l(w)$. Also note that we are restricting out definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws*. We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable* or *weak star irreducible*.

Example 3.2.1. Let $w \in \mathrm{FC}(B_4)$ and let $\overline{w} = s_0s_1s_0s_2s_3$ be a reduced expression for w as in Example 3.1.1. By Example 3.1.1 we know that w is left star reducible. Also, $tw = s_1s_0s_1s_0s_2s_3$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2s_3$. In addition, Example 3.1.1 showed that w is right star reducible. Also, $wt = s_0s_1s_0s_2s_3s_2$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is right weak star reducible by s_3 with respect to s_2 to $s_0s_1s_0s_2$. This implies that w is not non-cancellable.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall in Figure 3.1(a) we have a representation for w as described in Example 3.2.1 and an associated discussion about the reason for w being star reducible. Now in Figure 3.2 we can see that when we multiply w by s_1 we end up with a braid, highlighted in orange and hence $ws_1 \notin \mathrm{FC}(\Gamma)$. Here the same properties as described above for w to be star reducible must be visualized in a heap and when multiplying on the left or right by t a braid must appear.

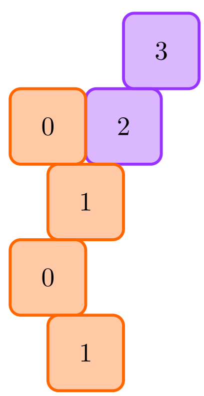


Figure 3.2: I don't know what to call this either.

Example 3.2.2. Let $w \in \mathrm{FC}(B_4)$ and let $\overline{w} = s_0s_1$ be a reduced expression for w. Note that w is left (respectively, right) star reducible by s_0 with respect to s_1 (respectively, by s_1 with respect to s_0). However, $s_1s_0s_1 \in \mathrm{FC}(B_4)$ (respectively, $s_0s_1s_0 \in \mathrm{FC}(B_4)$). Thus w is non-cancellable.

Chapter 4

Property-T and T-Avoiding

4.1 Property-T and T-Avoiding Elements

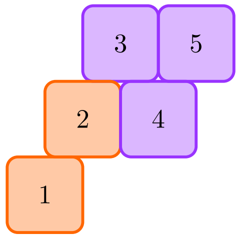
Introduction and Motivation for this section

We first begin by defining the notion of Property-T. Let (W, S) be a Coxeter system of type Γ and let $w \in W$ we say that w has *Property-T* if and only if there exists a reduced expression $\overline{w}$ such that $\overline{w} = stu$ or $\overline{w} = uts$ where $m(s,t) \geq 3$. That is, w has Property-T if there exists a reduced expression for w that begins with a product of non-commuting generators or ends with a product of non-commuting generators. It should be noted that by the symmetry of the definition if w has Property-T, then w^{-1} has Property-T.

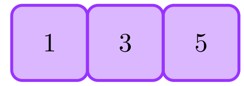
Example 4.1.1. Let $w \in W(A_5)$ and let $\overline{w} = s_1s_4s_2s_3s_5$. Note that applying a commutation to s_4s_5 results in $\overline{w}_1 = s_1s_2s_4s_3s_5$. Hence w has Property-T, since $m(s_1, s_2) = 3$ and there is a reduced expression for w which starts with s_1s_2.

Example 4.1.2. Let $w \in W(A_5)$ and let $\overline{w} = s_1s_3s_5$. It turns out that since w is a product of commuting generators there is no reduced expression for w that begins or ends with a pair of non-commuting generators. This implies that w does not have Property-T.

As with star reducible elements it may be helpful to visualize Property-T through heaps. Figure 4.1(a) provides a representation of $\overline{w}$ seen in Example 4.1.1 and Figure 4.1(b) provides a representation of $\overline{w}$ seen in Example 4.1.2. Notice that if we were to remove the block for s_1 in the bottom row of Figure 4.1(a) , we would have a new bottom row. However, in Figure 4.1(b), we are not able to remove able to remove any bricks and have a new brick come to the top or bottom row as the heap is just one row. Property-T provides the ability for us get a new row in our heap when we remove the top most or bottom most row from a specific reduced expression for w.



(a) Heap of an element with Property-T



(b) Heap of a T-Avoiding element

Figure 4.1: Heaps of an element with Property-T and a T-Avoiding element

An element $w \in W(\Gamma)$ is called *T-avoiding* if w and w^{-1} do not have Property-T. As seen in Example 4.1.2 an element $w \in W(\Gamma)$ will be T-avoiding if w is a product of commuting generators. We will call an element that is a product of commuting generators *trivially T-avoiding*. If w is T-avoiding and not a product of commuting generators, we will say that w is *non-trivially T-avoiding.*

Example 4.1.3. Let $w \in W(A_5)$ and let $\overline{w} = s_1 s_3 s_5$. Then by Example 4.1.2, we know that w is T-avoiding. Furthermore, since w is a product of commuting generators, w is trivially T-avoiding.

Example 4.1.4. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$. It turns out that w is non-trivially T-avoiding.

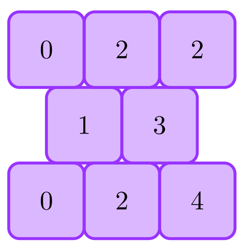


Figure 4.2: Heap of a non-trivially T-Avoiding element in $W(\widetilde{C}_4)$.

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