A Study of T-Avoiding Elements in Coxeter Groups

Taryn Laird

Northern Arizona University
Department of Mathematics and Statistics

NAU Thesis Defense
April 29, 2016

Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

where $m(s,t) \geq 2$ for all $s \neq t$.

Comment

Since s and t are involutions, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\begin{array}{lcll}
m(s,t) = 2 & \Longrightarrow & st = ts & \big\} \quad \text{commutations} \\
m(s,t) = 3 & \Longrightarrow & sts = tst & \multirow{3}{*}{$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ braid relations} \\
m(s,t) = 4 & \Longrightarrow & stst = tsts & \\
 & \vdots & &
\end{array}$$

Definition

We can encode (W, S) with a unique Coxeter graph Γ having:

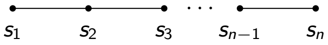
- vertex set S;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$;

Comments

- if $m(s, t) = 3$, we omit label.
- If s and t are not connected in Γ, then s and t commute.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Coxeter Groups of Type A

Coxeter groups of type A_n $(n \geq 1)$ are defined by:



Then $W(A_n)$ is generated by $\{s_1, s_2, \ldots, s_n\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

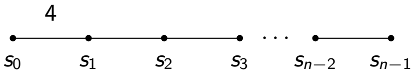
$W(A_n)$ is isomorphic to the symmetric group, Sym_{n+1}, under the correspondence

$$s_i \mapsto (i,\ i+1),$$

where $(i,\ i+1)$ is the adjacent transposition exchanging i and $i+1$.

Coxeter Groups of Type B

Coxeter groups of type B_n $(n \geq 2)$ are defined by:



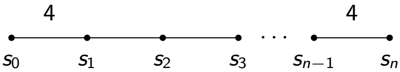
Then $W(B_n)$ is generated by $\{s_0, s_1, \ldots, s_{n-1}\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i,
2. $s_i s_j = s_j s_i$ if $|i-j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$ and $1 < i, j \leq n-1$,
4. $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

$W(B_n) \cong \mathrm{Sym}_n^B$ is a finite group of order $n!2^n$.

Coxeter Groups of Type Affine C

Coxeter groups of type $\widetilde{C}_n$ $(n \geq 2)$ are defined by:



Here, we see that $W(\widetilde{C}_n)$ is generated by $\{s_0, \dots, s_n\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j < n$,
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$ if $\{i, j\} = \{0, 1\}$ or $\{n - 1, n\}$.

$W(\widetilde{C}_n)$ is an infinite group.

Comment
We can obtain $W(A_n)$ and $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the appropriate generators and corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m}$ is called an expression for $w \in W$ if it is equal to w when considered as a group element.

We define the length of w, $\ell(w)$, to be the smallest m for which w is a product of m generators, such an expression is called reduced.

Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}.$$

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of commutations and braid moves.

Definition

We define $\mathrm{supp}(w)$ to be the set of generators appearing in any reduced expression for w. This is well-defined by Matsumoto's Theorem.

Definition

We define the left (respectively, right) descent set w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\} \qquad \mathcal{R}(W) := \{s \in S \mid l(ws) < l(w)\}$$

Example

Let $\overline{w} = s_2 s_1 s_2 s_3 s_1$ be a fixed expression for $w \in W(A_3)$. We see that

$$s_2 s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 s_1 = s_1 s_2 s_1 s_1 s_3 = s_1 s_2 s_3$$

Definition

Let (W, S) be a Coxeter system of type Γ. We say that $w \in W(\Gamma)$ is fully commutative (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements is denoted $\mathrm{FC}(\Gamma)$.

Theorem (Stembridge)

$w \in \mathrm{FC}(\Gamma)$ if and only if no reduced expression for w contains a braid.

Comment

It follows from Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements, while $W(A_n)$ and $W(B_n)$ do not.

Comment

The elements of $\mathrm{FC}(\widetilde{C}_n)$ are precisely those whose reduced expressions avoid the consecutive subwords $s_i s_j s_i$ for $m(s_i, s_j) = 3$, $s_0 s_1 s_0 s_1$, and $s_{n-1} s_n s_{n-1} s_n$.

Example

Let $\overline{w} = s_0 s_2 s_4 s_3 s_2 s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. We see that

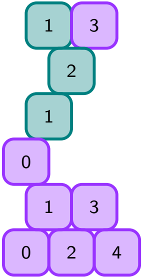
$$s_0 s_2 s_4 s_3 s_2 s_1 = s_0 s_4 s_2 s_3 s_2 s_1.$$

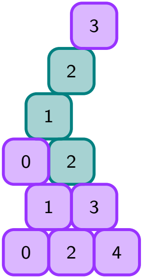
Since w has one of the forbidden consecutive subwords, w is not FC.

Every reduced expression $\overline{w}$ can be represented by a labeled partially ordered set (poset) called a heap, denoted $H(\overline{w})$. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators.

Example

Let $\overline{w} = s_4 s_5 s_1 s_0 s_2 s_4 s_1$ be a reduced expression for $w \in W(B_6)$.



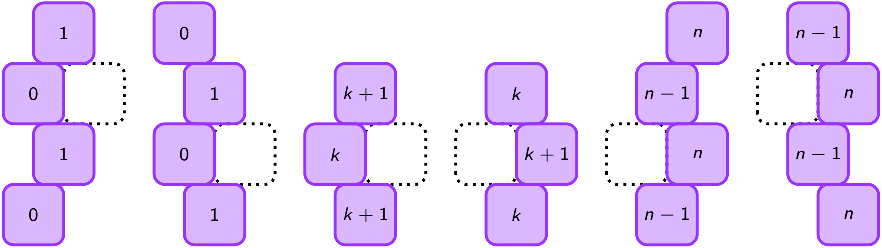


Theorem (Stembridge)

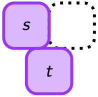
There is a unique heap for w if and only if w is FC.

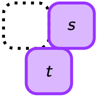
Lemma

Let $w \in \mathrm{FC}(\widetilde{C}_n)$. Then $H(w)$ can not contain any of the following convex subheaps



Definition
We define w to be left star reducible by s with respect to t if $m(s,t) \geq 3$, $s \in \mathcal{L}(w)$ and $t \in \mathcal{L}(sw)$. Analogous definition for right star reducible.





Definition
We define $W(\Gamma)$ to be star reducible if every element of $\mathrm{FC}(\Gamma)$ can be reduced to a product of commuting generators via a sequence of star reductions.

Theorem (Green)
There is a complete list of star reducible Coxeter systems. These include Coxeter systems of type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_n$ ($n \geq 3$ and n even), and type $\widetilde{C}_n$ ($n \geq 3$ and n odd).

Definition

We define w to have Property T if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s,t) \geq 3$.

We say w is T-avoiding if w does not have Property T.

Proposition

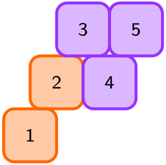
A product of commuting generators is T-avoiding.

Definition

We define w to be a trivial T-avoiding element if w is a product of commuting generators. Otherwise, we say w is a non-trivial T-avoiding element.

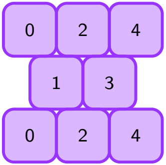
Example

Let $\overline{w} = s_5 s_3 s_2 s_4 s_1$ be a reduced expression for $w \in W(A_5)$.



Example

Let $\overline{w} = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$ be a reduced expression for $w \in W(\widetilde{C}_4)$.



Theorem (Fan, Green)

If n is odd and $n \geq 2$, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. If n is even and $n \geq 2$, then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

Conjecture

The only non-trivial T-avoiding elements of $W(\widetilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

Corollary

There are no non-trivial T-avoiding elements in $W(A_n)$.

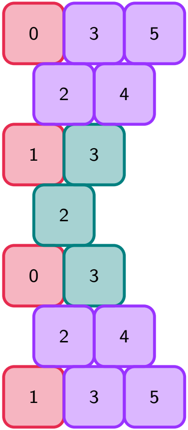
Theorem (Laird?)

There are no non-trivial T-avoiding elements in $W(I_2(m))$.

Known Classifications

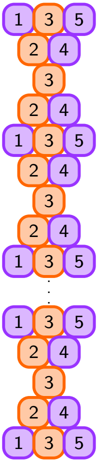
Theorem (Gern)

The only non-trivial T-avoiding elements in $W(D_n)$ have this pattern:



Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

The only non-trivial T-avoiding elements in F_5 are stacks of bowties:

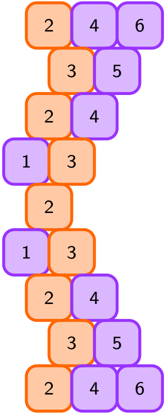


Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no non-trivial T-avoiding elements in F_4.

Comment

Classifying non-trivial T-avoiding elements in F_n for $n \geq 6$ gets very difficult.



Since $W(B_n) \cong \mathrm{Sym}_n^B$, we can write $w \in W(B_n)$ as a signed permutation

$$[w(1), w(2), \ldots, w(n)]$$

where we write a bar underneath a number in place of a negative sign.

Definition

We say that w has the signed consecutive pattern *abc* if there is some $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same relative order as $(|a|, |b|, |c|)$ and such that $\mathrm{sgn}(w(i)) = \mathrm{sgn}(a)$, $\mathrm{sgn}(w(i+1)) = \mathrm{sgn}(b)$, and $\mathrm{sgn}(w(i+2)) = \mathrm{sgn}(c)$.

We say that w avoids the signed consecutive pattern *abc* if there is no such i as above.

Example

Let $\overline{w} = s_0 s_1 s_3 s_4 s_5 s_2$ be a reduced expression for $w \in B_6$. We see that

$$\begin{aligned}
[1,2,3,4,5,6] &= [\underline{1},2,3,4,5,6] \\
&= [2,\underline{1},3,4,5,6] \\
&= [2,\underline{1},4,3,5,6] \\
&= [2,\underline{1},4,5,3,6] \\
&= [2,\underline{1},4,5,6,3] \\
&= [2,4,\underline{1},5,6,3]
\end{aligned}$$

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(B_n)$.

123	$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$	$\underline{12}3$	$\underline{1}2\underline{3}$	$1\underline{23}$	$\underline{123}$
132	$\underline{1}32$	$1\underline{3}2$	$13\underline{2}$	$\underline{13}2$	$\underline{1}3\underline{2}$	$1\underline{32}$	$\underline{132}$
213	$\underline{2}13$	$2\underline{1}3$	$21\underline{3}$	$\underline{21}3$	$\underline{2}1\underline{3}$	$2\underline{13}$	$\underline{213}$
231	$\underline{2}31$	$2\underline{3}1$	$23\underline{1}$	$\underline{23}1$	$\underline{2}3\underline{1}$	$2\underline{31}$	$\underline{231}$
312	$\underline{3}12$	$3\underline{1}2$	$31\underline{2}$	$\underline{31}2$	$\underline{3}1\underline{2}$	$3\underline{12}$	$\underline{312}$
321	$\underline{3}21$	$3\underline{2}1$	$32\underline{1}$	$\underline{32}1$	$\underline{3}2\underline{1}$	$3\underline{21}$	$\underline{321}$

Through a series of lemmas we were able to determine if a reduced product ends or begins with *st* given that *w* contains a certain signed consecutive pattern.

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

Theorem (Laird)

If n is odd, then there are no non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$.

Theorem (Laird)

If n is even, then the only non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$ are sandwich stacks.

Comment

Recall that Coxeter systems of Type D_n and F_n have non-trivial T-avoiding elements that are not FC. Also Coxeter systems of Type $\widetilde{A}_n$ and $\widetilde{C}_n$ for appropriate choice of n have non-trivial T-avoiding elements that are FC.

In all of the examples we have seen so far the non-trivial T-avoiding elements are either only FC or only not FC.