# A Study of T-Avoiding Elements in Coxeter Groups

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### Definition

A Coxeter system consists of a group $W$ (called a Coxeter group) generated by a set $S$ of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

where $m(s,t) \geq 2$ for all $s \neq t$.

### Comment

Since $s$ and $t$ are involutions, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\begin{array}{lcll}
m(s,t) = 2 & \Longrightarrow & st = ts & \} \quad \text{commutations} \\
m(s,t) = 3 & \Longrightarrow & sts = tst & \multirow{3}{*}{$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ braid relations} \\
m(s,t) = 4 & \Longrightarrow & stst = tsts & \\
 & \vdots & &
\end{array}$$

### Definition

We can encode $(W, S)$ with a unique Coxeter graph $\Gamma$ having:

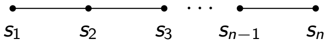
- vertex set $S$;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$;

### Comments

- if $m(s, t) = 3$, we omit label.
- If $s$ and $t$ are not connected in $\Gamma$, then $s$ and $t$ commute.
- Given $\Gamma$, we can uniquely reconstruct the corresponding $(W, S)$.

# Coxeter groups of type $A$

Coxeter groups of type $A_n$ ($n \geq 1$) are defined by:




Then $W(A_n)$ is generated by $\{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

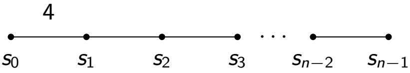
1. $s_i^2 = 1$ for all $i$,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

$W(A_n)$ is isomorphic to the symmetric group, $Sym_{n+1}$, under the correspondence

$$s_i \mapsto (i,\ i+1),$$

where $(i,\ i+1)$ is the adjacent transposition exchanging $i$ and $i+1$.

Coxeter groups of type $B_n$ ($n \geq 2$) are defined by:



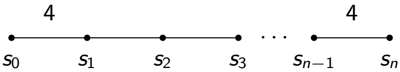

Then $W(B_n)$ is generated by $\{s_1, s_2, \cdots, s_{n-1}\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all $i$,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j \leq n$,
4. $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

$W(B_n)$ is a finite group of order $n!2^n$ (wreath product of $\mathbb{Z}_2$ and the symmetric group).

# Coxeter groups of type affine $C$

Coxeter groups of type $\widetilde{C}_n$ ($n \geq 2$) are defined by:




Here, we see that $W(\widetilde{C}_n)$ is generated by $\{s_0, \cdots, s_n\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all $i$,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j < n + 1$,
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$ if $\{i, j\} = \{0, 1\}$ or $\{n - 1, n\}$.

$W(\widetilde{C}_n)$ is an infinite group.

**Comment**

We can obtain $W(A_n)$ and $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the appropriate generators and corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for $w \in W$ if it is equal to $w$ when considered as a group element.

If $m$ is minimal, it is a reduced expression, and the length of $w$ is $\ell(w) := m$.

Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for $w$, we represent it as

$$\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}.$$

### Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of commutations and braid moves.

### Definition

We define $\mathrm{supp}(w)$ to be the set of generators appearing in any reduced expression for $w$. This is well defined by Matsumoto's theorem.

### Definition

We define the left descent set $w$ as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

### Example

Let $\overline{w} = s_2 s_1 s_2 s_3 s_1$ be a fixed expression for $w \in W(A_3)$. We see that

$$s_2 s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 s_1 = s_1 s_2 s_1 s_1 s_3 = s_1 s_2 s_3$$

### Definition

Let $(W, S)$ be a Coxeter system of type $\Gamma$. We say that $w \in W(\Gamma)$ is fully commutative (FC) if any two reduced expressions for $w$ can be transformed into each other via iterated commutations. The set of FC elements is denoted $\mathrm{FC}(\Gamma)$.

### Theorem (Stembridge)

$w \in \mathrm{FC}(\Gamma)$ if and only if no reduced expression for $w$ contains a braid.

### Comment

It follows from Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements, while $W(A_n)$ and $W(B_n)$ do not.

### Comment

The elements of $\mathrm{FC}(\widetilde{C}_n)$ are precisely those whose reduced expressions avoid the consecutive subwords $s_i s_j s_i$ for $m(s_i, s_j) = 3$, $s_0 s_1 s_0 s_1$, and $s_{n-1} s_n s_{n-1} s_n$.

### Example

Let $\overline{w} = s_0 s_2 s_4 s_3 s_2 s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. We see that

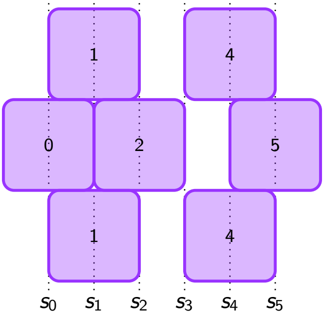
$$s_0 s_2 s_4 s_3 s_2 s_1 = s_0 s_4 s_2 s_3 s_2 s_1.$$

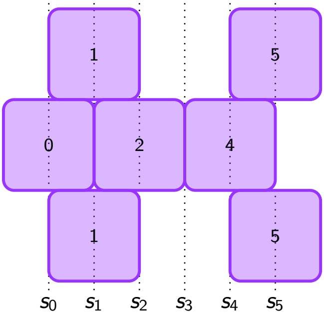
Since $w$ has one of the forbidden consecutive subwords, $w$ is not FC.

Every reduced expression $\overline{w}$ can be represented with a labeled partially ordered set (poset) called a heap, denoted $H(\overline{w})$. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators.

### Example

Let $\overline{w} = s_4 s_5 s_1 s_0 s_2 s_4 s_1$ be a reduced expression for $w \in W(B_6)$.



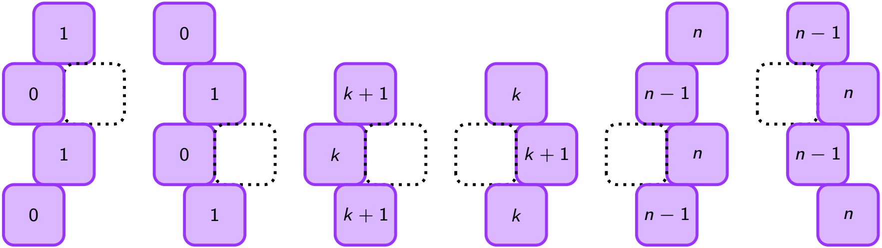



### Theorem (Stembridge)

There is a unique heap for $w$ if and only if $w$ is FC.

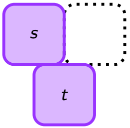
### Lemma

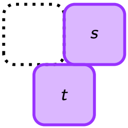
Let $w \in \mathrm{FC}(\widetilde{C}_n)$. Then $H(w)$ can not contain any of the following convex subheaps



**Definition**

We define $w$ to be left star reducible by $s$ with respect to $t$ if $m(s,t) \geq 3$, $s \in \mathcal{L}(w)$ and $t \in \mathcal{L}(sw)$.




**Definition**

We define $W(\Gamma)$ to be star reducible if every element of FC($\Gamma$) is star reducible to a product of commuting generators.

**Theorem (Green)**

Coxeter systems of type $A_n$ $(n \geq 1)$, type $B_n$ $(n \geq 2)$, type $D_n$ $(n \geq 4)$, type $F_n$ $(n \geq 4)$, type $H_n$ $(n \geq 2)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_n$ $(n \geq 3$ and $n$ even $)$, type $\widetilde{C}_n$ $(n \geq 3$ and $n$ odd $)$, type $\widetilde{E}_6$, or type $\widetilde{F}_5$, are star reducible.

### Definition

We define $w$ to have Property T if and only if there exists a reduced product for $w$ such that $w = stu$ or $w = uts$ where $m(s, t) \geq 3$.
We say $w$ is T-avoiding if $w$ does not have Property T.

### Proposition

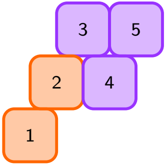
A product of commuting generators is T-avoiding.

### Definition

We define $w$ to be trivially T-avoiding if $w$ is a product of commuting generators.
Otherwise, we say $w$ is non-trivially T-avoiding.

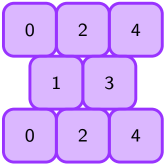
### Example

Let $\overline{w}_1 = s_5 s_3 s_2 s_4 s_1$ be a reduced expression for $w \in W(A_5)$.




### Example

Let $\overline{w}_2 = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$ be a reduced expression for $w \in W(\widetilde{C}_4)$.



**Theorem (Fan)**

If $n$ is odd, and $n \geq 2$ there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. If $n$ is even, and $n \geq 2$ then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

**Conjecture**

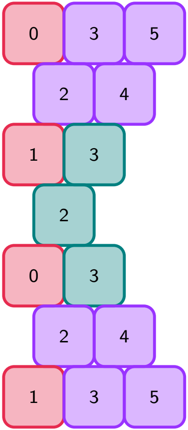
The only non-trivial T-avoiding elements of $W(\widetilde{A}_n)$ for $n$ odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

**Theorem**

There are no non-trivial T-avoiding elements in $W(A_n)$.

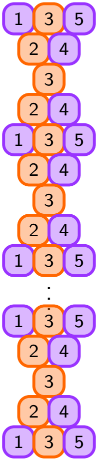
### Theorem (Gern)

There are non-trivial T-avoiding elements in $W(D_n)$.



**Theorem (Cross, Ernst, Hills-Kimball, Quaranta)**

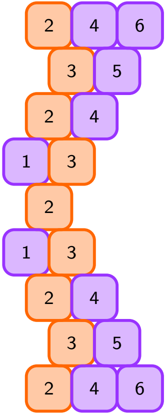
The only non-trivial T-avoiding elements in $F_5$ are stacks of bowties.



Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no non-trivial T-avoiding elements in $F_4$.

Classifying non-trivial T-avoiding elements in $F_n$ for $n \geq 6$ gets very difficult.



Theorem

There are no non-trivial T-avoiding elements in $W(I_2(m))$.

Since $W(B_n) \cong \mathrm{Sym}_n^B$, we can write $w \in W(B_n)$ as a signed permutation

$$[w(1), w(2), \ldots, w(n)]$$

where we write a bar underneath a number in place of a negative sign.

**Definition**

We

**Theorem (Laird)**

There are no non-trivial T-avoiding elements in $W(B_n)$.