Star Operations

Star Operations and Non-Cancellable ElementsStar

The notion of star operations was originally introduced by Kazhdan and Lusztig in Kazhdan1979 for simply laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s,t \in S$), and was later generalized to all Coxeter systems in Lusztig1985. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as star operations. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to define only the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

redDana will you make sure that this definition is correct. Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of noncommuting generators whose product has order m. Let $w \in W(\Gamma)$ such that $s \in L(w)$. We define w to be left star reducible by s with respect to t if there exists $t \in L(sw)$. We analogously define w to be right star reducible by s with respect to t. Observe that if $m(s,t) \geq 3$, then w is left (respectively, right) star reducible if and only if there is a reduced expression for w such that $\overline{w} = stv$ (respectively, $\overline{w} = vts$). We say that w is star reducible if it is either left or right star reducible.

example Let $w \in W(B_4)$ and let $\overline{w} = s_0 s_1 s_0 s_2 s_3$ be a reduced expression for w. We see that w is left star reducible by s_0 with respect to s_1 to $s_1 s_0 s_2 s_3$, since $m(s_0, s_1) = 4$ and $s_0 \in L(w)$ while $s_1 \in L(s_0 w)$. Also w is right star reducible by s_3 with respect to s_2 to $s_0 s_1 s_0 s_2$, since $m(s_2, s_3) = 3$ and $s_3 \in R(w)$ and $s_2 \in R(w s_3)$.