THE MEANING OF LIFE

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A Thesis

Submitted in Partial Fulfillment

of the Requirements for the Degree of

Master of Science

in Mathematics

Northern Arizona University

May 2016

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ABSTRACT

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Everything you always wanted to know will be discussed.

Contents

List of Figures

Chapter 1

Preliminaries

1.1 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [10]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

(1) vertex set S and

(2) edges $\{s, t\}$ for each $m(s,t) \geq 3$ labeled by its corresponding bond strength $m(s,t)$.

Since $m(s,t) = 3$ occurs frequently, it is customary to omit this label. Note that s and t are not connected by a single edge in the graph if and only if $m(s,t) = 2$. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. That is, given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. If (W,S) is a Coxeter system with corresponding Coxeter graph Γ, we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. Also, the Coxeter system (W,S) is said to be *irreducible* if and only if Γ is connected. If the graph Γ is disconnected, the connected components correspond to factors in a direct product of the corresponding Coxeter groups [10].

Example 1.1.1.

(a) The Coxeter system of type A_n is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group $W(A_n)$ with generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i-j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} under the correspondence $s_i \mapsto (i, i+1)$, where $(i, i+1)$ is the adjacent transposition that swaps i and $i+1$.

(b) The Coxeter system of type B_n is given by the graph in Figure 1.1(b). We can construct the corresponding Coxeter group $W(B_n)$ with generating set $S(B_n) = \{s_0, s_1, \ldots, s_{n-1}\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i-j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$ for $i, j \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to Sym_n^B, where Sym_n^B is the group of signed permutations on the set $\{1, 2, \ldots, n\}$.

(c) The Coxeter system of type $\widetilde{C}_n$ is seen in Figure 1.2(d). We can construct the corresponding Coxeter group $W(\widetilde{C}_n)$ with generating set $S(\widetilde{C}_n) = \{s_0, s_1, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i-j| > 1$ for $i \in \{0, 2, \ldots, n\}$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

Note that $W(\widetilde{C}_n)$ has $n+1$ generators.

The Coxeter graphs given in Figure 1.1 correspond to the collection of irreducible finite Coxeter groups, while the Coxeter graphs given in Figure 1.2 are the so-called irreducible *affine Coxeter groups*, which are infinite [10]. Note that $W(\widetilde{C}_n)$ is one of the affine groups making it infinite. The irreducible affine Coxeter systems are unique in that if a vertex is removed along with the corresponding edges from the Coxeter graph, the newly created graph will result in a Coxeter system with a finite Coxeter group.

Given a Coxeter system (W, S), a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent it as $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$. The following theorem tells us more about how reduced expressions for a given group element are related.

Theorem 1.1.2 (Matsumoto, [6]). Let (W, S) be a Coxeter system. If $w \in W$, then given a reduced expression for w we can obtain every other reduced expression for w by a sequence of braid moves and commutations of the form

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. $\square$

It follows from Matsumoto's Theorem that if a generator s appears in a reduced expression for $w \in W$, then s appears in all reduced expressions for w. Let $w \in W$ and define the *support* of w, denoted $\operatorname{supp}(w)$, to be the set of all generators that appear in any reduced expression for w. If $\operatorname{supp}(w) = S$, we say that w has *full support*.

Given $w \in W$ and a fixed reduced expression w for w, any subsequence of w is called a *subexpression* of w. We will refer to a subexpression consisting of a consecutive subsequence of w as a *subword* of w. If $u, v \in W(\Gamma)$, we say that the product of group elements uv is *reduced* if $l(uv) = l(u) + l(v)$.

Example 1.1.3. Let $w \in W(A_7)$ and let $\mathsf{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be a fixed expression for w. Then we have

$$\begin{aligned} s_7s_2s_4s_5s_3s_2s_3s_6 &= s_7s_4s_2s_5s_3s_2s_3s_6 \\ &= s_7s_4s_5s_2s_3s_2s_3s_6 \\ &= s_7s_4s_5s_3s_2s_3s_3s_6 \\ &= s_7s_4s_5s_3s_2s_6, \end{aligned}$$

where the purple highlighted text corresponds to a commutation, the teal highlighted text corresponds to a braid move, and the red highlighted text corresponds to cancellation. This shows that the original expression w is not reduced. However, it turns out that $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\text{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a reduced expression for w that begins (respectively, ends) with s.

Example 1.1.4. The following list consists of all reduced expressions some $w \in W(B_4)$:

$$\begin{array}{ll} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{array}$$

We see that $l(w) = 5$ and w has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

1.2 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. Following [13], we define a relation $\sim$ on the set of reduced expressions for w. Let w_1 and w_2 be two reduced expressions for w. We define $\mathsf{w}_1 \sim \mathsf{w}_2$ if we can obtain w_2 from w_1 by applying a single commutation move of the form $st \mapsto ts$ where $m(s,t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative* (FC).

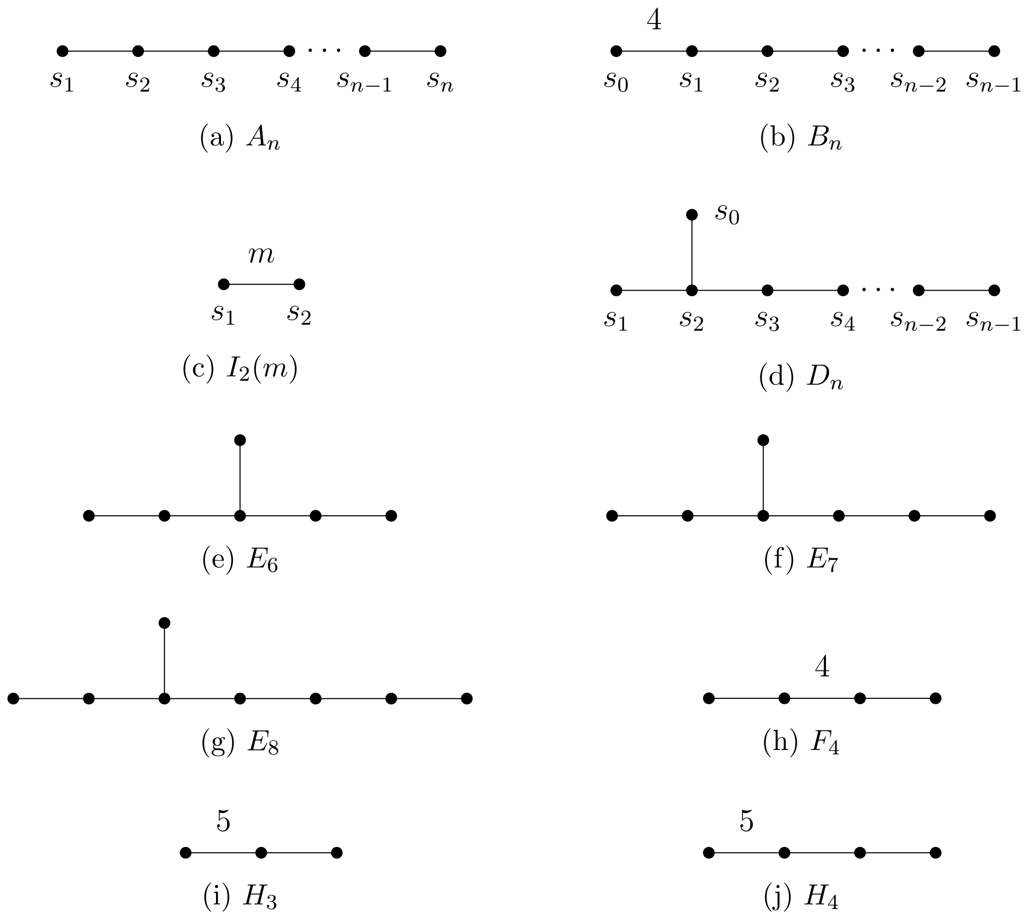


Figure 1.1: Coxeter graphs corresponding to the irreducible finite Coxeter sytems.

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$, and a starting reduced expression for w, observe that the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for w, but the following result due to Stembridge [13] states that when w is FC, performing commutations is the only possible way to obtain another reduced expression for w.

Theorem 1.2.1 (Stembridge, [13])**.** Let (W, S) be a Coxeter system. An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all $m(s,t) \geq 3$. $\square$

In other words, w is FC if and only if no reduced expression provides the opportunity to apply a braid move. In particular, for a Coxeter group of type B_n an element is FC if it does not contain the subwords $s_0s_1s_0s_1$, $s_1s_0s_1s_0$, $s_ks_{k+1}s_k$, and $s_{k+1}s_ks_{k+1}$

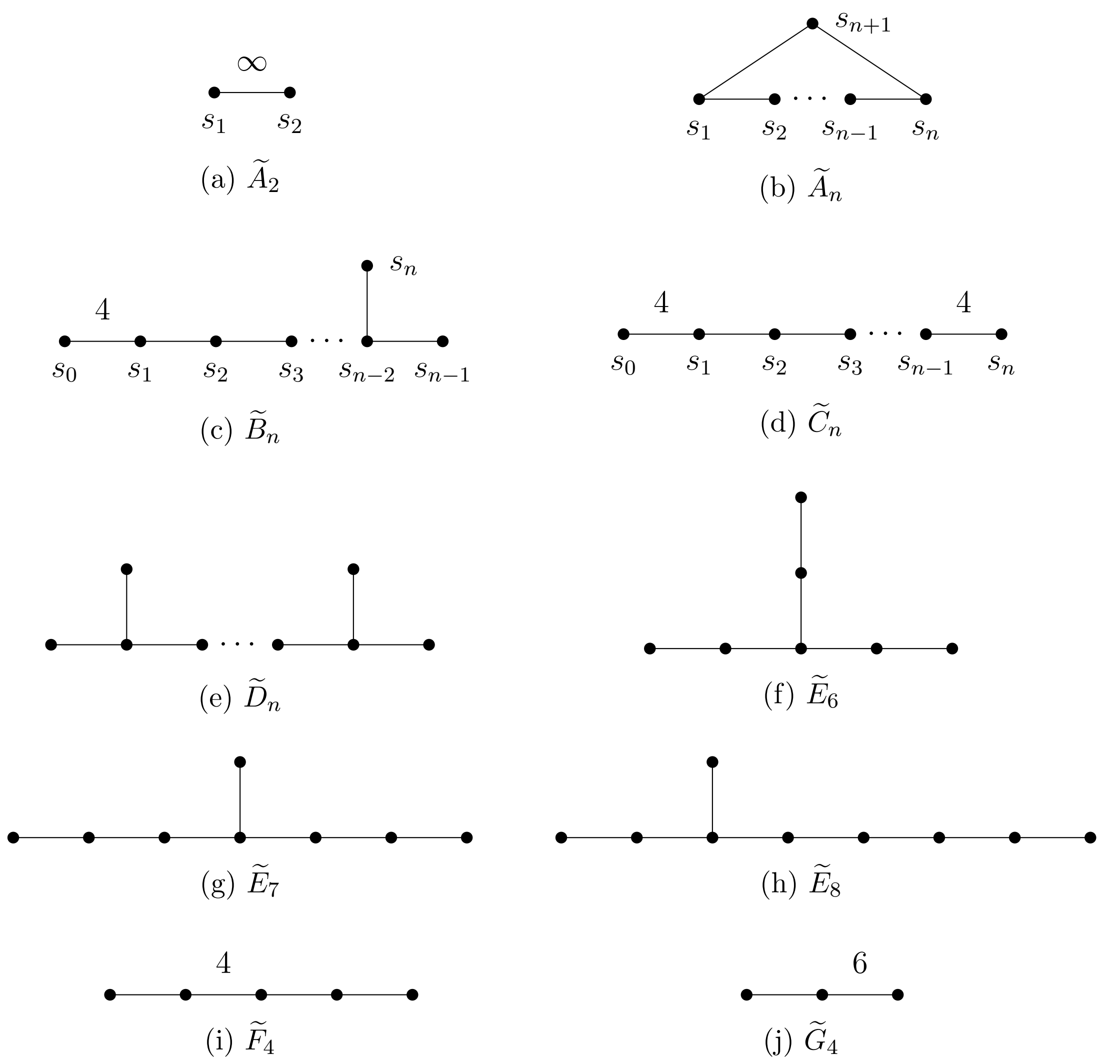


Figure 1.2: Coxeter graphs corresponding to the irreducible affine Coxeter systems.

where $0 < k < n-1$. In a Coxeter group of type $\widetilde{C}_n$, an element is FC if it does not contain the subwords seen above and does not contain the subwords $s_{n-1}s_ns_{n-1}s_n$ and $s_ns_{n-1}s_ns_{n-1}$.

Example 1.2.2. Let $w \in W(\widetilde{C}_4)$ and let $\mathsf{w} = s_0s_1s_2s_0s_3s_1$ be a reduced expression for w. We see that

$$s_0s_1s_2s_0s_3s_1 = s_0s_1s_0s_2s_3s_1 = s_0s_1s_0s_2s_1s_3,$$

where the purple highlighted text indicates applying a commutation. Although it is not immediately obvious, these are all of the reduced expressions for the given w. Note that there is no possible way to perform a braid move. Hence w is FC.

Example 1.2.3. Let $\mathsf{w} = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. Applying the commutation $s_4s_2 = s_2s_4$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_1s_0s_4s_1s_3s_5s_4s_2s_6$, which is in the same commutation class as w. However, applying the braid move $s_2s_3s_2 = s_3s_2s_3$, we obtain another reduced expression $\mathsf{w} = s_1s_3s_2s_3s_4s_0$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. It turns out that w has exactly two commutation classes, one containing w_1 and w_2 and another containing w_3. So w is not FC by Theorem 1.2.1.

Example 1.2.4. Let $w \in W(\widetilde{C}_4)$ and let $\mathsf{w} = s_0s_1s_2s_0s_1s_2$ be a reduced expression for w. We see that

$$s_0s_1s_3s_0s_1s_2 = s_0s_1s_0s_3s_1s_2 = s_0s_1s_0s_1s_3s_2,$$

where the purple highlighted text indicates applying a commutation and the orange highlighted text indicates applying a braid move. Thus w is not FC by Theorem 1.2.1.

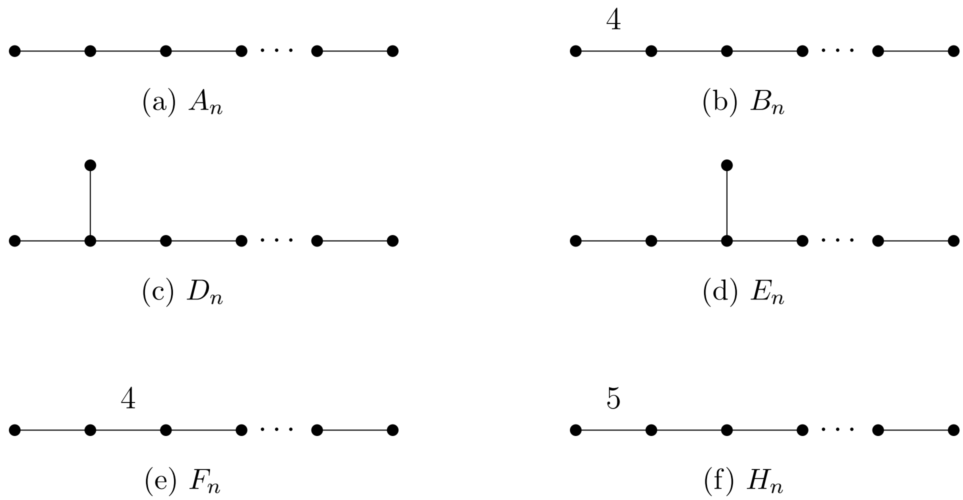
Stembridge classified the Coxeter systems that contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. Both $W(A_n)$ and $W(B_n)$ are finite Coxeter groups, and thus are FC-finite. On the other hand, $W(\widetilde{C}_n)$ is infinite and happens to also contain infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, $W(E_n)$ for $n \geq 9$ (see Figure 1.3) is infinite, but contains only finitely many FC elements.

Theorem 1.2.5 (Stembridge, [13])**.** The FC-finite irreducible Coxeter systems are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. □

The irreducible FC-finite Coxeter graphs are given in Figure 1.3. Note that in Figure 1.1 we classified the irreducible finite Coxeter systems, since these are finite Coxeter groups it is clear that they will have a finite number of FC elements. However, we have not yet encountered the Coxeter groups determined by graphs in Figures 1.3(d) for $n \geq 9$, 1.3(e) for $n \geq 5$, 1.3(f) for $n \geq 5$. All of these Coxeter systems are infinite for sufficiently large n, yet contain only finitely many FC elements.

1.3 Heaps

We now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a



(a) A_n (b) B_n (c) D_n (d) E_n (e) F_n (f) H_n

Figure 1.3: Coxeter graphs corresponding to the irreducible FC-finite Coxeter systems.

heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps for straight line Coxeter groups found in [1], [3], and [13].

Let (W, S) be a Coxeter system of type Γ. Suppose $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ is a fixed reduced expression for $w \in W(\Gamma)$. As in [13], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, since w is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of w, where i is labeled by s_{x_i}. Note that for simplicity we are omitting the labels of the underlying poset but retaining the labels of the corresponding generators.

It follows from [13] that heaps are well-defined up to commutation class. That is, given two reduced expressions w_1 and w_2 for $w \in W$ that are in the same commutation class, the heaps for w_1 and w_2 will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then w has one commutation class, and thus w has a unique heap. Conversely, if w_1 and w_2 are in different commutation classes, then the heap for w_1 will be distinct from the heap from w_2.

Example 1.3.1. Let $\mathsf{w} = s_6 s_4 s_2 s_5 s_3 s_1 s_4 s_0 s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that w is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $9 \lessdot 8$ since $8 < 9$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 1.4.

Let w be a reduced expression for an element $w \in W(\widetilde{C}_n)$. As in [1] and [3] we can represent a heap for w as a set of lattice points embedded in $\{0, 1, 2, \ldots, n\} \times \mathbb{N}$. To

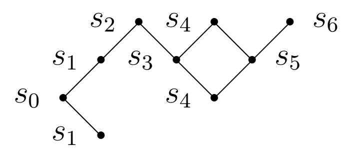


Figure 1.4: Labeled hasse diagram for the heap of an element in $\mathrm{FC}(\widetilde{C}_6)$.

do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \ldots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of w in such a way that:

(1) An entry with coordinates (x, y) is labeled s_i (or i) in the heap if and only if $x = i$;

(2) If an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight line Coxeter graph with the appropriate adjustments made to the label set and assignment of coordinates. Specifically, for type A_n our label set is $\{1, 2, \ldots, n\}$ and for type B_n our label set is $\{0, 1, \ldots, n-1\}$.

In the case of any straight line Coxeter graph it follows from the definition that (x, y) covers (x', y') in the heap if and only if $x = x' \pm 1$, $y' < y$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries in the top correspond to the generators occurring in the left descent set of the corresponding reduced expression.

Let w be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice representation of the heap poset in $\{0, 1, 2, \ldots n\} \times \mathbb{N}$ described in the preceding paragraphs via $H(\mathsf{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ and we refer to $H(w)$ as the heap of w. Note that we will use the same notation for heaps in Coxeter groups of all types with straightline Coxeter graphs.

Let $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_i} at a level that is *above* the level of the point labeled by s_{x_j}. Because generators in a Coxeter graph that are not adjacent do commute, points whose x-coordinates differ by more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners in such a way that if one entry covers another the squares overlap halfway. In addition, we will also label each square for s_i with i.

There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap for each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

Example 1.3.2. Let $\mathsf{w} = s_6s_4s_2s_5s_3s_1s_4s_0s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.3.1. Figure 1.5 shows a possible lattice point representation for $H(w)$. Since w is FC this is the unique heap representation for w. Because $H(w)$ is the unique heap we can obtain $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) from the blocks that are exposed in the top (respectively, bottom) of the heap. Note that if we were to draw the heap differently the exposures we are looking for would change.

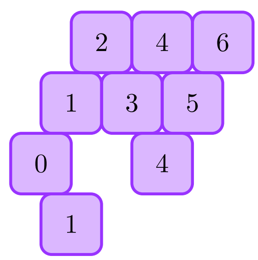
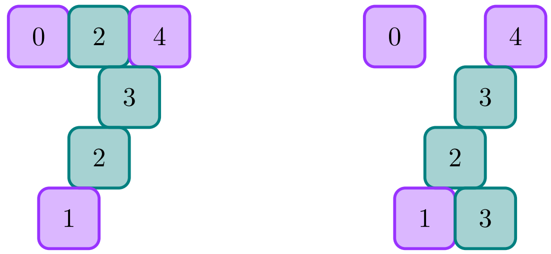


Figure 1.5: A lattice point representation for the heap of an FC element in $W(\widetilde{C}_6)$.

Example 1.3.3. Let $\mathsf{w}_1 = s_0s_2s_4s_3s_2s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. Applying the commutation move $s_4s_2 \mapsto s_2s_4$, we can obtain another reduced expression for w, namely w_2, which is in the same commutation class as w_1 and hence has the same heap. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w}_3 = s_0s_4s_3s_2s_3s_1$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. Representations of $H(\mathsf{w}_1)$, $H(\mathsf{w}_2)$, and $H(\mathsf{w}_3)$ are seen in Figure 1.6, where the braid relation is colored in teal.

It will be extremely useful for us to be able to quickly determine whether a heap corresponds to an element in $\mathrm{FC}(B_n)$ and $\mathrm{FC}(\widetilde{C}_n)$. The next proposition is a special case of [13, Proposition 3.3] and follows quickly when one considers the consecutive subwords that are impermissible in reduced expressions for elements in $\mathrm{FC}(B_n)$ and $\mathrm{FC}(\widetilde{C}_n)$ as discussed in Section 1.2.



(a) $H(\mathsf{w}_1)$ and $H(\mathsf{w}_2)$.

(b) $H(\mathsf{w}_3)$

Figure 1.6: Two heaps of a non-FC element in $W(\widetilde{C}_4)$

Theorem 1.3.4. If $w \in \mathrm{FC}(\widetilde{C}_n)$, then $H(w)$ cannot contain any of the configurations seen in Figure 1.7, where $0 < k < n-1$ and we use a square with a dotted boundary to emphasize that no element of the heap occupies the corresponding position. $\square$

Since $W(B_n)$ is parabolic subgroup of $W(\widetilde{C}_n)$, we can use Figure 1.7 to classify the impermissible configurations for elements of $\mathrm{FC}(B_n)$. The impermissible configurations for elements of $\mathrm{FC}(B_n)$ are those seen in Figures 1.7(a), 1.7(b) 1.7(c), and 1.7(d).

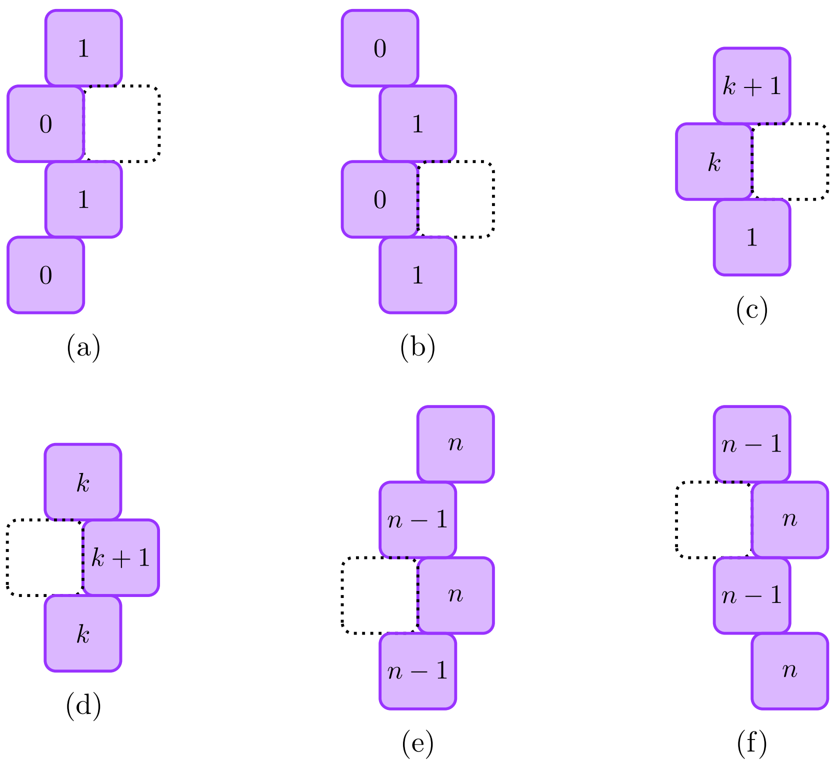


Figure 1.7: Impermissible configurations for heaps of $\mathrm{FC}(\widetilde{C}_n)$.

Chapter 2

Star Operations and Non-Cancellable Elements

2.1 Star Operations

The notion of a star operation was originally introduced by Kazhdan and Lusztig in [11] for simply-laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s, t \in S$), and was later generalized to all Coxeter systems in [12]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to only define the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We define w to be *left star reducible by s with respect to t* if there exists $t \in \mathcal{L}(sw)$. We analogously define w to be *right star reducible by s with respect to t*. Observe that w is left (respectively, right) star reducible if and only if $w = stu$ (respectively, $w = uts$), where the product on the right hand side of the equation is reduced and $m(s,t) \geq 3$. We say that w is *star reducible* if it is either left or right star reducible.

Example 2.1.1. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. We see that w is left star reducible by s_0 with respect to s_1 to $s_1s_0s_2$ since $m(s_0,s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0w)$. Also, w is not right star reducible since s_1 is not in $\mathcal{R}(w)$ when w is multiplied on the right by either s_0 or s_2.

It may be helpful to visualize star reductions in terms of heaps. Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Suppose w is left star reducible by s with respect to t. Then there exists a heap for w where the block for s is fully exposed to the top. Removing the block for s off of the top allows for t to now be fully exposed to the top. Similarly if w is right star reducible by s with respect to t, then there exists a heap for w where the block for s is fully exposed to the bottom. Removing the block for s off of the bottom allows for t to now be fully exposed to the bottom. This implies that if a heap for $w \in W(\Gamma)$ has this property, then w is star reducible. The following example illustrates the above concept.

Example 2.1.2. Let $\mathsf{w} = s_0s_1s_0s_2s_3$ be a reduced expression for $w \in W(B_4)$. Note that w if FC. By Example 2.1.1 we know that w is left star reducible by s_0 with respect to s_1. In Figure 2.1(a), we see the heap for w. Notice that the block for s_0 is fully exposed to the top of the heap. Removing the block for s_0 gives the heap in Figure 2.1(b). Notice that the block for s_1 is now fully exposed to the top of the heap. However, notice that the block for s_0 and s_2 are fully exposed to the bottom. In removing either of these we are unable to fully expose s_1 to the bottom. Thus we can see that w is not right star reducible. If w was not FC then we would not be able to say that w is not right star reducible as there could be a different heap for w in which we were able to fully expose an element that was previously blocked.

(a) (b)

Figure 2.1: Visualization of Example 2.1.1.

Using the notion of star reduction we are now able to introduce the concept of a star reducible Coxeter group. We say that a Coxeter group $W(\Gamma)$, or it's Coxeter graph Γ, is *star reducible* if every element of $\mathrm{FC}(\Gamma)$ is star reducible to a product of commuting generators. That is, $W(\Gamma)$ is star reducible if when we apply star reductions repeatedly to $w \in \mathrm{FC}(\Gamma)$, eventually we obtain a product of commuting generators. Using heaps to visualize a star reducible Coxeter group given a heap in $\mathrm{FC}(\Gamma)$, we are able to systematically remove a fully exposed block from the top or bottom of the heap and have a block that was previously not fully exposed become fully exposed until we are left with a heap that is one row in height.

In [8], Green classified all star reducible Coxeter groups.

Theorem 2.1.3 (Green, [8])**.** Let (W, S) be a Coxeter system of type Γ. Then (W, S) is star reducible if an only if each component of Γ is either a complete graph with labels $m(s,t) \geq 3$, or is one of the following types: type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type H_n $(n \geq 2)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_{n-1}$ $(n \geq 3$ and n odd $)$, type $\widetilde{C}_{n-1}$ $(n \geq 4$ and n even $)$, type $\widetilde{E}_6$, or type $\widetilde{F}_5$. $\square$

In [9], Green defined star reducible Coxeter groups to have *Property F*. We go a bit farther and say that an element of $FC(\Gamma)$ that is star reducible to a product of commuting generators has Property F. In the same paper, Green defined a Coxeter group with *Property S* to be a Coxeter group in which each element $w \in W(\Gamma) \backslash \mathrm{FC}(\Gamma)$ is star reducible to an element w for which either $\mathcal{L}(w)$ or $\mathcal{R}(w)$ contains a pair of noncommuting generators. Again we extend the definition to a specific element in the $W(\Gamma)$. That is, $w \in W(\Gamma) \backslash \mathrm{FC}(\Gamma)$ has Property S if w is star reducible to a product of non-commuting generators in $\mathcal{L}(w)$ or $\mathcal{R}(w)$.

2.2 Non-Cancellable Elements

We now introduce the concept of weak star reducible, which is related to the notion of cancellable in [4]. Let (W, S) be a Coxeter system of type Γ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators. If $w \in \mathrm{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t to sw* if

(1) w is left star reducible by s with respect to t, and

(2) $tw \notin \mathrm{FC}(W)$.

Notice that condition (2) implies that $l(tw) > l(w)$. Also note that we are restricting out definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws*. We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable* or *weak star irreducible.* Notice that from this we know that weak star reducible implies star reducible. However, star reducible does not imply weak star reducible.

Example 2.2.1. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$ as in Example 2.1.1. From Example 2.1.1 we know that w is left star reducible. However, $tw = s_1s_0s_1s_0s_2$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2$. In addition, Example 2.1.1 showed that w is not right star reducible and hence w is not right weak star reducible. However, since w is left weak star reducible we know that w is not non-cancellable.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall that in Section 2.1 we described what a star reduction looks like in a heap and what a star reducible heap looks like. Since the definition of weak star reducible includes that a heap is star reducible we again need to have those properties. In addition, for a heap to be weak star reducible when adding the block that becomes fully exposed when a block is removed from the heap must create a braid in the heap forcing the new heap to not be FC. That is, one of the impermissible configurations seen in Section 1.3 will appear.

Example 2.2.2. Let $\mathsf{w} = s_0 s_1 s_0 s_2$ be a reduce expression for $w \in W(B_4)$ as in Example 2.2.1. From Example 2.2.1, we know that w is left weak star reducible. Recall in Figure 2.1 the heap for w was seen along with what it star reduced to. In Figure 2.2 we see that adding s_1 to the top of the heap creates a braid which is highlighted in orange.

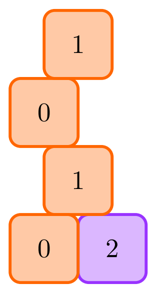


Figure 2.2: Visualization of a weak star reducible element of $\mathrm{FC}(B_4)$.

Example 2.2.3. Let $w \in \mathrm{FC}(B_4)$ and let $\mathsf{w} = s_0 s_1$ be a reduced expression for w. Note that w is left (respectively, right) star reducible by s_0 with respect to s_1 (respectively, by s_1 with respect to s_0). However, $s_1 s_0 s_1 \in \mathrm{FC}(B_4)$ (respectively, $s_0 s_1 s_0 \in \mathrm{FC}(B_4)$). Thus w is non-cancellable. Visually the heap appears in Figure 2.3. Clearly when s_0 is added to the bottom of the heap, the new heap is still in $\mathrm{FC}(B_4)$ and the same can be said when s_1 is added to the top of the heap.

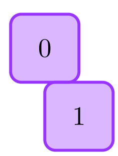


Figure 2.3: Visualization of a non-cancellable element of $\mathrm{FC}(B_4)$.

Chapter 3

Property T and T-Avoiding Elements

3.1 Property T

Dana will you think about the connection of Star Reducible and Property T

As mentioned in Section 2.1 Green classified all star reducible Coxeter groups. In [8], Green utilizes the following theorem to help classify the star reducible Coxeter groups.

Theorem 3.1.1 (Green, [8])**.** Let (W, S) be a star reducible Coxeter system of type Γ, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s,t, u with $m(s,t) \neq 2$, $m(t,v) \neq 2$, and $m(s,u) = 2$:

(1) w is a product of commuting generators;

(2) w has a reduced product beginning with stu;

(3) w has a reduced product ending in uts;

(4) w has a reduced product beginning with $svtu$. □

In the following discussion we will give name to elements that exhibit the properties above. We first begin by defining the notion of Property T which is motivated by Items (2) and (3) above. Let (W, S) be a Coxeter system of type Γ and let $w \in W$. We say that w has *Property T* if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s,t) \geq 3$. That is, w has Property T if there exists a reduced expression for w that begins or ends with a product of non-commuting generators.

Example 3.1.2. Let $w \in W(A_5)$ with reduced expression $\mathsf{w}_1 = s_1s_4s_2s_3s_5$. At first glance it may appear that w does not have Property T, since both s_1 and s_4 commute as well as s_3 and s_5. However, note that applying a commutation to s_4s_2 results in $\mathsf{w}_2 = s_1s_2s_4s_3s_5$. Hence w has Property T, since $m(s_1, s_2) = 3$ and there is a reduced expression for w that begins with s_1s_2.

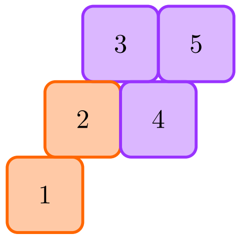
Example 3.1.3. Let $w \in W(A_5)$ with reduced expression $\mathsf{w} = s_1s_3s_5$. It turns out that since w is a product of commuting generators there is no reduced expression for w that begins or ends with a pair of non-commuting generators. This implies that w does not have Property T.

As with star reducible elements it may be helpful to visualize Property T through heaps. In Figure 3.1 we see the heap representation of an element with Property T, where the (reduced product) references u in the definition of Property T. It is important to note that if the group element we are evaluating for Property T is not FC, then we must consider all heap representations for the element before concluding that an element does not have Property T. Similar to star reducible in heaps we see that Property T can be thought of as removing a fully exposed block from the top or bottom of a heap and having a new block become fully exposed.

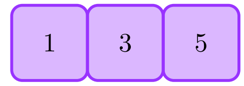
Figure 3.1: A visual representation of Property T.

Example 3.1.4. Let $\mathsf{w}_1 = s_1s_4s_2s_3s_5$ be a reduced expression for $w_1 \in W(A_5)$ as seen in Example 3.1.2 and let $\mathsf{w}_2 = s_1s_3s_5$ be a reduced expression for $w_2 \in W(A_5)$. In Figure 3.2(a) we see the heap for w_1. Note that we can see Property T in the bottom of the heap highlighted in orange. In Figure 3.2(b) we see the heap for w_2. Note that as the heap is only one row and w_2 is FC, it is clear that w_2 does not have Property T.

An element $w \in W(\Gamma)$ is called *T-avoiding* if w does not have Property T. We will call an element that is a product of commuting generators *trivially T-avoiding*. The reason behind this comes in the following theorem. If w is T-avoiding and not a product of commuting generators, we will say that w is *non-trivially T-avoiding.*



(a) Heap of an element with Property T



(b) Heap of a T-Avoiding element

Figure 3.2: Heaps of an element with Property T and a T-Avoiding element

Theorem 3.1.5. Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$ such that w is a product of commuting generators. Then w is T-avoiding. $\square$

Visually this is seen in Example 3.2(b) elements in $W(\Gamma)$ that are products of commuting generators are always going to be one row in the heap. This implies that we are not able to remove generators and have elements come into rows that they were previously in as the heap is only one row and there can be no lateral movement when we remove bricks.

Example 3.1.6. Let $w \in W(A_5)$ and let $\mathsf{w} = s_1 s_3 s_5$. Then by Example 3.1.3, we know that w is T-avoiding and since w is a product of commuting generators, w is trivially T-avoiding.

Example 3.1.7. Let $w \in W(\widetilde{C}_4)$ with reduced expression $\mathsf{w} = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$. It turns out that w is FC and non-trivially T-avoiding. The heap for w is seen if Figure 3.3. Notice that no matter which block we remove that is fully exposed to the top of the heap no new element becomes fully exposed. The same applies to the bottom exposure. Hence there is not a single block that can be removed from the top that allows a new element to become fully exposed in the top or bottom of the heap. Thus, w is non-trivially T-avoiding.

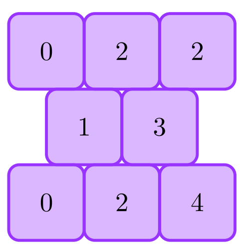


Figure 3.3: Heap of a non-trivially T-Avoiding element in $W(\widetilde{C}_4)$.

Referring back to Green's classification of what elements in star reducible Coxeter groups look like, we see that Item (1) corresponds to an element w being trivially T-avoiding, Items (2) and (3) refer to the element w having Property T at the beginning and end respectively, and Item (4) refers to an element being non-trivially T-avoiding if no reduced expressions for the element exhibit Items (2) and (3). It is not clear that non-trivially T-avoiding elements exist. However, in star reducible Coxeter groups, every FC element is star reducible to a product of commuting generators, which implies that no FC element can be non-trivially T-avoiding. For example, as will be seen in the following sections, the Coxeter groups of type A_n and B_n have no non-trivial T-avoiding elements, while the Coxeter group of type D_n does have non-trivial T-avoiding elements.

We will now extend the definition of Property T to Coxeter groups. Let (W, S) be a Coxeter group of type Γ. We say that $W(\Gamma)$ has *Property T* if all elements in $W(\Gamma)$ that are not the products of commuting generators have Property T. It is clear that if a Coxeter group has Property T, then it also has Property S. It remains an open question whether or not a Coxeter group with Property S, also has Property T, but we conjecture this is true. Dana I don't know if I agree with our conjecture anymore upon retooling definitions today. I feel like Property S implies that a Coxeter group has all of it's elements star reducible to a braid? If this is the case, then by a remark following the definition of Property S in [9], the classification of the T-avoiding elements for any connected, nonbranching Coxeter graph of finite or affine type, except the Coxeter system of type $\widetilde{F}_4$, would be complete.

One thing to notice here is that all Coxeter groups have trivial T-avoiding elements as they all contain products of commuting generators. As a result of this we will avoid mentioning them in the following classification. The more interesting non-trivial T-avoiding elements do not appear in all Coxeter groups. The remainder of this thesis discusses what is currently know regarding T-avoiding elements in the irreducible finite Coxeter groups and the irreducible affine Coxeter groups, and we classify the T-avoiding elements in Coxeter groups of types B_n and $\widetilde{C}_n$. In the next few sections we will summarize what is known about the T-avoiding elements in Coxeter groups of types $\widetilde{A}_n$, A_n, D_n, F_n, and $I_2(m)$.

3.2 T-Avoiding Elements in Types $\widetilde{A}_n$ and A_n

We start by classifying the T-avoiding elements in Coxeter groups of type $\widetilde{A}_n$ and A_n. We first classify non-trivial T-avoiding elements in $W(\widetilde{A}_n)$.

Theorem 3.2.1. If $n \geq 2$ and n is even, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. Otherwise, if $n \geq 2$ and n is even then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

Proof. This is [5, Proposition 3.1.2.]. □

The previous theorem implies that when n is even the Coxeter group of type $\widetilde{A}_n$, has no non-trivial T-avoiding elements. However, while n is odd, the Coxeter group of type $\widetilde{A}_n$ has non-trivial T-avoiding elements. The classification seen in [5] did not specifically classify the non-trivial T-avoiding elements for type $\widetilde{A}_n$ for n odd. Since $W(\widetilde{A}_n)$ for n odd is not star reducible we know that the non-trivial T-avoiding elements could be FC. The following is our conjecture regarding what the non-trivial T-avoiding elements are in $W(\widetilde{A}_n)$ for n odd.

Conjecture 3.2.2. The only non-trivial T-avoiding elements in $W(\widetilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

Notice that the above non-trivial T-avoiding elements are FC. As stated in the conjecture we believe that these are the only non-trivial T-avoiding elements. However, this remains an open problem. We now proceed with the classification of T-avoiding elements in Coxeter groups of type A_n.

Corollary 3.2.3. Then there are no non-trivially T-avoiding elements in $W(A_n)$.

Proof. Since $W(A_n)$ is a parabolic subgroup of $W(\widetilde{A}_n)$ this is a consequence of [5, Proposition 3.1.2.]. Specifically, we can obtain the Coxeter graph of type A_n from the Coxeter graph of type $\widetilde{A}_n$ for n even by removing the appropriate number of vertices and edges. From this we can see that if $W(A_n)$ was to have non-trivial T-avoiding elements, this would imply that $W(\widetilde{A}_n)$ for n even would also have non-trivial T-avoiding elements as well. Thus $W(A_n)$ can not have bad elements. □

3.3 T-Avoiding Elements in Type D_n

In this section we will classify the non-trivial T-avoiding elements in the Coxeter group of type D_n. Recall that $W(D_n)$ is a star reducible Coxeter group and as a result of this any non-trivial T-avoiding element will not be fully commutative.

Theorem 3.3.1. There are non-trivial T-avoiding elements in $W(D_n)$ for $n \geq 4$.

Proof. This is a consequence of [7, Section 2.2]. □

In addition, to showing that there are non-trivial T-avoiding elements in type D_n, Gern also classified the non-trivial T-avoiding elements as well. The following is his classification translated into heaps. Once we figure out the heap include it here. For the full details regarding his classification see [7]. Note that in his classification, Gern refers to non-trivially T-avoiding elements as "bad."

3.4 T-Avoiding Elements in Type F_n

In this section we classify what is known regarding the non-trivial T-avoiding elements in the Coxeter groups of type F_n for $n \geq 4$. Note that all of the following results are unpublished. Recall that $W(F_n)$ is a star reducible Coxeter group so any non-trivial T-avoiding element in $W(F_n)$ will not be FC.

We start with the Coxeter group of type F_5. In 2012, Cross, Ernst, Hills-Kimball, and Quaranta classified all non-trivial T-avoiding elements in the following theorem.

Theorem 3.4.1. An element in F_5 is non-trivially T-avoiding if and only if it is a stack of bowties. $\square$

The above theorem references a stack of bowties. This refers to what the heap of the given non-trivial T-avoiding element looks like. We first restrict our attention to a single bowtie, this is seen in Figure 3.4. Note that in this figure, the orange blocks correspond to the elements that have bond strength 4 (Figure 3.4(a)). Since the element is not FC we also wished to highlight the braid which is seen in the center of the element as colored in teal in Figure 3.4(b). In stacking the single bowties together, we get the stack of bowties referenced in the theorem which is seen in Figure 3.5. As a result of the classification in F_5, Cross et al. were also able to classify the non-trivial T-avoiding elements in F_4. The following is their classification.

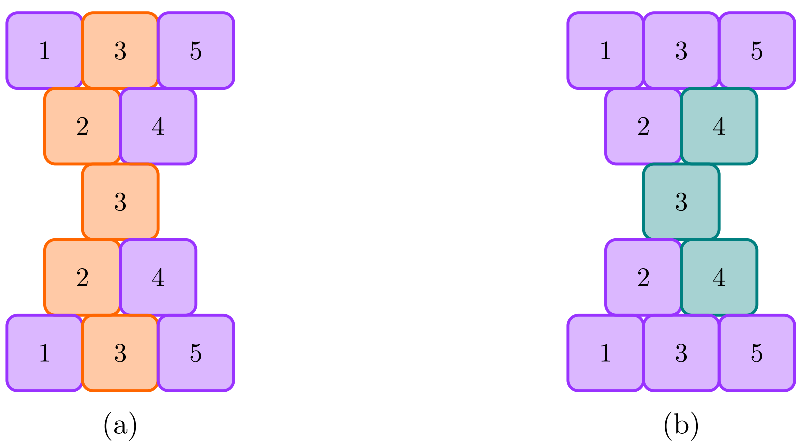


Figure 3.4: A single bowtie in $W(F_5)$.

Corollary 3.4.2. There are no non-trivial T-avoiding elements in F_4. $\square$

As a result of their work, Cross et al. conjectured that in Coxeter groups of type F_n for $n \geq 5$, an element is non-trivially T-avoiding if and only if it is a stack of bowties multiplied by a product of commuting generators. In 2013, Gilbertson and

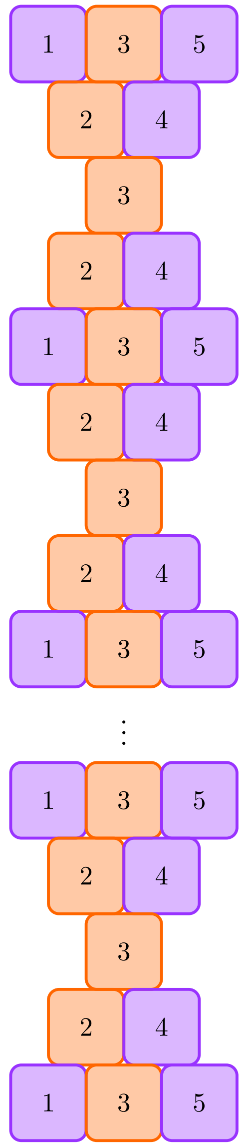


Figure 3.5: A stack of bowties in $W(F_5)$.

Ernst worked with this conjecture and quickly found out that it was incorrect. The heap seen in Figure 3.6 corresponds to a non-trivial T-avoiding element in F_6. It turns out that like the bowties discussed above these elements can also be stacked to create and infinite number of non-trivial T-avoiding elements as well. In addition, as n gets large there are a number of things that can be altered in the group element that result in the element being non-trivially T-avoiding. This leads us to believe that there are potentially even more non-trivially T-avoiding elements in $W(F_n)$ for $n \geq 6$. From this we conjecture that the classification of T-avoiding elements in Coxeter systems of type F_n for $n \geq 6$ gets complicated very quickly. Classifying T-avoiding elements in $W(F_n)$ for $n \geq 5$ remains an open problem.

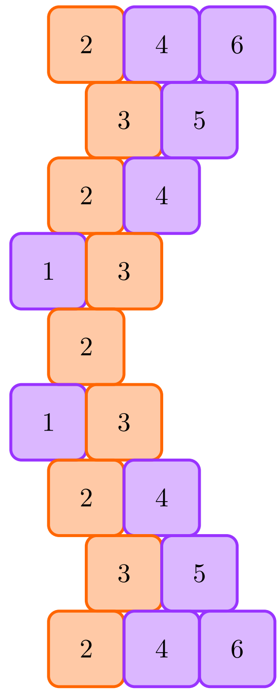


Figure 3.6: A non-trivial T-avoiding element in $W(F_6)$

3.5 T-Avoiding Elements in Type $I_2(m)$

We next will classify the T-avoiding elements in Coxeter groups of type $I_2(m)$. Note that in Coxeter groups of type $I_2(m)$, the only products of commuting generators have length 1. The following is the classification of non-trivial T-avoiding elements. Although the following is a quick result, we believe that the result does not already appear in literature.

Theorem 3.5.1. The Coxeter group $W(I_2(m))$ has no non-trivial T-avoiding elements.

Proof. The graph for the Coxeter group of type $I_2(m)$ appears in Figure 1.1(c). Note that the graph consists of two vertices, namely, s_1 and s_2, and a single edge with weight $m(s,t)$. First recall that $W(I_2(m))$ is a star reducible Coxeter group. This implies that any non-trivial T-avoiding elements in $W(I_2(m))$ must not be FC, ss all of the FC elements have Property T. The only non-FC element in $W(I_2(m))$ is the element of length $m(s,t)$ which has exactly two reduced expressions consisting of alternating products of s_1 and s_2. Cleary, this element begins with and ends with a product of noncommuting generators. Thus, this element has Property T. Hence $W(I_2(m))$ has no non-trivial T-avoiding elements. $\square$

Chapter 4

T-Avoiding Elements in Types B_n and $\widetilde{C}_n$

In this section we classify the T-Avoiding elements in Coxeter groups of type B_n and type $\widetilde{C}_n$. We start by introducing necessary tools for type B_n and finish with a proof of the classification in B_n. We then conclude with the classification of $\widetilde{C}_n$. Note that the proof for Coxeter systems of type B_n closely follows the classification of T-avoiding elements of Type D_n seen in [7].

4.1 Tools for the Classification

Recall from Example 1.1.1 that we can represent each element $w \in W(B_n)$ as a member of the signed permutation group. As a result we can write $w \in W(B_n)$ using one-line notation

$$w = [w(1), w(2), \ldots, w(n-1), w(n)]$$

where we write a bar underneath a number in place of a negative sign in order to simplify notation. This is obtained from the Coxeter group in the following way. We identify $s_i \in S(B_n)$ via

$$s_i = [1, 2, \ldots i-1, i+1, i, i+2, \ldots, n-1, n]$$

and we identify $s_0 \in S(B_n)$ via

$$s_0 = [\underline{1}, 2 \ldots, n].$$

Further $w(-i) = -w(i)$ for $i \in \{1, 2, \ldots, n\}$.

Example 4.1.1. Let $w \in W(B_6)$ with a given reduced expression $\mathsf{w} = s_0 s_1 s_3 s_4 s_5 s_2$. Then we can write $w = [2, 4, \underline{1}, 5, 6, 3]$.

It will be useful to easily determine what what happens to the window notation of a given element $w \in W(B_n)$ when we multiply on the right or left by $s_i \in S(B_n)$. The following Proposition allows us to do this.

Proposition 4.1.2. Let $w \in W(B_n)$ with corresponding signed permutation

$$w = [w(1), w(2), \ldots w(n)].$$

Suppose $s_i \in S(B_n)$. If $i \geq 1$, then multiplying w on the right by s_i has the effect of interchanging $w(i)$ and $w(i+1)$. Multiplying on the left by s_i has the effect of interchanging the entries whose absolute values are i and $i+1$.

If $i = 0$, then multiplying w on the right by s_i has the effect of switching the sign of $w(1)$. Multiplying w on the left by s_i has the effect of switching the sign of the entry whose absolute value is 1.

Proof. This follows from [2, Section 8.1 and A3.1]. $\square$

Given the one line notation for an element $w \in W(B_n)$ we can easily calculate the left and right descent sets of w. The following proposition explains how.

Proposition 4.1.3 (Björner, [2])**.** Let $w \in W(B_n)$. Then

$$\mathcal{R}(w) = \{s_i \in S : w(i) > w(i+1)\}$$

where w(0)=0 by definition.

Proof. This is, [2] Proposition 8.1.2. $\square$

We now will introduce the concept of signed pattern avoidance which will help with the classification of the T-avoiding elements in the Coxeter group of type B_n. This notion was first introduced in [7]. Let $w \in W(B_n)$. We say that w *avoids the consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(w(i),\ w(i+1),\ w(i+2))$ is in the same relative order as (a, b, c). We say that w *avoids the signed consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same consecutive order as $(|a|, |b|, |c|)$ and such that $\mathrm{sign}(w(i)) = \mathrm{sign}(a)$, $\mathrm{sign}(w(i+1)) = \mathrm{sign}(b)$, and $\mathrm{sign}(w(i+2)) = \mathrm{sign}(c)$.

Example 4.1.4. Let $w \in W(B_4)$ with signed permutation

$$w = [\underline{2}, 4, \underline{1}, 3].$$

We see that w has the signed consecutive pattern $\underline{2}3\underline{1}$, since $(|w(1)|, |w(2)|, |w(3)|)$ are in the same relative order as $(|-2|, |3|, |-1|)$, and $\mathrm{sign}(w(1)) = \mathrm{sign}(-2)$, $\mathrm{sign}(w(2)) = \mathrm{sign}(3)$, and $\mathrm{sign}(w(3)) = \mathrm{sign}(-1)$. However, w avoids the signed consecutive pattern $1\underline{2}3$.

4.2 Classification of T-Avoiding Elements in Type B_n

In this section we will classify the T-avoiding elements in Coxeter groups of type B_n. As in the previous classifications seen in Chapter 3 $W(B_n)$ has trivial T-avoiding elements as all Coxeter groups contain elements that are products of commuting generators. This leaves us to classify any non-trivial T-avoiding elements in $W(B_n)$. The following is our classification.

Theorem 4.2.1. There are no non-trivial T-avoiding elements in $W(B_n)$.

In order to prove this we will use the notion of signed pattern avoidance seen above. Before we prove this theorem we first need some preparatory lemmas.

Lemma 4.2.2. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then w has a reduced expression ending in sts if and only if w has the consecutive pattern 321.

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [2]. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+1)$. By 4.1.3 $s_i, s_{i+1} \in \mathcal{R}(w)$. By Tyson's reference to simply laced coxeter group stuff 1.2.1 w ends in $s_i s_{i+1} s_{i+2}$.

Conversely, suppose w ends in $s_i s_i + 1 s_i$. This implies that either $w_I = s_i s_{i+1}$ or $w_I = s_{i+1} s_i$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by 4.1.3. Thus w has the consecutive pattern 321. Therefore, w has a reduced expression ending in sts if and only if w has the consecutive pattern 321. □

Corollary 4.2.3. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then w has a reduced expression beginning with sts if and only if w^{-1} has the consecutive pattern 321.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. We know that w has no reduced expressions beginning with sts if and only if w^{-1} has no reduced expression ending with sts which by Theorem 4.2.3 happens only if w^{-1} avoids the consecutive pattern 321. □

Lemma 4.2.4. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then w has a reduced expression ending in st if and only if w has the consecutive pattern 231.

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [2]. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$. By 4.1.3 $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by s_{i+1} we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, this implies that $s_i \in \mathcal{R}(ws_{i+1})$. This implies w has a reduced expression that ends in $s_i s_{i+1}$.

Conversely, suppose that w has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence w has the consecutive pattern 231. Therefore, w has a reduced expression ending in st if and only if w has the consecutive pattern 231. □

Corollary 4.2.5. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then w has a reduced expression beginning with st if and only if w^{-1} has the consecutive pattern 231.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. We know that w has no reduced expressions beginning with st if and only if w^{-1} has no reduced expression ending with st which by Theorem 4.2.3 happens only if w^{-1} avoids the consecutive pattern 231. □

Lemma 4.2.6. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then w has a reduced expression ending in ts if and only if w has the consecutive pattern 312.

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [2]. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. By 4.1.3 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by s_i we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$. By above $w(i) > w(i+2)$, and by 4.1.3 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that w has a reduced expression ending in $s_{i+1} s_i$.

Conversely suppose w ends in a reduced expression with $s_{i+1} s_i$ Then $w_I = s_{i+1} s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, w has the consecutive pattern 312. Therefore, w has a reduced expression ending in ts if and only if w has the consecutive pattern 312. □

Corollary 4.2.7. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then w has a reduced expression beginning with ts if and only if w^{-1} has the consecutive pattern 312.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. We know that w has no reduced expressions beginning with ts if and only if w^{-1} has no reduced expression ending with ts which by Theorem 4.2.3 happens only if w^{-1} avoids the consecutive pattern 312. □

Lemma 4.2.8. Let $w \in W(B_n)$. Then w has a reduced expression ending in $s_1 s_0$ if and only if $w(0) > w(1)$ and $-w(1) > w(2)$.

Proof. Suppose $w \in W(B_n)$ such that w ends with $s_1 s_0$. Then $s_0 \in \mathcal{R}(w)$ and $s_1 \in \mathcal{R}(ws_0)$. This implies that $ws_0(1) > ws_0(2)$ by 4.1.3. We see that $ws_0(1) = w(-1) = -w(1)$ and $ws_0(2) = 2$. Hence $-w(1) = ws_0(1) > ws_0(2) = w(2)$. Further, since $s_0 \in \mathcal{R}(w)$, we see that $w(0) > w(1)$.

Conversely, suppose $w \in W(B_n)$ such that $w(0) > w(1)$ and $-w(1) > w(2)$. Since $w(0) > w(1)$ so $s_0 \in \mathcal{R}(w)$. Multiplying on the right by s_0 we see that $ws_0(1) = -w(1)$ and $ws_0(2) = w(2)$. Note that since $ws_0(1) = -w(1) > w(2) = ws_0(2)$, $s_1 \in \mathcal{R}(ws_0)$. Thus w ends with $s_1 s_0$. Therefore, w has a reduced expression ending in $s_1 s_0$ if and only if $w(0) > w(1)$ and $-w(1) > w(2)$. □

Corollary 4.2.9. Let $w \in W(B_n)$. Then w has a reduced expression beginning in $s_0 s_1$ if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$.

Proof. Let $w \in W(B_n)$. We know that w has no reduced expressions beginning in $s_0 s_1$ if and only if w^{-1} has no reduced expressions ending in $s_0 s_1$. By Lemma 4.2.8 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$. □

Lemma 4.2.10. Let $w \in W(B_n)$. Then w has a reduced expression ending in $s_0 s_1$ if and only if $w(0) > w(2)$ and $w(1) > w(2)$.

Proof. Suppose $w \in W(B_n)$ such that w ends with $s_0 s_1$. Then $s_1 \in \mathcal{R}(w)$ and $s_0 \in \mathcal{R}(ws_1)$. Then $ws_1(0) > ws_1(1)$. We see that $ws_1(0) = 0$ and $ws_1(1) = w(2)$. This implies that $0 = ws_1(0) > ws_1(1) = 2$. Further, since $s_1 \in \mathcal{R}(w)$ this implies that $w(1) > w(2)$. Thus if w ends with $s_0 s_1$, then $w(1) > w(2)$ and $w(0) > w(2)$.

Conversely, suppose $w \in W(B_n)$ such that $w(1) > w(2)$ and $w(0) > w(2)$. This implies that $s_1 \in \mathcal{R}(W)$. Multiplying w on the right by s_1 we see that $ws_1(0) = w(0)$ and $ws_1(1) = w(2)$. Note that since $ws_1(0) = w(0) > w(2) = ws_1(1)$, $s_0 \in \mathcal{R}(ws_1)$. Thus w ends with $s_0 s_1$. Therefore, w has a reduced expression ending in $s_0 s_1$ if and only if $w(1) > w(2)$ and $w(0) > w(2)$. □

Corollary 4.2.11. Let $w \in W(B_n)$. Then w has a reduced expression beginning in $s_1 s_0$ if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$.

Proof. Let $w \in W(B_n)$. We know that w has no reduced expressions beginning in $s_1 s_0$ if and only if w^{-1} has no reduced expressions ending in $s_1 s_0$. By Lemma 4.2.8 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$. □

Lemma 4.2.12. Let $w \in W(B_n)$ such that each entry for w in the one-line notation is positive and both w and w^{-1} avoid the consecutive patterns 321, 231, and 312, then w is a product of commuting generators.

Proof. This is [7, Lemma 2.2.9]. $\square$

Lemma 4.2.13. Let $w \in W(B_n)$ be trivially T-avoiding and let $i \in \{1, 2, \ldots, n\}$. Then w satisfies the following conditions:

(1) $w(j) > \min(\{w(i-1), w(i)\})$ for all $j > i$;

(2) $w(k) < \max(\{w(i-1), w(i)\})$ for all $k < i-1$;

(3) if $w(i), w(i+1) > 0$, then $w(j) > 0$ for all $j \geq i$;

(4) if $w(i), w(i+1) < 0$, then $w(j) < 0$ for all $j \leq i+1$.

Proof. Suppose there is some least $j > i$ such that $w(j) \leq \min(\{w(i-1), w(i)\})$. Note that $j > i$ so $j \neq i$, and $j \neq i-1$ so $w(j) < \min(\{w(i-1), w(i)\})$. Note that j is the least so $w(j-2) \geq \min(\{w(i+1), w(i)\}) > w(j)$. This implies that either $w(j-1) > w(j-2) > w(j)$ or $w(j-2) > w(j-1) > w(j)$, which implies w has the consecutive pattern 231 or 321 which is a contradiction to w being a non-trivial T-avoiding element by Lemmas 4.2.2 and 4.2.6. Thus proving (1).

Suppose there exists a maximal $k < i-1$ such that $w \geq \max(\{w(i-1), w(i)\})$. Note that $k < i-1$ so $k \neq i$ and $k \neq i-1$. Then $w(k) > \max(\{w(i-1), w(i)\})$. Since k is maximal then $w(k+1) \leq \max(\{w(i-1), w(i)\})$ and $w(k+2) \leq \max(\{w(i-1), w(i)\})$. This implies that either $w(k+2) < w(k+1) < w(k)$ or $w(k+1) < w(k+2) < w(k)$, which implies w has the consecutive patter 321 or 312 which is a contradiction to w being a non-trivial T-avoiding element by Lemmas 4.2.2 and 4.2.4. Thus proving (2).

It is easy to see that assertion (1) implies (3) and assertion (2) implies (4). $\square$

Lemma 4.2.14. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}31$. Then w has Property T.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}31$.

Case 1: Suppose w has the one-line notation $w = [\underline{2}, 3, 1]$. This implies that $w = s_1 s_0 s_2$. Clearly, w begins with a product of non-commuting generators. Thus w has Property T.

Case 2: Suppose that w has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: 31 ± 2, 32 ± 1, or 21 ± 3. We know that b, c must be positive since they are positive in w and we also know that $b > c$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case 3: Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 1\underline{2}3$, $\pm 2\underline{1}3$, and $\pm 3\underline{1}2$. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}31$, then w has Property T. $\square$

Lemma 4.2.15. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}3\underline{1}$. Then w has Property T.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}3\underline{1}$.

Case 1: Suppose w has the one-line notation $w = [\underline{2}, 3, \underline{1}]$. This implies that $w = s_0 s_1 s_0 s_2$. Clearly, w begins with a product of non-commuting generators. Thus w has Property T.

Case 2: Suppose that w has the one-line notation $w = [\underline{a}, b, \underline{c}, *, \ldots, *]$ where $\underline{a}, b, \underline{c}$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$. We now consider the signed consecutive pattern that can arise involving $b, \underline{c}, *$. The following are the possibilities for the signed consecutive pattern that can arise: $3\underline{1} \pm 2$, $3\underline{2} \pm 1$, or $2\underline{1} \pm 3$. We know that b must be positive since it is positive in w, c must be negative since it is negative in w, and we also know that $|b| > |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case 3: Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, \underline{c}]$ where $\underline{a}, b, \underline{c}$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 1\underline{2}3$, $\pm 2\underline{1}3$, and $\pm 3\underline{1}2$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}3\underline{1}$, then w has Property T. $\square$

Lemma 4.2.16. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}23$. Then w has Property T or is a trivial T-avoiding element.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}23$.

Case 1: Suppose w has the one-line notation $w = [\underline{1}23]$. This implies that $w = s_0$. Clearly, w is a trivial T-avoiding element as it is a single generator.

Case 2: Suppose that w has the one-line notation $w = [\underline{\mathrm{a}}, b, c, *, \ldots, *]$ where $\underline{\mathrm{a}}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: 23 ± 1, 13 ± 2, or 12 ± 3. We know that b, c, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case 3: Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{\mathrm{a}}, b, c]$ where $\underline{\mathrm{a}}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $*, \underline{\mathrm{a}}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 3\underline{1}2$, $\pm 2\underline{1}3$, and $\pm 1\underline{2}3$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}23$, then w has Property T or is a trivial T-avoiding element. □

Lemma 4.2.17. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}32$. Then w has Property T or is a trivial T-avoiding element.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}32$.

Case 1: Suppose w has the one-line notation $w = [\underline{1}32]$. This implies that $w = s_0 s_2$. Clearly, w is a trivial T-avoiding element as it is a single generator.

Case 2: Suppose that w has the one-line notation $w = [\underline{\mathrm{a}}, b, c, *, \ldots, *]$ where $\underline{\mathrm{a}}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: 23 ± 1, 13 ± 2, or 12 ± 3. We know that b, c, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case 3: Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{\mathrm{a}}, b, c]$ where $\underline{\mathrm{a}}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $*, \underline{\mathrm{a}}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 3\underline{1}2$, $\pm 2\underline{1}3$, and $\pm 3\underline{2}1$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}23$, then w has Property T or is a trivial T-avoiding element. $\square$

We can now prove Theorem 4.2.1.

Proof. Suppose that $w \in W(B_n)$ is a non-trivial T-avoiding element. There are $2^3 \cdot 3!$ possible choices of signed consecutive patterns for $w(1)w(2)w(3)$ where $w = [w(1), w(2), w(3), *, \ldots, *]$.

123	$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$	$\underline{12}3$	$\underline{1}2\underline{3}$	$1\underline{23}$	$\underline{123}$
132	$\underline{1}32$	$1\underline{3}2$	$13\underline{2}$	$\underline{13}2$	$\underline{1}3\underline{2}$	$1\underline{32}$	$\underline{132}$
213	$\underline{2}13$	$2\underline{1}3$	$21\underline{3}$	$\underline{21}3$	$\underline{2}1\underline{3}$	$2\underline{13}$	$\underline{213}$
231	$\underline{2}31$	$2\underline{3}1$	$23\underline{1}$	$\underline{23}1$	$\underline{2}3\underline{1}$	$2\underline{31}$	$\underline{231}$
312	$\underline{3}12$	$3\underline{1}2$	$31\underline{2}$	$\underline{31}2$	$\underline{3}1\underline{2}$	$3\underline{12}$	$\underline{312}$
321	$\underline{3}21$	$3\underline{2}1$	$32\underline{1}$	$\underline{32}1$	$\underline{3}2\underline{1}$	$3\underline{21}$	$\underline{321}$

We can use Lemma 4.2.2 and Corollary 4.2.3 to eliminate the signed consecutive patterns highlighted in turquoise. We can use Lemma 4.2.6 and Corollary 4.2.5 to eliminate the signed consecutive patterns highlighted in red. We can use Lemma 4.2.4 and Corollary 4.2.7 to eliminate the consecutive patterns highlighted in green. We can use Lemma 4.2.8 and Corollary 4.2.9 to eliminate the signed consecutive patterns highlighted in yellow. We can use Lemma 4.2.10 and Corrollary 4.2.11 to eliminate signed consecutive patterns highlighted in brown. We can use Lemma 4.2.12 to elminate the signed consecutive patterns highlighted in blue. We can use Lemmas 4.2.14 and 4.2.15 to eliminate signed consecutive patterns highlighted in purple. Finally we can use Lemmas 4.2.16 and 4.2.17 to eliminate signed consecutive patterns highlighted in orange. Since all of the above patterns are eliminated as possibilities for $w(1)w(2)w(3)$ and there are no other signed consecutive patterns that are possible for these positions, w can not be a trivial T-avoiding element in the Coxeter group of type B. Therefore, there are no non-trivial T-avoiding elements in $W(B_n)$. $\square$

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