Classification of the T-avoiding elements in Coxeter groups of type F

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Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S : s^2 = 1, \quad (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$m(s,t) = 2 \quad \Longrightarrow \quad st = ts \quad \} \quad \text{short braid relations}$$

$$\left.\begin{array}{lcl} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{long braid relations}$$

Coxeter groups can be thought of as generalized reflection groups.

Definition

We can encode (W, S) with a unique Coxeter graph X having:

1. vertex set S;
2. edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$.

Comments

- Typically labels of $m(s, t) = 3$ are omitted.
- Edges correspond to non-commuting pairs of generators.
- Given X, we can uniquely reconstruct the corresponding (W, S).

Type A

Example

The Coxeter group of type A_3 is defined by the graph below.

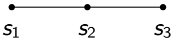


Figure: Coxeter graph of type A_3.

Then $W(A_3)$ is subject to:

- $s_i^2 = 1$ for all i
- $s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3$
- $s_1 s_3 = s_3 s_1$

In this case, $W(A_3)$ is isomorphic to the symmetric group Sym_4 under the correspondence

$$s_1 \leftrightarrow (1\ 2), \quad s_2 \leftrightarrow (2\ 3), \quad s_3 \leftrightarrow (3\ 4).$$

Example

The Coxeter group of type B_4 is defined by the graph below.

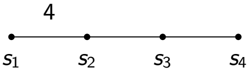


Figure: Coxeter graph of type B_4.

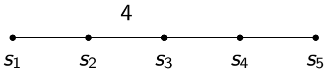
Then $W(B_4)$ is subject to:

- $s_i^2 = 1$ for all i
- $s_2 s_3 s_2 = s_3 s_2 s_3, \quad s_3 s_4 s_3 = s_4 s_3 s_4$
- $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$
- Non-connected nodes commute

In this case, $W(B_4)$ is isomorphic to the group that rearranges and flips 3 coins.

Example

The Coxeter group of type F_5 is defined by the graph below.



Then $W(F_5)$ is subject to:

- $s_i^2 = 1$ for all i
- $s_1 s_2 s_1 = s_2 s_1 s_2$; $\quad s_3 s_4 s_3 = s_4 s_3 s_4$; $\quad s_4 s_5 s_4 = s_5 s_4 s_5$
- $s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2$
- Non-connected nodes commute

F_4 is a finite group, however F_n for $n \geq 5$ is an infinite group.

Definition

A word $s_{x_1}s_{x_2}\cdots s_{x_m} \in S^*$ is called an expression for $w \in W$ if it is equal to w when considered as a group element. If m is minimal, it is a reduced expression.

Example

Consider the expression $s_1s_3s_2s_1s_2$ for an element $w \in W(A_3)$. Note that

$$s_1s_3s_2s_1s_2 = s_1s_3s_1s_2s_1 = s_3s_1s_1s_2s_1 = s_3s_2s_1.$$

Therefore, $s_1s_3s_2s_1s_2$ is not reduced. However, the expression on the right is reduced.

Theorem (Matsumoto/Tits)

Any two reduced expressions for $w \in W$ differ by a sequence of braid relations.

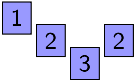
Heaps

One way of representing reduced expressions is via heaps. Fix a reduced expression $s_{x_1} s_{x_2} \cdots s_{x_m}$ for $w \in W$ (for any straight line Coxeter graph). Loosely speaking, the heap for this expression is a set of lattice points, one for each s_{x_i}, embedded in $\mathbb{N} \times \mathbb{N}$ such that:

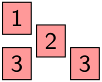
- The node corresponding to s_{x_i} has vertical component equal to $n + 1 - x_i$ (smaller numbers at the top),
- If $i < j$ and s_{x_i} does not commute with s_{x_j}, then s_{x_i} occurs to the left of s_{x_j}.

Example

Consider $s_1 s_2 s_3 s_2$, $s_1 s_3 s_2 s_3$, and $s_3 s_1 s_2 s_3$, which are all reduced expressions of the same element in A_3. It turns out, there are two distinct heaps:



and



Comment

If two reduced expressions differ by a sequence of short braid relations (i.e., commutations), then they have the same heap.

Definition

We say that $w \in W$ has Property T iff some reduced expression begins or ends with a product of non-commuting generators. That is,

$$w = \begin{array}{cc} \boxed{s} & \\ & \boxed{t} \end{array} (\text{other crap}) \qquad \text{or} \qquad w = (\text{other crap}) \begin{array}{cc} \boxed{t} & \\ & \boxed{s} \end{array}$$

Definition

We say that w is T-avoiding iff w does not have Property T.

Proposition

Products of commuting generators are T-avoiding.

Question

Are there other elements besides products of commuting generators that are T-avoiding?

Definition

An element is classified as bad iff it is T-avoiding, but *not* a product of commuting generators.

Theorem (Cormier, Ernst, Goldenberg, Kelly, Malbon)

In types A and B, there are no bad elements. In other words, $w \in W$ is T-avoiding iff w is a product of commuting generators.

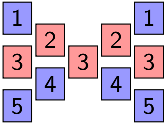
Comment

The answer isn't so simple in other Coxeter groups. In particular, there are bad elements in types $\widetilde{C}$ (Ernst) and D (Tyson Gern).

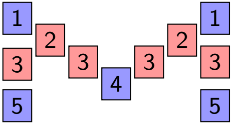
Proposition (Cross, Ernst, Hills-Kimball, Quaranta)

The following reduced expressions are bad elements in F_5:

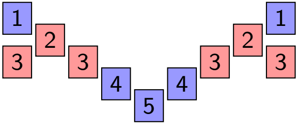
Bowtie



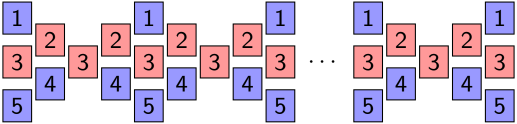
M



Seagull



We can convert Bowties $\leftrightarrow$ M's $\leftrightarrow$ Seagulls via braid relations. These expressions represent the same group element. We will restrict our attention to the bowties. We can also stack bowties to create infinitely many bad elements in F_5.



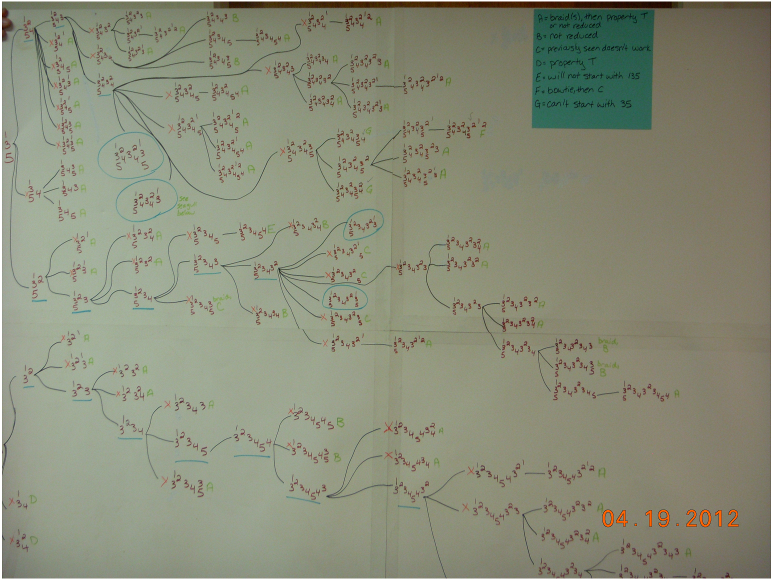
Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

An element is T-avoiding in F_5 iff it is a product of commuting generators or a stack of bowties.

Sketch of Proof

($\Leftarrow$) Easy.
($\Rightarrow$) Hard. Here's an outline.

- Every bad element must begin or end with 135 or 13.
- If w is bad, then w begins and ends with a bowtie, M, or seagull.


A = braid(s), then property T or not reduced
B = not reduced
C = previously seen doesn't work
D = property T
E = will not start with 135
F = bowtie, then C
G = can't start with 35
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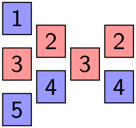
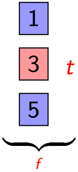
Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

An element is T-avoiding in F_5 iff it is a product of commuting generators or a stack of bowties.

Sketch of Proof

($\Leftarrow$) Easy.
($\Rightarrow$) Hard. Here's an outline.

- Every bad element must begin or end with 135 or 13.
- If w is bad, then w begins and ends with a bowtie, M, or seagull.
- Let $b =$ 

. If $w = b^k$ 

t is bad, then f is bad, as well.

$\hookrightarrow$ We proved the contrapositive: If f has Property T, then w also has Property T.

Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no bad elements in F_4. That is, the only T-avoiding elements in F_4 are products of commuting generators.

Conjecture

An element is T-avoiding in F_n for $n \geq 5$ iff it is a product of commuting generators or a stack of bowties times products of commuting generators. In other words, there are no new bad elements in F_n for $n \geq 6$.

THANK YOU!