A Study of T-Avoiding Elements in Coxeter Groups

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Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

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$$\left.\begin{array}{lcl} m(s,t) = 3 & \implies & sts = tst \\ m(s,t) = 4 & \implies & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

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We can encode (W, S) with a unique Coxeter graph Γ having:

- vertex set S;
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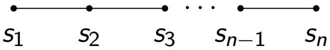
Comments

- if $m(s, t) = 3$, we omit label.
- If s and t are not connected in Γ, then s and t commute.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Coxeter groups of type A

Coxeter groups of type A_n ($n \geq 1$) are defined by:



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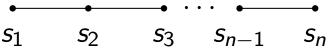

Then $W(A_n)$ is generated by $\{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
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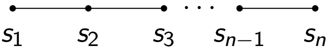

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$W(A_n)$ is isomorphic to the symmetric group, Sym_{n+1}, under the correspondence

$$s_i \mapsto (i,\ i+1),$$

where $(i,\ i+1)$ is the adjacent transposition exchanging i and $i+1$.

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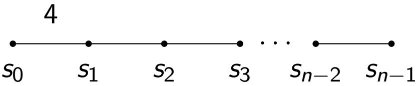

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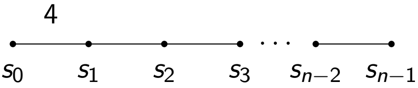

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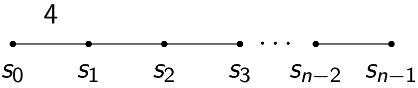

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$W(B_n)$ is a finite group of order $n!2^n$ (wreath product of $\mathbb{Z}_2$ and the symmetric group).

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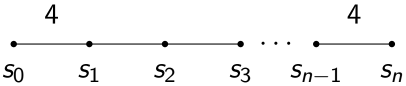
Coxeter groups of type $\widetilde{C}_n$ ($n \geq 2$) are defined by:



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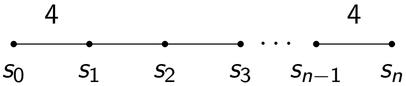

Here, we see that $W(\widetilde{C}_n)$ is generated by $\{s_0, \cdots, s_n\}$ and is subject to defining relations

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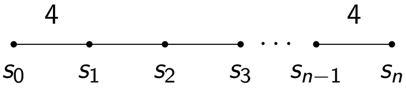

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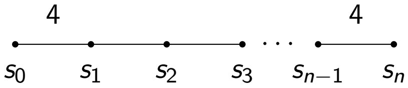

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Comment

We can obtain $W(A_n)$ and $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the appropriate generators and corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

Reduced expressions

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Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{w} = s_{x_1}s_{x_2}\cdots s_{x_m}.$$

Matsumoto's Theorem and Support

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This implies that $\overline{w}$ was not reduced. However, it turns out that $s_1 s_2 s_3$ is a reduced expression for w. Then $\mathrm{supp}(w) = \{s_1 s_2 s_3\}$ and $\ell(w) = 3$.

Definition

Let (W, S) be a Coxeter system of type Γ. We say that $w \in W(\Gamma)$ is fully commutative (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements is denoted $\mathrm{FC}(\Gamma)$.

Theorem (Stembridge)

$w \in \mathrm{FC}(\Gamma)$ if and only if no reduced expression for w contains a braid.

Comment

It follows from Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements, while $W(A_n)$ and $W(B_n)$ do not.

Comment

The elements of $\mathrm{FC}(\widetilde{C}_n)$ are precisely those whose reduced expressions avoid the consecutive subwords $s_i s_j s_i$ for $m(s_i, s_j) = 3$, $s_0 s_1 s_0 s_1$, and $s_{n-1} s_n s_{n-1} s_n$.

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Example

Let $\overline{w} = s_0 s_2 s_4 s_3 s_2 s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. We see that

$$s_0 s_2 s_4 s_3 s_2 s_1 = s_0 s_4 s_2 s_3 s_2 s_1.$$

Since w has one of the forbidden consecutive subwords, w is not FC.

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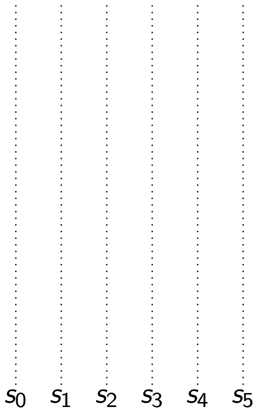
Example

Let $\overline{w} = s_4 s_5 s_1 s_0 s_2 s_4 s_1$ be a reduced expression for $w \in W(B_6)$.



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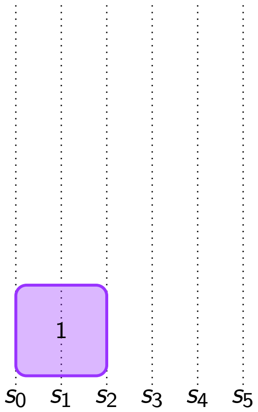

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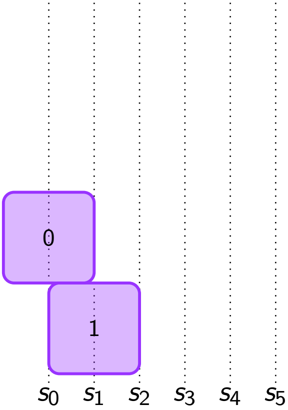

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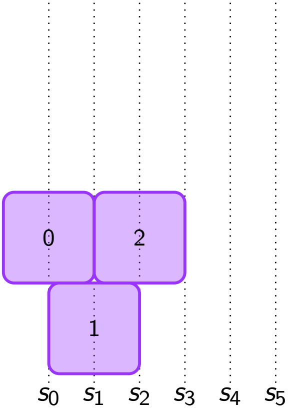

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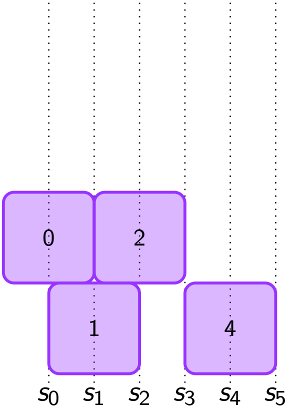

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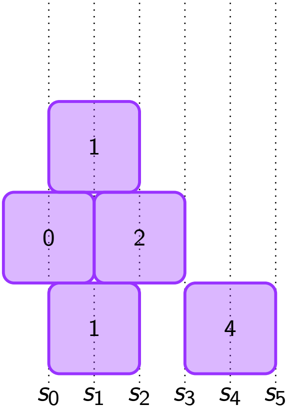

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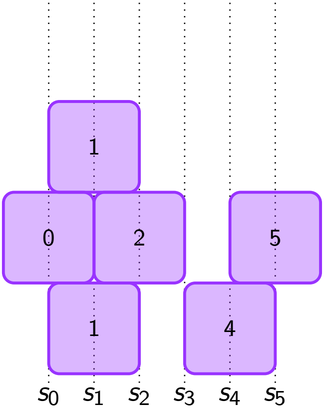

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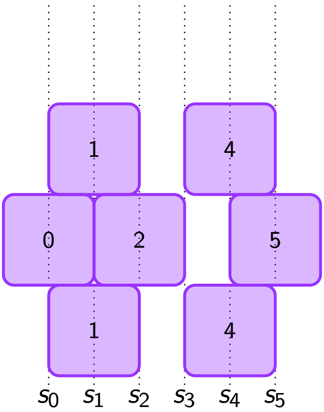

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