A STUDY OF T-AVOIDING ELEMENTS OF COXETER GROUPS

By Taryn Laird

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Approved:

Dana Ernst, Ph.D., Chair

Michael Falk, Ph.D.

Nandor Sieben, Ph.D.

ABSTRACT

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Coming soon. We are finishing writing everything else first so we know the appropriate roadmap to be in this part.

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Chapter 1

Preliminaries

1.1 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [10]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

(1) vertex set S and

(2) edges $\{s,t\}$ for each $m(s,t) \geq 3$ labeled by its corresponding bond strength.

Since $m(s,t) = 3$ occurs frequently, it is customary to omit this label. Note that s and t are not connected by a single edge in the graph if and only if $m(s,t) = 2$.

There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. That is, given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. If (W, S) is a Coxeter system with corresponding Coxeter graph Γ, we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. Also, the Coxeter system (W, S) is said to be *irreducible* if and only if Γ is connected. If the graph Γ is disconnected, the connected components correspond to factors in a direct product of the corresponding Coxeter groups [10]. The Coxeter graphs given in Figure 1.1 correspond to the Coxeter systems that will be primarily addressed in this thesis. Notice here that the vertices are labeled with the corresponding generators to provide context when talking about the different generating sets $S(\Gamma)$.

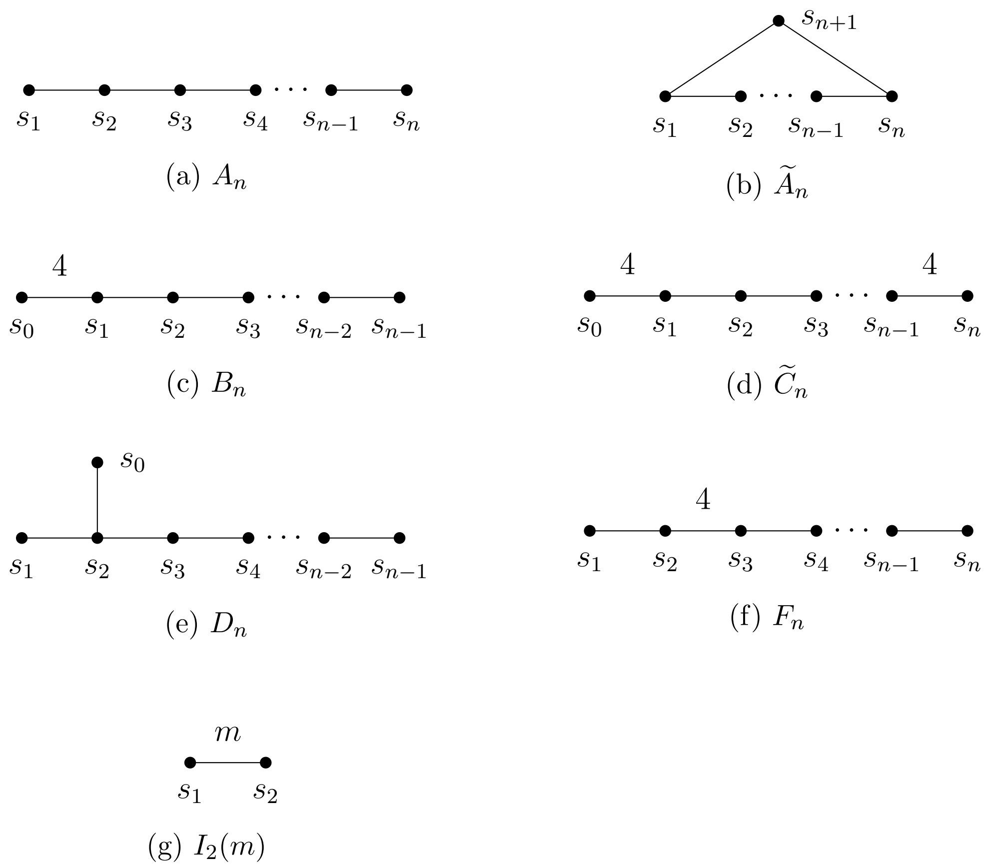


Figure 1.1: A few labeled Coxeter graphs.

Example 1.1.1.

(a) The Coxeter system of type A_n is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group $W(A_n)$ with generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} under the correspondence $s_i \mapsto (i, i+1)$, where $(i, i+1)$ is the adjacent transposition that swaps i and $i+1$.

(b) The Coxeter system of type B_n is given by the graph in Figure 1.1(c). We can construct the corresponding Coxeter group $W(B_n)$ with generating set $S(B_n) = \{s_0, s_1, \ldots, s_{n-1}\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to Sym_n^B, where Sym_n^B is the group of signed permutations on the set $\{1, 2, \ldots, n\}$.

(c) The Coxeter system of type $\widetilde{C}_n$ is seen in Figure 1.1(d). We can construct the corresponding Coxeter group $W(\widetilde{C}_n)$ with generating set $S(\widetilde{C}_n) = \{s_0, s_1, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$ for $i \in \{0, 2, \ldots, n\}$;

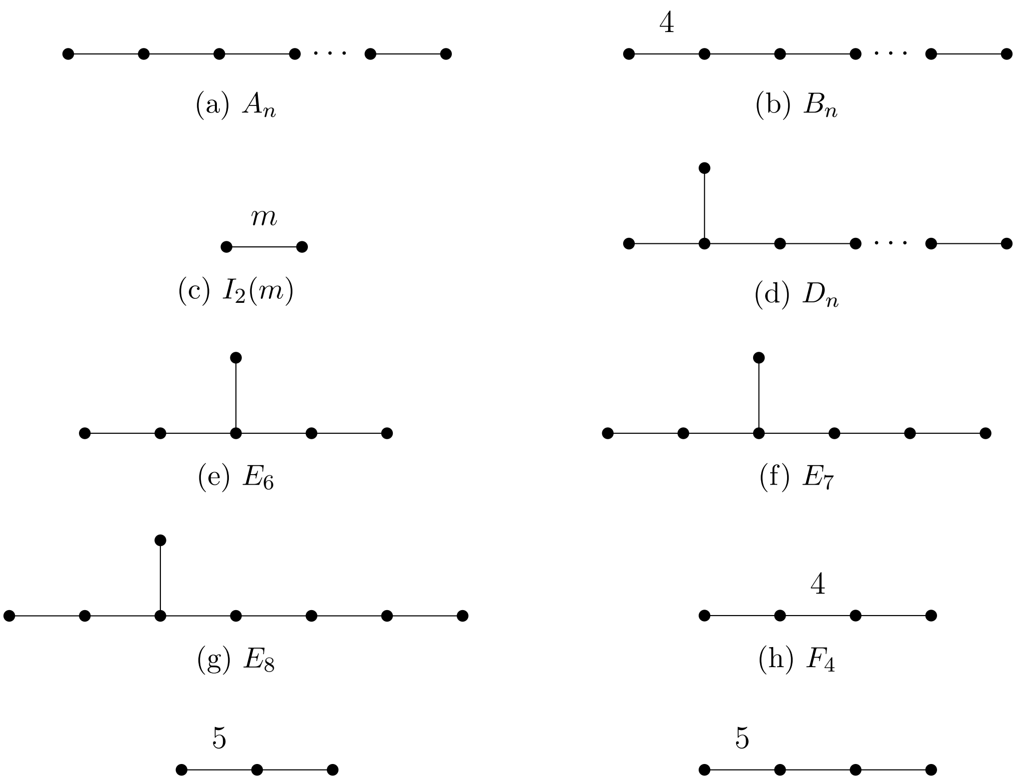
(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

Note that $W(\widetilde{C}_n)$ has $n + 1$ generators.

The Coxeter graphs given in Figure 1.2 correspond to the collection of irreducible finite Coxeter systems, whose corresponding Coxeter groups are finite, while the Coxeter graphs given in Figure 1.3 are the so-called irreducible *affine Coxeter systems*, which are infinite [10]. Note that $W(B_n)$ is one of the irreducible finite Coxeter groups so it is finite while $W(\widetilde{C}_n)$ is one of the affine groups making it infinite. The irreducible affine Coxeter systems are unique in that if a vertex is removed along with the corresponding edges from the Coxeter graph, the newly created graph will result in a Coxeter system with a finite Coxeter group.



(a) A_n (b) B_n (c) $I_2(m)$ (d) D_n (e) E_6 (f) E_7 (g) E_8 (h) F_4 (i) H_3 (j) H_4

Figure 1.2: Irreducible finite Coxeter systems.

Given a Coxeter system (W, S), a word $s_{x_1}s_{x_2}\cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_m}$ using (sans serif font). If $u, v \in W(\Gamma)$, we say that the product of

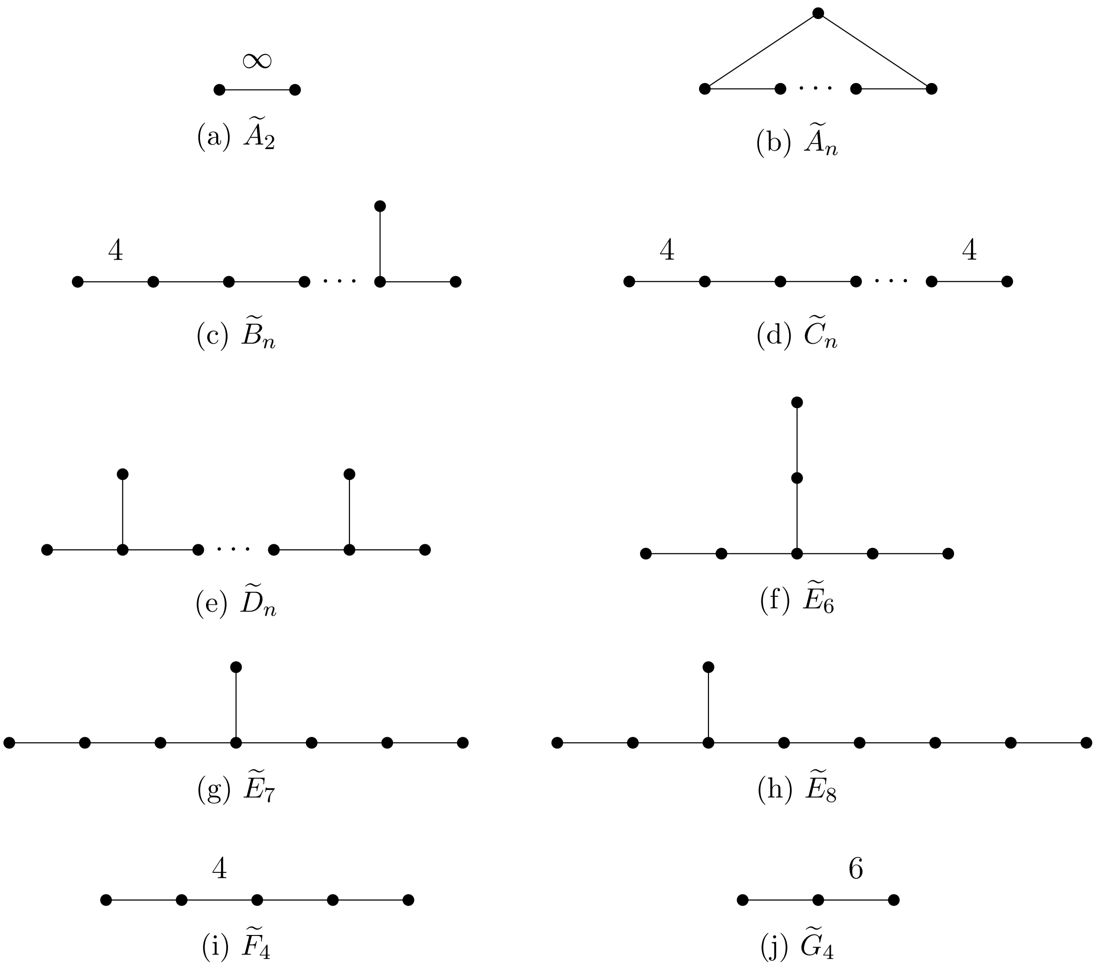


Figure 1.3: Irreducible affine Coxeter systems.

group elements uv is *reduced* if $l(uv) = l(u) + l(v)$. The following theorem tells us more about how reduced expressions for a given group element are related.

Theorem 1.1.2 (Matsumoto, [7])**.** Let (W, S) be a Coxeter system. If $w \in W$, then given a reduced expression for w we can obtain every other reduced expression for w by a sequence of braid moves and commutations of the form

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. $\square$

It follows from Matsumoto's Theorem that if a generator s appears in a reduced expression for $w \in W$, then s appears in all reduced expressions for w. Let $w \in W$

and define the *support* of w, denoted $\mathrm{supp}(w)$, to be the set of all generators that appear in any reduced expression for w. If $\mathrm{supp}(w) = S$, we say that w has *full support*.

Given $w \in W$ and a fixed reduced expression w for w, any subsequence of w is called a *subexpression* of w. We will refer to a subexpression consisting of a consecutive subsequence of w as a *subword* of w.

Example 1.1.3. Let $w \in W(A_7)$ and let $\mathsf{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be a fixed expression for w. Then we have

$$\begin{aligned} s_7{\color{purple}s_2s_4}s_5s_3s_2s_3s_6 &= s_7s_4{\color{purple}s_2s_5}s_3s_2s_3s_6 \\ &= s_7s_4s_5{\color{teal}s_2s_3s_2}s_3s_6 \\ &= s_7s_4s_5s_3s_2{\color{red}s_3s_3}s_6 \\ &= s_7s_4s_5s_3s_2s_6, \end{aligned}$$

where the purple highlighted text corresponds to a commutation, the teal highlighted text corresponds to a braid move, and the red highlighted text corresponds to cancellation. This shows that the original expression w is not reduced. However, it turns out that $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\mathrm{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a reduced expression for w that begins (respectively, ends) with s.

Example 1.1.4. The following list consists of all reduced expressions some $w \in W(B_4)$:

$$\begin{array}{ll} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{array}$$

We see that $l(w) = 5$ and w has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

1.2 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. Following [13], we define a relation $\sim$ on the set of reduced expressions for w. Let w_1 and w_2 be two reduced expressions for w. We define $\mathsf{w}_1 \sim \mathsf{w}_2$ if we can obtain w_2 from w_1 by applying a single commutation move of the form $st \mapsto ts$ where $m(s,t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative* (FC).

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$, and a starting reduced expression for w, observe that the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for w, but the following result due to Stembridge [13] states that when w is FC, performing commutations is the only possible way to obtain another reduced expression for w.

Theorem 1.2.1 (Stembridge, [13])**.** Let (W, S) be a Coxeter system. An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all $m(s,t) \geq 3$. $\square$

In other words, w is FC if and only if no reduced expression provides the opportunity to apply a braid move. For example, for a Coxeter system of type B_n an element is FC if no reduced expression contains the subwords $s_0s_1s_0s_1$, $s_1s_0s_1s_0$, $s_ks_{k+1}s_k$, and $s_{k+1}s_ks_{k+1}$ where $0 < k < n-1$. In a Coxeter group of type $\widetilde{C}_n$, an element is FC if no reduced expression for the element contains the subwords seen above and does not contain the subwords $s_{n-1}s_ns_{n-1}s_n$ and $s_ns_{n-1}s_ns_{n-1}$.

Example 1.2.2. Let $w \in W(\widetilde{C}_4)$ and let $\mathsf{w} = s_0s_1s_2s_0s_3s_1$ be a reduced expression for w. Although it is not immediately obvious, there is no possible way to perform a braid move in w. Hence w is FC.

Example 1.2.3. Let $\mathsf{w}_1 = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. Applying the commutation $s_4s_2 \mapsto s_2s_4$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_1s_0s_4s_1s_3s_5s_4s_2s_6$, which is in the same commutation class as w. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w} = s_1s_3s_2s_3s_4s_0$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. Since w has at least two commutation classes, one containing w_1 and w_2 and another containing w_3, w is not FC by Theorem 1.2.1.

Stembridge classified the Coxeter systems that contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. Both $W(A_n)$ and $W(B_n)$ are finite

Coxeter groups, and thus are FC-finite. On the other hand, $W(\widetilde{C}_n)$ is infinite and happens to also contain infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, $W(E_n)$ for $n \geq 9$ (see Figure 1.4) is infinite, but contains only finitely many FC elements.

Theorem 1.2.4 (Stembridge, [13])**.** The irreducible FC-finite Coxeter systems are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. □

The irreducible FC-finite Coxeter graphs are given in Figure 1.4. Note that the irreducible finite Coxeter systems given in Figure 1.2 certainly have only a finite number of FC elements. We have not yet encountered the Coxeter groups determined by graphs in Figures 1.4(d) for $n \geq 9$, 1.4(e) for $n \geq 5$, 1.4(f) for $n \geq 5$. All of these Coxeter systems are infinite for sufficiently large n, yet contain only finitely many FC elements.

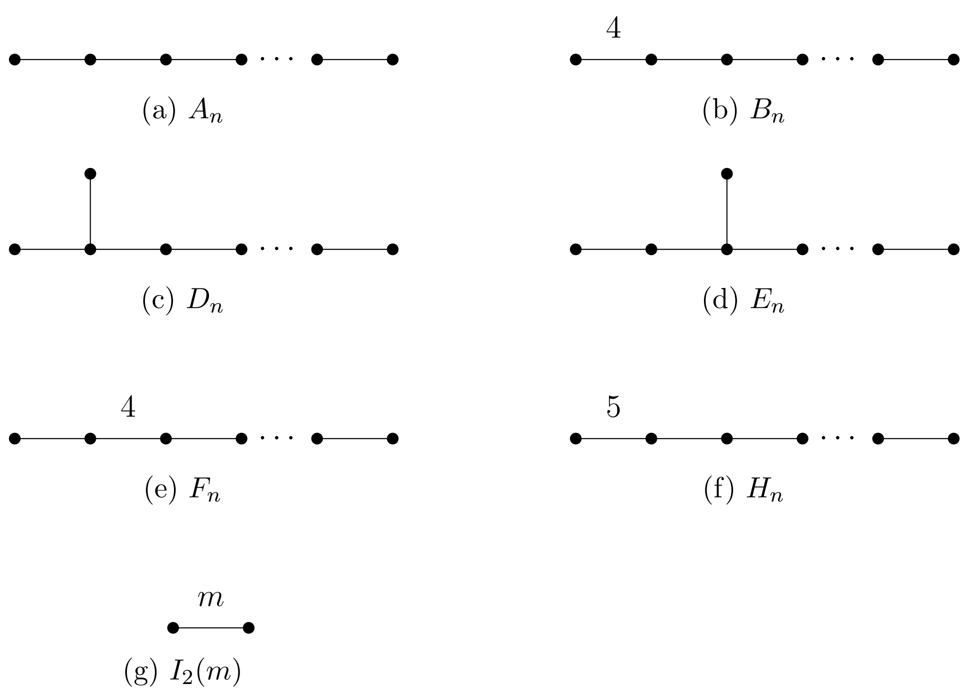


Figure 1.4: Irreducible FC-finite Coxeter systems.

1.3 Heaps

We now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps for straight line Coxeter groups found in [1], [3], and [13].

Let (W, S) be a Coxeter system of type Γ. Suppose $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ is a fixed reduced expression for $w \in W(\Gamma)$. As in [13], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, since w is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of w, where i is labeled by s_{x_i}. Note that for simplicity we are omitting the labels of the underlying poset yet retaining the labels of the corresponding generators.

It follows from [13] that heaps are well-defined up to commutation class. That is, given two reduced expressions w_1 and w_2 for $w \in W$ that are in the same commutation class, the heaps for w_1 and w_2 will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then w has one commutation class, and thus w has a unique heap. Conversely, if w_1 and w_2 are in different commutation classes, then the heap for w_1 will be distinct from the heap for w_2.

Example 1.3.1. Let $\mathsf{w} = s_6 s_4 s_2 s_5 s_3 s_1 s_4 s_0 s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that w is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $9 \lessdot 8$ since $8 < 9$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 1.5.

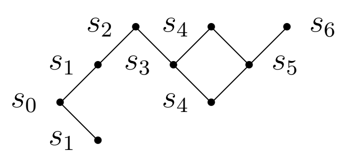


Figure 1.5: Labeled hasse diagram for the heap of an element in $\mathrm{FC}(\widetilde{C}_6)$.

Let w be a reduced expression for an element $w \in W(\widetilde{C}_n)$. As in [1] and [3] we can represent a heap for w as a set of lattice points embedded in $\{0, 1, 2, \ldots, n\} \times \mathbb{N}$. To do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \ldots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of w in such a way that:

(1) An entry with coordinates (x, y) is labeled s_i (or i) in the heap if and only if $x = i$;

(2) If an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight line Coxeter graph with the appropriate adjustments made to the label set and assignment of coordinates. Specifically, for type A_n our label set is $\{1, 2, \ldots, n\}$ and for type B_n our label set is $\{0, 1, \ldots, n-1\}$.

In the case of any straight line Coxeter graph it follows from the definition that (x, y) covers (x', y') in the heap if and only if $x = x' \pm 1$, $y' < y$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries fully exposed to the top (respectively, bottom) correspond to the generators occurring in the left (respectively, right) descent set of the corresponding reduced expression.

Let w be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice representation of the heap poset in $\{0, 1, 2, \ldots n\} \times \mathbb{N}$ described in the preceding paragraphs via $H(\mathsf{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ and we refer to $H(w)$ as the heap of w. Note that we will use the same notation for heaps in Coxeter groups of all types with straightline Coxeter graphs.

Let $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_i} at a level that is *above* the level of the point labeled by s_{x_j}. Because generators in a Coxeter graph that are not adjacent do commute, points whose x-coordinates differ by more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners (called a block) in such a way that if one entry covers another the block overlap halfway. In addition, we will also label each square for s_i with i.

There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap for each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

Example 1.3.2. Let $\mathsf{w} = s_6 s_4 s_2 s_5 s_3 s_1 s_4 s_0 s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.3.1. Figure 1.6 shows a possible lattice point representation for $H(w)$. Since w is FC this is the unique heap representation for w. Because $H(w)$ is the unique heap we can obtain $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) from the blocks that are fully exposed to the top (respectively, bottom) of the heap. We see that

$\mathcal{L}(w) = \{s_2, s_4, s_6\}$ and $\mathcal{R}(w) = \{s_1, s_4\}$.

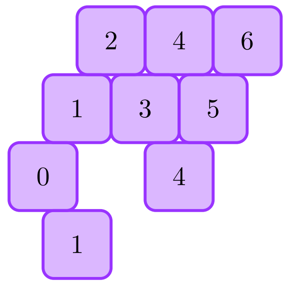
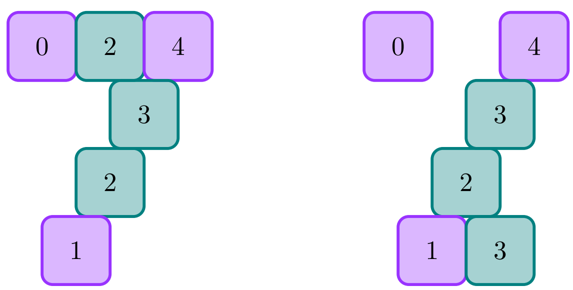


Figure 1.6: A lattice point representation for the heap of an FC element in $W(\widetilde{C}_6)$.

Example 1.3.3. Let $\mathsf{w}_1 = s_0s_2s_4s_3s_2s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. Applying the commutation move $s_2s_4 \mapsto s_4s_2$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_0s_4s_2s_3s_2s_1$, which is in the same commutation class as w_1, and hence has the same heap. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w}_3 = s_0s_4s_3s_2s_3s_1$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. Representations of $H(\mathsf{w}_1), H(\mathsf{w}_2)$, and $H(\mathsf{w}_3)$ are seen in Figure 1.7, where the braid relation is colored in teal.



(a) $H(\mathsf{w}_1)$ and $H(\mathsf{w}_2)$. (b) $H(\mathsf{w}_3)$

Figure 1.7: Two heaps of a non-FC element in $W(\widetilde{C}_4)$.

It will be extremely useful for us to be able to quickly determine whether a heap corresponds to an element in $\mathrm{FC}(B_n)$ or $\mathrm{FC}(\widetilde{C}_n)$. The next proposition is a special case of [13, Proposition 3.3] and follows quickly when one considers the consecutive subwords that are impermissible in reduced expressions for elements in $\mathrm{FC}(B_n)$ and $\mathrm{FC}(\widetilde{C}_n)$ as discussed in Section 1.2.

Theorem 1.3.4. If $w \in \mathrm{FC}(\widetilde{C}_n)$, then $H(w)$ cannot contain any of the configurations seen in Figure 1.8, where $0 < k < n-1$ and we use a square with a dotted boundary to emphasize that no element of the heap may occupy the corresponding position. $\square$

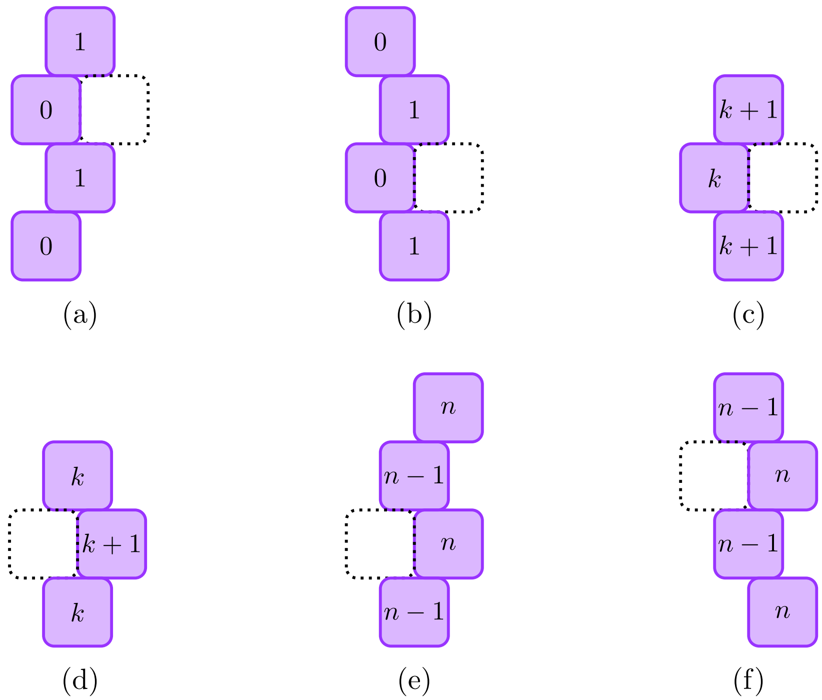


Figure 1.8: Impermissible configurations for heaps of $\mathrm{FC}(\widetilde{C}_n)$.

Since $W(B_n)$ is a parabolic subgroup of $W(\widetilde{C}_n)$, we can use Figure 1.8 to classify the impermissible configurations for elements of $\mathrm{FC}(B_n)$. In particular, the impermissible configurations for elements of $\mathrm{FC}(B_n)$ are those seen in Figures 1.8(a), 1.8(b) 1.8(c), and 1.8(d).

Chapter 2

Star Reductions and Property T

2.1 Star Operations

The notion of a star operation was originally introduced by Kazhdan and Lusztig in [11] for simply-laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s,t \in S$), and was later generalized to all Coxeter systems in [12]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to only define the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We define w to be *left star reducible by s with respect to t* if there exists $t \in \mathcal{L}(sw)$. We analogously define w to be *right star reducible by s with respect to t*. Observe that w is left (respectively, right) star reducible if and only if $w = stu$ (respectively, $w = uts$), where the product on the right hand side of the equation is reduced and $m(s,t) \geq 3$. We say that w is *star reducible* if it is either left or right star reducible.

Example 2.1.1. Let $\mathsf{w} = s_0 s_1 s_0 s_2$ be a reduced expression for $w \in W(B_4)$. We see that w is left star reducible by s_0 with respect to s_1 to $s_1 s_0 s_2$ since $m(s_0, s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0 w)$. Notice that w is FC and $\mathcal{R}(w) = \{s_2, s_0\}$. We see that $ws_2 = s_0 s_1 s_0$ and $ws_0 = s_0 s_1 s_2$. Note that in both instances $s_1 \notin \mathcal{R}(ws_2) = \{s_0\}$ and $s_1 \notin \mathcal{L}(ws_0) = \{s_2\}$. Because of this w is not right star reducible.

It may be helpful to visualize star reductions in terms of heaps. Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Suppose w is left star reducible by s with respect to t. Then there exists a heap for w where the block for s is fully exposed to the top such that removing the block for s off of the top allows for t to now be fully exposed to the top of the heap. Similarly if w is right star reducible by s with respect to t, then there exists a heap for w where the block for s is fully exposed to the bottom of the heap. Removing the block for s off of the bottom allows for t to now be fully exposed to the bottom. Conversely, if a heap for $w \in W(\Gamma)$ has this property, then w is star reducible. In Figure 2.1 we see the two possible heap representation of an element that is left star reducible by s with respect to t, where the dotted square represents that there can not be an element there. That is, in order for w to be star reducible the block $s + 2$ in Figure 2.1(a) (respectively, $s - 2$ in Figure 2.1(b)) do not appear fully exposed to the top of the heap. Notice that flipping the heap upsidedown in Figure 2.1 will result in a heap that is right star reducible. It is important to note that if the group element we are evaluating for star reducibility is not FC, then we must consider all heap representations for the element before concluding that an element does not have star reducibility.

Figure 2.1: A visual representation of an element that is star reducible at the top.

The following example utilizes heaps to show that an element is star reducible.

Example 2.1.2. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. Note that w is FC. By Example 2.1.1 we know that w is left star reducible by s_0 with respect to s_1. In Figure 2.2(a), we see the heap for w. Notice that the block for s_0 is fully exposed to the top of the heap. Removing the block for s_0 gives the heap in Figure 2.2(b). Notice that the block for s_1 is now fully exposed to the top of the heap. However, notice that the blocks for s_0 and s_2 are fully exposed to the bottom. In removing either of these we are unable to fully expose s_1 to the bottom. Thus we can see that w is not right star reducible.

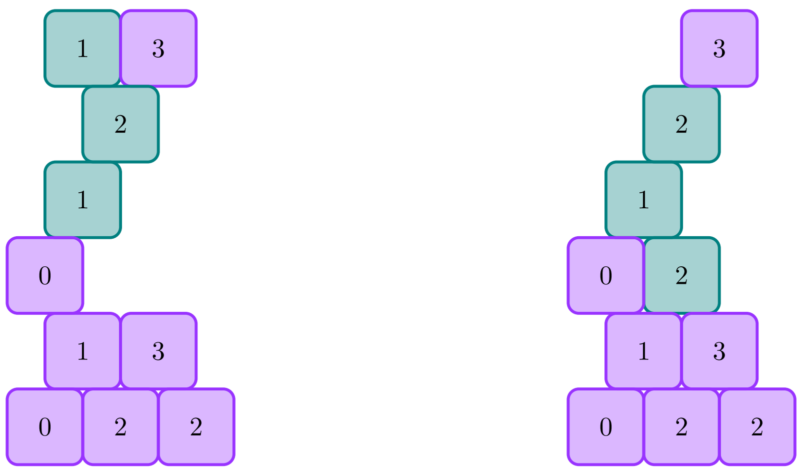
Notice that if w is not FC then we are not be able to say that w is not star reducible as there could be a different heap for w in which we are able to fully expose an element that was previously blocked.

Example 2.1.3. Let $\mathsf{w}_1 = s_3s_1s_2s_1s_0s_1s_3s_0s_2s_4$ be a reduced expression for $w \in W(\widetilde{C}_3)$. The heap for w is given in Figure 2.3(a), where we have highlighted a braid in teal. Notice that this heap appears to not be star reducible as if we were to remove

(a) (b)

Figure 2.2: Visualization of Example 2.1.1.

the block for s_1 or s_3 we would still not fully expose s_2 to the top of the heap. The same goes for fully exposing blocks in the bottom of the heap. However, when we perform the braid move resulting in the heap seen in Figure 2.3(b) it is now obvious that the element is star reducible. Thus when considering a non-FC element for star reducibility via the heap, it is very important to consider all heaps for that element.



(a) (b)

Figure 2.3: Visualization of Example 2.1.3

Using the notion of star reduction we are now able to introduce the concept of a star reducible Coxeter group. Let (W, S) be a Coxeter group of type Γ we say that $W(\Gamma)$ is *star reducible* if every element of FC(Γ) is star reducible to a product of commuting generators. That is, $W(\Gamma)$ is star reducible if when we apply star reductions on the left or right repeatedly to $w \in$ FC(Γ), eventually we obtain a product of commuting generators. Notice that we are restricting to just the FC elements in $W(\Gamma)$. Visually a star reducible Coxeter group can be thought of in the following way. Given a heap in FC(Γ), we are able to systematically remove a fully exposed block from the top or bottom of the heap and have a block that was previously not fully exposed become fully exposed until we are left with a heap that can be drawn as a single row.

In [9], Green classified all star reducible Coxeter groups.

Theorem 2.1.4 (Green, [9]). Let (W, S) be a Coxeter system of type Γ. Then (W, S) is star reducible if and only if each component of Γ is either a complete graph with labels $m(s,t) \geq 3$, or is one of the following types: type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type H_n $(n \geq 2)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_n$ $(n \geq 3$ and n even), type $\widetilde{C}_n$ $(n \geq 3$ and n odd), type $\widetilde{E}_6$, or type $\widetilde{F}_5$. □

2.2 Property T

In [9], Green utilizes the following theorem to help classify the star reducible Coxeter groups.

Theorem 2.2.1 (Green, [9]). Let (W, S) be a star reducible Coxeter system of type Γ, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s,t, u with $m(s,t) \neq 2$, $m(t,v) \neq 2$, and $m(s,u) = 2$:

(1) w is a product of commuting generators;

(2) w has a reduced product $w = stu$;

(3) w has a reduced product $w = uts$;

(4) w has a reduced product $w = svtu$. □

Notice that Items (2) and (3) indicate an element that is star reducible. Also notice that an element w that has the form of Item (1) does not meet the conditions of Items (2) and (3). In particular, w is not star reducible. Lastly, notice that if an element w is of the form of Item (4) and not of the form of Items (2) and (3), then w is not star reducible. Notice Items (2), (3), and (4) are not mutually exclusive.

Motivated by Items (1) and (4) above, we define the notions of Property T and T-avoiding. Let (W, S) be a Coxeter system of type Γ and let $w \in W$. We say that w has *Property T* if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s,t) \geq 3$. That is, w has Property T if there exists a reduced expression for w that begins or ends with a product of non-commuting generators. An element $w \in W(\Gamma)$ is called *T-avoiding* if w does not have Property T. This implies that a T-avoiding element is not star reducible.

Since elements that are star reducible also have Property T we already know how to visualize Property T in terms of heaps. It is clear that a product of commuting generators is T-avoiding, which we state as a theorem. Visually a product of commuting generators is a one row heap (as in Figure **??**), it is clear a one row heap will not portray the characteristic of Property T as seen in Figure 2.1.

Theorem 2.2.2. Let (W, S) be a Coxeter system of type Γ. If $w \in W(\Gamma)$ such that w is a product of commuting generators, then w is T-avoiding. □

We will call an element that is a product of commuting generators *trivially T-avoiding*. If w is T-avoiding and not a product of commuting generators, we will say that w is *non-trivially T-avoiding*. It is not clear that such elements exist. Referring back to Green's classification (Theorem 2.2.1) of what elements in star reducible Coxeter groups look like, we see that Item (1) corresponds to an element w being trivially T-avoiding, Items (2) and (3) refer to the element w having Property T at the beginning and end respectively, and Item (4) refers to an element being non-trivially T-avoiding provided no reduced expression for the element exhibits Items (2) and (3). In star reducible Coxeter systems, every FC element is star reducible to a product of commuting generators, which implies that no FC element can be non-trivially T-avoiding. For example, as will be seen in the following sections, the Coxeter groups of type A_n and B_n have no non-trivial T-avoiding elements, while the Coxeter group of type D_n does have non-trivial T-avoiding elements.

Example 2.2.3. Let $w \in W(A_5)$ with reduced expression $\mathsf{w} = s_1 s_3 s_5$. It turns out that since w is a product of commuting generators by Theorem 2.2.2 we know that w is trivially T-avoiding.

Example 2.2.4. Let $w \in W(A_5)$ with reduced expression $\mathsf{w}_1 = s_1 s_4 s_2 s_3 s_5$. At first glance it may appear that w does not have Property T since both s_1 and s_4 commute as well as s_3 and s_5. However, note that applying the commutation move $s_4 s_2 \mapsto s_2 s_4$ results in $\mathsf{w}_2 = s_1 s_2 s_4 s_3 s_5$. Hence w has Property T since $m(s_1, s_2) = 3$ and there is a reduced expression for w that begins with $s_1 s_2$. In Figure 2.4 we see the heap for w_1. Note that we can see Property T in the bottom of the heap highlighted in orange.

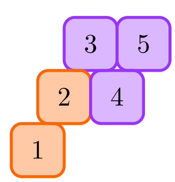


Figure 2.4: Heap of an element with Property T.

Example 2.2.5. Let $w \in W(\widetilde{C}_4)$ with reduced expression $\mathsf{w} = s_0s_2s_4s_1s_3s_0s_2s_4$. It turns out that w is FC and non-trivially T-avoiding. The heap for w is seen in Figure 2.5. Notice that no matter which block we remove that is fully exposed to the top of the heap no new element becomes fully exposed. The same applies to the bottom of the heap. Thus, w is non-trivially T-avoiding.

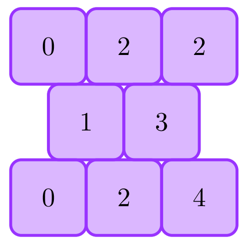


Figure 2.5: Heap of a non-trivially T-Avoiding element in $W(\widetilde{C}_4)$.

One thing to notice here is that all Coxeter groups have trivial T-avoiding elements as they all contain products of commuting generators. The more interesting non-trivial T-avoiding elements do not appear in all Coxeter groups. In Chapter 3 we will summarize what is known about the T-avoiding elements in Coxeter groups of types $\widetilde{A}_n$, A_n, D_n, F_n, and $I_2(m)$, and in Chapters 4 and 5 we classify the T-avoiding elements in Coxeter groups of types B_n and $\widetilde{C}_n$.

2.3 Non-Cancellable Elements

We now introduce the concept of weak star reducible, which is related to the notion of cancellable in [5]. Let (W, S) be a Coxeter system of type Γ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators. If $w \in \mathrm{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t to sw* if

(1) w is left star reducible by s with respect to t, and

(2) $tw \notin \mathrm{FC}(W)$.

Notice that condition (2) implies that $l(tw) > l(w)$. Also note that we are restricting out definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws*. We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable* or *weak star irreducible.* Notice that from this we know that weak star reducible implies star reducible. However, star reducible does not imply weak star reducible.

Example 2.3.1. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$ as in Example 2.1.1. From Example 2.1.1 we know that w is left star reducible. However, $tw = s_1s_0s_1s_0s_2$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2$. In addition, Example 2.1.1 showed that w is not right star reducible and hence w is not right weak star reducible. However, since w is left weak star reducible we know that w is not non-cancellable.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall that in Section 2.1 we described what a star reduction looks like in a heap and what a star reducible heap looks like. Since the definition of weak star reducible includes that a heap is star reducible we again need to have those properties. In addition, for a heap to be weak star reducible when adding the block that becomes fully exposed when a block is removed from the heap must create a braid in the heap forcing the new heap to not be FC. That is, one of the impermissible configurations seen in Section 1.3 will appear.

Example 2.3.2. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduce expression for $w \in W(B_4)$ as in Example 2.3.1. From Example 2.3.1, we know that w is left weak star reducible. Recall in Figure 2.2 the heap for w was seen along with what it star reduced to. In Figure 2.6 we see that adding s_1 to the top of the heap creates a braid which is highlighted in orange.

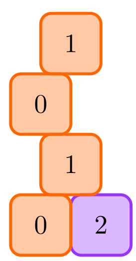


Figure 2.6: Heap of a weak star reducible element of $\mathrm{FC}(B_4)$.

Example 2.3.3. Let $w \in \mathrm{FC}(B_4)$ and let $\mathsf{w} = s_0s_1$ be a reduced expression for w. Note that w is left (respectively, right) star reducible by s_0 with respect to s_1 (respectively, by s_1 with respect to s_0). However, $s_1s_0s_1 \in \mathrm{FC}(B_4)$ (respectively, $s_0s_1s_0 \in \mathrm{FC}(B_4)$). Thus w is non-cancellable. Visually the heap appears in Figure 2.7. Clearly when s_0 is added to the bottom of the heap, the new heap is still in $\mathrm{FC}(B_4)$ and the same can be said when s_1 is added to the top of the heap.

In [3], Ernst classified the non-cancellable elements in Coxeter groups of type $W(B_n)$ and $W(\widetilde{C}_n)$. We will state part of the classification here as it is important

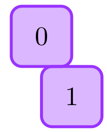


Figure 2.7: Heap of a non-cancellable element of $\mathrm{FC}(B_4)$.

to the development of the non-trivial T-avoiding elements in $W(\widetilde{C}_n)$. To see the full classification see [3, Sections 4.2 and 5].

Before we state the classification we first define a specific group element called a Type II element in $W(\widetilde{C}_n)$. We will refer to this element as a single *sandwich stack*, which is seen in Figure 2.8.

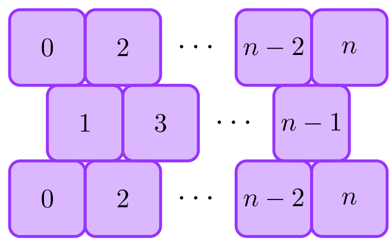


Figure 2.8: Heap of a single sandwich stack in $W(\widetilde{C}_n)$.

We can stack single sandwich stacks together and get a "stack of sandwich stacks." This is done by removing the top most layer of the heap and adding a new single bowtie to the stack. A stack of bowties is seen in Figure 2.9. These heaps are referenced in the classification of non-trivial T-avoiding elements in $W(\widetilde{C}_n)$.

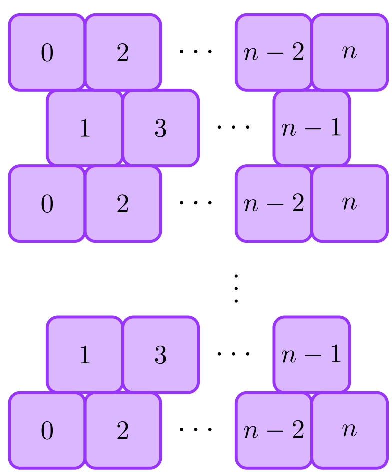


Figure 2.9: Heap of a stack of sandwich stacks in $W(\widetilde{C}_n)$.

In [3], Ernst classified the sandwich stack as a non-cancellable element. Ernst also classified two other types of non-cancellable elements. The first does not have full support in $W(\widetilde{C}_n)$ and the second clearly has Property T. Thus from now on we will only consider stacks of sandwich stacks.

Chapter 3

T-Avoiding Elements in Types $\widetilde{A}_n, A_n, D_n, F_n$, and $I_2(m)$

3.1 Types $\widetilde{A}_n$ and A_n

We start with T-avoiding elements in Coxeter systems of type $\widetilde{A}_n$ and A_n. We first focus on non-trivial T-avoiding elements in $W(\widetilde{A}_n)$.

Theorem 3.1.1. If $n \geq 2$ and n is even, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. Otherwise, if $n \geq 2$ and n is even then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

Proof. This is [6, Proposition 3.1.2.] after an appropriate translation of terminology. □

The classification seen in [6] did not specifically classify the non-trivial T-avoiding elements for type $\widetilde{A}_n$ for n odd. Since $W(\widetilde{A}_n)$ for n odd is not star reducible we know that the non-trivial T-avoiding elements could be FC. The following is our conjecture regarding what the non-trivial T-avoiding elements are in $W(\widetilde{A}_n)$ for n odd.

Conjecture 3.1.2. The only non-trivial T-avoiding elements in $W(\widetilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

Recall that $W(A_n)$, n even, is not a star reducible Coxeter group. Hence it makes sense that the T-avoiding elements in $W(\widetilde{A}_n)$, n odd, can be FC. Further, as $W(A_n)$ is a parabolic subgroup of $W(\widetilde{A}_n)$ and $W(A_n)$ is a star reducible Coxeter group, the FC non-trivial T-avoiding elements must have full support. First notice, that $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ is reduced, FC and has full support. In addition, w is in fact T-avoiding. Notice that the above non-trivial T-avoiding elements are FC. As stated in the conjecture we believe that these are the only non-trivial T-avoiding elements. However, it is not immediately obvious that there are not any non FC non-trivial T-avoiding elements. Classifying these non-trivial T-avoiding elements remains an open problem. We now proceed with discussion of T-avoiding elements in Coxeter groups of type A_n.

Corollary 3.1.3. There are no non-trivially T-avoiding elements in $W(A_n)$.

Proof. Notice that the Coxeter graph of type A_n can be obtained from the Coxeter graph of type $\widetilde{A}_k$, for $k > n$. This is done by removing the appropriate number of vertices and edges from the Coxeter graph of type $\widetilde{A}_k$. Since $W(\widetilde{A}_k)$, k even, has no non-trivial T-avoiding elements this forces $W(A_n)$ to not have non-trivial T-avoiding elements. Thus $W(A_n)$ can not have bad elements. □

3.2 Type D_n

In this section we will classify the T-avoiding elements in Coxeter systems of type D_n. Recall that $W(D_n)$ is a star reducible Coxeter group and as a result of this any potential non-trivial T-avoiding elements are not FC.

Theorem 3.2.1. The are non-trivial T-avoiding elements in $W(D_n)$ for $n \geq 4$.

Proof. This is a consequence of [8, Section 2.2]. For the full details regarding this classification see [8]. Note that in this classification, Gern refers to non-trivially T-avoiding elements as "bad." □

We now will classify these elements as seen in [8]. Before we do so we state interval notation useful to the classification from [8, Definition 2.3.1]. For $2 \leq i \leq j$ denote the element $s_i s_{i+1} \cdots s_{j-1} s_j$ by $[i, j]$. For $i \geq 3$, denote $s_1 s_3 s_4 \cdots s_i$ by $[1, i]$ and for $j \geq 2$ denote $s_1 s_2 s_3 \cdots s_j$ by $[0, j]$. If $0 \leq j < i$ and $i \geq 2$ define $[j, i] = [i, j]^{-1}$. Finally, for $i \leq -3$ and $j \geq 3$ denote $s_1 s_{i-1} s_{i-2} \cdots s_4 s_3 s_2 s_3 s_4 \cdots s_j$ by $[-i, j]$. The following is the classification for T-avoiding elements in $W(D_n)$.

Proposition 3.2.2. Let $w \in W(B_n)$ be non-trivially T-avoiding. Then w has the signed permutation notation

$$w_m = \begin{cases} [(-1)^{\frac{m}{2}}, \underline{m}, 3, \underline{m-2}, 5, \ldots, \underline{4}, m-1, \underline{2}, m+1, m+2, \ldots, n] & \text{if, } m \text{ is even} \\ [(-1)^{\frac{m-1}{2}}, \underline{m-1}, 3, \underline{m-3}, 5, \ldots, \underline{4}, m-2, \underline{2}, m, m+1, \ldots n] & \text{if, } m \text{ is odd} \end{cases}$$

Then $w = w_m u$ reduced for some $m \leq n$, where u is a product of mutually commuting generators such that $\text{supp}(u) \subset s_{m+2}, s_{m+3}, s_{m+4}, \ldots s_n$. Using the above notation, we can write w as

$$w_n = \begin{cases} [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n] & \text{if, } n \text{ is even} \\ [2,0][4,0]\cdots[m-2,0][m,0][m-k,m-2k]\cdots[m-1,m-2][m,m] & \text{if, } n \text{ is odd} \end{cases}$$

where $m = n - 1$ and
$$k = \begin{cases} \frac{n}{2} - 2 & \text{if, } n \text{ is even} \\ \frac{n-1}{2} - 2 & \text{if, } n \text{ is odd} \end{cases}$$
.

Proof. This is [8, Lemmas 2.2.18 and 2.3.4]. Although it is not immediately obvious w is reduced and not FC. □

In Figure 3.1, we see two different elements that are T-avoiding in $W(D_5)$. Notice that the blocks that are highlighted in red alternate, this prevents the teal highlighted braid from forcing its way to the top or the bottom of the heap.

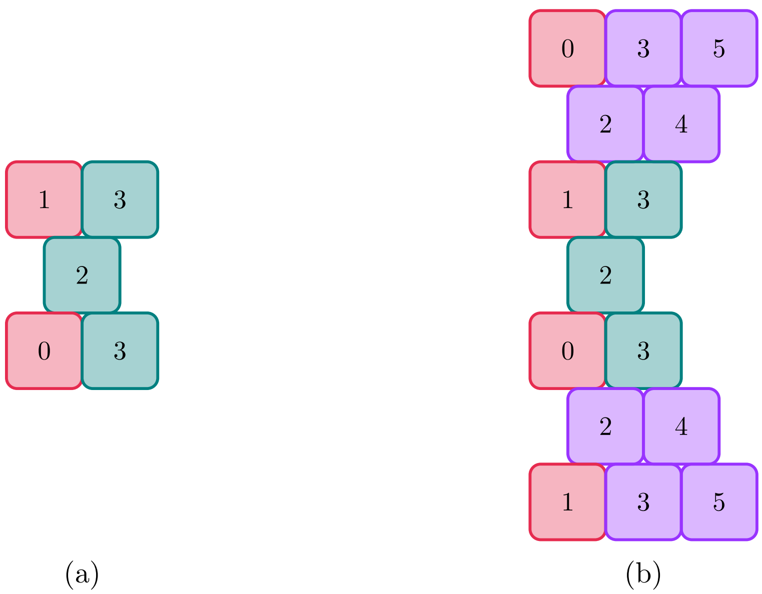
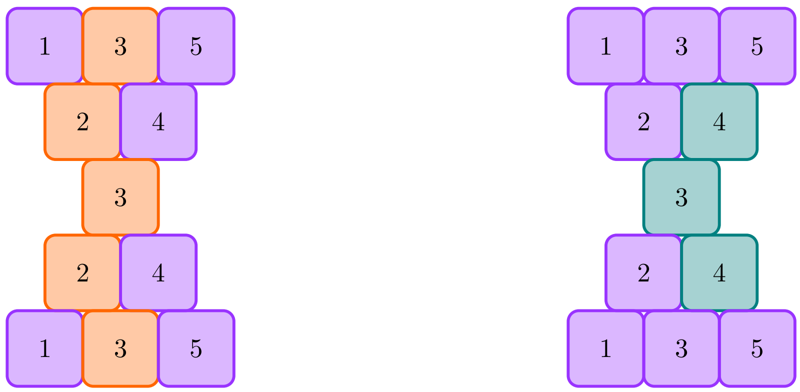


Figure 3.1: Visual representation of non-trivial T-avoiding elements in $W(D_5)$.

3.3 Type F_n

In this section we classify what is known regarding the T-avoiding elements in the Coxeter groups of type F_n for $n \geq 4$. Note that all of the following results are unpublished.

We start with the Coxeter system of type F_5. Recall that $W(F_5)$ is a star reducible Coxeter group so any non-trivial T-avoiding elements will not be FC. Before we begin the classification we introduce the notion of a specific element in $W(F_5)$ called a single *bowtie*, which is given by the heap in Figure 3.2. Note that in Figure 3.2(a), the orange blocks correspond to the elements that have bond strength 4. It turns out that the expression determined by the heap is in fact reduced, but it is not clear that it is not FC where we have highlighted the braid in teal in Figure 3.2(b). We can obtain a "stack of bowties" by removing the top most layer of the given heap for the bowtie and adding a new single bowtie to the stack, as seen in Figure 3.3. Similar to a single bowtie, the expression that corresponds to a stack of bowties is reduced and not FC. These heaps are referenced in the following unpublished theorem by Cross, Ernst, Hills-Kimball, and Quaranta in 2012 which classifies the T-avoiding elements in the Coxeter group of type F_5.



(a) (b)

Figure 3.2: Heap of a single bowtie in $W(F_5)$.

Theorem 3.3.1. The only non-trivial T-avoiding elements in $W(F_5)$ are stacks of bowties. □

As a result of the classification in type F_5, Cross et al. were also able to classify the T-avoiding elements in $W(F_4)$.

Corollary 3.3.2. There are no non-trivial T-avoiding elements in the Coxeter system of type F_4. □

Proof. Since there are no non-trivial T-avoiding elements in $W(F_5)$ that do not have full support, we know that there can not be any non-trivial T-avoiding elements in $W(F_4)$. Because if there were non-trivial T-avoiding elements they would also be non-trivially T-avoiding in $W(F_5)$. □

Cross et al. conjectured that in Coxeter systems of type F_n for $n \geq 5$, an element is non-trivially T-avoiding if and only if it is a stack of bowties multiplied by a product of commuting generators. In 2013, Gilbertson and Ernst worked with this conjecture

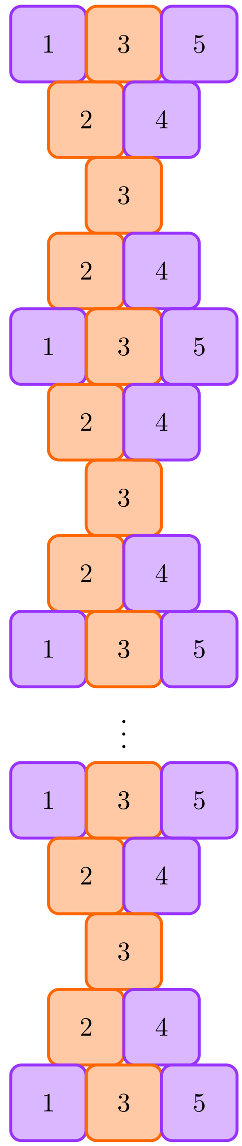


Figure 3.3: Heap of a stack of bowties in $W(F_5)$.

and quickly found it to be false. The heap seen in Figure 3.4 corresponds to a non-trivial T-avoiding element in F_6 that is not a bowtie. It turns out that like the bowties discussed above these elements can also be stacked to create an infinite number of non-trivial T-avoiding elements. In addition, as n gets large there are a number of things that can be altered that result in additional non-trivially T-avoiding elements. From this we conjecture that the classification of T-avoiding elements in Coxeter systems of type F_n for $n \geq 6$ gets complicated very quickly. Classifying T-avoiding elements in $W(F_n)$ for $n \geq 6$ remains an open problem.

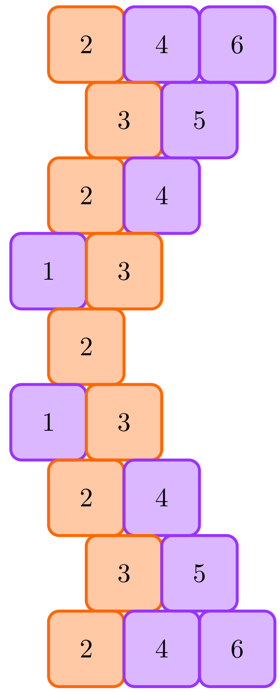


Figure 3.4: Heap of a non-trivial T-avoiding element in $W(F_6)$

3.4 Type $I_2(m)$

We next will classify the T-avoiding elements in Coxeter groups of type $I_2(m)$. Note that in Coxeter groups of type $I_2(m)$, the only products of commuting generators have length 1. Although the following is a quick result, we believe that it does not already appear in the literature.

Theorem 3.4.1. There are no non-trivial T-avoiding elements in the Coxeter system of type $I_2(m)$.

Proof. The graph for the Coxeter system of $I_2(m)$ appears in Figure 1.2(c). Note that the graph consists of two vertices, namely, s_1 and s_2, and a single edge with weight m. Also, recall that $W(I_2(m))$ is a star reducible Coxeter group. This implies that any non-trivial T-avoiding elements in $W(I_2(m))$ must not be FC, as all of the FC elements have Property T. The only non-FC element in $W(I_2(m))$ is the element of length m that has exactly two reduced expressions consisting of alternating products of s_1 and s_2. Cleary, this element begins and ends with a product of noncommuting generators. Thus, this element has Property T. Hence $W(I_2(m))$ has no non-trivial T-avoiding elements. $\square$

Chapter 4

T-Avoiding Elements in Types B_n

In this section we classify the T-Avoiding elements in Coxeter systems of type B_n. We start by introducing some combinatorial tools for type B_n and then finish with a proof of the classification in type B_n. Note that the proof for Coxeter systems of type B_n closely follows the classification of T-avoiding elements of type D_n seen in [8].

4.1 Tools for the Classification

Recall from Example 1.1.1 that $W(B_n) \cong \mathrm{Sym}_n^B$. This implies that we can represent each element $w \in W(B_n)$ as a signed permutation. That is we can write $w \in W(B_n)$ using one-line notation

$$w = [w(1), w(2), \dots, w(n-1), w(n)],$$

where we write a bar underneath a number in place of a negative sign in order to simplify notation. In particular, for each $s_i \in S(B_n)$, $i = 1, 2, \dots n-1$, we have

$$s_i = [1, 2, \dots i-1, i+1, i, i+2, \dots, n-1, n]$$

and we identify $s_0 \in S(B_n)$ via

$$s_0 = [\underline{1}, 2 \dots, n].$$

Further $w(-i) = -w(i)$ for $|i| \in \{1, 2, \dots, n\}$. The following propositions provide insight into what happens when we multiply a window notation by s_i on the right or the left.

Proposition 4.1.1. Let $w \in W(B_n)$ with corresponding signed permutation

$$w = [w(1), w(2), \dots w(n)].$$

Suppose $s_i \in S(B_n)$. If $i \geq 1$, then multiplying w on the right by s_i has the effect of interchanging $w(i)$ and $w(i+1)$.

If $i = 0$, then multiplying w on the right by s_i has the effect of switching the sign of $w(1)$.

Proof. This follows from [2, Section 8.1 and A3.1]. □

Proposition 4.1.2. Let $w \in W(B_n)$ with corresponding signed permutation

$$w = [w(1), w(2), \ldots w(n)].$$

Suppose $s_i \in S(B_n)$. If $i \geq 1$, then multiplying on the left by s_i has the effect of interchanging the entries whose absolute values are i and $i+1$.

If $i = 0$, then multiplying w on the left by s_i has the effect of switching the sign of the entry whose absolute value is 1.

Proof. This follows from [2, Section 8.1 and A3.1]. □

Suppose $w \in W(B_n)$ has reduced expression $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$. We construct the signed permutation of w from left to right as it is the easier way to multiply based upon the above propositions. We provide and example of this construction below.

Example 4.1.3. Let $w \in W(B_6)$ with a given reduced expression $\mathsf{w} = s_0 s_1 s_3 s_4 s_5 s_2$. Then we iteratively build the signed permutation as follows $s_0 = [-1, 2, 3, 4, 5, 6]$ by the definition of s_0. Next $s_0 s_1 = [2, -1, 3, 4, 5, 6]$, since multiplying by s_1 on the right hand side switches the element in position 1 and position 2. Repeating this we get $s_0 s_1 s_3 = [2, -1, 4, 3, 5, 6]$ and ultimately we end with $\mathsf{w} = s_0 s_1 s_3 s_4 s_5 s_2 = [2, 4, -1, 5, 6, 3]$. Notice that if we were to construct the signed permutation for w right to left, we would start with $s_2 = [1, 3, 2, 4, 5, 6]$ again this is just by definition. Next we would have $s_5 s_2 = [1, 3, 2, 4, 6, 5]$, which doesn't look much different from what we did above. However, $s_4 s_5 s_2 = [1, 3, 2, 5, 6, 4]$. Notice this time we were not able to just switch $w(i)$ and $w(i+1)$ instead we found 4 and 5 and switched their relative positions which is more difficult that constructing a window notation left to right, which is why we choose to construct left to right.

Given the one line notation for an element $w \in W(B_n)$ we can easily calculate the left and right descent sets of w. The following proposition explains how.

Proposition 4.1.4. Let $w \in W(B_n)$. Then

$$\mathcal{R}(w) = \{s_i \in S : w(i) > w(i+1)\}$$

where w(0)=0 by definition.

Proof. This is [2, Proposition 8.1.2]. □

We now will introduce the concept of signed pattern avoidance, which will help with the classification of the T-avoiding elements in Coxeter systems of type B_n. Our approach mimics the one found in [8]. Let $w \in W(B_n)$. We say that w *avoids the consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(w(i),\ w(i+1),\ w(i+2))$ is in the same relative order as (a, b, c), where $\{a, b, c\} = \{1, 2, 3\}$. We say that w *avoids the signed consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same consecutive order as $(|a|, |b|, |c|)$ and such that $\mathrm{sgn}(w(i)) = \mathrm{sgn}(a)$, $\mathrm{sgn}(w(i+1)) = \mathrm{sgn}(b)$, and $\mathrm{sgn}(w(i+2)) = \mathrm{sgn}(c)$.

Example 4.1.5. Let $w \in W(B_4)$ with signed permutation

$$w = [\underline{2}, 4, \underline{1}, 3].$$

We see that w has the signed consecutive pattern $\underline{2}3\underline{1}$, since $(|w(1)|, |w(2)|, |w(3)|)$ are in the same relative order as $(|-2|, |3|, |-1|)$, and $\mathrm{sgn}(w(1)) = \mathrm{sgn}(-2)$, $\mathrm{sgn}(w(2)) = \mathrm{sgn}(3)$, and $\mathrm{sgn}(w(3)) = \mathrm{sgn}(-1)$. However, w avoids the signed consecutive pattern $1\underline{2}3$.

Occasionally, we will need to factor a reduced expression w for $w \in W(B_n)$ in a specific manner. Let $I = \{s, t\}$ for $s, t \in S(B_n)$. Define w^I as the set of all $w \in W(B_n)$ such that $\mathcal{L}(w) \cap I = \emptyset$ and define $W_I = \langle s, t \rangle$. In [10], it is shown that any element $w \in W(B_n)$ can be written as $w = w_I w^I$ where $w_I \in W_I$ and $w^I \in W^I$.

4.2 Classification of T-Avoiding Elements in Type B_n

Time permitting we will streamline the corollaries into the lemma's themselves or as a remark at the end of the lemmas

In this section we will classify the T-avoiding elements in Coxeter groups of type B_n. Our main result in this section is Theorem 4.2.17. First we need some preparatory lemmas.

Lemma 4.2.1. Let $s, t \in S(B_n)$ such that $m(s, t) = 3$. Then w has a reduced expression ending in sts if and only if w has the consecutive pattern 321.

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$. Note that since $m(s_i, s_{i+1}) = 3$, $s_0 \notin I$. Observe that if w has a reduced expression ending in two non-commuting generators $s_i s_{i+1}$ or $s_{i+1} s_i$, then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+1)$. By Proposition 4.1.4 $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $W(B_n)$ is a simply laced Coxeter group, w ends in $s_i s_{i+1} s_{i+2}$.

Conversely, suppose w ends in $s_i s_{i+1} s_i$. This implies that either $w_I = s_i s_{i+1}$ or $w_I = s_{i+1} s_i$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by Proposition 4.1.4. Thus w has the consecutive pattern 321. Therefore, w has a reduced expression ending in sts if and only if w has the consecutive pattern 321. □

Corollary 4.2.2. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$. Then w has a reduced expression beginning with sts if and only if w^{-1} has the consecutive pattern 321.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s, t\}$. We know that w has no reduced expressions beginning with sts if and only if w^{-1} has no reduced expression ending with sts which by Theorem 4.2.2 happens only if w^{-1} avoids the consecutive pattern 321. □

Lemma 4.2.3. If $i \neq 0$, then w has a reduced expression ending in $s_i s_{i+1}$ if and only if w has the consecutive pattern 231.

Proof. Suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$. By Proposition 4.1.4 $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by s_{i+1} we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, this implies that $s_i \in \mathcal{R}(ws_{i+1})$. This implies w has a reduced expression that ends in $s_i s_{i+1}$.

Conversely, suppose that w has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence w has the consecutive pattern 231. Therefore, w has a reduced expression ending in st if and only if w has the consecutive pattern 231. □

Corollary 4.2.4. If $i \neq 0$, then w has a reduced expression beginning with $s_i s_{i+1}$ if and only if w^{-1} has the consecutive pattern 231.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s, t\}$. We know that w has no reduced expressions beginning with st if and only if w^{-1} has no reduced expression ending with st which by Theorem 4.2.2 happens only if w^{-1} avoids the consecutive pattern 231. □

Lemma 4.2.5. If $i \neq 0$, then w has a reduced expression ending in $s_{i+1} s_i$ if and only if w has the consecutive pattern 312.

Proof. Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. By Proposition 4.1.4 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by s_i we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$.

By above $w(i) > w(i+2)$, and by Proposition 4.1.4 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that w has a reduced expression ending in $s_{i+1}s_i$.

Conversely suppose w ends in a reduced expression with $s_{i+1}s_i$ Then $w_I = s_{i+1}s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, w has the consecutive pattern 312. Therefore, w has a reduced expression ending in ts if and only if w has the consecutive pattern 312. □

Corollary 4.2.6. If $i \neq 0$, then w has a reduced expression beginning with $s_{i+1}s_i$ if and only if w^{-1} has the consecutive pattern 312.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s, t\}$. We know that w has no reduced expressions beginning with ts if and only if w^{-1} has no reduced expression ending with ts which by Theorem 4.2.2 happens only if w^{-1} avoids the consecutive pattern 312. □

Lemma 4.2.7. Let $w \in W(B_n)$. Then w has a reduced expression ending in s_1s_0 if and only if $w(0) > w(1)$ and $-w(1) > w(2)$.

Proof. Suppose $w \in W(B_n)$ such that w ends with s_1s_0. Then $s_0 \in \mathcal{R}(w)$ and $s_1 \in \mathcal{R}(ws_0)$. This implies that $ws_0(1) > ws_0(2)$ by Proposition 4.1.4. We see that $ws_0(1) = w(-1) = -w(1)$ and $ws_0(2) = 2$. Hence $-w(1) = ws_0(1) > ws_0(2) = w(2)$. Further, since $s_0 \in \mathcal{R}(w)$, we see that $w(0) > w(1)$.

Conversely, suppose $w \in W(B_n)$ such that $w(0) > w(1)$ and $-w(1) > w(2)$. Since $w(0) > w(1)$ so $s_0 \in \mathcal{R}(w)$. Multiplying on the right by s_0 we see that $ws_0(1) = -w(1)$ and $ws_0(2) = w(2)$. Note that since $ws_0(1) = -w(1) > w(2) = ws_0(2)$, $s_1 \in \mathcal{R}(ws_0)$. Thus w ends with s_1s_0. Therefore, w has a reduced expression ending in s_1s_0 if and only if $w(0) > w(1)$ and $-w(1) > w(2)$. □

Corollary 4.2.8. Let $w \in W(B_n)$. Then w has a reduced expression beginning in s_0s_1 if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$.

Proof. Let $w \in W(B_n)$. We know that w has no reduced expressions beginning in s_0s_1 if and only if w^{-1} has no reduced expressions ending in s_0s_1. By Lemma 4.2.7 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$. □

Lemma 4.2.9. Let $w \in W(B_n)$. Then w has a reduced expression ending in s_0s_1 if and only if $w(0) > w(2)$ and $w(1) > w(2)$.

Proof. Suppose $w \in W(B_n)$ such that w ends with s_0s_1. Then $s_1 \in \mathcal{R}(w)$ and $s_0 \in \mathcal{R}(ws_1)$. Then $ws_1(0) > ws_1(1)$. We see that $ws_1(0) = 0$ and $ws_1(1) = w(2)$.

This implies that $0 = ws_1(0) > ws_1(1) = 2$. Further, since $s_1 \in \mathcal{R}(w)$ this implies that $w(1) > w(2)$. Thus if w ends with s_0s_1, then $w(1) > w(2)$ and $w(0) > w(2)$.

Conversely, suppose $w \in W(B_n)$ such that $w(1) > w(2)$ and $w(0) > w(2)$. This implies that $s_1 \in \mathcal{R}(W)$. Multiplying w on the right by s_1 we see that $ws_1(0) = w(0)$ and $ws_1(1) = w(2)$. Note that since $ws_1(0) = w(0) > w(2) = ws_1(1)$, $s_0 \in \mathcal{R}(ws_1)$. Thus w ends with s_0s_1. Therefore, w has a reduced expression ending in s_0s_1 if and only if $w(1) > w(2)$ and $w(0) > w(2)$. □

Corollary 4.2.10. Let $w \in W(B_n)$. Then w has a reduced expression beginning in s_1s_0 if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$.

Proof. Let $w \in W(B_n)$. We know that w has no reduced expressions beginning in s_1s_0 if and only if w^{-1} has no reduced expressions ending in s_1s_0. By Lemma 4.2.7 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$. □

Lemma 4.2.11. Let $w \in W(B_n)$ such that each entry for w in the one-line notation is positive and both w and w^{-1} avoid the consecutive patterns 321, 231, and 312, then w is a product of commuting generators.

Proof. This follows immediately from an appropriate translation of [8, Lemma 2.2.9]. □

Lemma 4.2.12. Let $w \in W(B_n)$ be trivially T-avoiding and let $i \in \{1, 2, \ldots, n\}$. Then w satisfies the following conditions:

(1) $w(j) > \min\{w(i-1), w(i)\}$ for all $j > i$;

(2) $w(k) < \max\{w(i-1), w(i)\}$ for all $k < i-1$;

(3) If $w(i), w(i+1) > 0$, then $w(j) > 0$ for all $j \geq i$;

(4) If $w(i), w(i+1) < 0$, then $w(j) < 0$ for all $j \leq i+1$.

Proof. Suppose there is some least $j > i$ such that $w(j) \leq \min\{w(i-1), w(i)\}$. Note that $j > i$ so $j \neq i$, and $j \neq i-1$ so $w(j) < \min\{w(i-1), w(i)\}$). Note that j is the least so $w(j-2) \geq \min\{w(i+1), w(i)\} > w(j)$. This implies that either $w(j-1) > w(j-2) > w(j)$ or $w(j-2) > w(j-1) > w(j)$, which implies w has the consecutive pattern 231 or 321 which is a contradiction to w being a non-trivial T-avoiding element by Lemmas 4.2.1 and 4.2.5. Thus proving (1).

Suppose there exists a maximal $k < i-1$ such that $w \geq \max\{w(i-1), w(i)\}$. Note that $k < i-1$ so $k \neq i$ and $k \neq i-1$. Then $w(k) > \max\{w(i-1), w(i)\}$. Since k is maximal then $w(k+1) \leq \max\{w(i-1), w(i)\}$ and $w(k+2) \leq \max\{w(i-1), w(i)\}$. This implies that either $w(k+2) < w(k+1) < w(k)$ or $w(k+1) < w(k+2) < w(k)$, which implies w has the consecutive pattern 321 or 312 which is a contradiction to w being a non-trivial T-avoiding element by Lemmas 4.2.1 and 4.2.3. Thus proving (2).

It is easy to see that assertion (1) implies (3) and assertion (2) implies (4). $\square$

Lemma 4.2.13. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}31$. Then w has Property T.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}31$.

Case (1): Suppose w has the one-line notation $w = [\underline{2}, 3, 1]$. This implies that $w = s_1 s_0 s_2$. Clearly, w begins with a product of non-commuting generators. Thus w has Property T.

Case (2): Suppose that w has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$, and $*$ indicates $w(i)$ for $i = 4, 5, \ldots n$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: 31 ± 2, 32 ± 1, or 21 ± 3. We know that b, c must be positive since they are positive in w and we also know that $b > c$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$, and $*$ indicates $w(i)$ for $i = 1, 2, \ldots, n-3$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 1\underline{2}3$, $\pm 2\underline{1}3$, and $\pm 3\underline{1}2$. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$, and $*$ indicates $w(i)$ for $|w(i)| \neq a, b, c$. Notice that w is a combination of Case (2) and Case (3). As a result we know that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}31$, then w has Property T. $\square$

Lemma 4.2.14. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}3\underline{1}$. Then w has Property T.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}3\underline{1}$.

Case (1): Suppose w has the one-line notation $w = [\underline{2}, 3, \underline{1}]$. This implies that $w = s_0 s_1 s_0 s_2$. Clearly, w begins with a product of non-commuting generators. Thus w has Property T.

Case (2): Suppose that w has the one-line notation $w = [\underline{a}, b, \underline{c}, *, \ldots, *]$ where $\underline{a}, b, \underline{c}$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$, and $*$ indicates $w(i)$ for $i = 4, 5, \ldots n$. We now consider the signed consecutive pattern that can arise involving $b, \underline{c}, *$. The following are the possibilities for the signed consecutive pattern that can arise: $3\underline{1} \pm 2$, $3\underline{2} \pm 1$, or $2\underline{1} \pm 3$. We know that b must be positive since it is positive in w, c must be negative since it is negative in w, and we also know that $|b| > |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, \underline{c}]$ where $\underline{a}, b, \underline{c}$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$, and $*$ indicates $w(i)$ for $i = 1, 2, \ldots, n-3$.. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 1\underline{2}3$, $\pm 2\underline{1}3$, and $\pm 3\underline{1}2$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, \underline{c}, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$, and $*$ indicates $w(i)$ for $|w(i)| \neq a, b, c$. Notice that w is a combination of Case (2) and Case (3). As a result we know that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}3\underline{1}$, then w has Property T. $\square$

Lemma 4.2.15. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}23$. Then w has Property T or is a trivial T-avoiding element.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}23$.

Case (1): Suppose w has the one-line notation $w = [\underline{1}23]$. This implies that $w = s_0$. Clearly, w is a trivial T-avoiding element as it is a single generator.

Case (2): Suppose that w has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$, and $*$ indicates $w(i)$ for $i = 4, 5, \ldots n$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: 23 ± 1, 13 ± 2, or 12 ± 3. We know that b, c, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$, and $*$ indicates $w(i)$ for

$i = 1, 2, \ldots, n-3$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 3\underline{1}2$, $\pm 2\underline{1}3$, and $\pm 1\underline{2}3$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$, and $*$ indicates $w(i)$ for $|w(i)| \neq a, b, c$. Notice that w is a combination of Case (2) and Case (3). As a result we know that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}23$, then w has Property T or is a trivial T-avoiding element. □

Lemma 4.2.16. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}32$. Then w has Property T or is a trivial T-avoiding element.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}32$.

Case (1): Suppose w has the one-line notation $w = [\underline{1}32]$. This implies that $w = s_0 s_2$. Clearly, w is a trivial T-avoiding element as it is a single generator.

Case (2): Suppose that w has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 3, 2$, and $*$ indicates $w(i)$ for $i = 4, 5, \ldots n$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: 23 ± 1, 13 ± 2, or 12 ± 3. We know that b, c, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 3, 2$, and $*$ indicates $w(i)$ for $i = 1, 2, \ldots, n-3$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 3\underline{1}2$, $\pm 2\underline{1}3$, and $\pm 3\underline{2}1$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 3, 2$, and $*$ indicates $w(i)$ for $w(i) \neq a, b, c$. Notice that w is a combination of Case (2) and Case (3). As a result we know that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}32$, then w has Property T or is a trivial T-avoiding element. $\square$

Theorem 4.2.17. There are no non-trivial T-avoiding elements in $W(B_n)$.

Proof. Suppose that $w \in W(B_n)$ is a non-trivial T-avoiding element. There are $2^3 \cdot 3!$ possible choices of signed consecutive patterns for $w(1)w(2)w(3)$ where $w = [w(1), w(2), w(3), *, \ldots, *]$.

123	$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$	$\underline{12}3$	$\underline{1}2\underline{3}$	$1\underline{23}$	$\underline{123}$
132	$\underline{1}32$	$1\underline{3}2$	$13\underline{2}$	$\underline{13}2$	$\underline{1}3\underline{2}$	$1\underline{32}$	$\underline{132}$
213	$\underline{2}13$	$2\underline{1}3$	$21\underline{3}$	$\underline{21}3$	$\underline{2}1\underline{3}$	$2\underline{13}$	$\underline{213}$
231	$\underline{2}31$	$2\underline{3}1$	$23\underline{1}$	$\underline{23}1$	$\underline{2}3\underline{1}$	$2\underline{31}$	$\underline{231}$
312	$\underline{3}12$	$3\underline{1}2$	$31\underline{2}$	$\underline{31}2$	$\underline{3}1\underline{2}$	$3\underline{12}$	$\underline{312}$
321	$\underline{3}21$	$3\underline{2}1$	$32\underline{1}$	$\underline{32}1$	$\underline{3}2\underline{1}$	$3\underline{21}$	$\underline{321}$

We can use Lemma 4.2.1 and Corollary 4.2.2 to eliminate the signed consecutive patterns highlighted in turquoise. We can use Lemma 4.2.5 and Corollary 4.2.4 to eliminate the signed consecutive patterns highlighted in red. We can use Lemma 4.2.3 and Corollary 4.2.6 to eliminate the consecutive patterns highlighted in green. We can use Lemma 4.2.7 and Corollary 4.2.8 to eliminate the signed consecutive patterns highlighted in yellow. We can use Lemma 4.2.9 and Corollary 4.2.10 to eliminate signed consecutive patterns highlighted in brown. We can use Lemma 4.2.11 to eliminate the signed consecutive patterns highlighted in blue. We can use Lemmas 4.2.13 and 4.2.14 to eliminate signed consecutive patterns highlighted in purple. Finally we can use Lemmas 4.2.15 and 4.2.16 to eliminate signed consecutive patterns highlighted in orange. Since all of the above patterns are eliminated as possibilities for $w(1)w(2)w(3)$ and there are no other signed consecutive patterns that are possible for these positions, w can not be a non-trivial T-avoiding element in the Coxeter group of type B. Therefore, there are no non-trivial T-avoiding elements in $W(B_n)$. $\square$

The upshot of Theorem 4.2.17 is that the only T-avoiding elements in Coxeter systems of Type B_n are products of commuting generators.

Chapter 5

T-Avoiding Elements in Type $\widetilde{C}_n$

5.1 Classification of T-Avoiding Elements in $W(\widetilde{C}_n)$

In this section we will classify the T-avoiding elements in Coxeter groups of type $\widetilde{C}_n$. Because, $W(A_n)$ and $W(B_n)$ are parabolic subgroups of $W(\widetilde{C}_n)$, this implies that if $W(\widetilde{C}_n)$ is to have any non-trivial T-avoiding elements they will have full support, because if they did not the problem is reduced to a cross product of $W(A_n)$ and $W(B_n)$ in some way. We will first show that there are no non-trivial T-avoiding elements that are not FC in $W(\widetilde{C}_n)$.

Theorem 5.1.1. There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$.

Proof. Let w be a reduced expression in $W(\widetilde{C}_n)$ such that w has full support and w does not have Property T. Consider all possible heaps for w and choose a heap a bottom most braid. That is, choose a heap where the braid is as low as possible in the heap, which means the generators below the braid are FC.

Case 1: Suppose the braid does not contain $0, 1$ or$n-1, n$. Subcase a: Suppose w has the fixed reduced product $w = us_ks_{k-1}s_ks_{k-1}s_{k-2}v$ or $w = us_ks_{k-1}s_ks_{k+1}s_{k-1}s_{k-2}v$ where v is fully commutative, and the braid is highlighted in teal. Applying the braid move we obtain the element $w = us_{k-1}s_ks_{k-1}s_{k-2}s_{k-1}v$ or $w = us_{k-1}s_ks_{k-1}s_{k-2}s_{k-1}s_{k+1}v$. Notice that the braid is now located next to v having moved closer to the bottom in the heap. This is a contradiction to choosing a heap with the lowest braid. Therefore w does not have the fixed reduced product $w = us_ks_{k-1}s_ks_{k-1}s_{k-2}v$ or $w = us_ks_{k-1}s_ks_{k+1}s_{k-1}s_{k-2}v$. Visually we see this in Figure 5.1, where s_{k+1} is represented as a striped block. Notice how there are two braids located in Figure 5.1(b), the braid that starts in purple and ends with teal and the braid that is fully highlighted in teal.

Subcase b: Suppose w has the fixed reduced product $w = us_ks_{k-1}s_ks_{k+1}v$ where v is FC and does not contain s_{k-2} and s_{k-1} in the left descent set. Again we have

(a) (b)

Figure 5.1: Visual representation of the heap configuration discussed in Case 1a.

highlighted the braid in teal. Applying the braid move we obtain an new fixed product $w = us_{k-1}s_ks_{k-1}s_{k+1}v$. Notice that the braid is now able to be written next to v whereas it previously was not. This again contradicts choosing a heap with the braid in the lowest location. Visually we see this in Figure 5.2. Notice how in Figure 5.2(b) the braid is located next to the block for s_{k+1} whereas in Figure 5.2(a) the braid is below the block for s_{k+1}.

(a) (b)

Figure 5.2: Visual representation of the heap configuration discussed in Case 1b.

Case 2: Suppose the braid contains 2 or $n-2$. Without loss of generality we will take the braid to contain 2 the other argument is symmetric to the one presented here. Notice that if the braid is of the form $s_2s_3s_2$ we are in the case above as a result we assume that the braid we refer to in the following subcases do not involve $s_2s_3s_2$ to start. Subcase a: Suppose w has the fixed reduced product $w = us_2s_1s_2s_0s_1s_0v$, where v is FC and does not contain s_2 in the left descent set. Again we highlight the braid for emphasis in teal. Applying the braid move we obtain the reduced product

$w = us_1s_2s_1s_0s_1s_0v$. Notice that the braid is now able to touch v as it wasn't before. This contradicts our original choice of heap and as a result we can not choose the reduced product $w = us_2s_1s_2s_0s_1s_0v$. Visually this is seen Figure 5.3. Notice how there are two braids located in Figure 5.3(b). The braid highlighted in orange did not appear in our original heap seen in Figure 5.3(a) and is lower in the heap than the original.

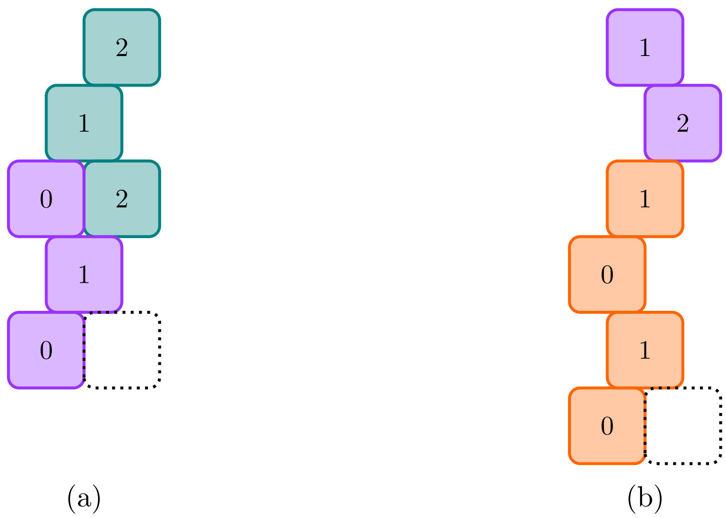


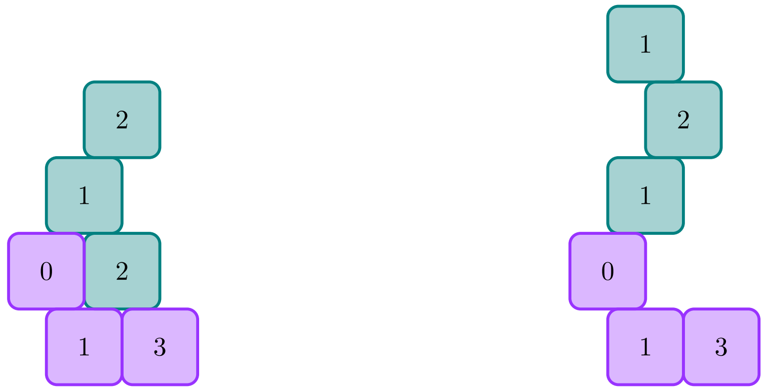
Figure 5.3: Visual representation of the heap configuration discussed in Case 2a.

Subcase b: Suppose w has the fixed reduced product $w = us_2s_1s_2s_0s_1s_3s_2v$ where v is FC. Again we have highlighted the braid in teal. Applying the braid move we end up with the reduced product $w = us_1s_2s_1s_0s_1s_3s_2v$. Notice this time the braid does not force a higher braid. Visually this appears in Figure 5.4. In Figures 5.4(a) and 5.4(b) we see that the braid actually moves higher in the heap.

Since we assumed that w does not have Property T, we know that u in the fixed reduced product that we have for w is non-trivial. That is, it contains some generators. Given our original reduced fixed expression for w we add a new row to our heap if we were to add s_0 to the new row we would have a braid appear higher in the heap so we will not add s_0. This forces us to add s_2 and we get the configuration seen here

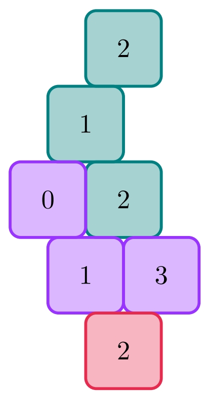
Again this can not be the top row in our heap so we must add another level in our heap. Notice that adding s_1 would create a braid in our heap so we will add s_3 however in doing so we will also need to add s_4. The resulting heap is seen here

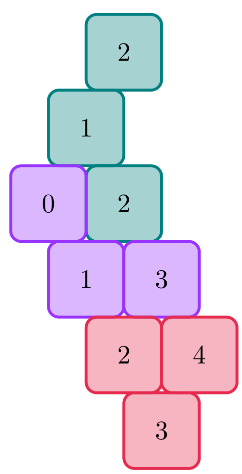
We once again have the same issue arise that this can not be the top level of our configuration as w would clearly have Property T on the top. Iterating this process



(a) (b)

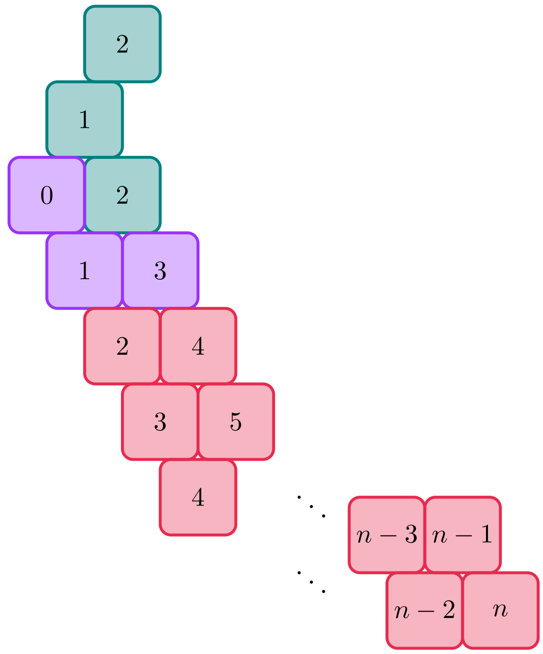
Figure 5.4: Visual representation of the heap configuration discussed in Case 2b.





we create the heap seen here

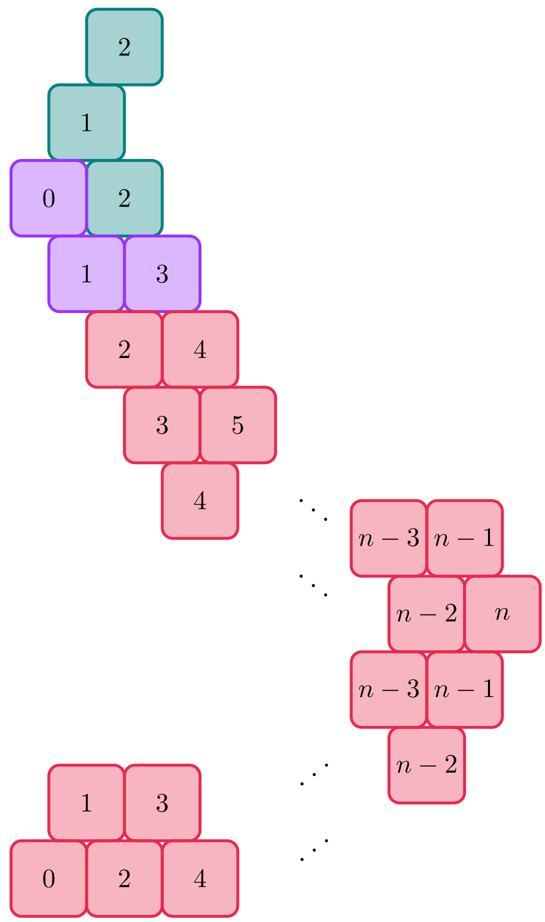
Notice that again if this were the bottom row of the heap we would have Property



T. Thus our construction can not be done. With this in mind we add s_{n-1} to the heap which will still have Property T. As a result we play this game again and create the heap seen here.

Recall that v is FC by assumption, where v is the red in the above heap. In [4, Lemma 3.3] Ernst classified that an FC element of this sort has the blank space in the middle filled in. This forces our heap to look like the one seen in Figure 5.5(a) where all of the blocks in the middle of the red v are filled in. As a result of this we now have s_0 in our heap. After applying the braid move to the teal braid in Figure 5.5(a). This leads to the heap seen in Figure 5.5(b) where a new orange braid appeared. This implies that for the fixed reduced product $w = us_2s_1s_2s_0s_1s_3s_2v$ there is a heap with a braid that is lower in the heap. A contradiction to the way in which we chose w. Thus w can not have the reduced expression $w = us_2s_1s_2s_0s_1s_3s_2v$.

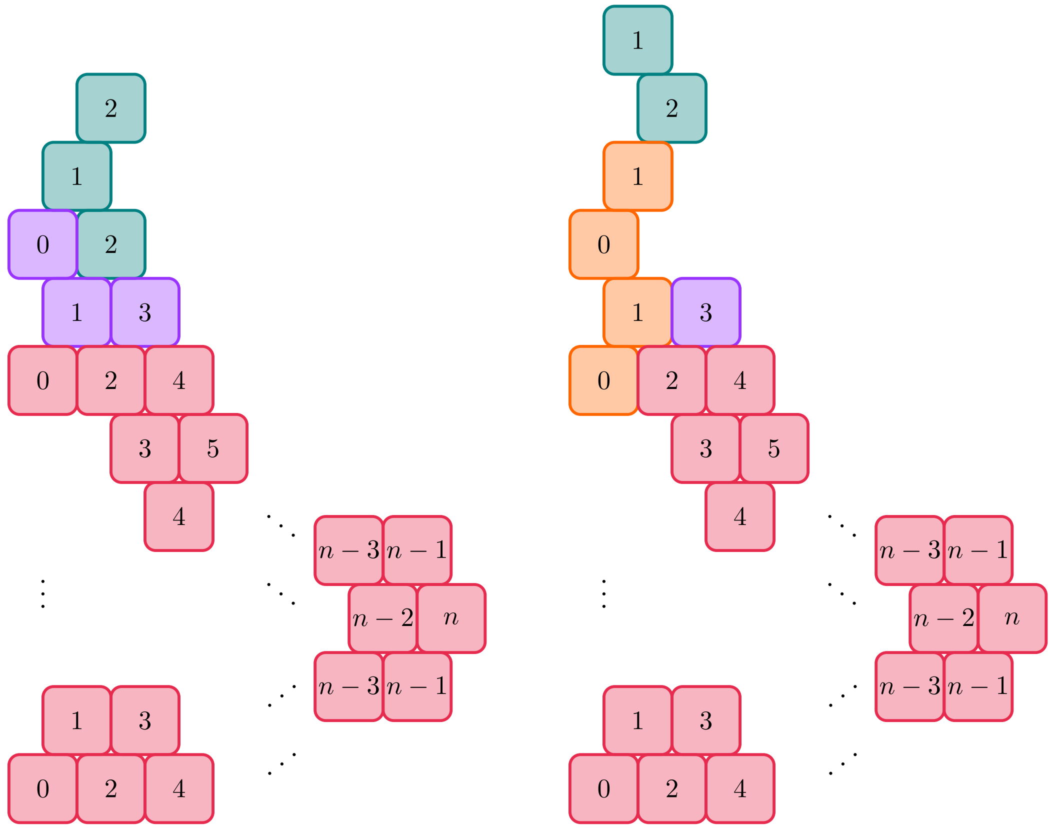
Case 3: Suppose the braid contains 1 or $n-1$. Without loss of generality we will assume the braid contains 1 as the other argument is symmetric to the one presented here. Subcase a: Suppose w has the reduced product $w = us_0s_1s_0s_1s_2v$ where v is fully commutative and does not contain s_0 in the left descent set. Notice that the braid is highlighted in orange. Applying the braid move leads to the reduced product $w = us_1s_0s_1s_0s_2v$. Notice that the braid is now able to be in the same level of the heap as s_2 whereas it previously was not. Visually this is seen in Figure 5.6. Notice how the braid in Figure 5.6(b) is located next to the block for s_2 whereas in Figure 5.6(a) the braid is stuck above the block for s_2. This is a contradiction to picking the heap



with the lowest braid.

Subcase b: Suppose w has the reduced product $w = us_1s_2s_1s_0v$ where v is fully commutative and does not contain s_2 in the left descent set. Notice that the braid is highlighted in teal. Applying the braid move leads to the reduced product $w = us_2s_1s_2s_0v$. Notice that the braid is now able to be located in the same level of the heap as s_0 whereas it previously was not. Visually this is seen in Figure 5.7. Notice how the braid in Figure 5.7(b) is located next to the block for s_0, but the braid in Figure 5.7(a) it is stuck above the block for s_0. This is a contradiction to the way in which we picked our heap. Thus w can not have the reduced product $w = us_1s_2s_1s_0v$.

Case 4: Suppose the braid contains 0 or n. Without loss of generality we will assume the braid contains 0 as the other argument is symmetric to the one presented here. Suppose w has the fixed reduced product $w = us_1s_0s_1s_0s_2s_1v$ where v is an FC element. Notice that we have highlighted the braid in orange. Applying the braid move we obtain the fixed reduced product $w = us_0s_1s_0s_1s_2s_1v$. In applying the braid the resulting reduced product now has the braid highlighted in teal. Notice that this



(a) (b)

(a) (b)

Figure 5.6: Visual representation of the heap configuration discussed in Case 3a.

braid is located next to v which is located lower in the heap than our original w. Visually we see this in Figure 5.8. We see in Figure 5.8(b) the braid in teal is located below the braid that stars in orange and ends with the s_1. It is clear that this braid is lower than the orange braid seen in Figure 5.8(a). Thus w can not have the reduced

(a) (b)

Figure 5.7: Visual representation of the heap configuration discussed in Case 3b.

(a) (b)

Figure 5.8: Visual representation of the heap configuration discussed in Case 4.

product $w = us_1s_0s_1s_0s_2s_1v$.

From this we see there is no possible way to find a reduced expression in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$ with full support and does not have Property T. Thus there are no non-trivial T-avoiding elements in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$. $\square$

We have now shown that there are no non-trivial T-avoiding elements in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$. We now proceed in a parity argument. We first will classify non-trivial T-avoiding elements in $W(\widetilde{C}_n)$ for n odd. First recall that $W(\widetilde{C}_n)$ for n odd is a Star reducible Coxeter group. This implies that there are no non-trivial T-avoiding FC elements in $W(\widetilde{C}_n)$ for n odd. This leads to the following Theorem.

Theorem 5.1.2. There are no non-trivial T-avoiding elements in the Coxeter group of type $\widetilde{C}_n$ for n odd.

Proof. Consider the Coxeter group of type $\widetilde{C}_n$. By Theorem 5.1.1 we know that $W(\widetilde{C}_n)$ contains no non-trivial T-avoiding elements that are not FC. Also since $W(\widetilde{C}_n)$

is a star reducible Coxeter group, we know that $W(\widetilde{C}_n)$ contains no non-trivial T-avoiding elements that are FC. Thus $W(\widetilde{C}_n)$ has no non-trivial T-avoiding elements. □

We next will classify the non-trivial T-avoiding elements in the Coxeter group of type $\widetilde{C}_n$ for n even. Recall that $W(\widetilde{C}_n)$ for n even is not a Star reducible Coxeter group. In Theorem 5.1.1 we showed that $W(\widetilde{C}_n)$ does not have non-trivial T-avoiding elements that are not FC. This leaves us with only the FC elements to check.

Theorem 5.1.3. The only non-trivial T-avoiding elements in $W(\widetilde{C}_n)$ for n odd are stacks of sandwich stacks.

Proof. Let $w \in W(\widetilde{C}_n)$. By Theorem 5.1.1, we know that w is an FC element. We can restrict our search down to non-cancellable elements as they are not star reducible. In Section 2.3 we classified the only non-cancellable element with full support is a stack of sandwich stacks. Thus the only non-trivial T-avoiding elements in $W(\widetilde{C}_n)$ for n odd are stacks of sandwich stacks. □

5.2 Future Work

As there were open problems interspersed all through out we restate them here and add a few more open problems. We have shown the classification of non-trivial T-avoiding elements in Coxeter groups of Type $\widetilde{A}_n, A_n, D_n, F_4, F_5$, and $I_2(m)$. The classification of non-trivial T-avoiding elements of $W(F_n)$ for $n \geq 6$.

We also mentioned several other groups in Figures 1.2 and 1.3. The classification of non-trivial T-avoiding elements in the Coxeter groups of type E_n. We do know that if these groups were to have non-trivial T-avoiding elements they would have to have full support as $W(B_n)$ is a parabolic subgroup of $W(E_n)$. The classification of non-trivial T-avoiding elements in the Coxeter groups of type H_n. Again we know that if this group is to have non-trivial T-avoiding elements they will have full support as $W(A_n)$ is a parabolic subgroup of $W(H_n)$.

A majority of the irreducible affine Coxeter systems are also open for classification. Specifically, Coxeter systems of type $\widetilde{B}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, and $\widetilde{G}_4$ do not have a classification. Similar to those above we know already that some of them have non-trivial T-avoiding elements ($W(\widetilde{D}_n)$) as parabolic subgroups of them have non-trivial T-avoiding elements but it remains an open question if there are anymore. Future work could include classifying the non-trivial T-avoiding elements of the Coxeter systems mentioned above.

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