A STUDY OF T-AVOIDING ELEMENTS OF COXETER GROUPS

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Chapter 1

Preliminaries

1.1 Introduction

In mathematics, one uses groups to study symmetry. In particular, a reflection group can be used to study the reflection and rotational symmetry of an object. A Coxeter group can be thought of as a generalized reflection group, where the group is generated by a set of elements of order two (i.e., reflections) and there are rules for how the generators interact with each other. Every element of a Coxeter group can be written as an expression in the generators, and if the number of generators in an expression (including multiplicity) is minimal, we say that the expression is reduced. Motivated by the desire to understand the Kazhdan–Lusztig theory of the Hecke algebra of the underlying Coxeter group, Green [9] classified the so-called star reducible Coxeter groups which have the property that all fully commutative elements (in the sense of Stembridge) can be sequentially reduced via star operations to a product of commuting generators. It turns out that in some Coxeter groups there are elements, called T-avoiding elements, which cannot be systematically dismantled in the way described above. More specifically an element w is called *T-avoiding* if w does not have a reduced expression beginning or ending with a pair of non-commuting generators. Clearly, a product of commuting generators is trivially T-avoiding. However, sometimes there are more interesting T-avoiding elements, which we will refer to as type II T-avoiding elements. Our motivation for studying the T-avoiding elements stems from the fact that computations involving the elements of the generalized Temperly–Lieb algebra for W that are indexed by T-avoiding elements is "well-behaved." In fact, knowing which elements correspond to T-avoiding elements often provides us with the base case for inductive arguments involving star operations.

In his PhD thesis [8], Gern classified the T-avoiding elements in Coxeter groups of type D_n. Unlike in types A_n and B_n, it turns out that the classification in type D_n includes non-trivial T-avoiding elements. The T-avoiding elements are rich in combinatorics and are interesting in their own right. The focus of this thesis is identifying T-avoiding elements in certain Coxeter groups.

This thesis is organized as follows. After necessary background information is presented in Section 1.2, we introduce the class of fully commutative elements in Section 1.3. Next in Section 1.3 we discuss a visual representation for elements of Coxeter groups, called heaps. In Section 2.1, we introduce the concept of a star reduction and star reducible Coxeter groups and in Section 2.2 we formally introduce the notion of a T-avoiding element. In Section 2.3 we define the notion of a non-cancellable element in Coxeter groups, as well as remark upon a specific family of non-cancellable elements in $W(\widetilde{C}_n)$. We then state classifications and conjectures regarding T-avoiding elements in several Coxeter groups in Chapter 3. All of these results, barring Section 3.4, are previously known. Chapters 4 and 5 contain the main results of this thesis, namely the classification of T-avoiding elements in Coxeter groups of types B_n and $\widetilde{C}_n$, which are new results. In Section 4.1, we introduce the necessary lemmas and definitions for the classification in Section 4.2, in which we show there are no type II T-avoiding elements in Coxeter groups of type B_n. In Section 5.1, we classify the type II T-avoiding elements in Coxeter groups of type $\widetilde{C}_n$. We conclude with some open questions in Section 5.2.

1.2 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s) \geq 2$ for $s \neq t$. If there is no relation between $s, t \in S$, then we define $m(s,t) = \infty$. However, in this thesis we assume that all $m(s,t)$ are finite. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [10]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

(1) vertex set S and

(2) edges $\{s, t\}$ for each $m(s, t) \geq 3$.

Each edge $\{s, t\}$ is labeled with its corresponding bond strength. Since $m(s, t) = 3$ occurs frequently, it is customary to omit this label. Note that s and t are not connected by an edge in the graph if and only if $m(s, t) = 2$. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. That is, given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. If (W, S) is a Coxeter system with corresponding Coxeter graph Γ, we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. Also, the Coxeter system (W, S) is said to be *irreducible* if and only if Γ is connected. If the graph Γ is disconnected, the connected components correspond to factors in a direct product of the corresponding Coxeter groups [10]. The Coxeter graphs given in Figure 1.1 correspond to the Coxeter systems that will be primarily addressed in this thesis. Notice that the vertices are labeled with the corresponding generators to provide context when talking about the different generating sets $S(\Gamma)$.

Example 1.2.1.

(a) The Coxeter system of type A_n is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group $W(A_n)$ with generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

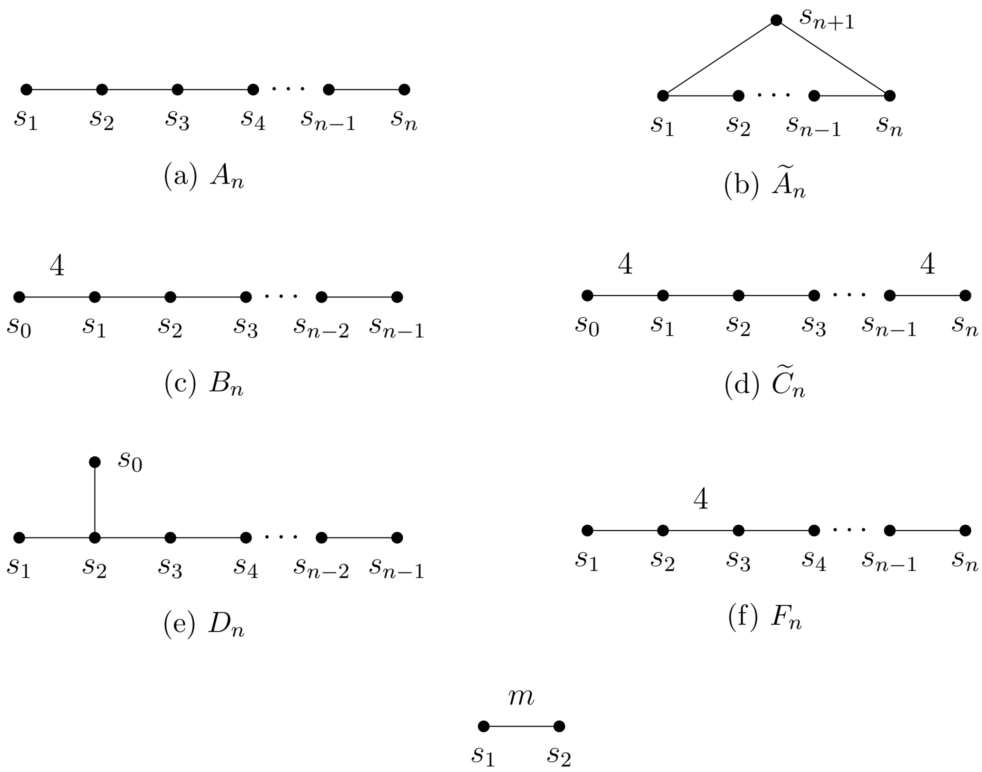
The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} under the correspondence $s_i \mapsto (i, i+1)$, where $(i, i+1)$ is the adjacent transposition that swaps i and $i + 1$.

(b) The Coxeter system of type B_n is given by the graph in Figure 1.1(c). We can construct the corresponding Coxeter group $W(B_n)$ with generating set $S(B_n) = \{s_0, s_1, \ldots, s_{n-1}\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \ldots, n - 1\}$;



(a) A_n (b) $\widetilde{A}_n$ (c) B_n (d) $\widetilde{C}_n$ (e) D_n (f) F_n (g) $I_2(m)$

Figure 1.1: Examples of a few named Coxeter graphs.

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to the group, Sym_n^B, of signed permutations on the set $\{1, 2, \ldots, n\}$. We discuss Sym_n^B in more detail in Section 4.1.

(c) The Coxeter system of type $\widetilde{C}_n$ is given by the graph in Figure 1.1(d). We can construct the corresponding Coxeter group $W(\widetilde{C}_n)$ with generating set $S(\widetilde{C}_n) = \{s_0, s_1, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$ for $i \in \{0, 2, \ldots, n\}$;

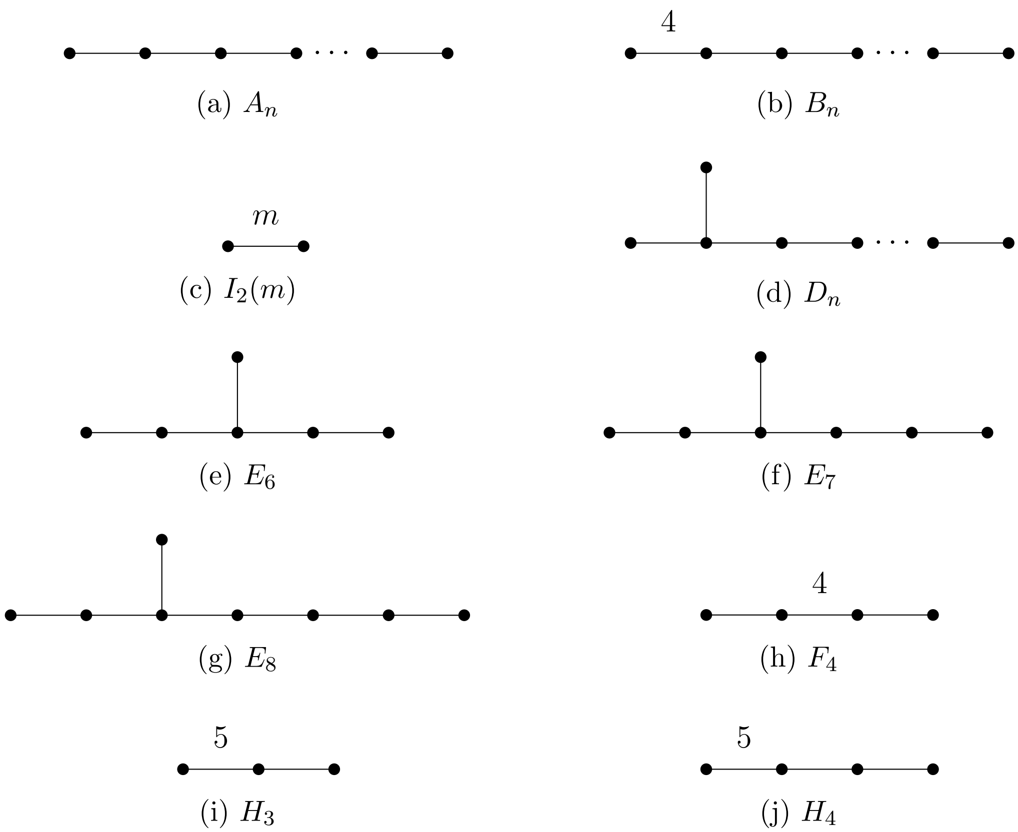
(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

Note that $W(\widetilde{C}_n)$ has $n+1$ generators.

The Coxeter graphs given in Figure 1.2 correspond to the collection of irreducible finite-type Coxeter systems, whose corresponding Coxeter groups are finite, while the Coxeter graphs given in Figure 1.3 are the so-called irreducible *affine Coxeter systems*, whose corresponding Coxeter groups are infinite [10]. Note that $W(B_n)$ is one of the irreducible finite Coxeter groups, so it is finite, while $W(\widetilde{C}_n)$ is one of the affine groups making it infinite. The irreducible affine Coxeter systems are unique in that if a vertex is removed along with the corresponding edges from the Coxeter graph, the newly created graph will result in a Coxeter system with a finite Coxeter group.



(a) A_n (b) B_n (c) $I_2(m)$ (d) D_n (e) E_6 (f) E_7 (g) E_8 (h) F_4 (i) H_3 (j) H_4

Figure 1.2: Irreducible finite-type Coxeter systems.

Given a Coxeter system (W, S), a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is

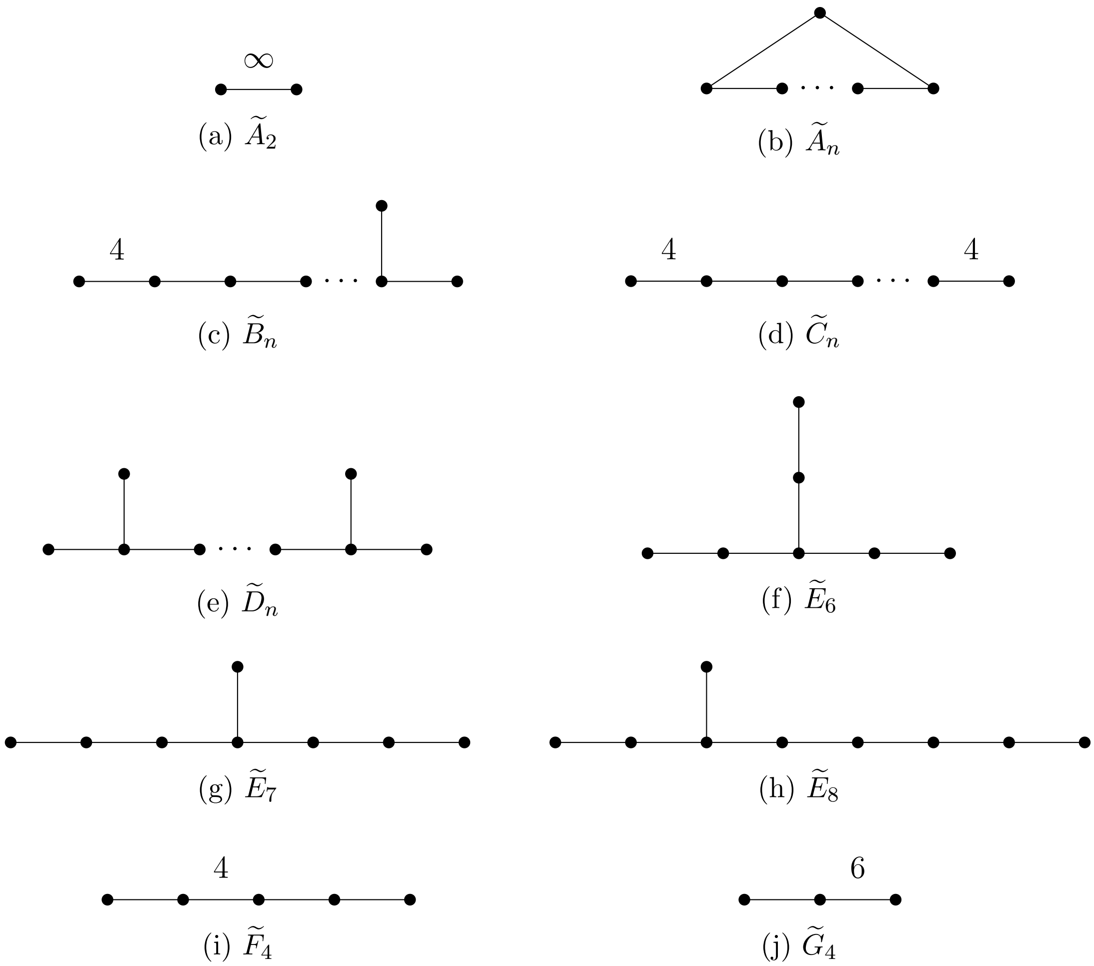


Figure 1.3: Irreducible affine Coxeter systems.

minimal among all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent it as $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_m}$ (using sans serif font). If $u, v \in W$, we say that the product uv is *reduced* if $l(uv) = l(u) + l(v)$. Matsumoto's Theorem, which follows, tells us more about how reduced expressions for a given group element are related.

Proposition 1.2.2 (Matsumoto, [7])**.** Let (W, S) be a Coxeter system. If $w \in W$, then given a reduced expression for w we can obtain every other reduced expression for w by a sequence of braid moves and commutations of the form

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. $\square$

It follows from Matsumoto's Theorem that if a generator s appears in a reduced expression for $w \in W$, then s appears in all reduced expressions for w. Let $w \in W$ and define the *support* of w, denoted $\mathrm{supp}(w)$, to be the set of all generators that appear in any reduced expression for w. If $\mathrm{supp}(w) = S$, we say that w has *full support*.

Given $w \in W$ and a fixed reduced expression w for w, any subsequence of w is called a *subexpression* of w. We will refer to a subexpression consisting of a consecutive subsequence of w as a *subword* of w.

Example 1.2.3. Let $\mathsf{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be an expression for $w \in W(A_7)$. Then we have

$$\begin{aligned} s_7s_2s_4s_5s_3s_2s_3s_6 &= s_7s_4s_2s_5s_3s_2s_3s_6 \\ &= s_7s_4s_5s_2s_3s_2s_3s_6 \\ &= s_7s_4s_5s_3s_2s_3s_3s_6 \\ &= s_7s_4s_5s_3s_2s_6, \end{aligned}$$

where the purple-highlighted text corresponds to a commutation, the teal-highlighted text corresponds to a braid move, and the red-highlighted text corresponds to cancellation. This shows that the original expression w is not reduced. However, it turns out that $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\mathrm{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let (W, S) be a Coxeter system and let $w \in W$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a reduced expression for w that begins (respectively, ends) with s.

Example 1.2.4. The following list consists of all reduced expressions for a particular $w \in W(B_4)$:

$$\begin{array}{cc} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{array}$$

We see that $l(w) = 5$ and w has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

Given a Coxeter system (W, S), for any subset $I \subset S$, define W_I to be the subgroup of W generated by all $s \in I$. Such a subgroup is called a *parabolic subgroup* of W.

1.3 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. Following [13], we define a relation $\sim$ on the set of reduced expressions for w. Let w_1 and w_2 be two reduced expressions for w. We define $\mathsf{w}_1 \sim \mathsf{w}_2$ if we can obtain w_2 from w_1 by applying a single commutation move of the form $st \mapsto ts$ where $m(s,t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative* (FC).

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$ and a starting reduced expression for w, observe that the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for w, but the following result due to Stembridge [13] states that when w is FC, performing commutations is the only possible way to obtain another reduced expression for w.

Proposition 1.3.1 (Stembridge, [13])**.** Let (W, S) be a Coxeter system. An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all

$m(s,t) \geq 3$. □

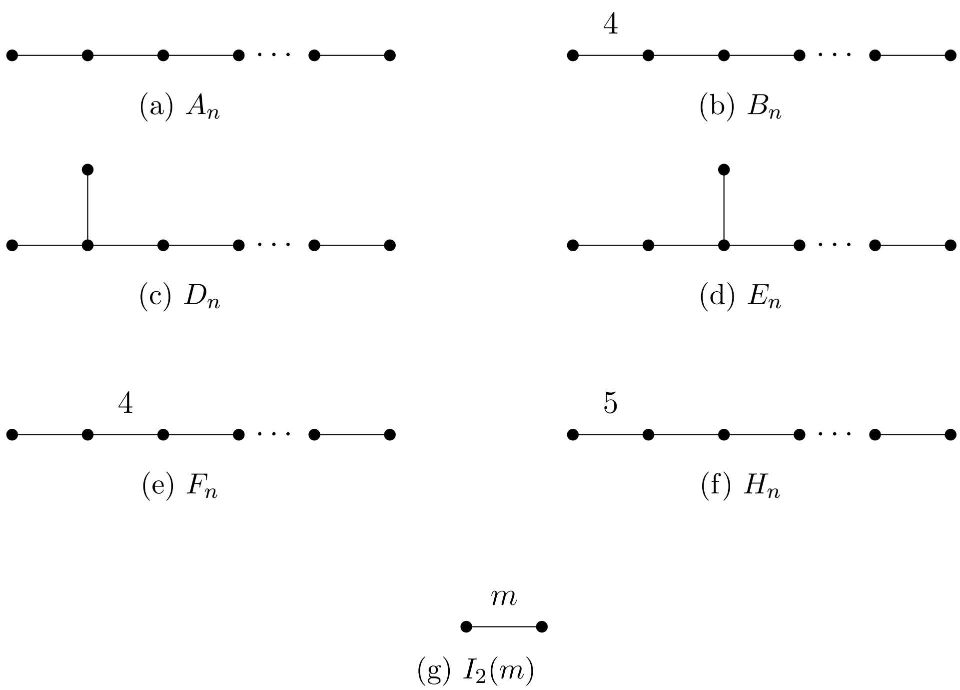
In other words, w is FC if and only if no reduced expression provides the opportunity to apply a braid move. For example, in a Coxeter system of type B_n an element is FC if no reduced expression contains the subwords $s_0s_1s_0s_1$, $s_1s_0s_1s_0$, $s_ks_{k+1}s_k$, and $s_{k+1}s_ks_{k+1}$ where $0 < k \leq n-2$. In a Coxeter system of type $\widetilde{C}_n$, an element is FC if no reduced expression for the element contains the subwords seen above with $0 < k \leq n-1$ and does not contain the subwords $s_{n-1}s_ns_{n-1}s_n$ and $s_ns_{n-1}s_ns_{n-1}$.

Example 1.3.2. Let $\mathsf{w}_1 = s_1s_0s_1s_3s_4s_5s_2s_4s_6$ be a reduced expression for $w \in W(\widetilde{C}_6)$. Applying the commutation $s_2s_4 \mapsto s_4s_2$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_1s_0s_1s_3s_4s_5s_4s_2s_6$, which is in the same commutation class as w_1. However, applying the braid move $s_4s_5s_4 \mapsto s_5s_4s_5$, we obtain another reduced expression $\mathsf{w}_3 = s_1s_0s_1s_3s_5s_4s_5s_2s_6$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. Since w has at least two commutation classes, one containing w_1 and w_2 and another containing w_3, w is not FC by Proposition 1.3.1.

Stembridge classified the Coxeter systems whose groups contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. Both $W(A_n)$ and $W(B_n)$ are finite Coxeter groups, and thus are FC-finite. On the other hand, $W(\widetilde{C}_n)$ is infinite and happens to also contain infinitely many FC elements. There exist infinite Coxeter groups that contain finitely many FC elements. For example, $W(E_n)$ for $n \geq 9$ (see Figure 1.4) is infinite, but contains only finitely many FC elements.

Proposition 1.3.3 (Stembridge, [13])**.** The irreducible FC-finite Coxeter systems are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. $\square$

The irreducible FC-finite Coxeter graphs are given in Figure 1.4. Note that the irreducible finite Coxeter systems given in Figure 1.2 certainly have only a finite number of FC elements. So the irreducible FC-finite Coxeter systems contain the irreducible finite-type Coxeter systems. However, notice there are a few graphs in Figure 1.2 that we have not yet encountered. Specifically, we have not yet encountered the Coxeter groups determined by graphs in Figures 1.4(d) for $n \geq 9$, 1.4(e) for $n \geq 5$, 1.4(f) for $n \geq 5$. All of these Coxeter systems have corresponding infinite groups for sufficiently large n, yet contain only finitely many FC elements.



(a) A_n (b) B_n (c) D_n (d) E_n (e) F_n (f) H_n (g) $I_2(m)$

Figure 1.4: Irreducible FC-finite Coxeter systems.

scale=0.455

We now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps for straight-line Coxeter groups found in [1], [3], and [13].

Let (W, S) be a Coxeter system of type Γ. Suppose $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_r}$ is a fixed reduced expression for $w \in W(\Gamma)$. As in [13], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, since w is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of w, where i is labeled by s_{x_i}. Note that for simplicity we are omitting the labels of the underlying poset yet retaining the labels of the corresponding generators.

It follows from [13] that heaps are well-defined up to commutation class. That is, given two reduced expressions w_1 and w_2 for $w \in W$ that are in the same commutation class, the heaps for w_1 and w_2 will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then w has one commutation class, and thus w has a unique heap. Conversely, if w_1 and w_2 are in different commutation classes, then the heap of w_1 will be distinct from the heap of w_2.

Example 1.3.4. Let $\mathsf{w} = s_6s_4s_2s_5s_3s_1s_4s_0s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that w is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $9 \lessdot 8$ since $8 < 9$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 1.5.

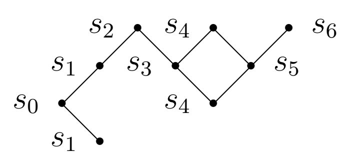


Figure 1.5: Labeled Hasse diagram for the heap of an element in $\mathrm{FC}(\widetilde{C}_6)$.

Let w be a reduced expression for an element $w \in W(\widetilde{C}_n)$. As in [1] and [3] we can represent a heap of w as a set of lattice points embedded in $\{0, 1, 2, \ldots, n\} \times \mathbb{N}$. To do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \ldots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of w in such a way that:

(1) An entry with coordinates (x, y) is labeled s_i (or i) in the heap if and only if $x = i$;

(2) If an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight-line Coxeter graph with the appropriate adjustments made to the label set and assignment of coordinates. Specifically, for type A_n our label set is $\{1, 2, \ldots, n\}$ and for type B_n our label set is $\{0, 1, \ldots, n-1\}$.

In the case of any straight-line Coxeter graph, it follows from the definition that (x, y) covers (x', y') in the heap if and only if $x = x' \pm 1$, $y' < y$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries fully exposed to the top (respectively, bottom) correspond to the generators occurring in the left (respectively, right) descent set of the corresponding reduced expression.

Let w be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice point representation of the heap poset in $\{0, 1, 2, \ldots n\} \times \mathbb{N}$ described in the preceding paragraphs via $H(\mathsf{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ and we refer to $H(w)$ as the heap of w. Note that we will use the same notation for heaps in Coxeter groups of all types with straight-line Coxeter graphs.

Let $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_i} at a level that is *above* the level of the point labeled by s_{x_j}. Because generators in a Coxeter graph that are not adjacent do commute, points whose x-coordinates differ by more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners (called a block) in such a way that if one entry covers another the blocks overlap halfway. In addition, we will also label each square for s_i with i.

There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap of each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

An important concept to this thesis is a block that is fully exposed to the top or bottom of the heap. We take *fully exposed* to the top (respectively, bottom) to mean that the top (respectively, bottom) edge of a heap block is not covered by any blocks above (respectively, below) in the heap. That is, there are no blocks that cover part of the top or bottom edge of the heap. Since there are multiple heap representations when $w \in W(\Gamma)$ is not FC, it is possible that a block that is fully exposed in one heap may not be fully exposed in a different heap representing w.

Example 1.3.5. Let $\mathsf{w} = s_6s_4s_2s_5s_3s_1s_4s_0s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.3.4. Figure 1.6 shows a possible lattice point representation for $H(\mathsf{w})$. Since w is FC this is the unique heap representation for w. Because w has a unique heap, we can obtain $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) from the blocks that are fully exposed to the top (respectively, bottom) of the heap. We see that $\mathcal{L}(w) = \{s_2, s_4, s_6\}$ and $\mathcal{R}(w) = \{s_1, s_4\}$.

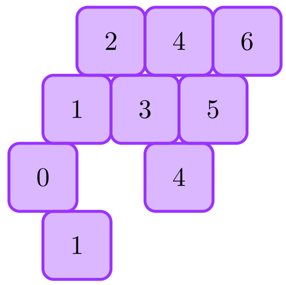


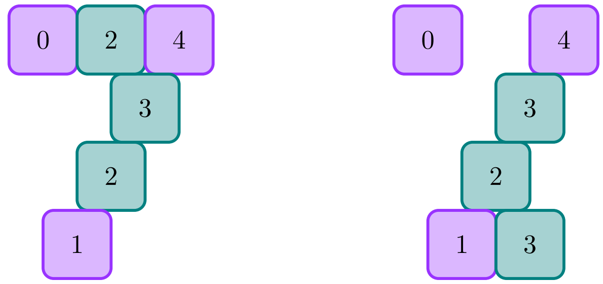
Figure 1.6: A lattice point representation for the heap of an FC element in $W(\widetilde{C}_6)$.

Example 1.3.6. Let $\mathsf{w}_1 = s_0s_2s_4s_3s_2s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. Applying the commutation move $s_2s_4 \mapsto s_4s_2$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_0s_4s_2s_3s_2s_1$, which is in the same commutation class as w_1, and hence has the same heap. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w}_3 = s_0s_4s_3s_2s_3s_1$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. Representations of $H(\mathsf{w}_1), H(\mathsf{w}_2)$, and $H(\mathsf{w}_3)$ are seen in Figure 1.7, where the braid relation is colored in teal. Notice that from the heaps we see that $\mathcal{L}(w) = \{s_0, s_2, s_4\}$ and $\mathcal{R} = \{s_1, s_3\}$. However, if we only had one heap or the other, we would miss some elements in the left and right descent sets as s_3 is not fully exposed to the bottom of the heap in Figure 1.7(a) and s_2 is not fully exposed to the top of the heap in Figure 1.7(b).

As for expressions, it will be helpful to have the notion of a subheap. Let $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_r}$ be a reduced expression for $w \in W(\Gamma)$. We define a heap H' to be a *subheap* of $H(\mathsf{w})$ if $H' = H(\mathsf{w}')$ where $\mathsf{w}' = s_{y_1}s_{y_2}\cdots s_{y_k}$ is a subexpression of w. We emphasize that the subexpression need not be a subword (i.e., a consecutive subexpression).

Recall that a subposet Q of P is called convex if $y \in Q$ whenever $x < y < z$ in P and $x, z \in Q$. We will refer to a subheap as a *convex subheap* if the underlying subposet is convex.

Example 1.3.7. Let $\mathsf{w} = s_3s_2s_1s_2s_5s_4s_6s_5$ be a reduced expression for $w \in W(\widetilde{C}_7)$. Now let $\mathsf{w}' = s_5s_4s_5$ be the subexpression of w that results from deleting all but the fifth, sixth, and last generators of w. Then the subheap $H(\mathsf{w}')$ is seen in Figure 1.8(a). However, $H(\mathsf{w}')$ is not convex since there is an entry in $H(w)$ labeled by s_6 occurring between the two consecutive



(a) $H(\mathsf{w}_1)$ and $H(\mathsf{w}_2)$. (b) $H(\mathsf{w}_3)$

Figure 1.7: Two heaps of a non-FC element in $W(\widetilde{C}_4)$.

occurrences of s_5 that does not occur in $H(w')$. However, if we do include the entry labeled by s_6, then we get the subheap seen in Figure 1.8(b), which is convex.

(a) (b)

Figure 1.8: Subheap and convex subheap of the heap for an element in $W(\widetilde{C}_7)$.

It will be extremely useful for us to be able to quickly determine whether a heap corresponds to an element in $\mathrm{FC}(B_n)$ or $\mathrm{FC}(\widetilde{C}_n)$. The next proposition is a special case of [13, Proposition 3.3] and follows quickly when one considers the consecutive subwords that are impermissible in reduced expressions for elements in $\mathrm{FC}(B_n)$ and $\mathrm{FC}(\widetilde{C}_n)$ as discussed in Section 1.3.

Proposition 1.3.8. Let (W, S) be a Coxeter system of type $\widetilde{C}_n$. If $w \in \mathrm{FC}(\widetilde{C}_n)$, then $H(w)$ cannot contain any of the configurations seen in Figure 1.9, where $0 < k < n-1$ and we use a square with a dotted boundary to emphasize that no element of the heap may occupy the corresponding position. $\square$

Since $W(B_n)$ is a parabolic subgroup of $W(\widetilde{C}_n)$, we can use Figure 1.9 to classify the impermissible configurations for elements of $\mathrm{FC}(B_n)$. In particular, the impermissible configurations for elements of $\mathrm{FC}(B_n)$ are those seen in Figures 1.9(a), 1.9(b) 1.9(c), and 1.9(d).

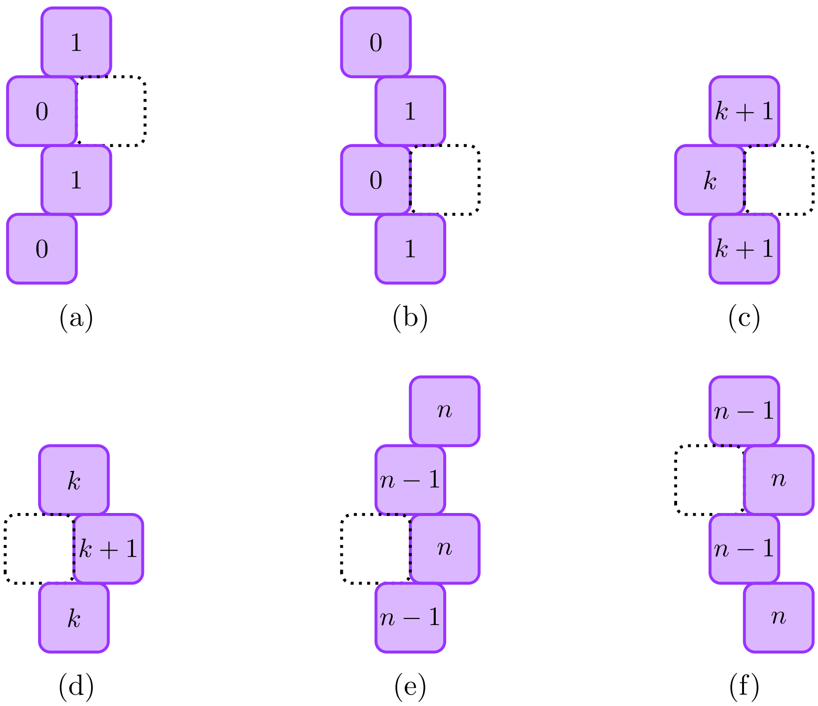


Figure 1.9: Impermissible configurations for heaps of $\mathrm{FC}(\widetilde{C}_n)$.

Chapter 2

Star Reductions and Property T

2.1 Star Reductions

The notion of a star operation was originally introduced by Kazhdan and Lusztig in [11] for simply-laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s, t \in S$), and was later generalized to all Coxeter systems in [12]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to only define the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We say w is *left star reducible by s with respect to t* if $m(s,t) \geq 3$, $s \in \mathcal{L}(w)$, and $t \in \mathcal{L}(sw)$. We analogously define w to be *right star reducible by s with respect to t*. Observe that w is left (respectively, right) star reducible if and only if $w = stu$ (respectively, $w = uts$), where the product on the right hand side of the equation is reduced and $m(s,t) \geq 3$. We say that w is *star reducible* if it is either left or right star reducible.

Example 2.1.1. Let $\mathsf{w} = s_0 s_1 s_0 s_2$ be a reduced expression for $w \in W(B_3)$. We see that w is left star reducible by s_0 with respect to s_1 to $s_1 s_0 s_2$ since $m(s_0, s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0 w)$. Notice that w is FC and $\mathcal{R}(w) = \{s_2, s_0\}$ since s_0 and s_2 commute. We see that $ws_2 = s_0 s_1 s_0$ and $ws_0 = s_0 s_1 s_2$. Note that in both instances $s_1 \notin \mathcal{R}(ws_2) = \{s_0\}$ and $s_1 \notin \mathcal{L}(ws_0) = \{s_2\}$. Because of this w is not right star reducible.

It may be helpful to visualize star reductions in terms of heaps. Let (W,S) be a Coxeter system with straight-line Coxeter graph Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Suppose w is left star reducible by s with respect to t. Then there exists a heap of w where the block for s is fully exposed to the top such that removing the block for s off of the top allows for t to now be fully exposed to the top of the heap. Similarly, if w is right star reducible by s with respect to t, then there exists a heap of w where the block for s is fully exposed to the bottom of the heap such that removing the block for s off of the bottom allows for t to now be fully exposed to the bottom. Conversely, if a heap of $w \in W(\Gamma)$ has this property, then w is star reducible. In Figure 2.1 we see the top portion of two possible heap representations of an element that is left star reducible by s with respect to t, where the dotted square indicates that no block may occupy this position. Notice that flipping the heap upside down in Figure 2.1 will result in a heap that is right star reducible. It is important to note that for non-FC group elements, when we are evaluating for star reducibility we must consider all heap representations for the element before concluding that it is not star reducible.

Figure 2.1: A visual representation of an element that is left star reducible by s with respect to t.

The following example utilizes heaps to show that an element is star reducible.

Example 2.1.2. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. Note that w is FC. By Example 2.1.1 we know that w is left star reducible by s_0 with respect to s_1. In Figure 2.2(a), we see the heap of w. Notice that the block for s_0 is fully exposed to the top of the heap. Removing the block for s_0 gives the heap in Figure 2.2(b). Notice that the block for s_1 is now fully exposed to the top of the heap. Hence, w is left star reducible by s_0 with respect to s_1. However, notice that the blocks for s_0 and s_2 are fully exposed to the bottom. In removing either of these blocks individually we are unable to fully expose s_1 to the bottom. Thus we can see that w is not right star reducible.

Notice that if w is not FC, then we are not be able to say that w is not star reducible when viewing a single heap as there could be a different heap for w in which we are able to fully expose a block that was previously blocked in a different heap.

Example 2.1.3. Let $\mathsf{w} = s_3s_1s_2s_1s_0s_1s_3s_0s_2s_4$ be a reduced expression for $w \in W(\widetilde{C}_3)$. The heap of w is given in Figure 2.3(a), where we have highlighted a braid in teal. Notice that this heap appears to not be star reducible since if we were to remove the block for

Figure 2.2: Visualization of Example 2.1.1.

s_1 or s_3 individually we would not fully expose s_2 to the top of the heap. The same goes for fully exposing blocks in the bottom of the heap. However, when we perform the braid move resulting in the heap seen in Figure 2.3(b) it is now obvious that the element is star reducible. Thus when considering a non-FC element for star reducibility via the heap, it is very important to consider all heaps for that element.

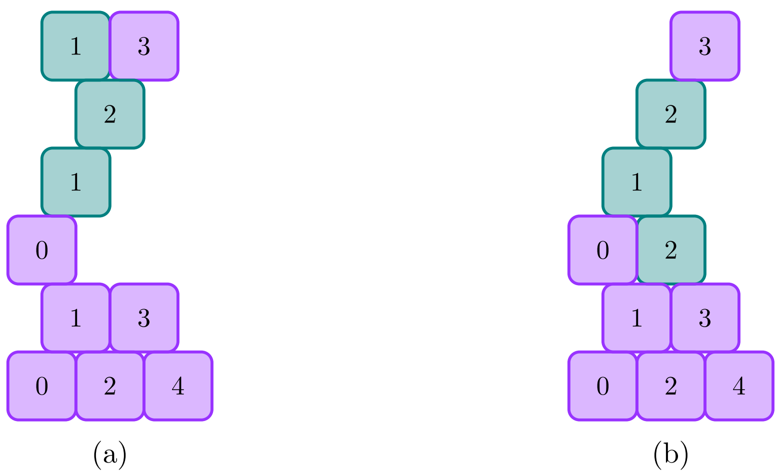


Figure 2.3: Visualization of Example 2.1.3

We say that $w \in W(\Gamma)$ is *star reducible to a product of commuting generators* if there is a sequence

$$w_1 = w \mapsto w_2 \mapsto \cdots \mapsto w_n$$

where for each $1 \leq i \leq n$, w_i is left star reducible or right star reducible to w_{i+1} with respect to some pair $\{s_i, t_i\}$, and w_n is a product of commuting generators. Using the notion of star reduction we are now able to introduce the concept of a star reducible Coxeter group. Let (W, S) be a Coxeter group of type Γ. We say that (W, S) or $W(\Gamma)$ is *star reducible* if every element of FC(Γ) is star reducible to a product of commuting generators. Notice that we are restricting to just the FC elements in $W(\Gamma)$. Visually a star reducible Coxeter group can be thought of in the following way. Given a heap in FC(Γ), we are able to systematically remove fully exposed blocks from the top or bottom of the heap and have a block that was previously not fully exposed become fully exposed until we are left with a heap that can be drawn as a single row.

In [9], Green classified all star reducible Coxeter groups.

Proposition 2.1.4 (Green, [9])**.** Let (W, S) be a Coxeter system of type Γ. Then (W, S) is star reducible if and only if each component of Γ is either a complete graph with labels $m(s,t) \geq 3$ or is one of the following types: type A_n ($n \geq 1$), type B_n ($n \geq 2$), type D_n ($n \geq 4$), type F_n ($n \geq 4$), type H_n ($n \geq 2$), type $I_2(m)$ ($m \geq 3$), type $\widetilde{A}_n$ ($n \geq 3$ and n even), type $\widetilde{C}_n$ ($n \geq 3$ and n odd), type $\widetilde{E}_6$, or type $\widetilde{F}_5$. $\square$

2.2 Property T

In [9], Green utilizes the following theorem to help classify the star reducible Coxeter groups.

Proposition 2.2.1 (Green, [9], Theorem 4.1)**.** Let (W, S) be a star reducible Coxeter system of type Γ, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s, t, u with $m(s,t) \neq 2$, $m(t,u) \neq 2$, and $m(s,u) = 2$:

(1) w is a product of commuting generators;

(2) w has a reduced product $w = stu$;

(3) w has a reduced product $w = uts$;

(4) w has a reduced product $w = sutv$. $\square$

Notice that Items (2) and (3) indicate an element that is left or right star reducible, respectively. Also notice that an element w that has the form of Item (1) does not meet the conditions of Items (2) and (3). In particular, w is not star reducible if it satisfies the condition of Item (1). Lastly, notice that if an element w is of the form of Item (4) and not of the form of Items (2) and (3), then w is not star reducible. Notice that Items (2), (3), and (4) are not mutually exclusive.

Motivated by Items (1) and (4) above, we define the notions of Property T and T-avoiding. Let (W, S) be a Coxeter system of type Γ and let $w \in W$. We say that w has *Property T* if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s,t) \geq 3$ and $u \in W$. That is, w has Property T if there exists a reduced expression for

w that begins or ends with a product of non-commuting generators. An element $w \in W(\Gamma)$ is called *T-avoiding* if w does not have Property T. This implies that a T-avoiding element is not star reducible.

Since elements that are star reducible also have Property T we already know how to visualize Property T in terms of heaps.

Visually a product of commuting generators can be made into a single row heap by pushing all the blocks into the same vertical position. It is clear that a single row heap will not portray the characteristic of Property T as seen in Figure 2.1 and thus a product of commuting generators is T-avoiding, which we state as a proposition.

Proposition 2.2.2. Let (W, S) be a Coxeter system of type Γ. If $w \in W(\Gamma)$ such that w is a product of commuting generators, then w is T-avoiding. □

We will call the identity or an element that is a product of commuting generators *type I T-avoiding*. If w is T-avoiding and not a of type I, we will say that w is *type II T-avoiding*. It is not clear that such elements exist. Referring back to Green's classification (Proposition 2.2.1) of what elements in star reducible Coxeter groups look like, we see that Item (1) corresponds to an element w being type I T-avoiding, Items (2) and (3) refer to the element w having Property T on the left and right, respectively and Item (4) refers to an element being type II T-avoiding provided no reduced expression for the element exhibits Items (2) and (3). In star reducible Coxeter systems, every FC element is star reducible to a product of commuting generators, which implies that no FC element can be type II T-avoiding in such groups. For example, as will be seen in Chapters 3 and 4, the Coxeter systems of type A_n and B_n have no type II T-avoiding elements, while the Coxeter systems of type D_n do.

Example 2.2.3. Let $\mathsf{w} = s_1 s_3 s_5$ be a reduced expression for $w \in W(A_5)$. Since w is a product of commuting generators, by Proposition 2.2.2 we know that w is type I T-avoiding.

Example 2.2.4. Let $\mathsf{w}_1 = s_5 s_3 s_2 s_4 s_1$ be a reduced expression for $w \in W(A_5)$. At first glance it may appear that w does not have Property T since both s_1 and s_4 commute as well as s_3 and s_5. However, note that applying the commutation move $s_4 s_2 \mapsto s_2 s_4$ results in $\mathsf{w}_2 = s_1 s_2 s_4 s_3 s_5$. Hence w has Property T since $m(s_1, s_2) = 3$ and there is a reduced expression for w that begins with $s_1 s_2$. In Figure 2.4 we see the heap of w. Note that we can see Property T in the bottom of the heap highlighted in orange. In addition to the orange highlighted subheap, w also has Property T with respect to s_3 and s_2 in the top of the heap, and s_4 and s_5 in the bottom of the heap.

Example 2.2.5. Let $\mathsf{w} = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$ be a reduced expression for $w \in W(\widetilde{C}_4)$. It turns out that w is FC and type II T-avoiding. The heap of w is seen in Figure 2.5. Notice that no

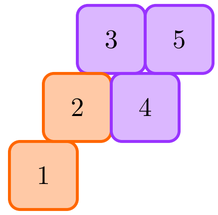


Figure 2.4: Heap of an element with Property T.

matter which block we remove that is fully exposed to the top of the heap no new element becomes fully exposed. The same applies to the bottom of the heap. Thus, w is type II T-avoiding.

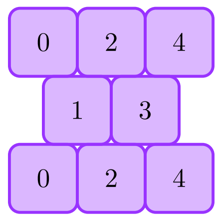


Figure 2.5: Heap of a type II T-avoiding element in $W(\widetilde{C}_4)$.

One thing to notice here is that all Coxeter groups have type I T-avoiding elements as the identity is type I T-avoiding and they also contain products of commuting generators, since individual elements of S are considered products of commuting generators. The more interesting type II T-avoiding elements do not appear in all Coxeter groups. In Chapter 3 we will summarize what is known about the T-avoiding elements in Coxeter systems of types $\widetilde{A}_n$, A_n, D_n, F_n, and $I_2(m)$, and in Chapters 4 and 5 we classify the T-avoiding elements in Coxeter systems of types B_n and $\widetilde{C}_n$.

2.3 Non-Cancellable Elements

We now introduce the concept of weak star reducible, which is related to the notion of cancellable in [5]. Let (W, S) be a Coxeter system of type Γ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators. If $w \in \mathrm{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t to sw* if

(1) w is left star reducible by s with respect to t, and

(2) $tw \notin \mathrm{FC}(\Gamma)$.

Notice that Condition (2) implies that $l(tw) > l(w)$. Also note that we are restricting our definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws*. We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable*. Notice that from this we know that weak star reducible implies star reducible. However, w being star reducible does not imply that w is weak star reducible.

Example 2.3.1. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. From Example 2.1.1 we know that w is left star reducible. However, $tw = s_1s_0s_1s_0s_2$, which is not in FC(B_4). Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2$. In addition, Example 2.1.1 showed that w is not right star reducible and hence w is not right weak star reducible.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall that in Section 2.1 we described what a star reduction looks like in terms of heap. Since the definition of weak star reducible includes that a heap is star reducible we again need to have those properties. In addition, for a heap to be weak star reducible, adding the block that becomes fully exposed when a block is removed from the heap must create a braid in the heap forcing the new larger heap to not be FC. That is, one of the impermissible configurations seen in Section 1.3 will appear at the top or bottom of the heap.

Example 2.3.2. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$ as in Example 2.3.1. Figure 2.6(a) shows the heap of w. Notice that in the heap we can clearly see that w is left star reducible by s_0 with respect to s_1. In Figure 2.6(b) we see that adding s_1 to the top of the heap creates a braid which is highlighted in orange. Therefore, w is left weak star reducible by s_0 with respect to s_1, to $\mathsf{w} = s_1s_0s_2$.

(a) Heap of w

(b) Heap of s_1w

Figure 2.6: Heap of a weak star reducible element of FC(B_4).

Example 2.3.3. Let $w \in \mathrm{FC}(B_4)$ and let $\mathsf{w} = s_0s_1$ be a reduced expression for w. Note that w is left (respectively, right) star reducible by s_0 with respect to s_1 (respectively, by s_1 with respect to s_0). However, $s_1s_0s_1 \in \mathrm{FC}(B_4)$ (respectively, $s_0s_1s_0 \in \mathrm{FC}(B_4)$). Visually the heap appears in Figure 2.7. Clearly when s_0 is added to the bottom of the heap, the new heap is still in $\mathrm{FC}(B_4)$ and the same can be said when s_1 is added to the top of the heap. Thus w is non-cancellable.

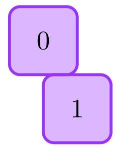


Figure 2.7: Heap of a non-cancellable element of $\mathrm{FC}(B_4)$.

In [3], Ernst classified the non-cancellable elements in Coxeter systems of type $W(B_n)$ and $W(\widetilde{C}_n)$. We will state part of the classification here as it is important to the development of the type II T-avoiding elements in $W(\widetilde{C}_n)$ for n odd. For the full classification see [3, Sections 4.2 and 5].

Before we state the classification we first define a specific group element in $W(\widetilde{C}_n)$ for n odd which we will refer to as a *sandwich stack*, an example of which is seen in Figure 2.8. Notice that this element has full support, is FC, and is type II T-avoiding.

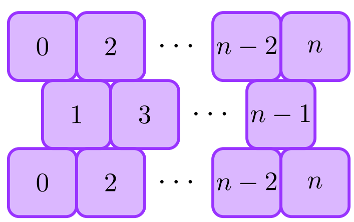


Figure 2.8: Heap of a single sandwich stack in $W(\widetilde{C}_n)$ for n odd.

We can extend this pattern to the heap seen in Figure 2.9. Like the smaller example above the element that corresponds to this heap has full support, is FC and is type II T-avoiding.

Remark 2.3.4. In Coxeter systems of type $\widetilde{C}_n$, the sandwich stacks are the only type II T-avoiding elements with full support. There are two other types of non-cancellable elements that were classified in [3]. The first does not have full support, which is important to our later classification and the second clearly has Property T.

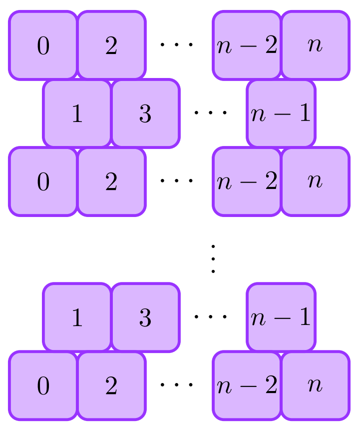


Figure 2.9: Heap of a sandwich stack in $W(\widetilde{C}_n)$ for n odd.