Conjugacy classes of cyclically fully commutative elements in Coxeter groups of type A

Brooke Fox

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Department of Mathematics & Statistics
Northern Arizona University

Definition

A Coxeter system consists of a group W (called a Coxeter group) together with a set S of generating involutions having presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Since s and t are involutions, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\left.\begin{array}{lcc} m(s,t) = 2 & \Longrightarrow & st = ts \end{array}\right\} \quad \text{commutations}$$

$$\left.\begin{array}{lcc} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

Definition

We can represent (W, S) with a unique Coxeter graph Γ having

(a) vertex set S and

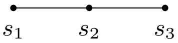
(b) edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$.

Comments

- Typically labels of $m(s, t) = 3$ are omitted.
- Edges correspond to non-commuting pairs of generators.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Example

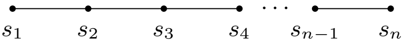
The Coxeter group of type A_3 is defined by the graph



Then $W(A_3)$ is subject to

- $s_i^2 = e$ for all i
- $s_1 s_2 s_1 = s_2 s_1 s_2, \;\; s_2 s_3 s_2 = s_3 s_2 s_3$
- $s_1 s_3 = s_3 s_1$.

In general, the Coxeter group of type A_n is defined by the graph



$W(A_n)$ is isomorphic to S_{n+1} under the correspondence

$$s_i \longleftrightarrow (i \ i+1).$$

Reduced expressions & Matsumoto's theorem

Definition

A word $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ (the free monoid) is called an expression for $w \in W$ if it is equal to w when considered as a group element. We will use $\mathsf{sans\ serif}$ to denote expressions. If m is minimal among all expressions for w, w is a reduced expression, and the length of w is $\ell(w) = m$.

Example

Consider the expression $\mathsf{w} = s_1 s_3 s_2 s_1 s_2$ for $w \in W(A_3)$. We see that

$$s_1 s_3 s_2 s_1 s_2 = s_1 s_3 s_1 s_2 s_1 = s_3 s_1 s_1 s_2 s_1 = s_3 s_2 s_1 \,.$$

Therefore, $s_1 s_3 s_2 s_1 s_2$ is not reduced. However, the expression on the right is reduced, so $\ell(w) = 3$.

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of commutations and braid relations.

Definition

We define $\mathrm{supp}(w)$ to be the set of generators appearing in any reduced expression for w. This is well-defined by Matsumoto's Theorem.

Definition

Given a reduced expression w for $w \in W$, we define a subexpression of w to be any expression obtained by deleting some subsequence from w. We will refer to a consecutive subexpression of w as a subword.

Example

Let W be the Coxeter group of type A_6 and let $w \in W$ have reduced expression $\mathsf{w} = s_1 s_3 s_4 s_2 s_5 s_6$. Then $s_1 s_4 s_6$ is a subexpression of w and $s_4 s_2 s_5$ is a subword.

Definition

A Coxeter element is an element $w \in W$ for which every generator appears exactly once in each reduced expression for w.

Example

Let W be the Coxeter group of type A_4. Then

$s_1 s_2 s_3 s_4$	$s_4 s_3 s_2 s_1$	$s_1 s_2 s_4 s_3$ $s_1 s_4 s_2 s_3$ $s_4 s_1 s_2 s_3$	$s_2 s_1 s_3 s_4$ $s_2 s_3 s_1 s_4$ $s_2 s_3 s_4 s_1$
$s_3 s_4 s_2 s_1$ $s_3 s_2 s_4 s_1$ $s_3 s_2 s_1 s_4$	$s_4 s_3 s_1 s_2$ $s_4 s_1 s_3 s_2$ $s_1 s_4 s_3 s_2$	$s_1 s_3 s_2 s_4$ $s_3 s_1 s_2 s_4$ $s_1 s_3 s_4 s_2$ $s_3 s_1 s_4 s_2$ $s_3 s_4 s_1 s_2$	$s_2 s_1 s_4 s_3$ $s_2 s_4 s_1 s_3$ $s_2 s_4 s_3 s_1$ $s_4 s_2 s_1 s_3$ $s_4 s_2 s_3 s_1$

are the Coxeter elements of W.

Definition

Let $w \in W$ have reduced expressions $\mathsf{w}_1, \mathsf{w}_2$. Then w_1 and w_2 are commutation equivalent if we can apply a sequence of commutations to w_1 to obtain w_2.

The corresponding equivalence classes are called commutation classes.

Example

Let W be the Coxeter group of type A_4 and let $w \in W$ have reduced expression

$$\mathsf{w} = s_1 s_2 s_3 s_2 s_4.$$

The commutation classes are

$$\{s_1 s_2 s_3 s_2 s_4, s_1 s_2 s_3 s_4 s_2\} \text{ and } \{s_1 s_3 s_2 s_3 s_4, s_3 s_1 s_2 s_3 s_4\}.$$

Definition

If w has exactly one commutation class, then we say that w is fully commutative, or just FC. The set of FC elements is denoted $\mathrm{FC}(\Gamma)$, where Γ is the corresponding Coxeter graph.

Theorem (Stembridge)

$w \in \mathrm{FC}(\Gamma)$ iff no reduced expression for w contains an opportunity to apply a braid relation.

Example

Let W be the Coxeter group of type A_5. Let $w \in W$ have reduced expression $\mathsf{w} = s_1 s_4 s_3 s_5 s_2 s_1 s_3 s_4$. Then we have

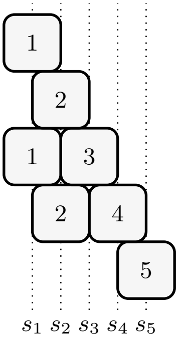
$$s_1 s_4 s_3 s_5 s_2 s_1 s_3 s_4 = s_1 s_4 s_5 s_3 s_2 s_1 s_3 s_4 = s_1 s_4 s_5 s_3 s_2 s_3 s_1 s_4.$$

So, w is not FC because there is opportunity to apply a braid relation.

We now introduce heaps through an example.

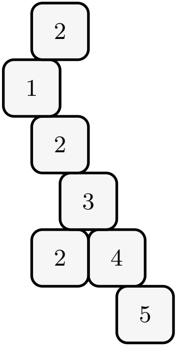
Example

Let W be the Coxeter group of type A_5 and let $\mathsf{w} = s_1 s_2 s_3 s_1 s_2 s_4 s_5$ be a reduced expression for $w \in W$.



Any element of the commutation class containing w has the heap above.

Another heap for w corresponding to $\mathsf{w}' = 2123245$ is



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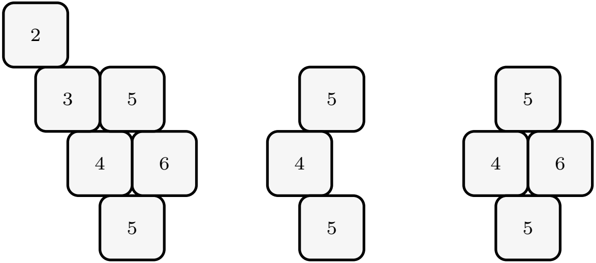
Proposition

There is a 1-1 correspondence between heaps and commutation classes. In particular, an element $w \in W$ is FC if and only if there is a unique heap for w.

Subheaps

Example

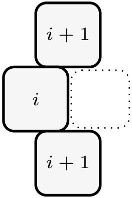
Let $w \in W(A_6)$ have reduced expression $\mathsf{w} = s_2 s_3 s_5 s_4 s_6 s_5$. Since there is no opportunity to apply a braid relation, w is FC, and so there is a unique heap.



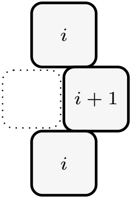
original heap a subheap a convex subheap

Proposition

Let $w \in \mathrm{FC}(A_n)$. Then $H(w)$ cannot contain either of the following convex subheaps



and



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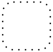
where $1 \leq i \leq n-1$ and  *is used to emphasize the absence of a block in the corresponding position in $H(w)$.*

Definition

Conjugating an expression by an initial generator results in a cyclic shift of the word:

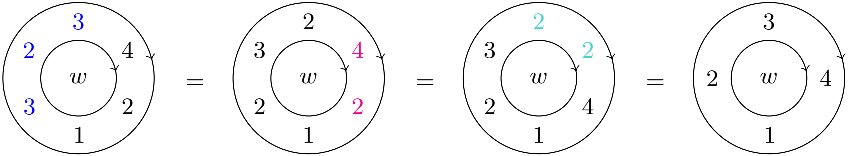
$$s_{x_1}(s_{x_1}s_{x_2}\cdots s_{x_m})s_{x_1} = s_{x_1}s_{x_1}s_{x_2}s_{x_3}\cdots s_{x_m}s_{x_1} = s_{x_2}s_{x_3}\cdots s_{x_m}s_{x_1}.$$

Let (W, S) be a Coxeter system and let w be a reduced expression for some $w \in W$. If every cyclic shift of w is a reduced expression for some element in W, then we say that w is cyclically reduced.

A group element $w \in W$ is cyclically reduced if every reduced expression for w is cyclically reduced. These are the group elements whose reduced expressions when written in a circle do not collapse in length.

Example

Consider the Coxeter group of type A_4. Let $w \in W$ have reduced expressions $\mathsf{w} = 342132$.



Thus, w is not cyclically reduced.

Question

Do two cyclically reduced expressions for conjugate group elements differ by a sequence of commutations, braid relations, and cyclic shifts?

Unfortunately the answer is "no" in general, but it is often true. Dana's research includes trying to understand when the answer is "yes."

Example

In $W(A_3)$, s_1 and s_2 do not differ by cyclic shifts, but

$$s_1 s_2 (s_1) s_2 s_1 = s_1 s_2 s_1 s_2 s_1 = s_1 s_1 s_2 s_1 s_1 = s_2\,.$$

Definition

Let W be a Coxeter group. We say that a conjugacy class C satisfies the cyclic version of Matsumoto's Theorem, or CVMT, if any two cyclically reduced expressions of elements in C differ by a sequence of commutations, braid relations, and cyclic shifts.

Note that the following result is the CVMT applied to Coxeter elements.

Theorem (Eriksson–Eriksson)

Let W be a Coxeter group and let c and c' be Coxeter elements. Then c and c' are conjugate iff c and c' are cyclically equivalent.

Unfortunately, FC-ness is not necessarily preserved under cyclic shifts. This motivates the following definition.

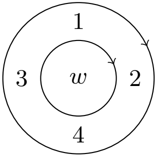
Definition

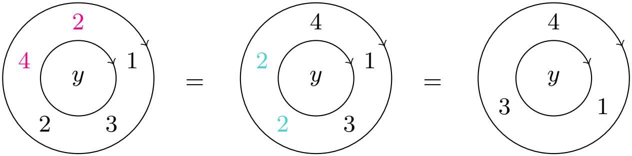
An element $w \in W$ is cyclically fully commutative, or CFC, if every cyclic shift of every reduced expression for w is a reduced expression for an FC element.

These are elements whose reduced expressions do not collapse and avoid braid relations when written in a circle.

Example

Let W be the Coxeter group of type A_4 and let $w, y \in W$ have reduced expressions $\mathsf{w} = 1243$ and $\mathsf{y} = 21324$, respectively. Then both w and y are FC, but, when we write each reduced expression in a circle, we have





So, w is CFC but y is not.

Proposition (Boothby, et al.)

Let $w \in W(A_n)$. Then w is CFC if and only if each generator in $\mathrm{supp}(w)$ *appears exactly once.*

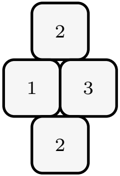
Example

Let W be the Coxeter group of type A_3. All of the CFC elements of W are

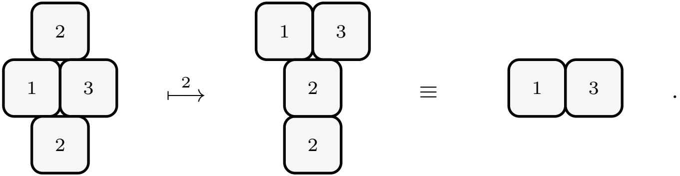
$$\begin{array}{ccccccc} e & 1 & 2 & 3 & 13 & 12 & 21 \\ 23 & 32 & 123 & 321 & 132 & 213 & \end{array}.$$

Example

The group element corresponding to the heap



is FC because there is no opportunity to apply a braid relation, but it is not CFC since



Definition

Let $w \in W(A_n)$ have reduced expression w and suppose w is commutation equivalent to a reduced expression that begins with i. Then a block labeled by i occurs at the top of the heap $H(\mathsf{w})$. A cyclic shift of $H(\mathsf{w})$ with respect to i is the heap that results from removing the block labeled by i from the top of the heap and appending it to the bottom.

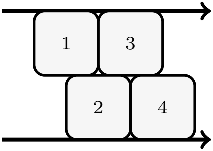
Let $w, w' \in \mathrm{CFC}(A_n)$. Then $H(w)$ and $H(w')$ are cyclically equivalent if $H(w)$ and $H(w')$ differ by a sequence of cyclic shifts of blocks.

Definition

We let $\widehat{H}(w)$ represent the equivalence class (generated by cyclic shifts) of cyclically equivalent CFC heaps, which we visualize by wrapping representatives on a cylinder. We call $\widehat{H}(w)$ a cylindrical heap.

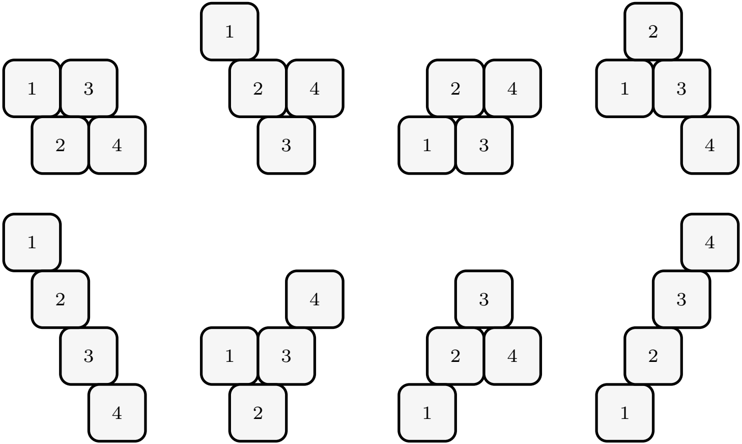
Example

Let $w \in \mathrm{CFC}(A_4)$ have reduced expression 1324. Then $\widehat{H}(w)$ can be represented by



.

The elements of the equivalence class $\widehat{H}(w)$ are



Recall that $W(A_n)$ is isomorphic to the symmetric group S_{n+1} via the mapping that sends i to the adjacent transposition $(i\ i+1)$.

If $w \in S_n$, then $[w(1)\ w(2) \cdots w(n)]$ is the one-line notation corresponding to w.

Example

Let W be the Coxeter graph of type A_4. Let $w \in W$ have reduced expression $\mathsf{w} = 12342$. Then the corresponding permutation in S_5 is

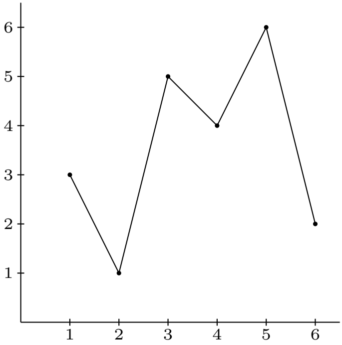
$$(12)(23)(34)(45)(23) = (1245).$$

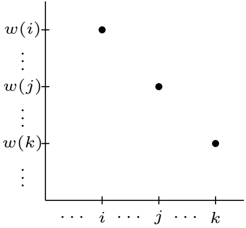
Then, in one-line notation, we have $(1245) = [24351]$.

A permutation line graph has line segments joining $(i, w(i))$ to $(i+1, w(i+1))$ for each $1 \leq i \leq n-1$.

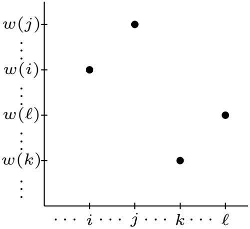
Example

Consider the permutation $w = [315462]$.





(a) The pattern 321



(b) The pattern 3412

Figure : The permutation line graphs of the 321 and 3412 patterns.

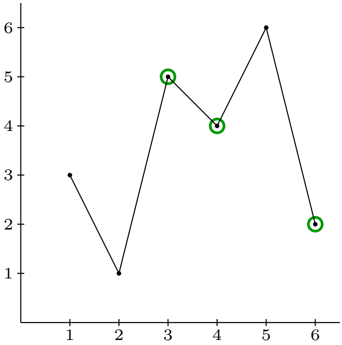
We say that w is 321-avoiding or 3412-avoiding if these patterns do not appear in the permutation line graph of w.

Example

Let $w \in W(A_5)$ have reduced expression $\mathsf{w} = 234513$. Then w corresponds to

$$(23)(34)(45)(56)(12)(34) = (13562)$$

in S_6. The one-line notation for w is $[315462]$. There is a 321 pattern in the one-line notation, but there is no 3412 pattern.



So, w is 3412-avoiding but not 321-avoiding.

Proposition (Billey)

An element $w \in W(A_n)$ is FC if and only if w is 321-avoiding.

Proposition (Boothby, et al.)

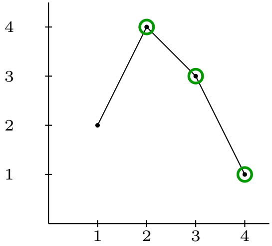
An element $w \in W(A_n)$ is CFC if and only if w is 321- *and* 3412-*avoiding.*

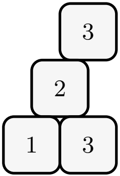
Example

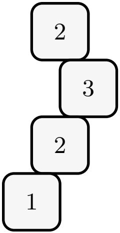
Let W be the Coxeter group of type A_3. Then $W \cong S_4$. Let $w \in W$ have reduced expression $\mathsf{w} = 3213$. Then, in cycle and one-line notations, we have that w corresponds to

$$(34)(23)(12)(34) = (124) = [2431]$$

in S_4.



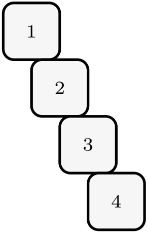




So, w is not FC and hence not CFC.

Example

Let $w \in W(A_4)$ have reduced expression $\mathsf{w} = 1234$. Then w is FC and the heap of w is



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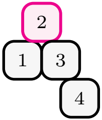
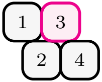
Then w corresponds to $(12)(23)(34)(45) = (12345)$.

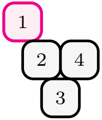
Conjecture



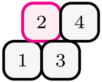
$\longmapsto (13452)$



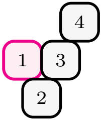
$\longmapsto (12453)$



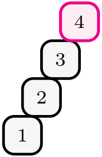
$\longmapsto (12354)$



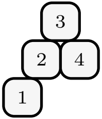
$\longmapsto (13542)$



$\longmapsto (14532)$



$\longmapsto (15432)$



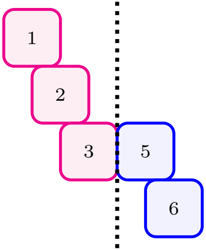
$\longmapsto (14532)$

Conjecture

Let $w \in W(A_n)$ correspond to a permutation with disjoint cycles $c_1, c_2, \ldots, c_k$ in S_{n+1}. Assume each c_j is written with the smallest number first. Then $w \in \mathrm{CFC}(A_n)$ if and only if each c_j has "connected support" and has at most one "direction change."

Example

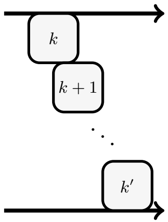
Consider the Coxeter group W of type A_6. Let $w \in \mathrm{CFC}(A_6)$ have reduced expression $\mathsf{w} = 12356$.



We say that $H(w)$ consists of a chunk of size 3 and a chunk of size 2.

Definition

A ring is a chunk wrapped on a cylinder.



Definition

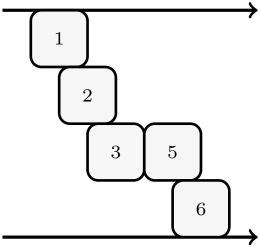
We say two rings are equivalent if they have the same number of blocks.

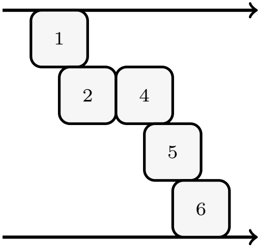
Definition

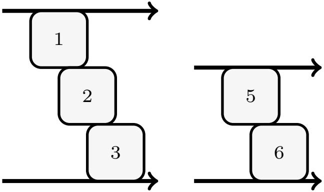
Two cylindrical heaps are ring equivalent if we can “slide” and “permute” rings of one cylindrical heap to obtain the other cylindrical heap.

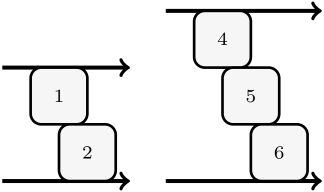
Example

Consider the Coxeter group W of type A_6. Let $w, y \in W$ have reduced expressions $\mathsf{w} = 12356$ and $\mathsf{y} = 12456$.

$\widehat{H}(w) =$ 

$\widehat{H}(y) =$ 





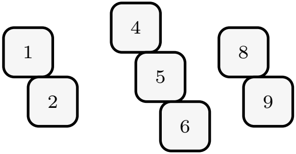
Conjugacy classes of CFC elements in $W(A_n)$

Theorem (Fox)

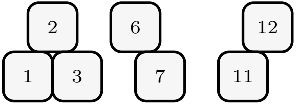
Two CFC elements are conjugate if and only if the corresponding cylindrical heaps are ring equivalent.

Example

Let W be the Coxeter group of type A_{12}. Then the element $w \in \mathrm{CFC}(A_{12})$ that corresponds to the heap



is conjugate to the group element $y \in \mathrm{CFC}(A_{12})$ that corresponds to the heap



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