SEVERAL REPRESENTATIONS OF MY FAVORITE OPEN PROBLEM

Department of Mathematics & Statistics Colloquium

Dana C. Ernst
Northern Arizona University
February 9, 2016

Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = 1, \quad (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comments

- The elements of S are distinct as group elements.
- $m(s,t)$ is the order of st.
- Coxeter groups can be thought of as generalized reflection groups.

Rewriting the relations

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$\left.\begin{array}{lcl} m(s,t) = 2 & \Longrightarrow & st = ts \end{array}\right\} \quad \text{short braid relations}$$

$$\left.\begin{array}{lcl} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{long braid relations}$$

This allows the replacement

$$\underbrace{sts\cdots}_{m(s,t)} \mapsto \underbrace{tst\cdots}_{m(s,t)}$$

in any word, which is called a commutation if $m(s,t) = 2$ and a braid move if $m(s,t) \geq 3$.

Definition

We can encode (W, S) with a unique Coxeter graph Γ having:

- vertex set S;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$.

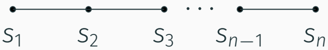
Comments

- Typically labels of $m(s, t) = 3$ are omitted.
- Edges correspond to non-commuting pairs of generators.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

EXAMPLE OF A COXETER GROUP

Example

The Coxeter group of type A_n is defined by the following graph.



Then $W(A_n)$ is subject to:

- $s_i^2 = 1$ for all i
- $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$
- $s_i s_j = s_j s_i$ if $|i-j| > 1$.

In this case, $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the correspondence $s_i \leftrightarrow (i, i+1)$.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an **expression** for $w \in W$ if it is equal to w when considered as a group element. If m is minimal, it is a **reduced expression**, and the **length** of w is $\ell(w) := m$.

Example

Consider the expression $s_1 s_3 s_2 s_1 s_2$ for an element $w \in W(A_3)$. Note that

$$s_1 s_3 s_2 s_1 s_2 = s_1 s_3 s_1 s_2 s_1 = s_3 s_1 s_1 s_2 s_1 = s_3 s_2 s_1 \,.$$

Therefore, $s_1 s_3 s_2 s_1 s_2$ is not reduced. However, the expression on the right is reduced, and so $\ell(w) = 3$.

Matsumoto's Theorem

Any two reduced expressions for $w \in W$ differ by a sequence of commutations & braid moves.

Theorem/Definition

Every finite Coxeter group contains a unique element of maximal length, which we refer to as the longest element and denote by w_0.

Comments

In the Coxeter group of type A_n:

- The longest element is the "reverse permutation":

$$w_0 = [n+1, n, \ldots, 2, 1]$$

- $\ell(w_0) = \binom{n+1}{2}$ (i.e., the nth triangular number).
- The number of reduced expressions of w_0 is known (Stanley).

Definition
Let $w \in W$ have reduced expressions $\overline{w_1}$ and $\overline{w_2}$. Then $\overline{w_1}$ and $\overline{w_2}$ are **commutation equivalent** if we can apply a sequence of commutations to $\overline{w_1}$ to obtain $\overline{w_2}$. The corresponding equivalence classes are called **commutation classes**.

Comments

- Claim: Studying commutation classes is a worthwhile endeavor.
- Applying a braid relation to a reduced expression will take you to a different commutation class. For each $w \in W$, this determines a graph called the **commutation graph** (vertices are commutation classes, edges correspond to braid moves).
- If W is finite, the longest element has more commutation classes than any other element in W.

EXAMPLE

When there is an interesting question involving Coxeter groups, we almost always begin by studying what happens in the type A_n situation (i.e., the symmetric group).

Let c_n be the number of commutation classes of the longest element w_0 in $W(A_n)$.

Example

The longest element w_0 in $W(A_3)$ has length 6 and is given by the permutation $[4,3,2,1] = (1,4)(2,3)$. It turns out that there are 16 distinct reduced expressions for w_0 while $c_3 = 8$.

$$\begin{array}{c} 321323 \\ 323123 \end{array} \quad \begin{array}{c} 312312 \\ 132312 \\ 312132 \\ 132132 \end{array} \quad 321232 \quad 232123 \quad \begin{array}{c} 123121 \\ 121321 \end{array} \quad \begin{array}{c} 231231 \\ 213231 \\ 231213 \\ 213213 \end{array} \quad 123212 \quad 212321$$

For brevity, we have written i in place of s_i.

Open Question

What is the number of commutation classes of the longest element in $W(A_n)$? That is, what is c_n?

Comments

- Problem was first introduced in 1992 by Knuth (but not using our current terminology).
- A more general version of the problem appears in a 1991 paper by Kapranov and Voevodsky.
- In 2006, Tenner explicitly states the open problem in terms of commutation classes.
- My advisor and academic brother (Hugh Denoncourt) became aware of the problem in 2007 via Brant Jones.
- Hugh spent a period of time obsessed with the problem (Heroin Hero).

OPEN PROBLEM

Comments (continued)

- NAU undergraduate math and physics major Dustin Story has been working on this problem all year.
- According to sequence A006245 of the OEIS, the first 10 values for c_n (starting at $n = 0$) are

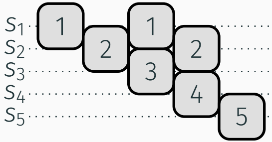
$$1, 1, 2, 8, 62, 908, 24698, 1232944, 112018190, 18410581880.$$

- To date, only the first 15 terms are known.
- The current best upper-bound for c_n was obtained by Felsner and Valtr in 2011. They prove that for sufficiently large n, $c_n \leq 2^{0.6571(n+1)^2}$. This bound is pretty awful.
- It turns out that the commutation classes of the longest element in $W(A_n)$ are in bijection with several interesting collections of mathematical objects. That is, c_n counts other cool stuff.

We now introduce heaps through an example.

Example

Let W be the Coxeter group of type A_5 and let $\overline{w} = s_1 s_2 s_3 s_1 s_2 s_4 s_5$ be a reduced expression for $w \in W$.



Any element of the commutation class containing $\overline{w}$ has the heap above.

Theorem (Stembridge)

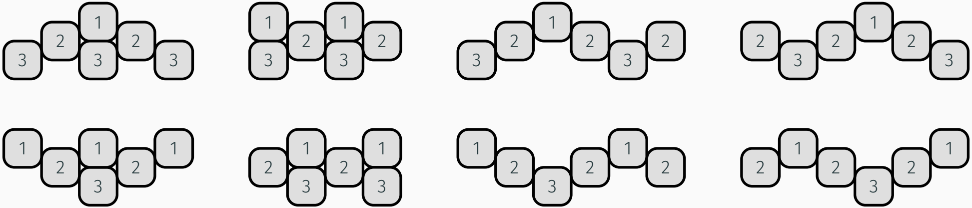
There is a 1-1 correspondence between heaps and commutation classes.

Corollary

The number of heaps for the longest element in $W(A_n)$ is c_n.

Example

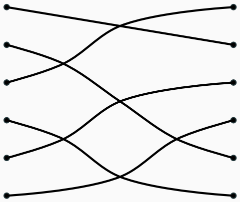
Here are the 8 heaps that correspond to the commutation classes for the longest element in $W(A_3)$.



One way of representing permutations is via string diagrams.

Example

Consider $\sigma = (1, 2, 5, 3)(4, 6)$.



Comment

When drawing a string diagram, we adopt the following conventions:

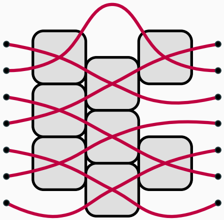
- No more than two strings cross each other at a given point.
- Strings are drawn to minimize crossings.

One often has many choices about how the strings are drawn. Loosely speaking, we say that two string diagrams are **equivalent** iff the relative arrangement of the crossings of the strings are the same.

Theorem

Up to equivalence, there is a 1-1 correspondence between string diagrams for a permutation in S_{n+1} and heaps for the corresponding permutation in $W(A_n)$. The points at which two strings cross correspond to blocks in a heap.

Example



Corollary

The number of string diagrams (up to equivalence) for the longest element in S_{n+1} is c_n.

Definition

An **arrangement of pseudolines** is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection. An arrangement is **simple** if no three pseudolines have a common point of intersection.

Corollary

The number of simple arrangements of $n+1$ pseudolines (up to equivalence) is c_n.

Definition

A comparator $[i:j]$ operates on a sequence of numbers $(x_1, \ldots, x_n)$ by replacing x_i and x_j respectively by $\min(x_i, x_j)$ and $\max(x_i, x_i)$.

A sorting network is a sequence of comparators that will sort any given sequence $(x_1, \ldots, x_n)$. That is, the successive comparators will produce an output sequence that always satisfies $x_1 \leq \cdots \leq x_n$. A sorting network is called primitive if its comparators all have the form $[i:i+1]$.

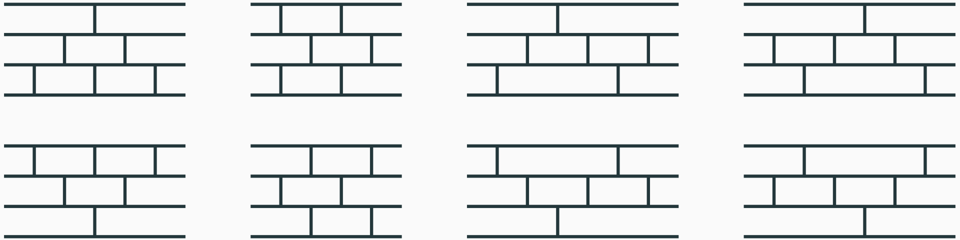
Theorem

A sequence of comparators is a sorting network iff it sorts the single permutation $[n, ..., 2, 1]$. A minimal primitive sorting network is equivalent to a sequence of adjacent transpositions $(i, i+1)$ that changes a sequence $(x_1, x_2, \ldots, x_n)$ into its reflection $(x_n, \ldots, x_2, x_1)$.

Primitive sorting networks can be represented with ladder diagrams (also called ladder lotteries, Amidakuji, or ghost legs).

Example

Here are the minimal ladder diagrams that correspond to the 8 primitive sorting networks on 4 elements.



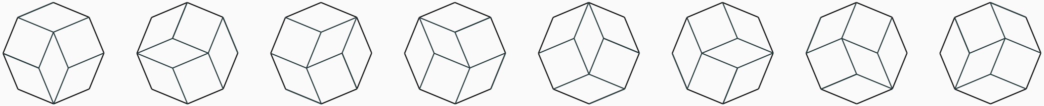
Theorem
There is a 1-1 correspondence between minimal primitive sorting networks on $n+1$ elements and heaps of the longest element in $W(A_n)$. Each rung in a ladder corresponds to a block in the heap.

Corollary
The number of minimal primitive sorting networks on $n+1$ elements is c_n.

It turns out that you can always tile a regular $2k$-gon using rhombi such that all side lengths of the rhombi and the $2k$-gon are the same.

Example

Here are the 8 distinct rhombic tilings of a regular octagon.



In this case, all rhombic tilings are rotation equivalent, but this is far from true in general.

By now, I'm sure you've seen this coming...

Theorem

There is a 1-1 correspondence between rhombic tilings of a regular $2(n+1)$-gon and heaps of the longest element in $W(A_n)$. Each tile corresponds to a block in the heap.

Corollary

The number of rhombic tilings of a regular $2(n+1)$-gon is c_n.

But there's more!

The number of commutation classes of the longest element is also related to the following.

- Uniform oriented matroids of rank 3.
- Condorcet domains (voting theory).
- Something about stability of quasicrystals (physics).

This academic year, Dustin and I have been working on attaining an improved upper bound for c_n. Our approach:

- We can obtain all possible string diagrams on $n+1$ strings by inserting a new string in all possible ways (up to equivalence) for every string diagram on n strings.
- In heap land, this is equivalent to "splitting" and "shifting" all the heaps for the longest element in $W(A_{n-1})$ and inserting a staircase of n blocks. This really will yield all the heaps for the longest element in $W(A_n)$.
- It turns out that our idea is related to cut paths.
- Strategy: Find a heap in $W(A_{n-1})$ with the greatest number of cut paths, count the cut paths, then multiply this number by the number of heaps of the longest element in $W(A_{n-1})$.

ATTEMPTS TO FIND A NEW UPPER BOUND

- Obvious answer: The even-odd sorting network (aka, brick sort) has the most cut paths of any other heap. Thanks to Nandor, we discovered a sequence on OEIS that turned us onto a paper by Galambos and Reiner that contains a nice formula that clearly counts what we wanted. They conjectured the same thing we did. Sweet!
- Our proposed upper bound kicks the crap out of the current best known. We're gonna be famous! ☺
- The problem is that our approach doesn't work. ☹
- It turns out that the even-odd network/heap doesn't have the greatest number of cut paths, which is a bit baffling.
- Danilov, Karzanov, and Koshevoy constructed a counterexample in S_{42}, which simultaneously disproved conjectures by Fishburn, by Monjardet, and by Galambos and Reiner.

OK, back to the drawing board.