A STUDY OF T-AVOIDING ELEMENTS OF COXETER GROUPS

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ABSTRACT

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Motivated by a desire to understand the Kazhdan-Lusztig theory of the Hecke algebra of the underlying Coxeter group, R.M. Green classified the so-called star reducible Coxeter groups, which have the property that all fully commutative elements (in the sense of Stembridge) can be sequentially reduced via star operations to a product of commuting generators. It turns out that in some Coxeter groups there are elements, called T-avoiding elements, which cannot be systematically dismantled in this way. More specifically an element w is called T-avoiding if w does not have a reduced expression beginning or ending with a pair of non-commuting generators. Clearly, a product of commuting generators is trivially T-avoiding. However, sometimes there are more interesting T-avoiding elements. We define two different types of T-avoiding elements, type I T-avoiding elements and type II T-avoiding elements. All Coxeter groups have type I T-avoiding elements. However, it has been shown that some Coxeter groups have type II T-avoiding elements and others do not. A natural question that arises from this is *which Coxeter groups have type II T-avoiding elements and which do not.*

In this thesis we state the already known results regarding T-avoiding elements in certain Coxeter groups. We also present a proof regarding the T-avoiding elements in Coxeter systems of types B_n and $\widetilde{C}_n$.

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Preliminaries

1.1 Introduction

In mathematics, one uses groups to study symmetry. In particular, a reflection group can be used to study the reflection and rotational symmetry of an object. A Coxeter group can be thought of as a generalized reflection group, where the group is generated by a set of elements of order two (i.e., reflections) and there are rules for how the generators interact with each other. Every element of a Coxeter group can be written as an expression in the generators, and if the number of generators in an expression (including multiplicity) is minimal, we say that the expression is reduced. Motivated by the desire to understand the Kazhdan–Lusztig theory of the Hecke algebra of the underlying Coxeter group, Green [9] classified the so-called star reducible Coxeter groups which have the property that all fully commutative elements (in the sense of Stembridge) can be sequentially reduced via star operations to a product of commuting generators. It turns out that in some Coxeter groups there are elements, called T-avoiding elements, which cannot be systematically dismantled in the way described above. More specifically an element w is called *T-avoiding* if w does not have a reduced expression beginning or ending with a pair of non-commuting generators. Clearly, a product of commuting generators is trivially T-avoiding. However, sometimes there are more interesting T-avoiding elements, which we will refer to as type II T-avoiding elements. Our motivation for studying the T-avoiding elements stems from the fact that computations involving the elements of the generalized Temperly–Lieb algebra for W that are indexed by T-avoiding elements is "well-behaved." In fact, knowing which elements correspond to T-avoiding elements often provides us with the base case for inductive arguments involving star operations.

In his PhD thesis [8], Gern classified the T-avoiding elements in Coxeter groups of type D_n. Unlike in types A_n and B_n, it turns out that the classification in type D_n includes non-trivial T-avoiding elements. The T-avoiding elements are rich in combinatorics and are interesting in their own right. The focus of this thesis is identifying T-avoiding elements in certain Coxeter groups.

This thesis is organized as follows. After necessary background information is presented in Section 1.2, we introduce the class of fully commutative elements in Section 1.3. Next in Section 1.3 we discuss a visual representation for elements of Coxeter groups, called heaps. In Section 2.1, we introduce the concept of a star reduction and star reducible Coxeter groups and in Section 2.2 we formally introduce the notion of a T-avoiding element. In Section 2.3 we define the notion of a non-cancellable element in Coxeter groups, as well as remark upon a specific family of non-cancellable elements in $W(\widetilde{C}_n)$. We then state classifications and conjectures regarding T-avoiding elements in several Coxeter groups in Chapter 3. All of these results, barring Section 3.4, are previously known. Chapters 4 and 5 contain the main results of this thesis, namely the classification of T-avoiding elements in Coxeter groups of types B_n and $\widetilde{C}_n$, which are new results. In Section 4.1, we introduce the necessary lemmas and definitions for the classification in Section 4.2, in which we show there are no type II T-avoiding elements in Coxeter groups of type B_n. In Section 5.1, we classify the type II T-avoiding elements in Coxeter groups of type $\widetilde{C}_n$. We conclude with some open questions in Section 5.2.

1.2 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s) \geq 2$ for $s \neq t$. If there is no relation between $s, t \in S$, then we define $m(s,t) = \infty$. However, in this thesis we assume that all $m(s,t)$ are finite. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [10]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

(1) vertex set S and

(2) edges $\{s, t\}$ for each $m(s, t) \geq 3$.

Each edge $\{s, t\}$ is labeled with its corresponding bond strength. Since $m(s, t) = 3$ occurs frequently, it is customary to omit this label. Note that s and t are not connected by an edge in the graph if and only if $m(s, t) = 2$. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. That is, given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. If (W, S) is a Coxeter system with corresponding Coxeter graph Γ, we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. Also, the Coxeter system (W, S) is said to be *irreducible* if and only if Γ is connected. If the graph Γ is disconnected, the connected components correspond to factors in a direct product of the corresponding Coxeter groups [10]. The Coxeter graphs given in Figure 1.1 correspond to the Coxeter systems that will be primarily addressed in this thesis. Notice that the vertices are the generators but to provide context when talking about the different generating sets $S(\Gamma)$ we have labeled them underneath with the specific generator they correspond to.

Example 1.2.1.

(a) The Coxeter system of type A_n is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group $W(A_n)$ with generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

   (1) $s_i^2 = e$ for all i;

   (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

   (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

   The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} under the correspondence $s_i \mapsto (i, i+1)$, where $(i, i+1)$ is the adjacent transposition that swaps i and $i + 1$.

(b) The Coxeter system of type B_n is given by the graph in Figure 1.1(c). We can construct the corresponding Coxeter group $W(B_n)$ with generating set $S(B_n) = \{s_0, s_1, \ldots, s_{n-1}\}$ and defining relations

   (1) $s_i^2 = e$ for all i;

   (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

   (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \ldots, n - 1\}$;

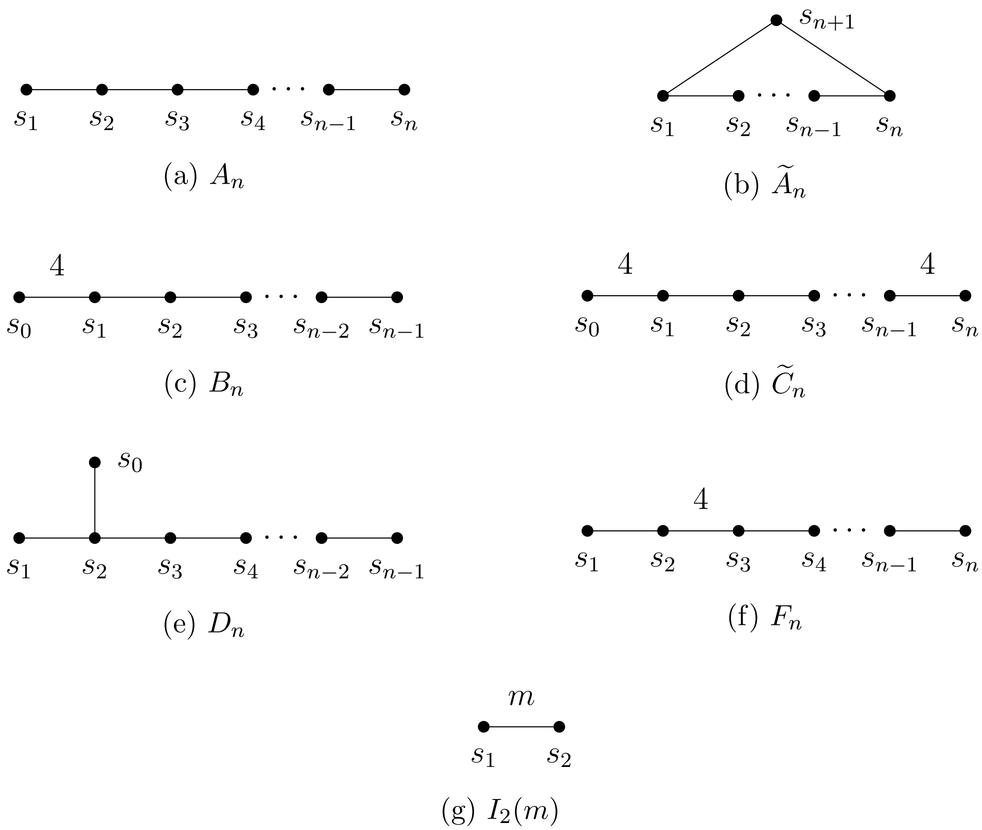


Figure 1.1: Examples of a few named Coxeter graphs.

(4) $s_0s_1s_0s_1 = s_1s_0s_1s_0$.

The Coxeter group $W(B_n)$ is isomorphic to the group, Sym_n^B, of signed permutations on the set $\{1, 2, \ldots, n\}$. We discuss Sym_n^B in more detail in Section 4.1.

(c) The Coxeter system of type $\widetilde{C}_n$ is given by the graph in Figure 1.1(d). We can construct the corresponding Coxeter group $W(\widetilde{C}_n)$ with generating set $S(\widetilde{C}_n) = \{s_0, s_1, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_is_j = s_js_i$ when $|i-j| > 1$ for $i \in \{0, 2, \ldots, n\}$;

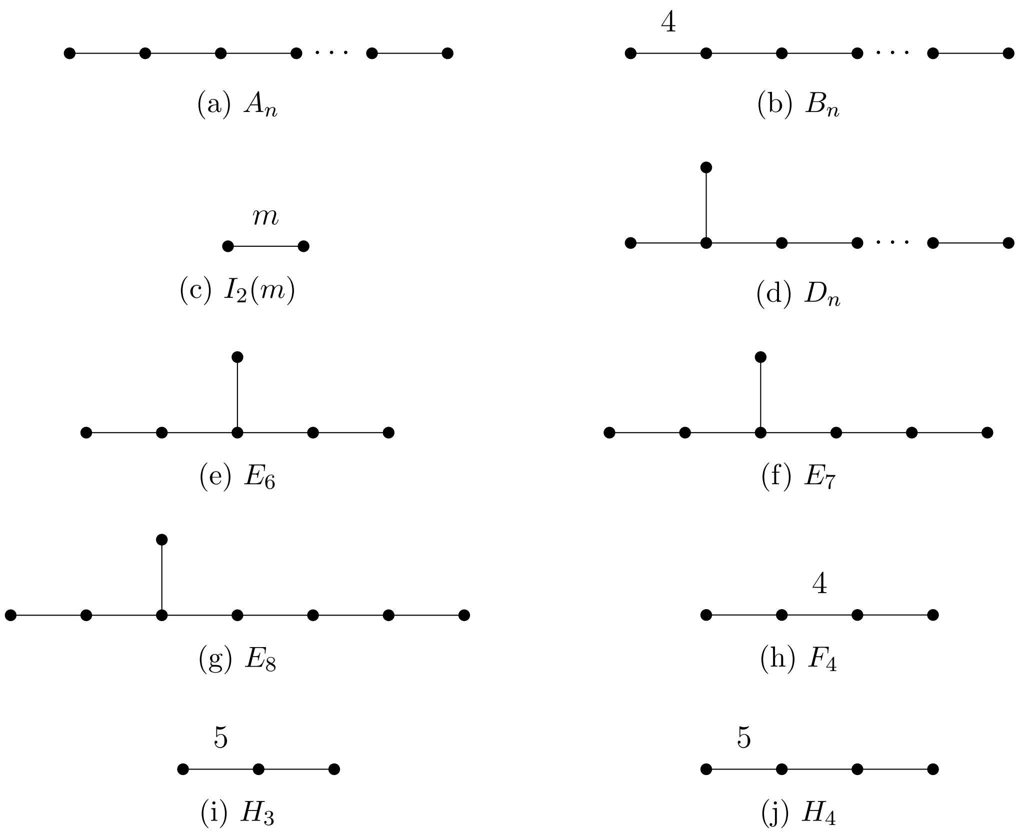
(3) $s_is_js_i = s_js_is_j$ when $|i-j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0s_1s_0s_1 = s_1s_0s_1s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

Note that $W(\widetilde{C}_n)$ has $n+1$ generators.

The Coxeter graphs given in Figure 1.2 correspond to the collection of irreducible finite-type Coxeter systems, whose corresponding Coxeter groups are finite, while the Coxeter graphs given in Figure 1.3 are the so-called irreducible *affine Coxeter systems*, whose corresponding Coxeter groups are infinite [10]. Note that $W(B_n)$ is one of the irreducible finite Coxeter groups, so it is finite, while $W(\widetilde{C}_n)$ is one of the affine groups making it infinite. The irreducible affine Coxeter systems are unique in that if a vertex is removed along with the corresponding edges from the Coxeter graph, the newly created graph will result in a Coxeter system with a finite Coxeter group.



(a) A_n (b) B_n (c) $I_2(m)$ (d) D_n (e) E_6 (f) E_7 (g) E_8 (h) F_4 (i) H_3 (j) H_4

Figure 1.2: Irreducible finite-type Coxeter systems.

Given a Coxeter system (W, S), a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is

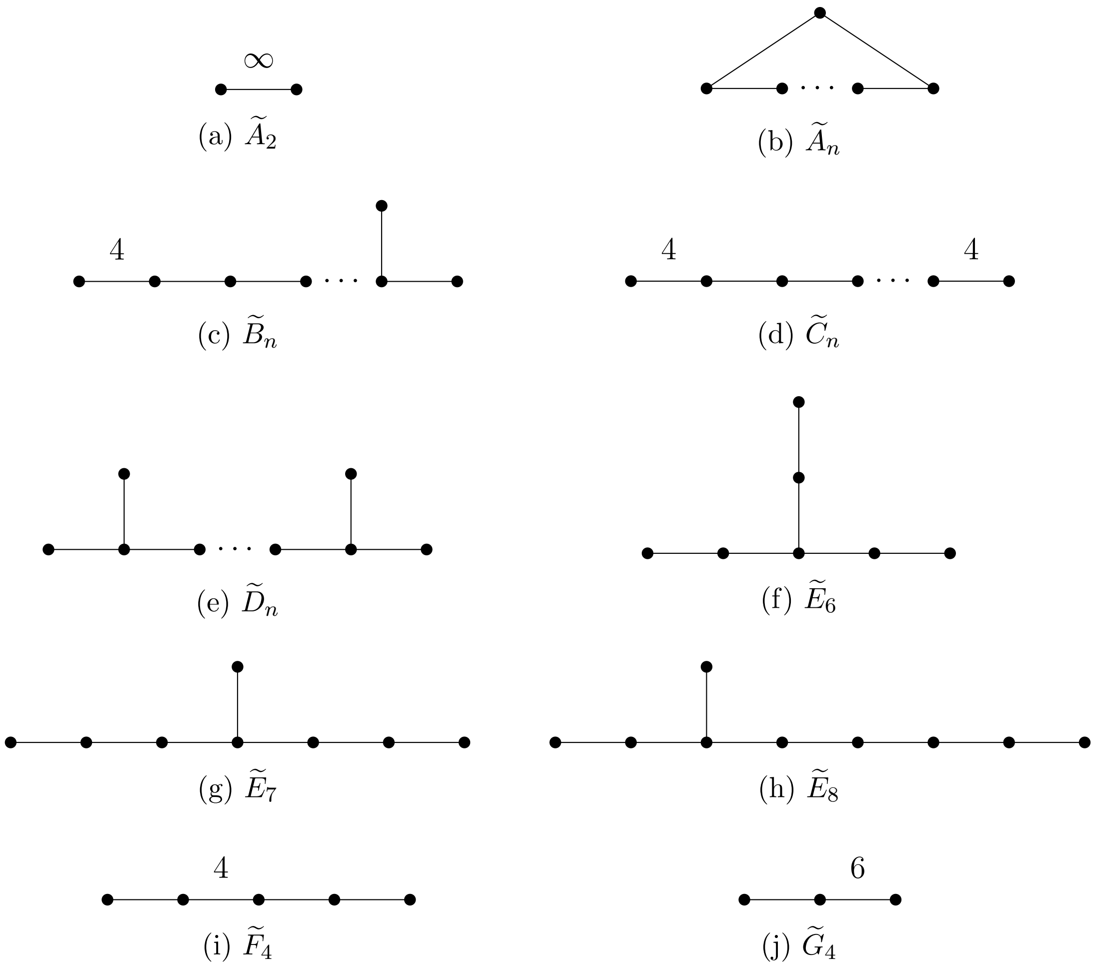


Figure 1.3: Irreducible affine Coxeter systems.

minimal among all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent it as $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ (using sans serif font). If $u, v \in W$, we say that the product uv is *reduced* if $l(uv) = l(u) + l(v)$. Matsumoto's Theorem, which follows, tells us more about how reduced expressions for a given group element are related.

Proposition 1.2.2 (Matsumoto, [7])**.** Let (W, S) be a Coxeter system. If $w \in W$, then given a reduced expression for w we can obtain every other reduced expression for w by a sequence of braid moves and commutations of the form

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. $\square$

It follows from Matsumoto's Theorem that if a generator s appears in a reduced expression for $w \in W$, then s appears in all reduced expressions for w. Let $w \in W$ and define the *support* of w, denoted $\operatorname{supp}(w)$, to be the set of all generators that appear in any reduced expression for w. If $\operatorname{supp}(w) = S$, we say that w has *full support*.

Given $w \in W$ and a fixed reduced expression w for w, any subsequence of w is called a *subexpression* of w. We will refer to a subexpression consisting of a consecutive subsequence of w as a *subword* of w.

Example 1.2.3. Let $\mathsf{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be an expression for $w \in W(A_7)$. Then we have

$$\begin{aligned} s_7{\color{purple}s_2s_4}s_5s_3s_2s_3s_6 &= s_7s_4{\color{purple}s_2s_5}s_3s_2s_3s_6 \\ &= s_7s_4s_5{\color{teal}s_2s_3s_2}s_3s_6 \\ &= s_7s_4s_5s_3s_2{\color{red}s_3s_3}s_6 \\ &= s_7s_4s_5s_3s_2s_6, \end{aligned}$$

where the purple-highlighted text corresponds to a commutation, the teal-highlighted text corresponds to a braid move, and the red-highlighted text corresponds to cancellation. This shows that the original expression w is not reduced. However, it turns out that $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\operatorname{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let (W, S) be a Coxeter system and let $w \in W$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a reduced expression for w that begins (respectively, ends) with s.

Example 1.2.4. The following list consists of all reduced expressions for a particular $w \in W(B_4)$:

$$\begin{array}{cc} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{array}$$

We see that $l(w) = 5$ and w has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

Given a Coxeter system (W, S), for any subset $I \subset S$, define W_I to be the subgroup of W generated by all $s \in I$. Such a subgroup is called a *parabolic subgroup* of W.

1.3 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. Following [13], we define a relation $\sim$ on the set of reduced expressions for w. Let w_1 and w_2 be two reduced expressions for w. We define $\mathsf{w}_1 \sim \mathsf{w}_2$ if we can obtain w_2 from w_1 by applying a single commutation move of the form $st \mapsto ts$ where $m(s,t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If there is a single commutation class for the set of reduced expressions for w, then we say that w is *fully commutative* (FC).

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$ and a starting reduced expression for w, observe that the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for w, but the following result due to Stembridge [13] states that when w is FC, performing commutations is the only possible way to obtain another reduced expression for w.

Proposition 1.3.1 (Stembridge, [13])**.** An element $w \in \mathrm{FC}(\Gamma)$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all $m(s,t) \geq 3$. $\square$

In other words, w is FC if and only if no reduced expression provides the opportunity to apply a braid move. For example, in a Coxeter system of type B_n an element is FC if no reduced expression contains the subwords $s_0s_1s_0s_1$, $s_1s_0s_1s_0$, $s_ks_{k+1}s_k$, and $s_{k+1}s_ks_{k+1}$ where $0 < k \leq n-2$. In a Coxeter system of type $\widetilde{C}_n$, an element is FC if no reduced expression for the element contains the subwords seen above with $0 < k \leq n-1$ and does not contain the subwords $s_{n-1}s_ns_{n-1}s_n$ and $s_ns_{n-1}s_ns_{n-1}$.

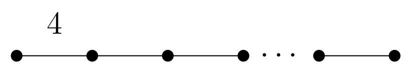
Example 1.3.2. Let $\mathsf{w}_1 = s_1s_0s_1s_3s_4s_5s_2s_4s_6$ be a reduced expression for $w \in W(\widetilde{C}_6)$. Applying the commutation $s_2s_4 \mapsto s_4s_2$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_1s_0s_1s_3s_4s_5s_4s_2s_6$, which is in the same commutation class as w_1. However, applying the braid move $s_4s_5s_4 \mapsto s_5s_4s_5$, we obtain another reduced expression $\mathsf{w}_3 = s_1s_0s_1s_3s_5s_4s_5s_2s_6$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class from w_1 and w_2. Since w has at least two commutation classes, one containing w_1 and w_2 and another containing w_3, w is not FC by Proposition 1.3.1.

Stembridge classified the Coxeter systems whose groups contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. Both $W(A_n)$ and $W(B_n)$ are finite Coxeter groups, and thus are FC-finite. On the other hand, $W(\widetilde{C}_n)$ is infinite and happens to also contain infinitely many FC elements. There exist infinite Coxeter groups that contain finitely many FC elements. For example, $W(E_n)$ for $n \geq 9$ (see Figure 1.4) is infinite, but contains only finitely many FC elements.

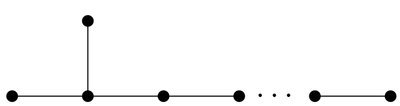
Proposition 1.3.3 (Stembridge, [13])**.** The irreducible FC-finite Coxeter systems are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. $\square$

The irreducible FC-finite Coxeter graphs are given in Figure 1.4. Note that the irreducible finite Coxeter systems given in Figure 1.2 certainly have only a finite number of FC elements. So the irreducible FC-finite Coxeter systems contain the irreducible finite-type Coxeter systems. However, notice there are a few graphs in Figure 1.2 that we have not yet encountered. Specifically, we have not yet encountered the Coxeter groups determined by graphs in Figures 1.4(d) for $n \geq 9$, 1.4(e) for $n \geq 5$, 1.4(f) for $n \geq 5$. All of these Coxeter systems have corresponding infinite groups for sufficiently large n, yet contain only finitely many FC elements.

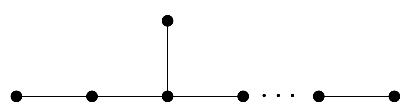
(a) A_n



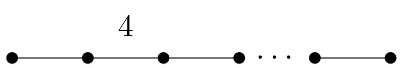
(b) B_n



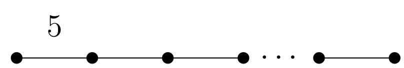
(c) D_n



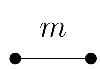
(d) E_n



(e) F_n



(f) H_n



(g) $I_2(m)$

Figure 1.4: Irreducible FC-finite Coxeter systems.

scale=0.455

We now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps for straight-line Coxeter groups found in [1], [3], and [13].

Let (W, S) be a Coxeter system of type Γ. Suppose $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ is a fixed reduced expression for $w \in W(\Gamma)$. As in [13], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, since w is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of w, where i is labeled by s_{x_i}. Note that for simplicity we are omitting the labels of the underlying poset yet retaining the labels of the corresponding generators.

It follows from [13] that heaps are well-defined up to commutation class. That is, given two reduced expressions w_1 and w_2 for $w \in W$ that are in the same commutation class, the heaps for w_1 and w_2 will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then w has one commutation class, and thus w has a unique heap. Conversely, if w_1 and w_2 are in different commutation classes, then the heap of w_1 will be distinct from the heap of w_2.

Example 1.3.4. Let $\mathsf{w} = s_6 s_4 s_2 s_5 s_3 s_1 s_4 s_0 s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that w is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $9 \lessdot 8$ since $8 < 9$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 1.5.

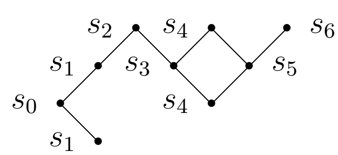


Figure 1.5: Labeled Hasse diagram for the heap of an element in $\mathrm{FC}(\widetilde{C}_6)$.

Let w be a reduced expression for an element $w \in W(\widetilde{C}_n)$. As in [1] and [3] we can represent a heap of w as a set of lattice points embedded in $\{0, 1, 2, \ldots, n\} \times \mathbb{N}$. To do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \ldots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of w in such a way that:

(1) An entry with coordinates (x, y) is labeled s_i (or i) in the heap if and only if $x = i$;

(2) If an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight-line Coxeter graph with the appropriate adjustments made to the label set and assignment of coordinates. Specifically, for type A_n our label set is $\{1, 2, \ldots, n\}$ and for type B_n our label set is $\{0, 1, \ldots, n-1\}$.

In the case of any straight-line Coxeter graph, it follows from the definition that (x, y) covers (x', y') in the heap if and only if $x = x' \pm 1$, $y' < y$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries fully exposed to the top (respectively, bottom) correspond to the generators occurring in the left (respectively, right) descent set of the corresponding reduced expression.

Let w be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice point representation of the heap poset in $\{0, 1, 2, \ldots n\} \times \mathbb{N}$ described in the preceding paragraphs via $H(\mathsf{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ and we refer to $H(w)$ as the heap of w. Note that we will use the same notation for heaps in Coxeter groups of all types with straight-line Coxeter graphs.

Let $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_i} at a level that is *above* the level of the point labeled by s_{x_j}. Because generators in a Coxeter graph that are not adjacent do commute, points whose x-coordinates differ by more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners (called a block) in such a way that if one entry covers another the blocks overlap halfway. In addition, we will also label each square for s_i with i.

There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap of each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

An important concept to this thesis is a block that is fully exposed to the top or bottom of the heap. We take *fully exposed* to the top (respectively, bottom) to mean that the top (respectively, bottom) edge of a heap block is not covered by any blocks above (respectively, below) in the heap. That is, there are no blocks that cover part of the top or bottom edge of the heap. Since there are multiple heap representations when $w \in W(\Gamma)$ is not FC, it is possible that a block that is fully exposed in one heap may not be fully exposed in a different heap representing w.

Example 1.3.5. Let $\mathsf{w} = s_6s_4s_2s_5s_3s_1s_4s_0s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.3.4. Figure 1.6 shows a possible lattice point representation for $H(\mathsf{w})$. Since w is FC this is the unique heap representation for w. Because w has a unique heap, we can obtain $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) from the blocks that are fully exposed to the top (respectively, bottom) of the heap. We see that $\mathcal{L}(w) = \{s_2, s_4, s_6\}$ and $\mathcal{R}(w) = \{s_1, s_4\}$.

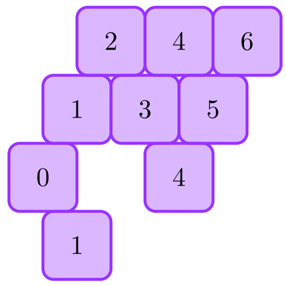


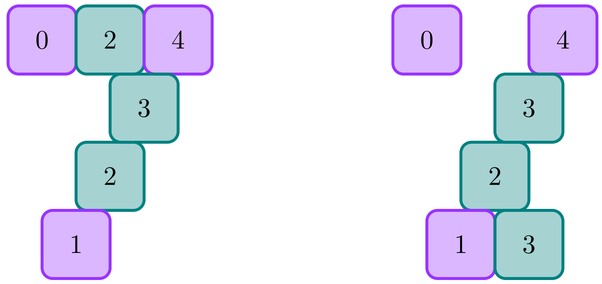
Figure 1.6: A lattice point representation for the heap of an FC element in $W(\widetilde{C}_6)$.

Example 1.3.6. Let $\mathsf{w}_1 = s_0s_2s_4s_3s_2s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. Applying the commutation move $s_2s_4 \mapsto s_4s_2$, we can obtain another reduced expression for w, namely $\mathsf{w}_2 = s_0s_4s_2s_3s_2s_1$, which is in the same commutation class as w_1, and hence has the same heap. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w}_3 = s_0s_4s_3s_2s_3s_1$. Note that since w_3 was obtained by applying a braid move, w_3 is in a different commutation class than w_1 and w_2. Representations of $H(\mathsf{w}_1), H(\mathsf{w}_2)$, and $H(\mathsf{w}_3)$ are seen in Figure 1.7, where the braid relation is colored in teal. Notice that from the heaps we see that $\mathcal{L}(w) = \{s_0, s_2, s_4\}$ and $\mathcal{R} = \{s_1, s_3\}$. However, if we only had one heap or the other, we would miss some elements in the left and right descent sets as s_3 is not fully exposed to the bottom of the heap in Figure 1.7(a) and s_2 is not fully exposed to the top of the heap in Figure 1.7(b).

As for expressions, it will be helpful to have the notion of a subheap. Let $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_r}$ be a reduced expression for $w \in W(\Gamma)$. We define a heap H' to be a *subheap* of $H(\mathsf{w})$ if $H' = H(\mathsf{w}')$ where $\mathsf{w}' = s_{y_1}s_{y_2}\cdots s_{y_k}$ is a subexpression of w. We emphasize that the subexpression need not be a subword (i.e., a consecutive subexpression).

Recall that a subposet Q of P is called convex if $y \in Q$ whenever $x < y < z$ in P and $x, z \in Q$. We will refer to a subheap as a *convex subheap* if the underlying subposet is convex.

Example 1.3.7. Let $\mathsf{w} = s_3s_2s_1s_2s_5s_4s_6s_5$ be a reduced expression for $w \in W(\widetilde{C}_7)$. Now let $\mathsf{w}' = s_5s_4s_5$ be the subexpression of w that results from deleting all but the fifth, sixth, and last generators of w. Then the subheap $H(\mathsf{w}')$ is seen in Figure 1.8(a). However, $H(\mathsf{w}')$ is not convex since there is an entry in $H(w)$ labeled by s_6 occurring between the two consecutive



(a) $H(\mathsf{w}_1)$ and $H(\mathsf{w}_2)$. (b) $H(\mathsf{w}_3)$

Figure 1.7: Two heaps of a non-FC element in $W(\widetilde{C}_4)$.

occurrences of s_5 that does not occur in $H(w')$. However, if we do include the entry labeled by s_6, then we get the subheap seen in Figure 1.8(b), which is convex.

(a) (b)

Figure 1.8: Subheap and convex subheap of the heap for an element in $W(\widetilde{C}_7)$.

It will be extremely useful for us to be able to quickly determine whether a heap corresponds to an element in $\mathrm{FC}(B_n)$ or $\mathrm{FC}(\widetilde{C}_n)$. The next proposition is a special case of [13, Proposition 3.3] and follows easily when one considers the consecutive subwords that are impermissible in reduced expressions for elements in $\mathrm{FC}(B_n)$ and $\mathrm{FC}(\widetilde{C}_n)$ as discussed in Section 1.3.

Proposition 1.3.8. Let (W, S) be a Coxeter system of type $\widetilde{C}_n$. If $w \in \mathrm{FC}(\widetilde{C}_n)$, then $H(w)$ cannot contain any of the configurations seen in Figure 1.9, where $0 < k < n-1$ and we use a square with a dotted boundary to emphasize that no element of the heap may occupy the corresponding position. $\square$

Since $W(B_n)$ is a parabolic subgroup of $W(\widetilde{C}_n)$, we can use Figure 1.9 to classify the impermissible configurations for elements of $\mathrm{FC}(B_n)$. In particular, the impermissible configurations for elements of $\mathrm{FC}(B_n)$ are those seen in Figures 1.9(a), 1.9(b) 1.9(c), and 1.9(d).

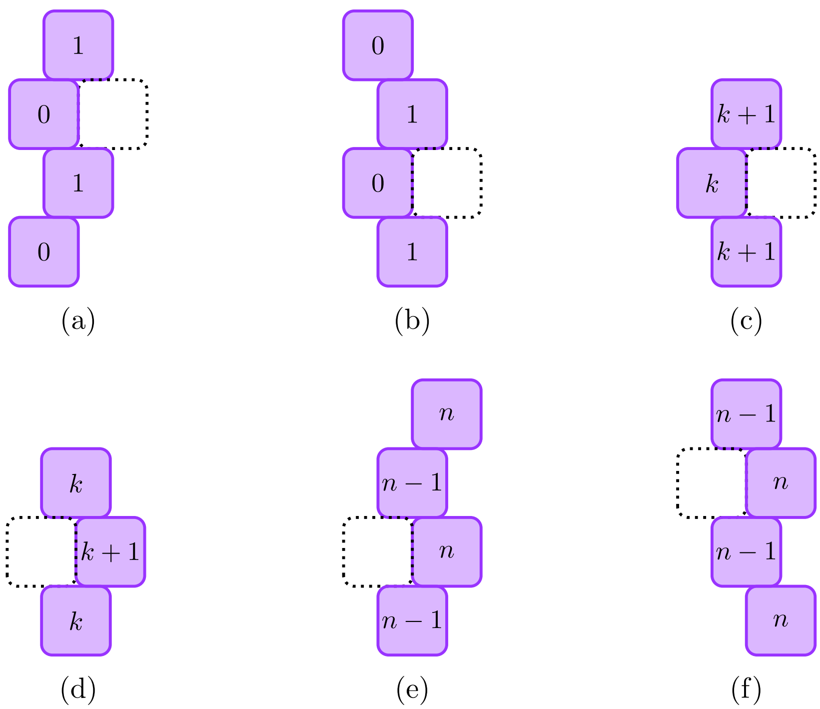


Figure 1.9: Impermissible configurations for heaps of $\mathrm{FC}(\widetilde{C}_n)$.

Chapter 2

Star Reductions and Property T

2.1 Star Reductions

The notion of a star operation was originally introduced by Kazhdan and Lusztig in [11] for simply-laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s,t \in S$), and was later generalized to all Coxeter systems in [12]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to only define the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We say w is *left star reducible by s with respect to t* if $m(s,t) \geq 3$, $s \in \mathcal{L}(w)$, and $t \in \mathcal{L}(sw)$. We analogously define w to be *right star reducible by s with respect to t*. Observe that w is left (respectively, right) star reducible if and only if $w = stu$ (respectively, $w = uts$), where the product on the right hand side of the equation is reduced and $m(s,t) \geq 3$. We say that w is *star reducible* if it is either left or right star reducible.

Example 2.1.1. Let $\mathsf{w} = s_0 s_1 s_0 s_2$ be a reduced expression for $w \in W(B_3)$. We see that w is left star reducible by s_0 with respect to s_1 to $s_1 s_0 s_2$ since $m(s_0, s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0 w)$. Notice that w is FC and $\mathcal{R}(w) = \{s_2, s_0\}$ since s_0 and s_2 commute. We see that $ws_2 = s_0 s_1 s_0$ and $ws_0 = s_0 s_1 s_2$. Note that in both instances $s_1 \notin \mathcal{R}(ws_2) = \{s_0\}$ and $s_1 \notin \mathcal{L}(ws_0) = \{s_2\}$. Because of this w is not right star reducible.

It may be helpful to visualize star reductions in terms of heaps. Let (W,S) be a Coxeter system with straight-line Coxeter graph Γ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Suppose w is left star reducible by s with respect to t. Then there exists a heap of w where the block for s is fully exposed to the top such that removing the block for s off of the top allows for t to now be fully exposed to the top of the heap. Similarly, if w is right star reducible by s with respect to t, then there exists a heap of w where the block for s is fully exposed to the bottom of the heap such that removing the block for s off of the bottom allows for t to now be fully exposed to the bottom. Conversely, if a heap of $w \in W(\Gamma)$ has this property, then w is star reducible. In Figure 2.1 we see the top portion of two possible heap representations of an element that is left star reducible by s with respect to t, where the dotted square indicates that no block may occupy this position. Notice that flipping the heap upside down in Figure 2.1 will result in a heap that is right star reducible. It is important to note that for non-FC group elements, when we are evaluating for star reducibility we must consider all heap representations for the element before concluding that it is not star reducible.

(a) (b)

Figure 2.1: A visual representation of an element that is left star reducible by s with respect to t.

The following example utilizes heaps to show that an element is star reducible.

Example 2.1.2. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. Note that w is FC. By Example 2.1.1 we know that w is left star reducible by s_0 with respect to s_1. In Figure 2.2(a), we see the heap of w. Notice that the block for s_0 is fully exposed to the top of the heap. Removing the block for s_0 gives the heap in Figure 2.2(b). Notice that the block for s_1 is now fully exposed to the top of the heap. Hence, w is left star reducible by s_0 with respect to s_1. However, notice that the blocks for s_0 and s_2 are fully exposed to the bottom. In removing either of these blocks individually we are unable to fully expose s_1 to the bottom. Thus we can see that w is not right star reducible.

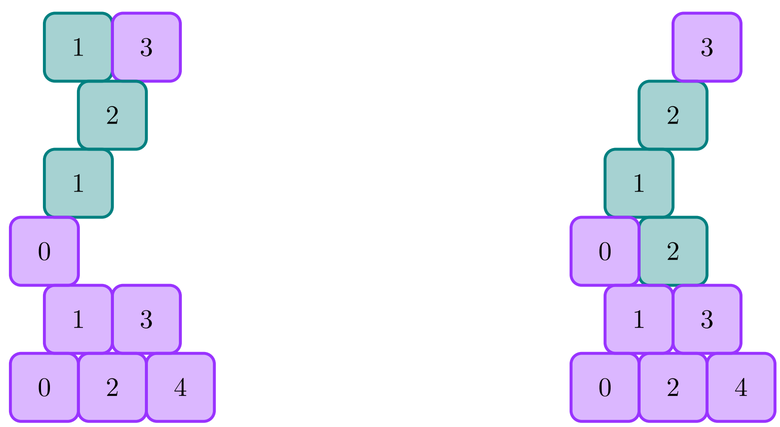
Notice that if w is not FC, then we are not be able to say that w is not star reducible when viewing a single heap as there could be a different heap for w in which we are able to fully expose a block that was previously blocked in a different heap.

Example 2.1.3. Let $\mathsf{w} = s_3s_1s_2s_1s_0s_1s_3s_0s_2s_4$ be a reduced expression for $w \in W(\widetilde{C}_3)$. The heap of w is given in Figure 2.3(a), where we have highlighted a braid in teal. Notice that this heap appears to not be star reducible since if we were to remove the block for

(a) (b)

Figure 2.2: Visualization of Example 2.1.1.

s_1 or s_3 individually we would not fully expose s_2 to the top of the heap. The same goes for fully exposing blocks in the bottom of the heap. However, when we perform the braid move resulting in the heap seen in Figure 2.3(b) it is now obvious that the element is star reducible. Thus when considering a non-FC element for star reducibility via the heap, it is very important to consider all heaps for that element.



(a) (b)

Figure 2.3: Visualization of Example 2.1.3

We say that $w \in W(\Gamma)$ is *star reducible to a product of commuting generators* if there is a sequence

$$w_1 = w \mapsto w_2 \mapsto \cdots \mapsto w_n$$

where for each $1 \leq i \leq n$, w_i is left star reducible or right star reducible to w_{i+1} with respect to some pair $\{s_i, t_i\}$, and w_n is a product of commuting generators. Using the notion of star reduction we are now able to introduce the concept of a star reducible Coxeter group. Let (W, S) be a Coxeter group of type Γ. We say that (W, S) or $W(\Gamma)$ is *star reducible* if every element of FC(Γ) is star reducible to a product of commuting generators. Notice that we are restricting to just the FC elements in $W(\Gamma)$. Visually a star reducible Coxeter group can be thought of in the following way. Given a heap in FC(Γ), we are able to systematically remove fully exposed blocks from the top or bottom of the heap and have a block that was previously not fully exposed become fully exposed until we are left with a heap that can be drawn as a single row.

In [9], Green classified all star reducible Coxeter groups.

Proposition 2.1.4 (Green, [9])**.** Let (W, S) be a Coxeter system of type Γ. Then (W, S) is star reducible if and only if each component of Γ is either a complete graph with labels $m(s,t) \geq 3$ or is one of the following types: type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type H_n $(n \geq 2)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_n$ $(n \geq 3$ and n even$)$, type $\widetilde{C}_n$ $(n \geq 3$ and n odd$)$, type $\widetilde{E}_6$, or type $\widetilde{F}_5$. $\square$

2.2 Property T

In [9], Green utilizes the following theorem to help classify the star reducible Coxeter groups.

Proposition 2.2.1 (Green, [9], Theorem 4.1)**.** Let (W, S) be a star reducible Coxeter system of type Γ, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s, t, u with $m(s,t) \neq 2$, $m(t,u) \neq 2$, and $m(s,u) = 2$:

(1) w is a product of commuting generators;

(2) w has a reduced product $w = stu$;

(3) w has a reduced product $w = uts$;

(4) w has a reduced product $w = sutv$. $\square$

Notice that Items (2) and (3) indicate an element that is left or right star reducible, respectively. Also notice that an element w that has the form of Item (1) does not meet the conditions of Items (2) and (3). In particular, w is not star reducible if it satisfies the condition of Item (1). Lastly, notice that if an element w is of the form of Item (4) and not of the form of Items (2) and (3), then w is not star reducible. Notice that Items (2), (3), and (4) are not mutually exclusive.

Motivated by Items (1) and (4) above, we define the notions of Property T and T-avoiding. Let (W, S) be a Coxeter system of type Γ and let $w \in W$. We say that w has *Property T* if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s,t) \geq 3$ and $u \in W$. That is, w has Property T if there exists a reduced expression for

w that begins or ends with a product of non-commuting generators. An element $w \in W(\Gamma)$ is called *T-avoiding* if w does not have Property T. This implies that a T-avoiding element is not star reducible.

Since elements that are star reducible also have Property T we already know how to visualize Property T in terms of heaps.

Visually a product of commuting generators be made into a single row heap by pushing all the blocks into the same vertical position. It is clear that a single row heap will not portray the characteristic of Property T as seen in Figure 2.1 and thus a product of commuting generators is T-avoiding, which we state as a proposition.

Proposition 2.2.2. Let (W, S) be a Coxeter system of type Γ. If $w \in W(\Gamma)$ such that w is a product of commuting generators, then w is T-avoiding. $\square$

We will call the identity or an element that is a product of commuting generators *type I T-avoiding*, which we abbreviate as T_1-avoiding. If w is T-avoiding and not a of type I, we will say that w is *type II* T-avoiding, which we abbreviate as T_2-avoiding. It is not clear that such elements exist. Referring back to Green's classification (Proposition 2.2.1) of what elements in star reducible Coxeter groups look like, we see that Item (1) corresponds to an element w being T_1-avoiding, Items (2) and (3) refer to the element w having Property T on the left and right, respectively and Item (4) refers to an element being T_2-avoiding provided no reduced expression for the element exhibits Items (2) and (3). In star reducible Coxeter systems, every FC element is star reducible to a product of commuting generators, which implies that no FC element can be T_2-avoiding in such groups. For example, as will be seen in Chapters 3 and 4, the Coxeter systems of type A_n and B_n have no T_2-avoiding elements, while the Coxeter systems of type D_n do.

Example 2.2.3. Let $\mathsf{w} = s_1s_3s_5$ be a reduced expression for $w \in W(A_5)$. Since w is a product of commuting generators, by Proposition 2.2.2 we know that w is T_1-avoiding.

Example 2.2.4. Let $\mathsf{w}_1 = s_5s_3s_2s_4s_1$ be a reduced expression for $w \in W(A_5)$. At first glance it may appear that w does not have Property T since both s_1 and s_4 commute as well as s_3 and s_5. However, note that applying the commutation move $s_4s_2 \mapsto s_2s_4$ results in $\mathsf{w}_2 = s_1s_2s_4s_3s_5$. Hence w has Property T since $m(s_1, s_2) = 3$ and there is a reduced expression for w that begins with s_1s_2. In Figure 2.4 we see the heap of w. Note that we can see Property T in the bottom of the heap highlighted in orange. In addition to the orange highlighted subheap, w also has Property T with respect to s_3 and s_2 in the top of the heap, and s_4 and s_5 in the bottom of the heap.

Example 2.2.5. Let $\mathsf{w} = s_0s_2s_4s_1s_3s_0s_2s_4$ be a reduced expression for $w \in W(\widetilde{C}_4)$. It turns out that w is FC and T_2-avoiding. The heap of w is seen in Figure 2.5. Notice that no

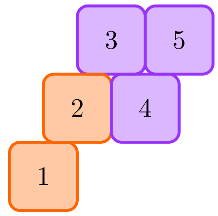


Figure 2.4: Heap of an element with Property T.

matter which block we remove that is fully exposed to the top of the heap no new element becomes fully exposed. The same applies to the bottom of the heap. Thus, w is T_2-avoiding.

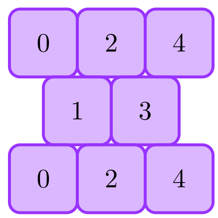


Figure 2.5: Heap of a T_2-avoiding element in $W(\widetilde{C}_4)$.

One thing to notice here is that all Coxeter groups have T_1-avoiding elements as the identity is T_1-avoiding and they also contain products of commuting generators, since individual elements of S are considered products of commuting generators. The more interesting T_2-avoiding elements do not appear in all Coxeter groups. In Chapter 3 we will summarize what is known about the T-avoiding elements in Coxeter systems of types $\widetilde{A}_n$, A_n, D_n, F_n, and $I_2(m)$, and in Chapters 4 and 5 we classify the T-avoiding elements in Coxeter systems of types B_n and $\widetilde{C}_n$.

2.3 Non-Cancellable Elements

We now introduce the concept of weak star reducible, which is related to the notion of cancellable in [5]. Let (W, S) be a Coxeter system of type Γ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators. If $w \in \mathrm{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t to sw* if

(1) w is left star reducible by s with respect to t, and

(2) $tw \notin \mathrm{FC}(\Gamma)$.

Notice that Condition (2) implies that $l(tw) > l(w)$. Also note that we are restricting our definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws*. We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable*. Notice that from this we know that weak star reducible implies star reducible. However, w being star reducible does not imply that w is weak star reducible.

Example 2.3.1. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. From Example 2.1.1 we know that w is left star reducible. However, $tw = s_1s_0s_1s_0s_2$, which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2$. In addition, Example 2.1.1 showed that w is not right star reducible and hence w is not right weak star reducible.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall that in Section 2.1 we described what a star reduction looks like in terms of heap. Since the definition of weak star reducible includes that a heap is star reducible we again need to have those properties. In addition, for a heap to be weak star reducible, adding the block that becomes fully exposed when a block is removed from the heap must create a braid in the heap forcing the new larger heap to not be FC. That is, one of the impermissible configurations seen in Section 1.3 will appear at the top or bottom of the heap.

Example 2.3.2. Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$ as in Example 2.3.1. Figure 2.6(a) shows the heap of w. Notice that in the heap we can clearly see that w is left star reducible by s_0 with respect to s_1. In Figure 2.6(b) we see that adding s_1 to the top of the heap creates a braid which is highlighted in orange. Therefore, w is left weak star reducible by s_0 with respect to s_1, to $\mathsf{w} = s_1s_0s_2$.

(a) Heap of w

(b) Heap of s_1w

Figure 2.6: Heap of a weak star reducible element of $\mathrm{FC}(B_4)$.

Example 2.3.3. Let $w \in \mathrm{FC}(B_4)$ and let $\mathsf{w} = s_0 s_1$ be a reduced expression for w. Note that w is left (respectively, right) star reducible by s_0 with respect to s_1 (respectively, by s_1 with respect to s_0). However, $s_1 s_0 s_1 \in \mathrm{FC}(B_4)$ (respectively, $s_0 s_1 s_0 \in \mathrm{FC}(B_4)$). Visually the heap appears in Figure 2.7. Clearly when s_0 is added to the bottom of the heap, the new heap is still in $\mathrm{FC}(B_4)$ and the same can be said when s_1 is added to the top of the heap. Thus w is non-cancellable.

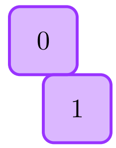


Figure 2.7: Heap of a non-cancellable element of $\mathrm{FC}(B_4)$.

In [3], Ernst classified the non-cancellable elements in Coxeter systems of type $W(B_n)$ and $W(\widetilde{C}_n)$. We will state part of the classification here as it is important to the development of the T_2-avoiding elements in $W(\widetilde{C}_n)$ for n odd. For the full classification see [3, Sections 4.2 and 5].

Before we state the classification we first define a specific group element in $W(\widetilde{C}_n)$ for n odd which we will refer to as a *sandwich stack*, an example of which is seen in Figure 2.8. Notice that this element has full support, is FC, and is T_2-avoiding.

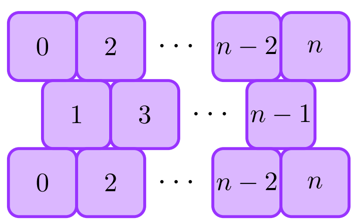


Figure 2.8: Heap of a single sandwich stack in $W(\widetilde{C}_n)$ for n odd.

We can extend this pattern to the heap seen in Figure 2.9. Like the smaller example above the element that corresponds to this heap has full support, is FC and is T_2-avoiding.

Remark 2.3.4. In Coxeter systems of type $\widetilde{C}_n$, the sandwich stacks are the only T_2-avoiding elements with full support. There are two other types of non-cancellable elements that were classified in [3]. The first does not have full support, which is important to our later classification and the second clearly has Property T.

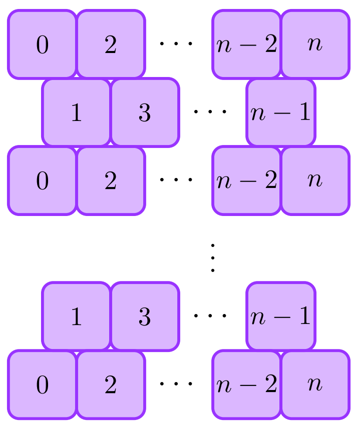


Figure 2.9: Heap of a sandwich stack in $W(\widetilde{C}_n)$ for n odd.

Chapter 3

T-Avoiding Elements in Types $\widetilde{A}_n, A_n, D_n, F_n$, and $I_2(m)$

In this chapter we classify the T-avoiding elements of Coxeter systems of types $\widetilde{A}_n, A_n, D_n, F_n$ and $I_2(m)$.

3.1 Types $\widetilde{A}_n$ and A_n

In this section we state the already known classification regarding T-avoiding elements in Coxeter systems of type $\widetilde{A}_n$ and A_n. We first focus on T_2-avoiding elements in $W(\widetilde{A}_n)$.

Proposition 3.1.1. If $n \geq 2$ and n is odd, then there are no T_2-avoiding elements in $W(\widetilde{A}_n)$. Otherwise, if $n \geq 2$ and n is even, then $W(\widetilde{A}_n)$ contains T_2-avoiding elements.

Proof. This is [6, Proposition 3.1.2] after a translation of terminology. □

The classification seen in [6] did not specifically classify the T_2-avoiding elements for type $\widetilde{A}_n$ for n even. Since $W(\widetilde{A}_n)$ for n even is not star reducible, the T_2-avoiding elements could be FC. The following is our conjecture regarding what the T_2-avoiding elements are in $W(\widetilde{A}_n)$ for n even.

Conjecture 3.1.2. The only T_2-avoiding elements in $W(\widetilde{A}_n)$ for n odd are of the form $\mathsf{w} = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

Recall that $W(\widetilde{A}_n)$, for n even, is not a star reducible Coxeter group. Hence it makes sense that the T-avoiding elements in $W(\widetilde{A}_n)$, for n even, can be FC. Further, as $W(A_n)$

is a parabolic subgroup of $W(\widetilde{A}_n)$ and $W(A_n)$ is a star reducible Coxeter group, any FC T_2-avoiding elements must have full support. First notice that

$$\mathsf{w} = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$$

is a reduced product, FC, and has full support. In addition, w is in fact T-avoiding. Since $W(\widetilde{A}_n)$ does not have a straight-line Coxeter graph, the heaps for $W(\widetilde{A}_n)$ are more appropriately viewed as three dimensional as a result providing visual representation for these heaps is difficult. However, we can envision the element above as a "castle turret" in which every block is in the wall or as a sandwich stack that has been made circular and the holes that appeared in the sandwich stack when it was rolled have been patched with an appropriate generator. As stated in the conjecture we believe that these are the only T_2-avoiding elements. However, it is not immediately obvious that there are no T_2-avoiding elements in $W(\widetilde{A}_n)\setminus\mathrm{FC}(\widetilde{A}_n)$. Classifying these T_2-avoiding elements remains an open problem. We now proceed with the classification of T-avoiding elements in Coxeter groups of type A_n.

Corollary 3.1.3. There are no T_2-avoiding elements in $W(A_n)$.

Proof. Notice that the Coxeter graph of type A_n can be obtained from the Coxeter graph of type $\widetilde{A}_k$, for $k > n$. This is done by removing the appropriate number of vertices and edges from the Coxeter graph of type $\widetilde{A}_k$. Since $W(\widetilde{A}_k)$ for k even has no T_2-avoiding elements this forces $W(A_n)$ to not have T_2-avoiding elements. Thus $W(A_n)$ does not have any T_2-avoiding elements. □

3.2 Type D_n

In this section we summarize the already known classification of the T-avoiding elements in Coxeter systems of type D_n seen in [8]. Recall that $W(D_n)$ is a star reducible Coxeter group and as a result any potential T_2-avoiding elements are not FC.

Proposition 3.2.1. There are T_2-avoiding elements in $W(D_n)$ for $n \geq 4$.

Proof. This is a consequence of [8, Section 2.2]. □

We now will classify these elements as seen in [8]. Before we do so we define interval notation useful to the classification from [8, Definition 2.3.1]. For $2 \leq i \leq j$ denote the element $s_i s_{i+1} \cdots s_{j-1} s_j$ by $[i, j]$. For $i \geq 3$, denote $s_1 s_3 s_4 \cdots s_i$ by $[1, i]$ and for $j \geq 2$ denote $s_1 s_2 s_3 \cdots s_j$ by $[0, j]$. If $0 \leq j < i$ and $i \geq 2$ define $[j, i] = [i, j]^{-1}$. Finally, for $i \leq -3$ and $j \geq 3$ denote $s_i s_{i-1} s_{i-2} \cdots s_4 s_3 s_2 s_3 s_4 \cdots s_j$ by $[-i, j]$. The following determines the classification for T-avoiding elements in $W(D_n)$.

Proposition 3.2.2. Let $w \in W(D_n)$ be T_2-avoiding. Then $w = w_n u$ (reduced product) for some $m \leq n$, where u is the identity or is a product of commuting generators such that $\mathrm{supp}(u) \subseteq \{s_{m+2}, s_{m+3}, s_{m+4}, \ldots, s_n\}$ and

$$w_n = \begin{cases} [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n] & n \text{ even} \\ [2,0][4,0]\cdots[m-2,0][m,0][m-k,m-2k]\cdots[m-1,m-2][m,m] & n \text{ odd} \end{cases}$$

where $m = n-1$ and

$$k = \begin{cases} \frac{n}{2} - 2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This is [8, Lemmas 2.2.18 and 2.3.4]. Although it is not immediately obvious, w_n is reduced and not FC. □

In Figure 3.1, we see two different elements that are T-avoiding in $W(D_5)$. Notice that the blocks that are highlighted in red alternate, this prevents the teal-highlighted braid from forcing its way to the top or the bottom of the heap. Due to the fork in the graph we must make slight alterations to heaps for $W(D_n)$. Specifically we allow s_0 and s_1 to occupy the same horizontal placement.

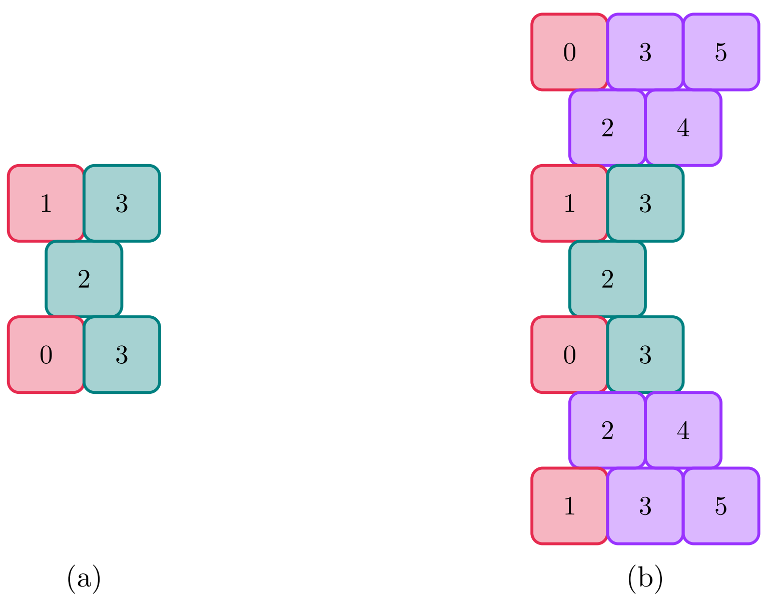


Figure 3.1: Visual representation of T_2-avoiding elements in $W(D_5)$.

3.3 Type F_n

In this section we state the known but unpublished classification of T-avoiding elements in Coxeter systems of type F_n for $n \geq 4$. Note that these results are not needed in Chapters 4 and 5.

We start with the Coxeter system of type F_5. Recall that $W(F_5)$ is a star reducible Coxeter group so any T_2-avoiding elements will not be FC. Before we begin the classification we introduce the notion of a specific element in $W(F_5)$ called a *bowtie*, which is given by the heap in Figure 3.2. Note that in Figure 3.2(a), the orange blocks correspond to the elements that have bond strength 4. It turns out that the expression determined by this heap is in fact reduced. Looking at the heap in Figure 3.2(b), we have highlighted a braid in teal. We can obtain a "stack of bowties" by removing the top most layer of the given heap of the bowtie and adding a new single bowtie to the stack. Doing this repeatedly results in the heap seen in Figure 3.3. Similar to a single bowtie, the expression that corresponds to a stack of bowties is reduced and not FC. These heaps are referenced in the following unpublished theorem by Cross, Ernst, Hills-Kimball, and Quaranta in 2012, which classifies the T-avoiding elements in the Coxeter systems of type F_5.

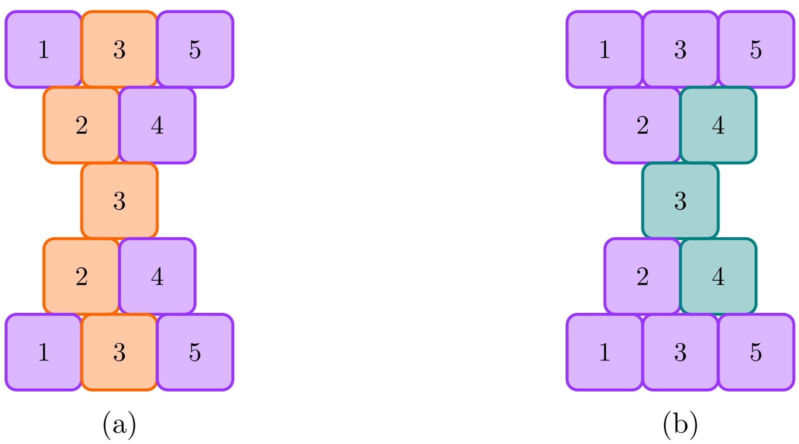


Figure 3.2: Heap of a single bowtie in $W(F_5)$.

Proposition 3.3.1. The only T_2-avoiding elements in $W(F_5)$ are stacks of bowties. □

As a result of the classification in type F_5, Cross et al. were also able to classify the T-avoiding elements in $W(F_4)$.

Corollary 3.3.2. There are no T_2-avoiding elements in the Coxeter system of type F_4. □

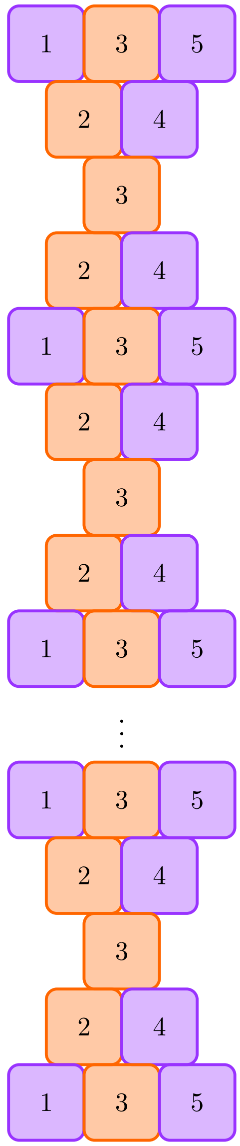


Figure 3.3: Heap of a stack of bowties in $W(F_5)$.

Proof. Since there are no T_2-avoiding elements in $W(F_5)$ that do not have full support, we know that there are not any T_2-avoiding elements in $W(F_4)$. Because if there were T_2-avoiding elements they would also be T_2-avoiding in $W(F_5)$. ☐

Cross et al. conjectured that in Coxeter systems of type F_n for $n \geq 5$, an element is T_2-avoiding if and only if it is a stack of bowties multiplied by a product of commuting generators. In 2013, Gilbertson and Ernst worked with this conjecture and quickly found it to be false. The heap seen in Figure 3.4 corresponds to a T_2-avoiding element in the Coxeter group of type F_6 that is not a bowtie. It turns out that like the bowties discussed above these elements can also be stacked to create an infinite number of T_2-avoiding elements. In addition, as n gets large there are a number of modifications that can be made that result in additional T_2-avoiding elements. From this we conjecture that the classification of T-avoiding elements in Coxeter systems of type F_n for $n \geq 6$ gets complicated very quickly. Classifying T-avoiding elements in $W(F_n)$ for $n \geq 6$ remains an open problem.

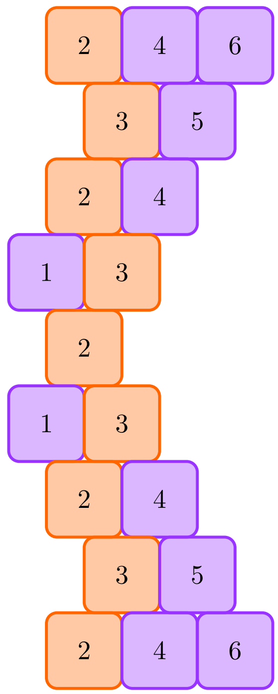


Figure 3.4: Heap of a T_2-avoiding element in $W(F_6)$.

3.4 Type $I_2(m)$

In this section, we next classify the T-avoiding elements in Coxeter systems of type $I_2(m)$. Note that in Coxeter systems of type $I_2(m)$, the only products of commuting generators have length 1. Although the following is a quick result, we believe that it does not already appear in the literature.

Theorem 3.4.1. There are no T_2-avoiding elements in Coxeter systems of type $I_2(m)$.

Proof. The graph for the Coxeter system of $I_2(m)$ appears in Figure 1.2(c). Note that the graph consists of two vertices, namely, s_1 and s_2, and a single edge with weight m. Also, recall that $W(I_2(m))$ is a star reducible Coxeter group. This implies that any T_2-avoiding elements in $W(I_2(m))$ must not be FC, as all of the FC elements have Property T or are T_1-avoiding. The only non-FC element in $W(I_2(m))$ is the element of length m that has exactly two reduced expressions consisting of alternating products of s_1 and s_2. Cleary, this element begins and ends with a product of noncommuting generators. Thus, this element has Property T. Hence $W(I_2(m))$ has no T_2-avoiding elements. $\square$

Chapter 4

T-Avoiding Elements in Type B_n

In this chapter we classify the T-avoiding elements in Coxeter systems of type B_n, a new result. We start by introducing some combinatorial tools for type B_n and then finish with a proof of the classification in type B_n. Note that the proof for Coxeter systems of type B_n closely follows the classification of T-avoiding elements of type D_n seen in [8].

4.1 Tools for the Classification

Recall from Example 1.2.1 that $W(B_n) \cong \mathrm{Sym}_n^B$ (also called the hyperoctahedral group). We define Sym_n^B to be the group of all bijections w of the set $\{-n, \ldots, 0, 1, 2, \ldots, n\}$ in itself such that

$$w(-a) = -w(a)$$

for all $a \in \{-n, -n-1, \ldots, 0, 1, 2, \ldots, n\}$ where the group operation is composition. For $w \in \mathrm{Sym}_n^B$ we write $w = [a_1, a_2, \ldots, a_n]$, to mean that $w(i) = a_i$ for $i \in \{1, 2, \ldots n\}$ and call this the signed permutation notation of w. That is we can write $w \in W(B_n)$ using signed permutation notation

$$w = [w(1), w(2), \ldots, w(n-1), w(n)],$$

where we write a bar underneath a number in place of a negative sign in order to simplify notation.

As a set of generators for Sym_n^B we take $S_B = \{s_1, s_2, \ldots, s_{n-1}, s_0\}$, where for each $i \in \{1, 2, \ldots n-1\}$, we have

$$s_i = [1, 2, \ldots i-1, i+1, i, i+2, \ldots, n-1, n]$$

and we identify s_0 with

$$s_0 = [\underline{1}, 2 \ldots, n].$$

Further $w(-i) = -w(i)$ for $|i| \in \{1, 2, \ldots, n\}$. The following propositions provide insight into what happens to a given one-line notation when we multiply by s_i on the right or the left.

Proposition 4.1.1. Let $w \in W(B_n)$ with corresponding signed permutation

$$w = [w(1), w(2), \ldots, w(n)].$$

Suppose $s_i \in S(B_n)$. If $i \geq 1$, then multiplying w on the right by s_i has the effect of interchanging $w(i)$ and $w(i+1)$ in the signed permutation notation. If $i = 0$, then multiplying w on the right by s_i has the effect of switching the sign of $w(1)$.

Proof. This follows from [2, Section 8.1 and A3.1]. □

Proposition 4.1.2. Let $w \in W(B_n)$ with corresponding signed permutation

$$w = [w(1), w(2), \ldots, w(n)].$$

Suppose $s_i \in S(B_n)$. If $i \geq 1$, then multiplying on the left by s_i has the effect of interchanging the entries whose absolute values are i and $i+1$ in the signed permutation notation. If $i = 0$, then multiplying w on the left by s_i has the effect of switching the sign of the entry whose absolute value is 1.

Proof. This follows from [2, Section 8.1 and A3.1]. □

Suppose $w \in W(B_n)$ has reduced expression $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$. We may construct the signed permutation of w from left to right as it is the easier way to multiply based upon the above propositions. We provide an example of this construction below.

Example 4.1.3. Let $w \in W(B_6)$ with a given reduced expression $\mathsf{w} = s_0 s_1 s_3 s_4 s_5 s_2$. Then we iteratively build the signed permutation as follows. First, $s_0 = [\underline{1}, 2, 3, 4, 5, 6]$ by definition. Next $s_0 s_1 = [2, \underline{1}, 3, 4, 5, 6]$ since multiplying by s_1 on the right hand side switches the values in position 1 and position 2. Repeating this we get $s_0 s_1 s_3 = [2, \underline{1}, 4, 3, 5, 6]$ and ultimately we end with $w = [2, 4, \underline{1}, 5, 6, 3]$.

Notice that if we were to construct the signed permutation for w from right to left, we would start with $s_2 = [1, 3, 2, 4, 5, 6]$. Next we would have $s_5 s_2 = [1, 3, 2, 4, 6, 5]$. However, $s_4 s_5 s_2 = [1, 3, 2, 5, 6, 4]$. Notice this time we were not able to just switch $w(i)$ and $w(i+1)$ instead we found 4 and 5 and switched their relative positions, which is more difficult than constructing the signed permutation notation notation left to right, which is why we choose to construct left to right.

Given the signed permutation notation for an element $w \in W(B_n)$ we can easily calculate the left and right descent sets of w. The following proposition explains how.

Proposition 4.1.4. Let $w \in W(B_n)$. Then

$$\mathcal{R}(w) = \{s_i \in S \mid w(i) > w(i+1)\}$$

where $w(0) = 0$ by definition.

Proof. This is [2, Proposition 8.1.2]. $\square$

We now will introduce the concept of signed pattern avoidance, which will help with the classification of the T-avoiding elements in Coxeter systems of type B_n. Our approach mimics the one found in [8]. Let $w \in W(B_n)$, and let $a, b, c \in \mathbb{Z}$. We say that w *contains the signed consecutive pattern abc* if there is some $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same relative order as $(|a|, |b|, |c|)$ and $\mathrm{sgn}(w(i)) = \mathrm{sgn}(a), \mathrm{sgn}(w(i+1)) = \mathrm{sgn}(b)$, and $\mathrm{sgn}(w(i+2)) = \mathrm{sgn}(c)$, where typically one takes $\{a, b, c\} = \{\pm 1, \pm 2, \pm 3\}$. We say that w *avoids the signed consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same consecutive order as $(|a|, |b|, |c|)$ and such that $\mathrm{sgn}(w(i)) = \mathrm{sgn}(a)$, $\mathrm{sgn}(w(i+1)) = \mathrm{sgn}(b)$, and $\mathrm{sgn}(w(i+2)) = \mathrm{sgn}(c)$.

Example 4.1.5. Let $w \in W(B_4)$ with signed permutation

$$w = [\underline{2}, 4, \underline{1}, 3].$$

We see that w has the signed consecutive pattern $\underline{2}3\underline{1}$, since $(|w(1)|, |w(2)|, |w(3)|)$ are in the same relative order as $(|-2|, |3|, |-1|)$, and $\mathrm{sgn}(w(1)) = \mathrm{sgn}(-2)$, $\mathrm{sgn}(w(2)) = \mathrm{sgn}(3)$, and $\mathrm{sgn}(w(3)) = \mathrm{sgn}(-1)$. However, w avoids the signed consecutive pattern $1\underline{2}3$.

Occasionally, we will need to factor $w \in W(B_n)$ in a specific manner. Let $I = \{s, t\}$ for $s, t \in S(B_n)$. Define W^I as the set of all $w \in W(B_n)$ such that $\mathcal{L}(w) \cap I = \emptyset$ and define $W_I = \langle s, t \rangle$. In [10], it is shown that any element $w \in W(B_n)$ can be written as $w = w^I w_I$ reduced where $w_I \in W_I$ and $w^I \in W^I$.

4.2 Classification of T-Avoiding Elements in Type B_n

In this section we will classify the T-avoiding elements in Coxeter systems of type B_n. Our main result in this section is Theorem 4.2.17. Notice that if $n = 2$, $W(B_2) \cong I_2(4)$, which by Section 3.4 we know has no T_2-avoiding elements. We proceed for $n \geq 3$. First we need some preparatory lemmas.

Lemma 4.2.1. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$. Then w has a reduced expression ending in sts if and only if w has the consecutive pattern 321.

Proof. Let $i \geq 1$, $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$. Note that since $m(s_i, s_{i+1}) = 3$, $s_0 \notin I$. Observe that if w has a reduced expression ending in the product of two non-commuting generators $s_i s_{i+1}$ or $s_{i+1} s_i$, then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\}$.

Suppose w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+2)$. By Proposition 4.1.4, $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $m(s_i, s_{i+1}) = 3$ and $s_i, s_{i+1} \in \mathcal{R}(w)$, w ends in $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$.

Conversely, suppose w ends in $s_i s_{i+1} s_i$. This implies that either $w_I = s_i s_{i+1} s_i$ or $w_I = s_{i+1} s_i s_{i+1}$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by Proposition 4.1.4. Thus w has the consecutive pattern 321. Therefore, w has a reduced expression ending in sts if and only if w has the consecutive pattern 321. $\square$

Corollary 4.2.2. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$. Then w has a reduced expression beginning with sts if and only if w^{-1} has the consecutive pattern 321.

Proof. Let $s, t \in S(B_n)$ such that $m(s,t) = 3$ and $s_0 \notin \{s,t\}$. We know that w has no reduced expressions beginning with sts if and only if w^{-1} has no reduced expression ending with sts which by Theorem 4.2.2 happens only if w^{-1} avoids the consecutive pattern 321. $\square$

Lemma 4.2.3. If $i \neq 0$, then w has a reduced expression ending in $s_i s_{i+1}$ if and only if w has the consecutive pattern 231.

Proof. Suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$. By Proposition 4.1.4, $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by s_{i+1} we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, which implies that $s_i \in \mathcal{R}(ws_{i+1})$, and hence w has a reduced expression that ends in $s_i s_{i+1}$.

Conversely, suppose that w has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence w has the consecutive pattern 231.

Therefore, w has a reduced expression ending in $s_i s_{i+1}$ if and only if w has the consecutive pattern 231. $\square$

Corollary 4.2.4. If $i \neq 0$, then w has a reduced expression beginning with $s_i s_{i+1}$ if and only if w^{-1} has the consecutive pattern 231.

Proof. Let $s_i, s_{i+1} \in S(B_n)$ such that $m(s_i, s_{i+1}) = 3$ and $s_0 \notin \{s_i, s_{i+1}\}$. We know that w has no reduced expressions beginning with $s_i s_{i+1}$ if and only if w^{-1} has no reduced expression ending with $s_i s_{i+1}$ which by Theorem 4.2.2 happens only if w^{-1} avoids the consecutive pattern 231. $\square$

Lemma 4.2.5. If $i \neq 0$, then w has a reduced expression ending in $s_{i+1}s_i$ if and only if w has the consecutive pattern 312.

Proof. Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. By Proposition 4.1.4 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by s_i we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$. By above $w(i) > w(i+2)$, and by Proposition 4.1.4 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that w has a reduced expression ending in $s_{i+1}s_i$.

Conversely suppose w ends in a reduced expression with $s_{i+1}s_i$ Then $w_I = s_{i+1}s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, w has the consecutive pattern 312.

Therefore, w has a reduced expression ending in $s_{i+1}s_i$ if and only if w has the consecutive pattern 312. $\square$

Corollary 4.2.6. If $i \neq 0$, then w has a reduced expression beginning with $s_{i+1}s_i$ if and only if w^{-1} has the consecutive pattern 312.

Proof. Let $s_i, s_{i+1} \in S(B_n)$ such that $m(s,t) = 3$ and $s_0 \notin \{s_i, s_{i+1}\}$. We know that w has no reduced expression beginning with $s_{i+1}s_i$ if and only if w^{-1} has no reduced expression ending with $s_{i+1}s_i$ which by Theorem 4.2.2 happens only if w^{-1} avoids the consecutive pattern 312. $\square$

Lemma 4.2.7. Let $w \in W(B_n)$. Then w has a reduced expression ending in $s_1 s_0$ if and only if $w(0) > w(1)$ and $-w(1) > w(2)$.

Proof. Suppose $w \in W(B_n)$ such that w ends with $s_1 s_0$. Then $s_0 \in \mathcal{R}(w)$ and $s_1 \in \mathcal{R}(ws_0)$. This implies that $ws_0(1) > ws_0(2)$ by Proposition 4.1.4. We see that $ws_0(1) = w(-1) = -w(1)$ and $ws_0(2) = 2$. Hence $-w(1) = ws_0(1) > ws_0(2) = w(2)$. Further, since $s_0 \in \mathcal{R}(w)$, we see that $w(0) > w(1)$.

Conversely, suppose $w \in W(B_n)$ such that $w(0) > w(1)$ and $-w(1) > w(2)$. Since $w(0) > w(1)$, we know that $s_0 \in \mathcal{R}(w)$. Multiplying on the right by s_0 we see that $ws_0(1) = -w(1)$ and $ws_0(2) = w(2)$. Note that since $ws_0(1) = -w(1) > w(2) = ws_0(2)$, $s_1 \in \mathcal{R}(ws_0)$. Thus w ends with $s_1 s_0$.

Therefore, w has a reduced expression ending in $s_1 s_0$ if and only if $w(0) > w(1)$ and $-w(1) > w(2)$. $\square$

Corollary 4.2.8. Let $w \in W(B_n)$. Then w has a reduced expression beginning in $s_0 s_1$ if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$.

Proof. Let $w \in W(B_n)$. We know that w has no reduced expressions beginning in $s_0 s_1$ if and only if w^{-1} has no reduced expressions ending in $s_0 s_1$. By Lemma 4.2.7 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$. ☐

Lemma 4.2.9. Let $w \in W(B_n)$. Then w has a reduced expression ending in $s_0 s_1$ if and only if $w(0) > w(2)$ and $w(1) > w(2)$.

Proof. Suppose $w \in W(B_n)$ such that w ends with $s_0 s_1$. Then $s_1 \in \mathcal{R}(w)$ and $s_0 \in \mathcal{R}(ws_1)$. Then $ws_1(0) > ws_1(1)$. We see that $ws_1(0) = 0$ and $ws_1(1) = w(2)$. This implies that $0 = ws_1(0) > ws_1(1) = 2$. Further, since $s_1 \in \mathcal{R}(w)$ this implies that $w(1) > w(2)$. Thus if w ends with $s_0 s_1$, then $w(1) > w(2)$ and $w(0) > w(2)$.

Conversely, suppose $w \in W(B_n)$ such that $w(1) > w(2)$ and $w(0) > w(2)$. This implies that $s_1 \in \mathcal{R}(W)$. Multiplying w on the right by s_1 we see that $ws_1(0) = w(0)$ and $ws_1(1) = w(2)$. Note that since $ws_1(0) = w(0) > w(2) = ws_1(1)$, $s_0 \in \mathcal{R}(ws_1)$. Thus w ends with $s_0 s_1$.

Therefore, w has a reduced expression ending in $s_0 s_1$ if and only if $w(1) > w(2)$ and $w(0) > w(2)$. ☐

Corollary 4.2.10. Let $w \in W(B_n)$. Then w has a reduced expression beginning in $s_1 s_0$ if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$.

Proof. Let $w \in W(B_n)$. We know that w has no reduced expressions beginning in $s_1 s_0$ if and only if w^{-1} has no reduced expressions ending in $s_1 s_0$. By Lemma 4.2.7 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$. ☐

Lemma 4.2.11. Let $w \in W(B_n)$ such that each entry for w in the signed permutation notation is positive and both w and w^{-1} avoid the consecutive patterns 321, 231, and 312. Then w is a product of commuting generators.

Proof. This follows from an appropriate translation of [8, Lemma 2.2.9]. ☐

Lemma 4.2.12. Let $w \in W(B_n)$ be T_1-avoiding and let $i \in \{1, 2, \ldots, n\}$. Then w satisfies all the following conditions:

(1) $w(j) > \min\{w(i-1), w(i)\}$ for all $j > i$;

(2) $w(k) < \max\{w(i-1), w(i)\}$ for all $k < i-1$;

(3) If $w(i), w(i+1) > 0$, then $w(j) > 0$ for all $j \geq i$;

(4) If $w(i), w(i+1) < 0$, then $w(j) < 0$ for all $j \leq i+1$.

Proof. Suppose there is some least $j > i$ such that $w(j) \leq \min\{w(i-1), w(i)\}$. Note that $j > i$ so $j \neq i$, and $j \neq i-1$ so $w(j) < \min\{w(i-1), w(i)\})$. Note that j is the least so $w(j-2) \geq \min\{w(i+1), w(i)\} > w(j)$. This implies that either $w(j-1) > w(j-2) > w(j)$ or $w(j-2) > w(j-1) > w(j)$, which implies w has the consecutive pattern 231 or 321, which is a contradiction to w being a T_1-avoiding element by Lemmas 4.2.1 and 4.2.5. Thus proving (1).

Suppose there exists a maximal $k < i-1$ such that $w \geq \max\{w(i-1), w(i)\}$. Note that $k < i-1$ so $k \neq i$ and $k \neq i-1$. Then $w(k) > \max\{w(i-1), w(i)\}$. Since k is maximal $w(k+1) \leq \max\{w(i-1), w(i)\}$ and $w(k+2) \leq \max\{w(i-1), w(i)\}$. This implies that either $w(k+2) < w(k+1) < w(k)$ or $w(k+1) < w(k+2) < w(k)$, which implies w has the consecutive pattern 321 or 312, which is a contradiction to w being a T_1-avoiding element by Lemmas 4.2.1 and 4.2.3. Thus proving (2).

It is easy to see that Assertion (1) implies (3) and Assertion (2) implies (4). □

Lemma 4.2.13. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}31$. Then w has Property T.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}31$.

Case (1): Suppose w has the signed permutation notation $w = [\underline{2}, 3, 1]$. This implies that $w = s_1 s_0 s_2$. Clearly, w begins with a product of non-commuting generators. Thus w has Property T.

Case (2): Suppose that w has the signed permutation notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{2}31$, and $*$ indicates unknown values for $w(i)$ for $i = 4, 5, \ldots, n$. We now consider the possible signed consecutive pattern $bc*$. The following are the possibilities: 312, $31\underline{2}$, 321, $32\underline{1}$, 213, or $21\underline{3}$. We know that b and c must be positive since they are positive in w and we also know that $b > c$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{2}31$, and $*$ indicates unknown values for $w(i)$ for $i = 1, 2, \ldots, n-3$. We now consider the possible signed consecutive pattern $*\underline{a}b$. The following are the possibilities: $1\underline{2}3$, $\underline{1}\underline{2}3$, $2\underline{1}3$, $\underline{2}\underline{1}3$, $3\underline{1}2$, or $\underline{3}\underline{1}2$. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, c, *, \ldots, *]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{2}31$, and $*$ indicates unknown values for $w(i)$ for $|w(i)| \neq a, b, c$. In this case we can apply either Case (2) or Case (3) and we can conclude that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}31$, then w has Property T. $\square$

Lemma 4.2.14. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}3\underline{1}$. Then w has Property T.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{2}3\underline{1}$.

Case (1): Suppose w has the signed permutation notation $w = [\underline{2}, 3, \underline{1}]$. This implies that $w = s_0 s_1 s_0 s_2$. Clearly, w begins with a product of non-commuting generators. Thus w has Property T.

Case (2): Suppose that w has the signed permutation notation $w = [\underline{a}, b, \underline{c}, *, \ldots, *]$ where $\underline{a}b\underline{c}$ corresponds to the signed consecutive pattern $\underline{2}3\underline{1}$, and $*$ indicates unknown values for $w(i)$ for $i = 4, 5, \ldots n$. We now consider the possible signed consecutive pattern $b\underline{c}*$. The following are the possibilities: $3\underline{1}2$, $3\underline{1}\underline{2}$, $3\underline{2}1$, $3\underline{2}\underline{1}$, $2\underline{1}3$, or $2\underline{1}\underline{3}$. We know that b must be positive since it is positive in w, c must be negative since it is negative in w, and we also know that $|b| > |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, \underline{c}]$ where $\underline{a}b\underline{c}$ corresponds to the signed consecutive pattern $\underline{2}3\underline{1}$, and $*$ indicates unknown values for $w(i)$ for $i = 1, 2, \ldots, n-3$. We now consider the possible signed consecutive pattern $*ab$. The following are the possibilities: $1\underline{2}3$, $\underline{1}\underline{2}3$, $2\underline{1}3$, $\underline{2}\underline{1}3$, $3\underline{1}2$, or $\underline{3}\underline{1}2$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, \underline{c}, *, \ldots, *]$ where $\underline{a}b\underline{c}$ corresponds to the signed consecutive pattern $\underline{2}3\underline{1}$, and $*$ indicates unknown values for $w(i)$ for $|w(i)| \neq a, b, c$. In this case we can apply either Case (2) or Case (3) and we can conclude that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}3\underline{1}$, then w has Property T. $\square$

Lemma 4.2.15. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}23$. Then either w has Property T or is a T_1-avoiding element.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}23$.

Case (1): Suppose w has the signed permutation notation $w = [\underline{1}23]$. This implies that $w = s_0$. Clearly, w is a T_1-avoiding element as it is a single generator.

Case (2): Suppose that w has the signed permutation notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{1}23$, and $*$ indicates unknown values for $w(i)$ for $i = 4, 5, \ldots n$. We now consider the possible signed consecutive patterns $bc*$. The following are the possibilities: 231, $23\underline{1}$, 132, $13\underline{2}$, 123, $12\underline{3}$. We know that b and c are positive, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{1}23$, and $*$ indicates unknown values for $w(i)$ for $i = 1, 2, \ldots, n-3$. We now consider the possible signed consecutive patterns $*\underline{a}b$. The following are the possibilities: $3\underline{1}2$, $\underline{31}2$, $2\underline{1}3$, $\underline{21}3$, $1\underline{2}3$, or $\underline{12}3$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9, all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, c, *, \ldots, *]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{1}23$, and $*$ indicates unknown values for $w(i)$ for $|w(i)| \neq a, b, c$. In this case we can apply either Case (2) or Case (3) and we can conclude that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}23$, then w has Property T or is a T_1-avoiding element. □

Lemma 4.2.16. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}32$. Then either w has Property T or is a T_1-avoiding element.

Proof. Let $w \in W(B_n)$ such that w has the consecutive pattern $\underline{1}32$.

Case (1): Suppose w has the signed permutation notation $w = [\underline{1}32]$. This implies that $w = s_0 s_2$. Clearly, w is a T_1-avoiding element as it is a product of commuting generators.

Case (2): Suppose that w has the signed permutation notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{1}32$, and $*$ indicates unknown values for $w(i)$ for $i = 4, 5, \ldots n$. We now consider the possible signed consecutive pattern $bc*$. The following are the possibilities: 231, $23\underline{1}$, 132, $13\underline{2}$, 123, or $12\underline{3}$. We know that b and c are positive, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.1, 4.2.3, and 4.2.7 all of these patterns imply that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (3): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{1}32$, and $*$ indicates unknown values for

$w(i)$ for $i = 1, 2, \ldots, n-3$. We now consider the possible signed consecutive pattern $*\underline{a}b$. The following are the possibilities: $3\underline{1}2$, $\underline{31}2$, $2\underline{1}3$, $\underline{21}3$, $3\underline{2}1$, or $\underline{32}1$. We know that a must be negative, b must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.3, 4.2.7, and 4.2.9 all of these patterns implies that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Case (4): Suppose that w has the signed permutation notation $w = [*, \ldots, *, \underline{a}, b, c, *, \ldots, *]$ where $\underline{a}bc$ corresponds to the signed consecutive pattern $\underline{1}32$, and $*$ indicates unknown values for $w(i)$ for $w(i) \neq a, b, c$. In this case we can apply either Case (2) or Case (3) and we can conclude that w ends or begins with a product of noncommuting generators. Thus w has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}32$, then w has Property T or is a T_1-avoiding element. $\square$

We now are ready to tackle one of the main results of this thesis.

Theorem 4.2.17. There are no T_2-avoiding elements in $W(B_n)$.

Proof. We proceed by contradiction. Suppose that $w \in W(B_n)$ is a T_2-avoiding element. There are $2^3 \cdot 3! = 48$ possible choices of signed consecutive patterns for $w(1)w(2)w(3)$ where $w = [w(1), w(2), w(3), *, \ldots, *]$. These 48 signed consecutive patterns are seen in the table below. We only consider these signed consecutive patterns in the first three entries of the signed permutation representation, as if we can eliminate all possibilities we have a contradiction to w being a T_2-avoiding element.

123	$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$	$\underline{12}3$	$\underline{1}2\underline{3}$	$1\underline{23}$	$\underline{123}$
132	$\underline{1}32$	$1\underline{3}2$	$13\underline{2}$	$\underline{13}2$	$\underline{1}3\underline{2}$	$1\underline{32}$	$\underline{132}$
213	$\underline{2}13$	$2\underline{1}3$	$21\underline{3}$	$\underline{21}3$	$\underline{2}1\underline{3}$	$2\underline{13}$	$\underline{213}$
231	$\underline{2}31$	$2\underline{3}1$	$23\underline{1}$	$\underline{23}1$	$\underline{2}3\underline{1}$	$2\underline{31}$	$\underline{231}$
312	$\underline{3}12$	$3\underline{1}2$	$31\underline{2}$	$\underline{31}2$	$\underline{3}1\underline{2}$	$3\underline{12}$	$\underline{312}$
321	$\underline{3}21$	$3\underline{2}1$	$32\underline{1}$	$\underline{32}1$	$\underline{3}2\underline{1}$	$3\underline{21}$	$\underline{321}$

We can use Lemma 4.2.1 and Corollary 4.2.2 to eliminate the signed consecutive patterns highlighted in turquoise. In addition, using Lemma 4.2.5 and Corollary 4.2.4 to eliminate the signed consecutive patterns highlighted in red. From Lemma 4.2.3 and Corollary 4.2.6 we eliminate the consecutive patterns highlighted in green. Using Lemma 4.2.7 and Corollary 4.2.8 we are able to eliminate the signed consecutive patterns highlighted in yellow. Also Lemma 4.2.9 and Corollary 4.2.10 show that w will not have the signed consecutive patterns highlighted in brown. We also use Lemma 4.2.11 to eliminate the signed consecutive patterns highlighted in blue. From Lemmas 4.2.13 and 4.2.14 we are able to eliminate signed consecutive patterns highlighted in purple. Finally, we can use Lemmas 4.2.15 and 4.2.16 to eliminate signed consecutive patterns highlighted in orange. Since all of the above patterns are eliminated as possibilities for $w(1)w(2)w(3)$ and there are no other signed consecutive patterns that are possible for these positions, and hence w is not a T_2-avoiding element in the Coxeter group of type B_n. $\square$

The upshot of Theorem 4.2.17 is that the only T-avoiding elements in Coxeter systems of type B_n are products of commuting generators and the identity.

Chapter 5

T-Avoiding Elements in Type $\widetilde{C}_n$

5.1 Classification of T-Avoiding Elements in Type $\widetilde{C}_n$

In this section we will classify the T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$, a new result. Since $W(A_n)$ and $W(B_n)$ are parabolic subgroups of $W(\widetilde{C}_n)$ and these groups have no T_2-avoiding elements, any T_2-avoiding elements of $W(\widetilde{C}_n)$ must have full support. We will first show that there are no T_2-avoiding elements that are not FC in $W(\widetilde{C}_n)$.

Before we begin the proof we must first define the notion of a pushed-down representation of a heap. First recall that there are potentially many ways to draw the lattice point representation of a heap, each differing by the amount of vertical space between blocks. We wish to fix one such representation. Let w be a reduced expression for $w \in W(\widetilde{C}_n)$. We construct the *pushed-down representation of a heap* of $H(\mathsf{w})$ by first giving all blocks fully exposed to the bottom the same vertical position, and then all blocks are as low as possible in the heap. Loosely speaking the heap has been constructed by placing all blocks in the lowest possible vertical position of the heap. Notice that we can now label the rows in the heap from bottom to top where the bottom-most row is row 1 and proceeding naturally upward from there.

We now define the height of a braid. Given the presence of a braid in the heap of an reduced expression w, we say that the *height of the braid* is the row number in which the uppermost block involved in the braid is located in the pushed-down representation. It is important to note that in the pushed-down representation a braid may not appear in consecutive rows. That is, some of the blocks may be lower in the heap and the braid may not be immediately apparent.

Example 5.1.1. Let $\mathsf{w} = s_0s_1s_3s_2s_1s_0s_1s_3$ be a reduced expression for $w \in W(C_3)$. The pushed-down representation of a heap of w is given in Figure 5.1. The height of the braid $s_1s_2s_1$ which is highlighted in teal in Figure 5.1 is 5 since the upper block for s_1 is located in the fifth row of the pushed-down representation heap. Notice that the block for s_3 can slide up higher in the heap. If we were to slide the block for s_3 up until it hits the block for s_2 we would obtain the braid $s_3s_2s_3$ in the heap. In this case, the height for the braid $s_3s_2s_3$ is also 5.

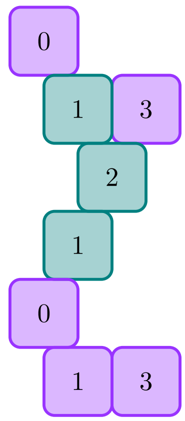
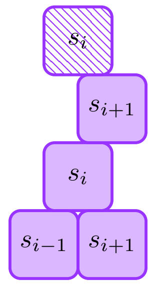


Figure 5.1: Pushed-down representation of a heap

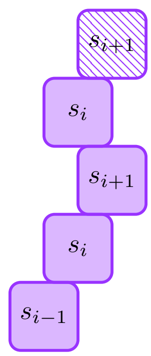
Theorem 5.1.2. There are no T_2-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

Proof. We proceed by contradiction. Let $w \in W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$ such that w has full support and w is T-avoiding. Consider all possible pushed-down representations for heaps of w. Choose a representation that has a minimal height braid among all braids appearing in all heaps for w and let k represent that minimum height. There may be a tie, in which case choose your favorite. Without loss of generality we will call the generators involved in the braid s_i, s_{i+1} where the bond strength is case specific and will be given in the following cases. In the following cases whenever we refer to a block being in a specific row, we are considering the pushed-down representation of the heap. However, in order to consider the braids that we are looking for we need to allow some flexibility when referring to the absolute vertical position of a given block. In the following cases, whenever we refer to a subheap of w they are assumed to be convex.

Case (1): Suppose the lowest braid occurs in the bottom-most row where $k = 3$ (respectively, $k = 4$) if $m(s_i, s_{i+1}) = 3$ (respectively, $m(s_i, s_{i+1}) = 4$). In this case, we are assuming that the braid is located in consecutive rows with the upper-most block in the above specified row and the lowest block in the heap located in the bottom-most row of the heap. Without loss of generality assume s_{i+1} is in the bottom-most row of the heap. Clearly, the block for s_{i-1} must be in the bottom-most row of the heap as well, otherwise w has Property T which is a contradiction to the original choice of w. From this restricting our view to the subheap of w that contains the braid we are considering we see that the heap of w has the following form



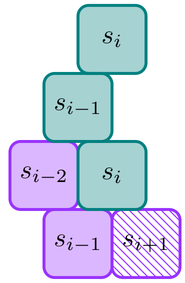
where the striped heap block represents the fourth block in the braid if $m(s_i, s_{i+1}) = 4$. Applying the braid move we get the subheap seen here



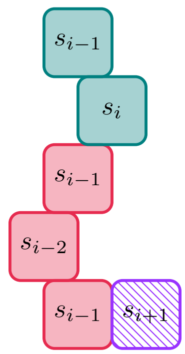
which clearly has Property T since s_{i-1} is now in the 1st row of the pushed-down representation. This is a contradiction to the way in which we chose w. T For the rest of the cases we will assume that $k \geq 4$.

Case (2): Suppose the braid has height k and assume the braid does not contain s_0, s_1, s_{n-1} or s_n. Without loss of generality assume s_i is in the kth row of the heap and if necessary we have brought the blocks for s_{i-1} and s_i up next to s_i in row k. We now consider what can be in the $(k-3)$th row of the heap in two cases.

Subcase (a): Assume that the block for s_{i-1} is in the heap in the $(k-3)$th row and we allow for the block for s_{i+1} to be in the same row as well, but it does not necessarily have to be. In the following pictures the block for s_{i+1} will be represented in a purple striped block to indicate that it could be present but it does not have to be. The following is the subheap that we are considering

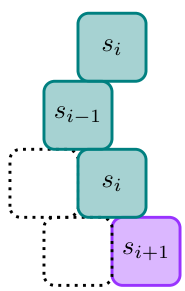


where we have highlighted the braid in teal. Notice that the block for s_{i-2} is present in the $(k-2)$th row. If it was not there, w would have had the braid $s_{i-1}s_is_{i-1}$ since $m(s_{i-1}, s_i) = 3$ and we would have had a heap with a lower braid to choose. Applying the braid move to the heap we get the following subheap

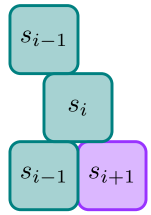


which has a new braid in it. This braid which we have highlighted in red for emphasis has height $k-1$. In applying the braid move we have obtained a subheap which has a braid that has a lower height than our original choice. This is a contradiction to the way in which we chose our heap.

Subcase (b): Assume that the block for s_{i+1} is in the $(k-3)$th row of the heap and the block for s_{i-1} does not appear in the $(k-3)$th row and s_{k-2} does not appear in the $(k-2)$th. The following is the subheap we are considering



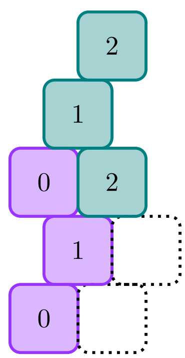
where the dotted square represents that no block may occupy this space and the braid is highlighted in teal. Applying the braid move in the subheap we obtain the following heap



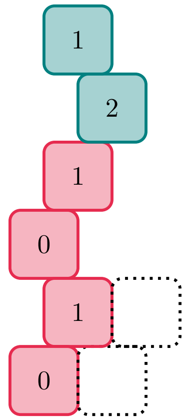
where the braid now occurs in the $(k-1)$th row. This contradicts the way in which we chose the heap of w. From this we gather that the braid must contain s_0, s_1, s_{n-1}, or s_n.

Case 3: Suppose the braid is located in the kth row and assume the braid contains s_2 or s_{n-2}. Without loss of generality we assume that the braid contains s_2 as the other argument is symmetric to the one presented here, and assume s_2 is in the kth row. Notice that if the braid contains $s_2s_3s_2$, we are in Case (2), as a result we assume our braid is not of the form $s_2s_3s_2$. Assume that if necessary the blocks s_1 and s_2 have been brought up next to s_2 in row k. We now consider what can be in the $(k-3)$th and $(k-4)$th rows of the heap in two cases.

Subcase (a): Assume the block for s_1 is in row $k-3$, and s_0 is in row $k-4$ but s_3 is not in the $(k-3)$th row and s_2 is not in the $(k-4)$th row. Then the subheap we are considering is as follows

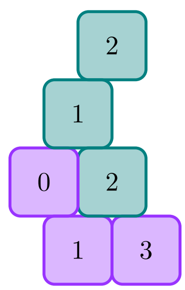


where the braid we are considering in the heap is highlighted in teal. Notice that the block for s_0 is in the $(k-2)$th row. It must be here, as if it was not w would not be reduced. Applying the braid move we get the following heap

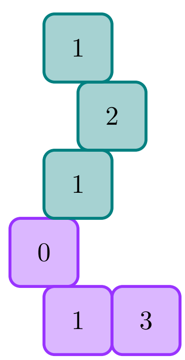


where a new braid has appeared highlighted in red. Notice that the height of this new braid is $k-1$. This braid is lower in the heap than our original braid. This is a contradiction to the original choice of heap.

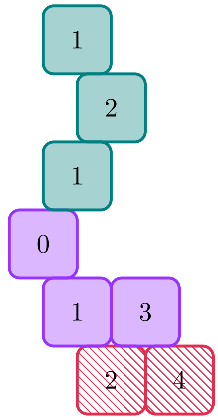
Subcase (b): Assume the block for s_0 is in the $(k-2)$th row, and s_1 and s_3 are in the $(k-3)$th row. Then the subheap we are considering is



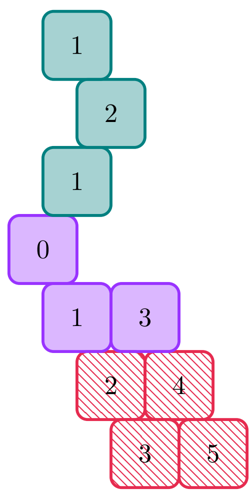
where the braid is highlighted in teal. Applying the braid move we get the following subheap



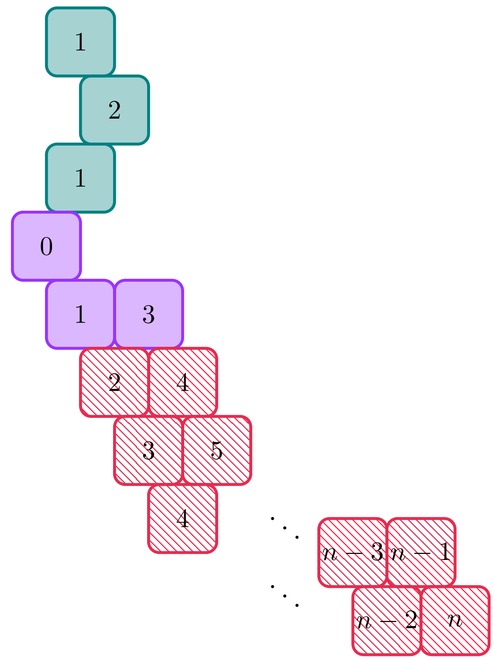
where no new braids have appeared, and in fact the original braid is higher now with height $k+1$. Notice however, that the $(k-3)$th row will not be the bottom row of our heap because then w would contain Property T a contradiction to our assumption. With this in mind we consider the $(k-4)$th row. Notice that s_0 will not appear in the $(k-4)$th row as we would have a lower braid. This implies that the $(k-4)$th row contains at least one of s_2 or s_4. With this in mind we represent this with the following subheap



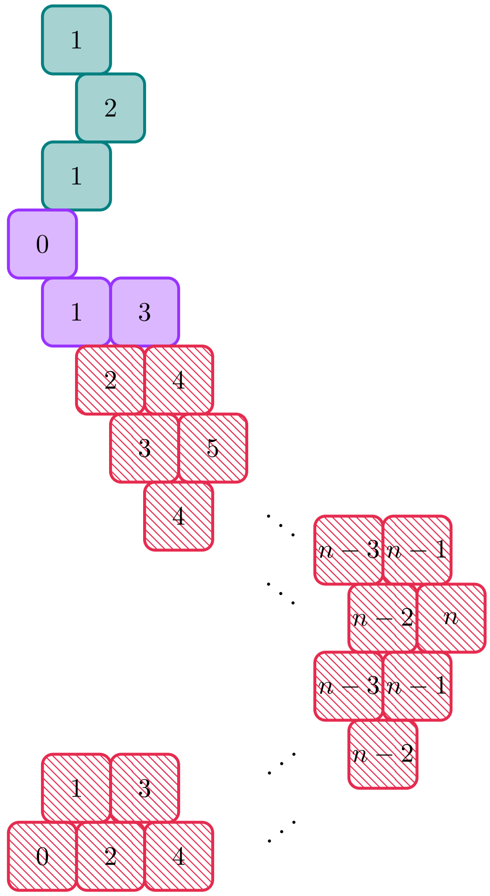
where we have highlighted the additions in red and striped blocks. Here these striped blocks represent that at least one of the blocks appear but possibly both. Notice that if this new row is row 1, then w would have Property T. Again, this implies that this will not be the bottom row of the heap. Repeating this process again we see that s_1 will not be in row $k-5$ since this would create a lower braid, thus we must have at least one of s_3 or s_5 in the $(k-5)$th row. We represent this with the following subheap



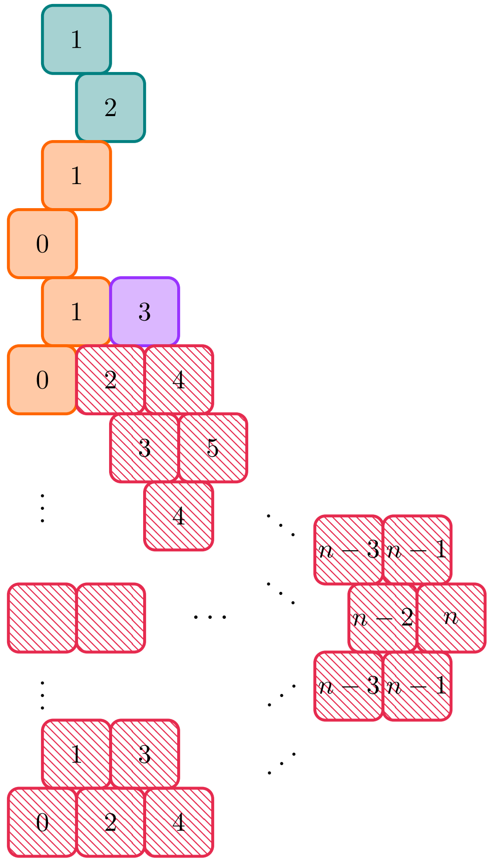
where again we have highlighted the additions in red and striped blocks. Again, if the $(k-5)$th row is the first row, then w would have Property T. This implies that the $(k-5)$th row is not the bottom-most row in our heap. Iterating this process we obtain the subheap as follows



where again we see that if the row containing the blocks for s_{n-2} and s_n corresponds to row 1, then w would have Property T. From this we obtain the following subheap



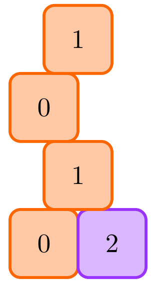
where the striped red blocks correspond to a portion of an FC element. Originally we said that the white inner triangle of this heap must be empty, however this contradicts [4, Lemma 3.3] which says that elements that have the triangle of white space must be filled completely in order to maintain that the heap is FC. That is, in the blank space between the purple blocks s_0, s_1 and the striped red blocks s_1, s_3 in the 2nd row every block will actually be present. This leads to the heap seen here



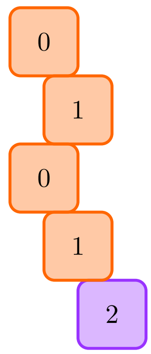
where we see that this has led to s_0 being located in the $(k-4)$th row. Notice that a new braid has now appeared in the heap. We have highlighted this in orange. This new braid has height $k-1$. This is a contradiction as the braid appears lower than the original braid we chose.

Case (4): Suppose the braid is located in the kth row and assume the braid contains s_1 or s_{n-1}. Without loss of generality we assume that the braid contains s_1, as the the other argument is symmetric to the one presented here and assume s_1 is in the kth row. Assume that, if necessary, the blocks that complete the braid have been brought up next to s_1 in the kth row. We now consider what can happen in the $(k-3)$th row and $(k-4)$th row in two cases. Notice that if the braid is $s_1s_2s_1$, then we are in Case 1, so assume the braid consists of s_0 and s_1.

Subcase (a): Assume the braid involves s_1 and s_0 and the block for s_2 is located in the $(k-3)$th row. Then the subheap we are considering follows here

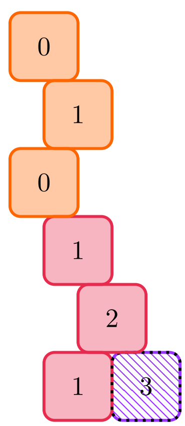


where the braid we are considering is highlighted in orange. Applying the braid move we get the following subheap



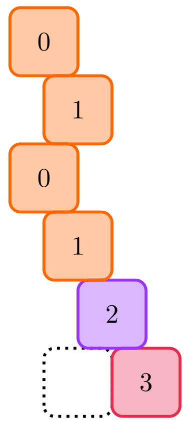
in which we see that the original braid is now located higher in the heap with height $k+1$. By Case (1), we know that the original heap we started with implies that s_0 and s_2 are located above row 1. This implies that the subheap has more rows underneath to fill in.

Subcase (i): We first consider if the block for s_1 is located in row $k-4$ and s_3 is allowed but not required to be there. This leads to the following heap

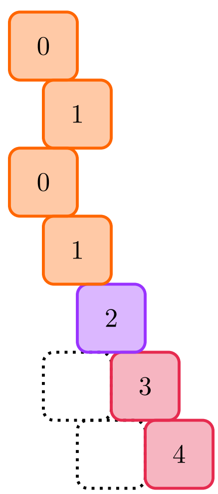


where a new braid appears which we have highlighted in red. This new braid has height $k-2$ which is lower than the height of the original braid that we chose. This is a contradiction to the way in which we chose w.

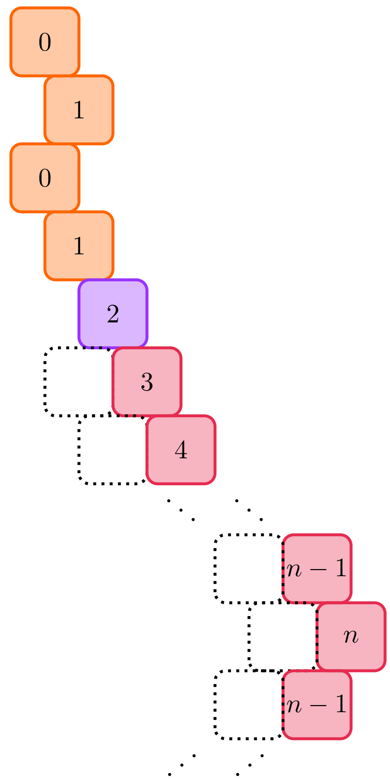
Subcase (ii): Now we consider if the block for s_3 is in the $(k-4)$th row and s_1 is not. This leads to the subheap seen here



where again there are no new braids present. However, the bottom row of the subheap is not the bottom row of the heap of w since otherwise w would have Property T. Using this notion we extend our heap to look like

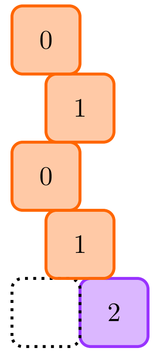


where again we see no new braids. Again, we know that the bottom row of the subheap is not the bottom row of the heap of w since otherwise w would have Property T. Iterating this process we obtain a heap that looks like

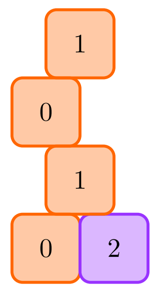


where the zig-zag pattern continues. That is, the red row of blocks continues without blocks above or below these blocks. Notice that if the zig-zag was to end before reaching s_0, the heap would have Property T, which is a contradiction to the way in which we chose w. Suppose that the zig-zag continues on after reaching s_0. Then reverting back to the original configuration of the subheap we are able to drop the block for s_0 down and create a lower braid. This is a contradiction to the way in which we chose the heap.

Case (5): Suppose the braid is located in the kth row and assume the braid contains s_0 or s_n. Without loss of generality we assume that the braid contains s_0 as the other argument is symmetric to the one presented here, and assume s_0 is in row k. We now consider what is in row $k-4$. Notice that s_0 can not be in the $(k-4)$th row as w would not be reduced. This implies that s_2 is in the $(k-4)$th row and we get the subheap below.



where we have highlighted the braid in orange. Applying the braid move we get the following subheap



which does not have any new braids. However, notice that the height of the braid is now $k-1$. This is a contradiction to our original assumption that the heap we started with contains the lowest braid.

Therefore, it follow that $W(\widetilde{C}_n)$ does not contain any not FC T_2-avoiding elements. $\square$

We will now classify the T_2-avoiding elements in $W(\widetilde{C}_n)$. We first classify T_2-avoiding elements in $W(\widetilde{C}_n)$ for n odd and then proceed to the classification for n even.

Theorem 5.1.3. If n is odd, then there are no T_2-avoiding elements in the Coxeter system of type $\widetilde{C}_n$.

Proof. Consider the Coxeter system of type $\widetilde{C}_n$. By Theorem 5.1.2 we know that $W(\widetilde{C}_n)$ contains no T_2-avoiding elements that are not FC. Recall $W(\widetilde{C}_n)$ is a star reducible Coxeter group, which implies that $W(\widetilde{C}_n)$ contains no T_2-avoiding elements that are FC. Thus as $W(\widetilde{C}_n)$ has no T_2-avoiding elements that are FC and no T_2-avoiding elements that are not FC, $W(\widetilde{C}_n)$ has no T_2-avoiding elements. $\square$

We next will classify the T_2-avoiding elements in the Coxeter system of type $\widetilde{C}_n$ for n even. Recall that $W(\widetilde{C}_n)$ for n even is not a star reducible Coxeter group. In Theorem 5.1.2

we showed that $W(\widetilde{C}_n)$ does not have T_2-avoiding elements that are not FC. This leaves us with only the FC elements to check.

Theorem 5.1.4. If n is even, then the only T_2-avoiding elements in $W(\widetilde{C}_n)$ are sandwich stacks.

Proof. Let $w \in W(\widetilde{C}_n)$. By Theorem 5.1.2, we know that w is an FC element. Further, we can restrict our search down to the subset of non-cancellable elements that are not star reducible. Specifically we can consider the non-cancellable elements that do not contain Property T. In Remark 2.3.4 we stated the classification of the only T_2-avoiding elements with full support. Recall these to be sandwich stacks. Thus the only T_2-avoiding elements in $W(\widetilde{C}_n)$ for n odd are sandwich stacks. $\square$

5.2 Future Work

In Sections 3.1–3.4, we relayed the known results involving T-avoiding elements in types $\widetilde{A}_n, A_n, D_n, F_4, F_5$, and proved results involving T-avoiding elements in type $I_2(m)$. It remains to be shown that the conjecture in Section 3.1 regarding the classification of the T_2-avoiding elements in type $\widetilde{A}_n$ holds. The classification of T_2-avoiding elements in Coxeter systems of type F_n for $n \geq 6$ still remains open.

We also mentioned several other Coxeter systems in Figures 1.2 and 1.3. The classification of T_2-avoiding elements in the Coxeter systems of type E_n remains an open problem. However, we do know that these groups have T_2-avoiding elements as $W(D_n)$ (which has T_2-avoiding elements) is a parabolic subgroup of $W(E_n)$. The classification of T_2-avoiding elements in the Coxeter systems of type H_n is also an open problem.

A majority of the irreducible affine Coxeter systems currently do not have a classification of the T_2-avoiding elements. Specifically, Coxeter systems of type $\widetilde{B}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, and $\widetilde{G}_4$ do not have a classification. Future work could include classifying the T_2-avoiding elements of the Coxeter systems mentioned above.

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