THE MEANING OF LIFE

By Tony Stark

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Approved:

Évariste Galois, Ph.D., Chair

John von Neumann, Ph.D.

Srinivasa Ramanujan, Ph.D.

ABSTRACT

THE MEANING OF LIFE

Tony Stark

Everything you always wanted to know will be discussed.

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Chapter 1

Preliminaries

1.1 Introduction

To be written once we know where everything is going to go.

1.2 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st [7]. We call $m(s,t)$ the *bond strength* of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation* and s and t commute. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *braid move* if $m(s,t) \geq 3$ or a *commutation* if $m(s,t) = 2$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

(1) vertex set S and

(2) edges $\{s, t\}$ for each $m(s, t) \geq 3$.

Each edge $\{s, t\}$ is labeled with its corresponding bond strength $m(s, t)$. Since $m(s, t) = 3$ occurs most frequently, it is customary to leave the edge unlabeled. If (W, S) is a Coxeter group with corresponding Coxeter system Γ, we may denote the group as $W(\Gamma)$ for clarity. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. Given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system. Note that s and t are not connected in the graph if and only if $m(s, t) = 2$. Also, the Coxeter group $W(\Gamma)$ is said to be *irreducible* if and only if Γ is connected. Otherwise, $W(\Gamma)$ is said to be *reducible*. Furthermore, if the graph is disconnected, the connected components correspond to factors in a direct product of irreducible Coxeter groups [7].

Example 1.2.1.

(a) The Coxeter graph of type A_n is given in Figure 1.1a. Given A_n, we can construct the corresponding Coxeter system with generating set $S = \{s_1, s_2, \ldots s_n\}$ and defining relations

  (1) $s_i^2 = e$ for all i;
  (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;
  (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

  The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} sending $s_i \mapsto (i, i+1)$, where s_i corresponds to the adjacent transposition $(i, i+1)$.

(b) The Coxeter graph of type B_n is given in Figure 1.1b. From B_n, we can construct the corresponding Coxeter system with generating set $S = \{s_0, s_1, \ldots s_{n-1}\}$ and defining relations

  (1) $s_i^2 = e$ for all i;
  (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;
  (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \ldots, n-1\}$;
  (4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

  The Coxeter group $W(B_n)$ is isomorphic to Sym_n^B, where Sym_n^B is the group of all signed permutations.

(c) The Coxeter graph of type $\widetilde{C}_n$ is seen in Figure 1.2d. From $\widetilde{C}_n$, we can construct the Coxeter group $W(\widetilde{C}_n)$ with generating set $S = \{s_0, s_1, \ldots s_n\}$ and defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_i$ when $|i-j| > 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

In Figures 1.1 and 1.2 all finite and infinite Coxeter graphs are given. The Coxeter graphs in Figure 1.1 correspond to the finite Coxeter groups, while the Coxeter graphs in Figure 1.2 are affine Coxeter groups and are infinite. This thesis will mainly focus upon $W(A_n)$, $W(B_n)$, and $W(\widetilde{C}_n)$. However, this thesis will also briefly touch upon $W(\widetilde{A}_n)$, and $W(F)$.

Given a Coxeter system (W, S), a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal over all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions to represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent it as $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$. The following theorem tells us more about how reduced expressions for a given group element are related.

Theorem 1.2.2 (Matsumoto, [5])**.** *If $w \in W$, then every reduced expression for w can be obtained by a sequence of braid moves and commutations of the form*

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. $\square$

It follows from Matsumoto's Theorem that if a generator s_i appears in a reduced expression for $w \in W$, then s_i appears in all reduced expressions for w. Let $w \in W$ and fix a reduced expression $\overline{w}$ for w. Then the *support* of w, denoted $\operatorname{supp}(\overline{w})$, is the set of all generators of that appear in $\overline{w}$. It follows from Matsumoto's Theorem that s appears in $\operatorname{supp}(\overline{w})$ if and only if s appears in the support of all reduced expressions for w. If $\operatorname{supp}(\overline{w}) = S$, we say that s has *full support*.

Given $w \in W$ and a fixed reduced expression $\overline{w}$ for w, any subsequence of $\overline{w}$ is called a *subexpression* of $\overline{w}$. We will refer to a string of consecutive generators from a reduced expression $\overline{w}$ as a *subword* of $\overline{w}$.

Example 1.2.3. Let $w \in W(A_7)$ and let $\overline{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be a fixed expression for w. Then we have

$$\begin{aligned} s_7 s_2 s_4 s_5 s_3 s_2 s_3 s_6 &= s_7 s_4 s_2 s_5 s_3 s_2 s_3 s_6 \\ &= s_7 s_4 s_5 s_2 s_3 s_2 s_3 s_6 \\ &= s_7 s_4 s_5 s_3 s_2 s_3 s_3 s_6 \\ &= s_7 s_4 s_5 s_3 s_2 s_6, \end{aligned}$$

where the blue highlighted text corresponds to a commutation, the orange highlighted text corresponds to a braid move, and the red highlighted text corresponds to two elements that create the identity. This shows that $\overline{w}$ is not reduced. However it turns out that, $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\text{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

It should be noted that $s \in \mathcal{L}(w)$ if and only if there is a reduced expression for w that begins with s and $s \in \mathcal{R}(w)$ if and only if there is a reduced expression for w that ends with s [?, ?]

Example 1.2.4. Let $w \in W(B_4)$ and let $\overline{w} = s_0s_1s_2s_1s_3$ be a reduced expression for w. Note that all reduced expressions for w are as follows

$$\begin{array}{cc} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{array}$$

We see that $l(w) = 5$, and w has full support. Note that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

1.3 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W$. Following [10], we define a relation $\sim$ on the set of reduced expressions for w. Let $\overline{w}_1$ and $\overline{w}_2$ be two reduced expressions for w. We define $\overline{w}_1 \sim \overline{w}_2$ if we can obtain $\overline{w}_2$ from $\overline{w}_1$ by applying a single commutation move of the form $st \mapsto ts$ where $m(s,t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence

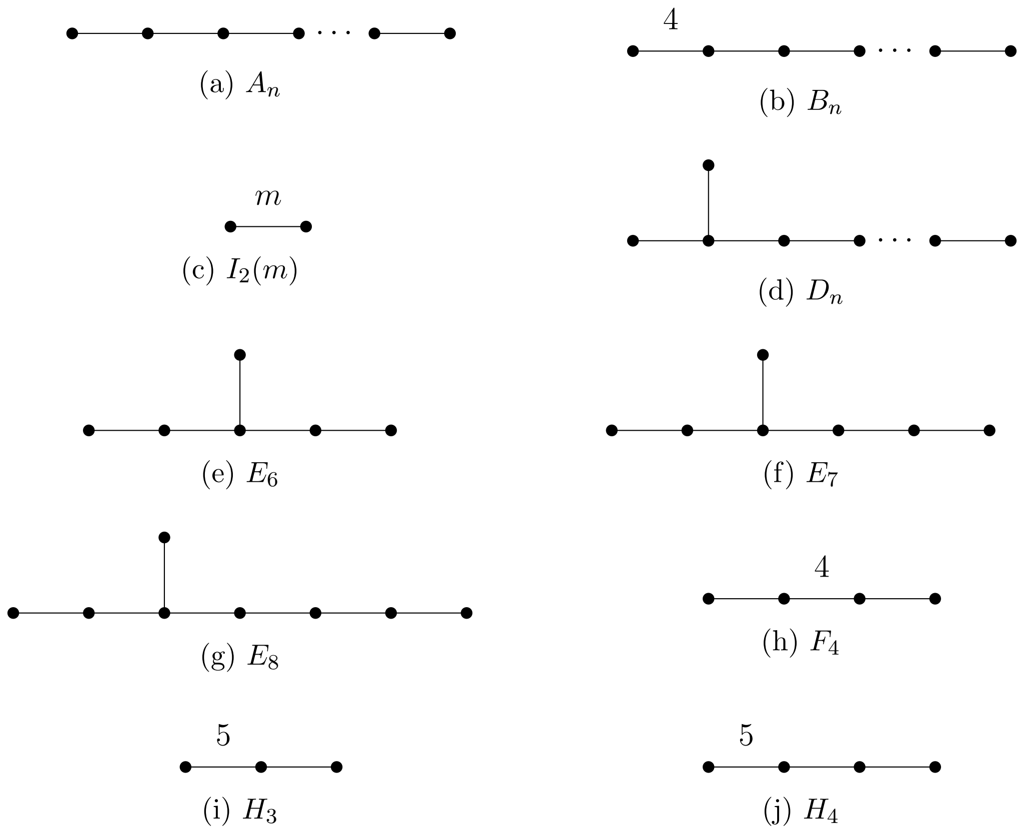


Figure 1.1: Coxeter graphs corresponding to the finite Coxeter groups.

class under $\approx$ is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative*, (FC).

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. We say that a reduced expression $\overline{w}$ is FC if it is a reduced expression for $w \in \mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$ and some starting expression for w, observe that the definition of FC states that to obtain all the reduced expressions for w, one only needs to perform commutations. The following theorem tells us that when w is FC, performing commutations is the only possible way to another reduced expression for w.

Theorem 1.3.1 (Stembridge, [10])**.** *An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all $s_i \neq s_j$ when $m(s,t) \geq 3$.*

$\square$

Example 1.3.2. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0s_1s_2s_0s_3s_1$ be a reduced expression

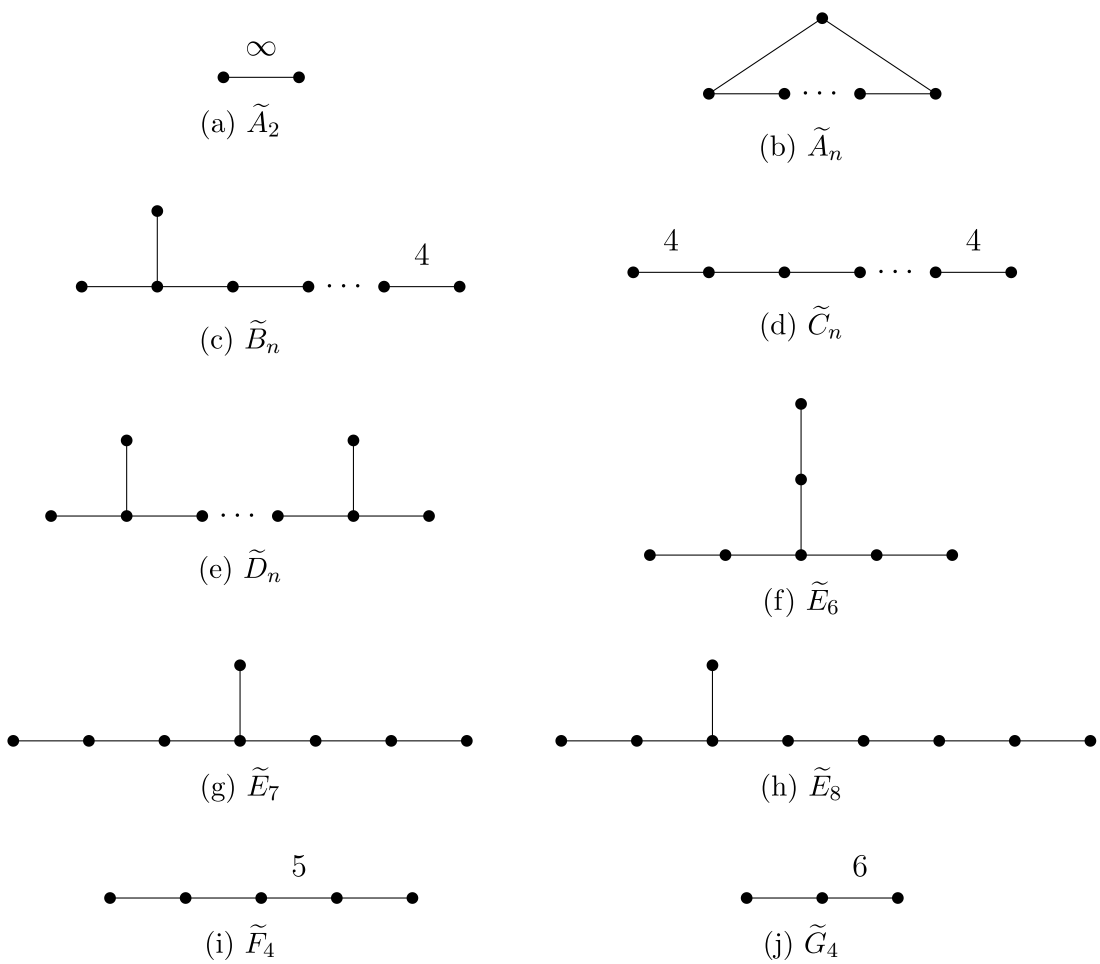


Figure 1.2: Coxeter graphs corresponding to the infinite Coxeter groups

for w. We see that

$$s_0s_1s_2s_0s_3s_1 = s_0s_1s_0s_2s_3s_1 = s_0s_1s_0s_2s_1s_3,$$

where the purple indicates applying a commutation. Note that there is no possible way to perform a braid move. Hence w is FC.

Example 1.3.3. Let $w \in W(\widetilde{C}_4)$ and let $\overline{w} = s_0s_1s_2s_0s_1s_2$ be a reduced expression for w. We see that

$$s_0s_1s_3s_0s_1s_2 = s_0s_1s_0s_3s_1s_2 = s_0s_1s_0s_1s_3s_2,$$

where the purple indicates applying a commutation and the blue indicates applying a braid move. Thus w is not FC since a braid move can be applied.

Example 1.3.4. Let $\overline{w} = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ for $w \in \mathrm{FC}(\widetilde{C}_6)$. Applying the commutation $s_4s_2 = s_2s_4$, we can obtain another reduced expression for w, namely $\overline{w}_2 = s_1s_0s_4s_1s_3s_5s_4s_2s_6$ which is in the same commutation class as $\overline{w_1}$. However, applying the braid move $s_2s_3s_2 = s_3s_2s_3$, we obtain another reduced expression $\overline{w_3} = s_1s_3s_2s_3s_4s_0$. Note that since $\overline{w}_3$ was obtained by applying a braid move, $\overline{w_3}$ is in a different commutation class than $\overline{w_1}$ and $\overline{w_2}$. It turns out that the w has exactly two commutation classes, one containing $\overline{w}_1$ and $\overline{w}_2$ and another containing $\overline{w}_3$.

Stembridge classified the irreducible Coxeter groups that contain a finite number of fully commutative elements, the so-called *FC-finite Coxeter groups*. This thesis is mainly concerned with $W(A_n)$, $W(B_n)$, $W(\widetilde{C}_n)$. Both $W(A_n)$, $W(B_n)$ are finite Coxeter groups, and thus are FC finite. On the other hand $W(\widetilde{C}_n)$ is infinite and has infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, E_n for $n \geq 9$ which is seen in Figure 1.2 is infinite, but contain only finitely many fully commutative elements.

Theorem 1.3.5 (Stembridge, [10])**.** *The FC-finite irreducible Coxeter groups are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. The corresponding Coxeter graphs are shown in Figures 1.1 and 1.2.* □

1.4 Heaps

We can now discuss another representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps of straight line Coxeter groups in [1], [3] and [10].

Let (W, S) be a Coxeter system. Suppose $\overline{w} = s_{x_1}s_{x_2}\cdots s_{x_r}$ is a fixed reduced expression for $w \in W$. As in [10], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular since $\overline{w}$ is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of $\overline{w}$, where i is labeled by s_{x_i}. Note that for simplicity we are omitting the labels of the underlying poset but retaining the labels of the corresponding generators.

It follows from [10] that heaps are well-defined up to commutation class. That is, given two reduced expressions $\overline{w}$ and $\overline{w'}$ for $w \in W$ that are in the same commutation class, then the heaps for $\overline{w}$ and $\overline{w'}$ will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then w has a one commutation class, and thus w has a unique heap.

Example 1.4.1. Let $\overline{w} = s_1 s_0 s_4 s_1 s_3 s_5 s_2 s_4 s_6$ for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that $\overline{w}$ is indexed by $\{1,2,3,4,5,6,7,8,9\}$. As an example, $1 \lessdot 0$ since $0 < 1$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 1.3.

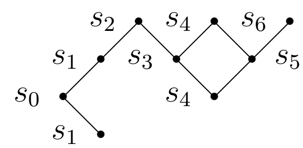


Figure 1.3: Labeled Hasse Diagram for the heap of an element of $\mathrm{FC}(\widetilde{C}_n)$

Let $\overline{w}$ be a reduced expression for an element in $w \in W(\widetilde{C}_n)$. As in [1] and [3] we can represent a heap for $\overline{w}$ as a set of lattice points embedded in $\{0,1,2,\dots,n\} \times \mathbb{N}$. To do so, we assign coordinates (not unique) $(x,y) \in \{0,1,2,\dots,n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of $\overline{w}$ in such a way that:

(1) An entry with coordinates (x,y) is labeled s_i (or i) in the heap if and only if $x = i$;

(2) If an entry with coordinates (x,y) is greater than an entry with coordinates (x',y') in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight line Coxeter graph with the appropriate adjustments made to the lattice point set and assignments of coordinates. In the case of any straight line Coxeter graph it follows from the definition that (x,y) covers (x',y') in the heap if and only if $x = x' \pm 1$, $y > y'$, and there are no entries (x'',y'') such that $x'' \in \{x,x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries in the top correspond to the generators occurring in the right of the corresponding reduced expression.

Let $\overline{w}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice representation of the heap poset in $\{0,1,2,\dots n\} \times n$ described in the preceding paragraphs via $H(\overline{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ (note the absence of sans serif font) and we refer to $H(w)$ as the heap of w. As above the necessary adjustments to the lattice representation will result in a general result for heaps of all straight line coxeter graphs.

Given a heap, there are many possible coordinate assignments, yet the x-coordinates will be fixed for each entry will be fixed for all of them. In particular, two entries labeled by the same generator will only differ by the amount of vertical space between them while they will maintain the same horizontal position to adjacent entries in the heap.

Let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$ then we must place the point labeled by s_{x_j} at a level that is *above* the level of the point labeled by s_{x_i}. Because generators in a Coxeter graph that are not adjacent do commute, points whose x-coordinates differ more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners in such a way that if one entry covers another the squares overlap halfway. In addition, we will also label each square with i representing the generator s_i.

Example 1.4.2. Let $\overline{w} = s_1 s_0 s_4 s_1 s_3 s_5 s_2 s_4 s_6$ for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.4.1. Figure 1.4 shows a possible lattice point representation for $H(w)$.

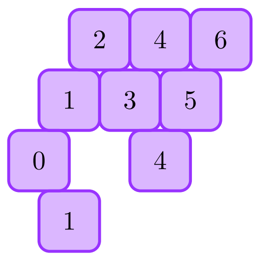
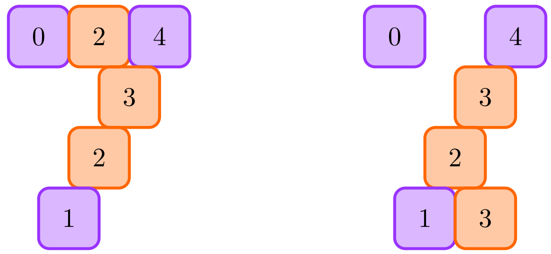


Figure 1.4: A possible lattice point representation of for $H(w)$

Example 1.4.3. Let $\overline{w}_1 = s_1 s_2 s_3 s_4 s_2 s_0$ be a reduced expression for $w \in W(\widetilde{C}_n)$. Applying the commutation $s_4 s_2 = s_2 s_4$, we can obtain another reduced expression for w, namely $\overline{w}_2$ which is in the same commutation class as $\overline{w_1}$ and hence has the same heap. However, applying the braid move $s_2 s_3 s_2 = s_3 s_2 s_3$, we obtain another reduced expression $\overline{w_3} = s_1 s_3 s_2 s_3 s_4 s_0$. Note that since $\overline{w}_3$ was obtained by applying a braid move, $\overline{w_3}$ is in a different commutation class than $\overline{w_1}$ and $\overline{w_2}$. Representations of $H(\overline{w}_1), H(\overline{w}_2)$ and $H(\overline{w_3})$ are seen in Figure 1.5 where the braid relation is colored in orange.

When w is FC, we wish to make a canonical choice for the representation of $H(w)$ by assembling the entries in a particular way. To do so, we position all of the entries corresponding to elements in $\mathcal{L}(w)$ in the same vertical position, and all of the remaining elements should be positioned as high as possible in the lattice point representation. For example, the representation in Figure 1.4 is the canonical representation for w. Note that our canonical representation of heaps corresponds



(a) $H(\overline{w}_1)$ and $H(\overline{w}_2)$ (b) $H(\overline{w}_3)$

Figure 1.5: Two Heaps of a non-FC element

to Cartier-Foata normal form for monomials [2, 6]. When illustrating heaps we will stick to the canonical choice, and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

1.5 Star Operations and Non-Cancellable Elements

The notion of star operations was originally introduced by Kazhdan and Lusztig in [8] for simply laced Coxeter systems (i.e. $m(s,t) \leq 3$ for all $s, t \in S$), and was later generalized to all Coxeter systems in [9]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group W, then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to define only the star operations which decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$ be a pair of noncommuting generators whose product has order m. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We define w to be *left star reducible by s with respect to t* if there exists $t \in \mathcal{L}(sw)$. We analogously define w to be *right star reducible by s with respect to t*. Observe that if $m(s,t) \geq 3$, then w is left (respectively, right) star reducible if and only if there is a reduced expression for w such that $\overline{w} = stv$ (respectively, $\overline{w} = vts$). We say that w is *star reducible* if it is either left or right star reducible.

Example 1.5.1. Let $w \in W(B_4)$ and let $\overline{w} = s_0s_1s_0s_2s_3$ be a reduced expression for w. We see that w is left star reducible by s_0 with respect to s_1 to $s_1s_0s_2s_3$, since $m(s_0,s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ and $s_1 \in \mathcal{L}(s_0w)$. Also w is right star reducible by s_3 with respect to s_2 to $s_0s_1s_0s_2$, since $m(s_2,s_3) = 3$ and $s_3 \in \mathcal{R}(w)$ and $s_2 \in \mathcal{R}(ws_3)$.

We now introduce the concept of weak star reducible which is related to the notion of cancellable in [4]. Let (W,S) be a Coxeter system of type Γ and let $I = \{s,t\} \subseteq S$

be a pair of noncommuting generators of the Coxeter group $W(\Gamma)$. If $w \in \mathrm{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t to sw* if

(1) w is left star reducible by s with respect to t;

(2) and $tw \notin \mathrm{FC}(W)$.

Notice that (2) implies that $l(tw) > l(w)$. Also note that we are restricting out definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws*. We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable* or *weak star irreducible*.

Example 1.5.2. Let $w \in \mathrm{FC}(B_4)$ and let $\overline{w} = s_0s_1s_0s_2s_3$ be a reduced expression for w as in Example 1.5.1. By Example 1.5.1 we know that w is left star reducible. Also, $tw = s_1s_0s_1s_0s_2s_3$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2s_3$. In addition, Example 1.5.1 showed that w is right star reducible. Also, $wt = s_0s_1s_0s_2s_3s_2$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that w is right weak star reducible by s_3 with respect to s_2 to $s_0s_1s_0s_2$. This implies that w is not non-cancellable.

Example 1.5.3. Let $w \in \mathrm{FC}(B_4)$ and let $\overline{w} = s_0s_1s_2$ be a reduced expression for w. Note that w is left (respectively, right) star reducible by s_0 with respect to s_1 (respectively, by s_2 with respect to s_1). However, $s_1s_2 \in \mathrm{FC}(B_4)$ (respectively, $s_0s_1 \in \mathrm{FC}(B_4)$). Thus w is non-cancellable.

Using the notion of star reducible we are now able to introduce the concept of a star reducible Coxeter group. We say that a Coxeter group $W(\Gamma)$, or it's Coxeter graph Γ, is *star reducible* if every element of $\mathrm{FC}(\Gamma)$ is star reducible to a product of commuting generators. That is, $W(\Gamma)$ is star reducible if when we apply star operations repeatedly to $w \in \mathrm{FC}(\Gamma)$ eventually w is a product of commuting generators. In[6], Green classified all star reducible Coxeter groups. The Coxeter groups $W(A_n)$, $W(B_n)$ and $W(\widetilde{C}_n)$ are star reducible. However, $W(A_n)$ and $W(B_n)$ don't have non-trivial T-avoiding elements, while $W(\widetilde{C}_n)$ in one parity does have non-trivial T-avoiding elements.

Theorem 1.5.4. *Let $W(\Gamma)$ be a Coxeter group with (finite) generating set S. Then $W(\Gamma)$ is star reducible if an only if each component of Γ is either a complete graph with labels $m(s,t) \geq 3$, or is one of the following types: type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type H_n $(n \geq 2)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_{n-1}$ $(n \geq 3$ and n odd $)$, type $\widetilde{C}_{n-1}$ $(n \geq 4$ and n even $)$, type $\widetilde{E}_6$ or type $\widetilde{F}_5$.* $\square$

Chapter 2

My Cool Stuff

2.1 Classification of T-Avoiding Elements in Type B

An introduction should go here regarding the awesome sauce that is to follow.

And we probably need some other stuff to go here but Sarah S. told me to work on typing page 1 today.

Proposition 2.1.1 (Björner, cite when in Mendalay)**.** *Let $w \in W(B_n)$. Then*

$$\mathcal{R}(w) = \{s_i \in S : w(i) > w(i+1)\}$$

where w(0)=0 by definition.

Proof. This is, cite when in Mendelay Proposition 8.1.2. □

Lemma 2.1.2. *Let $s, t \in S$ such that $m(s,t) = 3$. Then w has a reduced expression ending in sts if and only if w has the consecutive pattern* 321*.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in cite BB when in Mendelay. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

($\Rightarrow$) Suppose w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+1)$. By 2.1.1 $s_i, s_{i+1} \in \mathcal{R}(w)$. By Tyson's reference to simply laced coxeter group stuff 1.2.1 w ends in $s_i s_{i+1} s_{i+2}$. ($\Leftarrow$) Suppose w ends in $s_i s_i + 1 s_i$. This implies that either $w_I = s_i s_{i+1}$ or $w_I = s_{i+1} s_i$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by 2.1.1. Thus w has the consecutive pattern 321. Therefore, w has a reduced expression ending in sts if and only if w has the consecutive pattern 321. □

Lemma 2.1.3. *Let $s, t \in S$ such that $m(s,t) = 3$. Then w has a reduced expression ending in st if and only if w has the consecutive pattern* 231*.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in cite BB when in Mendelay. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

($\Rightarrow$) Suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$. By 2.1.1 $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by s_{i+1} we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, this implies that $s_i \in \mathcal{R}(ws_{i+1})$. This implies w has a reduced expression that ends in $s_i s_{i+1}$. ($\Leftarrow$) Suppose that w has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence w has the consecutive pattern 231. Therefore, w has a reduced expression ending in st if and only if w has the consecutive pattern 231. □

Lemma 2.1.4. *Let $s, t \in S$ such that $m(s,t) = 3$. Then w has a reduced expression ending in ts if and only if w has the consecutive pattern* 312.

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in cite BB when in Mendelay. Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

($\Rightarrow$) Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. By 2.1.1 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by s_i we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$. By above $w(i) > w(i+2)$, and by 2.1.1 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that w has a reduced expression ending in $s_{i+1}s_i$. ($\Leftarrow$) Conversely suppose w ends in a reduced expression with $s_{i+1}s_i$ Then $w_I = s_{i+1}s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, w has the consecutive pattern 312. Therefore, w has a reduced expression ending in ts if and only if w has the consecutive pattern 312. □

Chapter 3

Title of Chapter 3

Slightly boring, but still intelligent. I think.

Chapter 4

Title of Chapter 4

In this chapter, we'll say the most intelligent stuff anyone has ever said.

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