Investigations of T-avoiding elements of Coxeter groups

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Happy First Contact Day!

Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of elements of order 2 having presentation

$$W = \langle S : s^2 = 1, \quad (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Since s and t are their own inverses, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$\left.\begin{array}{lcl} m(s,t) = 2 & \Longrightarrow & st = ts \end{array}\right\} \quad \text{commutations}$$

$$\left.\begin{array}{lcl} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

Coxeter groups can be thought of as generalized reflection groups.

Definition

We can encode (W, S) with a unique Coxeter graph X having:

1. vertex set S;
2. edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$.

Comments

- Typically labels of $m(s, t) = 3$ are omitted.
- Edges correspond to non-commuting pairs of generators.
- If there is no edge between a pair of generators they commute.
- Given X, we can uniquely reconstruct the corresponding (W, S).

Type A

Example

The Coxeter group of type A_3 is defined by the graph below.

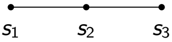


Figure: Coxeter graph of type A_3.

Then $W(A_3)$ is subject to:

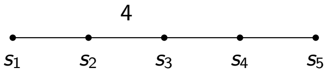
- $s_i^2 = 1$ for all i
- $s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3$
- $s_1 s_3 = s_3 s_1$

In this case, $W(A_3)$ is isomorphic to the symmetric group Sym_4 under the correspondence

$$s_1 \leftrightarrow (1\ 2), \quad s_2 \leftrightarrow (2\ 3), \quad s_3 \leftrightarrow (3\ 4).$$

Example

The Coxeter group of type F_5 is defined by the graph below.



Then $W(F_5)$ is subject to:

- $s_i^2 = 1$ for all i
- $s_1 s_2 s_1 = s_2 s_1 s_2$; $s_3 s_4 s_3 = s_4 s_3 s_4$; $s_4 s_5 s_4 = s_5 s_4 s_5$
- $s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2$
- Non-connected generators commute

F_4 is a finite group, however F_n for $n \geq 5$ is an infinite group.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for $w \in W$ if it is equal to w when considered as a group element. If m is minimal, it is a reduced expression.

Example

Consider the expression $s_1 s_3 s_2 s_1 s_2 s_3$ for an element $w \in W(A_3)$. Note that

$$s_1 s_3 s_2 s_1 s_2 s_3 = s_1 s_3 s_1 s_2 s_1 s_3 = s_3 s_1 s_1 s_2 s_1 s_3 = \underbrace{s_3 s_2 s_1 s_3 = s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1}_{\text{reduced}}.$$

Therefore, $s_1 s_3 s_2 s_1 s_2$ is not reduced. However, the expression on the right is reduced.

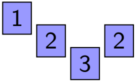
Theorem (Matsumoto/Tits)

Any two reduced expressions for $w \in W$ differ by a sequence of braid relations and commutations.

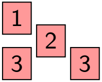
One way of representing reduced expressions is via heaps. Fix a reduced expression $s_{x_1} s_{x_2} \cdots s_{x_m}$ for $w \in W$ (for any straight line Coxeter graph). Loosely speaking, the heap for this expression is a set of lattice points, one for each s_{x_i}, embedded in $\mathbb{N} \times \mathbb{N}$, subject to contraints illustrated by example.

Example

Consider $s_1 s_2 s_3 s_2$, $s_1 s_3 s_2 s_3$, and $s_3 s_1 s_2 s_3$, which are all reduced expressions of the same element in A_3. It turns out, there are two distinct heaps:



and



Comment

If two reduced expressions differ by a sequence of commutations, then they have the same heap.

Property T and T-avoiding

Definition

We say that $w \in W$ has Property T iff some reduced expression begins or ends with a product of non-commuting generators. That is,

$$w = \boxed{s}\,\boxed{t}\ (\text{other crap}) \qquad \text{or} \qquad w = (\text{other crap})\ \boxed{t}\,\boxed{s}$$

Definition

We say that w is T-avoiding iff w does not have Property T.

Proposition

Products of commuting generators are T-avoiding.

Question

Are there other elements besides products of commuting generators that are T-avoiding?

Definition

An element is classified as bad iff it is T-avoiding, but *not* a product of commuting generators.

Theorem (Cormier, Ernst, Goldenberg, Kelly, Malbon)

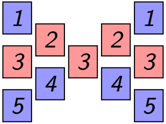
In types A and B, there are no bad elements. In other words, $w \in W$ is T-avoiding iff w is a product of commuting generators.

Comment

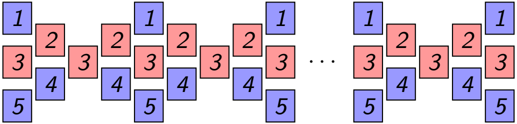
The answer isn't so simple in other Coxeter groups. In particular, there are bad elements in types $\widetilde{C}$ (Ernst) and D (Tyson Gern).

Proposition (Cross, Ernst, Hills-Kimball, Quaranta)

The following heap (called a bowtie) corresponds to a bad element in F_5:



We can also stack bowties to create infinitely many bad elements in F_5.



Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

An element is T-avoiding in F_5 *iff it is a product of commuting generators or a stack of bowties.*

Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no bad elements in F_4*. That is, the only T-avoiding elements in* F_4 *are products of commuting generators.*

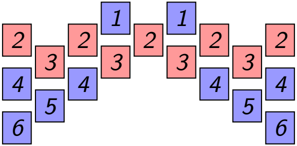
Conjecture (Cross, Ernst, Hills-Kimball, Quaranta)

An element is T-avoiding in F_n *for* $n \geq 5$ *iff it is a product of commuting generators or a stack of bowties. In other words, there are no new bad elements in* F_n *for* $n \geq 6$.

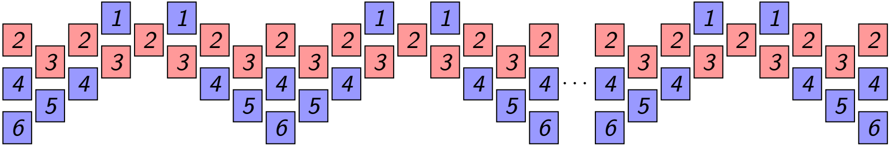
However...

Proposition (Ernst, Gilbertson)

The following heap corresponds to a bad element in F_6*:*

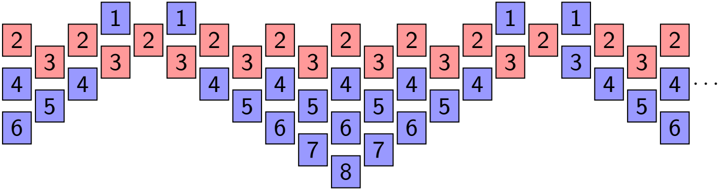


As in F_5*, we can stack these elements to create infinitely many bad elements in* F_6*.*



If n *is even, we can create bad elements in* F_n *using a similar construction. (However, when* n *is large, the outer walls of each heap block do not need to be the same size.)*

Stacking in F_n with $n > 5$ gets more complicated.



Question

Will adding more 1's give us more T-avoiding elements?

Open questions

- If n is even, are there other bad elements in F_n that we have not thought of? Proof?
- We have noticed that when n is large and even, we can insert some extra 1's. How awful can this get?
- What happens with F_n when n is odd and larger than 5?
- What happens in other types?

Thank You!