Preliminaries

Introduction In mathematics, one uses groups to study symmetry. In particular, a reflection group can be used to study the reflection and rotational symmetry of an object. A Coxeter group can be thought of as a generalized reflection group, where the group is generated by a set of elements of order two (i.e., reflections) and there are rules for how the generators interact with each other. Every element of a Coxeter group can be written as an expression in the generators, and if the number of generators in an expression (including multiplicity) is minimal, we say that the expression is reduced.

Kazhdan–Lusztig polynomials arise in the context of Hecke algebras associated to Coxeter groups. The computation of these polynomials is very difficult even for relatively small groups. Motivated by the desire to understand the Kazhdan–Lusztig theory of the Hecke algebra of the underlying Coxeter group, Green Green2006a classified the so-called star reducible Coxeter groups which have the property that all fully commutative elements (in the sense of Stembridge) can be sequentially reduced via star operations to a product of commuting generators.

It turns out that in some Coxeter groups there are elements, called T-avoiding elements, which cannot be systematically dismantled in the way described above. More specifically an element w is called T-avoiding if w does not have a reduced expression beginning or ending with a pair of non-commuting generators. Clearly, a product of commuting generators is trivially T-avoiding. However, sometimes there are more interesting T-avoiding elements, which we will refer to as type 2 T-avoiding elements.

Our interest in T-avoiding elements is motivated by a desire to compute Kahzdan–Lusztig polynomials, denoted $P_{x,w}$, where x and w are elements of a fixed Coxeter group. A bound on the degree of $P_{x,w}$ is known, but in general it is not known when this bound is achieved. Of particular interest are the coefficients $\mu(x,w)$ that appear when the maximum degree of $P_{x,w}$ is attained. The polynomials $P_{x,w}$ and coefficients $\mu(x,w)$ are determined through recurrence relations, however no closed form is known for calculating either in an efficient matter. Calculations involving type 2 T-avoiding elements are generally more difficult as the descent sets of these elements have undesirable properties. In addition, knowing which elements are T-avoiding often provides us with the base case for inductive arguments involving star operations.

In his PhD thesis Gern2013a, Gern classified the T-avoiding elements in Coxeter groups of type D_n. Unlike in types A_n and B_n, it turns out that the classification in type D_n includes type 2 T-avoiding elements. The T-avoiding elements are rich in combinatorics and are interesting in their own right. The focus of this thesis is classifying T-avoiding elements in certain Coxeter groups.

This thesis is organized as follows. After necessary background information is presented in Section sec:coxeter, we introduce the class of fully commutative elements in Section sec:FC. Next in Section sec:Heaps we discuss a visual representation for elements of Coxeter groups, called heaps. In Section sec:star, we introduce the concept of a star reduction and star reducible Coxeter groups and in Section sec:Tavoid we formally introduce the notion of a T-avoiding element. In Section sec:noncancel we define non-cancellable elements in Coxeter groups, as well as remark upon a specific family of non-cancellable elements in $W(_n)$ when n is odd. We then state classifications and conjectures regarding T-avoiding elements in several Coxeter groups in Chapter chap:TandTavoid. All of the results in Chapter chap:TandTavoid, barring Section sec:tavoidI, are previously known. Chapter chap:Cn contains the main results of this thesis, namely the classification of T-avoiding elements in Coxeter groups of types $_n$ and B_n, which are new results. Section sec:Btools introduces signed permutations and signed patterns in the context of Coxeter systems of type B_n. We characterize Property T and T-avoiding in Coxeter systems of type B_n in terms of consecutive signed patterns in Section sec:TAB. We conclude with some open questions in Section sec:open.

Coxeter Systemssec:coxeter

A Coxeter system is a pair (W,S) consisting of a finite set S of generating involutions and a group W, called a Coxeter group, with presentation $\mathrm{W} = \langle S \mid (st)^{m(s,t)} = e \rangle, where e is the identity,$m(s,t) $=$ $1 if and only if$ s = t, andm(s,t) = m(t,s) ≥ 2 for $s \neq t$. If there is no relation between $s,t \in S$, then we define $m(s,t) = \infty$. However, in this thesis we assume that all $m(s,t)$ are finite. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st Humphreys1990. We call $m(s,t)$ the bond strength of s and t.

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as equationbraid
$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)}$$