# I DON'T KNOW WHAT TO CALL IT

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# ABSTRACT

## I DON'T KNOW WHAT TO CALL IT

Taryn Laird

Everything you always wanted to know will be discussed.

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# Chapter 1

# Preliminaries

## 1.1 Coxeter Systems

A *Coxeter system* is a pair $(W, S)$ consisting of a finite set $S$ of generating involutions and a group $W$, called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where $e$ is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of $S$ are distinct as group elements and that $m(s,t)$ is the order of $st$ [10]. We call $m(s,t)$ the *bond strength* of $s$ and $t$.

Since $s$ and $t$ are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)} \tag{1.1}$$

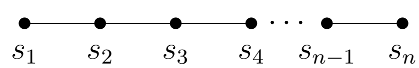
with $m(s,t) \geq 2$ factors. If $m(s,t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s,t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts\cdots}_{m(s,t)}$ with $\underbrace{tst\cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

We can represent a Coxeter system $(W, S)$ with a unique *Coxeter graph* $\Gamma$ having

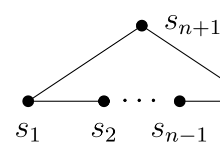
(1) vertex set $S$ and

(2) edges $\{s,t\}$ for each $m(s,t) \geq 3$ labeled by its corresponding bond strength $m(s,t)$ .

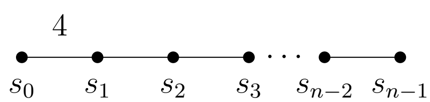
Since $m(s,t) = 3$ occurs frequently, it is customary to omit this label. Note that $s$ and $t$ are not connected by a single edge in the graph if and only if $m(s,t) = 2$. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. That is, given a Coxeter graph $\Gamma$, we can uniquely reconstruct the corresponding Coxeter system. If $(W,S)$ is a Coxeter system with corresponding Coxeter graph $\Gamma$, we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. Also, the Coxeter system $(W,S)$ is said to be *irreducible* if and only if $\Gamma$ is connected. If the graph $\Gamma$ is disconnected, the connected components correspond to factors in a direct product of the corresponding Coxeter groups [10]. The Coxeter graphs given in Figure 1.1 correspond to the Coxeter systems that will be primarily addressed in this thesis. Notice here that the vertices are labeled with the corresponding generators to provide context when talking about the different generating sets $S(\Gamma)$.



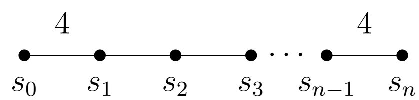

(a) $A_n$



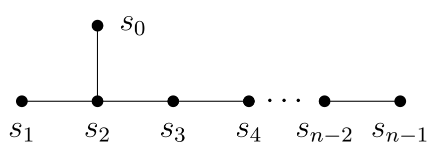

(b) $\widetilde{A}_n$



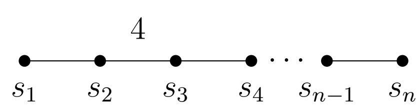

(c) $B_n$



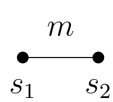

(d) $\widetilde{C}_n$




(e) $D_n$




(f) $F_n$




(g) $I_2(m)$

Figure 1.1: A few labeled Coxeter graphs.

**Example 1.1.1.**

(a) The Coxeter system of type $A_n$ is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group $W(A_n)$ with generating set $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all $i$;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group $\mathrm{Sym}_{n+1}$ under the correspondence $s_i \mapsto (i, i+1)$, where $(i, i+1)$ is the adjacent transposition that swaps $i$ and $i+1$.

(b) The Coxeter system of type $B_n$ is given by the graph in Figure 1.1(c). We can construct the corresponding Coxeter group $W(B_n)$ with generating set $S(B_n) = \{s_0, s_1, \ldots, s_{n-1}\}$ and defining relations

(1) $s_i^2 = e$ for all $i$;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to $\mathrm{Sym}_n^B$, where $\mathrm{Sym}_n^B$ is the group of signed permutations on the set $\{1, 2, \ldots, n\}$.

(c) The Coxeter system of type $\widetilde{C}_n$ is seen in Figure 1.1(d). We can construct the corresponding Coxeter group $W(\widetilde{C}_n)$ with generating set $S(\widetilde{C}_n) = \{s_0, s_1, \ldots, s_n\}$ and defining relations

(1) $s_i^2 = e$ for all $i$;

(2) $s_i s_j = s_j s_i$ when $|i - j| > 1$ for $i \in \{0, 2, \ldots, n\}$;

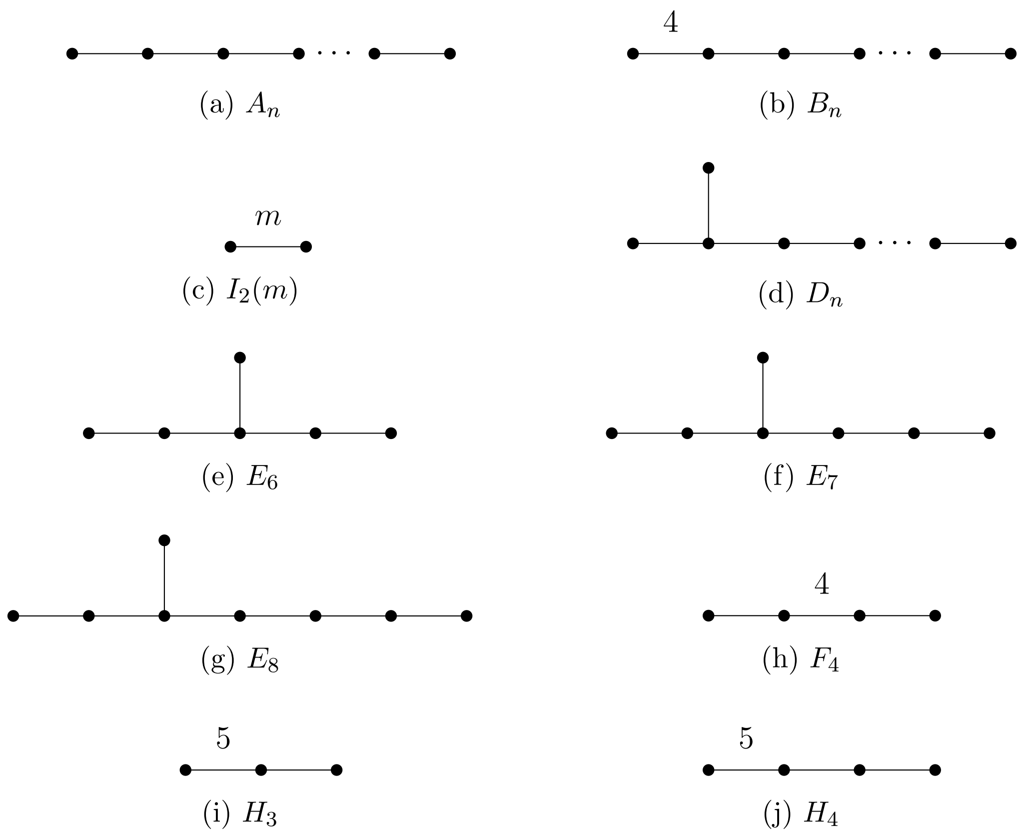
(3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i \in \{1, 2, \ldots, n-1\}$;

(4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;

(5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

Note that $W(\widetilde{C}_n)$ has $n+1$ generators.

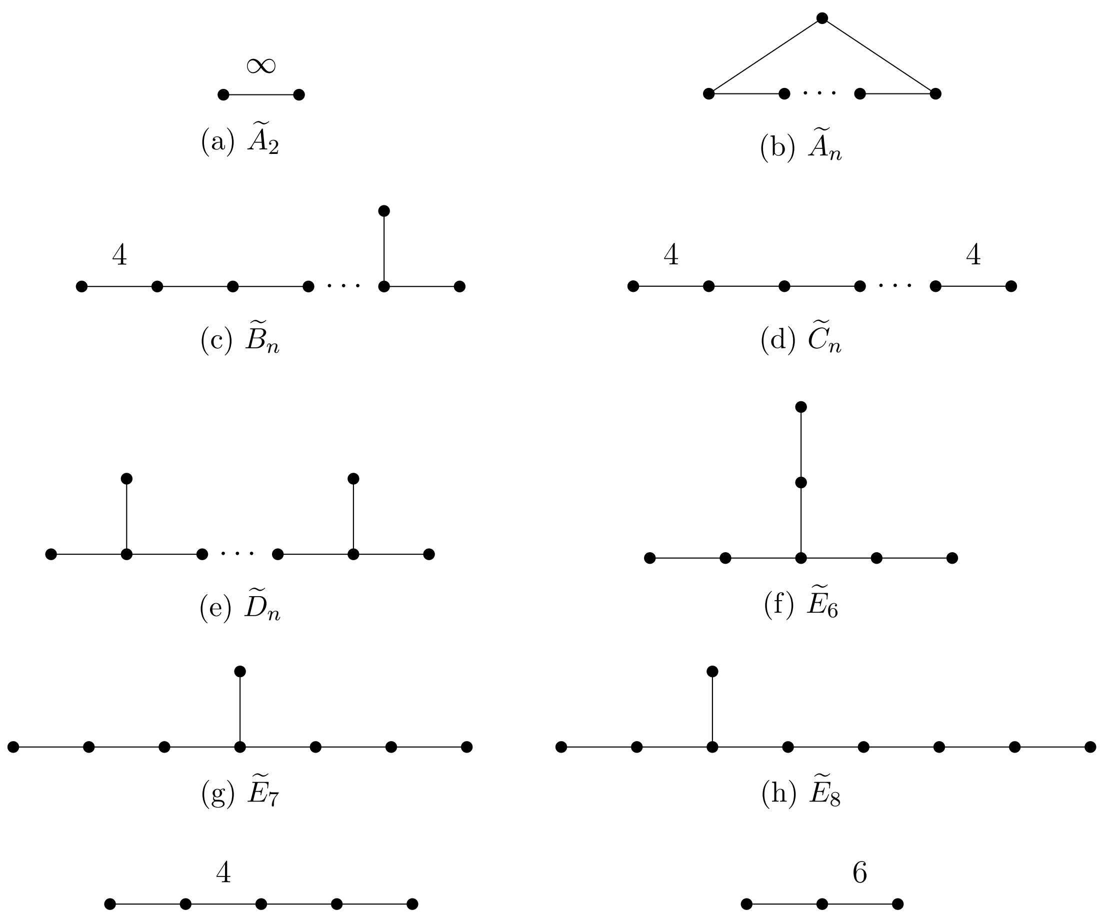
The Coxeter graphs given in Figure 1.2 correspond to the collection of irreducible finite Coxeter systems, whose corresponding Coxeter groups are finite, while the Coxeter graphs given in Figure 1.3 are the so-called irreducible *affine Coxeter systems*, which are infinite [10]. Note that $W(B_n)$ is one of the irreducible finite Coxeter groups so it is finite while $W(\widetilde{C}_n)$ is one of the affine groups making it infinite. The irreducible affine Coxeter systems are unique in that if a vertex is removed along with the corresponding edges from the Coxeter graph, the newly created graph will result in a Coxeter system with a finite Coxeter group.




(a) $A_n$ (b) $B_n$ (c) $I_2(m)$ (d) $D_n$ (e) $E_6$ (f) $E_7$ (g) $E_8$ (h) $F_4$ (i) $H_3$ (j) $H_4$

Figure 1.2: Irreducible finite Coxeter systems.

Given a Coxeter system $(W, S)$, a word $s_{x_1}s_{x_2}\cdots s_{x_m}$ in the free monoid $S^*$ on $S$ is called an *expression* for $w \in W$ if it is equal to $w$ when considered as a group element. If $m$ is minimal among all expressions for $w$, the corresponding word is called a *reduced expression* for $w$. In this case, we define the *length* of $w$ to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_m}$. using sans serif font. If $u, v \in W(\Gamma)$, we say that the product of group elements $uv$ is *reduced* if $l(uv) = l(u) + l(v)$. The following theorem tells us more about how reduced expressions for a given group element are related.




(a) $\widetilde{A}_2$ (b) $\widetilde{A}_n$ (c) $\widetilde{B}_n$ (d) $\widetilde{C}_n$ (e) $\widetilde{D}_n$ (f) $\widetilde{E}_6$ (g) $\widetilde{E}_7$ (h) $\widetilde{E}_8$ (i) $\widetilde{F}_4$ (j) $\widetilde{G}_4$

Figure 1.3: Irreducible affine Coxeter systems.

**Theorem 1.1.2** (Matsumoto, [7])**.** Let $(W, S)$ be a Coxeter system. If $w \in W$, then given a reduced expression for $w$ we can obtain every other reduced expression for $w$ by a sequence of braid moves and commutations of the form

$$\underbrace{sts\cdots}_{m(s,t)} \to \underbrace{tst\cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s,t) \geq 2$. $\square$

It follows from Matsumoto's Theorem that if a generator $s$ appears in a reduced expression for $w \in W$, then $s$ appears in all reduced expressions for $w$. Let $w \in W$ and define the *support* of $w$, denoted $\mathrm{supp}(w)$, to be the set of all generators that appear in any reduced expression for $w$. If $\mathrm{supp}(w) = S$, we say that $w$ has *full support*.

Given $w \in W$ and a fixed reduced expression $\mathsf{w}$ for $w$, any subsequence of $\mathsf{w}$ is called a *subexpression* of $\mathsf{w}$. We will refer to a subexpression consisting of a consecutive subsequence of $\mathsf{w}$ as a *subword* of $\mathsf{w}$.

**Example 1.1.3.** Let $w \in W(A_7)$ and let $\mathsf{w} = s_7s_2s_4s_5s_3s_2s_3s_6$ be a fixed expression for $w$. Then we have

$$\begin{aligned} s_7{\color{purple}s_2s_4}s_5s_3s_2s_3s_6 &= s_7s_4{\color{purple}s_2s_5}s_3s_2s_3s_6 \\ &= s_7s_4s_5{\color{teal}s_2s_3s_2}s_3s_6 \\ &= s_7s_4s_5s_3s_2{\color{red}s_3s_3}s_6 \\ &= s_7s_4s_5s_3s_2s_6, \end{aligned}$$

where the purple highlighted text corresponds to a commutation, the teal highlighted text corresponds to a braid move, and the red highlighted text corresponds to cancellation. This shows that the original expression $\mathsf{w}$ is not reduced. However, it turns out that $s_7s_4s_5s_3s_2s_6$ is reduced. Thus $l(w) = 6$ and $\mathrm{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let $(W, S)$ be a Coxeter system of type $\Gamma$ and let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of $w$ as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a reduced expression for $w$ that begins (respectively, ends) with $s$.

**Example 1.1.4.** The following list consists of all reduced expressions some $w \in W(B_4)$:

$$\begin{array}{cc} s_0s_1s_2s_1s_3 & s_0s_2s_1s_2s_3 \\ s_0s_1s_2s_3s_1 & s_2s_0s_1s_2s_3 \end{array}$$

We see that $l(w) = 5$ and $w$ has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

## 1.2 Fully Commutative Elements

Let $(W, S)$ be a Coxeter system of type $\Gamma$ and let $w \in W(\Gamma)$. Following [13], we define a relation $\sim$ on the set of reduced expressions for $w$. Let $\mathsf{w}_1$ and $\mathsf{w}_2$ be two reduced expressions for $w$. We define $\mathsf{w}_1 \sim \mathsf{w}_2$ if we can obtain $\mathsf{w}_2$ from $\mathsf{w}_1$ by applying a single commutation move of the form $st \mapsto ts$ where $m(s,t) = 2$. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*. If $w$ has a single commutation class, then we say that $w$ is *fully commutative* (FC).

The set of FC elements of $W(\Gamma)$ is denoted by $\mathrm{FC}(\Gamma)$. Given some $w \in \mathrm{FC}(\Gamma)$, and a starting reduced expression for $w$, observe that the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for $w$, but the following result due to Stembridge [13] states that when $w$ is FC, performing commutations is the only possible way to obtain another reduced expression for $w$.

**Theorem 1.2.1** (Stembridge, [13])**.** Let $(W, S)$ be a Coxeter system. An element $w \in W$ is FC if and only if no reduced expression for $w$ contains $\underbrace{sts\cdots}_{m(s,t)}$ as a subword for all $m(s,t) \geq 3$. $\square$

In other words, $w$ is FC if and only if no reduced expression provides the opportunity to apply a braid move. For example, for a Coxeter system of type $B_n$ an element is FC if no reduced expression contains the subwords $s_0s_1s_0s_1$, $s_1s_0s_1s_0$, $s_ks_{k+1}s_k$, and $s_{k+1}s_ks_{k+1}$ where $0 < k < n-1$. In a Coxeter group of type $\widetilde{C}_n$, an element is FC if no reduced expression for the element contains the subwords seen above and does not contain the subwords $s_{n-1}s_ns_{n-1}s_n$ and $s_ns_{n-1}s_ns_{n-1}$.

**Example 1.2.2.** Let $w \in W(\widetilde{C}_4)$ and let $\mathsf{w} = s_0s_1s_2s_0s_3s_1$ be a reduced expression for $w$. Although it is not immediately obvious, there is no possible way to perform a braid move in $w$. Hence $w$ is FC.

**Example 1.2.3.** Let $\mathsf{w}_1 = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. Applying the commutation $s_4s_2 \mapsto s_2s_4$, we can obtain another reduced expression for $w$, namely $\mathsf{w}_2 = s_1s_0s_4s_1s_3s_5s_4s_2s_6$, which is in the same commutation class as $\mathsf{w}$. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w} = s_1s_3s_2s_3s_4s_0$. Note that since $\mathsf{w}_3$ was obtained by applying a braid move, $\mathsf{w}_3$ is in a different commutation class than $\mathsf{w}_1$ and $\mathsf{w}_2$. Since $w$ has at least two commutation classes, one containing $\mathsf{w}_1$ and $\mathsf{w}_2$ and another containing $\mathsf{w}_3$, $w$ is not FC by Theorem 1.2.1.

Stembridge classified the Coxeter systems that contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. Both $W(A_n)$ and $W(B_n)$ are finite Coxeter groups, and thus are FC-finite. On the other hand, $W(\widetilde{C}_n)$ is infinite and happens to also contain infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, $W(E_n)$ for $n \geq 9$ (see Figure 1.4) is infinite, but contains only finitely many FC elements.

**Theorem 1.2.4** (Stembridge, [13])**.** The irreducible FC-finite Coxeter systems are of type $A_n$ with $n \geq 1$, $B_n$ with $n \geq 2$, $D_n$ with $n \geq 4$, $E_n$ with $n \geq 6$, $F_n$ with $n \geq 4$, $H_n$ with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. $\square$

The irreducible FC-finite Coxeter graphs are given in Figure 1.4. Note that the irreducible finite Coxeter systems given in Figure 1.2 certainly have only a finite number of FC elements. We have not yet encountered the Coxeter groups determined by graphs in Figures 1.4(d) for $n \geq 9$, 1.4(e) for $n \geq 5$, 1.4(f) for $n \geq 5$. All of these Coxeter systems are infinite for sufficiently large $n$, yet contain only finitely many FC elements.

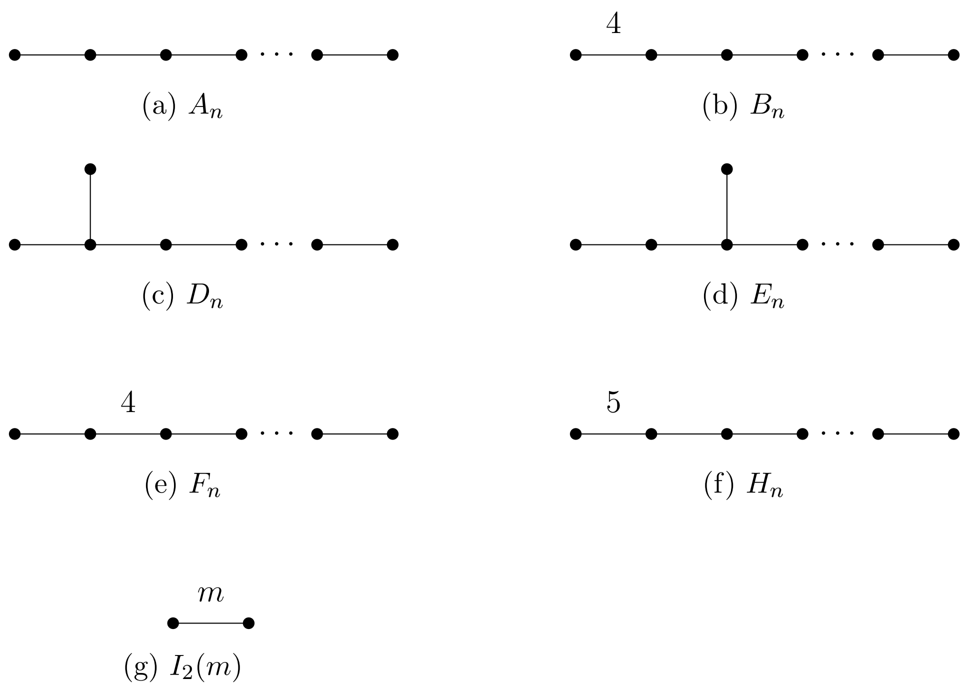



Figure 1.4: Irreducible FC-finite Coxeter systems.

## 1.3 Heaps

We now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps for straight line Coxeter groups found in [1], [3], and [13].

Let $(W, S)$ be a Coxeter system of type $\Gamma$. Suppose $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ is a fixed reduced expression for $w \in W(\Gamma)$. As in [13], we define a partial ordering on the indices $\{1, 2, \ldots, r\}$ by the transitive closure of the relation $\lessdot$ defined via $j \lessdot i$ if $i < j$ and $s_{x_i}$ and $s_{x_j}$ do not commute. In particular, since $\mathsf{w}$ is reduced, $j \lessdot i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of $\mathsf{w}$, where $i$ is labeled by $s_{x_i}$. Note that for simplicity we are omitting the labels of the underlying poset yet retaining the labels of the corresponding generators.

It follows from [13] that heaps are well-defined up to commutation class. That is, given two reduced expressions $\mathsf{w}_1$ and $\mathsf{w}_2$ for $w \in W$ that are in the same commutation class, the heaps for $\mathsf{w}_1$ and $\mathsf{w}_2$ will be equal. In particular, if $w \in \mathrm{FC}(\Gamma)$, then $w$ has one commutation class, and thus $w$ has a unique heap. Conversely, if $\mathsf{w}_1$ and $\mathsf{w}_2$ are in different commutation classes, then the heap for $\mathsf{w}_1$ will be distinct from the heap for $\mathsf{w}_2$.

**Example 1.3.1.** Let $\mathsf{w} = s_6 s_4 s_2 s_5 s_3 s_1 s_4 s_0 s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$. We see that $\mathsf{w}$ is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $9 \lessdot 8$ since $8 < 9$ and $s_0$ and $s_1$ do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 1.5.

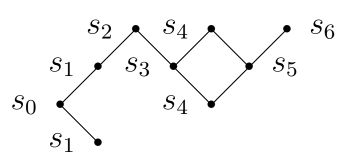



Figure 1.5: Labeled hasse diagram for the heap of an element in $\mathrm{FC}(\widetilde{C}_6)$.

Let $\mathsf{w}$ be a reduced expression for an element $w \in W(\widetilde{C}_n)$. As in [1] and [3] we can represent a heap for $\mathsf{w}$ as a set of lattice points embedded in $\{0, 1, 2, \ldots, n\} \times \mathbb{N}$. To do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \ldots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of $\mathsf{w}$ in such a way that:

(1) An entry with coordinates $(x, y)$ is labeled $s_i$ (or $i$) in the heap if and only if $x = i$;

(2) If an entry with coordinates $(x, y)$ is greater than an entry with coordinates $(x', y')$ in the heap then $y > y'$.

Although the above is specific to $W(\widetilde{C}_n)$, the same construction works for any straight line Coxeter graph with the appropriate adjustments made to the label set and assignment of coordinates. Specifically, for type $A_n$ our label set is $\{1, 2, \ldots, n\}$ and for type $B_n$ our label set is $\{0, 1, \ldots, n-1\}$.

In the case of any straight line Coxeter graph it follows from the definition that $(x, y)$ covers $(x', y')$ in the heap if and only if $x = x' \pm 1$, $y' < y$, and there are no entries $(x'', y'')$ such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries at the top (respectively, bottom) correspond to the generators occurring in the left (respectively, right) descent set of the corresponding reduced expression.

Let $\mathsf{w}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. We denote the lattice representation of the heap poset in $\{0, 1, 2, \ldots n\} \times \mathbb{N}$ described in the preceding paragraphs via $H(\mathsf{w})$. If $w$ is FC, then the choice of reduced expression for $w$ is irrelevant and we will often write $H(w)$ and we refer to $H(w)$ as the heap of $w$. Note that we will use the same notation for heaps in Coxeter groups of all types with straightline Coxeter graphs.

Let $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_r}$ be a reduced expression for $w \in W(\widetilde{C}_n)$. If $s_{x_i}$ and $s_{x_j}$ are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by $s_{x_i}$ at a level that is *above* the level of the point labeled by $s_{x_j}$. Because generators in a Coxeter graph that are not adjacent do commute, points whose $x$-coordinates differ by more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners in such a way that if one entry covers another the squares overlap halfway. In addition, we will also label each square for $s_i$ with $i$.

There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap for each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

**Example 1.3.2.** Let $\mathsf{w} = s_6 s_4 s_2 s_5 s_3 s_1 s_4 s_0 s_1$ be a reduced expression for $w \in \mathrm{FC}(\widetilde{C}_6)$ as seen in Example 1.3.1. Figure 1.6 shows a possible lattice point representation for $H(w)$. Since $w$ is FC this is the unique heap representation for $w$. Because $H(w)$ is the unique heap we can obtain $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) from the blocks that are exposed in the top (respectively, bottom) of the heap. From $H(w)$ we see that $\mathcal{L}(w) = \{s_2, s_4, s_6\}$ since these are the bricks that are fully exposed to the top and $\mathcal{R}(w) = \{s_1, s_4\}$ as these are the bricks that are fully exposed to the bottom of the heap.

**Example 1.3.3.** Let $\mathsf{w}_1 = s_0 s_2 s_4 s_3 s_2 s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$.

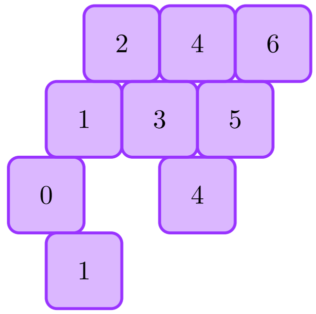
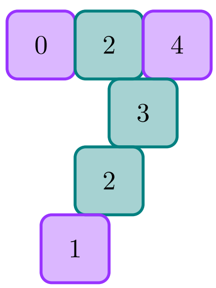
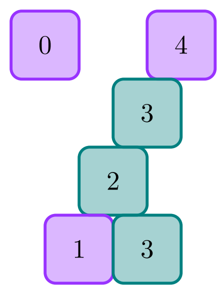



Figure 1.6: A lattice point representation for the heap of an FC element in $W(\widetilde{C}_6)$.

Applying the commutation move $s_2s_4 \mapsto s_4s_2$, we can obtain another reduced expression for $w$, namely $\mathsf{w}_2 = s_0s_4s_2s_3s_2s_1$, which is in the same commutation class as $\mathsf{w}_1$, and hence has the same heap. However, applying the braid move $s_2s_3s_2 \mapsto s_3s_2s_3$, we obtain another reduced expression $\mathsf{w}_3 = s_0s_4s_3s_2s_3s_1$. Note that since $\mathsf{w}_3$ was obtained by applying a braid move, $\mathsf{w}_3$ is in a different commutation class than $\mathsf{w}_1$ and $\mathsf{w}_2$. Representations of $H(\mathsf{w}_1), H(\mathsf{w}_2)$, and $H(\mathsf{w}_3)$ are seen in Figure 1.7, where the braid relation is colored in teal.




(a) $H(\mathsf{w}_1)$ and $H(\mathsf{w}_2)$.




(b) $H(\mathsf{w}_3)$

Figure 1.7: Two heaps of a non-FC element in $W(\widetilde{C}_4)$.

It will be extremely useful for us to be able to quickly determine whether a heap corresponds to an element in $\mathrm{FC}(B_n)$ or $\mathrm{FC}(\widetilde{C}_n)$. The next proposition is a special case of [13, Proposition 3.3] and follows quickly when one considers the consecutive subwords that are impermissible in reduced expressions for elements in $\mathrm{FC}(B_n)$ and $\mathrm{FC}(\widetilde{C}_n)$ as discussed in Section 1.2.

**Theorem 1.3.4.** If $w \in \mathrm{FC}(\widetilde{C}_n)$, then $H(w)$ cannot contain any of the configurations seen in Figure 1.8, where $0 < k < n-1$ and we use a square with a dotted boundary to emphasize that no element of the heap may occupy the corresponding position. $\square$

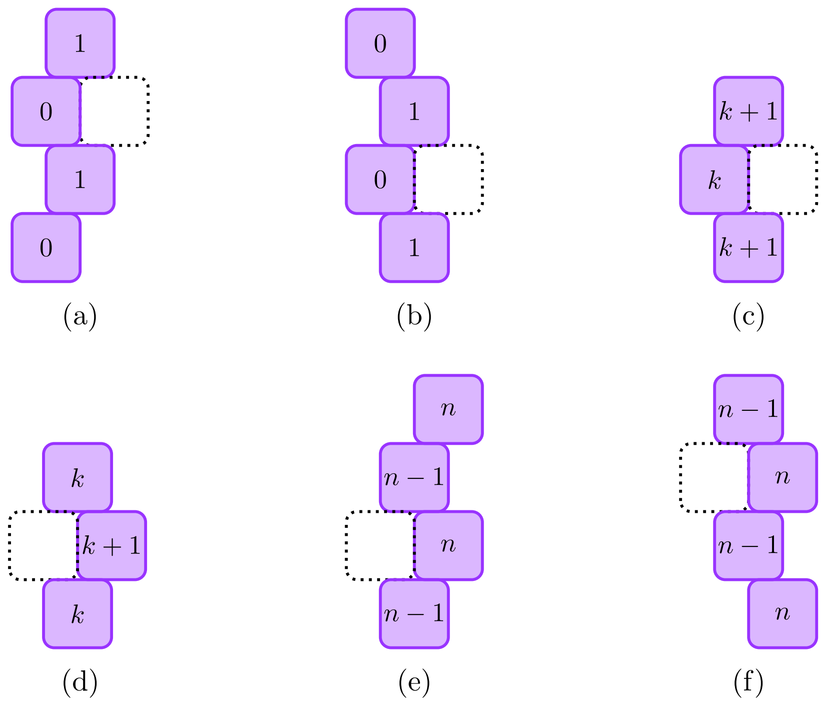



Figure 1.8: Impermissible configurations for heaps of $\mathrm{FC}(\widetilde{C}_n)$.

Since $W(B_n)$ is parabolic subgroup of $W(\widetilde{C}_n)$, we can use Figure 1.8 to classify the impermissible configurations for elements of $\mathrm{FC}(B_n)$. In particular, the impermissible configurations for elements of $\mathrm{FC}(B_n)$ are those seen in Figures 1.8(a), 1.8(b) 1.8(c), and 1.8(d).

# Chapter 2

# Star Operations and Property T

## 2.1 Star Operations

The notion of a star operation was originally introduced by Kazhdan and Lusztig in [11] for simply-laced Coxeter systems (i.e., $m(s,t) \leq 3$ for all $s,t \in S$), and was later generalized to all Coxeter systems in [12]. If $I = \{s,t\}$ is a pair of non-commuting generators of a Coxeter group $W$, then $I$ induces four partially defined maps from $W$ to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to only define the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let $(W,S)$ be a Coxeter system of type $\Gamma$ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We define $w$ to be *left star reducible by $s$ with respect to $t$* if there exists $t \in \mathcal{L}(sw)$. We analogously define $w$ to be *right star reducible by $s$ with respect to $t$*. Observe that $w$ is left (respectively, right) star reducible if and only if $w = stu$ (respectively, $w = uts$), where the product on the right hand side of the equation is reduced and $m(s,t) \geq 3$. We say that $w$ is *star reducible* if it is either left or right star reducible.

**Example 2.1.1.** Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. We see that $w$ is left star reducible by $s_0$ with respect to $s_1$ to $s_1s_0s_2$ since $m(s_0,s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0w)$. Notice that $w$ if FC. We see that $ws_2 = s_0s_1s_0$ and $ws_0 = s_0s_1s_2$. Note that in both instances $s_1 \notin \mathcal{R}(ws_2)$ and $s_1 \notin \mathcal{L}(ws_0)$. Because of this $w$ is not right star reducible.

It may be helpful to visualize star reductions in terms of heaps. Let $(W,S)$ be a Coxeter system of type $\Gamma$ and let $I = \{s,t\} \subseteq S$ be a pair of generators with $m(s,t) \geq 3$. Suppose $w$ is left star reducible by $s$ with respect to $t$. Then there exists a heap for $w$ where the block for $s$ is fully exposed to the top. Removing the block for $s$ off of the top allows for $t$ to now be fully exposed to the top. Similarly if $w$ is right star reducible by $s$ with respect to $t$, then there exists a heap for $w$ where the block for $s$ is fully exposed to the bottom. Removing the block for $s$ off of the bottom allows for $t$ to now be fully exposed to the bottom. Conversely, if a heap for $w \in W(\Gamma)$ has this property, then $w$ is star reducible. In Figure 2.1 we see the heap representation of an element that is left star reducible, where the dotted brick represents that there can not be an element there. Notice that flipping the heap upside in Figure 2.1 will result in a heap being right star reducible. It is important to note that if the group element we are evaluating for Property T is not FC, then we must consider all heap representations for the element before concluding that an element does not have Property T.


(a) (b)

Figure 2.1: A visual representation of an element with Property T at the top.

The following example utilizes heaps to show that an element is star reducible.

**Example 2.1.2.** Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$. Note that $w$ is FC. By Example 2.1.1 we know that $w$ is left star reducible by $s_0$ with respect to $s_1$. In Figure 2.2(a), we see the heap for $w$. Notice that the block for $s_0$ is fully exposed to the top of the heap. Removing the block for $s_0$ gives the heap in Figure 2.2(b). Notice that the block for $s_1$ is now fully exposed to the top of the heap. However, notice that the block for $s_0$ and $s_2$ are fully exposed to the bottom. In removing either of these we are unable to fully expose $s_1$ to the bottom. Thus we can see that $w$ is not right star reducible.

Notice that if $w$ is not FC then we are not be able to say that $w$ is not star reducible as there could be a different heap for $w$ in which we are able to fully expose an element that was previously blocked.

**Example 2.1.3.** Let $\mathsf{w}_1 = s_3s_1s_2s_1s_0s_1s_3s_0s_2s_4$ be a reduced expression for $w \in W(\widetilde{C}_3)$. The heap for $\mathsf{w}$ is given in Figure 2.3(a) where we have highlighted the braid in teal. Notice that this heap appears to not be star reducible as if we were to remove the brick for $s_1$ or $s_3$ we would still not fully expose $s_2$ to the top of the heap. The same goes for fully exposing bricks in the bottom of the heap. However, when we perform the braid operation resulting in the heap seen in Figure 2.3(b) it is now


Figure 2.2: Visualization of Example 2.1.1.

obvious that the element is star reducible. Thus when considering a non-FC element for star reducibility via the heap, it is very important to consider all heaps for that element.

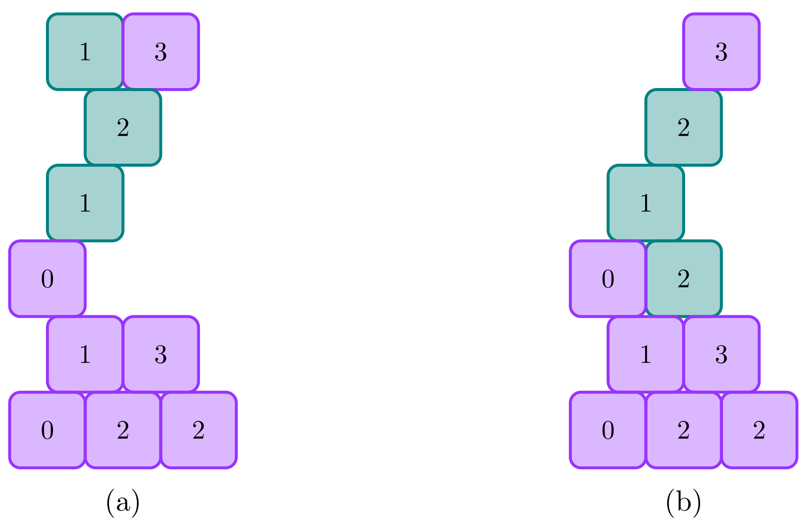



Figure 2.3: Visualization of Example 2.1.3

Using the notion of star reduction we are now able to introduce the concept of a star reducible Coxeter group. We say that a Coxeter group $W(\Gamma)$, or its Coxeter graph $\Gamma$, is *star reducible* if every element of $\mathrm{FC}(\Gamma)$ is star reducible to a product of commuting generators. That is, $W(\Gamma)$ is star reducible if when we apply star reductions repeatedly to $w \in \mathrm{FC}(\Gamma)$, eventually we obtain a product of commuting generators. Notice that in the previous definition we are restricting the elements that must be star reducible to a product of commuting generators to just the FC elements in $W(\Gamma)$. Visually a star reducible Coxeter group can be thought of in the following way, given a heap in $\mathrm{FC}(\Gamma)$, we are able to systematically remove a fully exposed block from the top or bottom of the heap and have a block that was previously not fully exposed become fully exposed until we are left with a heap that is one row in height.

In [9], Green classified all star reducible Coxeter groups.

**Theorem 2.1.4** (Green, [9])**.** Let $(W, S)$ be a Coxeter system of type $\Gamma$. Then $(W, S)$ is star reducible if and only if each component of $\Gamma$ is either a complete graph with labels $m(s,t) \geq 3$, or is one of the following types: type $A_n$ ($n \geq 1$), type $B_n$ ($n \geq 2$), type $D_n$ ($n \geq 4$), type $F_n$ ($n \geq 4$), type $H_n$ ($n \geq 2$), type $I_2(m)$ ($m \geq 3$), type $\widetilde{A}_{n-1}$ ($n \geq 3$ and $n$ odd ), type $\widetilde{C}_{n-1}$ ($n \geq 4$ and $n$ even ), type $\widetilde{E}_6$, or type $\widetilde{F}_5$. □

## 2.2 Property T

As previously mentioned Green classified all star reducible Coxeter groups. In [9], Green utilizes the following theorem to help classify the star reducible Coxeter groups.

**Theorem 2.2.1** (Green, [9])**.** Let $(W, S)$ be a star reducible Coxeter system of type $\Gamma$, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s,t, u with $m(s,t) \neq 2$, $m(t,v) \neq 2$, and $m(s,u) = 2$:

(1) $w$ is a product of commuting generators;

(2) $w$ has a reduced product $w = stu$;

(3) $w$ has a reduced product $w = uts$;

(4) $w$ has a reduced product $w = svtu$. □

In the following discussion we will give name to elements that exhibit the properties above. However, first notice that Items (2) and (3) refer to an element that is star reducible. While Item (4) refers to an element that is not star reducible provided no reduced expression for the element exhibits Items (2) and (3).

We first begin by defining the notion of Property T which is motivated by Items (2) and (3) above. Let $(W, S)$ be a Coxeter system of type $\Gamma$ and let $w \in W$. We say that $w$ has *Property T* if and only if there exists a reduced product for $w$ such that $w = stu$ or $w = uts$ where $m(s,t) \geq 3$. That is, $w$ has Property T if there exists a reduced expression for $w$ that begins or ends with a product of non-commuting generators.

Recall the definition of star reducible from Section 2.1. Notice that if an element in a Coxeter group is star reducible, then it has Property T. The difference be between the two definitions is that when we call an element star reducible we are referring to being able to dismantle the heap in a specified manner. Whereas, when we refer to an element having Property T, we are referring to the element having a pair of non-commuting generators in the top or bottom of the heap. Since elements that are star reducible also have Property T we already know how to visualize Property T in terms of heaps. Recall that an element is star reducible if we can remove a fully exposed block from the top or bottom of a heap and have a new block become fully exposed.

An element $w \in W(\Gamma)$ is called *T-avoiding* if $w$ does not have Property T. This implies that a T-avoiding element is not star reducible.

**Theorem 2.2.2.** Let $(W, S)$ be a Coxeter system of type $\Gamma$. If $w \in W(\Gamma)$ such that $w$ is a product of commuting generators, then $w$ is T-avoiding. $\square$

We will call an element that is a product of commuting generators *trivially T-avoiding.* It is clear that a product of commuting generators is T-avoiding, which we state as a theorem. Visually a product of commuting generators is a one row heap (as in Figure 2.4(b)), it is clear a one row heap will not portray the characteristic of Property T as seen in Figure 2.1. If $w$ is T-avoiding and not a product of commuting generators, we will say that $w$ is *non-trivially T-avoiding.*

**Example 2.2.3.** Let $w \in W(A_5)$ with reduced expression $\mathsf{w} = s_1s_3s_5$. It turns out that since $w$ is a product of commuting generators by Theorem 2.2.2 we know that $w$ is trivially T-avoiding.

**Example 2.2.4.** Let $w \in W(A_5)$ with reduced expression $\mathsf{w}_1 = s_1s_4s_2s_3s_5$. At first glance it may appear that $w$ does not have Property T, since both $s_1$ and $s_4$ commute as well as $s_3$ and $s_5$. However, note that applying a commutation move $s_4s_2 \mapsto s_2s_4$ results in $\mathsf{w}_2 = s_1s_2s_4s_3s_5$. Hence $w$ has Property T, since $m(s_1, s_2) = 3$ and there is a reduced expression for $w$ that begins with $s_1s_2$.

**Example 2.2.5.** Let $\mathsf{w}_1 = s_1s_4s_2s_3s_5$ be a reduced expression for $w_1 \in W(A_5)$ as seen in Example 2.2.4 and let $\mathsf{w}_2 = s_1s_3s_5$ be a reduced expression for $w_2 \in W(A_5)$. In Figure 2.4(a) we see the heap for $w_1$. Note that we can see Property T in the bottom of the heap highlighted in orange. In Figure 2.4(b) we see the heap for $w_2$. Note that as the heap is only one row and $w_2$ is FC, it is clear that $w_2$ does not have Property T.

**Example 2.2.6.** Let $w \in W(\widetilde{C}_4)$ with reduced expression $\mathsf{w} = s_0s_2s_4s_1s_3s_0s_2s_4$. It turns out that $w$ is FC and non-trivially T-avoiding. The heap for $w$ is seen in Figure 2.5. Notice that no matter which block we remove that is fully exposed to the top of the heap no new element becomes fully exposed. The same applies to the bottom of the heap. Thus, $w$ is non-trivially T-avoiding.


(a) Heap of an element with Property T (b) Heap of a T-Avoiding element

Figure 2.4: Heaps of an element with Property T and a T-Avoiding element

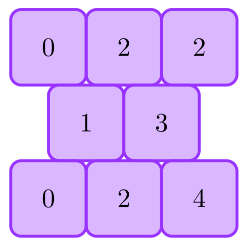



Figure 2.5: Heap of a non-trivially T-Avoiding element in $W(\widetilde{C}_4)$.

Referring back to Green's classification (Theorem 2.2.1) of what elements in star reducible Coxeter groups look like, we see that Item (1) corresponds to an element $w$ being trivially T-avoiding, Items (2) and (3) refer to the element $w$ having Property T at the beginning and end respectively, and Item (4) refers to an element being non-trivially T-avoiding provided no reduced expression for the element exhibits Items (2) and (3). In star reducible Coxeter groups, every FC element is star reducible to a product of commuting generators, which implies that no FC element can be non-trivially T-avoiding. For example, as will be seen in the following sections, the Coxeter groups of type $A_n$ and $B_n$ have no non-trivial T-avoiding elements, while the Coxeter group of type $D_n$ does have non-trivial T-avoiding elements.

One thing to notice here is that all Coxeter groups have trivial T-avoiding elements as they all contain products of commuting generators. As a result of this we will avoid mentioning them in the following classification. The more interesting non-trivial T-avoiding elements do not appear in all Coxeter groups. Chapters 3 and 4 discusse what is currently known regarding T-avoiding elements in the irreducible finite Coxeter groups and the irreducible affine Coxeter groups. In Chapter 3 we will summarize what is known about the T-avoiding elements in Coxeter groups of types $\widetilde{A}_n$, $A_n$, $D_n$, $F_n$, and $I_2(m)$, and in Chapter 4 we classify the T-avoiding elements in Coxeter groups of types $B_n$ and $\widetilde{C}_n$.

## 2.3 Non-Cancellable Elements

We now introduce the concept of weak star reducible, which is related to the notion of cancellable in [5]. Let $(W, S)$ be a Coxeter system of type $\Gamma$ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators. If $w \in \mathrm{FC}(\Gamma)$, then $w$ is *left weak star reducible by $s$ with respect to $t$ to $sw$* if

(1) $w$ is left star reducible by $s$ with respect to $t$, and

(2) $tw \notin \mathrm{FC}(W)$.

Notice that condition (2) implies that $l(tw) > l(w)$. Also note that we are restricting out definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by $s$ with respect to $t$ to $ws$*. We say that $w$ is *weak star reducible* if $w$ is either left or right weak star reducible. Otherwise, we say that $w$ is *non-cancellable* or *weak star irreducible*. Notice that from this we know that weak star reducible implies star reducible. However, star reducible does not imply weak star reducible.

**Example 2.3.1.** Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduced expression for $w \in W(B_4)$ as in Example 2.1.1. From Example 2.1.1 we know that $w$ is left star reducible. However, $tw = s_1s_0s_1s_0s_2$ which is not in $\mathrm{FC}(B_4)$. Thus, we see that $w$ is left weak star reducible by $s_0$ with respect to $s_1$ to $s_1s_0s_2$. In addition, Example 2.1.1 showed that $w$ is not right star reducible and hence $w$ is not right weak star reducible. However, since $w$ is left weak star reducible we know that $w$ is not non-cancellable.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall that in Section 2.1 we described what a star reduction looks like in a heap and what a star reducible heap looks like. Since the definition of weak star reducible includes that a heap is star reducible we again need to have those properties. In addition, for a heap to be weak star reducible when adding the block that becomes fully exposed when a block is removed from the heap must create a braid in the heap forcing the new heap to not be FC. That is, one of the impermissible configurations seen in Section 1.3 will appear.

**Example 2.3.2.** Let $\mathsf{w} = s_0s_1s_0s_2$ be a reduce expression for $w \in W(B_4)$ as in Example 2.3.1. From Example 2.3.1, we know that $w$ is left weak star reducible. Recall in Figure 2.2 the heap for $w$ was seen along with what it star reduced to. In Figure 2.6 we see that adding $s_1$ to the top of the heap creates a braid which is highlighted in orange.

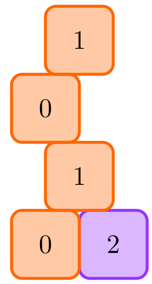



Figure 2.6: Visualization of a weak star reducible element of $\mathrm{FC}(B_4)$.

**Example 2.3.3.** Let $w \in \mathrm{FC}(B_4)$ and let $\mathsf{w} = s_0 s_1$ be a reduced expression for $w$. Note that $w$ is left (respectively, right) star reducible by $s_0$ with respect to $s_1$ (respectively, by $s_1$ with respect to $s_0$). However, $s_1 s_0 s_1 \in \mathrm{FC}(B_4)$ (respectively, $s_0 s_1 s_0 \in \mathrm{FC}(B_4)$). Thus $w$ is non-cancellable. Visually the heap appears in Figure 2.7. Clearly when $s_0$ is added to the bottom of the heap, the new heap is still in $\mathrm{FC}(B_4)$ and the same can be said when $s_1$ is added to the top of the heap.

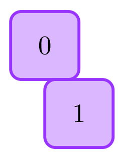



Figure 2.7: Visualization of a non-cancellable element of $\mathrm{FC}(B_4)$.

# Chapter 3

# Classification of Non-Trivial T-Avoiding Elements in Types $\widetilde{A}_n, A_n, D_n, F_n$, and $I_2(m)$

## 3.1 T-Avoiding Elements in Types $\widetilde{A}_n$ and $A_n$

We start by classifying the T-avoiding elements in Coxeter groups of type $\widetilde{A}_n$ and $A_n$. We first classify non-trivial T-avoiding elements in $W(\widetilde{A}_n)$.

**Theorem 3.1.1.** If $n \geq 2$ and $n$ is even, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. Otherwise, if $n \geq 2$ and $n$ is even then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

*Proof.* This is [6, Proposition 3.1.2.]. □

The previous theorem implies that when $n$ is even the Coxeter group of type $\widetilde{A}_n$, has no non-trivial T-avoiding elements. However, while $n$ is odd, the Coxeter group of type $\widetilde{A}_n$ has non-trivial T-avoiding elements. The classification seen in [6] did not specifically classify the non-trivial T-avoiding elements for type $\widetilde{A}_n$ for $n$ odd. Since $W(\widetilde{A}_n)$ for $n$ odd is not star reducible we know that the non-trivial T-avoiding elements could be FC. The following is our conjecture regarding what the non-trivial T-avoiding elements are in $W(\widetilde{A}_n)$ for $n$ odd.

**Conjecture 3.1.2.** The only non-trivial T-avoiding elements in $W(\widetilde{A}_n)$ for $n$ odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

Notice that the above non-trivial T-avoiding elements are FC. As stated in the conjecture we believe that these are the only non-trivial T-avoiding elements. However, this remains an open problem. We now proceed with the classification of T-avoiding elements in Coxeter groups of type $A_n$.

**Corollary 3.1.3.** Then there are no non-trivially T-avoiding elements in $W(A_n)$.

*Proof.* Since $W(A_n)$ is a parabolic subgroup of $W(\widetilde{A}_n)$ this is a consequence of [6, Proposition 3.1.2.]. Specifically, we can obtain the Coxeter graph of type $A_n$ from the Coxeter graph of type $\widetilde{A}_n$ for $n$ even by removing the appropriate number of vertices and edges. From this we can see that if $W(A_n)$ was to have non-trivial T-avoiding elements, this would imply that $W(\widetilde{A}_n)$ for $n$ even would also have non-trivial T-avoiding elements as well. Thus $W(A_n)$ can not have bad elements. □

## 3.2 T-Avoiding Elements in Type $D_n$

In this section we will classify the non-trivial T-avoiding elements in the Coxeter group of type $D_n$. Recall that $W(D_n)$ is a star reducible Coxeter group and as a result of this any non-trivial T-avoiding element are not FC.

**Theorem 3.2.1.** The only non-trivial T-avoiding elements in $W(D_n)$ for $n \geq 4$ are given by the heaps Once we figure out the heap include it here..

*Proof.* This is a consequence of [8, Section 2.2]. For the full details regarding his classification see [8]. Note that in his classification, Gern refers to non-trivially T-avoiding elements as "bad." □

## 3.3 T-Avoiding Elements in Type $F_n$

In this section we classify what is known regarding the non-trivial T-avoiding elements in the Coxeter groups of type $F_n$ for $n \geq 4$. Note that all of the following results are unpublished.

We start with the Coxeter group of type $F_5$. Recall that $W(F_5)$ is a star reducible Coxeter group so any non-trivial T-avoiding elements will not be FC. Before we begin the classification we introduce the notion of a specific element in $W(F_5)$. We call this element a single *bowtie*, which is seen in Figure 3.1. Note that in this heap, the orange blocks correspond to the elements that have bond strength 4 (Figure 3.1(a)). Note that the element is also not FC and we have highlighted the braid in teal which is seen in Figure 3.1(b). In stacking the single bowties together, we get a "stack of bowties." This is done by removing the top most layer of the heap and adding a new single bowtie to the stack, a stack of bowties is seen in Figure 3.2. These heaps are referenced in the following theorem which classifies all non-trivial T-avoiding elements in the following in the Coxeter group of type $F_5$ completed by Cross, Ernst, Hills-Kimball, and Quaranta in 2012.

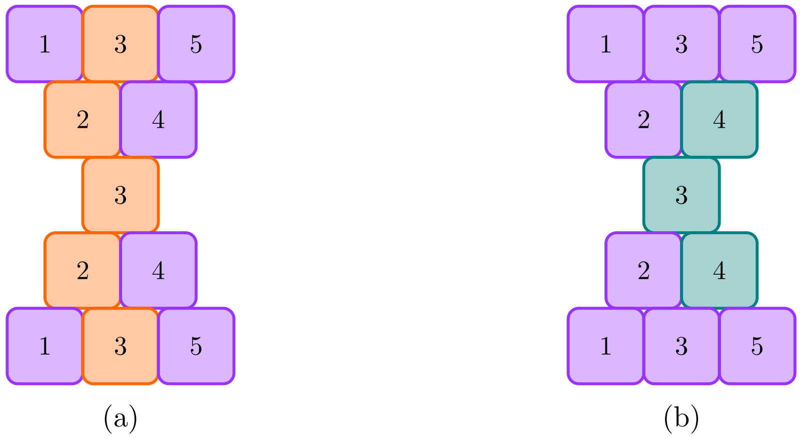



Figure 3.1: A single bowtie in $W(F_5)$.

**Theorem 3.3.1.** The only non-trivial T-avoiding elements in $W(F_5)$ are stacks of bowties. □

As a result of the classification in type $F_5$, Cross et al. were also able to classify the non-trivial T-avoiding elements in $W(F_4)$.

**Corollary 3.3.2.** There are no non-trivial T-avoiding elements in $F_4$. □

As a result of their work, Cross et al. conjectured that in Coxeter groups of type $F_n$ for $n \geq 5$, an element is non-trivially T-avoiding if and only if it is a stack of bowties multiplied by a product of commuting generators. In 2013, Gilbertson and Ernst worked with this conjecture and quickly found out that it was incorrect. The heap seen in Figure 3.3 corresponds to a non-trivial T-avoiding element in $F_6$. It turns out that like the bowties discussed above these elements can also be stacked to create an infinite number of non-trivial T-avoiding elements. In addition, as $n$ gets large there are a number of things that can be altered that result in additional non-trivially T-avoiding elements. From this we conjecture that the classification of T-avoiding elements in Coxeter systems of type $F_n$ for $n \geq 6$ gets complicated very quickly. Classifying T-avoiding elements in $W(F_n)$ for $n \geq 6$ remains an open problem.

## 3.4 T-Avoiding Elements in Type $I_2(m)$

We next will classify the T-avoiding elements in Coxeter groups of type $I_2(m)$. Note that in Coxeter groups of type $I_2(m)$, the only products of commuting generators have length 1. Although the following is a quick result, we believe that it does not already appear in the literature.

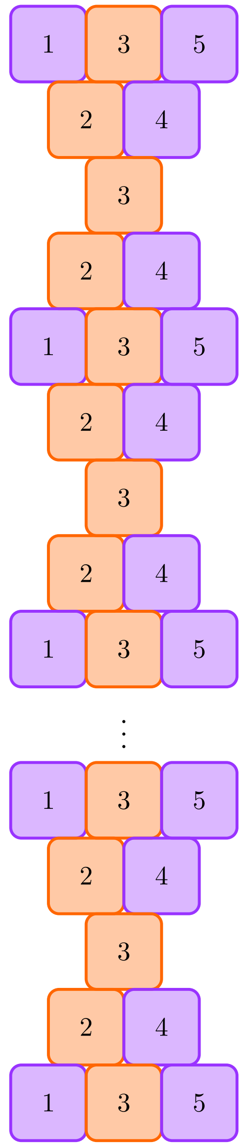



Figure 3.2: A stack of bowties in $W(F_5)$.

**Theorem 3.4.1.** The Coxeter group $W(I_2(m))$ has no non-trivial T-avoiding elements.

*Proof.* The graph for the Coxeter group of type $I_2(m)$ appears in Figure 1.2(c). Note that the graph consists of two vertices, namely, $s_1$ and $s_2$, and a single edge with weight $m$. Also, recall that $W(I_2(m))$ is a star reducible Coxeter group. This implies that any non-trivial T-avoiding elements in $W(I_2(m))$ must not be FC, ss all of the FC

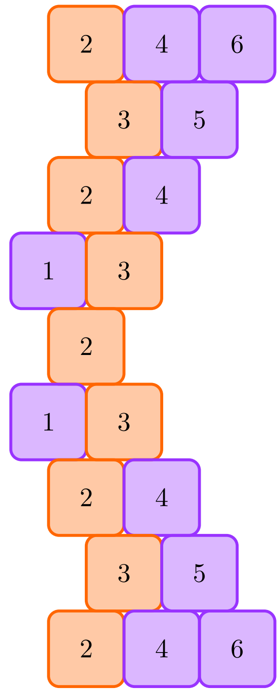



Figure 3.3: A non-trivial T-avoiding element in $W(F_6)$

elements have Property T. The only non-FC element in $W(I_2(m))$ is the element of length $m$ which has exactly two reduced expressions consisting of alternating products of $s_1$ and $s_2$. Cleary, this element begins and ends with a product of noncommuting generators. Thus, this element has Property T. Hence $W(I_2(m))$ has no non-trivial T-avoiding elements. $\square$

# Chapter 4

# T-Avoiding Elements in Types $B_n$ and $\widetilde{C}_n$

In this section we classify the T-Avoiding elements in Coxeter groups of type $B_n$ and type $\widetilde{C}_n$. We start by introducing necessary tools for type $B_n$ and finish with a proof of the classification in $B_n$. We then conclude with the classification of $\widetilde{C}_n$. Note that the proof for Coxeter systems of type $B_n$ closely follows the classification of T-avoiding elements of Type $D_n$ seen in [8].

## 4.1 Tools for the Classification

Recall from Example 1.1.1 that we can represent each element $w \in W(B_n)$ as a member of the signed permutation group. As a result we can write $w \in W(B_n)$ using one-line notation

$$w = [w(1), w(2), \ldots, w(n-1), w(n)]$$

where we write a bar underneath a number in place of a negative sign in order to simplify notation. This is obtained from the Coxeter group in the following way. We identify $s_i \in S(B_n)$ via

$$s_i = [1, 2, \ldots i-1, i+1, i, i+2, \ldots, n-1, n]$$

and we identify $s_0 \in S(B_n)$ via

$$s_0 = [\underline{1}, 2 \ldots, n].$$

Further $w(-i) = -w(i)$ for $i \in \{1, 2, \ldots, n\}$.

**Example 4.1.1.** Let $w \in W(B_6)$ with a given reduced expression $\mathsf{w} = s_0 s_1 s_3 s_4 s_5 s_2$. Then we can write $w = [2, 4, \underline{1}, 5, 6, 3]$.

It will be useful to easily determine what what happens to the window notation of a given element $w \in W(B_n)$ when we multiply on the right or left by $s_i \in S(B_n)$. The following Proposition allows us to do this.

**Proposition 4.1.2.** Let $w \in W(B_n)$ with corresponding signed permutation

$$w = [w(1), w(2), \ldots w(n)].$$

Suppose $s_i \in S(B_n)$. If $i \geq 1$, then multiplying $w$ on the right by $s_i$ has the effect of interchanging $w(i)$ and $w(i+1)$. Multiplying on the left by $s_i$ has the effect of interchanging the entries whose absolute values are $i$ and $i+1$.

If $i = 0$, then multiplying $w$ on the right by $s_i$ has the effect of switching the sign of $w(1)$. Multiplying $w$ on the left by $s_i$ has the effect of switching the sign of the entry whose absolute value is 1.

*Proof.* This follows from [2, Section 8.1 and A3.1]. $\square$

Given the one line notation for an element $w \in W(B_n)$ we can easily calculate the left and right descent sets of $w$. The following proposition explains how.

**Proposition 4.1.3** (Björner, [2])**.** Let $w \in W(B_n)$. Then

$$\mathcal{R}(w) = \{s_i \in S : w(i) > w(i+1)\}$$

where w(0)=0 by definition.

*Proof.* This is, [2] Proposition 8.1.2. $\square$

We now will introduce the concept of signed pattern avoidance which will help with the classification of the T-avoiding elements in the Coxeter group of type $B_n$. This notion was first introduced in [8]. Let $w \in W(B_n)$. We say that $w$ *avoids the consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(w(i),\ w(i+1),\ w(i+2))$ is in the same relative order as $(a, b, c)$. We say that $w$ *avoids the signed consecutive pattern abc* if there is no $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same consecutive order as $(|a|, |b|, |c|)$ and such that $\text{sign}(w(i)) = \text{sign}(a)$, $\text{sign}(w(i+1)) = \text{sign}(b)$, and $\text{sign}(w(i+2)) = \text{sign}(c)$.

**Example 4.1.4.** Let $w \in W(B_4)$ with signed permutation

$$w = [\underline{2}, 4, \underline{1}, 3].$$

We see that $w$ has the signed consecutive pattern $\underline{2}3\underline{1}$, since $(|w(1)|, |w(2)|, |w(3)|)$ are in the same relative order as $(|-2|, |3|, |-1|)$, and $\text{sign}(w(1)) = \text{sign}(-2)$, $\text{sign}(w(2)) = \text{sign}(3)$, and $\text{sign}(w(3)) = \text{sign}(-1)$. However, $w$ avoids the signed consecutive pattern $1\underline{2}3$.

## 4.2 Classification of T-Avoiding Elements in Type $B_n$

In this section we will classify the T-avoiding elements in Coxeter groups of type $B_n$. Specifically we will classify the non-trivial T-avoiding elements in $W(B_n)$. The following is our classification.

**Theorem 4.2.1.** There are no non-trivial T-avoiding elements in $W(B_n)$.

In order to prove this we will use the notion of signed pattern avoidance seen above. Before we prove this theorem we first need some preparatory lemmas.

**Lemma 4.2.2.** Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then $w$ has a reduced expression ending in $sts$ if and only if $w$ has the consecutive pattern 321.

*Proof.* Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [2]. Observe that if $w$ has a reduced expression ending in two non-commuting generators $s_i, s_{i+1}$ in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose $w$ has the consecutive pattern 321. Then there is some $i$ such that $w(i) > w(i+1) > w(i+1)$. By 4.1.3 $s_i, s_{i+1} \in \mathcal{R}(w)$. By Tyson's reference to simply laced coxeter group stuff 1.2.1 $w$ ends in $s_i s_{i+1} s_{i+2}$.

Conversely, suppose $w$ ends in $s_i s_i + 1 s_i$. This implies that either $w_I = s_i s_{i+1}$ or $w_I = s_{i+1} s_i$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by 4.1.3. Thus $w$ has the consecutive pattern 321. Therefore, $w$ has a reduced expression ending in $sts$ if and only if $w$ has the consecutive pattern 321. □

**Corollary 4.2.3.** Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then $w$ has a reduced expression beginning with $sts$ if and only if $w^{-1}$ has the consecutive pattern 321.

*Proof.* Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. We know that $w$ has no reduced expressions beginning with $sts$ if and only if $w^{-1}$ has no reduced expression ending with $sts$ which by Theorem 4.2.3 happens only if $w^{-1}$ avoids the consecutive pattern 321. □

**Lemma 4.2.4.** Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then $w$ has a reduced expression ending in $st$ if and only if $w$ has the consecutive pattern 231.

*Proof.* Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [2]. Observe that if $w$ has a reduced expression ending in two non-commuting generators $s_i, s_{i+1}$ in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose that $w$ has the consecutive pattern 231. Then there is some $i$ such that $w(i+1) > w(i) > w(i+2)$. By 4.1.3 $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by $s_{i+1}$ we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, this implies that $s_i \in \mathcal{R}(ws_{i+1})$. This implies $w$ has a reduced expression that ends in $s_i s_{i+1}$.

Conversely, suppose that $w$ has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence $w$ has the consecutive pattern 231. Therefore, $w$ has a reduced expression ending in $st$ if and only if $w$ has the consecutive pattern 231. □

**Corollary 4.2.5.** Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then $w$ has a reduced expression beginning with $st$ if and only if $w^{-1}$ has the consecutive pattern 231.

*Proof.* Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. We know that $w$ has no reduced expressions beginning with $st$ if and only if $w^{-1}$ has no reduced expression ending with $st$ which by Theorem 4.2.3 happens only if $w^{-1}$ avoids the consecutive pattern 231. □

**Lemma 4.2.6.** Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then $w$ has a reduced expression ending in $ts$ if and only if $w$ has the consecutive pattern 312.

*Proof.* Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [2]. Observe that if $w$ has a reduced expression ending in two non-commuting generators $s_i, s_{i+1}$ in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose that $w$ has the consecutive pattern 312. Then there is some $i$ such that $w(i) > w(i+2) > w(i+1)$. By 4.1.3 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by $s_i$ we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$. By above $w(i) > w(i+2)$, and by 4.1.3 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that $w$ has a reduced expression ending in $s_{i+1} s_i$.

Conversely suppose $w$ ends in a reduced expression with $s_{i+1} s_i$ Then $w_I = s_{i+1} s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, $w$ has the consecutive pattern 312. Therefore, $w$ has a reduced expression ending in $ts$ if and only if $w$ has the consecutive pattern 312. □

**Corollary 4.2.7.** Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. Then $w$ has a reduced expression beginning with $ts$ if and only if $w^{-1}$ has the consecutive pattern 312.

*Proof.* Let $s, t \in S(B_n)$ such that $m(s,t) = 3$, and $s_0 \notin \{s,t\}$. We know that $w$ has no reduced expressions beginning with $ts$ if and only if $w^{-1}$ has no reduced expression ending with $ts$ which by Theorem 4.2.3 happens only if $w^{-1}$ avoids the consecutive pattern 312. $\square$

**Lemma 4.2.8.** Let $w \in W(B_n)$. Then $w$ has a reduced expression ending in $s_1s_0$ if and only if $w(0) > w(1)$ and $-w(1) > w(2)$.

*Proof.* Suppose $w \in W(B_n)$ such that $w$ ends with $s_1s_0$. Then $s_0 \in \mathcal{R}(w)$ and $s_1 \in \mathcal{R}(ws_0)$. This implies that $ws_0(1) > ws_0(2)$ by 4.1.3. We see that $ws_0(1) = w(-1) = -w(1)$ and $ws_0(2) = 2$. Hence $-w(1) = ws_0(1) > ws_0(2) = w(2)$. Further, since $s_0 \in \mathcal{R}(w)$, we see that $w(0) > w(1)$.

Conversely, suppose $w \in W(B_n)$ such that $w(0) > w(1)$ and $-w(1) > w(2)$. Since $w(0) > w(1)$ so $s_0 \in \mathcal{R}(w)$. Multiplying on the right by $s_0$ we see that $ws_0(1) = -w(1)$ and $ws_0(2) = w(2)$. Note that since $ws_0(1) = -w(1) > w(2) = ws_0(2)$, $s_1 \in \mathcal{R}(ws_0)$. Thus $w$ ends with $s_1s_0$. Therefore, $w$ has a reduced expression ending in $s_1s_0$ if and only if $w(0) > w(1)$ and $-w(1) > w(2)$. $\square$

**Corollary 4.2.9.** Let $w \in W(B_n)$. Then $w$ has a reduced expression beginning in $s_0s_1$ if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$.

*Proof.* Let $w \in W(B_n)$. We know that $w$ has no reduced expressions beginning in $s_0s_1$ if and only if $w^{-1}$ has no reduced expressions ending in $s_0s_1$. By Lemma 4.2.8 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(1)$ and $-w^{-1}(1) > w^{-1}(2)$. $\square$

**Lemma 4.2.10.** Let $w \in W(B_n)$. Then $w$ has a reduced expression ending in $s_0s_1$ if and only if $w(0) > w(2)$ and $w(1) > w(2)$.

*Proof.* Suppose $w \in W(B_n)$ such that $w$ ends with $s_0s_1$. Then $s_1 \in \mathcal{R}(w)$ and $s_0 \in \mathcal{R}(ws_1)$. Then $ws_1(0) > ws_1(1)$. We see that $ws_1(0) = 0$ and $ws_1(1) = w(2)$. This implies that $0 = ws_1(0) > ws_1(1) = 2$. Further, since $s_1 \in \mathcal{R}(w)$ this implies that $w(1) > w(2)$. Thus if $w$ ends with $s_0s_1$, then $w(1) > w(2)$ and $w(0) > w(2)$.

Conversely, suppose $w \in W(B_n)$ such that $w(1) > w(2)$ and $w(0) > w(2)$. This implies that $s_1 \in \mathcal{R}(W)$. Multiplying $w$ on the right by $s_1$ we see that $ws_1(0) = w(0)$ and $ws_1(1) = w(2)$. Note that since $ws_1(0) = w(0) > w(2) = ws_1(1)$, $s_0 \in \mathcal{R}(ws_1)$. Thus $w$ ends with $s_0s_1$. Therefore, $w$ has a reduced expression ending in $s_0s_1$ if and only if $w(1) > w(2)$ and $w(0) > w(2)$. $\square$

**Corollary 4.2.11.** Let $w \in W(B_n)$. Then $w$ has a reduced expression beginning in $s_1s_0$ if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$.

*Proof.* Let $w \in W(B_n)$. We know that $w$ has no reduced expressions beginning in $s_1s_0$ if and only if $w^{-1}$ has no reduced expressions ending in $s_1s_0$. By Lemma 4.2.8 we know that this occurs if and only if $w^{-1}(0) > w^{-1}(2)$ and $w^{-1}(1) > w^{-1}(2)$. $\square$

**Lemma 4.2.12.** Let $w \in W(B_n)$ such that each entry for $w$ in the one-line notation is positive and both $w$ and $w^{-1}$ avoid the consecutive patterns 321, 231, and 312, then $w$ is a product of commuting generators.

*Proof.* This is [8, Lemma 2.2.9]. $\square$

**Lemma 4.2.13.** Let $w \in W(B_n)$ be trivially T-avoiding and let $i \in \{1, 2, \ldots, n\}$. Then $w$ satisfies the following conditions:

(1) $w(j) > \min(\{w(i-1), w(i)\})$ for all $j > i$;

(2) $w(k) < \max(\{w(i-1), w(i)\})$ for all $k < i-1$;

(3) if $w(i), w(i+1) > 0$, then $w(j) > 0$ for all $j \geq i$;

(4) if $w(i), w(i+1) < 0$, then $w(j) < 0$ for all $j \leq i+1$.

*Proof.* Suppose there is some least $j > i$ such that $w(j) \leq \min(\{w(i-1), w(i)\})$. Note that $j > i$ so $j \neq i$, and $j \neq i-1$ so $w(j) < \min(\{w(i-1), w(i)\})$. Note that $j$ is the least so $w(j-2) \geq \min(\{w(i+1), w(i)\}) > w(j)$. This implies that either $w(j-1) > w(j-2) > w(j)$ or $w(j-2) > w(j-1) > w(j)$, which implies $w$ has the consecutive pattern 231 or 321 which is a contradiction to $w$ being a non-trivial T-avoiding element by Lemmas 4.2.2 and 4.2.6. Thus proving (1).

Suppose there exists a maximal $k < i-1$ such that $w \geq \max(\{w(i-1), w(i)\})$. Note that $k < i-1$ so $k \neq i$ and $k \neq i-1$. Then $w(k) > \max(\{w(i-1), w(i)\})$. Since $k$ is maximal then $w(k+1) \leq \max(\{w(i-1), w(i)\})$ and $w(k+2) \leq \max(\{w(i-1), w(i)\})$. This implies that either $w(k+2) < w(k+1) < w(k)$ or $w(k+1) < w(k+2) < w(k)$, which implies $w$ has the consecutive patter 321 or 312 which is a contradiction to $w$ being a non-trivial T-avoiding element by Lemmas 4.2.2 and 4.2.4. Thus proving (2).

It is easy to see that assertion (1) implies (3) and assertion (2) implies (4). $\square$

**Lemma 4.2.14.** Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{2}31$. Then $w$ has Property T.

*Proof.* Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{2}31$.

Case 1: Suppose $w$ has the one-line notation $w = [\underline{2}, 3, 1]$. This implies that $w = s_1 s_0 s_2$. Clearly, $w$ begins with a product of non-commuting generators. Thus $w$ has Property T.

Case 2: Suppose that $w$ has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: $31 \pm 2$, $32 \pm 1$, or $21 \pm 3$. We know that $b, c$ must be positive since they are positive in $w$ and we also know that $b > c$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Case 3: Suppose that $w$ has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 1\underline{2}3$, $\pm 2\underline{1}3$, and $\pm 3\underline{1}2$. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}31$, then $w$ has Property T. □

**Lemma 4.2.15.** Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{2}3\underline{1}$. Then $w$ has Property T.

*Proof.* Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{2}3\underline{1}$.

Case 1: Suppose $w$ has the one-line notation $w = [\underline{2}, 3, \underline{1}]$. This implies that $w = s_0 s_1 s_0 s_2$. Clearly, $w$ begins with a product of non-commuting generators. Thus $w$ has Property T.

Case 2: Suppose that $w$ has the one-line notation $w = [\underline{a}, b, \underline{c}, *, \ldots, *]$ where $\underline{a}, b, \underline{c}$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$. We now consider the signed consecutive pattern that can arise involving $b, \underline{c}, *$. The following are the possibilities for the signed consecutive pattern that can arise: $3\underline{1} \pm 2$, $3\underline{2} \pm 1$, or $2\underline{1} \pm 3$. We know that $b$ must be positive since it is positive in $w$, $c$ must be negative since it is negative in $w$, and we also know that $|b| > |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Case 3: Suppose that $w$ has the one-line notation $w = [*, \ldots, *, \underline{a}, b, \underline{c}]$ where $\underline{a}, b, \underline{c}$ correspond to the signed consecutive pattern $\underline{2}, 3, \underline{1}$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 1\underline{2}3$, $\pm 2\underline{1}3$, and $\pm 3\underline{1}2$. We know that $a$ must be negative, $b$ must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{2}3\underline{1}$, then $w$ has Property T. □

**Lemma 4.2.16.** Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{1}23$. Then $w$ has Property T or is a trivial T-avoiding element.

*Proof.* Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{1}23$.

Case 1: Suppose $w$ has the one-line notation $w = [\underline{1}23]$. This implies that $w = s_0$. Clearly, $w$ is a trivial T-avoiding element as it is a single generator.

Case 2: Suppose that $w$ has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: $23 \pm 1$, $13 \pm 2$, or $12 \pm 3$. We know that $b, c$, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Case 3: Suppose that $w$ has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 3\underline{1}2$, $\pm 2\underline{1}3$, and $\pm 1\underline{2}3$. We know that $a$ must be negative, $b$ must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}23$, then $w$ has Property T or is a trivial T-avoiding element. □

**Lemma 4.2.17.** Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{1}32$. Then $w$ has Property T or is a trivial T-avoiding element.

*Proof.* Let $w \in W(B_n)$ such that $w$ has the consecutive pattern $\underline{1}32$.

Case 1: Suppose $w$ has the one-line notation $w = [\underline{1}32]$. This implies that $w = s_0 s_2$. Clearly, $w$ is a trivial T-avoiding element as it is a single generator.

Case 2: Suppose that $w$ has the one-line notation $w = [\underline{a}, b, c, *, \ldots, *]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{1}, 2, 3$. We now consider the signed consecutive pattern that can arise involving $b, c, *$. The following are the possibilities for the signed consecutive pattern that can arise: $23 \pm 1$, $13 \pm 2$, or $12 \pm 3$. We know that $b, c$, and we also know that $|b| < |c|$ by the original signed consecutive pattern. Note that by Lemmas 4.2.2, 4.2.4, and 4.2.8 all of these patterns imply that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Case 3: Suppose that $w$ has the one-line notation $w = [*, \ldots, *, \underline{a}, b, c]$ where $\underline{a}, b, c$ correspond to the signed consecutive pattern $\underline{2}, 3, 1$. We now consider the signed consecutive pattern that can arise involving $*, \underline{a}, b$. The following are the possibilities for the signed consecutive pattern that can arise: $\pm 3\underline{1}2$, $\pm 2\underline{1}3$, and $\pm 3\underline{2}1$. We know that $a$ must be negative, $b$ must be positive and $|a| < |b|$ by the original signed permutation. Note that by Lemmas 4.2.4, 4.2.8, and 4.2.10 all of these patterns implies that $w$ ends or begins with a product of noncommuting generators. Thus $w$ has Property T.

Therefore, if $w \in W(B_n)$ contains the consecutive pattern $\underline{1}23$, then $w$ has Property T or is a trivial T-avoiding element. $\square$

We can now prove Theorem 4.2.1.

*Proof.* Suppose that $w \in W(B_n)$ is a non-trivial T-avoiding element. There are $2^3 \cdot 3!$ possible choices of signed consecutive patterns for $w(1)w(2)w(3)$ where $w = [w(1), w(2), w(3), *, \ldots, *]$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $123$ | $\underline{1}23$ | $1\underline{2}3$ | $12\underline{3}$ | $\underline{12}3$ | $\underline{1}2\underline{3}$ | $1\underline{23}$ | $\underline{123}$ |
| $132$ | $\underline{1}32$ | $1\underline{3}2$ | $13\underline{2}$ | $\underline{13}2$ | $\underline{1}3\underline{2}$ | $1\underline{32}$ | $\underline{132}$ |
| $213$ | $\underline{2}13$ | $2\underline{1}3$ | $21\underline{3}$ | $\underline{21}3$ | $\underline{2}1\underline{3}$ | $2\underline{13}$ | $\underline{213}$ |
| $231$ | $\underline{2}31$ | $2\underline{3}1$ | $23\underline{1}$ | $\underline{23}1$ | $\underline{2}3\underline{1}$ | $2\underline{31}$ | $\underline{231}$ |
| $312$ | $\underline{3}12$ | $3\underline{1}2$ | $31\underline{2}$ | $\underline{31}2$ | $\underline{3}1\underline{2}$ | $3\underline{12}$ | $\underline{312}$ |
| $321$ | $\underline{3}21$ | $3\underline{2}1$ | $32\underline{1}$ | $\underline{32}1$ | $\underline{3}2\underline{1}$ | $3\underline{21}$ | $\underline{321}$ |

We can use Lemma 4.2.2 and Corollary 4.2.3 to eliminate the signed consecutive patterns highlighted in turquoise. We can use Lemma 4.2.6 and Corollary 4.2.5 to eliminate the signed consecutive patterns highlighted in red. We can use Lemma 4.2.4 and Corollary 4.2.7 to eliminate the consecutive patterns highlighted in green. We can use Lemma 4.2.8 and Corollary 4.2.9 to eliminate the signed consecutive patterns highlighted in yellow. We can use Lemma 4.2.10 and Corrollary 4.2.11 to eliminate signed consecutive patterns highlighted in brown. We can use Lemma 4.2.12 to eliminate the signed consecutive patterns highlighted in blue. We can use Lemmas 4.2.14 and 4.2.15 to eliminate signed consecutive patterns highlighted in purple. Finally we can use Lemmas 4.2.16 and 4.2.17 to eliminate signed consecutive patterns highlighted in orange. Since all of the above patterns are eliminated as possibilities for $w(1)w(2)w(3)$ and there are no other signed consecutive patterns that are possible for these positions, $w$ can not be a non-trivial T-avoiding element in the Coxeter group of type B. Therefore, there are no non-trivial T-avoiding elements in $W(B_n)$. $\square$

## 4.3 Classification of T-Avoiding Elements in $W(\widetilde{C}_n)$

In this section we will classify the T-avoiding elements in Coxeter groups of type $\widetilde{C}_n$. Because, $W(A_n)$ and $W(B_n)$ are parabolic subgroups of $W(\widetilde{C}_n)$, this implies that if $W(\widetilde{C}_n)$ is to have any non-trivial T-avoiding elements they will have full support, because if they did not the problem is reduced to a cross product of $W(A_n)$ and $W(B_n)$ in some way. We will first show that there are no non-trivial T-avoiding elements that are not FC in $W(\widetilde{C}_n)$.

**Theorem 4.3.1.** There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$.

*Proof.* Let $w$ be a reduced expression in $W(\widetilde{C}_n)$ such that $w$ has full support and $w$ does not have Property T. Consider all possible heaps for $w$ and choose a heap a bottom most braid. That is, choose a heap where the braid is as low as possible in the heap, which means the generators below the braid are FC.

Case 1: Suppose the braid does not contain $0, 1$ or$n-1, n$. Subcase a: Suppose $w$ has the fixed reduced product $w = us_k s_{k-1} s_k s_{k-1} s_{k-2} v$ or $w = us_k s_{k-1} s_k s_{k+1} s_{k-1} s_{k-2} v$ where $v$ is fully commutative, and the braid is highlighted in teal. Applying the braid move we obtain the element $w = us_{k-1} s_k s_{k-1} s_{k-2} s_{k-1} v$ or $w = us_{k-1} s_k s_{k-1} s_{k-2} s_{k-1} s_{k+1} v$. Notice that the braid is now located next to $v$ having moved closer to the bottom in the heap. This is a contradiction to choosing a heap with the lowest braid. Therefore $w$ does not have the fixed reduced product $w = us_k s_{k-1} s_k s_{k-1} s_{k-2} v$ or $w = us_k s_{k-1} s_k s_{k+1} s_{k-1} s_{k-2} v$. Visually we see this in Figure 4.1. Notice how there are two braids located in Figure 4.1(b), the braid that starts in purple and ends with teal and the braid that is fully highlighted in teal.

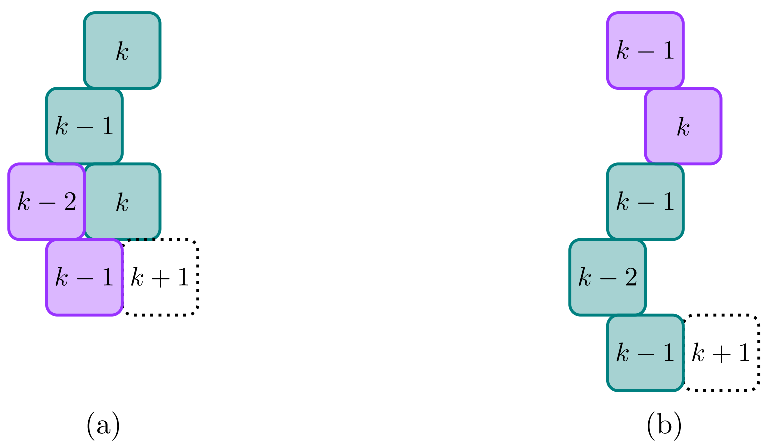



Figure 4.1: Visual representation of the heap configuration discussed in Case 1a.

Subcase b: Suppose $w$ has the fixed reduced product $w = us_k s_{k-1} s_k s_{k+1} v$ where $v$ is FC and does not contain $s_{k-2}$ and $s_{k-1}$ in the left descent set. Again we have highlighted the braid in teal. Applying the braid move we obtain an new fixed product $w = us_{k-1} s_k s_{k-1} s_{k+1} v$. Notice that the braid is now able to be written next to $v$ whereas it previously was not. This again contradicts choosing a heap with the braid in the lowest location. Visually we see this in Figure 4.2. Notice how in Figure 4.2(b) the braid is located next to the block for $s_{k+1}$ whereas in Figure 4.2(a) the braid is below the block for $s_{k+1}$.

Case 2: Suppose the braid contains 2 or $n-2$. Without loss of generality we will take the braid to contain 2 the other argument is symmetric to the one presented

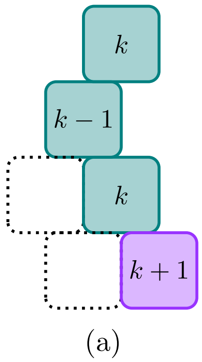



Figure 4.2: Visual representation of the heap configuration discussed in Case 1b.

here. Notice that if the braid is of the form $s_2s_3s_2$ we are in the case above as a result we assume that the braid we refer to in the following subcases do not involve $s_2s_3s_2$ to start. Subcase a: Suppose $w$ has the fixed reduced product $w = us_2s_1s_2s_0s_1s_0v$, where $v$ is FC and does not contain $s_2$ in the left descent set. Again we highlight the braid for emphasis in teal. Applying the braid move we obtain the reduced product $w = us_1s_2s_1s_0s_1s_0v$. Notice that the braid is now able to touch $v$ as it wasn't before. This contradicts our original choice of heap and as a result we can not choose the reduced product $w = us_2s_1s_2s_0s_1s_0v$. Visually this is seen Figure 4.3. Notice how there are two braids located in Figure 4.3(b). The braid highlighted in orange did not appear in our original heap seen in Figure 4.3(a) and is lower in the heap than the original.

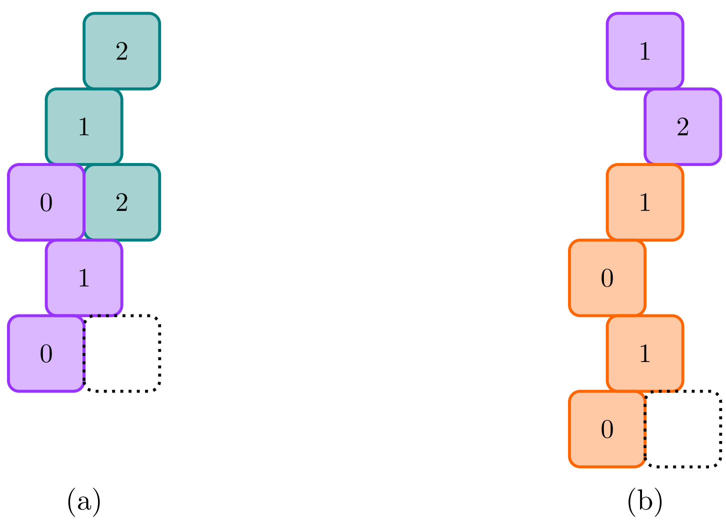



Figure 4.3: Visual representation of the heap configuration discussed in Case 2a.

Subcase b: Suppose $w$ has the fixed reduced product $w = us_2s_1s_2s_0s_1s_3s_2v$ where $v$ is FC. Again we have highlighted the braid in teal. Applying the braid move we end up with the reduced product $w = us_1s_2s_1s_0s_1s_3s_2v$. Notice this time the braid does not force a higher braid. Visually this appears in Figure 4.4. In Figures 4.4(a) and 4.4(b) we see that the braid actually moves higher in the heap.

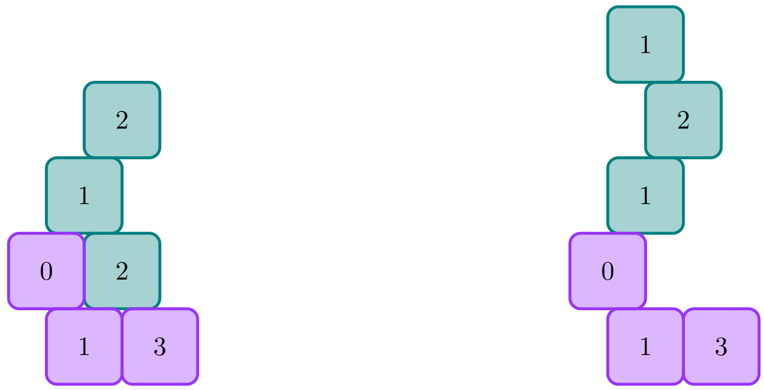
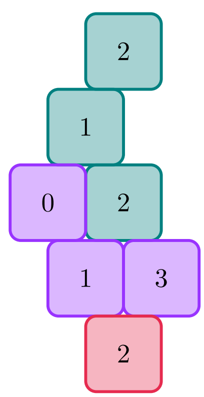
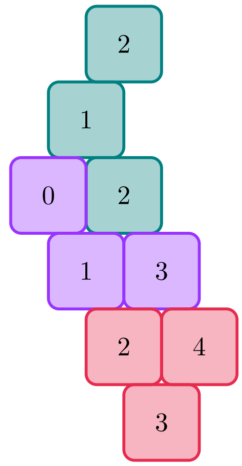



Figure 4.4: Visual representation of the heap configuration discussed in Case 2b.

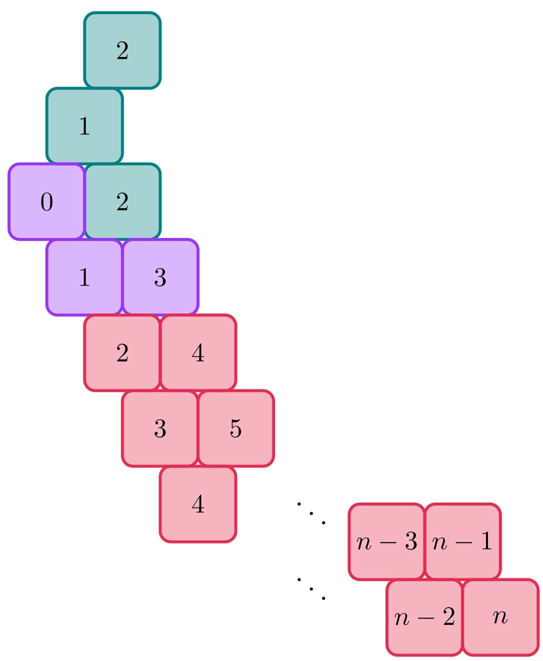
Since we assumed that $w$ does not have Property T, we know that $u$ in the fixed reduced product that we have for $w$ is non-trivial. That is, it contains some generators. Given our original reduced fixed expression for $w$ we add a new row to our heap if we were to add $s_0$ to the new row we would have a braid appear higher in the heap so we will not add $s_0$. This forces us to add $s_2$ and we get the configuration seen here



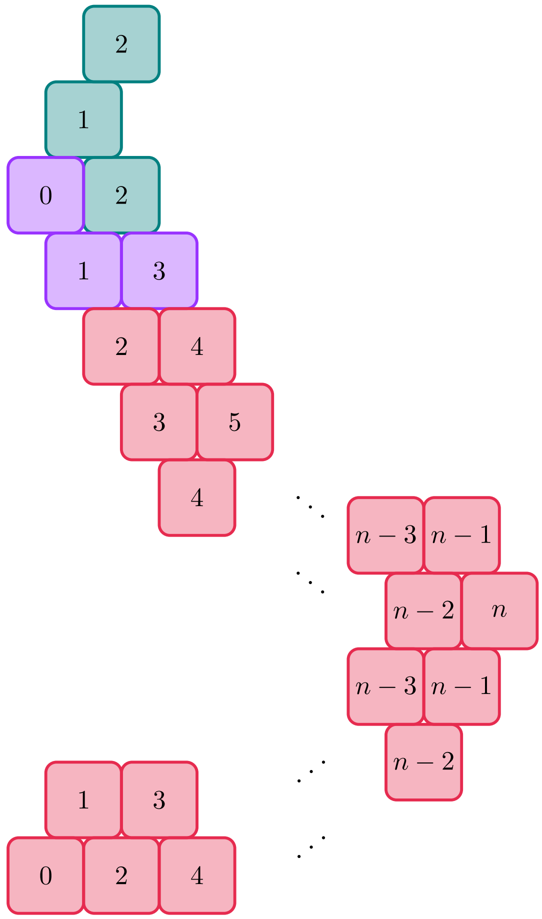

Again this can not be the top row in our heap so we must add another level in our heap. Notice that adding $s_1$ would create a braid in our heap so we will add $s_3$ however in doing so we will also need to add $s_4$. The resulting heap is seen here



We once again have the same issue arise that this can not be the top level of our configuration as $w$ would clearly have Property T on the top. Iterating this process we create the heap seen here

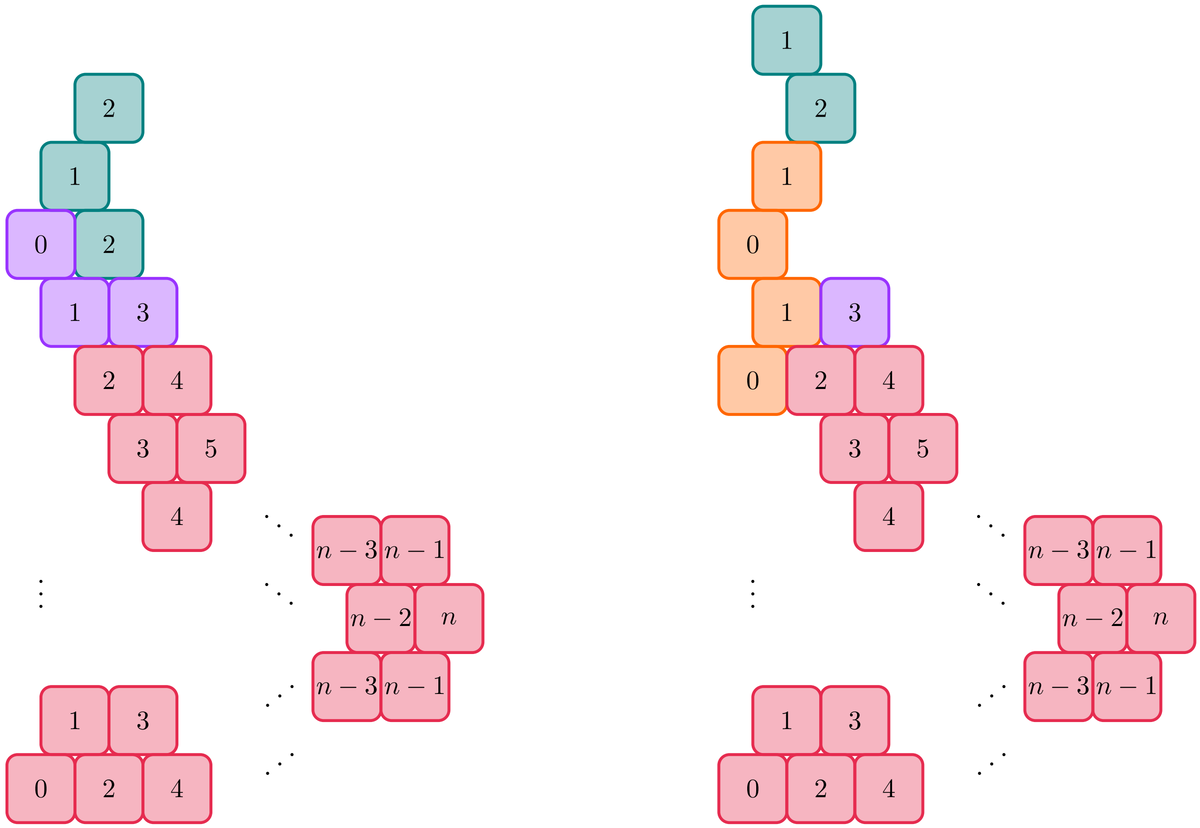


Notice that again if this were the bottom row of the heap we would have Property T. Thus our construction can not be done. With this in mind we add $s_{n-1}$ to the heap which will still have Property T. As a result we play this game again and create the heap seen here.




Recall that $v$ is FC by assumption, where $v$ is the red in the above heap. In [4, Lemma 3.3] Ernst classified that an FC element of this sort has the blank space in the middle filled in. This forces our heap to look like the one seen in Figure 4.5(a) where all of the blocks in the middle of the red v are filled in. As a result of this we now have $s_0$ in our heap. After applying the braid move to the teal braid in Figure 4.5(a). This leads to the heap seen in Figure 4.5(b) where a new orange braid appeared. This implies that for the fixed reduced product $w = us_2s_1s_2s_0s_1s_3s_2v$ there is a heap with a braid that is lower in the heap. A contradiction to the way in which we chose $w$. Thus $w$ can not have the reduced expression $w = us_2s_1s_2s_0s_1s_3s_2v$.

Case 3: Suppose the braid contains 1 or $n-1$. Without loss of generality we will assume the braid contains 1 as the other argument is symmetric to the one presented here. Subcase a: Suppose $w$ has the reduced product $w = us_0s_1s_0s_1s_2v$ where $v$ is fully commutative and does not contain $s_0$ in the left descent set. Notice that the braid is highlighted in orange. Applying the braid move leads to the reduced product $w = us_1s_0s_1s_0s_2v$. Notice that the braid is now able to be in the same level of the heap




(a) (b)

as $s_2$ whereas it previously was not. Visually this is seen in Figure 4.6. Notice how the braid in Figure 4.6(b) is located next to the block for $s_2$ whereas in Figure 4.6(a) the braid is stuck above the block for $s_2$. This is a contradiction to picking the heap with the lowest braid.


(a) (b)

Figure 4.6: Visual representation of the heap configuration discussed in Case 3a.

Subcase b: Suppose $w$ has the reduced product $w = us_1s_2s_1s_0v$ where $v$ is fully commutative and does not contain $s_2$ in the left descent set. Notice that the braid is highlighted in teal. Applying the braid move leads to the reduced product $w = us_2s_1s_2s_0v$. Notice that the braid is now able to be located in the same level of the heap as $s_0$ whereas it previously was not. Visually this is seen in Figure 4.7. Notice how the braid in Figure 4.7(b) is located next to the block for $s_0$, but the braid in Figure 4.7(a) it is stuck above the block for $s_0$. This is a contradiction to the way in which we picked our heap. Thus $w$ can not have the reduced product $w = us_1s_2s_1s_0v$.


Figure 4.7: Visual representation of the heap configuration discussed in Case 3b.

Case 4: Suppose the braid contains 0 or $n$. Without loss of generality we will assume the braid contains 0 as the other argument is symmetric to the one presented here. Suppose $w$ has the fixed reduced product $w = us_1s_0s_1s_0s_2s_1v$ where $v$ is an FC element. Notice that we have highlighted the braid in orange. Applying the braid


Figure 4.8: Visual representation of the heap configuration discussed in Case 4.

move we obtain the fixed reduced product $w = us_0s_1s_0s_1s_2s_1v$. In applying the braid the resulting reduced product now has the braid highlighted in teal. Notice that this braid is located next to $v$ which is located lower in the heap than our original $w$. Visually we see this in Figure 4.8. We see in Figure 4.8(b) the braid in teal is located below the braid that stars in orange and ends with the $s_1$. It is clear that this braid is lower than the orange braid seen in Figure 4.8(a). Thus $w$ can not have the reduced product $w = us_1s_0s_1s_0s_2s_1v$.

From this we see there is no possible way to find a reduced expression in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$ with full support and does not have Property T. Thus there are no non-trivial T-avoiding elements in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$. □

We have now shown that there are no non-trivial T-avoiding elements in $W(\widetilde{C}_n)/\mathrm{FC}(\Gamma)$. We now proceed in a parity argument. We first will classify non-trivial T-avoiding elements in $W(\widetilde{C}_n)$ for $n$ odd. First recall that $W(\widetilde{C}_n)$ for $n$ odd is a Star reducible Coxeter group. This implies that there are no non-trivial T-avoiding FC elements in $W(\widetilde{C}_n)$ for $n$ odd. This leads to the following Theorem.

**Theorem 4.3.2.** There are no non-trivial T-avoiding elements in the Coxeter group of type $\widetilde{C}_n$ for $n$ odd.

*Proof.* Consider the Coxeter group of type $\widetilde{C}_n$. By Theorem 4.3.1 we know that $W(\widetilde{C}_n)$ contains no non-trivial T-avoiding elements that are not FC. Also since $W(\widetilde{C}_n)$ is a star reducible Coxeter group, we know that $W(\widetilde{C}_n)$ contains no non-trivial T-avoiding elements that are FC. Thus $W(\widetilde{C}_n)$ has no non-trivial T-avoiding elements. □

We next will classify the non-trivial T-avoiding elements in the Coxeter group of type $\widetilde{C}_n$ for $n$ even. Recall that $W(\widetilde{C}_n)$ for $n$ even is not a Star reducible Coxeter group. In Theorem 4.3.1 we showed that $W(\widetilde{C}_n)$ does not have non-trivial T-avoiding elements that are not FC. This leaves us with only the FC elements to check.

**Theorem 4.3.3.** There are non-trivial T-avoiding elements in the Coxeter group of type $\widetilde{C}_n$.

*Proof.* Let $\mathsf{w} = s_0s_2\cdots s_{n-3}s_{n-1}(s_1s_3\cdots s_{n-2}s_0s_2\cdots s_{n-3}s_{n-1})^k$ be a reduced expression for $w \in W(\widetilde{C}_n)$. □

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