

THE MEANING OF LIFE

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ABSTRACT

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Everything you always wanted to know will be discussed.

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Chapter 1

Preliminaries

1.1 Coxeter Systems

A *Coxeter system* is a pair (W, S) consisting of a finite set S of generating involutions and a group W , called a *Coxeter group*, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s, t) = 1$ if and only if $s = t$, and $m(s, t) = m(t, s)$. It turns out that the elements of S are distinct as group elements and that $m(s, t)$ is the order of st [10]. We call $m(s, t)$ the *bond strength* of s and t .

Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = e$ can be rewritten as

$$\underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)} \quad (1.1)$$

with $m(s, t) \geq 2$ factors. If $m(s, t) = 2$, then $st = ts$ is called a *commutation relation*. Otherwise, if $m(s, t) \geq 3$, then the relation in (1.1) is called a *braid relation*. Replacing $\underbrace{sts \cdots}_{m(s,t)}$ with $\underbrace{tst \cdots}_{m(s,t)}$ will be referred to as a *commutation* if $m(s, t) = 2$ and a *braid move* if $m(s, t) \geq 3$.

We can represent a Coxeter system (W, S) with a unique *Coxeter graph* Γ having

- (1) vertex set S and
- (2) labeled edges $\{s, t\}$ for each $m(s, t) \geq 3$ with labeled with its corresponding bond strength $m(s, t)$.

Since $m(s, t) = 3$ occurs frequently, it is customary to omit this label. Note that s and t are not connected in the graph if and only if $m(s, t) = 2$. There is a one-to-one correspondence between Coxeter systems and Coxeter graphs. That is, given a Coxeter graph Γ , we can uniquely reconstruct the corresponding Coxeter system. If (W, S) is a Coxeter system with corresponding Coxeter graph Γ , we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for clarity. Also, the Coxeter system (W, S) is said to be *irreducible* if and only if Γ is connected. Further, if the graph Γ is disconnected, the connected components correspond to factors in a direct product of the corresponding Coxeter groups [10].

Example 1.1.1.

- (a) The Coxeter system of type A_n is given by the graph in Figure 1.1(a). We can construct the corresponding Coxeter group $W(A_n)$ with generating set $S(A_n) = \{s_1, s_2, \dots, s_n\}$ and defining relations

- (1) $s_i^2 = e$ for all i ;
- (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;
- (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group Sym_{n+1} under the correspondence $s_i \mapsto (i, i + 1)$, where $(i, i + 1)$ is the adjacent transposition that swaps i and $i + 1$.

- (b) The Coxeter system of type B_n is given by the graph in Figure 1.1(b). We can construct the corresponding Coxeter group $W(B_n)$ with generating set $S(B_n) = \{s_0, s_1, \dots, s_{n-1}\}$ and defining relations

- (1) $s_i^2 = e$ for all i ;
- (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$;
- (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i, j \in \{1, 2, \dots, n - 1\}$;
- (4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The Coxeter group $W(B_n)$ is isomorphic to Sym_n^B , where Sym_n^B is the group of signed permutations on the set $\{1, 2, \dots, n\}$.

- (c) The Coxeter system of type \tilde{C}_n is seen in Figure 1.2(d). We can construct the corresponding Coxeter group $W(\tilde{C}_n)$ with generating set $S(\tilde{C}_n) = \{s_0, s_1, \dots, s_n\}$ and defining relations

- (1) $s_i^2 = e$ for all i ;

- (2) $s_i s_j = s_j s_i$ when $|i - j| > 1$ for $i \in \{1, 2, \dots, n - 1\}$;
- (3) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ for $i \in \{1, 2, \dots, n - 1\}$;
- (4) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$;
- (5) $s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$.

Note that $W(\tilde{C}_n)$ has $n + 1$ generators.

The Coxeter graphs given in Figure 1.1 correspond to the collection of irreducible finite Coxeter groups, while the Coxeter graphs given in Figure 1.2 are the irreducible affine Coxeter groups, which are infinite [10]. The irreducible affine Coxeter systems are unique in that if a vertex is removed along with the corresponding edges from the Coxeter graph, the newly created graph will result in a finite Coxeter group. This thesis will focus on Coxeter groups of type B_n and \tilde{C}_n .

Given a Coxeter system (W, S) , a word $s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* on S is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w , the corresponding word is called a *reduced expression* for w . In this case, we define the *length* of w to be $l(w) := m$. Each element $w \in W$ may have multiple reduced expressions that represent it. If we wish to emphasize a specific, possibly reduced, expression for $w \in W$ we will represent it as $\bar{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$. The following theorem tells us more about how reduced expressions for a given group element are related.

Theorem 1.1.2 (Matsumoto, [7]). *Let (W, S) be a Coxeter system. If $w \in W$, then given a reduced expression for w we can obtain every other reduced expression for w by a sequence of braid moves and commutations of the form*

$$\underbrace{sts \cdots}_{m(s,t)} \rightarrow \underbrace{tst \cdots}_{m(s,t)}$$

where $s, t \in S$ and $m(s, t) \geq 2$. □

It follows from Matsumoto's Theorem that if a generator s appears in a reduced expression for $w \in W$, then s appears in all reduced expressions for w . Let $w \in W$ and define the *support* of w , denoted $\text{supp}(w)$, to be the set of all generators that appear in any reduced expression for w . If $\text{supp}(w) = S$, we say that w has *full support*.

Given $w \in W$ and a fixed reduced expression \bar{w} for w , any subsequence of \bar{w} is called a *subexpression* of \bar{w} . We will refer to a subexpression consisting of a consecutive subsequence of \bar{w} as a *subword* of \bar{w} .

Example 1.1.3. Let $w \in W(A_7)$ and let $\bar{w} = s_7 s_2 s_4 s_5 s_3 s_2 s_3 s_6$ be a fixed expression for w . Then we have

$$\begin{aligned}
 s_7 s_2 s_4 s_5 s_3 s_2 s_3 s_6 &= s_7 s_4 s_2 s_5 s_3 s_2 s_3 s_6 \\
 &= s_7 s_4 s_5 s_2 s_3 s_2 s_3 s_6 \\
 &= s_7 s_4 s_5 s_3 s_2 s_3 s_3 s_6 \\
 &= s_7 s_4 s_5 s_3 s_2 s_6,
 \end{aligned}$$

where the **blue** highlighted text corresponds to a commutation, the **teal** highlighted text corresponds to a braid move, and the **red** highlighted text corresponds to cancellation. This shows that \bar{w} is not reduced. However, it turns out that $s_7 s_4 s_5 s_3 s_2 s_6$ is reduced. Thus $l(w) = 6$ and $\text{supp}(w) = \{s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let (W, S) be a Coxeter system of type Γ and let $w \in W(\Gamma)$. We define the *left descent set* and *right descent set* of w as follows:

$$\mathcal{L}(w) := \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

In [2] it is shown that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a reduced expression for w that begins (respectively, ends) with s .

Example 1.1.4. The following list consists of all reduced expressions for a given element $w \in W(B_4)$.

$$\begin{array}{cc}
 s_0 s_1 s_2 s_1 s_3 & s_0 s_2 s_1 s_2 s_3 \\
 s_0 s_1 s_2 s_3 s_1 & s_2 s_0 s_1 s_2 s_3
 \end{array}$$

We see that $l(w) = 5$ and w has full support. Also, we see that $\mathcal{L}(w) = \{s_0, s_2\}$ while $\mathcal{R}(w) = \{s_1, s_3\}$.

1.2 Fully Commutative Elements

Let (W, S) be a Coxeter system of type Γ and let $w \in W$. Following [13], we define a relation \sim on the set of reduced expressions for w . Let \bar{w}_1 and \bar{w}_2 be two reduced expressions for w . We define $\bar{w}_1 \sim \bar{w}_2$ if we can obtain \bar{w}_2 from \bar{w}_1 by applying a single commutation move of the form $st \mapsto ts$ where $m(s, t) = 2$. Now, define the equivalence relation \approx by taking the reflexive transitive closure of \sim . Each equivalence class under \approx is called a *commutation class*. If w has a single commutation class, then we say that w is *fully commutative* (FC).

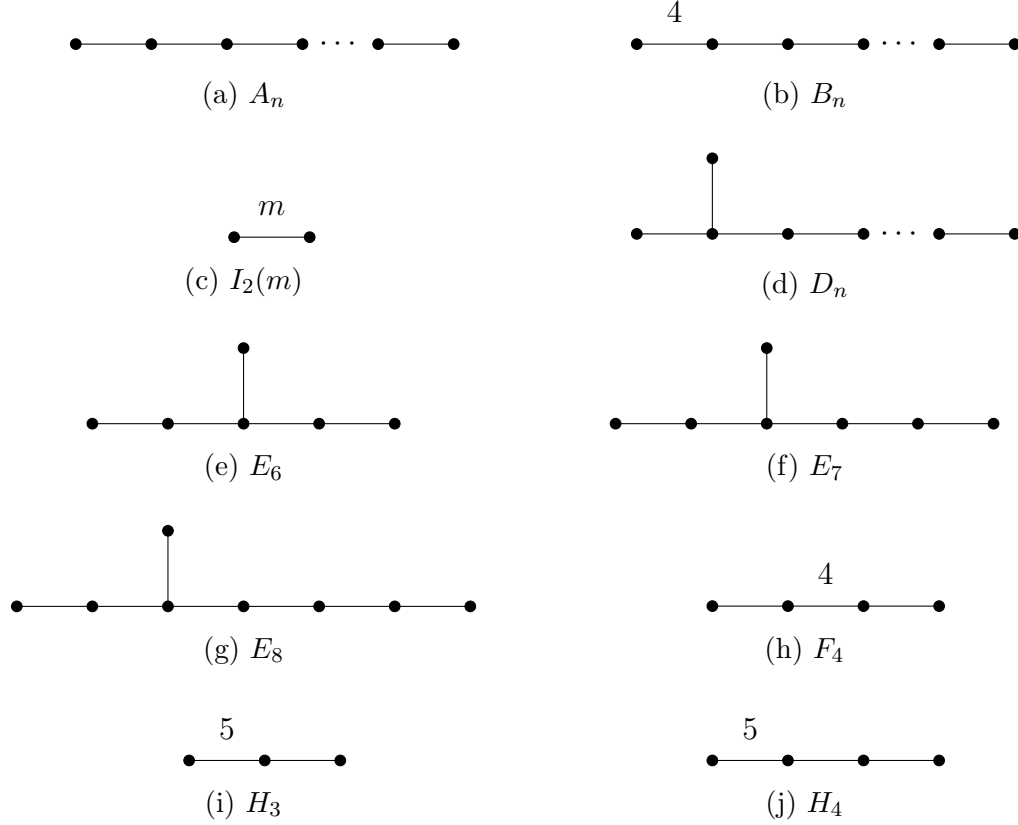


Figure 1.1: Coxeter graphs corresponding to the irreducible finite Coxeter systems.

The set of FC elements of $W(\Gamma)$ is denoted by $\text{FC}(\Gamma)$. Given some $w \in \text{FC}(\Gamma)$, and a starting reduced expression for w , observe the definition of FC states that one only needs to perform commutations to obtain all reduced expressions for w , but the following result due to Stembridge [13] states that when w is FC performing commutations is the only possible way to obtain another reduced expression for w .

Theorem 1.2.1 (Stembridge, [13]). *Let (W, S) be a Coxeter system. An element $w \in W$ is FC if and only if no reduced expression for w contains $\underbrace{sts \cdots}_{m(s,t)}$ as a subword for all when $m(s, t) \geq 3$.* \square

In other words, w is FC if and only if we never have the opportunity to apply a braid move.

Example 1.2.2. Let $w \in W(\tilde{C}_4)$ and let $\bar{w} = s_0 s_1 s_2 s_0 s_3 s_1$ be a reduced expression for w . We see that

$$s_0 s_1 \textcolor{violet}{s_2} s_0 s_3 s_1 = s_0 s_1 s_0 s_2 \textcolor{violet}{s_3} s_1 = s_0 s_1 s_0 s_2 s_1 s_3,$$

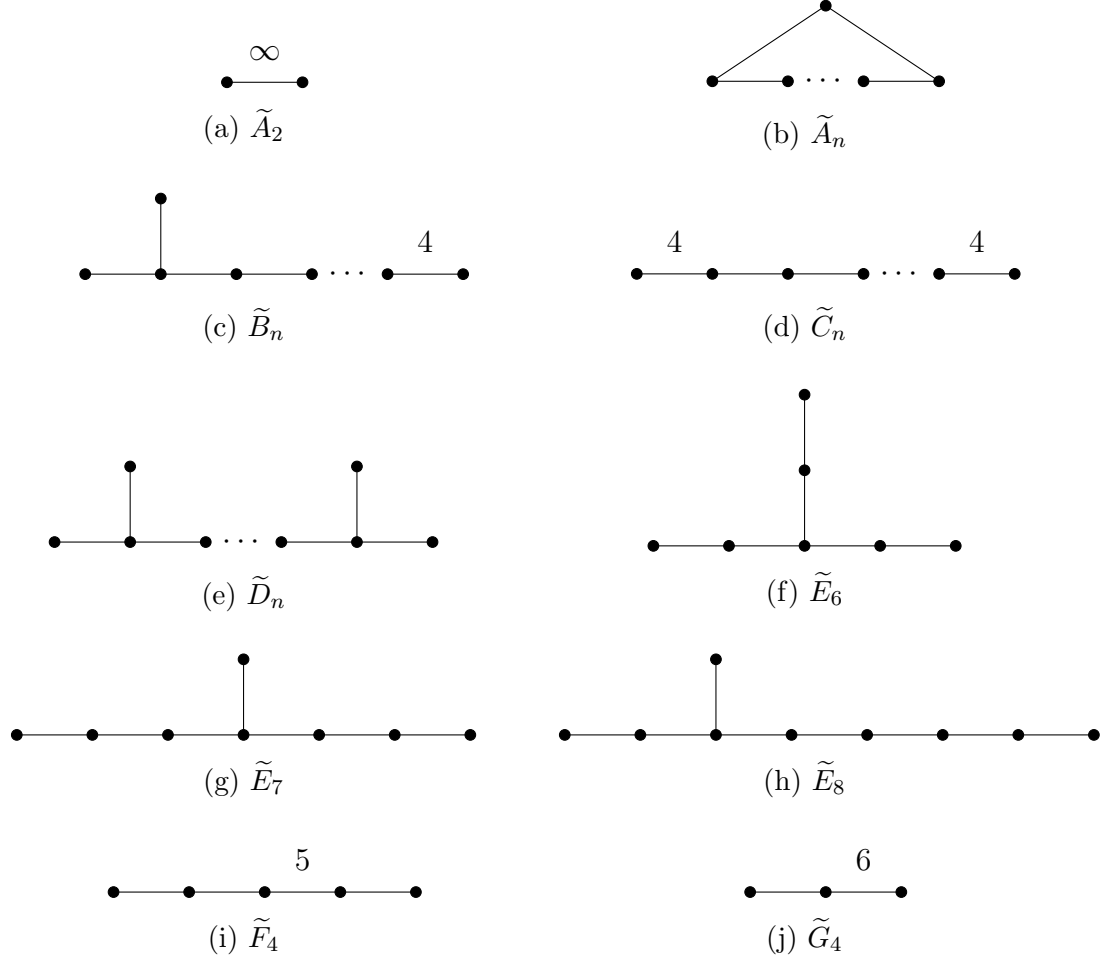


Figure 1.2: Coxeter graphs corresponding to the irreducible affine Coxeter systems.

where the **purple** indicates applying a commutation. Note that there is no possible way to perform a braid move. Hence w is FC.

Example 1.2.3. Let $\bar{w} = s_1 s_0 s_4 s_1 s_3 s_5 s_2 s_4 s_6$ be a reduced expression for $w \in \text{FC}(\tilde{C}_6)$. Applying the commutation $s_4 s_2 = s_2 s_4$, we can obtain another reduced expression for w , namely $\bar{w}_2 = s_1 s_0 s_4 s_1 s_3 s_5 s_4 s_2 s_6$ which is in the same commutation class as \bar{w}_1 . However, applying the braid move $s_2 s_3 s_2 = s_3 s_2 s_3$, we obtain another reduced expression $\bar{w}_3 = s_1 s_3 s_2 s_3 s_4 s_0$. Note that since \bar{w}_3 was obtained by applying a braid move, \bar{w}_3 is in a different commutation class than \bar{w}_1 and \bar{w}_2 . It turns out w has exactly two commutation classes, one containing \bar{w}_1 and \bar{w}_2 and another containing \bar{w}_3 . So w is not FC.

Example 1.2.4. Let $w \in W(\tilde{C}_4)$ and let $\bar{w} = s_0 s_1 s_2 s_0 s_1 s_2$ be a reduced expression

for w . We see that

$$s_0 s_1 s_3 s_0 s_1 s_2 = s_0 s_1 s_0 s_3 s_1 s_2 = s_0 s_1 s_0 s_1 s_3 s_2,$$

where the **purple** indicates applying a commutation and the **orange** indicates applying a braid move. Thus w is not FC by Theorem 1.2.1.

Stembridge classified the irreducible Coxeter Systems that contain a finite number of FC elements, the so-called *FC-finite Coxeter groups*. This thesis is mainly concerned with $W(A_n)$, $W(B_n)$, and $W(\tilde{C}_n)$. Both $W(A_n)$ and $W(B_n)$ are finite Coxeter groups, and thus are FC-finite. On the other hand, $W(\tilde{C}_n)$ is infinite and has infinitely many FC elements. However, there exist some infinite Coxeter groups that contain finitely many FC elements. For example, E_n for $n \geq 9$ (see Figure 1.3) is infinite, but contains only finitely many FC elements.

Theorem 1.2.5 (Stembridge, [13]). *The FC-finite irreducible Coxeter systems are of type A_n with $n \geq 1$, B_n with $n \geq 2$, D_n with $n \geq 4$, E_n with $n \geq 6$, F_n with $n \geq 4$, H_n with $n \geq 3$, and $I_2(m)$ with $5 \leq m < \infty$. \square*

The irreducible FC-finite Coxeter graphs are given in Figure 1.3. Note that we have already encountered some of the FC-finite Coxeter groups in Figure 1.1. Since these are finite Coxeter groups it is clear that they will have a finite number of FC elements. However, we haven't yet encountered the Coxeter groups determined by graphs in Figures 1.3(d), 1.3(e), 1.3(f). All of these Coxeter systems are infinite for large n , yet contain only finitely many FC elements.

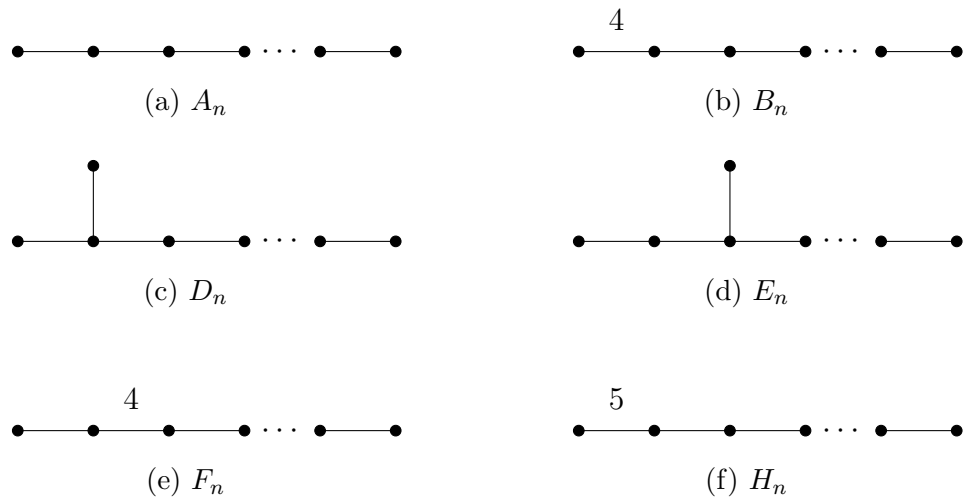


Figure 1.3: Coxeter graphs corresponding to the irreducible FC-finite Coxeter systems.

Chapter 2

Heaps

2.1 Heaps

We now discuss a visual representation of Coxeter group elements. Each reduced expression can be associated with a labeled partially ordered set (poset) called a heap. Heaps provide a visual representation of a reduced expression while preserving the relations among the generators. We follow the development of heaps for straight line Coxeter groups found in [1], [4], and [13].

Let (W, S) be a Coxeter system of type Γ . Suppose $\bar{w} = s_{x_1}s_{x_2}\cdots s_{x_r}$ is a fixed reduced expression for $w \in W$. As in [13], we define a partial ordering on the indices $\{1, 2, \dots, r\}$ by the transitive closure of the relation $j \triangleleft i$ if $i < j$ and s_{x_i} and s_{x_j} do not commute. In particular, since \bar{w} is reduced, $j \triangleleft i$ if $s_{x_i} = s_{x_j}$ by transitivity. This partial order is referred to as the *heap* of \bar{w} , where i is labeled by s_{x_i} . Note that for simplicity we are omitting the labels of the underlying poset but retaining the labels of the corresponding generators.

It follows from [13] that heaps are well-defined up to commutation class. That is, given two reduced expressions \bar{w} and \bar{w}' for $w \in W$ that are in the same commutation class, then the heaps for \bar{w} and \bar{w}' will be equal. In particular, if $w \in \text{FC}(\Gamma)$, then w has a one commutation class, and thus w has a unique heap. Conversely, if \bar{w} and \bar{w}' are in different commutation classes, then there will not be a unique heap representation for w .

Example 2.1.1. Let $\bar{w} = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ be a reduced expression for $w \in \text{FC}(\tilde{C}_6)$. We see that \bar{w} is indexed by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As an example, $1 \triangleleft 0$ since $0 < 1$ and s_0 and s_1 do not commute. The labeled Hasse diagram for the heap poset is seen in Figure 2.1.

Let \bar{w} be a reduced expression for an element in $w \in W(\tilde{C}_n)$. As in [1] and [4] we can represent a heap for \bar{w} as a set of lattice points embedded in $\{0, 1, 2, \dots, n\} \times \mathbb{N}$.

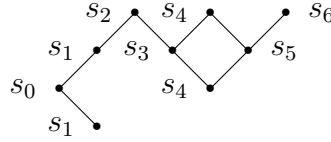


Figure 2.1: Labeled hasse diagram for the heap of an element in $\text{FC}(\tilde{C}_6)$.

To do so, we assign coordinates (not unique) $(x, y) \in \{0, 1, 2, \dots, n\} \times \mathbb{N}$ to each entry of the labeled Hasse diagram for the heap of \bar{w} in such a way that:

- (1) An entry with coordinates (x, y) is labeled s_i (or i) in the heap if and only if $x = i$;
- (2) If an entry with coordinates (x, y) is greater than an entry with coordinates (x', y') in the heap then $y > y'$.

Although the above is specific to $W(\tilde{C}_n)$, the same construction works for any straight line Coxeter graph with the appropriate adjustments made to the label set and assignment of coordinates. Specifically for type A_n our label set is $\{1, 2, \dots, n\}$ and for type B_n our label set is $\{0, 1, \dots, n-1\}$.

In the case of any straight line Coxeter graph it follows from the definition that (x, y) covers (x', y') in the heap if and only if $x = x' \pm 1$, $y > y'$, and there are no entries (x'', y'') such that $x'' \in \{x, x'\}$ and $y' < y'' < y$. This implies that we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. The lattice point representation can help us visualize arguments that are potentially complex. Note that in our heaps the entries in the top correspond to the generators occurring in the right descent set of the corresponding reduced expression.

Let \bar{w} be a reduced expression for $w \in W(\tilde{C}_n)$. We denote the lattice representation of the heap poset in $\{0, 1, 2, \dots, n\} \times \mathbb{N}$ described in the preceding paragraphs via $H(\bar{w})$. If w is FC, then the choice of reduced expression for w is irrelevant and we will often write $H(w)$ and we refer to $H(w)$ as the heap of w . Note that we will use the same notation for heaps in Coxeter groups of all types.

When w is FC, we wish to make a canonical choice for the representation of $H(w)$ by assembling the entries in a particular way. To do so, we position all of the entries corresponding to elements in $\mathcal{L}(w)$ in the same vertical position, and all of the remaining elements should be positioned as high as possible in the lattice point representation. For example, the representation in Figure 2.2 is the canonical representation for w . Note that our canonical representation of heaps corresponds to Cartier-Foata normal form for monomials [3, 9]. There are potentially many ways to illustrate a heap of an arbitrary reduced expression, each differing by the vertical

placement of the blocks. For example, we can place blocks in vertical positions as high as possible, as low as possible, or some combination of low/high. In this thesis, we choose what we view to be the best representation of the heap for each example and when illustrating the heaps of arbitrary reduced expressions we will discuss the relative position of the entries but never the absolute coordinates.

Let $\bar{w} = s_{x_1}s_{x_2}\cdots s_{x_r}$ be a reduced expression for $w \in W(\tilde{C}_n)$. If s_{x_i} and s_{x_j} are adjacent generators in the Coxeter graph with $i < j$, then we must place the point labeled by s_{x_j} at a level that is *above* the level of the point labeled by s_{x_i} . Because generators in a Coxeter graph that are not adjacent do commute, points whose x -coordinates differ by more than one can slide past each other or land in the same level. To emphasize the covering relations of the lattice point representation we will enclose each entry in the heap in a square with rounded corners in such a way that if one entry covers another the squares overlap halfway. In addition, we will also label each square for s_i with i .

Example 2.1.2. Let $\bar{w} = s_1s_0s_4s_1s_3s_5s_2s_4s_6$ be a reduced expression for $w \in \text{FC}(\tilde{C}_6)$ as seen in Example 1.4.1. Figure 2.2 shows a possible lattice point representation for $H(w)$.

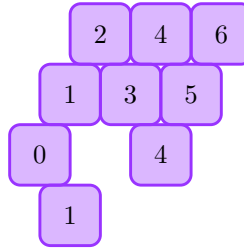


Figure 2.2: A possible lattice point representation for the heap of an FC element in $W(\tilde{C}_6)$.

Example 2.1.3. Let $\bar{w}_1 = s_1s_2s_3s_4s_2s_0$ be a reduced expression for $w \in W(\tilde{C}_4)$. Applying the commutation $s_4s_2 = s_2s_4$, we can obtain another reduced expression for w , namely \bar{w}_2 which is in the same commutation class as \bar{w}_1 and hence has the same heap. However, applying the braid move $s_2s_3s_2 = s_3s_2s_3$, we obtain another reduced expression $\bar{w}_3 = s_1s_3s_2s_3s_4s_0$. Note that since \bar{w}_3 was obtained by applying a braid move, \bar{w}_3 is in a different commutation class than \bar{w}_1 and \bar{w}_2 . Representations of $H(\bar{w}_1)$, $H(\bar{w}_2)$, and $H(\bar{w}_3)$ are seen in Figure 2.3 where the braid relation is colored in orange.

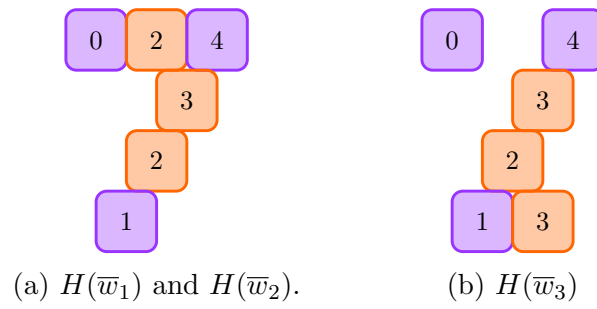


Figure 2.3: Two heaps of a non-FC element in $W(\tilde{C}_4)$

Chapter 3

Star Operations

3.1 Star Operations

The notion of star operations was originally introduced by Kazhdan and Lusztig in [11] for simply laced Coxeter systems (i.e., $m(s, t) \leq 3$ for all $s, t \in S$), and was later generalized to all Coxeter systems in [12]. If $I = \{s, t\}$ is a pair of non-commuting generators of a Coxeter group W , then I induces four partially defined maps from W to itself, known as *star operations*. A star operation, when it is defined, increases or decreases the length of an element to which it is applied by 1. For our purposes it is enough to only define the star operations that decrease the length of an element by 1, and as a result we will not develop the notion in full generality.

Let (W, S) be a Coxeter system of type Γ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators whose product has order m . Let $w \in W(\Gamma)$ such that $s \in \mathcal{L}(w)$. We define w to be *left star reducible by s with respect to t* if there exists $t \in \mathcal{L}(sw)$ and $m(s, t) \geq 3$. We analogously define w to be *right star reducible by s with respect to t* . Observe that if $m(s, t) \geq 3$, then w is left (respectively, right) star reducible if and only if there is a reduced expression for w such that $\bar{w} = stv$ (respectively, $\bar{w} = vts$). We say that w is *star reducible* if it is either left or right star reducible.

Example 3.1.1. Let $w \in W(B_4)$ and let $\bar{w} = s_0 s_1 s_0 s_2 s_3$ be a reduced expression for w . We see that w is left star reducible by s_0 with respect to s_1 to $s_1 s_0 s_2 s_3$, since $m(s_0, s_1) = 4$ and $s_0 \in \mathcal{L}(w)$ while $s_1 \in \mathcal{L}(s_0 w)$. Also w is right star reducible by s_3 with respect to s_2 to $s_0 s_1 s_0 s_2$, since $m(s_2, s_3) = 3$ and $s_3 \in \mathcal{R}(w)$ and $s_2 \in \mathcal{R}(ws_3)$.

It may be helpful to visualize star reductions in terms of heaps. Figure 3.1(a) represents $H(\bar{w})$. Note that we can see s_0 is in the left descent set of w since s_0 is in the bottom row of the heap. Furthermore, multiplying on the left by s_0 we get the heap in Figure 3.1(b). Again, since s_1 is in the bottom row of the heap, $s_1 \in \mathcal{L}(s_1 w)$.

In Figure 3.1(a) we also see that s_3 is in the right descent set of w since $s_3 \in \mathcal{R}(w)$. Multiplying on the left by s_3 we can see that s_2 would be in the top level of the heap so $s_2 \in \mathcal{R}(ws_2)$. From this we can interpret visually an element $w \in W(\Gamma)$ is left star reducible (respectively, right star reducible) if there exists a heap that we can pull a block off the bottom row of the heap (respectively, top of the heap) and a new block that wasn't previously in the bottom row (respectively, top row) is now in the bottom row (respectively, top row) of the heap.

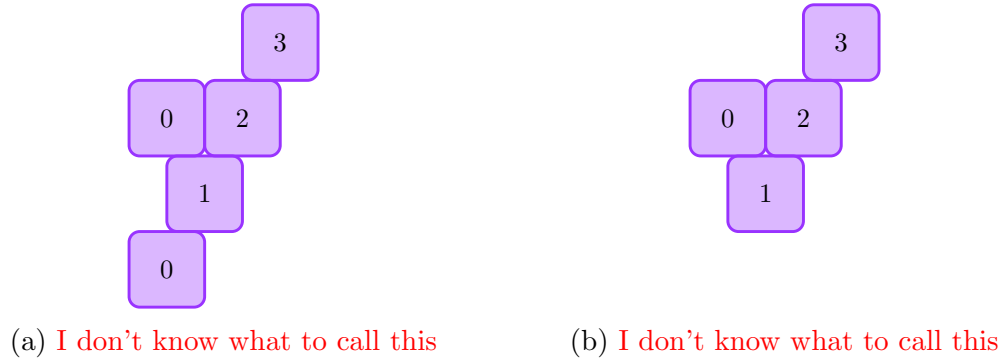


Figure 3.1: Visualization of Example 3.1.1

Using the notion of star reduction we are now able to introduce the concept of a star reducible Coxeter group. We say that a Coxeter group $W(\Gamma)$, or its Coxeter graph Γ , is *star reducible* if every element of $\text{FC}(\Gamma)$ is star reducible to a product of commuting generators. That is, $W(\Gamma)$ is star reducible if when we apply star operations repeatedly to $w \in \text{FC}(\Gamma)$, eventually we obtain a product of commuting generators. In [9], Green classified all star reducible Coxeter groups.

Theorem 3.1.2 (Green, [9]). *Let $W(\Gamma)$ be a Coxeter group with (finite) generating set S . Then $W(\Gamma)$ is star reducible if and only if each component of Γ is either a complete graph with labels $m(s, t) \geq 3$, or is one of the following types: type A_n ($n \geq 1$), type B_n ($n \geq 2$), type D_n ($n \geq 4$), type F_n ($n \geq 4$), type H_n ($n \geq 2$), type $I_2(m)$ ($m \geq 3$), type \tilde{A}_{n-1} ($n \geq 3$ and n odd), type \tilde{C}_{n-1} ($n \geq 4$ and n even), type \tilde{E}_6 or type \tilde{F}_5 . \square*

3.2 Non-Cancellable Elements

We now introduce the concept of weak star reducible, which is related to the notion of cancellable in [5]. Let (W, S) be a Coxeter system of type Γ and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators of the Coxeter group $W(\Gamma)$. If $w \in \text{FC}(\Gamma)$, then w is *left weak star reducible by s with respect to t* to sw if

- (1) w is left star reducible by s with respect to t and;
- (2) $tw \notin \text{FC}(W)$.

Notice that (2) implies that $l(tw) > l(w)$. Also note that we are restricting our definition of weak star reducible to the set of FC elements of $W(\Gamma)$. We analogously define *right weak star reducible by s with respect to t to ws* . We say that w is *weak star reducible* if w is either left or right weak star reducible. Otherwise, we say that w is *non-cancellable* or *weak star irreducible*.

Example 3.2.1. Let $w \in \text{FC}(B_4)$ and let $\bar{w} = s_0s_1s_0s_2s_3$ be a reduced expression for w as in Example 3.1.1. By Example 3.1.1 we know that w is left star reducible. Also, $tw = s_1s_0s_1s_0s_2s_3$ which is not in $\text{FC}(B_4)$. Thus, we see that w is left weak star reducible by s_0 with respect to s_1 to $s_1s_0s_2s_3$. In addition, Example 3.1.1 showed that w is right star reducible. But, $wt = s_0s_1s_0s_2s_3s_2$ which is not in $\text{FC}(B_4)$. Thus, w is right weak star reducible by s_3 with respect to s_2 to $s_0s_1s_0s_2$. This implies that w is not non-cancellable.

Again it might be useful to visualize the concept of weak star reducible in terms of heaps. Recall in Figure 3.1(a) we have a representation for w as described in Example 3.2.1 and an associated discussion about the reason for w being star reducible. Now in Figure 3.2 we can see that when we multiply w by s_1 we end up with a braid, highlighted in orange and hence $ws_1 \notin \text{FC}(\Gamma)$. Here the same properties as described above for w to be star reducible must be visualized in a heap and when multiplying on the left or right by t a braid must appear.

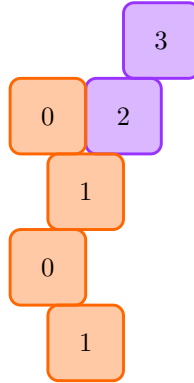


Figure 3.2: I don't know what to call this either.

Example 3.2.2. Let $w \in \text{FC}(B_4)$ and let $\bar{w} = s_0s_1$ be a reduced expression for w . Note that w is left (respectively, right) star reducible by s_0 with respect to s_1

(respectively, by s_1 with respect to s_0). However, $s_1s_0s_1 \in \text{FC}(B_4)$ (respectively, $s_0s_1s_0 \in \text{FC}(B_4)$). Thus w is non-cancellable.

Chapter 4

Property T and T-Avoiding

4.1 Property T and T-Avoiding Elements

Introduction and Motivation for this section

We first begin by defining the notion of Property T. Let (W, S) be a Coxeter system of type Γ and let $w \in W$ we say that w has *Property T* if and only if there exists a reduced expression \bar{w} such that $\bar{w} = stu$ or $\bar{w} = uts$ where $m(s, t) \geq 3$. That is, w has Property T if there exists a reduced expression for w that begins with a product of non-commuting generators or ends with a product of non-commuting generators. It should be noted that by the symmetry of the definition if w has Property T, then w^{-1} has Property T.

Example 4.1.1. Let $w \in W(A_5)$ and let $\bar{w} = s_1 s_4 s_2 s_3 s_5$. Note that applying a commutation to $s_4 s_5$ results in $\bar{w}_1 = s_1 s_2 s_4 s_3 s_5$. Hence w has Property T, since $m(s_1, s_2) = 3$ and there is a reduced expression for w which starts with $s_1 s_2$.

Example 4.1.2. Let $w \in W(A_5)$ and let $\bar{w} = s_1 s_3 s_5$. It turns out that since w is a product of commuting generators there is no reduced expression for w that begins or ends with a pair of non-commuting generators. This implies that w does not have Property T.

As with star reducible elements it may be helpful to visualize Property T through heaps. Figure 4.1(a) provides a representation of \bar{w} seen in Example 4.1.1 and Figure 4.1(b) provides a representation of \bar{w} seen in Example 4.1.2. Notice that if we were to remove the block for s_1 in the bottom row of Figure 4.1(a), we would have a new bottom row. However, in Figure 4.1(b), we are not able to remove any bricks and have a new brick come to the top or bottom row as the heap is just one row. Property T provides the ability for us get a new row in our heap when we remove the top most or bottom most row from a specific reduced expression for w .

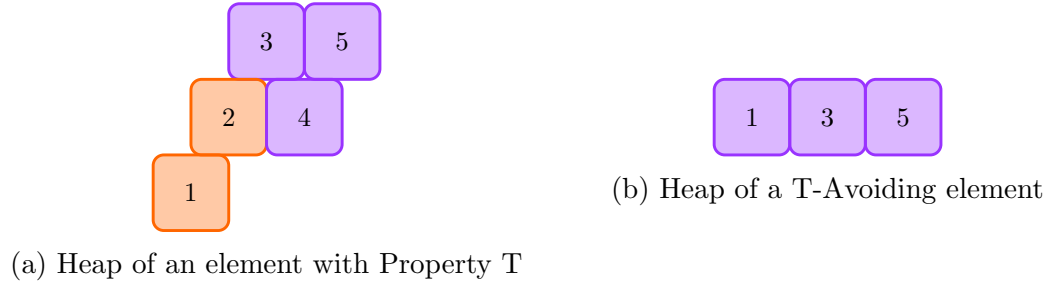


Figure 4.1: Heaps of an element with Property T and a T-Avoiding element

An element $w \in W(\Gamma)$ is called *T-avoiding* if w and w^{-1} do not have Property T. As seen in Example 4.1.2 an element $w \in W(\Gamma)$ will be T-avoiding if w is a product of commuting generators. We will call an element that is a product of commuting generators *trivially T-avoiding*. If w is T-avoiding and not a product of commuting generators, we will say that w is *non-trivially T-avoiding*.

Example 4.1.3. Let $w \in W(A_5)$ and let $\bar{w} = s_1 s_3 s_5$. Then by Example 4.1.2, we know that w is T-avoiding. Furthermore, since w is a product of commuting generators, w is trivially T-avoiding.

Example 4.1.4. Let $w \in W(\tilde{C}_4)$ and let $\bar{w} = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$. It turns out that w is non-trivially T-avoiding.

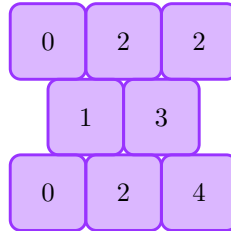


Figure 4.2: Heap of a non-trivially T-Avoiding element in $W(\tilde{C}_4)$.

One thing to notice here is that all Coxeter groups have trivial T-avoiding elements as they all contain products of commuting generators. The more interesting non-trivial T-avoiding elements don't appear in all Coxeter groups. The remainder of this thesis classifies what is known regarding non-trivial T-avoiding elements in the irreducible finite Coxeter groups and the irreducible affine Coxeter groups.

As mentioned in Section 3.1 Green classified all star reducible Coxeter groups. It is interesting to note that in a star reducible Coxeter group there are no non-trivially T-avoiding FC elements. In [9], Green utilizes the following theorem to help classify the star reducible Coxeter groups.

Theorem 4.1.5 (Green, [9]). *Let W be a star reducible Coxeter group, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s, t, u with $m(s, t) \neq 2$, $m(t, u) \neq 2$, and $m(s, u) = 2$:*

- (1) *w is a product of commuting generators;*
- (2) *w has a reduced expression beginning with st ;*
- (3) *w has a reduced expression ending in ts ;*
- (4) *w has a reduced expression beginning with sut .* □

We see that Item (1) corresponds to the element w being trivially T-avoiding, Items (2) and (3) refer to the element w having Property T at the beginning and end respectively, and Item (4) corresponds to the element w being T-avoiding. Notice that this implies that some Coxeter groups will have elements that are non-trivially T-avoiding while others may not. For example, as will be seen in the following sections, the Coxeter groups of type A_n and B_n have no non-trivial T-avoiding elements, while the Coxeter group of type D_n does have non-trivial T-avoiding elements. The following sections and chapters classify T-avoiding elements in different Coxeter groups.

4.2 T-Avoiding Elements in Type \tilde{A}_n and A_n

We start by classifying the non-trivial T-avoiding elements in Coxeter groups of type \tilde{A}_n and A_n .

Theorem 4.2.1. *Let $W = W(\tilde{A}_n)$. If $n \geq 2$ and n is even, then there are no non-trivial T-avoiding elements in W . Otherwise, if $n \geq 2$ and n is odd then W contains non-trivial T-avoiding elements.*

Proof. This is [6, Proposition 3.1.2]. □

Although Green and Fan stated their results in terms of the more general type \tilde{A}_n we can state the result as a corollary in type A_n .

Corollary 4.2.2. *Let $W = W(A_n)$. Then there are no non-trivially T-avoiding elements in W .*

Proof. This is a consequence of [6, Proposition 3.1.2.]. That is, we know that $W(A_n)$ is a parabolic subgroup of $W(\tilde{A}_n)$. Since this is true if $W(A_n)$ were to have non-trivial T-avoiding elements, then both parities of $W(\tilde{A}_n)$ would also have non-trivial T-avoiding elements which by above we know that the even parity of $W(\tilde{A}_n)$ does not have non-trivial T-avoiding elements. □

This implies that in the Coxeter group of type \tilde{A}_n , when n is even there are no non-trivial T-avoiding elements, while when n is odd, there are non-trivial T-avoiding elements. One interesting observation to go along with this is $W(\tilde{A}_n)$ is star reducible for n even but not star reducible for n odd. [6] did not specifically classify the non-trivial T-avoiding elements for type \tilde{A}_n for n odd. Before we state our conjecture as to what the non-trivial T-Avoiding elements are in $W(\tilde{A}_n)$ we will discuss shortly what the heaps look like. Because the Coxeter graph for type \tilde{A}_n is not a straight line Coxeter graph, the heaps can no longer be 2 dimensional and now must take on a 3 dimensional representation. The nature of the Coxeter graph of type \tilde{A}_n leads us to represent heaps in the sense of a turret on a castle where the bricks are stacked in a circular manner. We now continue with our conjectured classification. Because type A_n does not contain any non-trivial T-avoiding elements we know that the non-trivial T-avoiding elements in $W(\tilde{A}_n)$ for n odd must have full support. We conjecture that the only non-trivial T-avoiding elements are castle turrets that have no missing blocks in the walls. However, this remains an open problem to be proved.

4.3 T-Avoiding Elements in Type D_n

In this section we will classify the non-trivial T-avoiding elements in the Coxeter group of type D_n . Recall that $W(D_n)$ is a star reducible Coxeter group and as a result of this any non-trivial T-avoiding element will not be fully commutative.

Theorem 4.3.1. *Let $W = W(D_n)$. Then W contains non-trivial T-avoiding elements.*

Proof. This is a consequence of [8, Section 2.2]. □

In addition, to showing that there are non-trivial T-avoiding elements in type D_n Gern also classified the non-trivial T-avoiding elements as well. The following is his classification translated into heaps. **Once we figure out the heap include it here.** For the full details regarding his classification see [8]. Note that in his classification Gern refers to non-trivially T-avoiding elements as "bad."

4.4 T-Avoiding Elements in Type F_n

In this section we state what is known regarding the non-trivial T-avoiding elements in the Coxeter groups of type F_n for $n \geq 4$. Note that all of the following results are unpublished.

We start with the Coxeter group of type F_5 .

4.5 T-Avoiding Elements in Type $I_2(m)$

We next will classify the non-trivial T-avoiding elements in Coxeter groups of type $I_2(m)$. Although the following is a quick result, this is the first time that we believe it has been written.

Theorem 4.5.1. *Let $W = W(I_2(m))$. Then W has no non-trivial T-avoiding elements.*

Proof. Let $W = W(I_2(m))$. Recall that the Coxeter graph for Type $I_2(m)$ consists of 2 vertices, s, t , and an edge with weight $m(s, t)$. Since s and t are involutions this implies that the elements in W must alternate between s and t . This implies that the fully commutative elements in W are just s and t . The rest of the elements in W must be alternating products of s and t . By definition $m(s, t) > 3$ and hence all elements in W that are not products of commuting generators have Property T. Hence W has no non-trivial T-avoiding elements. \square

As mentioned before the Coxeter group of type $I_2(m)$ is a star reducible Coxeter group. Similar to the Coxeter group of type A_n and \widetilde{A}_n for n odd both star reducible Coxeter groups, $W(I_2(m))$ has no non-trivial T-avoiding elements.

Chapter 5

Type-B

5.1 Classification of T-Avoiding Elements in Type B

An introduction should go here regarding the awesome sauce that is to follow.

And we probably need some other stuff to go here but Sarah S. told me to work on typing page 1 today.

Proposition 5.1.1 (Björner, [cite when in Mendelay](#)). *Let $w \in W(B_n)$. Then*

$$\mathcal{R}(w) = \{s_i \in S : w(i) > w(i+1)\}$$

where $w(0)=0$ by definition.

Proof. This is, [cite when in Mendelay](#) Proposition 8.1.2. □

Lemma 5.1.2. *Let $s, t \in S$ such that $m(s, t) = 3$. Then w has a reduced expression ending in sts if and only if w has the consecutive pattern 321.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [cite BB when in Mendelay](#). Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

(\Rightarrow) Suppose w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+2)$. By 5.1.1 $s_i, s_{i+1} \in \mathcal{R}(w)$. By [Tyson's reference to simply laced coxeter group stuff 1.2.1](#) w ends in $s_i s_{i+1} s_{i+2}$. (\Leftarrow) Suppose w ends in $s_i s_{i+1} s_i$. This implies that either $w_I = s_i s_{i+1}$ or $w_I = s_{i+1} s_i$ which implies that $s_i, s_{i+1} \in \mathcal{R}(w)$. Since $s_i, s_{i+1} \in \mathcal{R}(w)$, we see that $w(i) > w(i+1) > w(i+2)$ by 5.1.1. Thus w has the consecutive pattern 321. Therefore, w has a reduced expression ending in sts if and only if w has the consecutive pattern 321. □

Lemma 5.1.3. *Let $s, t \in S$ such that $m(s, t) = 3$. Then w has a reduced expression ending in st if and only if w has the consecutive pattern 231.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [cite BB when in Mendelay](#). Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

(\Rightarrow) Suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$. By 5.1.1 $s_{i+1} \in \mathcal{R}(w)$. Now multiplying on the right by s_{i+1} we see that $ws_{i+1}(i+1) = w(i+2)$ and $ws_{i+1}(i) = w(i)$. We know that $w(i+2) < w(i)$, this implies that $s_i \in \mathcal{R}(ws_{i+1})$. This implies w has a reduced expression that ends in $s_i s_{i+1}$. (\Leftarrow) Suppose that w has a reduced expression ending in $s_i s_{i+1}$. Then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$. Thus we have that $w(i+1) > w(i) > w(i+2)$. Hence w has the consecutive pattern 231. Therefore, w has a reduced expression ending in st if and only if w has the consecutive pattern 231. \square

Lemma 5.1.4. *Let $s, t \in S$ such that $m(s, t) = 3$. Then w has a reduced expression ending in ts if and only if w has the consecutive pattern 312.*

Proof. Let $i \geq 1$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in 2.2.4 in [cite BB when in Mendelay](#). Observe that if w has a reduced expression ending in two non-commuting generators s_i, s_{i+1} in some order then we have $w_I \in \{s_i s_{i+1}, s_{i+1} s_i\}$.

(\Rightarrow) Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. By 5.1.1 we see that $s_i \in \mathcal{R}(w)$. Multiplying on the right by s_i we get $ws_i(i+1) = w(i)$ and $ws_i(i+2) = w(i+2)$. By above $w(i) > w(i+2)$, and by 5.1.1 $s_{i+1} \in \mathcal{R}(ws_i)$. This implies that w has a reduced expression ending in $s_{i+1} s_i$. (\Leftarrow) Conversely suppose w ends in a reduced expression with $s_{i+1} s_i$. Then $w_I = s_{i+1} s_i$. We see that $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$. From this we have $w(i) > w(i+2)$, so $w(i) > w(i+2) > w(i+1)$. Hence, w has the consecutive pattern 312. Therefore, w has a reduced expression ending in ts if and only if w has the consecutive pattern 312. \square

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