T-AVOIDING ELEMENTS OF COXETER GROUPS

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A **Coxeter system** is a pair (W, S) consisting of a distinguished (finite) set S of generating involutions and a group W, called a **Coxeter group**, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements, and that $m(s,t)$ is the order of st. Since the elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written as

$$\underbrace{sts\cdots}_{m(s,t)} = \underbrace{tst\cdots}_{m(s,t)}$$

with $m(s,t) \geq 2$ factors on each side of the equation.

Given a Coxeter system (W, S), a word $s_{x_1}s_{x_2}\cdots s_{x_m}$ in the free monoid S^* is called an **expression** for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w, the corresponding word is called a **reduced expression** for w. In this case, we define the **length** of w to be $\ell(w) = m$. Each element $w \in W$ can have several different reduced expressions that represent it. A product $w_1w_2\cdots w_r$ with $w_i \in W$ is called **reduced** if $\ell(w_1w_2\cdots w_r) = \sum \ell(w_i)$.

Given a Coxeter system (W, S), the associated **Coxeter graph** is the graph Γ with vertex set S and edges $\{s,t\}$ labeled with $m(s,t)$ for all $m(s,t) \geq 3$. If $m(s,t) = 3$, it is customary to leave the corresponding edge unlabeled. Given a Coxeter graph Γ, we can uniquely reconstruct the corresponding Coxeter system (W, S). In this case, we say that (W, S), or just W, is of type Γ. If (W, S) is of type Γ, for emphasis, we may write W and S as $W(\Gamma)$ and $S(\Gamma)$, respectively. Note that generators s and t are connected by an edge in the Coxeter graph Γ if and only if s and t do not commute.

A few of the well-known Coxeter graphs are given in Figure 1. As an example, consider the Coxeter graph of type A_n. This determines the Coxeter system $(W(A_n), S(A_n))$ having $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ as the distinguished generating set and defining relations

(1) $s_is_i = e$ for all i;
(2) $s_is_j = s_js_i$ when $|i-j| > 1$;
(3) $s_is_js_i = s_js_is_j$ when $|i-j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the correspondence $s_i \mapsto (i, i+1)$.

Suppose (W, S) is a Coxeter system. We say that $w \in W$ has **Property T** if and only if $w = stu$ or $w = uts$, where the product is reduced and $m(s,t) \geq 3$. In other words, a Coxeter group element has Property T if it has a reduced expression that begin or ends with a product of non-commuting generators. In turn, we say that w is **T-avoiding** if and only if w does not have Property T.

It is clear that if w is a product of commuting generators, then w is T-avoiding. If w is T-avoiding but not a product of commuting generators, then we say that w is **bad**.

The central focus of this thesis is to address the following questions:

(1) Which Coxeter systems contain bad elements?
(2) If a Coxeter system contains bad elements, what form do they take?

In other words, we would like to classify the T-avoiding elements of Coxeter groups. It is highly unlikely that we will attain a complete classification. Instead, we will take an ad hoc approach by studying specific Coxeter systems in turn.

We will now briefly summarize what is currently known about T-avoiding elements. During the 2010–2011 academic year, D.C. Ernst mentored J. Cormier, Z. Goldenberg, J. Kelly, and C. Malbon at Plymouth State University on a research project aimed at exploring the T-avoiding elements in Coxeter systems of types

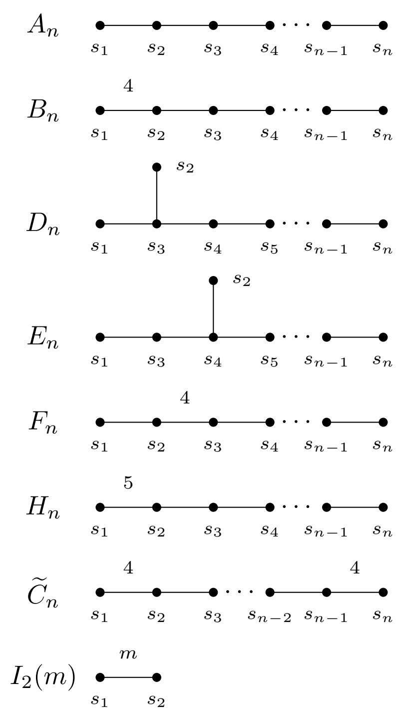

FIGURE 1. Several well-known Coxeter graphs.

A_n and B_n. The research team discovered that there are no bad elements in either system. That is, the only T-avoiding elements in Coxeter systems of types A_n and B_n are products of commuting generators. In the case of type A_n, their results are a reformulation of known results, but with a much simpler proof. Unfortunately, the students never finished writing up their results for publication and the proof in the case of type B_n had been forgotten.

During the 2011–2012 academic year, Ernst mentored R. Cross, K. Hills-Kimball, and C. Quaranta at PSU on a project focused on the T-avoiding elements in Coxeter groups of type F_n. In particular, the team successfully classified the T-avoiding elements in the infinite Coxeter system of type F_5, as well as the finite Coxeter system of type F_4. Both types included bad elements. At the time, the students conjectured that their classification holds more generally for arbitrary F_n.

In the Spring of 2013, Ernst worked with S. Gilbertson at Northern Arizona University on extending the results obtained by Cross, Hills-Kimball, and Quaranta the previous year. The initial goal was to prove that there were no new T-avoiding elements (other than multiplying by products of commuting generators) in type F_n for $n \geq 6$. However, Gilbertson discovered that this is horribly wrong. It appears that the classification of T-avoiding elements in higher ranks gets more and more complicated. Gilbertson and Ernst conjectured a classification of the T-avoiding elements in type F_6, but were unable to complete the proof. The problem remains open.

In his PhD thesis, T. Gern classified the T-avoiding elements in Coxeter groups of type D_n. Unlike in types A_n and B_n, it turns out that the classification in type D_n includes bad elements.

In the Spring of 2015, T. Laird worked with Ernst on a semester-long research project involving T-avoiding elements of various Coxeter systems. The major accomplishment of our work was the re-discovery of the proof of the classification of the T-avoiding elements in type B_n. In addition, we conjectured a classification of the T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$ and put most of the pieces of the proof together.

In this thesis, we propose to do the following. First, we will summarize the currently known classifications of T-avoiding elements. This will include our versions of the proofs for the classifications involving types A_n and B_n. Next, we will attempt to complete the proof of the classification in type $\widetilde{C}_n$. As time allows, we will tackle other Coxeter systems. In particular, we will explore the T-avoiding elements in systems of type E_n and see if we can make any additional headway in the case of type F_n.