# A Study of T-Avoiding Elements in Coxeter Groups

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## **Coxeter Systems**

#### Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

where  $m(s,t) \ge 2$  for all  $s \ne t$ .

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 } commutations  $m(s,t) = 3 \implies sts = tst$   $m(s,t) = 4 \implies stst = tsts$  } braid relations

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We can encode (W, S) with a unique Coxeter graph  $\Gamma$  having:

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## Comments

- if m(s, t) = 3, we omit label.
- If s and t are not connected in Γ, then s and t commute.
- Given  $\Gamma$ , we can uniquely reconstruct the corresponding (W, S).

# **Coxeter Groups of Type** *A*

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  $s_2$   $s_3$   $s_{n-1}$   $s_n$ 

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Then  $W(A_n)$  is generated by  $\{s_1, s_2, \dots, s_n\}$  and is subject to defining relations

- 1.  $s_i^2 = e$  for all i,
- 2.  $s_i s_j = s_j s_i$  if |i j| > 1,
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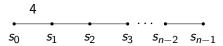
 $W(A_n)$  is isomorphic to the symmetric group,  $Sym_{n+1}$ , under the correspondence

$$s_i \mapsto (i, i+1),$$

where (i, i+1) is the adjacent transposition exchanging i and i+1.

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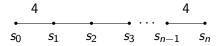
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 $W(B_n)$  is a finite group of order  $n!2^n$  (wreath product of  $\mathbb{Z}_2$  and the symmetric group).

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#### Comment

We can obtain  $W(A_n)$  and  $W(B_n)$  from  $W(\widetilde{C}_n)$  by removing the appropriate generators and corresponding relations. In fact, we can obtain  $W(B_n)$  in two ways.

# **Reduced Expressions**

#### Definition

A word  $s_{x_1}s_{x_2}\cdots s_{x_m}$  is called an expression for  $w\in W$  if it is equal to w when considered as a group element.

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Given  $w \in W$ , if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{W} = s_{x_1} s_{x_2} \cdots s_{x_m}.$$

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Any two reduced expressions for  $w \in W$  differ by a sequence of commutations and braid moves.

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We define the left descent set (respectively, right) w as follows:

$$\mathcal{L}(w) := \{ s \in S \mid I(sw) < I(w) \}$$
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## Example

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# **Fully Commutative Elements**

## Definition

Let (W, S) be a Coxeter system of type  $\Gamma$ . We say that  $w \in W(\Gamma)$  is fully commutative (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements is denoted FC( $\Gamma$ ).

## Theorem (Stembridge)

 $w \in FC(\Gamma)$  if and only if no reduced expression for w contains a braid.

#### Comment

It follows from Stembridge that  $W(\widetilde{C}_n)$  contains an infinite number of FC elements, while  $W(A_n)$  and  $W(B_n)$  do not.

# Fully Commutative Elements

## Comment

The elements of  $FC(\widetilde{C}_n)$  are precisely those whose reduced expressions avoid the consecutive subwords  $s_i s_j s_i$  for  $m(s_i, s_j) = 3$ ,  $s_0 s_1 s_0 s_1$ , and  $s_{n-1} s_n s_{n-1} s_n$ .

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## Example

Let  $\overline{w} = s_0 s_2 s_4 s_3 s_2 s_1$  be a reduced expression for  $w \in W(\widetilde{C}_4)$ . We see that

$$s_0 s_2 s_4 s_3 s_2 s_1 = s_0 s_4 s_2 s_3 s_2 s_1.$$

Since w has one of the forbidden consecutive subwords, w is not FC.

# Heaps

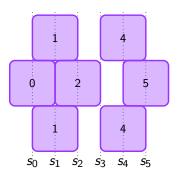
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Let  $\overline{w} = s_4 s_5 s_1 s_0 s_2 s_4 s_1$  be a reduced expression for  $w \in W(B_6)$ .

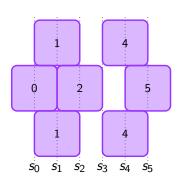


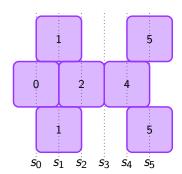
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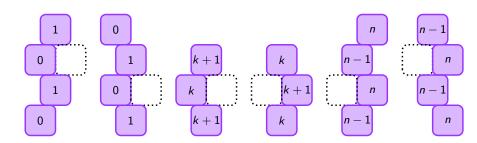


## Theorem (Stembridge)

There is a unique heap for w if and only if w is FC.

## Lemma

Let  $w \in FC(\widetilde{C}_n)$ . Then H(w) can not contain any of the following convex subheaps



#### Definition

We define w to be left star reducible by s with respect to t if  $m(s,t) \geq 3$ ,  $s \in \mathcal{L}(w)$  and  $t \in \mathcal{L}(sw)$ . Analogous definition for right star reducible.

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#### Definition

We define  $W(\Gamma)$  to be star reducible if every element of  $FC(\Gamma)$  is can be reduced to a product of commuting generators via a sequence of star reductions.

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#### Definition

We define  $W(\Gamma)$  to be star reducible if every element of FC( $\Gamma$ ) is can be reduced to a product of commuting generators via a sequence of star reductions.

## Theorem (Green)

There is a complete list of star reducible Coxeter systems. These include Coxeter systems of type  $A_n$   $(n \ge 1)$ , type  $B_n$   $(n \ge 2)$ , type  $D_n$   $(n \ge 4)$ , type  $F_n$   $(n \ge 4)$ , type  $I_2(m)$   $I_2($ 

### Property T

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We define w to have Property T if and only if there exists a reduced product for w such that w = stu or w = uts where  $m(s, t) \ge 3$ .

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We say w is T-avoiding if w does not have Property T.

### Proposition

A product of commuting generators is T-avoiding.

#### Definition

We define w to be a trivial T-avoiding element if w is a product of commuting generators. Otherwise, we say w is a non-trivially T-avoiding element.

## Examples of Property T and T-avoiding

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# Classification of T-Avoiding Elements: Already Known $A_n$

### Theorem (Fan, Green)

If n is odd and  $n \geq 2$ , then there are no non-trivial T-avoiding elements in  $W(\widetilde{A}_n)$ . If n is even and  $n \geq 2$ , then  $W(\widetilde{A}_n)$  contains non-trivial T-avoiding elements.

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#### Conjecture

The only non-trivial T-avoiding elements of  $W(A_n)$  for n odd are of the form  $w=(s_0s_2\cdots s_{n-2}s_ns_1s_3\cdots s_{n-3}s_{n-1})^k$  for  $k\in\mathbb{Z}^+$ .

# Classification of T-Avoiding Elements: Already Known $\widetilde{A}_n$

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### Corollary

There are no non-trivial T-avoiding elements in  $W(A_n)$ .

### Theorem (Laird?)

There are no non-trivial T-avoiding elements in  $W(I_2(m))$ .

## Classification of T-Avoiding Elements : Already Known $D_n$

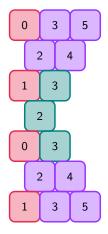
Theorem (Gern)

The only non-trivial T-avoiding elements in  $W(D_n)$  have this pattern

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## Classification of T-Avoiding Elements: Already Known $F_n$

Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

The only non-trivial T-avoiding elements in  $F_5$  are stacks of bowties.

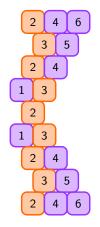
## Classification of T-Avoiding Elements: Already Known $F_n$

Corollary (Cross, Ernst, Hills-Kimball, Quaranta) There are no non-trivial T-avoiding elements in  $F_4$ .

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Classifying non-trivial T-avoiding elements in  $F_n$  for  $n \ge 6$  gets very difficult.

## Signed Permutation Representation

Since  $W(B_n)\cong \operatorname{Sym}_n^B$ , we can write  $w\in W(B_n)$  as a signed permutation

$$[w(1), w(2), \ldots, w(n)]$$

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#### Definition

We say that w has the signed consecutive pattern abc if there is some  $i \in \{1,2,\ldots,n-2\}$  such that (|w(i)|,|w(i+1)|,|w(i+2)|) is in the same consecutive order as (|a|,|b|,|c|) and such that  $\operatorname{sgn}(w(i)) = \operatorname{sgn}(a), \operatorname{sgn}(w(i+1)) = \operatorname{sgn}(b),$  and  $\operatorname{sgn}(w(i+2)) = \operatorname{sgn}(c)$ . We say that w avoids the signed consecutive pattern abc if there is no such i.

### Example

Let  $\overline{w} = s_0 s_1 s_3 s_4 s_5 s_2$  be a reduced expression for  $w \in B_6$ . We see that

$$[1,2,3,4,5,6] = [\underline{1},2,3,4,5,6]$$
$$= [2,\underline{1},3,4,5,6]$$

## Classification of T-Avoiding Elements: $B_n$

### Theorem (Laird)

There are no non-trivial T-avoiding elements in  $W(B_n)$ .

123	<u>1</u> 23	1 <u>2</u> 3	12 <u>3</u>	<u>12</u> 3	<u>1</u> 2 <u>3</u>	1 <u>23</u>	<u>123</u>
132	<u>1</u> 32	1 <u>3</u> 2	13 <u>2</u>	<u>13</u> 2	<u>1</u> 3 <u>2</u>	1 <u>32</u>	<u>132</u>
213	<u>2</u> 13	2 <u>1</u> 3	21 <u>3</u>	<u>21</u> 3	<u>2</u> 1 <u>3</u>	2 <u>13</u>	<u>213</u>
231	<u>2</u> 31	2 <u>3</u> 1	23 <u>1</u>	<u>23</u> 1	<u>2</u> 3 <u>1</u>	2 <u>31</u>	<u>231</u>
312	<u>3</u> 12	3 <u>1</u> 2	31 <u>2</u>	<u>31</u> 2	<u>3</u> 1 <u>2</u>	3 <u>12</u>	<u>312</u>
321	<u>3</u> 21	3 <u>2</u> 1	32 <u>1</u>	<u>32</u> 1	<u>3</u> 2 <u>1</u>	3 <u>21</u>	<u>321</u>

Through a series of lemmas we were to determine if a reduced product ends or begins with *st* given that *w* contains a certain signed consecutive pattern.

# Classification of T-Avoiding Elements: $C_n$

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## Classification of T-Avoiding Elements: $C_n$

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### Theorem (Laird)

If n is even, then the only non-trivial T-avoiding elements in Coxeter systems of type  $\widetilde{C}_n$  are sandwich stacks.