A Study of T-Avoiding Elements in Coxeter Groups

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NAU Thesis Defense
April 29, 2016

Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

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$$\left.\begin{array}{lcl} m(s,t) = 3 & \implies & sts = tst \\ m(s,t) = 4 & \implies & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

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We can encode (W, S) with a unique Coxeter graph Γ having:

- vertex set S;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$;

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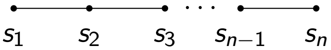
Comments

- if $m(s, t) = 3$, we omit label.
- If s and t are not connected in Γ, then s and t commute.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Coxeter groups of type A_n ($n \geq 1$) are defined by:



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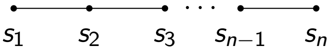

Then $W(A_n)$ is generated by $\{s_1, s_2, \ldots, s_n\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
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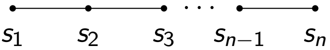

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$W(A_n)$ is isomorphic to the symmetric group, Sym_{n+1}, under the correspondence

$$s_i \mapsto (i,\ i+1),$$

where $(i,\ i+1)$ is the adjacent transposition exchanging i and $i+1$.

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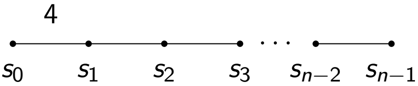

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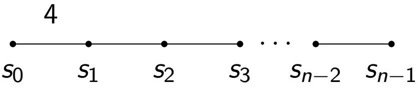

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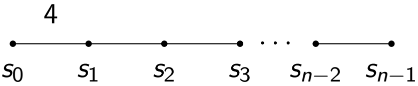

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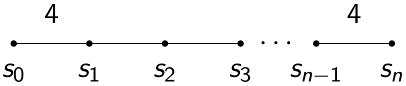
$W(B_n)$ is a finite group of order $n!2^n$ (wreath product of $\mathbb{Z}_2$ and the symmetric group).

Coxeter Groups of Type Affine C

Coxeter groups of type $\widetilde{C}_n$ ($n \geq 2$) are defined by:



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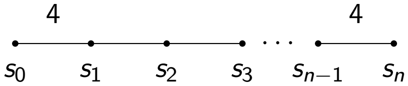

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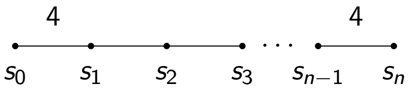

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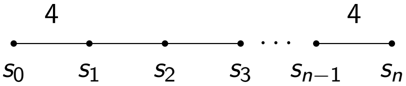

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Comment

We can obtain $W(A_n)$ and $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the appropriate generators and corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

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Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{w} = s_{x_1}s_{x_2}\cdots s_{x_m}.$$

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of commutations and braid moves.

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Let $\overline{w} = s_2 s_1 s_2 s_3 s_1$ be a fixed expression for $w \in W(A_3)$. We see that

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Let (W,S) be a Coxeter system of type Γ. We say that $w \in W(\Gamma)$ is fully commutative (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements is denoted $\mathrm{FC}(\Gamma)$.

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Comment

It follows from Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements, while $W(A_n)$ and $W(B_n)$ do not.

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The elements of $\mathrm{FC}(\widetilde{C}_n)$ are precisely those whose reduced expressions avoid the consecutive subwords $s_i s_j s_i$ for $m(s_i, s_j) = 3$, $s_0 s_1 s_0 s_1$, and $s_{n-1} s_n s_{n-1} s_n$.

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Example

Let $\overline{w} = s_0 s_2 s_4 s_3 s_2 s_1$ be a reduced expression for $w \in W(\widetilde{C}_4)$. We see that

$$s_0 s_2 s_4 s_3 s_2 s_1 = s_0 s_4 s_2 s_3 s_2 s_1.$$

Since w has one of the forbidden consecutive subwords, w is not FC.

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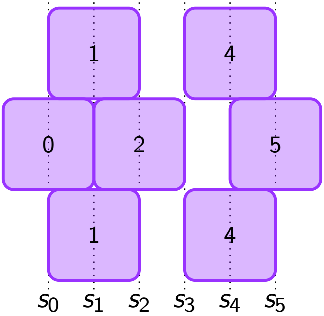

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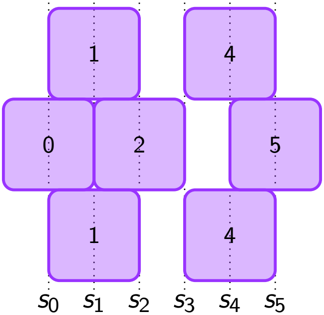
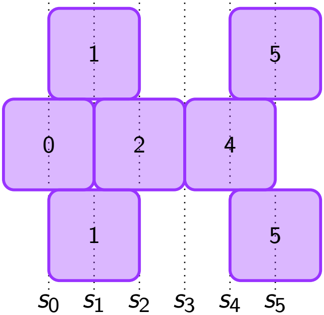


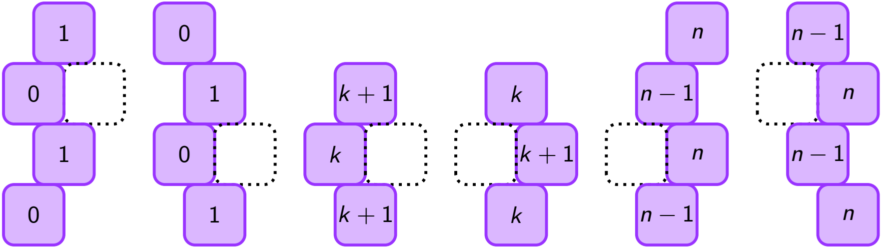


Theorem (Stembridge)

There is a unique heap for w if and only if w is FC.

Lemma

Let $w \in \mathrm{FC}(\widetilde{C}_n)$. Then $H(w)$ can not contain any of the following convex subheaps



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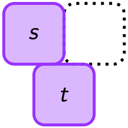

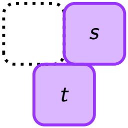

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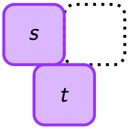

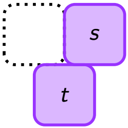

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Theorem (Green)

There is a complete list of star reducible Coxeter systems. These include Coxeter systems of type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_n$

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Proposition

A product of commuting generators is T-avoiding.

Definition

We define w to be a trivial T-avoiding element if w is a product of commuting generators. Otherwise, we say w is a non-trivially T-avoiding element.

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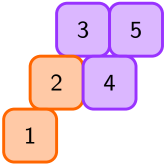

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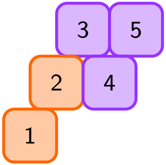

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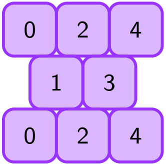

Classification of T-Avoiding Elements: Already Known $\widetilde{A}_n$

Theorem (Fan, Green)

If n is odd and $n \geq 2$, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. If n is even and $n \geq 2$, then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

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Conjecture

The only non-trivial T-avoiding elements of $W(\widetilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

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If n is odd and $n \geq 2$, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. If n is even and $n \geq 2$, then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

Conjecture

The only non-trivial T-avoiding elements of $W(\widetilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

Corollary

There are no non-trivial T-avoiding elements in $W(A_n)$.

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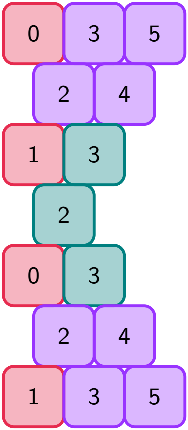
Theorem (Laird?)

There are no non-trivial T-avoiding elements in $W(I_2(m))$.

Classification of T-Avoiding Elements : Already Known D_n

Theorem (Gern)

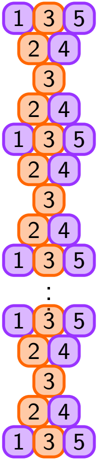
The only non-trivial T-avoiding elements in $W(D_n)$ have this pattern



Classification of T-Avoiding Elements: Already Known F_n

Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

The only non-trivial T-avoiding elements in F_5 are stacks of bowties.



Classification of T-Avoiding Elements: Already Known F_n

Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no non-trivial T-avoiding elements in F_4.

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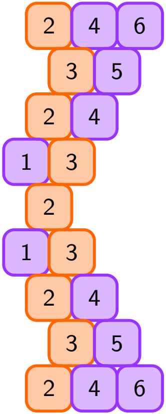
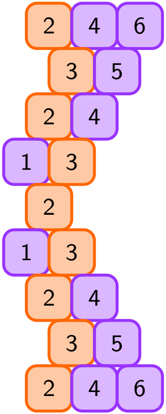


Classification of T-Avoiding Elements: Already Known F_n

Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no non-trivial T-avoiding elements in F_4.



Classifying non-trivial T-avoiding elements in F_n for $n \geq 6$ gets very difficult.

Signed Permutation Representation

Since $W(B_n) \cong \mathrm{Sym}_n^B$, we can write $w \in W(B_n)$ as a signed permutation

$$[w(1), w(2), \ldots, w(n)]$$

where we write a bar underneath a number in place of a negative sign.

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where we write a bar underneath a number in place of a negative sign.

Definition

We say that w has the signed consecutive pattern *abc* if there is some $i \in \{1, 2, \dots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same consecutive order as $(|a|, |b|, |c|)$ and such that $\mathrm{sgn}(w(i)) = \mathrm{sgn}(a)$, $\mathrm{sgn}(w(i+1)) = \mathrm{sgn}(b)$, and $\mathrm{sgn}(w(i+2)) = \mathrm{sgn}(c)$. We say that w avoids the signed consecutive pattern *abc* if there is no such i.

Example

Let $\overline{w} = s_0 s_1 s_3 s_4 s_5 s_2$ be a reduced expression for $w \in B_6$. We see that

$$\begin{aligned}[1,2,3,4,5,6] &= [\underline{1},2,3,4,5,6] \\ &= [2,\underline{1},3,4,5,6]\end{aligned}$$

Classification of T-Avoiding Elements: B_n

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(B_n)$.

123	$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$	$\underline{12}3$	$\underline{1}2\underline{3}$	$1\underline{23}$	$\underline{123}$
132	$\underline{1}32$	$1\underline{3}2$	$13\underline{2}$	$\underline{13}2$	$\underline{1}3\underline{2}$	$1\underline{32}$	$\underline{132}$
213	$\underline{2}13$	$2\underline{1}3$	$21\underline{3}$	$\underline{21}3$	$\underline{2}1\underline{3}$	$2\underline{13}$	$\underline{213}$
231	$\underline{2}31$	$2\underline{3}1$	$23\underline{1}$	$\underline{23}1$	$\underline{2}3\underline{1}$	$2\underline{31}$	$\underline{231}$
312	$\underline{3}12$	$3\underline{1}2$	$31\underline{2}$	$\underline{31}2$	$\underline{3}1\underline{2}$	$3\underline{12}$	$\underline{312}$
321	$\underline{3}21$	$3\underline{2}1$	$32\underline{1}$	$\underline{32}1$	$\underline{3}2\underline{1}$	$3\underline{21}$	$\underline{321}$

Through a series of lemmas we were to determine if a reduced product ends or begins with st given that w contains a certain signed consecutive pattern.

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

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If n is odd, then there are no non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$.

Classification of T-Avoiding Elements: $\widetilde{C}_n$

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

Theorem (Laird)

If n is odd, then there are no non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$.

Theorem (Laird)

If n is even, then the only non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$ are sandwich stacks.