A Study of T-Avoiding Elements in Coxeter Groups

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Colloquium
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Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle$$

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$$\begin{aligned}
m(s,t) = 2 &\implies st = ts \quad \Big\} \quad \text{commutations} \\
m(s,t) = 3 &\implies sts = tst \\
m(s,t) = 4 &\implies stst = tsts \\
&\vdots
\end{aligned} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \quad \text{braid relations}$$

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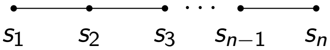
Comments

- If $m(s, t) = 3$, we omit label.
- If s and t are not connected in Γ, then s and t commute.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Coxeter groups of type A_n ($n \geq 1$) are defined by:



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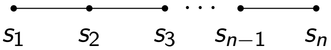

Then $W(A_n)$ is generated by $\{s_1, s_2, \ldots, s_n\}$ and is subject to defining relations

1. $s_i^2 = e$ for all i,
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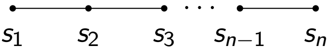

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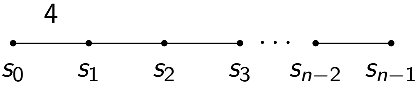
$W(A_n)$ is isomorphic to the symmetric group, Sym_{n+1}, under the correspondence

$$s_i \mapsto (i,\ i+1),$$

where $(i,\ i+1)$ is the adjacent transposition exchanging i and $i+1$.

Coxeter groups of type B_n ($n \geq 2$) are defined by:



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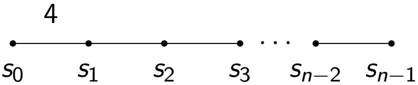

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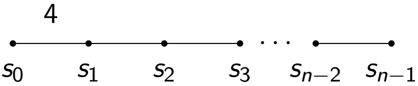

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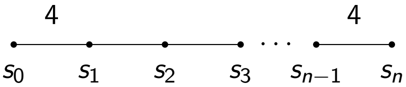
$W(B_n) \cong \mathrm{Sym}_n^B$ is a finite group of order $n!2^n$.

Coxeter Groups of Type Affine C

Coxeter groups of type $\widetilde{C}_n$ ($n \geq 2$) are defined by:



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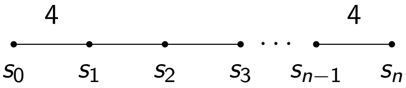

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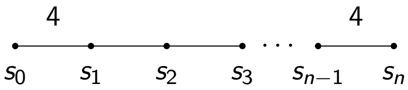

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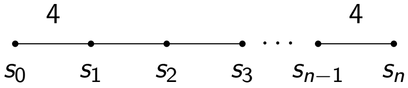

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Comment

We can obtain $W(A_n)$ and $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the appropriate generators and corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

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Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{w} = s_{x_1}s_{x_2}\cdots s_{x_m}.$$

Matsumoto's Theorem and Support

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Any two reduced expressions for $w \in W$ differ by a sequence of commutations and braid moves.

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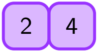

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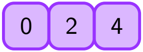

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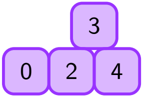

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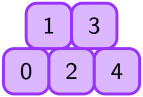

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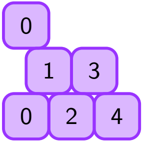

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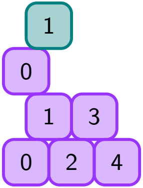

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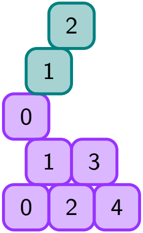

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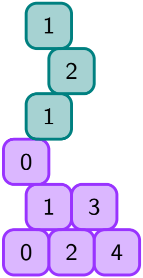

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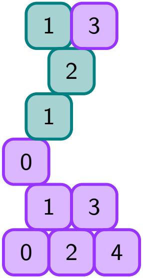

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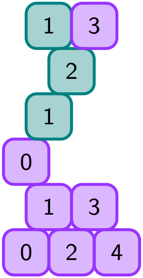

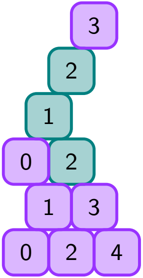

Definition

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Comment

It follows from Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements, while $W(A_n)$ and $W(B_n)$ do not.

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The elements of $\mathrm{FC}(\widetilde{C}_n)$ are precisely those whose reduced expressions avoid the consecutive subwords $s_i s_j s_i$ for $m(s_i, s_j) = 3$, $s_0 s_1 s_0 s_1$, and $s_{n-1} s_n s_{n-1} s_n$.

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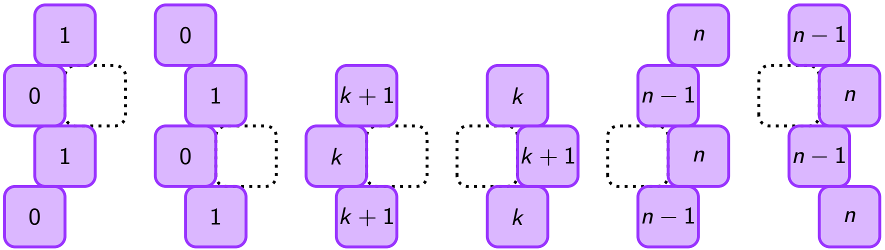
Lemma

Let $w \in \mathrm{FC}(\widetilde{C}_n)$. Then $H(w)$ can not contain any of the following convex subheaps



Theorem (Stembridge)

There is a unique heap for w if and only if w is FC.

Example

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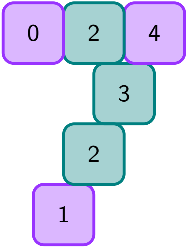
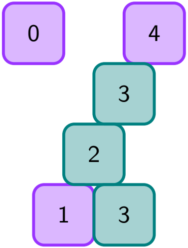
$$s_0 s_2 s_4 s_3 s_2 s_1 = s_0 s_4 s_2 s_3 s_2 s_1.$$

Since w has one of the forbidden consecutive subwords, w is not FC.





Definition

We define w to be left star reducible by s with respect to t if $m(s,t) \geq 3$, $s \in \mathcal{L}(w)$ and $t \in \mathcal{L}(sw)$. Analogous definition for right star reducible.

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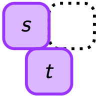

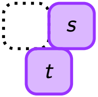

Star Reductions

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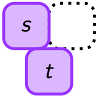
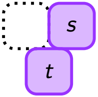


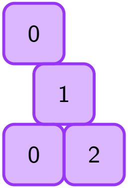


Definition

We define $W(\Gamma)$ to be star reducible if every element of $\mathrm{FC}(\Gamma)$ can be reduced to a product of commuting generators via a sequence of star reductions.

Example

Let $\overline{w} = s_0 s_1 s_0 s_2$ be a reduced expression for $w \in W(B_4)$.



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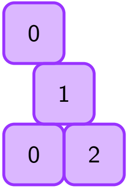

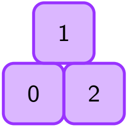

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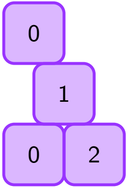

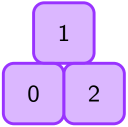

Theorem (Green)

There is a complete list of star reducible Coxeter systems. These include Coxeter systems of type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type F_n $(n \geq 4)$, type $I_2(m)$ $(m \geq 3)$, type $\widetilde{A}_n$ ($n \geq 3$ and n even), and type $\widetilde{C}_n$ ($n \geq 3$ and n odd).

Definition

We define w to have Property T if and only if there exists a reduced product for w such that $w = stu$ or $w = uts$ where $m(s, t) \geq 3$.

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Proposition

A product of commuting generators is T-avoiding.

Definition

We define w to be a trivial T-avoiding element if w is a product of commuting generators. Otherwise, we say w is a non-trivial T-avoiding element.

Example

Let $\overline{w} = s_5 s_3 s_2 s_4 s_1$ be a reduced expression for $w \in W(A_5)$.

Examples of Property T and T-avoiding

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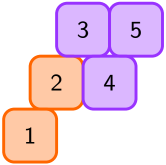

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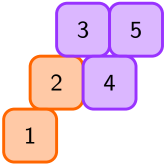
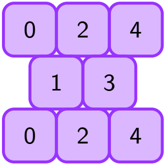


Example

Let $\overline{w} = s_0 s_2 s_4 s_1 s_3 s_0 s_2 s_4$ be a reduced expression for $w \in W(\widetilde{C}_4)$.



Theorem (Fan, Green)

If n is odd and $n \geq 2$, then there are no non-trivial T-avoiding elements in $W(\widetilde{A}_n)$. If n is even and $n \geq 2$, then $W(\widetilde{A}_n)$ contains non-trivial T-avoiding elements.

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Conjecture

The only non-trivial T-avoiding elements of $W(\widetilde{A}_n)$ for n odd are of the form $w = (s_0 s_2 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-3} s_{n-1})^k$ for $k \in \mathbb{Z}^+$.

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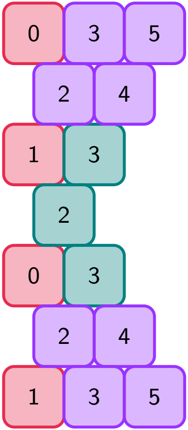
There are no non-trivial T-avoiding elements in $W(A_n)$.

Theorem (Laird?)

There are no non-trivial T-avoiding elements in $W(I_2(m))$.

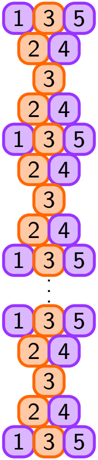
Theorem (Gern)

The only non-trivial T-avoiding elements in $W(D_n)$ have this pattern:



Theorem (Cross, Ernst, Hills-Kimball, Quaranta)

The only non-trivial T-avoiding elements in F_5 are stacks of bowties:



Corollary (Cross, Ernst, Hills-Kimball, Quaranta)

There are no non-trivial T-avoiding elements in F_4.

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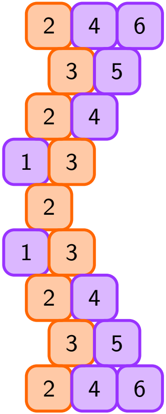
Comment

Classifying non-trivial T-avoiding elements in F_n for $n \geq 6$ gets very difficult.



Signed Permutation Representation

Since $W(B_n) \cong \mathrm{Sym}_n^B$, we can write $w \in W(B_n)$ as a signed permutation

$$[w(1), w(2), \ldots, w(n)]$$

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Definition

We say that w has the signed consecutive pattern abc if there is some $i \in \{1, 2, \ldots, n-2\}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same relative order as $(|a|, |b|, |c|)$ and such that $\mathrm{sgn}(w(i)) = \mathrm{sgn}(a)$, $\mathrm{sgn}(w(i+1)) = \mathrm{sgn}(b)$, and $\mathrm{sgn}(w(i+2)) = \mathrm{sgn}(c)$.

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We say that w avoids the signed consecutive pattern abc if there is no such i as above.

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$$\begin{aligned}[1,2,3,4,5,6] &= [\underline{1},2,3,4,5,6] \\ &= [2,\underline{1},3,4,5,6]\end{aligned}$$

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Classification of T-Avoiding Elements: B_n

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(B_n)$.

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123	$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$	$\underline{12}3$	$\underline{1}2\underline{3}$	$1\underline{23}$	$\underline{123}$
132	$\underline{1}32$	$1\underline{3}2$	$13\underline{2}$	$\underline{13}2$	$\underline{1}3\underline{2}$	$1\underline{32}$	$\underline{132}$
213	$\underline{2}13$	$2\underline{1}3$	$21\underline{3}$	$\underline{21}3$	$\underline{2}1\underline{3}$	$2\underline{13}$	$\underline{213}$
231	$\underline{2}31$	$2\underline{3}1$	$23\underline{1}$	$\underline{23}1$	$\underline{2}3\underline{1}$	$2\underline{31}$	$\underline{231}$
312	$\underline{3}12$	$3\underline{1}2$	$31\underline{2}$	$\underline{31}2$	$\underline{3}1\underline{2}$	$3\underline{12}$	$\underline{312}$
321	$\underline{3}21$	$3\underline{2}1$	$32\underline{1}$	$\underline{32}1$	$\underline{3}2\underline{1}$	$3\underline{21}$	$\underline{321}$

Through a series of lemmas we were able to determine if a reduced product ends or begins with *st* given that *w* contains a certain signed consecutive pattern.

Classification of T-Avoiding Elements: $\widetilde{C}_n$

Theorem (Laird)

There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

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There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

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If n is odd, then there are no non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$.

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There are no non-trivial T-avoiding elements in $W(\widetilde{C}_n) \setminus \mathrm{FC}(\widetilde{C}_n)$.

Theorem (Laird)

If n is odd, then there are no non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$.

Theorem (Laird)

If n is even, then the only non-trivial T-avoiding elements in Coxeter systems of type $\widetilde{C}_n$ are sandwich stacks.

Interesting Fact

Comment

Recall that Coxeter systems of Type D_n and F_n have non-trivial T-avoiding elements that are not FC. Also Coxeter systems of Type $\widetilde{A}_n$ and $\widetilde{C}_n$ for appropriate choice of n have non-trivial T-avoiding elements that are FC.

In all of the examples we have seen so far the non-trivial T-avoiding elements are either only FC or only not FC.