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A comparison of model-based and data-driven controller tuning

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SUMMARY

In many industrial applications, finding a model from physical laws that is both simple and reliable for control design is a hard and time-consuming undertaking. When a set of input/output measurements is available, one can derive the controller directly from data, without relying on the knowledge of the physics. In the scientific literature, two main approaches have been proposed for control system design from data. In the 'model-based' approach, a model of the system is first derived from data and then a controller is computed-based on the model. In the 'data-driven' approach, the controller is directly computed from data. In this work, the previous approaches are compared from a novel perspective. The main finding of the paper is that, although from the standard perspective of parameter variance analysis the model-based approach is always statistically more efficient, the data-driven controller might outperform the model-based solution for what concerns the final control cost. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the last decade, the progress of data-acquisition technology has made it easy and straightforward to collect a large amount of measurements from industrial plants. The use of data as an alternative to physical knowledge to design fixed-order controllers, for example PID, has attracted more and more interest throughout the years, because it is often cheaper and less time-consuming. Specifically, two main approaches have been studied in the scientific literature.

In the 'model-based' approach, a model of the plant is identified from data and used to compute the fixed-order controller satisfying some user-defined requirements. As an example, in model reference control, the identified model is used to design a controller that minimizes the model reference criterion, either algebraically or through optimization, and a controller order reduction step is performed (if needed) before implementation. However, this controller is not necessarily optimal when connected to the plant, and the control performance is limited by modeling errors.

In the 'data-driven' controller tuning approach, the controller is directly derived from input/output (I/O) data. These techniques have been proposed to avoid the problem of under-modeling and to facilitate the design of fixed-order controllers, both iteratively [1–4] and non-iteratively [5–7]. Specifically, in non-iterative approaches, stability can be guaranteed [7], and because the controller parameter estimation problem is convex for most interesting controller structures, the global optimum can be found. Various application examples (*e.g.*, [8, 9]) have shown that critical control problems can be dealt with by using a data-driven method. However, it can be debated whether

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similar results can be obtained if the same amount of data is available for system identification and a model-based controller tuning approach is used.

In the context of system identification, it has been shown that an indirect approach consisting of two optimization steps is statistically efficient [10]. As a matter of fact, according to the invariance principle of maximum likelihood (ML) estimators, an estimator of a function of the model parameter estimates is asymptotically efficient if the model parameter estimate is statistically efficient. Translating these results to the specific case of controller tuning, arguments have been put forward in favor of model-based approaches [11]. In fact, on the basis of the translation of the previous results to controller estimation, it can be argued that an efficient model-based approach is optimal and will therefore achieve equivalent or better results than data-driven approaches that are not statistically efficient.

Analysis of the accuracy of controller estimates is limited both for data-driven and model-based approaches, and a quantitative comparison confirming the argument given previously is lacking. One of the problems in performing such an analysis is that the achieved performance of model-based controller tuning methods strongly depends on the modeling technique that is used. If an identified parametric model is used, the control performance depends on the identification approach and the resulting amount of under-modeling. Furthermore, the order of the controller depends in general on the order of the identified model. In practice, bounds on the modeling error can be defined, but the exact amount of under-modeling will be unknown and problem dependent.

In this paper, a model-based controller tuning approach based on the invariance principle of ML estimators is proposed that allows for a comparison of the asymptotic variance of the controller parameter estimate with the accuracy achieved by data-driven approaches. A high-order model is identified using ML estimation (in this step, the modeling error can be assumed negligible), and the controller parameters are estimated using an L_2 approach, under the assumption that the control objective is achievable. According to the arguments set out previously, this approach achieves the Cramér–Rao lower bound [11]. Moreover, this method can fairly be compared with non-iterative data-driven control (in this work, the Correlation-based Tuning, CbT [7], will be accounted for) as both approaches are based on convex optimization only. However, from the perspective of control design, *the variance analysis of the controller parameters is only an intermediate step* toward the evaluation of the methods. In fact, the real final objective is the control cost achieved by the designed controller.

In this work, the accuracy of this final control objective is analyzed. By doing so, a more direct analysis of the performance will be carried out. The main conclusions of this paper are the following:

- If the model structure is perfectly known and the model order is low, the model-based approach is theoretically always the best in terms of statistical performance, as argued in [11];
- If the model structure is not completely known and/or a high-order model is identified using an ML estimator as indicated previously, the data-driven approach can statistically outperform the model-based solution in terms of the control cost, even if the variance of the parameters remains larger.
- Because in the real world the model structure is never *perfectly* known and under-modeling cannot be avoided with a low-order model, the data-driven approach may give better results in real applications.

The analysis in this study is limited to one specific data-driven and model-based method; only stable systems and open-loop experiments are considered and it is assumed that the control objective is achievable. Generalization of the conclusions of this study is not straightforward; however, the analysis shows that results on reduced-order system identification cannot be directly extended to data-driven controller tuning and that, as a consequence, data-driven methods can outperform a model-based approach.

The remainder of the paper is as follows. Preliminaries and notation are given in Section 2. The model-based and the data-driven methods used in the paper for fixed-order model reference design are described in Section 3. The main results on accuracy analysis are presented in Section 4. A simulation example is used in Section 5 to illustrate the theoretical observations on the benchmark system

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introduced in [12]. The previous approaches are tested on a real experimental setup, by using the System Identification Toolbox [13] for the ML approach, in Section 6. Finally, Section 7 concludes the paper.

2. PRELIMINARIES

2.1. The approximate model reference control problem

Consider the stable linear SISO plant $G(q^{-1})$, where q^{-1} denotes the backward shift operator. Specifications for the controlled plant are given as a reference model $M(q^{-1})$. In the following, it is assumed that $M \neq 1$. The backward shift operator will be omitted in the sequel for convenience. The control objective is to design the controller $K(\rho)$, parameterized through ρ, such that the closed-loop system resembles the reference model M. This can be achieved by minimizing the two norms of the difference between the reference model and the achieved closed-loop system:

$$J_{mr}(\rho) = \left\| M - \frac{K(\rho)G}{1 + K(\rho)G} \right\|_2^2 \tag{1}$$

In the following, the reference model as well as the controller structure is assumed to be given. Notice that when no model of the plant is available, the definition of an adequate reference model may be challenging. If constrained optimization is used to guarantee stability [7], an iterative procedure to define an adequate control objective can be performed off-line. Alternatively, a very simple (even nonparametric) model can be used for this purpose. Notice that in this second case, the design is still data-driven, because the model is not directly used for controller design. A thorough discussion on the choice of M can be found in [14].

In this paper, the controller structure is chosen linear in the parameters,

$$K(q^{-1}, \rho) = \beta^T(q^{-1})\rho, \quad \rho \in \mathcal{D}_K \subseteq \mathbb{R}^{n_\rho} \tag{2}$$

where the set $\mathcal{D}_K$ is compact and

$$\beta(q^{-1}) = [\beta_1(q^{-1}), \cdots, \beta_{n_\rho}(q^{-1})]^T \tag{3}$$

is a vector of size n_ρ of linear discrete-time transfer operators (in general an orthogonal basis). Only the cases where $K(\rho)$ is stable or it contains an integrator if $M(1) = 1$ will be considered.

The ideal controller K^* can be defined indirectly by G and M as

$$K^* = \frac{M}{G(1-M)} \tag{4}$$

that always exists because $M \neq 1$. Notice that K^* might be of very high order; it might not stabilize the plant internally, and it might be noncausal.

Notice that the model reference criterion (1) is non-convex with respect to ρ. An approximation that is convex for linearly parameterized controllers (2) can be defined using the reference model, as follows. The ideal sensitivity function is given by

$$\frac{1}{1 + K^*G} = 1 - M$$

Note that this function is causal (as well as the reference model M) independent of the causality of K^*. Recalling (4), the model reference criterion (1) can be expressed as:

$$J_{mr}(\rho) = \left\| \frac{K^*G - K(\rho)G}{(1 + K^*G)(1 + K(\rho)G)} \right\|_2^2 \tag{5}$$

Approximation of $1/(1 + GK(\rho))$ by $1 - M$, the ideal sensitivity function, leads to the following approximation of the model reference criterion:

$$J(\rho) = \left\| \frac{K^*G - K(\rho)G}{(1 + K^*G)^2} \right\|_2^2 = \|(1-M)[M - K(\rho)(1-M)G]\|_2^2 \tag{6}$$

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The quality of this approximation of $J_{mr}(\rho)$ is discussed in [6]. Notice that with the selected parameterization, $J(\rho)$ is a quadratic function of ρ and its global optimizer can be easily found using the least squares techniques.

The optimal controller is defined as $K_o = K(\rho_o)$ with

$$\rho_o = \arg\min_{\rho \in \mathcal{D}_K} J(\rho) \tag{7}$$

In practice, if the controller order is fixed according to (2), the objective is not necessarily achievable and $K^* \notin \{K(\rho)\}$, $K_o \neq K^*$ and $J(\rho_o) > 0$. To allow for analysis of the accuracy of the estimated controller parameters, it is assumed that

A1 The objective can be achieved, that is $K^* \in \{K(\rho)\}$. Therefore, it holds that $K_o = K(\rho_o) = K^*$ and $J(\rho_o) = 0$.

2.2. *System identification*

Assume that a set of input, $r(t)$, and output data, $y(t)$, with data length N is available from an open-loop experiment. Suppose that the output is generated as:

$$y(t) = G(q^{-1})r(t) + v(t) \tag{8}$$

where $v(t)$ is the measurement noise.

From the point of view of system identification, many different approaches can be employed to identify the system dynamics. In this paper, an Finite Impulse Response (FIR) model $\hat{G}$ of G will be identified, as the optimization is convex and does not require any prior knowledge on the system structure, except for the length of its impulse response (that however can be inferred from data, if the energy of noise is low).

Introduce the impulse response $g(t)$ of G and $\theta_o = [g(0)\dots g(n-1)]^T$, where n is the length of the impulse response, such that $g(t) \approx 0$, $t \geqslant n$. Note now that (8) can be rewritten as $y(t) \approx \psi^T(t)\theta_o + v(t)$, where

$$\psi(t) = [r(t)\dots r(t-n+1)]^T$$

An FIR estimate of G of length n is given by:

$$\hat{\theta} = \left[\frac{1}{N}\sum_{t=1}^{N}\psi(t)\psi^T(t)\right]^{-1}\frac{1}{N}\sum_{t=1}^{N}\psi(t)y(t) \tag{9}$$

Assume now that

A2 The measurement noise $v(t)$ is uncorrelated with $r(t)$.
A3 The measurement noise can be represented as $v(t) = He(t)$, where $e(t)$ is a zero-mean white noise signal with variance σ^2 and bounded fourth moments. H and H^{-1} are stable filters.
A4 $r(t)$ is persistently exciting of order n and $(1-M)^2G$ has no zero on the imaginary axis.
A5 The FIR model order is such that $n \geqslant n_\rho$.

The estimate (9) provides a unique solution, if **A4** is satisfied, given by $\hat{\theta} = \theta_o + \tilde{\theta}$, where

$$\tilde{\theta} = \left[\frac{1}{N}\sum_{t=1}^{N}\psi(t)\psi^T(t)\right]^{-1}\frac{1}{N}\sum_{t=1}^{N}\psi(t)v(t) \tag{10}$$

This estimate is consistent: $\lim_{N\to\infty}\hat{\theta} = \theta_o$, w.p.1, [15]. Moreover, if $v(t)$ is white, (9) is an ML estimator and the Cramér–Rao lower bound for the variance is achieved. When this is the case, the following principle (Theorem 5.1.1 [16]) holds, regarding all quantities derived from $\hat{\theta}$.

Invariance principle of ML estimation: Let $f : \Theta \to \Omega$ be a function mapping $\theta \in \Theta \in \mathbf{R}^n$ to an interval $\Omega \in \mathbf{R}^m$, with $m \leqslant n$. The invariance principle of ML estimation then states that if $\hat{\theta}$ is a ML estimator of θ, then $f(\hat{\theta})$ is a ML estimator of $f(\theta)$.

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3. MODEL REFERENCE CONTROL DESIGN FROM DATA

3.1. The correlation approach

Consider the scheme in Figure 1, when $v = 0$ and the reference signal $r(t) = u(t)$, with $u(t)$ a white noise of unit variance. This scheme can be used to derive the optimal controller without using any explicit mathematical model of the process.

As a matter of fact, the most important observation at the basis of the CbT rationale is that, in the noiseless setting, the error signal $\varepsilon_c(t, \rho)$ can be directly computed from I/O data as follows:

$$\varepsilon_c(t,\rho) = Mr(t) - (1-M)K(\rho)Gr(t) = Mu(t) - (1-M))K(\rho)y(t)$$

and, assuming **A1** holds, the minimizer of the two norms of $\varepsilon_c(t,\rho)$ is exactly K_o.

When data are collected in a noisy environment, the method resorts to the correlation approach to identify the controller. Specifically, an extended instrumental variable $\zeta(t)$ correlated with $u(t)$ and uncorrelated with $v(t)$ is introduced to decorrelate the error signal $\varepsilon_c(t)$ and $u(t)$. $\zeta(t)$ is defined as

$$\zeta(t) = [u(t+l), \ldots, u(t), \ldots, u(t-l)]^T \tag{11}$$

where l is a sufficiently large integer. The correlation function is defined as

$$f_{N,l}(\rho) = \frac{1}{N}\sum_{t=1}^{N} \zeta(t)\varepsilon_c(t,\rho) \tag{12}$$

and the correlation criterion as

$$J_{N,l}(\rho) = f_{N,l}^T(\rho) f_{N,l}(\rho) \tag{13}$$

In [7], it has been proven that

$$\lim_{N,l\to\infty, l/N\to 0} J_{N,l}(\rho) = J(\rho) \tag{14}$$

for any sufficiently exciting input sequence, if data in $\zeta(t)$ are prefiltered by $L_c(q^{-1})$, defined as

$$L_c(e^{-j\omega}) = \frac{1 - M(e^{-j\omega})}{\Phi_u(\omega)} \tag{15}$$

where $\Phi_u(\omega)$ denotes the spectral density of $u(t)$. Notice that such a prefilter may be noncausal but it can be implemented off-line.

The optimal controller is then defined as $K_{CbT} = K(\hat{\rho}_{CbT})$ with

$$\hat{\rho}_{CbT} = \arg\min_{\rho\in\mathcal{D}_K} J_{N,l}(\rho) \tag{16}$$

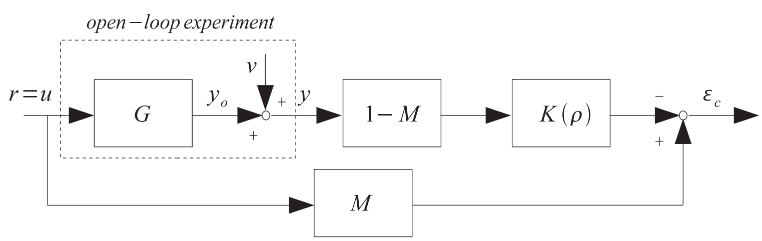

Figure 1. Tuning scheme for Correlation-based Tuning.

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3.2. Model-based model reference control

If a model $\hat{G}$ of the system is available, a model reference controller $\hat{K}$ can be computed as

$$\hat{K} = \frac{M}{\hat{G}(1-M)}$$

However, in any model reference method, this might lead to a high-order controller that may destabilize the system if M is not minimum phase. $\mathcal{H}_2$ control theory can be used to compute a full-order model reference controller followed by a controller order reduction technique to compute a fixed-order controller. The accuracy of the final (fixed-order) controller is difficult to compute.

An alternative design of a fixed-order controller by minimization of the model reference criterion (1) approximated using the model $\hat{G}$ leads to a non-convex optimization approach. The quality of this controller estimate will depend on the initial values of the optimization variables and a fair comparison with data-driven approaches based on convex optimization is not possible. In this paper, the approximate control criterion (6) used in the data-driven approaches is therefore considered to develop a model-based approach that is comparable with the data-driven approaches.

Specifically, the approximate model reference criterion (6) can be approximated using the model $\hat{G}$ of the plant G, by minimizing $\left\|(1-M)[M-K(\rho)(1-M)\hat{G}\right\|$ over ρ. Because a parametric model is available, a simulated output sequence can be generated. This sequence can then be used to approximate the control criterion. This approach has also been used in model reduction, that is [10, 17].

In the following, a high-order parametric model $\hat{G}$ parametrized through $\hat{\theta}$ with an FIR structure is used together with the impulse excitation signal $\delta(t)$ to generate a simulated impulse response sequence, $y_{\hat{\theta}}(t) = \hat{G}\delta(t)$. This simulated output can be used to minimize the approximate model reference criterion

$$\hat{\rho}_{\hat{\theta}} = \arg\min_{\rho \in \mathcal{D}_K} J_{mb}(\rho, \hat{\theta}) \tag{17}$$

$$J_{mb}(\rho, \hat{\theta}) = \frac{1}{N_\delta} \sum_{t=1}^{N_\delta} \left(s(t) - K(\rho)(1-M)^2 y_{\hat{\theta}}(t)\right)^2 \tag{18}$$

where $s(t)$ is the impulse response of $(1-M)M$, that is $s(t) = (1-M)M\delta(t)$ and the number of generated samples $N_\delta \geqslant n$. The error can be written as:

$$s(t) - K(\rho)(1-M)^2 y_{\hat{\theta}}(t) = s(t) - \phi_{\hat{\theta}}^T(t)\rho \tag{19}$$

where the regression vector $\phi_{\hat{\theta}}(t)$ is given by

$$\phi_{\hat{\theta}}(t) = \beta(1-M)^2 y_{\hat{\theta}}(t) = \beta(1-M)^2 G\delta(t) + \beta(1-M)^2 \Delta G\delta(t) \triangleq \phi_o(t) + \tilde{\phi}_{\hat{\theta}}(t) \tag{20}$$

and $\Delta G = \hat{G} - G$. The minimizer of (18) is given by

$$\hat{\rho}_{\hat{\theta}} = \left[\frac{1}{N_\delta}\sum_{t=1}^{N_\delta} \phi_{\hat{\theta}}(t)\phi_{\hat{\theta}}^T(t)\right]^{-1} \frac{1}{N_\delta}\sum_{t=1}^{N_\delta} \phi_{\hat{\theta}}(t)s(t) \tag{21}$$

For simplicity, from now on, let $N_\delta = N$ without loss of generality.

Proposition 1

Assume that **A1, A2, A3, A4, A5** are satisfied and let $N > n$. Then, if the FIR model $\hat{\theta}$ is estimated according to (9) and the controller parameters $\hat{\rho}_\theta$ according to (21),

$$\lim_{N\to\infty} \hat{\rho}_{\hat{\theta}} = \rho_o, \text{ w.p.1.}$$

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Proof
The noise-free signal $s(t)$ can be written as $s(t) = \phi_{\hat{\theta}}^T(t)\rho_o - \tilde{\phi}_{\hat{\theta}}(t)\rho_o$, the estimation error is given by

$$\hat{\rho}_{\hat{\theta}} - \rho_o = -\left[\frac{1}{N}\sum_{t=1}^{N}\phi_{\hat{\theta}}(t)\phi_{\hat{\theta}}^T(t)\right]^{-1}\frac{1}{N}\sum_{t=1}^{N}\phi_{\hat{\theta}}(t)\tilde{\phi}_{\hat{\theta}}^T(t)\rho_o \tag{22}$$

Because $\lim_{N\to\infty}\hat{\theta} = \theta_o$, a continuous function of this variable $f(\hat{\theta})$ converges w.p.1 to $f(\theta_o)$ ([18], page 450). Consequently $\lim_{N\to\infty}\tilde{\phi}_{\hat{\theta}}(t) = 0$, w.p.1, the regressor converges to the noise-free regressor, $\lim_{N\to\infty}\phi_{\hat{\theta}}(t) = \phi_o(t)$, w.p.1, and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\phi_{\hat{\theta}}(t)\phi_{\hat{\theta}}^T(t) = R_o, \quad \text{w.p.1} \tag{23}$$

with R_o defined as

$$R_o = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\phi_o(t)\phi_o^T(t) \tag{24}$$

This matrix has full rank since $N \geqslant n$ and **A4, A5** hold. It follows that $\lim_{N\to\infty}(\hat{\rho}_{\hat{\theta}} - \rho_o) = 0$, w.p.1, which completes the proof. □

4. ACCURACY ANALYSIS

4.1. Variance analysis

For the correlation approach, if **A1** holds, the error between the estimated controller parameters $\hat{\rho}_{CbT}$ and the optimal controller parameters ρ_o is asymptotically normally distributed and the asymptotic covariance matrix of $\sqrt{N}(\hat{\rho}_{CbT} - \rho_o)$ is given by [19]:

$$P_c = \sigma^2(Q^TQ)^{-1}Q^TSQ(Q^TQ)^{-1} \tag{25}$$

where

$$Q = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\zeta(t)\phi_o^T(t)$$

$$S = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}[H^*\zeta(t)][H^*\zeta(t)]^T$$

and $H^* = K^*(1-M)H$.

For model-based control, the accuracy of the estimate $\hat{\rho}_{\hat{\theta}}$ of (21) clearly depends on the accuracy of the estimate of the model parameters $\hat{\theta}$ defined in (9). However, the invariance principle of ML estimation provides a condition on $\hat{\theta}$ that assures that $\hat{\rho}_\theta$ is statistically efficient. As a matter of fact, according to the invariance principle, $\hat{\rho}_\theta$ is an ML estimator of ρ_o if $\hat{\theta}$ is an ML estimator of θ_o. If the measurement noise is white (*i.e.*, $H = 1$), the FIR estimate $\hat{\theta}$ is an ML estimator, whose variance corresponds to the Cramér–Rao bound, and also $\hat{\rho}_\theta$ is a ML estimate. Specifically, the Cramér-Rao bound for the function $f(\theta)$ of the ML estimate θ is given by

$$\frac{\partial f(\theta)}{\partial\theta}P_\theta\frac{\partial f(\theta)}{\partial\theta}$$

where P_θ is the Cramér–Rao bound for the estimate $\hat{\theta}$ [20]. The best variance that can be achieved thus depends on the function $f(\theta)$. Results from asymptotic analysis in system identification can be used to calculate the Cramér–Rao bound for $f(\theta)$ as illustrated by the following Proposition.

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Proposition 2
Assume that $N > n$. Then, if $\hat{\theta}$ is estimated according to (9) and $\hat{\rho}_{\hat{\theta}}$ according to (21), $\sqrt{N}(\hat{\rho}_{\hat{\theta}} - \rho_o)$ is asymptotically normally distributed with covariance matrix P_{mb}:

$$P_{mb} = \sigma^2 R_o^{-1} C R_o^{-1} \tag{26}$$

where C is defined as

$$C = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [H^* \phi_o(t)][H^* \phi_o(t)]^T \tag{27}$$

Proof
The proof is based on Theorem 9.1 of [15]. It can be shown that the estimate $\hat{\rho}_\theta$ satisfies the assumptions of Theorem 9.1 of [15]. A complete proof can be found in [21]. □

According to the previous analysis, the given model-based control design method using any full-order model is statistically efficient if the noise is white, whereas the proposed data-driven technique is not, for any H. In the following, it will be shown that this does not imply that the model-based approach achieves better control performance.

4.2. The control objective

The main idea behind data-driven methods is that the model of the system to control is only an intermediate step toward the final controller tuning phase, and therefore, it might be better to directly focus on the final objective, to avoid the risk of losing some information in under-modeling. According to this mindset, *also the variance of the parameters is only an intermediate step* toward the evaluation of what happens to the control criterion when data are noisy and N is large, but finite.

In this subsection, the effect of noise will be assessed on the capability of the control design criteria of estimating (6). The estimate will be shown to be biased when N is large but finite and therefore, the average model-matching error will be greater than zero even when **A1** holds. It will be also shown that, from this point of view, *a criterion based on an ML estimator of the model is not always statistically better, in terms of the control cost (6), than data-driven design*.

Concerning CbT, the following result holds, already proven in [7].

Proposition 3
For large N, the expected value of the correlation criterion (13) is as follows.

$$\mathbb{E}\left[J_{N,l}(\rho)\right] \approx J(\rho) + \frac{\sigma^2(2l+1)}{2\pi N} \int_{-\pi}^{\pi} \frac{|1-M|^4 \, |K(\rho)|^2 \, |H|^2}{\Phi_u(\omega)} \, d\omega \tag{28}$$

Proof
See [7]. □

The same approach applied to model reference control using model-based formula (21) gives the following bias for the control cost for large and finite N.

Proposition 4
For large N, the expected value of the model-based cost function (21) is as follows.

$$\mathbb{E}\left[J_{mb}(\rho)\right] \approx J(\rho) + \frac{\sigma^2 n}{2\pi N} \int_{-\pi}^{\pi} \frac{|1-M|^4 \, |K(\rho)|^2 \, |H|^2}{\Phi_u(\omega)} \, d\omega \tag{29}$$

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Proof
Consider again the model-based control cost (18) and define $\Delta G = G - \hat{G}$. Because $\hat{y}_\theta(t) = \hat{G}\delta(t)$, where $\delta(t)$ is the discrete-time impulse, the control cost is given by

$$\begin{aligned} J_{mb}(\rho) =& \frac{1}{N}\sum_{t=1}^{N}\left(s(t) - K(\rho)(1-M)^2\hat{y}_\theta(t)\right)^2 = \frac{1}{N}\sum_{t=1}^{N}\left(s(t) - K(\rho)(1-M)^2 G\delta(t)\right)^2 + \\ &+ \frac{1}{N}\sum_{t=1}^{N}\left(K(\rho)(1-M)^2\Delta G\delta(t)\right)^2 \\ &+ \frac{2}{N}\sum_{t=1}^{N}\left(s(t) - K(\rho)(1-M)^2 G\delta(t)\right)\left(K(\rho)(1-M)^2\Delta G\delta(t)\right) \end{aligned}$$

Notice that the first term of the sum is a (noiseless) consistent estimator of $J(\rho)$. Because the estimate of G is consistent, that is $\mathbb{E}[\Delta G] = 0$, then the expectation of $J_{mb}(\rho)$ becomes

$$\mathbb{E}[J_{mb}(\rho)] = J(\rho) + \frac{1}{N}\sum_{t=1}^{N}\mathbb{E}\left[\left(K(\rho)(1-M)^2\Delta G\delta(t)\right)^2\right]$$

and its Parseval counterpart is

$$\mathbb{E}[J_{mb}(\rho)] = J(\rho) + \frac{1}{2\pi}\int_{-\pi}^{\pi}|1-M|^4\,|K(\rho)|^2\,\mathbb{E}\left[|\Delta G|^2\right]\Phi_\delta(\omega)d\omega \tag{30}$$

In the literature [15], it is well known that for high-order models, the following approximation holds

$$\mathbb{E}\left[|\Delta G|^2\right] \approx \frac{n}{N}|H|^2\sigma^2\Phi_u^{-1}(\omega)$$

Moreover, being δ an impulse, $\Phi_\delta(\omega) = 1,\ \forall\omega$ and therefore (29) holds, which completes the proof. $\square$

Propositions 3 and 4 indicate that:

- Both the data-driven and the model-based criteria $J_{N,l}$ and J_{mb} are biased and the bias depends on ρ;
- The bias is composed by an integral term (equal in (29) and (28)) and a coefficient that is different in the two cases;
- Depending on l/n, the bias will be larger in one case or in the other.

The results of the previous analysis will be commented upon in detail in the next subsection.

4.3. Discussion

The results of the last subsection are interesting as they evaluate the average behavior of the model-based and the data-driven controllers from a different view than standard statistical analysis. This new perspective highlights some critical points that should be evaluated before drawing final conclusions about the comparison of model-based and data-driven approaches.

In standard statistical analysis, the performance of an estimator that is asymptotically consistent is evaluated by means of the asymptotic variance. The method that achieves the lowest asymptotic variance is usually considered to be the best estimator. If such an evaluation, combined with the invariance principle of ML estimation, is applied to the controller design methods discussed in this paper, the model-based design approach (that achieves optimal asymptotic variance) can be considered the best estimator. However, the results in Propositions 3 and 4 show that the expectation of the final control criterion is *lower* in the data-driven case, when the model is high-order and $n > 2l + 1$. The reason for this discrepancy is that the analysis based on asymptotic variance does not take into account the other factors affecting the final control criterion, that is l and n. These design parameters offer a trade-off between the minimizer of the real criterion to minimize, that is J, and the

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minimizer of a bias term that is null if $\rho = 0$. Notice that the case of $n > 2l + 1$ is all but unlikely in real world applications. As a matter of fact, l should be close to the length of the impulse response of $M - KG(1 - M)$, which is unknown. However, standing on the assumption that it is possible to match most of M with K, the choice of l equal to the length of the impulse response of M is sufficient. For the condition $n > 2l + 1$ to be satisfied, it is then sufficient that the settling time of the FIR model $\hat{G}$ is larger than that of M or $\hat{G}$ is low-damped (see the example in Section 5).

In standard practice of model-based design, when the system is complex and a low-order model is not sufficient to accurately describe the I/O dynamics, one may think that increasing the order is the best way to find a good model. For what was said previously, one of the main conclusions of this paper is that *this is not generally true if the model has to be used for control design*. A data-driven method, that does not depend on a model of the system, might be a better solution instead.

Furthermore, it should also be considered that the 'order' of a real system is a badly defined concept. Every model is only an approximation of the real world. It follows that the data-driven method might outperform the model-based method also when the model is low order. In the following sections, it will be shown that this might happen even when the model error is very small (see the numerical example) and when standard procedures for system identification are followed (see the experimental example).

Finally, to complete the comparison, the following remarks should be made. Firstly, the data-driven approach is convex if the controller is linearly parameterized, whereas in the model-based approach, both the model and the controller need to be linearly parameterized to obtain convexity. Secondly, the model-based approach requires both the system and the noise model to be correctly parameterized to achieve a statistically efficient estimate, whereas the results on statistical performance of data-driven methods does not require H to be either parameterized or computed. These observations make data-driven techniques appealing for the practical use.

5. NUMERICAL EXAMPLE

5.1. *The benchmark system*

The flexible transmission system proposed as a benchmark in [12] was used in [6, 22] and [23] to illustrate data-driven controller tuning approaches. The same example is used here. The plant is given by the discrete-time model

$$G(q^{-1}) = \frac{0.28q^{-3} + 0.51q^{-4}}{1 - 1.42q^{-1} + 1.59q^{-2} - 1.32q^{-3} + 0.89q^{-4}}$$

The controller structure is given as

$$K(\rho) = \frac{\rho_1 + \rho_2 q^{-1} + \rho_3 q^{-2} + \rho_4 q^{-3} + \rho_5 q^{-4} + \rho_6 q^{-5}}{1 - q^{-1}}$$

Pseudo-random binary sequence (PRBS) signals with unity amplitude are used as input to the system, $r(t)$. The output of the plant is disturbed by zero-mean white noise $v(t)$. Results are given for $N = 1000$, sampling time $T_s = 50\ ms$, and increasing length of the instrumental variable l. A Monte Carlo simulation with 100 experiments is performed, by using a different noise realization for each experiment, for a signal-to-noise ratio of 10 in terms of standard deviation. The noise realizations are the same for all methods. The reference model is defined as

$$M(q^{-1}) = \frac{K(\rho_o)G}{1 + K(\rho_o)G} \tag{31}$$

with

$$\rho_o = [0.2045, -0.2715, 0.2931, -0.2396, 0.1643, 0.0084]^T \tag{32}$$

The optimal controller $K(\rho_o) \in \{K(\rho)\}$ and the objective can be achieved. In Figure 2, the impulse response of G and M are illustrated. Because the number of nonzero samples is (almost) 180 for G

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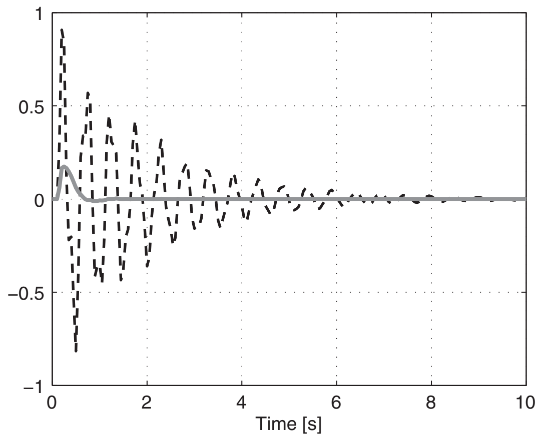

Figure 2. Impulse response of G (dashed) and M (solid).

and (almost) 35 for M, an FIR model with $n = 180$ is used in the model-based approach, whereas for CbT, $l = 35$ is selected.

The results of the 100 Monte Carlo runs for the model-based design using an FIR model with $n = 180$, for CbT with $l = 35$ and for CbT with $l = 130$ are summarized in Table I. Two estimates of $\mathbb{E}[J(\rho)]$ and $\mathbb{E}[J_{mr}(\rho)]$ are calculated, respectively, as

$$V_c = \frac{1}{100}\sum_{i=1}^{100} J(\hat{\rho}^{(i)}) \tag{33}$$

and

$$V_{mr} = \frac{1}{100}\sum_{i=1}^{100} J_{mr}(\hat{\rho}^{(i)}) \tag{34}$$

where $\hat{\rho}^{(i)}$ is the controller parameter vector at the i^{th} Monte Carlo run, and the average trace of the parameter variance

$$V_t = \frac{1}{100}\sum_{i=1}^{100} tr\left\{var[\hat{\rho}^{(i)}]\right\} \tag{35}$$

is also given. For comparison, the performance achieved using low-order models estimated using the Output Error (OE) approach is finally presented.

As predicted by the theory of Propositions 3 and 4, the average of the cost criterion is lower in the data-driven case when $n > 2l + 1$, even if the parameter variance is larger. When l is overestimated, for example when $l = 130$ and $n < 2l + 1$, the variance of the model-based design remains smaller, but now the average of the control cost is also lower than that for the data-driven design.

Table I. Achieved performance (33), (34), and (35) over 100 runs for model-based and data-driven design.

	MB				CbT	
	OE(2,4,3)	OE(2,4,4)	OE(1,4,3)	FIR: $n = 180$	$l = 35$	$l = 130$
V_t ($\times 10^{-3}$)	0.3676	0.3896	0.0144	0.7378	3.0578	0.9646
V_c	0.0064	0.0688	0.1332	0.0573	0.0375	0.0586
V_{mr}	0.0064	0.0759	0.1424	0.0575	0.0376	0.0587

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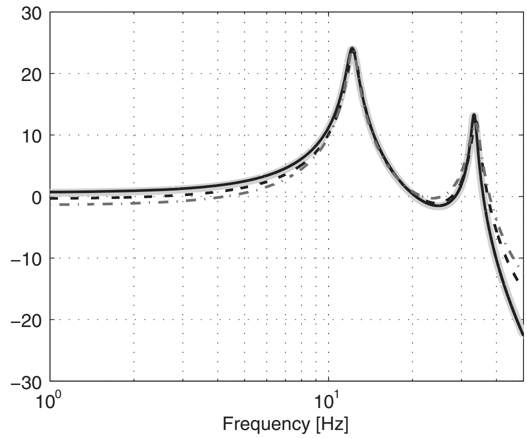

Figure 3. Output error modeling of G: magnitude of the frequency response of the real plant (thick gray line), of the OE(2,4,3) model (solid black line), of the OE(1,4,3) model (dashed black line), and of the OE(2,4,4) model (dash-dotted gray line).

If both the model structure (OE) and the model order are known, the low-order model-based solution outperforms the data-driven approach (note that in this case, because the order of the real system $n = 4$ is low, the result of Proposition 4 no longer holds). However, this does not mean that the model-based approach is more suitable in practice. In the real world, a 'full-order model' does not exist and any description is by definition an approximation. The results presented in Table I show that even a small under-modeling error may jeopardize the control performance. The case where an OE model with the right number of poles and the right relative degree but without zeros introduces a modeling error that is very likely in practice. As a matter of fact, note that the physics usually suggest the order of the model but not the exact number of zeros, especially in discrete time. The identified model is very similar to the 'real' system, as illustrated in Figure 3, and the user may believe that this is an accurate description of the system, but the resulting controller does not yield good control performance. The same observations can be made for the case where the relative degree is 4 instead of 3 (only one more than the 'real' system).

The average of the achieved original model reference criterion J_{mr} is reported to show that the approximate criterion J is a good approximation and that therefore the conclusions hold for the original model reference criterion, even if the analysis has been carried out with respect to the convexified one. The results show that J_{mr} and J are very similar for the FIR and the CbT approaches as well as the low-order model approach when no under-modeling is present. In the case of under-modeling in the model-based approach, the approximation is less good because (18) depends on the model (and not on system) dynamics. As a result, the model reference control cost (1) is larger than (6), which further encourages the use of a data-driven technique.

This example then shows that:

- Standard, statistically efficient model-based approaches achieve better performance than the data-driven solution considered in this paper only if the correct model structure and order are used;
- The data-driven approach can outperform a statistically efficient model-based solution based on a high-order model (if $n > 2l + 1$);
- The data-driven approach can outperform a statistically efficient model-based solution in case of (slight) under-modeling.

6. EXPERIMENTAL TEST

The effectiveness of the proposed approach is demonstrated experimentally on the torsional setup shown in Figure 4, already used to present the CbT theory in [7]. The setup consists of three disks connected by a torsionally flexible shaft. The shaft is driven by a Brushless Servo Motor, whereas

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Figure 4. Torsional setup, ECP Model 205.

the angular displacement of the top disk is measured by an encoder and expressed in degrees. The sampling time is 60 ms.

A set of periodic open-loop data is collected using a zero-mean PRBS input of 255 samples. Five periods of input and output measurements are used for controller design. The controller structure is fixed as a $7th$-order FIR filter. Because the input is the shaft torque and the output the disk position signal, the plant is expected to have at least one integral action and the reference model needs to have unity static gain. Specifically, the model

$$M(q^{-1}) = \frac{0.0765q^{-1}}{(1-0.7q^{-1})^2(1-0.15q^{-1})}$$

has been selected.

As already mentioned, a reasonable reference model can be chosen either by exploiting some (mild) information about the dominant plant dynamics or by means of iterative off-line procedures. In this example, a plot of the open-loop frequency response as estimated from the identification data is sufficient to define the desired bandwidth of M. This information will not be needed to tune the controller parameters.

In this example, three methods are compared. Two of them are the CbT, where $\zeta(t)$ is defined as in (11) with $l = 35$, and the model-based controller design using a high-order FIR model, where the order of FIR is equal to the number of samples in the PRBS signal, that is $n = 255$. Notice that, unlike the simulation example in the previous section, the third method can be neither the 'full-order' model-based design nor a 'reduced-order' model-based design, as in real-life, the 'real order' of a system is a vague (and therefore often abused) concept.

The third method employed to tune the $7th$-order FIR controller is a low-order model-based design, where the model is selected using the best tools of system identification, according to the state of the art. For the complete procedure, see the Appendix. The resulting model is an output error OE(5,6,2). The fit of this model with the estimated frequency response, computed as the ratio of the fast Fourier transforms of the I/O signals, is shown in Figure 5.

Because no 'real' plant G is available to compute the final performance, the methods are evaluated using a closed-loop experiment for each controller while feeding the loop with a PRBS reference

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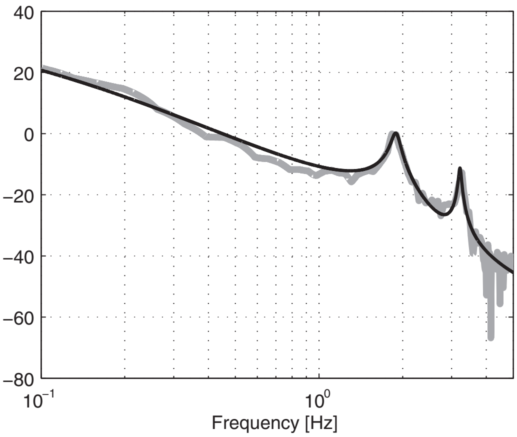

Figure 5. Output error modeling of the torsional plant in Figure 4: magnitude of the frequency response of the real plant estimated from the dataset used for identification (thick gray line) and of the frequency response of the OE(5,6,2) model (solid black line) given by the SYSID procedure given in the Appendix.

Table II. Achieved performance (36) for model-based and data-driven design in the experimental example.

	MB		CbT
	OE(5,6,2)	FIR: $n = 255$	$l = 35$
V_{exp}	1.0915×10^{-3}	5.0942×10^{-3}	0.8486×10^{-3}

signal of 255 samples. From these tests, good estimators of the expected value of J can be computed. Specifically, in this paper, the indicator

$$V_{exp} = \frac{1}{255} \sum_{t=1}^{255} \left(y_\rho(t) - y_M(t) \right)^2 \tag{36}$$

will be employed, where y_M is the output given by the reference model M and y_ρ is the measured output of the closed-loop system with a given controller in the loop. In Table II, the values of (36) for the three considered methods are shown.

Surprisingly, the data-driven method is not only better than the high-order model-based design, as predicted by the theory of this paper because $n >> 2l + 1$, but it also outperforms the model-based control method using a low-order model. This is nothing but another confirmation that in the real world, *no 'full-order model' of a system exists*. This result of the comparison with the low-order model-based method is indeed not general but application-dependent. However, because even small modeling errors may significantly affect the final control performance (see the previous example), it could be concluded that the data-driven approach is a good candidate for controller design in many practical problems.

7. CONCLUSIONS

In this paper, the accuracy of data-driven non-iterative controller tuning is compared with the accuracy of a model-based approach using a ML estimator, in the case where the control objective can be achieved. Data-driven non-iterative controller tuning approaches lead to a nonstandard identification problem, where estimates are consistent also if the control objective cannot be achieved, but they are statistically not optimal [24], that is the Cramér–Rao bound is not reached. It could therefore be argued that, from a statistical point of view, it is always better to first identify a model and then design a model-based controller. However, this assessment of the statistical properties does not look at the final control objective.

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In this paper, it has been shown that the expected value of the final control cost is biased and the bias depends not only on the variance of the controller parameters, but also on some parameters. Specifically, in CbT, the bias is affected by the length of the instrumental variable, whereas in the model-based approach, it is influenced by the model order. It might therefore happen that for large but finite number of data, a data-driven approach achieves a lower control cost than a statistically efficient model-based approach, as illustrated in the proposed numerical example. In the paper, it is also shown that, when applied to real systems, also the best model found via standard identification techniques can be outperformed by a data-driven method. This is because in a real setup, a 'full-order' model does not exists and every description is by definition an approximation of the reality.

The comparison in this paper is clearly limited, as one data-driven method and one specific model-based technique are considered. Furthermore, it is assumed that the control objective can be achieved. This will not be the case in practice, and the performance of different methods will be strongly case dependent. The results of this paper do show that the conclusion from [10] that 'it is never better to estimate the (low-order) model directly from data, compared to estimating it via L_2 model reduction of a high-order FIR model' is true for reduced-order system identification but *does not hold for controller tuning*.

Future work will extend the present analysis to unstable systems and closed-loop identification. An interesting direction of research would be to perform the same comparison between model-based and data-driven filtering [25] or model-based and data-driven fault detection [26].

APPENDIX: IDENTIFICATION OF THE TORSIONAL SYSTEM

In the scientific literature, several methods have been proposed to derive a mathematical description of the behavior of a physical system, starting from empirical observations; see, among others, the reference books [15, 18, 19] and [27]. In this work, the model of the torsional system will be identified using the well-known Prediction Error Method described in [15] and, more specifically, the commands of the System Identification Toolbox for Matlab [13]. This software is a widespread tool for the mathematical description of linear dynamical systems using data and can be seen as the state of the art in most of the industrial practice.

The procedure is as follows.

- Firstly, 5 periods of a 255-sample PRBS signal (see Section 6) are used to feed the system. The output is then collected and the trends are removed.
- The order of the system is estimated using the autoregressive with external input (ARX) method [13] and the order giving the lowest prediction loss function is selected. In the employed experiment, the model order minimizer of the loss function is 6.
- The selected order is checked via analysis of the standard deviation of poles and zeros of autoregressive-moving-average with external input (ARMAX) models of different orders. Specifically, by analyzing in detail the maps of poles and zeros (*e.g.*, using the command '*pzmap*'), it is easy to check if some of them are likely to cancel each other. In the specific case, one cancelation is very likely if the order 7 is identified and two cancelations are evident if the order 8 is selected. Therefore, the order 6 is confirmed by this test.
- The delay (or relative degree) of the system is estimated using the command '*present*' on an ARMAX model of order 6 with unitary delay. The absolute value of the first coefficient of the numerator of the model is within its standard deviation and this coefficient can be set to zero. The final model delay is then 2.
- Different model structures of order 6 are evaluated using the previous dataset, suitably partitioned in an identification dataset (3 periods) and a validation dataset (2 periods). In terms of fitting of the frequency response, good results are given by ARMAX and Box–Jenkins models. In terms of fitting (*i.e.*, using the command '*compare*'), the best choices seem to be ARX or OE. The correlation analysis performed via the command '*resid*' validates only the OE model. It is concluded that the validated OE model of order 6 with a relative degree of 2 provides an appropriate description of the dynamics of the EPC system.

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