Chapter 9: Nonparametric Methods

- t-tests assume Normality ("parametric")
  - ↳ but only for small n (CLT for large n)
- What if you have small n and non-normal data?
  - ↳ use non-parametric methods (work for all data, not just normal)
- Also, not all data are cardinal
- Cardinal variable: Numeric variable where distances b/t points make sense
  - ↳ e.g. body weight
- Ordinal variable: Order matters, but not the specific numbers
  - ↳ E.g. visual acuity 20-20 v.s. 20-30 v.s. 20-40
  - ↳ E.g. Likert scales
    - 1 = very strongly disagree
    - 2 = disagree
    - etc...
- Means/variances not meaningful for ordinal data.
- Nominal variable: categorical variable with no ordering
  - ↳ e.g. death classification, cancer, cardiovascular disease, etc...

- Methods here for (i) ordinal data or (ii) cardinal data with normality violations.

- **Sign Test.** (Non-parametric version of paired t-test)

Suppose we just know (or use) that for two paired observations (A, B), that either A > B, A < B, or A = B

Ex.) Two ointments (A and B). Randomly apply one to left arm and other to right. See which ointment produced more redness for each person.

For $n = 45$, saw 22 with $A < B$
18 with $A > B$
5 with $A = B$

- X_i = redness on arm A
Y_i = redness on arm B
$d_i = X_i - Y_i$
Δ = median(d_i)
↑ population median, not sample median

$H_0: \Delta = 0$
$H_A: \Delta \neq 0$

d_i not observed, only observe $d_i > 0$
< 0
$= 0$

Let $n = \#(d_i > 0) + \#(d_i < 0)$

exclude $d_i = 0$

Let $X = \# d_i > 0$

If H_0 were true, $X \sim \text{Binom}(n, \frac{1}{2})$

Why: "Success" $= d_i > 0$
$Pr(\text{"Success"}) = Pr(d_i > 0) = Pr(d_i > \text{median}(d_i)) = \frac{1}{2}$

if H_0 true

definition of median

- So just use binomial methods (Normal or exact)

Ex.) $\frac{1}{n} X \sim N\left(\frac{1}{2}, \frac{1}{n}\frac{1}{2}(1-\frac{1}{2})\right) = N\left(\frac{1}{2}, \frac{1}{4n}\right)$

if H_0 true

$$\frac{\frac{1}{40} \cdot 22 - \frac{1}{2}}{\sqrt{1/(4 \cdot 40)}} = 0.6325$$

(can also do continuity correction)

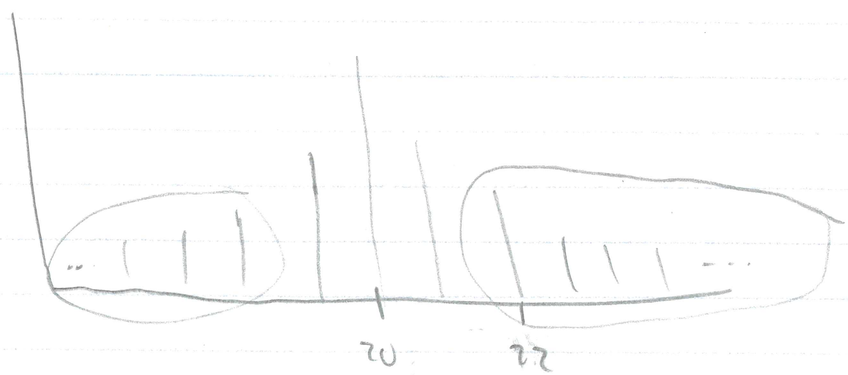
2 · pnorm(-0.6325) = 0.5271

prop.test(x = 22, n = 40, p = 1/2)

p-value = 0.6353

- Or do exact method with binom.test()

p-value = Pr(as extreme or more extreme than what we saw | H_0)



binom.test(x = 22, n = 40, p = $\frac{1}{2}$) = 0.6353

Wilcoxin Signed Rank Test

↳ Also an alternative to the paired t-test

↳ More powerful than the sign test

Idea: Still use d_i, but take into account the rank of the magnitude:

- Observe d_i = −10, −7, −6, −5, 1, 2, 3, 4

↳ techincally #($d_i > 0$) = #($d_i < 0$)

↳ But negative differences are way more negative

Idea: ① Rank observations from 1 to n in terms of $|d_i|$ (smallest to largest)

$\uparrow$ absolute value

② Sum ranks s.t. $d_i > 0$

$$\begin{array}{lcccccccc} d_i & -10, & -7, & -6, & -5, & 1, & 2, & 3, & 4 \\ |d_i| & 8 & 7 & 6 & 5 & 1 & 2 & 3 & 4 \end{array}$$

$\underbrace{1 + 2 + 3 + 4}_{\text{sum}} = 10$

If $H_0: \Delta = 0$ is true, then

$$E[R] = \frac{n(n+1)}{4} \qquad R = \text{rank sum}$$

$$Var(R) = \frac{n(n+1)(2n+1)}{24}$$

$R \sim N(E(R), Var(R))$ for large n

Compare R to null distribution to get p-value.

- Exact methods exist when n is small
- Variations exist when there are ties in d_i
- Note: Null is really that $\Delta = 0$ <u>and</u> d_i are symmetric (though possibly non-normal)
  - Might reject H_0 if $\Delta = 0$ and d_i are skewed.
  - So really only testing H_0: $\Delta = 0$ if d_i are symmetric (checkable via histogram)
  - Otherwise, use sign test.

$\boxed{\text{Wilcoxon Signed Rank test in R}}$

- <u>Wilcoxin Rank-Sum Test</u> (AKA Mann-Whitney U-test)

↑ Nonparametric alternative to two-sample t-test

- Want to test if distribution is shifted in one group or the other

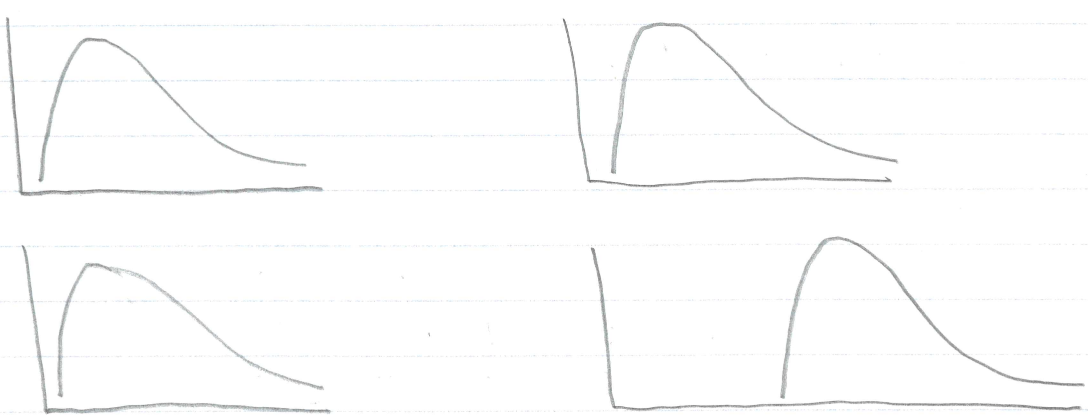
- Let F_1 be the CDF of group 1

  Let F_2 be the CDF of group 2

H_0: $F_1 = F_2$

H_1: $F_1(x) = F_2(x+\Delta)$ for some $\Delta \neq 0$

H_0 H_1



↑ Assume distribution is same in each group, except one is shifted over (for H_0)

- Procedure: Rank all values (<u>not</u> magnitude like before).

Add up ranks in one group.

Let R_1 = sum of ranks in group 1

$$\left.\begin{aligned} E[R_1] &= \frac{n_1(n_1+n_2+1)}{2} \\ Var(R_1) &= \frac{n_1 n_2 (n_1+n_2+1)}{12} \end{aligned}\right] \text{ under } H_0$$

By CLT, for large n_1, $R_1 \sim N(E(R_1), Var(R_1))$

So compare to this distribution to get a p-value

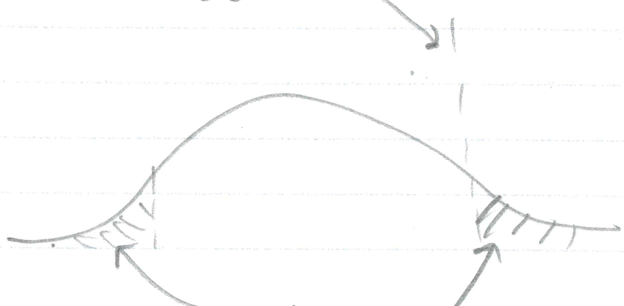
- Eg.) $X = -3, -1, 0, 1, 3$

$Y = -2, 2, 4$

Sample	X	Y	X	X	X	Y	X	Y
Value	-3	-2	-1	0	1	2	3	4
Rank	1	2	3	4	5	6	7	8

add

20



p-value

Intuition (for $n_1 = n_2$): Expect $R_1 \approx R_2$, so if $R_1 \gg R_2$ or $R_1 \ll R_2$, then reject H_0.

But $R_1 + R_2 = \sum_{i=1}^{n_1+n_2} i = \frac{(n_1+n_2)(n_1+n_2+1)}{2}$

So just need to look at distribution of R_1

- For small n, an exact distribution of R_1 (when H_0 is true) is available.
- Modifications exist when there are ties.
- Can use this to compare ordinal data

E.g.) Visual Acuity for individuals with dominant form of retinitis pigmentosa (RP) vs. visual acuity for sex-linked RP.

Visual Acuity: 20-20, 20-25, 20-30, ...

$\boxed{\text{Wilcoxon Rank-Sum in R}}$

$\boxed{\text{Permutation tests in R}}$