Financial Engineering and Risk Management

Mean Variance Optimization

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Overview

- Assets and portfolios
- Quantifying random asset and portfolio returns: mean and variance
- Mean-variance optimal portfolios
- Efficient frontier
- Sharpe ratio and Sharpe optimal portfolios
- Market portfolio
- Capital Asset Pricing Model

Assets and portfolios

Asset $\equiv$ anything we can purchase

- Random price $P(t)$
- Random gross return $R(t) = \frac{P(t+1)}{P(t)}$
- Random net return: $r(t) = R(t) - 1 = \frac{P(t+1)-P(t)}{P(t)}$

Total wealth $W > 0$ distributed over d assets

- w_i = dollar amount in asset i: $w_i > 0 \equiv$ long, $w_i < 0 \equiv$ short
- Net return on a position $\boldsymbol{w}$

$$r_w(t) = \frac{\sum_{i=1}^d R_i(t) w_i - \sum_{i=1}^d w_i}{W} = \frac{\sum_{i=1}^d r_i(t) w_i}{\sum_{i=1}^d w_i} = \sum_{i=1}^d r_i(t) \cdot \underbrace{\frac{w_i}{W}}_{x_i}$$

- portfolio vector $\boldsymbol{x} = (x_1, \ldots, x_d)$: each component can be +ve/-ve
  - x_i = fraction invested in asset $i \Rightarrow \sum_{i=1}^d x_i = 1$

How does one deal with randomness?

Random net return on the portfolio $r_x = \sum_{i=1}^{d} r_i x_i$

How does one "quantify" random returns ?

- Maximize expected return $\mathbb{E}[r_x]$?
- Should one worry about spread around the mean?
- How does one quantify the spread?

Random returns on assets and portfolios

Parameters defining asset returns

- Mean of asset returns: $\mu_i = \mathbb{E}[r_i(t)]$
- Variance of asset returns: $\sigma_i^2 = \mathbf{var}(r_i(t))$
- Covariance of asset returns: $\sigma_{ij} = \mathbf{cov}\big(r_i(t), r_j(t)\big) = \rho_{ij}\sigma_i\sigma_j$
- Correlation of asset returns $\rho_{ij} = \mathbf{cor}\big(r_i(t), r_j(t)\big)$

All parameters assumed to be constant over time.

Parameters defining portfolio returns

- Expected return on a portfolio $\boldsymbol{x} = (x_1, \ldots, x_d)^\top$

$$\mu_x = \mathbb{E}[r_x(t)] = \sum_{i=1}^{d} \mathbb{E}[r_i(t)]x_i = \sum_{i=1}^{d} \mu_i x_i$$

- Variance of the return on portfolio $\boldsymbol{x}$:

$$\sigma_x^2 = \mathbf{var}(r_x(t)) = \mathbf{var}\Big(\sum_{i=1}^{d} r_i x_i\Big) = \sum_{i=1}^{d}\sum_{j=1}^{d} \mathbf{cov}\big(r_i(t), r_j(t)\big)x_i x_j$$

Example

$d = 2$ assets with Normally distributed returns $\mathcal{N}(\mu, \sigma^2)$

$$r_1 \sim \mathcal{N}(1, 0.1) \qquad r_2 \sim \mathcal{N}(2, 0.5) \qquad \mathbf{cor}(r_1, r_2) = \quad 0.25$$

Parameters

$$\mu_1 = 1 \qquad \mu_2 = 2$$
$$\sigma_1^2 = \mathbf{var}(r_1) = 0.1 \qquad \sigma_2^2 = \mathbf{var}(r_2) = 0.5$$
$$\sigma_{12} = \mathbf{cov}(r_1, r_2) = \mathbf{cor}(r_1, r_2)\sigma_1\sigma_2 = \quad 0.25\sqrt{0.05} = 0.0559$$

Portfolio: $(x, 1-x)$

$$\mu_x = \sum_{i=1}^{d} \mu_i x_i = x + 2(1-x)$$

$$\begin{aligned}
\sigma_x^2 &= \sum_{i,j=1}^{d} \sigma_{ij} x_i x_j = \sum_{i=1}^{d} \sigma_i^2 x_i^2 + 2 \sum_{j>i} \sigma_{ij} x_i x_j \\
&= 0.1x^2 + 0.5(1-x)^2 + 2(0.0559)x(1-x)
\end{aligned}$$

Diversification reduces uncertainty

d assets each with $\mu_i \equiv \mu$, $\sigma_i \equiv \sigma$, $\rho_{ij} = 0$ for all $i \neq j$

Two different portfolios

- $\boldsymbol{x} = (1, 0, \ldots, 0)^\top$: everything invested in asset 1
- $\boldsymbol{y} = \frac{1}{d}(1, 1, \ldots, 1)^\top$: equal investment in all assets.

Expected returns of the two portfolios

- $\mu_x = \mathbb{E}[\sum_{i=1}^d \mu_i x_i] = \mu_1 = \mu$
- $\mu_y = \mathbb{E}[\sum_{i=1}^d \mu_i y_i] = \frac{1}{d} \sum_{i=1}^d \mu_i = \mu$

Both have the same expected return!

Variance of returns of the two portfolios

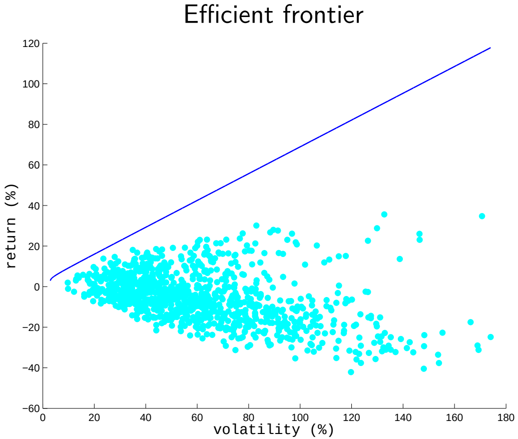
- $\sigma_x^2 = \mathbf{var}(\sum_{i=1}^d r_i x_i) = \sigma^2$
- $\sigma_y^2 = \mathbf{var}(\sum_{i=1}^d r_i y_i) = \sum_{i=1}^d \sigma^2 (\frac{1}{d})^2 = \frac{\sigma^2}{d}$

Diversified portfolio has a much lower variance!

Markowitz mean-variance portfolio selection

Markowitz (1954) proposed that

- "Return" of a portfolio $\equiv$ Expected return μ_x
- "Risk" of a portfolio $\equiv$ volatility σ_x



Efficient frontier $\equiv$ max return for a given risk

How does one characterize the efficient frontier?

How does one compute efficient/optimal portfolios?

Mean variance formulations

- Minimize risk ensuring return $\geq$ target return

$$\begin{array}{rl} \min_x & \sigma_x^2 \\ \text{s.t.} & \mu_x \geq r \end{array} \qquad \equiv \qquad \begin{array}{rl} \min_x & \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \\ \text{s.t.} & \sum_{i=1}^d \mu_i x_i \geq r \\ & \sum_{i=1}^d x_i = 1. \end{array}$$

- Maximize return ensuring risk $\leq$ risk budget

$$\begin{array}{rl} \max_x & \mu_x \\ \text{s.t.} & \sigma_x^2 \leq \bar{\sigma}^2 \end{array} \qquad \equiv \qquad \begin{array}{rl} \max_x & \sum_{i=1}^d \mu_i x_i \\ \text{s.t.} & \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \leq \bar{\sigma}^2, \\ & \sum_{i=1}^d x_i = 1. \end{array}$$

- Maximize a risk-adjusted return

$$\begin{array}{rl} \max_x & \mu_x - \tau \sigma_x^2 \end{array} \qquad \equiv \qquad \begin{array}{rl} \max_x & \sum_{i=1}^d \mu_i x_i - \tau \left(\sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \right) \\ \text{s.t.} & \sum_{i=1}^d x_i = 1. \end{array}$$

$\tau \equiv$ risk-aversion parameter

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Efficient frontier

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Mean-variance for 2-asset market

$d = 2$ assets

- Asset 1: mean return μ_1 and variance σ_1^2
- Asset 2: mean return μ_2 and variance σ_2^2
- Correlation between asset returns ρ

Portfolio: $(x, 1-x)$

$$\mu_x = \sum_{i=1}^{d} \mu_i x_i = \mu_1 x + \mu_2 (1-x)$$

$$\begin{aligned} \sigma_x^2 &= \sum_{i,j=1}^{d} \sigma_{ij} x_i x_j = \sum_{i=1}^{d} \sigma_i^2 x_i^2 + 2 \sum_{j>i} \sigma_{ij} x_i x_j \\ &= \sigma_1^2 x^2 + \sigma_2^2 (1-x)^2 + 2\rho\sigma_1\sigma_2 x(1-x) \end{aligned}$$

Mean-variance for 2-asset market

Minimize risk formulation for the mean-variance portfolio selection problem

$$\begin{array}{rl} \min_x & \sigma_1^2 x^2 + \sigma_2^2 (1-x)^2 + 2\rho\sigma_1\sigma_2 x(1-x) \\ \text{s.t.} & \mu_1 x + \mu_2 (1-x) \; = \; r \end{array}$$

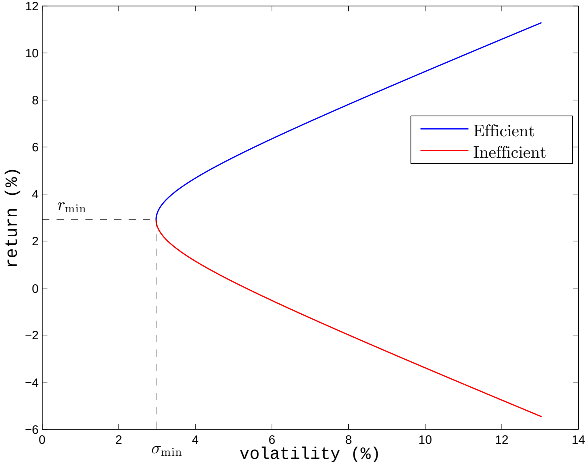
Expected return constraint: $x = \frac{r-\mu_2}{\mu_1-\mu_2}$

Variance:

$$\begin{aligned} \sigma_r^2 &= \sigma_1^2 \left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)^2 + \sigma_2^2 \left(\frac{\mu_1-r}{\mu_1-\mu_2}\right)^2 + 2\rho\sigma_1\sigma_2 \left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)\left(\frac{\mu_1-r}{\mu_1-\mu_2}\right) \\ &= ar^2 + br + c \end{aligned}$$

Explicit expression for the variance as a function of target return r.

Efficient frontier



Only the top half is efficient! why did we get the bottom?

How does one solve the d asset problem?

Computing the optimal portfolio

Mean-variance portfolio selection problem

$$\begin{aligned} \sigma^2(r) = \quad \min_x \quad & \textstyle\sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^d \mu_i x_i = r \\ & \textstyle\sum_{i=1}^d x_i = 1. \end{aligned}$$

Form the Lagrangian with Lagrange multipliers u and v

$$L = \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j - v \left(\sum_{i=1}^d \mu_i x_i - r \right) - u \left(\sum_{i=1}^d x_i - 1 \right)$$

Setting $\frac{\partial L}{\partial x_i} = 0$ for $i = 1, \ldots, d$ gives d equations

$$2 \sum_{j=1}^d \sigma_{ij} x_j - v\mu_i - u = 0, \quad \text{for all } i = 1, \ldots d \quad (*)$$

Can solve the $d+2$ equations in $d+2$ variables: $x_1, \ldots, x_d$, u and v.

Theorem. A portfolio $\boldsymbol{x}$ is mean-variance optimal if, and only if, it is feasible and there exists u and v satisfying $(*)$.

Computing the optimal portfolio

Matrix formulation

$$\underbrace{\begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & \dots & 2\sigma_{1d} & -\mu_1 & -1 \\ 2\sigma_{21} & 2\sigma_{22} & \dots & 2\sigma_{2d} & -\mu_2 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2\sigma_{d1} & 2\sigma_{d2} & \dots & 2\sigma_{dd} & -\mu_d & -1 \\ \mu_1 & \mu_2 & \dots & \mu_d & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \\ v \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r \\ 1 \end{bmatrix}}_{\boldsymbol{b}}$$

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \\ v \\ u \end{bmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

Two fund theorem

Fix two different target returns: $r_1 \neq r_2$

Suppose

- $\boldsymbol{x}^{(1)} = (x_1^{(1)}, \ldots, x_d^{(1)})^\top$ optimal for r_1: Lagrange multipliers (v_1, u_1)
- $\boldsymbol{x}^{(2)} = (x_1^{(2)}, \ldots, x_d^{(2)})^\top$ optimal for r_2: Lagrange multipliers (v_2, u_2)

Consider any other return r

- Choose $\beta = \frac{r - r_1}{r_2 - r_1}$
- Consider the position: $\boldsymbol{y} = (1-\beta)\boldsymbol{x}^{(1)} + \beta \boldsymbol{x}^{(2)}$

$\boldsymbol{y}$ is a portfolio

$$\sum_{i=1}^{d} y_i = (1-\beta)\sum_{i=1}^{d} x_i^{(1)} + \beta \sum_{i=1}^{d} x_i^{(2)} = (1-\beta) + \beta = 1$$

$\boldsymbol{y}$ is feasible for target return r

$$\sum_{i=1}^{d} \mu_i y_i = (1-\beta)\sum_{i=1}^{d} \mu_i x_i^{(1)} + \beta \sum_{i=1}^{d} \mu_i x_i^{(2)} = (1-\beta)r_1 + \beta r_2 = r$$

Two fund theorem (contd)

Set $v = (1-\beta)v_1 + \beta v_2$ and $u = (1-\beta)u_1 + \beta u_2$.

$$\begin{aligned}
2\sum_{j=1}^{d} \sigma_{ij} y_j - v\mu_i - u &= \sum_{j=1}^{d} 2\sigma_{ij}((1-\beta)x_j^{(1)} + \beta x_j^{(2)}) \\
&\quad - \mu_i((1-\beta)v_1 + \beta v_2) - ((1-\beta)u_1 + \beta u_2) \\
&= (1-\beta)\Big(2\sum_{j=1}^{d} \sigma_{ij} x_j^{(1)} - v_1\mu_i - u_1\Big) \\
&\quad + \beta\Big(2\sum_{j=1}^{d} \sigma_{ij} x_j^{(2)} - v_2\mu_i - u_2\Big) = 0
\end{aligned}$$

$\boldsymbol{y}$ is optimal for target return r!

Theorem All efficient portfolios can be constructed by diversifying between any two efficient portfolios with different expected returns.

Why are there so many funds in the market?

Efficient frontier

The optimal portfolio for target return r

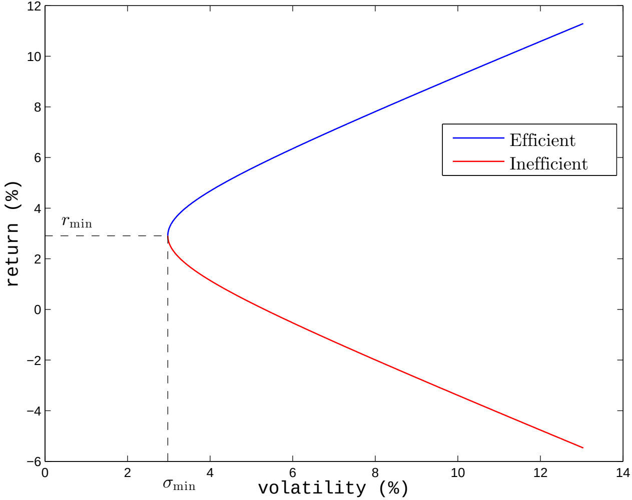
$$
\begin{aligned}
\boldsymbol{y}^* &= \left(\frac{r_2 - r}{r_2 - r_1}\right)\boldsymbol{x}^{(1)} + \left(\frac{r - r_1}{r_2 - r_1}\right)\boldsymbol{x}^{(2)} \\
&= r\underbrace{\left(\frac{\boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}}{r_2 - r_1}\right)}_{\boldsymbol{g}} + \underbrace{\left(\frac{r_2\boldsymbol{x}^{(1)} - r_1\boldsymbol{x}^{(2)}}{r_2 - r_1}\right)}_{\boldsymbol{h}} \\
y_i^* &= rg_i + h_i, \qquad i = 1, \ldots, d.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sigma^2(r) &= \sum_{i=1}^{d}\sum_{j=1}^{d}\sigma_{ij}(rg_i + h_i)(rg_j + h_j) \\
&= r^2\Big(\sum_{i=1}^{d}\sum_{j=1}^{d}\sigma_{ij}g_ig_j\Big) + 2r\Big(\sum_{i=1}^{d}\sum_{j=1}^{d}\sigma_{ij}g_ih_j\Big) + \Big(\sum_{i=1}^{d}\sum_{j=1}^{d}\sigma_{ij}h_ih_j\Big)
\end{aligned}
$$

The d-asset frontier has the same structure as the 2-asset frontier.

Efficient frontier



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Mean-variance with a risk-free asset

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Mean Variance with a risk-free asset

New asset: pays net return r_f with no risk (deterministic return)

x_0 = fraction invested in the risk-free asset

Mean-variance problem: x_0 does not contribute to risk.

$$\begin{array}{ll} \text{max} & \left(r_f x_0 + \sum_{i=1}^d \mu_i x_i\right) - \tau\Big(\sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \Big) \\ \text{s. t.} & x_0 + \sum_{i=1}^d x_i = 1. \end{array}$$

Only meaningful for $r \geq r_f$

Substituting $x_0 = 1 - \sum_{i=1}^d x_i$ we get

$$\text{max} \quad r_f + \sum_{i=1}^d (\mu_i - r_f) x_i - \tau\Big(\sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \Big)$$

$\hat{\mu}_i = \mu_i - r_f$ = excess return of asset i

Mean-variance optimal portfolio

Taking derivatives we get

$$\hat{\mu}_i - 2\tau \sum_{j=1}^{d} \sigma_{ij} x_j = 0, \quad i = 1, \ldots, d.$$

Matrix formulation

$$2\tau \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \ldots & \sigma_{dd} \end{bmatrix}}_{\boldsymbol{V}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_d \end{bmatrix}}_{\hat{\boldsymbol{\mu}}} \quad \Rightarrow \quad \boldsymbol{x}(\tau) = \frac{1}{2\tau} \boldsymbol{V}^{-1} \hat{\boldsymbol{\mu}}$$

The family of frontier portfolios as a function of τ:

$$\left\{ \left(1 - \sum_{i=1}^{d} x_i(\tau),\ \boldsymbol{x}(\tau) \right) : \tau \geq 0 \right\}$$

One-fund theorem

The positions in the risky assets in the frontier portfolio

$$\boldsymbol{x} = \frac{1}{2\tau}\boldsymbol{V}^{-1}\hat{\boldsymbol{\mu}}$$

do not add up to 1.

Define a portfolio of risky assets by dividing $\boldsymbol{x}$ by the sum of its components.

$$\boldsymbol{s}^* = \left(\frac{1}{\sum_{i=1}^{d} x_i}\right)\boldsymbol{x} = \left(\frac{1}{\frac{1}{2\tau}\sum_{i=1}^{d}(\boldsymbol{V}^{-1}\hat{\boldsymbol{\mu}})_i}\right)\left(\frac{1}{2\tau}\boldsymbol{V}^{-1}\hat{\boldsymbol{\mu}}\right)$$

The portfolio $\boldsymbol{s}^*$ is independent of τ! Since $\sum_{i=1}^{d} x_i = 1 - x_0$, $\boldsymbol{x} = (1 - x_0)\boldsymbol{s}^*$.

Family of frontier portfolios $= \{(x_0, (1 - x_0)\boldsymbol{s}^*) : x_0 \in \mathbb{R}\}$

Theorem All efficient portfolios in a market with a risk-free asset can be constructed by diversifying between the risk-less asset and the single portfolio $\boldsymbol{s}^*$.

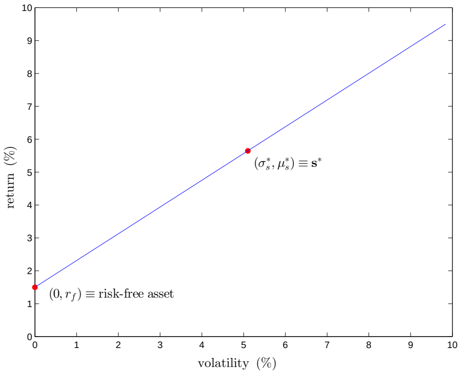
Efficient frontier with risk-free asset

Return and risk of portfolio $\boldsymbol{s}^*$: $\mu_s^* = \sum_{i=1}^d \mu_i s_i^*$, $\quad \sigma_s^* = \sqrt{\sum_{i=1}^d \sum_{j=1} \sigma_{ij}^2 s_i^* s_j^*}$

Return on a generic frontier portfolio: x_0 in risk-free and $(1 - x_0)$ in $\boldsymbol{s}^*$

$$\mu_x = x_0 r_f + (1 - x_0)\mu_s^* \qquad \sigma_x = (1 - x_0)\sigma_s^*$$

Efficient Frontier



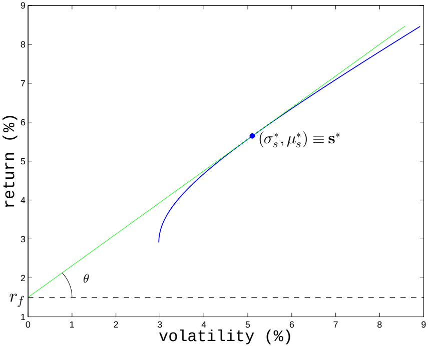
Straight line with an intercept r_f at $\sigma = 0$ and slope

$$m = \frac{\mu_s - r_f}{\sigma_s}$$

How does this relate to the frontier with only risky assets?

Does the portfolio $\boldsymbol{s}^*$ have an economic interpretation?

Efficient frontier with risk-free asset



s^* must be an efficient risky portfolio

The efficient frontier with a risk-free asset must be tangent to the efficient frontier with only risky assets.

The portfolio s^* maximizes the angle θ or equivalently

$$\tan(\theta) = \frac{\sum_{i=1}^{d} \mu_i x_i - r_f}{\sqrt{\sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} x_i x_j}} = \frac{\text{expected excess return}}{\text{volatility}}$$

Sharpe Ratio

Definition. The Sharpe ratio of a portfolio or an asset is the ratio of the expected excess return to the volatility. The Sharpe optimal portfolio is a portfolio that maximizes the Sharpe ratio.

The portfolio $\boldsymbol{s}^*$ is a Sharpe optimal portfolio

$$\boldsymbol{s}^* = \underset{\left\{\boldsymbol{x}: \sum_{i=1}^{d} x_i = 1\right\}}{\operatorname{argmax}} \left\{ \frac{\mu_x - r_f}{\sigma_x} \right\}$$

Investors diversify between the risk-free asset and the Sharpe optimal portfolio.

The investment in the various risky assets are in fixed proportions ... prices/returns should be correlated! This insight leads to the Capital Asset Pricing Model.

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Capital Asset Pricing Model

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Market Portfolio

Definition. Let C_i, $i = 1, \ldots, d$, denote the market capitalization of the d assets. Then the market portfolio $\boldsymbol{x}^{(m)}$ is defined as follows.

$$x_i^{(m)} = \frac{C_i}{\sum_{j=1}^{d} C_j}, \quad i = 1, \ldots, d.$$

Let μ_m denote the expected net rate of return on the market portfolio, and let σ_m denote the volatility of the market portfolio.

Suppose all investors in the market are mean-variance optimizers. Then all of them invest in the Sharpe optimal portfolio $\boldsymbol{s}^*$. Let

$$\begin{aligned} w^{(k)} &= \text{wealth of the } k\text{-th investor} \\ x_0^{(k)} &= \text{fraction of wealth of the } k\text{-investor in the risk-free asset} \end{aligned}$$

Then

$$C_i = \sum_k w^{(k)}(1 - x_0^{(k)}) s_i^*$$

The market portfolio $\boldsymbol{x}^{(m)}$ = Sharpe optimal portfolio $\boldsymbol{s}^*$!

Capital Market Line

Capital market line is another name for the efficient frontier with risk-free asset

Recall: Efficient frontier = line though the points $(0, r_f)$ and (σ_m, μ_m)

Slope of the capital market line

$$m_{\text{CML}} = \frac{\mu_m - r_f}{\sigma_m} = \text{maximum achievable Sharpe ratio}$$

m_{CML} is frequently called the price of risk. It is used to compare projects.

Example. Suppose the price of a share of an oil pipeline venture is $\$875$. It is expected to yield $\$1000$ in one year, but the volatility $\sigma = 40\%$. The current interest rate $r_f = 5\%$, the expected rate of return on the market portfolio $\mu_m = 17\%$ and the volatility of the market $\sigma_m = 12\%$. Is the oil pipeline worth considering?

$$r_{oil} = \frac{1000}{875} - 1 = 14\% \ll \bar{r} = r_f + \left(\frac{\mu_m - r_f}{\sigma_m}\right)\sigma = 45\%$$

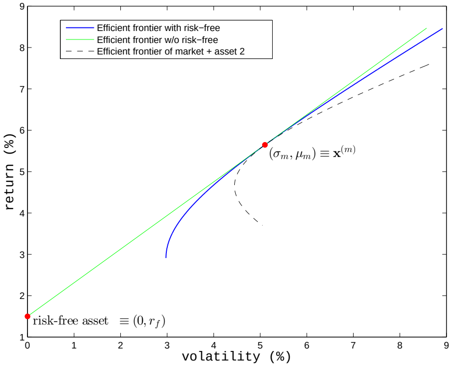
Not worth considering!

Inferring asset returns from market returns

An asset is a portfolio: asset $j \equiv \boldsymbol{x}^{(j)} = (0, \ldots, 1, \ldots, 0)^\top$, 1 in the j-th position.

Diversify between $\boldsymbol{x}^{(j)}$ and market portfolio $\boldsymbol{x}^{(m)}$: $\gamma \boldsymbol{x}^{(j)} + (1-\gamma)\boldsymbol{x}^{(m)}$

- return $\mu_\gamma = \gamma \mu_j + (1-\gamma)\mu_m$
- volatility $\sigma_\gamma = \sqrt{\gamma^2 \sigma_j^2 + (1-\gamma)^2 \sigma_m^2 + 2\sigma_{jm}\gamma(1-\gamma)}$



All three curves are tangent at (σ_m, r_m)

Slope of the capital market line

$$m_{\mathsf{CML}} = \frac{\mu_m - r_f}{\sigma_m}$$

Slope of the frontier generated by asset j and market portfolio $\boldsymbol{x}^{(m)}$

$$\frac{d\mu_\gamma}{d\sigma_\gamma} = \frac{\frac{d\mu_\gamma}{d\gamma}}{\frac{d\sigma_\gamma}{d\gamma}} = \frac{\mu_j - \mu_m}{\frac{\gamma\sigma_j^2 - (1-\gamma)\sigma_m^2 + (1-\gamma)\sigma_{jm} - \gamma\sigma_{jm}}{\sqrt{\gamma^2\sigma_j^2 + (1-\gamma)^2\sigma_m^2 + 2\sigma_{jm}\gamma(1-\gamma)}}}$$

$$\left.\frac{d\mu_\gamma}{d\sigma_\gamma}\right|_{\gamma=0} = \frac{\mu_j - \mu_m}{\frac{\sigma_{jm} - \sigma_m^2}{\sigma_m}}$$

Equating slopes at $\gamma = 0$ we get the following result:

$$\mu_j - r_f = \underbrace{\left(\frac{\sigma_{jm}}{\sigma_m^2}\right)}_{\text{beta of asset } j} (\mu_m - r_f)$$

This pricing formula is called the Capital Asset Pricing Model (CAPM).

Connecting CAPM to regression

Regress the excess return $r_j - r_f$ of asset j on the excess market return $r_m - r_f$

$$(r_j - r_f) = \alpha + \beta(r_m - r_f) + \epsilon_j$$

Parameter estimates

- coefficient $\beta_j = \frac{\mathbf{cov}(r_j - r_f, r_m - r_f)}{\mathbf{var}(r_m - r_f)} = \frac{\sigma_{jm}}{\sigma_m^2}$
- intercept $\alpha_j = (\mathbb{E}[r_j] - r_f) - \beta(\mathbb{E}[r_m] - r_f) = (\mu_j - r_f) - \beta(\mu_m - r_f)$.
- residuals ϵ_j and $(r_m - r_f)$ are uncorrelated, i.e. $\mathbf{cor}(\epsilon_j, r_m - r_f) = 0$.
- CAPM implies that $\alpha_j = 0$ for all assets.
- Effective relation: $r_j - r_f = \beta_j(r_m - r_f) + \epsilon_j$

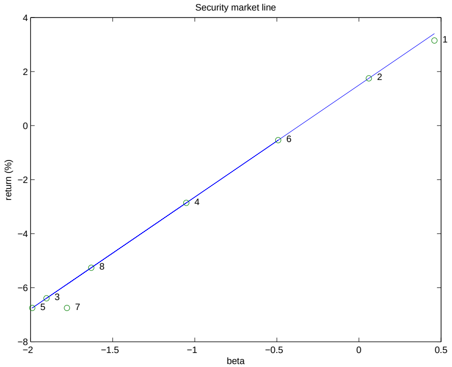
Decomposition of risk

$$\begin{aligned}
\mathbf{var}(r_j - r_f) &= \beta_j^2 \mathbf{var}(r_m - r_f) + \mathbf{var}(\epsilon) \\
\sigma_j^2 &= \underbrace{\beta_j^2 \sigma_m^2}_{\text{market risk}} + \underbrace{\mathbf{var}(\epsilon)}_{\text{residual risk}}
\end{aligned}$$

Only compensated for taking on market risk and not residual risk

Security Market Line

Plot of the historical returns on an asset vs $r_f + \beta(\mu_m - r_f)$



The assets are labeled in the order they appears in the spreadsheet.

All assets should lie on the security line if CAPM holds. So why the discrepancy?

Assumptions underlying CAPM

- All investors have identical information about the uncertain returns.
- All investors are mean-variance optimizers (or the returns are Normal)
- The markets are in equilibrium.

Leveraging deviations from the security market line

- Jensen's index or alpha

$$\alpha = (\hat{\mu}_j - r_f) - \beta_j(\hat{\mu}_m - r_f)$$

  hold long if positive, short otherwise
- Sharpe ratio of a stock

$$s_j = \frac{\hat{\mu}_j - r_f}{\hat{\sigma}_j}$$

  hold long if $> m_{\text{MCM}}$, short otherwise.

CAPM as a pricing formula

Suppose the payoff from an investment in 1yr is X. What is the fair price for this investment.

Let $r_X = \frac{X}{P} - 1$ denote the net rate of return on X. The beta of X is given by

$$\beta_X = \frac{\mathbf{cov}(r_X, r_m)}{\sigma_m^2} = \frac{1}{P}\frac{\mathbf{cov}(X, r_m)}{\sigma_m^2}$$

Suppose CAPM holds. Then $\mu_X = \mathbb{E}[r_X]$ must lie on the security market line, i.e.

$$\begin{aligned} \mu_X &= r_f + \beta_X(r_m - r_f) \\ \frac{\mathbb{E}[X]}{P} - 1 &= r_f + \frac{1}{P}\frac{\mathbf{cov}(X, r_m)}{\mathbf{var}(r_m)}(\mu_m - r_f) \end{aligned}$$

Rearranging terms:

$$P = \frac{\mathbb{E}[X]}{1 + r_f} - \frac{\mathbf{cov}(X, r_m)}{(1 + r_f)\mathbf{var}(r_m)}(\mu_m - r_f)$$