Divide and Conquer

1. Cut whole problem into parts
2. Assemble solutions to subproblems into solution to the whole problem.

Why is Divide and Conquer faster?

The Karatsuba Algrothim:

$$
\begin{array}{r}
{}^{1\,2\,2}\quad\\
1\,2\,3\,4\\
\times\ 5\,6\,7\,8\\
\hline
9\,8\,7\,2\\
8\,6\,3\,8\,0\\
7\,4\,0\,4\,0\,0\\
+\ 6\,1\,7\,0\,0\,0\,0\\
\hline
7\,0\,0\,6\,6\,5\,2
\end{array}
$$

n rows

$n \cdot n + n$ steps

$= n^2$

Need new algrothim to multiply two numbers:

Time: $O(n^{1.59})$

← n digits →

a-high	a-low
b-high	b-low

$\frac{n}{2}$ $\frac{n}{2}$

Note: When you do $a_h \cdot b_l + a_l \cdot b_h$

$$
\begin{array}{cccc}
 & a_h \cdot b_l & a_l \cdot b_l & \\
+ & a_h \cdot b_h & a_l \cdot b_h & \\
\hline
a_h \cdot b_h & (a_h \cdot b_l + a_l \cdot b_h) & & a_l \cdot b_l \\
2 & 3 & 4 & 1
\end{array}
$$

	a-low	a-high
b-high	4	2
b-low	1	3

⇒ Clean up 3 at the price of one.

Trick:

$$\begin{array}{rll} & a_h \cdot b_l & a_l \cdot b_h \\ + & a_h \cdot b_h & a_l \cdot b_l \\ \hline a_h \cdot b_h & (a_h \cdot b_l + a_l \cdot b_l) & a_l \cdot b_h \end{array}$$

Each op is $\frac{n}{2} \cdot \frac{n}{2}$

Worst-Case Time Complexity

1. Formula:

$$T_w(n) = 3 \cdot T_w\left(\frac{n}{2}\right) + O(n)$$

2. Guess

$$T_w(n) = O(n^x) = c \cdot n^x \quad \text{for some } c > 0$$

I.H. $\forall i < n,\ T_w(i) \leq c \cdot i^2$

3. Check

$$T_w(n) = 3 \cdot T_w\left(\frac{n}{2}\right) + O(n)$$

$$c \cdot n^x \geq 3 \cdot c \cdot \left(\frac{n}{2}\right)^x + a \cdot n$$

$$c \cdot n^x \geq 3 \cdot c \cdot \frac{n^x}{2^x} + a \cdot n$$

$$1 \geq 3 \cdot \frac{1}{2^x} + \frac{a \cdot n}{c \cdot n^x}$$

$$1 \geq 3 \cdot \frac{1}{2^x}$$

$$2^x \geq 3$$

$$x \geq \log_2(3)$$

$$x \geq 1.59$$

So worst-case $O(n^{1.59})$