

Divide and Conquer

1. Cut whole problem into parts
2. Assemble solutions to subproblems into solution to the whole problem.

Why is Divide and Conquer faster?

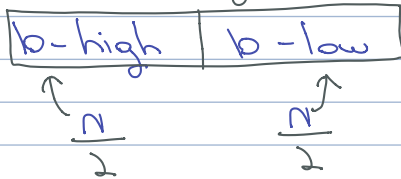
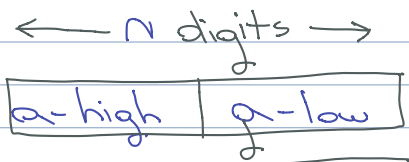
The Karatsuba Algorithm:

$$\begin{array}{r}
 \overset{1}{1} \overset{2}{2} \overset{2}{3} \overset{4}{4} \\
 \times \underset{5}{5} \underset{6}{6} \underset{7}{7} \underset{8}{8} \\
 \hline
 9 \ 8 \ 7 \ 2 \quad \uparrow \\
 8 \ 6 \ 3 \ 8 \ 0 \quad \uparrow \text{ rows} \\
 7 \ 4 \ 0 \ 4 \ 0 \ 0 \quad \uparrow \\
 + \ 6 \ 1 \ 7 \ 0 \ 0 \ 0 \ 0 \quad \downarrow \\
 \hline
 7 \ 0 \ 0 \ 6 \ 6 \ 5 \ 2
 \end{array}$$

$n \cdot n + n \text{ steps}$
 $= n^2$

Need new algorithm to multiply two numbers:

Time: $O(n^{1.59})$

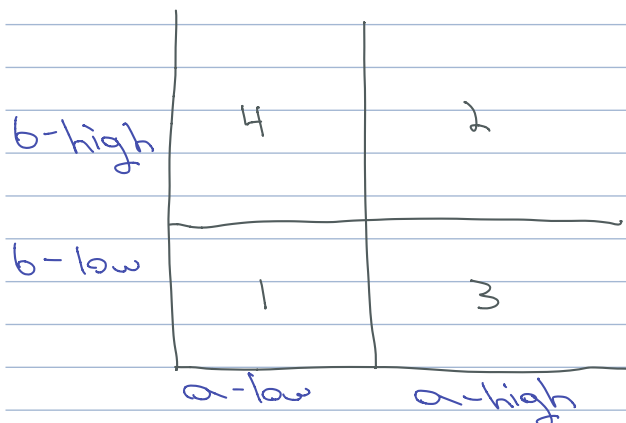


Note: When you do $a_h \cdot b_l + a_l \cdot b_h$

$a_h \cdot b_l \quad a_l \cdot b_l$

+ $a_h \cdot b_h \quad a_l \cdot b_h$

$a_h \cdot b_h$	$(a_h \cdot b_l + a_l \cdot b_h)$	$a_l \cdot b_l$
2	3	4



⇒ Clean up 3 at the price of one.

Trick:

$$\begin{array}{r} a_h \cdot b_l \quad a_l \cdot b_h \\ + \quad a_h \cdot b_h \quad a_l \cdot b_l \\ \hline a_h \cdot b_h \quad (a_h \cdot b_l + a_l \cdot b_l) \quad a_l \cdot b_h \end{array}$$

Each op is $\frac{n}{2} \cdot \frac{n}{2}$

Worst-Case Time Complexity

1. Formula:

$$T_w(n) = 3 \cdot \frac{T_w(n)}{2} + O(n)$$

2. Guess

$$T_w(n) = O(n^x) = c \cdot n^x \quad \text{for some } c > 0$$

$$\text{I.H. } \forall i < n, T_w(i) \leq c \cdot i^2$$

3. Check

$$T_w(n) = 3 \cdot \frac{T_w(n)}{2} + O(n)$$

$$c \cdot n^x \geq 3 \cdot c \cdot \left(\frac{n}{2}\right)^x + a \cdot n$$

$$c \cdot n^x \geq 3 \cdot c \cdot \frac{n^x}{2^x} + a \cdot n$$

$$1 \geq 3 \cdot \frac{1}{2^x} + \frac{a \cdot n}{c \cdot n^x}$$

$$1 \geq 3 \cdot \frac{1}{2^x}$$

$$2^x \geq 3$$

$$x \geq \log_2(3)$$

$$x \geq 1.59$$

So worst-case $O(n^{1.59})$