1-STEP GROWTH RATE

1. FTCS

2D FTCS:

Formula: $U_{a,b}^{n+1} = U_{a,b}^{n} + C(U_{a+1,b}^{n} + U_{a-1,b}^{n} + U_{a,b+1}^{n} + U_{a,b-1}^{n} - 4U_{a,b}^{n})$

Assume $U_{a,b}^{n} = G^n e^{ija\Delta x} e^{ikb\Delta y}, j = 1,2...N, k = 1,2...N$

Let $F = e^{ija\Delta x} e^{ikb\Delta y}$

Substitution into FTCS formula gives us:

$$G^{n+1}F = G^n F + CG^n(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} - 4F)$$

$$\frac{G^{n+1}}{G^n} = 1 + C(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} - 4)$$

$$\frac{G^{n+1}}{G^n} = 1 + C[2\cos(j\Delta x) + 2\cos(k\Delta y) - 4]$$

The worst case is when the cosine factors are both -1, and the growth becomes $|1-8C| \le 1$. Solving for the condition on C, we get $-1 \le 1-8C \le 1$. Finally, $0 < C \le \frac{1}{4}$.

3D FTCS:

Formula:

$$U_{a,b,c}^{n+1} = U_{a,b,c}^{n} + C(U_{a+1,b,c}^{n} + U_{a-1,b,c}^{n} + U_{a,b+1,c}^{n} + U_{a,b-1,c}^{n} + U_{a,b,c+1}^{n} + U_{a,b,c-1}^{n} - 6U_{a,b,c}^{n})$$

Assume $U_{a,b,c}^{n} = G^n e^{ija\Delta x} e^{ikb\Delta y} e^{ilc\Delta z}, j = 1,2...N, k = 1,2...N, l = 1,2...N$

Let $F = e^{ija\Delta x} e^{ikb\Delta y} e^{ilc\Delta z}$

Substitution into FTCS formula gives us:

$$G^{n+1}F = G^n F + CG^n(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} + Fe^{il\Delta z} + Fe^{-il\Delta z} - 6F)$$

$$\frac{G^{n+1}}{G^n} = 1 + C(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} + e^{il\Delta z} + e^{-il\Delta z} - 6)$$

$$\frac{G^{n+1}}{G^n} = 1 + C[2\cos(j\Delta x) + 2\cos(k\Delta y) + 2\cos(l\Delta z) - 6]$$

The worst case is when the cosine factors are all -1, and the growth becomes $|1-12C| \le 1$. Solving for the condition on C, we get $-1 \le 1-12C \le 1$. Finally,

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$0 < C \le \frac{1}{6}$.

2. BECS

1D BECS:

Formula: $U_a^{n+1} = U_a^n + C(U_{a+1}^{n+1} + U_{a-1}^{n+1} - 2U_a^{n+1})$

Assume $U_a^n = G^n e^{ija\Delta x}, j = 1,2...N$

Let $F = e^{ija\Delta x}$

Substitution into BECS formula gives us:

$G^{n+1}F = G^n F + CG^{n+1}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} - 2F)$

$[1 - C(e^{ij\Delta x} + e^{-ij\Delta x} - 2)]G^{n+1} = G^n$

$\frac{G^{n+1}}{G^n} = \frac{1}{1 - C(e^{ij\Delta x} + e^{-ij\Delta x} - 2)} = \frac{1}{1 - C[2\cos(j\Delta x) - 2]}$

$0 \le -[2\cos(j\Delta x) - 2] \le 4$

As long as C is greater than 0, it is unconditionally stable.

$\frac{1}{4C+1} \le \left|\frac{G^{n+1}}{G^n}\right| < 1$

2D BECS:

Formula: $U_{a,b}^{n+1} = U_{a,b}^n + C(U_{a+1,b}^{n+1} + U_{a-1,b}^{n+1} + U_{a,b+1}^{n+1} + U_{a,b-1}^{n+1} - 4U_{a,b}^{n+1})$

Assume $U_{a,b}^n = G^n e^{ija\Delta x} e^{ikb\Delta y}, j = 1,2...N, k = 1,2...N$

Let $F = e^{ija\Delta x} e^{ikb\Delta y}$

Substitution into BECS formula gives us:

$G^{n+1}F = G^n F + CG^{n+1}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} - 4F)$

$\frac{G^{n+1}}{G^n} = \frac{1}{1 - C(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} - 4)}$

$\frac{G^{n+1}}{G^n} = \frac{1}{1 - C[2\cos(j\Delta x) + 2\cos(k\Delta y) - 4]}$

Same as 1D case, unconditionally stable.

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$$\frac{1}{8C+1} \le \left|\frac{G^{n+1}}{G^n}\right| < 1$$

3D BECS:

Formula:

$$U^{n+1}_{a,b,c} = U^n_{a,b,c} + C(U^{n+1}_{a+1,b,c} + U^{n+1}_{a-1,b,c} + U^{n+1}_{a,b+1,c} + U^{n+1}_{a,b-1,c} + U^{n+1}_{a,b,c+1} + U^{n+1}_{a,b,c-1} - 6U^{n+1}_{a,b,c})$$

Assume $U^n_{a,b,c} = G^n e^{ija\Delta x} e^{ikb\Delta y} e^{ilc\Delta z}$, $j = 1,2...N, k = 1,2...N, l = 1,2...N$

Let $F = e^{ija\Delta x} e^{ikb\Delta y} e^{ilc\Delta z}$

Substitution into BECS formula gives us:

$$G^{n+1}F = G^n F + CG^{n+1}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} + Fe^{il\Delta z} + Fe^{-il\Delta z} - 6F)$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - C(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} + e^{il\Delta z} + e^{-il\Delta z} - 6)}$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - C[2\cos(j\Delta x) + 2\cos(k\Delta y) + 2\cos(l\Delta z) - 6]}$$

Same as 1D and 2D cases, unconditionally stable.

$$\frac{1}{12C+1} \le \left|\frac{G^{n+1}}{G^n}\right| < 1$$

3. Crank-Nicolson

1D CN:

Formula: $U^{n+1}_a = U^n_a + \frac{C}{2}(U^{n+1}_{a+1} + U^{n+1}_{a-1} - 2U^{n+1}_a + U^n_{a+1} + U^n_{a-1} - 2U^n_a)$

Assume $U^n_a = G^n e^{ija\Delta x}$, $j = 1,2...N$

Let $F = e^{ija\Delta x}$

Substitution into CN formula gives us:

$$G^{n+1}F = G^n F + \frac{C}{2}G^{n+1}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} - 2F) + \frac{C}{2}G^n(Fe^{ij\Delta x} + Fe^{-ij\Delta x} - 2F)$$

$$[1 - \frac{C}{2}(e^{ij\Delta x} + e^{-ij\Delta x} - 2)]G^{n+1} = [1 + \frac{C}{2}(e^{ij\Delta x} + e^{-ij\Delta x} - 2)]G^n$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + C[\cos(j\Delta x) - 1]}{1 - C[\cos(j\Delta x) - 1]}$$

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$$-2 \le \cos(j\Delta x) - 1 \le 0$$

$$1 + C[\cos(j\Delta x) - 1] < 1 - C[\cos(j\Delta x) - 1]$$

$$0 < \left|\frac{G^{n+1}}{G^n}\right| < 1$$

It is unconditionally stable.

2D CN:

Formula:

$$U_{a,b}^{n+1} = U_{a,b}^{n} + \frac{C}{2}(U_{a+1,b}^{n+1} + U_{a-1,b}^{n+1} + U_{a,b+1}^{n+1} + U_{a,b-1}^{n+1} - 4U_{a,b}^{n+1} + U_{a+1,b}^{n} + U_{a-1,b}^{n} + U_{a,b+1}^{n} + U_{a,b-1}^{n} - 4U_{a,b}^{n})$$

Assume $U_{a,b}^{n} = G^n e^{ija\Delta x} e^{ikb\Delta y}, j = 1,2...N, k = 1,2...N$

Let $F = e^{ija\Delta x} e^{ikb\Delta y}$

Substitution into CN formula gives us:

$$G^{n+1}F = G^n F + \frac{C}{2}G^{n+1}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} - 4F)$$
$$+ \frac{C}{2}G^n(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} - 4F)$$

$$[1 - \frac{C}{2}(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} - 4)]G^{n+1} = [1 + \frac{C}{2}(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} - 4)]G^n$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + C[\cos(j\Delta x) + \cos(k\Delta y) - 2]}{1 - C[\cos(j\Delta x) + \cos(k\Delta y) - 2]}$$

$$0 < \left|\frac{G^{n+1}}{G^n}\right| < 1$$

Same as 1D case, unconditionally stable.

3D CN:

Formula:

$$U_{a,b,c}^{n+1} = U_{a,b,c}^{n} + \frac{C}{2}(U_{a+1,b,c}^{n+1} + U_{a-1,b,c}^{n+1} + U_{a,b+1,c}^{n+1} + U_{a,b-1,c}^{n+1} + U_{a,b,c+1}^{n+1} + U_{a,b,c-1}^{n+1} - 6U_{a,b,c}^{n+1}$$
$$+ U_{a+1,b,c}^{n} + U_{a-1,b,c}^{n} + U_{a,b+1,c}^{n} + U_{a,b-1,c}^{n} + U_{a,b,c+1}^{n} + U_{a,b,c-1}^{n} - 6U_{a,b,c}^{n})$$

Assume $U_{a,b,c}^{n} = G^n e^{ija\Delta x} e^{ikb\Delta y} e^{ilc\Delta z}, j = 1,2...N, k = 1,2...N, l = 1,2...N$

Let $F = e^{ija\Delta x} e^{ikb\Delta y} e^{ilc\Delta z}$

Substitution into CN formula gives us:

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$$G^{n+1}F = G^{n}F + \frac{C}{2}G^{n+1}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} + Fe^{il\Delta z} + Fe^{-il\Delta z} - 6F)$$
$$+ \frac{C}{2}G^{n}(Fe^{ij\Delta x} + Fe^{-ij\Delta x} + Fe^{ik\Delta y} + Fe^{-ik\Delta y} + Fe^{il\Delta z} + Fe^{-il\Delta z} - 6F)$$

$$\frac{G^{n+1}}{G^{n}} = \frac{1 + \frac{C}{2}(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} + e^{il\Delta z} + e^{-il\Delta z} - 6)}{1 - \frac{C}{2}(e^{ij\Delta x} + e^{-ij\Delta x} + e^{ik\Delta y} + e^{-ik\Delta y} + e^{il\Delta z} + e^{-il\Delta z} - 6)}$$

$$\frac{G^{n+1}}{G^{n}} = \frac{1 + C[\cos(j\Delta x) + \cos(k\Delta y) + \cos(l\Delta z) - 3]}{1 - C[\cos(j\Delta x) + \cos(k\Delta y) + \cos(l\Delta z) - 3]}$$

$$0 < \left|\frac{G^{n+1}}{G^{n}}\right| < 1$$

Same as 1D and 2D cases, unconditionally stable.

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