Robotic Localization and Mapping - Miniset 1
Basic Probability & Homogeneous Transformations

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Probability Review

PR-Q1. For each of the following functions specify whether the function is a valid probability mass functions (PMF), a valid probability density function (PDF), or not a valid PMF or PDF. For each function that is not a valid PMF/PDF, specify the axiom that is violated.

a)

$$P(X=x) = \begin{cases} 0.2 & \text{if } x = 1 \\ 0.4 & \text{if } x = 2 \\ 0.5 & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

b)

$$P(X=x) = \begin{cases} 0.3 & \text{if } x = 1 \\ 0.4 & \text{if } x = 2 \\ 0.3 & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

c)

$$P(X=x) = \begin{cases} 0.8 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2 \\ -0.1 & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

d)

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 0.5 \\ 2 & \text{if } 0.5 < x \leq 0.75 \\ 0 & \text{otherwise.} \end{cases}$$

e)
Any $f_X(x)$ where, given

$$a < b < c,$$

$$\int_a^c f_X(x) \neq \int_a^b f_X(x) + \int_b^c f_X(x)$$

f)

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

PR-Q2. A joint PMF/PDF encodes the joint probability that multiple random variables will take on certain values at the same time. Figure 1 encodes one possible joint PMF for two random variables Study (representing the amount of time you spend studying) and Grade (representing your grade). Lets pretend that this data pmf was estimated by collecting and analyzing data from an arbitrary math class.

	Study = 2hr	Study = 4 hr	Study = 6 hr
Grade = A	0.04	0.1	0.3
Grade = B	0.07	0.18	0.1
Grade = C	0.15	0.05	0.01

Figure 1: Joint probability mass function for the random variables study and grade.

Using this joint probability distribution, answer the following questions:

a) If we randomly pick a student, what is the likelihood (or probability) that the chosen student got an A and studied for 6 hours?

b) If we randomly pick another student, what is the likelihood (or probability) that the chosen student got a C and studied for 6 hours?

c) Now lets say we are only interested in the student's grade. We can summarize information from the joint distribution so that we can focus on the variable we care about. This is called **marginalization**. In doing so, we remove one of the random variables from the distribution to get a new distribution that summarizes or includes the information about the removed random variable. We call this **marginalizing out** a random variable. Marginalize out Study so that we get a distribution over Grade alone that tells us what grade we are likely to achieve (while taking into account all possible study times). State both the marginal distribution and the "marginalization" equation from the slides that best describes what you did to obtain it. What is the marginal or overall probability of getting an A?

d) We can also use a joint pmf to look at the likelihood of one random variable given another. This is called **conditioning**. State the discrete version of the definition of conditional probability, then use it to find the distribution over possible grades given that the student studied for 6 hours. Write out this conditional probability distribution. What is the probability of getting an A given that the student studied for 6 hours?

PR-Q3. State the definition of independence for two random events. Now state the definition of conditional independence. Show (by example) that independence does not imply conditional independence.

PR-Q4. Bayes rule/formula is very useful tool that enables us to rewrite a conditional probability in terms of other conditional and marginal distributions that are sometimes easier to work with. A common form of the Bayes rule is:

$$P(x|y) = \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)}.$$

Derive the Bayes rule from the definition of a conditional probabilities and the law of total probability.

Rotation Matrices

RM-Q1. Show[1] that the dot product of two free vectors is the same regardless of the coordinate frame in which they are represented. Note that $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1^\top \mathbf{v}_2$.

RM-Q2. Show[1] that the length of a free vector is not changed by rotation, that is, that $||\mathbf{v}|| = ||R\mathbf{v}||$.

RM-Q3. Show[1] that the distance between points is not changed by rotation, that is, that $||\mathbf{p}_1 - \mathbf{p}_2|| = ||R\mathbf{p}_1 - R\mathbf{p}_2||$.

RM-Q4. If[1] a matrix R satisfies $R^\top R = I$, show that the column (and row) vectors or R are of unit length and mutually perpendicular.

RM-Q5. If[1] a matrix R satisfies $R^\top R = I$, then

a) Show that $\det R = \pm 1$. Hint: use properties of determinants.

b) Show that $\det R = +1$ if we restrict ourselves to right-handed coordinate frames. Hint: use the definition of cross product and the fact that cross product takes two vectors and returns a third vector that follows the right hand rule.

[1]Problems selected from Spong, Hutchinson, and Vidyasagar, *Robot Modeling and Control*, Chapter 2.

RM-Q6. A[1] **group** is a set X together with an operation $*$ defined on that set such that

- $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
- $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$.
- For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$.

Show that $\mathrm{SO}(n)$ with the operation of matrix multiplication is a group.

RM-Q7. Suppose[1] A is a $2x2$ rotation matrix. In other words $A^\top A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

RM-Q8. Verify[1] the following equations (where $R_{z,\alpha}$ is a rotation around the z-axis by α degrees):

a) $R_{z,0} = I$
b) $R_{z,\theta} R_{z,\phi} = R_{z,\theta+\phi}$
c) $(R_{z,\theta})^{-1} = R_{z,-\theta}$

RM-Q9. Consider the following sequence of 3D rotations:

1. Rotate by θ around the current z-axis.
2. Rotate by ψ around the current y-axis.
3. Rotate by ϕ around the current x-axis.

Write the matrix product that will give the resulting rotation matrix. (You do not need to perform matrix multiplication).

RM-Q10. Consider the following sequence of 3D rotations:

1. Rotate by ϕ around the world x-axis.
2. Rotate by θ around the current z-axis.
3. Rotate by ψ around the world x-axis.

Write the matrix product that will give the resulting rotation matrix. (You do not need to perform matrix multiplication).

Homogeneous Transformations

HT-Q1. Consider[1] the diagram in Figure 2. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. Coordinate frame 1 is fixed to the edge of the table as shown. A cube measuring 20 cm on one side is placed in the center of the table with coordinate frame 2 established as the center of the cube. A camera is situated directly above the center of block at a height of 2 meters above the table with coordinate frame 3 as shown. Find the homogeneous transformations relating these frames to the base frame of the robot (coordinate frame 0). Find the homogeneous transformation relating frame 2 to the camera frame (frame 3).

HT-Q2. In[1] the last problem, suppose that, after the camera is calibrated, it is rotated 90^o about z_3. Recompute the above coordinate transformations.

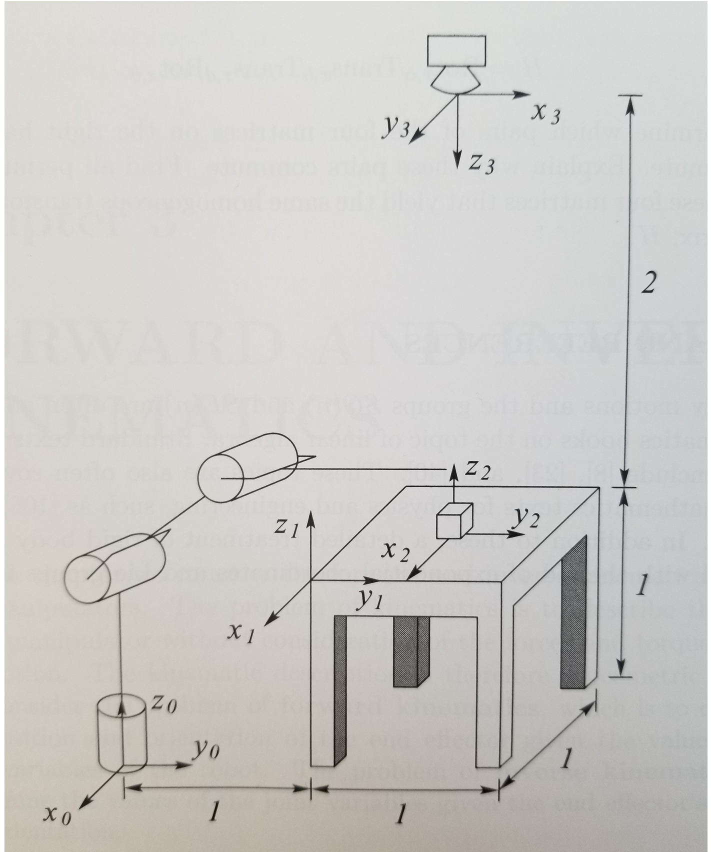

Figure 2: Figure 2-14 from Spong et al. *Robot Modeling and Control*

HT-Q3. If[1] the block on the table is rotated 90^o about z_2 and moved so that its center has coordinates $[0, 0.8, 0.1]^\top$ (in meters) relative to frame 1, compute the homogeneous transformation that relates the block frame to the camera frame and the transformation that relates the camera frame to the block frame.