conventions

Conventions and useful identities

Form field

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \dots dx^{\mu_p}, \quad A_{\mu_1 \dots \mu_p} = A_{[\mu_1 \dots \mu_p]}, \quad |A|^2 = \frac{1}{p!} A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p} = \frac{1}{p!} A^2$$

$$(dA)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$$

$$(A^{(p)} \wedge B^{(q)})_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} A^{(p)}_{[\mu_1 \dots \mu_p} B^{(q)}_{\nu_1 \dots \nu_q]}$$

We will use $\epsilon_{\mu_1 \dots \mu_p}$ to denote the Levi-Civita tensor and $\varepsilon_{\mu_1 \dots \mu_p}$ to denote the flat-space fully antisymmetrized symbol.

$$\epsilon_{\mu_1 \dots \mu_D} = \sqrt{|g|} \varepsilon_{\mu_1 \dots \mu_D}, \quad \varepsilon_{0,1,\dots,(D-2),(D-1)} = 1,$$

$$\epsilon_{\mu_1 \dots \mu_p \lambda_1 \dots \lambda_{D-p}} \epsilon^{\nu_1 \dots \nu_p \lambda_1 \dots \lambda_{D-p}} = (-1)^{[t]} p! (D-p)! \delta_{\mu_1 \dots \mu_p}{}^{\nu_1 \dots \nu_p}$$

$$(*A)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{D-p}}{}^{\nu_1 \dots \nu_p} A_{\nu_1 \dots \nu_p},$$

Useful Identities

One can then show that

$$*(*A) = (-1)^{[t]} (-1)^{p(D-p)} A,$$

$$(*F) \wedge *(*F) = (-1)^{[t]} F \wedge *F, \quad |*F|^2 = (-1)^{[t]} |F|^2$$

as well as

$$\frac{1}{(D-p-1)!} (*F)_{\mu \dots} (*F)_{\nu}{}^{\dots} = (-1)^{[t]} \left[\frac{1}{p!} F^2 g_{\mu\nu} - \frac{1}{(p-1)!} F_{\mu \dots} F_{\nu}{}^{\dots} \right]$$

Derivation:

$$\frac{1}{(D-p-1)!}(*F)_{\mu\ldots}(*F)_{\nu}{}^{\cdots} = \frac{1}{(D-p-1)!}\frac{1}{p!p!}\epsilon_{\mu\mu_1\ldots\mu_p\lambda_1\ldots\lambda_{D-p-1}}\epsilon_{\nu}{}^{\nu_1\ldots\nu_p\lambda_1\ldots\lambda_{D-p-1}}F^{\mu_1\ldots}$$

$$= \frac{p+1}{p!}(-1)^{[t]}g_{\nu\rho}\delta^{[\rho\nu_1\ldots\nu_p]}_{[\mu\mu_1\ldots\mu_p]}F^{\mu_1\ldots\mu_p}F_{\nu_1\ldots\nu_p}$$

$$\delta_{[\mu\mu_1\ldots\mu_p]}{}^{[\rho\nu_1\ldots\nu_p]}$$

has in total $(p+1)!(p+1)!$ terms and is thus normalized by such number. Within those terms, $(p+1)p!p!$ of which give

$$\delta_{\mu}{}^{\rho}\delta_{\mu_1\ldots\mu_p}{}^{\nu_1\ldots\nu_p}$$

equivalent and $(p+1)(p)p!p!$ give

$$-\delta_{\mu_1}{}^{\rho}\delta_{\mu\ldots\mu_p}{}^{\nu_1\ldots\nu_p}$$

equivalent, so we find

$$\frac{1}{(D-p-1)!}(*F)_{\mu\ldots}(*F)_{\nu}{}^{\cdots} = (-1)^{[t]}\left[\frac{1}{p!}F^2 g_{\mu\nu} - \frac{1}{(p-1)!}F_{\mu\ldots}F_{\nu}{}^{\cdots}\right].$$