conventions

Conventions and useful identities

Form field

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \dots dx^{\mu_p}, \quad A_{\mu_1 \dots \mu_p} = A_{[\mu_1 \dots \mu_p]}, \quad |A|^2 = \frac{1}{p!} A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p} = \frac{1}{p!} A^2 \tag{1}$$

$$(dA)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$$

$$(A^{(p)} \wedge B^{(q)})_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p! q!} A^{(p)}_{[\mu_1 \dots \mu_p} B^{(q)}_{\nu_1 \dots \nu_q]} \tag{2}$$

We will use $\epsilon_{\mu_1 \dots \mu_p}$ to denote the Levi-Civita tensor and $\varepsilon_{\mu_1 \dots \mu_p}$ to denote the flat-space fully antisymmetrized symbol.

$$\epsilon_{\mu_1 \dots \mu_D} = \sqrt{|g|} \varepsilon_{\mu_1 \dots \mu_D}, \quad \varepsilon_{0,1,\dots,(D-2),(D-1)} = 1, \tag{3}$$

$$\epsilon_{\mu_1 \dots \mu_p \lambda_1 \dots \lambda_{D-p}} \epsilon^{\nu_1 \dots \nu_p \lambda_1 \dots \lambda_{D-p}} = (-1)^{[t]} p! (D-p)! \delta_{\mu_1 \dots \mu_p}{}^{\nu_1 \dots \nu_p}$$

$$(*A)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{D-p}}{}^{\nu_1 \dots \nu_p} A_{\nu_1 \dots \nu_p}, \tag{4}$$

Useful Identities

One can then show that

$$* (*A) = (-1)^{[t]} (-1)^{p(D-p)} A,$$

$$(*F) \wedge * (*F) = (-1)^{[t]} F \wedge * F, \quad |*F|^2 = (-1)^{[t]} |F|^2$$

as well as

$$\frac{1}{(D-p-1)!} (*F)_{\mu \dots} (*F)_{\nu}{}^{\dots} = (-1)^{[t]} \left[\frac{1}{p!} F^2 g_{\mu\nu} - \frac{1}{(p-1)!} F_{\mu \dots} F_{\nu}{}^{\dots} \right] \tag{5}$$

Derivation:

$$\begin{aligned} \frac{1}{(D-p-1)!} (*F)_{\mu \dots} (*F)_{\nu}{}^{\dots} &= \frac{1}{(D-p-1)!} \frac{1}{p! p!} \epsilon_{\mu \mu_1 \dots \mu_p \lambda_1 \dots \lambda_{D-p-1}} \epsilon_{\nu}{}^{\nu_1 \dots \nu_p \lambda_1 \dots \lambda_{D-p-1}} F^{\mu_1 \dots \mu_p} F_{\nu_1 \dots \nu_p} \\ &= \frac{p+1}{p!} (-1)^{[t]} g_{\nu\rho} \delta^{[\rho \nu_1 \dots \nu_p]}_{[\mu \mu_1 \dots \mu_p]} F^{\mu_1 \dots \mu_p} F_{\nu_1 \dots \nu_p} \end{aligned} \tag{6}$$

$$\delta_{[\mu \mu_1 \dots \mu_p]}{}^{[\rho \nu_1 \dots \nu_p]}$$

has in total $(p+1)!(p+1)!$ terms and is thus normalized by such number. Within those terms, $(p+1)p!p!$ of which give

$$\delta_{\mu}{}^{\rho}\delta_{\mu_1\ldots\mu_p}{}^{\nu_1\ldots\nu_p}$$

equivalent and $(p+1)(p)p!p!$ give

$$-\delta_{\mu_1}{}^{\rho}\delta_{\mu\ldots\mu_p}{}^{\nu_1\ldots\nu_p}$$

equivalent, so we find

$$\frac{1}{(D-p-1)!}(*F)_{\mu\ldots}(*F)_{\nu}{}^{\cdots} = (-1)^{[t]}\left[\frac{1}{p!}F^2 g_{\mu\nu} - \frac{1}{(p-1)!}F_{\mu\ldots}F_{\nu}{}^{\cdots}\right]. \tag{7}$$