

review

A review of brane solutions

EQMs

This follows [1](#).

In D dimensions, for the Einstein-Hilbert action coupled to various n -forms with no dilaton,

$$S = \int d^D x \sqrt{-g} \left[R - \sum_n \frac{1}{2} \frac{1}{n!} F_n^2 \right] \quad (1)$$

The Einstein equations are

$$\begin{aligned} R_{MN} = S_{MN} &= \sum_n \frac{1}{2(n-1)!} \left[(F_n)_M \dots (F_n)_N \dots - \frac{n-1}{n(D-2)} (F_n)^2 g_{MN} \right] \\ &= \frac{1}{2} \sum_n \left[\frac{1}{(n-1)!} F_M \dots F_N \dots - \frac{n-1}{D-2} \left(\frac{1}{n!} F^2 \right) g_{MN} \right], \end{aligned} \quad (2)$$

$$dF = 0.$$

We look for solutions that are d -dimensional Minkowski space transverse to $(D-d)$ -dimensional Euclidean space. So our coordinates are split into $x^M = (x^\mu, y^m)$, where x^μ for $\mu = 0, 1, \dots, d-1$ are coordinates on the Minkowski space, and y^m for $m = d, d+1, \dots, D-1$ are embedding coordinates on the $D-d$ dimensional Euclidean space.

Our conventions for form fields are given in [conventions](#).

Brane-solution ansatz and its implications

We usually solve with the ansatz for a ($p=d-1$)-brane:

$$ds_D^2 = H^a(r) \eta_{\mu\nu} dx^\mu dx^\nu + H^b(r) \delta_{mn} dy^m dy^n, \quad r \equiv \sqrt{y^m y^m}. \quad (3)$$

where H is a harmonic function in $(D-d)$ -dimensions, usually taken to be

$$H = 1 + \frac{Q}{r^{\tilde{d}}}, \quad \tilde{d} = D - d - 2 \quad (4)$$

Then the Ricci tensor is found to be

$$R_{\mu\nu} = -\eta_{\mu\nu}(H')^2 H^{a-b-2} \left[\frac{a(ad + b\tilde{d} - 2)}{4} \right] \quad (5)$$

$$\begin{aligned} R_{mn} &= \frac{(H')^2}{4H^2} \left[-b(ad + b\tilde{d} - 2)\delta_{mn} - \frac{y^m y^n}{r^2} \left[ad(a - b) - (ad + b\tilde{d})(b + 2) \right] \right] \\ &\quad + \frac{H'}{rH} \frac{(ad + b\tilde{d})}{2} \left[-\delta_{mn} + \frac{y^m y^n}{r^2} (2 + \tilde{d}) \right] \end{aligned} \quad (6)$$

Usually one adopts the ansatz that $A_{n-1} \propto H$, hence $F_n \propto H'$. Then S_{MN} will be exactly quadratic in H' up to some powers of H :

$$S_{MN} \sim H^{(\dots)}(H')^2.$$

which means in order to satisfy the Einstein equation, R_{MN} should be exactly quadratic in H' . This demands

$$b = -ad/\tilde{d},$$

and we can write down a simplified Ricci tensor

$$R_{\mu\nu} = \frac{a}{2}(H')^2 H^{a-b-2} \eta_{\mu\nu} = \frac{a}{2}(H')^2 H^{\frac{ad+(a-2)(D-d-2)}{(D-d-2)}} \eta_{\mu\nu}, \quad (7)$$

$$R_{mn} = \frac{(H')^2}{4H^2} \left[2b\delta_{mn} - ad(a - b) \frac{y^m y^n}{r^2} \right] = -\frac{ad}{2(D - d - 2)} \frac{(H')^2}{H^2} \left[\delta_{mn} + \frac{a(D - 2)}{2} \right]$$

Electric ansatz

For a F_{d+1} form field, the electric ansatz can be given by

$$(F_{d+1})_{m\mu_0\mu_1\dots\mu_{d-1}} = c\varepsilon_{\mu_0\mu_1\dots\mu_{d-1}} H^{-1+\frac{ad}{2}} \partial_m H \quad (9)$$

which satisfies $dF_{d+1} = 0$. The exponent $-1 + \frac{ad}{2}$ is required by matching the scaling in H on both sides of $R_{mn} = S_{mn}$. One can then solve to find that

$$a = -\frac{2}{d}, \quad b = \frac{2}{\tilde{d}}, \quad c^2 = \frac{2(D - 2)}{d\tilde{d}}. \quad (10)$$

There are then no free parameters, and we end up with the usual electric brane solutions as described in [2](#).

Magnetic ansatz

From the form field identities given in [conventions](#), one can show a relation between the form field and its Hodge dual

$$S_{MN}|_{*F} = -(-1)^{[t]} S_{MN}|_F.$$

So for spacetime with an odd number of timelike directions, the Hodge dual of the electric solution is the magnetic solution. Performing the duality transformation, we find that for a $D - d - 1$ form $*F$, it supports the $d - 1$ brane ansatz with

$$(*F)_{m_1 \dots m_{D-d-1}} = -c \epsilon_{m_1 \dots m_{D-d-1} m} \partial_m H.$$

From this we conclude that a $(d-1)$ -brane can either be supported by a $(d+1)$ -form or a $(D-d-1)$ -form.

References

1. K. S. Stelle, *BPS branes in supergravity, ICTP Summer School in High-Energy Physics and Cosmology* (1997), [arXiv:hep-th/9803116](https://arxiv.org/abs/hep-th/9803116).
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2. K. S. Stelle, *BPS branes in supergravity, ICTP Summer School in High-Energy Physics and Cosmology* (1997), [arXiv:hep-th/9803116](https://arxiv.org/abs/hep-th/9803116).
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