A review of brane solutions

EQMs

This follows [1].

In D dimensions, for the Einstein-Hilbert action coupled to various n-forms with no dilaton,

$$S = \int d^D x \sqrt{-g} \left[R - \sum_n \frac{1}{2} \frac{1}{n!} F_n^2 \right] \tag{1}$$

The Einstein equations are

$$
\begin{aligned}
R_{MN} = S_{MN} &= \sum_n \frac{1}{2(n-1)!} \left[(F_n)_{M\ldots}(F_n)_N{}^{\cdots} - \frac{n-1}{n(D-2)}(F_n)^2 g_{MN} \right] \\
&= \frac{1}{2} \sum_n \left[\frac{1}{(n-1)!} F_{M\ldots} F_{\ddot{N}} - \frac{n-1}{D-2} \left(\frac{1}{n!} F^2 \right) g_{MN} \right],
\end{aligned}
\tag{1}
$$

$$dF = 0.$$

We look for solutions that are d-dimensional Minkowski space transverse to $(D-d)$-dimensional Euclidean space. So our coordinates are split into $x^M = (x^\mu, y^m)$, where x^μ for $\mu = 0, 1, ..., d-1$ are coordinates on the Minkowski space, and y^m for $m = d, d+1, ..., D-1$ are embedding coordinates on the $D-d$ dimensional Euclidean space.

Our conventions for form fields are given in conventions.

Brane-solution ansatz and its implications

We usually solve with the ansatz for a ($p=d-1$)-brane:

$$ds_D^2 = H^a(r) \eta_{\mu\nu} dx^\mu dx^\nu + H^b(r) \delta_{mn} dy^m dy^n, \quad r \equiv \sqrt{y^m y^m}. \tag{1}$$

where H is a harmonic function in $(D-d)$-dimensions, usually taken to be

$$H = 1 + \frac{Q}{r^{\tilde{d}}}, \quad \tilde{d} = D - d - 2 \tag{1}$$

Then the Ricci tensor is found to be

$$R_{\mu\nu} = -\eta_{\mu\nu}(H')^2 H^{a-b-2} \left[\frac{a(ad + b\tilde{d} - 2)}{4} \right. \tag{1}$$

$$R_{mn} = \frac{(H')^2}{4H^2}\left[-b(ad+b\tilde{d}-2)\delta_{mn} - \frac{y^m y^n}{r^2}\left[ad(a-b)-(ad+b\tilde{d})(b+2)\right]\right]$$
$$+ \frac{H'}{rH}\frac{(ad+b\tilde{d})}{2}\left[-\delta_{mn} + \frac{y^m y^n}{r^2}(2+\tilde{d})\right] \tag{1}$$

Usually one adopts the ansatz that $A_{n-1} \propto H$, hence $F_n \propto H'$. Then S_{MN} will be exactly quadratic in H' up to some powers of H:

$$S_{MN} \sim H^{(\ldots)}(H')^2.$$

which means in order to satisfy the Einstein equation, R_{MN} should be exactly quadratic in H'. This demands

$$b = -ad/\tilde{d},$$

and we can write down a simplified Ricci tensor

$$R_{\mu\nu} = \frac{a}{2}(H')^2 H^{a-b-2}\eta_{\mu\nu} = \frac{a}{2}(H')^2 H^{\frac{ad+(a-2)(D-d-2)}{(D-d-2)}}\eta_{\mu\nu}, \tag{1}$$

$$R_{mn} = \frac{(H')^2}{4H^2}\left[2b\delta_{mn} - ad(a-b)\frac{y^m y^n}{r^2}\right] = -\frac{ad}{2(D-d-2)}\frac{(H')^2}{H^2}\left[\delta_{mn} + \frac{a(D-2)}{2}\frac{y^m y^n}{r^2}\right]$$

Electric ansatz

For a F_{d+1} form field, the electric ansatz can be given by

$$(F_{d+1})_{m\mu_0\mu_1\ldots\mu_{d-1}} = c\varepsilon_{\mu_0\mu_1\ldots\mu_{d-1}}H^{-1+\frac{ad}{2}}\partial_m H \tag{1}$$

which satisfies $dF_{d+1} = 0$. The exponent $-1 + \frac{ad}{2}$ is required by matching the scaling in H on both sides of $R_{mn} = S_{mn}$. One can then solve to find that

$$a = -\frac{2}{d}, \quad b = \frac{2}{\tilde{d}}, \quad c^2 = \frac{2(D-2)}{d\tilde{d}}. \tag{1}$$

There are then no free parameters, and we end up with the usual electric brane solutions as described in [2].

Magnetic ansatz

From the form field identities given in conventions, one can show a relation between the form field and its Hodge dual

$$S_{MN}|_{*F} = -(-1)^{[t]}S_{MN}|_F.$$

So for spacetime with an odd number of timelike directions, the Hodge dual of the electric solution is the magnetic solution. Performing the duality transformation, we find that for a $D-d-1$ form $*F$, it supports the $d-1$ brane ansatz with

$$(*F)_{m_1\ldots m_{D-d-1}} = -c\varepsilon_{m_1\ldots m_{D-d-1}m}\partial_m H.$$

From this we conclude that a $(d-1)$-brane can either be supported by a $(d+1)$-form or a $(D-d-1)$-form.

References

1. K. S. Stelle, *BPS branes in supergravity, ICTP Summer School in High-Energy Physics and Cosmology* (1997), arXiv:hep-th/9803116.↵

2. K. S. Stelle, *BPS branes in supergravity, ICTP Summer School in High-Energy Physics and Cosmology* (1997), arXiv:hep-th/9803116.↵