Townsend II.2.

$$(\mathcal{L}_\xi g)_{\mu\nu} \to [\mathcal{L}_\xi (\Lambda^2 g)]_{\mu\nu}$$

$$= \xi^\alpha{}_{,\mu} (\Lambda^2 g_{\alpha\nu}) + \xi^\alpha{}_{,\nu} (\Lambda^2 g_{\alpha\mu}) + \xi^\alpha (\Lambda^2 g_{\mu\nu})_{,\alpha}$$

$$= \Lambda^2 [\mathcal{L}_\xi g]_{\mu\nu} + \xi^\alpha 2\Lambda (\partial_\alpha \Lambda)\, g_{\mu\nu}$$

$$= \xi^\alpha 2\Lambda (\partial_\alpha \Lambda)\, g_{\mu\nu}$$

$$\Uparrow$$

$$\Omega^2$$

Now consider $x^\mu \to x^\mu + \alpha \xi^\mu(x)$, $e \to e + \frac{1}{4}\alpha e g^{\mu\nu} (\mathcal{L}_\xi g)_{\mu\nu}$

expand $e^{-1} \to ?$ to $O(\alpha^2)$

$$e^{-1}(\alpha) = e^{-1}(\alpha=0) + -1\, e^{-2}(\alpha=0) \left.\frac{\partial e(\alpha)}{\partial \alpha}\right|_{\alpha=0} \alpha + O(\alpha^2)$$

$$= e^{-1}(\alpha=0) - \frac{\alpha}{e^2(\alpha=0)} \frac{1}{4} g^{\mu\nu} e (\mathcal{L}_\xi g)_{\mu\nu} + O(\alpha^2)$$

$$\Rightarrow e^{-1} \to e^{-1}\left[1 - \frac{\alpha}{4} g^{\mu\nu} (\mathcal{L}_\xi g)_{\mu\nu}\right]$$

ξ conformal killing $\Rightarrow$ $$= e^{-1}\left[1 - \frac{\alpha}{4} g^{\mu\nu} (\Omega^2 g_{\mu\nu})\right]$$

From $x^\mu \to x^{\mu\prime}$, $e \to e'$ we have the variations of them to $O(\alpha^2)$ the variations of them independently:

$x \to x'$: $$\delta I = + \frac{\alpha}{2} \int d\lambda \, e^{-1} \dot{x}^\mu \dot{x}^\nu (\mathcal{L}_\xi g)_{\mu\nu}$$ recall (2.18), which I have checked already

$$= \frac{\alpha}{2} \int d\lambda \, e^{-1} \dot{x}^\mu \dot{x}^\nu \Omega^2 g_{\mu\nu} = \frac{\alpha}{2} \int d\lambda \, \dot{x}^\mu \dot{x}_\mu \Omega^2$$

$e \to e'$: $$\delta I = -\frac{\alpha}{2} \frac{1}{4} \int d\lambda \, e^{-1} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} g^{ab} (\Omega^2 g_{ab})$$

$$= -\frac{\alpha}{2} \frac{1}{4} \int d\lambda \, e^{-1} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} \Omega^2 g^a{}_a$$

$$= -\frac{\alpha}{2} \frac{1}{4} \int d\lambda \, e^{-1} \dot{x}^\mu \dot{x}_\mu \Omega^2 g^a{}_a$$

They cancel if $g^a{}_a = 4$, otherwise I'm off by a factor, ☹.

ξ being a continuous symmetry $\Rightarrow$ ξ is associated with a conserved current & conserved charge as indicated by Noether's thm.

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