4 Hooft Exercise 5.2 pg 21.

$$\frac{d}{d\sigma}\left(x^{\mu}(\tilde{\sigma})\right) = \frac{dx^{\mu}}{d\tilde{\sigma}}\frac{d\tilde{\sigma}}{d\sigma}$$

⇒ Geodesic Eq:

$$\frac{d}{d\sigma}\left[\frac{dx^{\mu}}{d\tilde{\sigma}}\frac{d\tilde{\sigma}}{d\sigma}\right] + \Gamma^{\mu}_{\kappa\lambda}\frac{dx^{\kappa}}{d\tilde{\sigma}}\frac{dx^{\lambda}}{d\tilde{\sigma}}\left(\frac{d\tilde{\sigma}}{d\sigma}\right)^2 = 0.$$

$$\frac{d^2x^{\mu}}{d\tilde{\sigma}^2}\left(\frac{d\tilde{\sigma}}{d\sigma}\right)^2 + \frac{dx^{\mu}}{d\tilde{\sigma}}\frac{d^2\tilde{\sigma}}{d\sigma^2} + \Gamma^{\mu}_{\kappa\lambda}\frac{dx^{\kappa}}{d\tilde{\sigma}}\frac{dx^{\lambda}}{d\tilde{\sigma}}\left(\frac{d\tilde{\sigma}}{d\sigma}\right)^2 = 0$$

$$\Rightarrow \frac{dx^{\mu}(\tilde{\sigma})}{d\tilde{\sigma}}\left[\frac{d^2\tilde{\sigma}}{d\sigma^2}\right]\left[\frac{d\sigma}{d\tilde{\sigma}}\right]^2 = -\frac{d^2x^{\mu}(\tilde{\sigma})}{d\tilde{\sigma}^2} - \Gamma^{\mu}_{\kappa\lambda}\frac{dx^{\kappa}}{d\tilde{\sigma}}\frac{dx^{\lambda}}{d\tilde{\sigma}}.$$

$\tilde{\sigma}(\sigma)$ is arbitrary, so let $\dfrac{d^2\tilde{\sigma}}{d\sigma^2} = 0$, then

the right handside vanish, we left with the condition

$$\frac{dx^{\mu}(\tilde{\sigma})}{d\tilde{\sigma}}\left[\frac{d\sigma}{d\tilde{\sigma}}\right]^2 \times 0 = 0$$

This allows us to set $\left[\dfrac{d\sigma}{d\tilde{\sigma}}\right]^2 = \alpha(\tilde{\sigma})$ to anything?

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