Derivation of $i\frac{da}{dt} = [a, H]$. Heisenberg picture time derivative

In schrodinger picture, $\psi(t_0+t) = e^{-iHt}\psi(t_0)$, moving time dependence onto operators:

$$a(t_0+t) = e^{iHt}\, a(t_0)\, e^{-iHt}$$

$$\Rightarrow \quad a(t_0+dt) = e^{iHdt}\, a(t_0)\, e^{-iHdt}$$

$$\approx (1+iHdt)\, a\, (1-iHdt)$$

$$= (a + iHdt\, a)(1-iHdt)$$

$$= a + i(Hdt)a - i\,a(Hdt) + O(dt^2)$$

$$\Rightarrow \quad a(t_0+dt) - a(t_0) = i\,dt\,[H, a] + O(dt^2)$$

$$\frac{a(t_0+dt) - a(t_0)}{dt} = i[H, a]$$

$$\boxed{\frac{da}{dt} = i[H, a] \qquad i\frac{d}{dt}a = [a, H]}$$

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