$$\varphi - \varphi_0 = \int_{u_0}^{u} du \left(\frac{E^2 - 1}{J^2} + \frac{2M}{J^2} u - u^2 \right)^{-1/2}$$

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exercise 12.1
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can be solved via

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2+bx+c} \right| & a > 0 \\ \frac{1}{\sqrt{-a}} \sin^{-1} \frac{-2ax - b}{\sqrt{b^2 - 4ac}} & a < 0 \end{cases}$$

$$\Rightarrow \int_{u_0}^{u} du \left(\cdots \right)^{-1/2} = \sin^{-1} \frac{2u - \frac{2M}{J^2}}{\sqrt{\left(\frac{2M}{J^2}\right)^2 + 4\left(\frac{E^2-1}{J^2}\right)}}$$

$$= \boxed{\sin^{-1} \frac{\left(2u - \frac{2M}{J^2}\right) J}{\sqrt{\frac{2M}{J^2} - 4(E^2 - 1)}}}$$

In the Newtonian Limit, r is large, so $\frac{1}{r}$ small,

$u = \frac{1}{r}$ is small, u^3 is $(\text{small})^3$.

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