Hassani
10.4

$$\frac{d}{dz}(\ln z) = \frac{1}{z} \qquad \ln(z) = \ln(x+iy)$$

$\ln(x+iy)$ defined as $\exp[\ln(x+iy)] = x+iy$.

$$\exp[\ln(x+iy)] = \exp[u(x,y) + iv(x,y)]$$

$$= e^{u(x,y)} e^{iv(x,y)} = x+iy.$$

$$e^{u(x,y)} = |x+iy| = \sqrt{x^2+y^2}$$

$$\boxed{u(x,y) = \ln\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad e^{iv(x,y)} = \frac{x+iy}{\sqrt{x^2+y^2}} = \cos[v(x,y)] + i\sin[v(x,y)]$$

$$\Rightarrow \quad \frac{i\sin[v(x,y)]}{\cos[v(x,y)]} = \frac{iy}{x}$$

$$\tan[v(x,y)] = y/x$$

$$\boxed{v(x,y) = \tan^{-1}[y/x]}$$

$$\frac{d}{dz}(\ln z) = \frac{d}{dz}\left[u(x,y) + i\,v(x,y)\right]$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial x}\left[\ln\sqrt{x^2+y^2}\right] + i\,\frac{\partial}{\partial x}\left[\tan^{-1}\frac{y}{x}\right]$$

$$= \frac{1}{\sqrt{x^2+y^2}}\,\frac{x}{\sqrt{x^2+y^2}} + i\left[\frac{1}{1+\frac{y^2}{x^2}}\right]\left[\frac{-y}{x^2}\right]$$

$$= \frac{x}{\left(\sqrt{x^2+y^2}\right)^2} + i\,\frac{(-y)}{x^2+y^2}$$

$$= (x - iy)/(x^2+y^2)$$

$$= \frac{1}{x+iy} = \boxed{\frac{1}{z}}$$

Davidson Cheng

2.18.2024