Goldstein

1.8. $\dfrac{dL'}{dq_k} = \dfrac{dL}{dq_k} + \dfrac{d}{dq_k}\dfrac{d}{dt}F$

$\dfrac{d}{dt}\dfrac{dL'}{d\dot{q}_k} = \dfrac{d}{dt}\dfrac{dL}{d\dot{q}_k} + \dfrac{d}{dt}\dfrac{d}{d\dot{q}_k}\dfrac{d}{dt}F$.

$\dfrac{dL}{dq_k} = \dfrac{d}{dt}\dfrac{dL}{d\dot{q}_k}$,   it remains to show

$$\dfrac{d}{dq_k}\dfrac{d}{dt}F = \dfrac{d}{dt}\dfrac{d}{d\dot{q}_k}\dfrac{d}{dt}F.$$

Consider $\dfrac{d}{dt}\dfrac{d}{d\dot{q}_k} = \dfrac{d}{dX}\left[\dfrac{d}{d\dot{q}_k}\right]\dfrac{dX}{dt}$,

Letting $X = q_k$, it becomes

$$\dfrac{d}{dq_k}\dfrac{d}{d\dot{q}_k}\dfrac{dq_k}{dt} = \dfrac{d}{dq_k}\dfrac{d}{d\dot{q}_k}\dot{q}_k$$

$$= \dfrac{d}{dq_k}.$$

Thus $\dfrac{d}{dq_k}\left[\dfrac{d}{dt}F\right] = \left[\dfrac{d}{dt}\dfrac{d}{d\dot{q}_k}\right]\left[\dfrac{d}{dt}F\right]$.

$\Updownarrow$

$\dfrac{d}{dq_k}$