Schutz

7.6. The Geodesic is given by $\nabla_{\vec{P}} \vec{P} = 0$.

This is obtained by the affine reparameterization of the particle's trajectory $\vec{U}(\gamma)$, where γ is the proper time.

Recall $\nabla_{\vec{U}} \vec{V} = \dfrac{dx^{\alpha}}{d\gamma} V^{\beta}{}_{;\alpha} = V^{\alpha} V^{\beta}{}_{;\alpha}$.

under affine reparameterization $\gamma \rightarrow \dfrac{\gamma}{m}$,

$$V = \dfrac{dx^{\alpha}}{d\gamma} \rightarrow \dfrac{dx^{\alpha}}{d(\gamma/m)} = m\dfrac{dx^{\alpha}}{d\gamma} = m\vec{U} = \vec{P}.$$

So such reparameterization $\gamma \rightarrow \dfrac{\gamma}{m}$ can be written with the original variable γ with the particle's trajectory $\vec{U}(\gamma)$ replaced by $\vec{P}(\gamma)$. So the geodesic equation is replaced as.

$$P^{\alpha} V^{\beta}{}_{;\alpha} = 0, \quad \text{for } V^{\beta} = P^{\beta}, \text{ this}$$

is $\quad P^{\alpha} P^{\beta}{}_{;\alpha} = 0$.