Schutz 6.39 (a) (i) $[\vec{U}, \vec{V}]^\alpha = U^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta U^\alpha$

$[\vec{V}, \vec{U}] = V^\beta \nabla_\beta U - U^\beta \nabla_\beta V^\alpha = \boxed{-[\vec{U}, \vec{V}]^\alpha}$

(ii) $[\vec{U}, \vec{V}]^\alpha = U^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta U^\alpha.$

$= U^\beta \left[ V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \right] - V^\beta \left[ U^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} U^\mu \right]$

$= U^\beta V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu U^\beta - V^\beta U^\alpha_{,\beta} - \boxed{\Gamma^\alpha_{\mu\beta} U^\mu V^\beta}$

use symmetry
of $\Gamma$
and relabel
dummy indices

$\Gamma^\alpha_{\mu\beta} U^\beta V^\mu$

$= \boxed{U^\beta V^\alpha_{,\beta} - V^\beta U^\alpha_{,\beta}}$

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Schutz 6.39 (b)   $[\vec{U}, f\vec{V}] = U^\beta \nabla_\beta (fV^\alpha) - fV^\beta (\nabla_\beta U^\alpha)$

$$= U^\beta (\nabla_\beta f) V^\alpha + U^\beta f (\nabla_\beta V^\alpha) - f V^\beta \nabla_\beta U^\alpha$$

$$= f\left[ U^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta U^\alpha \right] + V^\alpha U^\beta \nabla_\beta f$$

$$= \boxed{f[\vec{U}, \vec{V}] + \vec{V}(\vec{U} \cdot \nabla f).}$$

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Schutz 6.39 (c)

By Defn, $\pounds_{\vec{U}}[\tilde{\omega}(\vec{V})] = (\pounds_{\vec{U}}\tilde{\omega})(\vec{V}) + \tilde{\omega}(\pounds_{\vec{U}}\vec{V})$,

$\Rightarrow (\pounds_{\vec{U}}\tilde{\omega})(\vec{V}) = \pounds_{\vec{U}}[\tilde{\omega}(\vec{V})] - \tilde{\omega}(\pounds_{\vec{U}}\vec{V})$

$$= \mathcal{L}_{\vec{U}}[w_\alpha V^\alpha] - w_\alpha(U^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta U^\alpha)$$

$$= U^\beta \nabla_\beta[w_\alpha V^\alpha] - w_\alpha U^\beta(\nabla_\beta V^\alpha) + w_\alpha V^\beta(\nabla_\beta U^\alpha)$$

$$= U^\beta(\nabla_\beta w_\alpha)V^\alpha + \cancel{U^\beta w_\alpha(\nabla_\beta V^\alpha)} - \cancel{w_\alpha U^\beta(\nabla_\beta V^\alpha)}$$

$$+ w_\alpha V^\beta(\nabla_\beta U^\alpha)$$

$$= U^\beta(\nabla_\beta w_\alpha)V^\alpha + w_\beta(\nabla_\alpha U^\beta)V^\alpha \qquad \text{(relabel dummy index)}$$

$$= \boxed{(U^\beta \nabla_\beta w_\alpha + w_\beta \nabla_\alpha U^\beta)V^\alpha}$$

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