Griffiths.

(2) ~~~~ 7.34. $\mathcal{M}_1 \mathcal{M}_2^*$

$$= \frac{g_e^4}{(p_1-p_3)^2(p_1-p_4)^2}[\bar{u}_3\gamma^\mu u_1][\bar{u}_4\gamma_\mu u_2][\bar{u}_4\gamma^\nu u_1]^*[\bar{u}_3\gamma_\nu u_2]^*$$

$$\cancel{= (\cdots)[\bar{u}_2\gamma_\mu u_2][u_2^*\gamma^0\gamma^0\gamma_\nu^\dagger\gamma^0 u_3]}$$

$$= (\cdots)[\bar{u}_3\gamma^\mu u_1][u_1^*\gamma^0\gamma^0\gamma^{\nu\dagger}\gamma^0 u_4][\bar{u}_4\gamma_\mu u_2][u_2^*\gamma^0\gamma^0\gamma_\nu^\dagger\gamma^0 u_3]$$

$$= (\cdots)\bar{u}_3\gamma^\mu(u_1\bar{u}_1)\gamma^\nu(u_4\bar{u}_4)\gamma_\mu(u_2\bar{u}_2)\gamma_\nu u_3$$

$$= (\cdots)\bar{u}_3\gamma^\mu \not{p}_1\gamma^\nu \not{p}_4\gamma_\mu \not{p}_2\gamma_\nu u_3$$

$$\boxed{\text{summing over all spin: } (\cdots)\,\mathrm{Tr}[\gamma^\mu \not{p}_1\gamma^\nu \not{p}_4\gamma_\mu \not{p}_2\gamma_\nu \not{p}_3]}$$

$$\mathcal{M}_2\mathcal{M}_1^* = (\cdots)\bar{u}_4\gamma^\mu(u_1\bar{u}_1)\gamma^\nu(u_3\bar{u}_3)\gamma_\mu(u_2\bar{u}_2)\gamma_\nu u_4$$

$$= (\cdots)\bar{u}_4\gamma^\mu \not{p}_1\gamma^\nu \not{p}_3\gamma_\mu \not{p}_2\gamma_\nu u_4$$

$$\boxed{\text{summing over all spin: } (\cdots)\,\mathrm{Tr}[\gamma^\mu \not{p}_1\gamma^\nu \not{p}_3\gamma_\mu \not{p}_2\gamma_\nu p_4]}$$

$$\langle|\mathcal{M}_1|^2\rangle = (\cdots)[\bar{u}_3\gamma^\mu u_1][\bar{u}_3\gamma^\nu u_1]^*[\bar{u}_4\gamma_\mu u_2][\bar{u}_4\gamma_\nu u_2]^*$$

$$= \boxed{(\cdots)\cdot\mathrm{Tr}[\gamma^\mu \not{p}_1\gamma^\nu \not{p}_3] \times \mathrm{Tr}[\gamma_\mu \not{p}_2\gamma_\nu \not{p}_4]}$$ by Casmir.

Similarly, $\langle |\mathcal{M}_2|^2 \rangle$

$$= (\cdots) [\bar{u}_4 \gamma^\mu u_1][\bar{u}_4 \gamma^\nu u_1]^* [\bar{u}_3 \gamma_\mu u_2][\bar{u}_3 \gamma_\nu u_2]^*$$

$$= \boxed{\begin{aligned} &(\cdots) \,\mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4] \\ &\quad \times \mathrm{Tr}[\gamma_\mu \not{p}_2 \gamma_\nu \not{p}_3] \end{aligned}}$$

We now compute the traces:

$$\mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_3]$$

$$= \mathrm{Tr}[\gamma^\mu \gamma^\alpha p_{1\alpha} \gamma^\nu \gamma^\beta p_{4\beta} \gamma_\mu \gamma^\sigma p_{2\sigma} \gamma_\nu \gamma^\lambda p_{3\lambda}].$$

$$= \mathrm{Tr}[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma_\mu \gamma^\sigma \gamma_\nu \gamma^\lambda (p_{1\alpha} p_{4\beta} p_{2\sigma} p_{3\lambda})].$$

$$= \mathrm{Tr}[\gamma^\alpha \gamma^\nu \gamma^\beta (\gamma_\mu \gamma^\sigma \gamma_\nu \gamma^\lambda \gamma^\mu)(p_{1\alpha} p_{4\beta} p_{2\sigma} p_{3\lambda})]$$

$$= \mathrm{Tr}[\gamma^\alpha \gamma^\nu \gamma^\beta (-2\gamma^\lambda \gamma_\nu \gamma^\sigma)(\cdots)]$$

$$= \mathrm{Tr}[-2\gamma^\beta \gamma^\lambda (\gamma_\nu \gamma^\sigma \gamma^\alpha \gamma^\nu)(\cdots)]$$

$$= \mathrm{Tr}[-2\gamma^\beta \gamma^\lambda (4g^{\sigma\alpha})(\cdots)].$$

$$= \mathrm{Tr}[-8\gamma^\beta \gamma^\lambda g^{\sigma\alpha} p_{1\alpha} p_{4\beta} p_{2\sigma} p_{3\lambda}]$$

$$= \mathrm{Tr}[-8\gamma^\beta p_{4\beta} \gamma^\lambda p_{3\lambda}.(p_1 \cdot p_2)].$$

$$= \boxed{\mathrm{Tr}(-8(p_1 \cdot p_2)\not{p}_4 \not{p}_3)}. = -8(p_1 \cdot p_2)4(p_4 \cdot p_3).$$

Similarly, $\mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4]$

$$= \boxed{\mathrm{Tr}(-8(p_1 \cdot p_2)\not{p}_3 \not{p}_4).} = -8(p_1 \cdot p_2)4(p_3 \cdot p_4).$$

$$\mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3] \times \mathrm{Tr}[\gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4]$$

$$= \mathrm{Tr}[\gamma^\mu \gamma^\alpha p_{1\alpha} \gamma^\nu \gamma^\beta p_{3\beta}] \times (\cdots)$$

$$= \mathrm{Tr}[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta] p_{1\alpha} p_{3\beta}$$

$$= 4(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\nu}) p_{1\alpha} p_{3\beta} \times (\cdots)$$

$$= 4(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_3^\mu p_1^\nu)$$
$$\times 4(p_{2\mu} p_{4\nu} - g_{\mu\nu}(p_2 \cdot p_4) + p_{4\mu} p_{2\nu})$$

$$= 16[(p_1 \cdot p_2)(p_3 \cdot p_4) - \cancel{(p_1 \cdot p_3)(p_2 \cdot p_4)} + (p_1 \cdot p_4)(p_3 \cdot p_2)$$
$$\cancel{+ 4(p_1 \cdot p_3)(p_2 \cdot p_4)}$$
$$- \cancel{(p_2 \cdot p_4)(p_1 \cdot p_3)} + \cancel{4(p_1 \cdot p_3)(p_2 \cdot p_4)} - \cancel{(p_2 \cdot p_4)(p_1 \cdot p_3)}$$
$$+ (p_2 \cdot p_3)(p_1 \cdot p_4) - \cancel{(p_1 \cdot p_3)(p_2 \cdot p_4)} + (p_3 \cdot p_4)(p_1 \cdot p_2)]$$

$$= \boxed{16[2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3)]}$$

This implies $\mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4] \times \mathrm{Tr}[\gamma_\mu \not{p}_2 \gamma_\nu \not{p}_3]$

$$= \boxed{16[2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_3)(p_2 \cdot p_4)]}$$

$$\langle |\mathcal{M}|^2 \rangle = \langle |\mathcal{M}_1|^2 \rangle + \langle |\mathcal{M}_2|^2 \rangle - \langle \mathcal{M}_1 \mathcal{M}_2^* \rangle - \langle \mathcal{M}_2 \mathcal{M}_1^* \rangle .$$

$$\cancel{= \left[\frac{g_e^4}{4(p_1-p_3)^4} + \frac{g_e^4}{4(p_1-p_4)^4} \right]}$$

$$= \frac{g_e^4}{4(p_1-p_3)^4} \, 32 \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right]$$

$$+ \frac{g_e^4}{4(p_1-p_4)^4} \, 32 \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) \right]$$

$$- \frac{g_e^4}{(p_1-p_3)^2 (p_1-p_4)^2} \left[-8 (p_1 \cdot p_2) 4 (p_3 \cdot p_4) \right] \times 2 .$$

$$= \frac{g_e^4}{\cancel{4}} \left[\frac{\cancel{32}^{\,8} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right]}{(p_1-p_3)^4} \right.$$

$$+ \frac{\cancel{32}^{\,8} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) \right]}{(p_1-p_4)^4}$$

$$\left. + \frac{\cancel{64}^{\,16} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) \right]}{(p_1-p_3)^2 (p_1-p_4)^2} \right] .$$

Griffiths.

7.25. Compute $\displaystyle\sum_{\text{all spin}} [\bar{v}(a)\Gamma_1 v(b)][\bar{v}(a)\Gamma_2 v(b)]^*$,

$\displaystyle\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 v(b)][\bar{u}(a)\Gamma_2 v(b)]^*$,

$\displaystyle\sum_{\text{all spins}} [\bar{v}(a)\Gamma_1 u(b)][\bar{v}(a)\Gamma_2 u(b)]^*$

with Casimir's trick.

$[\bar{v}(a)\Gamma_1 v(b)][\bar{v}(a)\Gamma_2 v(b)]^*$

$= \bar{v}(a)\Gamma_1 v(b) \underbrace{v^*(b)\gamma^0}\gamma^0 \Gamma_2^* \gamma^0 v(a)$

$= \bar{v}(a)\Gamma_1 v(b)\bar{v}(b)\bar{\Gamma}_2 v(a).$

Sum over all states $\Rightarrow \displaystyle\sum_a \bar{v}(a)\Gamma_1\left\{\sum_b v(b)\bar{v}(b)\right\}\bar{\Gamma}_2 v(a)$

$= \displaystyle\sum_a \bar{v}(a)\left[\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right]v(a).$

$= \displaystyle\sum_a \bar{v}(a)_i \, Q_{ji} \, v(a)_j$

$= \displaystyle\cancel{\sum_a}\sum_a \cancel{\mathrm{Tr}}\left[\bar{v}(a)_i v(a)_j\right] Q_{ji}$

$= \displaystyle\sum_a \mathrm{Tr}\left[[\bar{v}(a)v(a)]_{kj}\, Q_{ji}\right].$

$= \mathrm{Tr}\left[\displaystyle\sum_a(\bar{v}(a)v(a))_{kj}\, Q_{ji}\right].$

$$\mathrm{Tr}\left[\sum_a (\bar{v}(a)\, v(a))_{kj}\, Q_{ji}\right]$$

$$= \boxed{\mathrm{Tr}\left[(\not{p}_a - m_a c)_{kj}\left[\Gamma_1(\not{p}_b - m_b c)\bar{\Gamma}_2\right]_{ji}\right]}.$$

More succinctly, $\mathrm{Tr}\left[(\not{p}_a - m_a c)\Gamma_1(\not{p}_b - m_b c)\bar{\Gamma}_2\right]$

$$\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 v(b)][\bar{u}(a)\Gamma_2 v(b)]^*$$

$$= \sum \bar{u}_a \Gamma_1 v_b v_b^* \gamma^0\gamma^0 \Gamma_2^* \gamma^0 u(a)$$

$$= \sum \bar{u}_a \Gamma_1 v_b \bar{v}_b \bar{\Gamma}_2 u_a.$$

$$= \sum_a \bar{u}_a \left\{\sum_b \Gamma_1 v_b \bar{v}_b \bar{\Gamma}_2\right\} u_a.$$

$$= \sum_a \bar{u}_a \left[\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right] u_a$$

$$= \sum_a \bar{u}_{a_i} \left[\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right]_{ij} u_{a_j}.$$

$$= \sum_a \delta_{jk} (\bar{u}_a u_a)_{ji} \left[\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right]_{ik}$$

$$= \mathrm{Tr}\left\{\left[\sum_a (\bar{u}_a u_a)\right]_{ji} \left[\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right]_{ik}\right\}$$

$$= \boxed{\mathrm{Tr}\left\{(\not{p}_a + mc)_{ji}\left(\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right)_{ik}\right\}}$$

$$\sum_{\text{all spins}} [\bar{v}_a T_1 u_b][\bar{v}_a T_2 u_b]^*$$

$$= \sum \bar{v}_a T_1 u_b u_b^* \gamma^0 \gamma^0 T_2^* \gamma^0 v_a$$

$$= \sum_a \bar{v}_a T_1 \left\{ \sum_b u_b \bar{u}_b \right\} \bar{T}_2 v_a$$

$$= \sum_a \bar{v}_a T_1 (\not{p}_b + m_b c) \bar{T}_2 v_a$$

$$= \sum_a \bar{v}_{a_i} \left[T_1 (\not{p}_b + m_b c) \bar{T}_2 \right]_{ij} v_{a_j}$$

$$= \sum_a \left[\bar{v}_{a_i} v_{a_j} \right] \left[T_1 (\not{p}_b + m_b c) \bar{T}_2 \right]_{ij}$$

$$= \sum_a [\bar{v}_a v_a]_{ik} \delta_{kj} [T_1 (\not{p}_b + m_c) \bar{T}_2]_{ij}$$

$$= \sum_a \delta_{kj} [\bar{v}_a v_a]_{ik} [T_1 (\not{p}_b + m_c) \bar{T}_2]_{ij}$$

$$= \cancel{\sum_a} \mathrm{Tr} \left\{ \sum_a [\bar{v}_a v_a]_{ik} [T_1 (\not{p}_b + m_c) \bar{T}_2]_{ij} \right\}$$

$$= \boxed{\mathrm{Tr} \left\{ (\not{p}_a - m_a c)(T_1 (\not{p}_b + m_b c) \bar{T}_2) \right\}}$$