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2.(a) From linear Algebra, the inverse of M is

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T$$

where C_{ij} denote the cofactor at i-th row, j-th column.

Let $|M| = 1$, we have

$$M^{-1} = \begin{pmatrix} M_{22} & -M_{21} \\ -M_{12} & M_{11} \end{pmatrix}^T = \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}.$$

$$M^\dagger = M^{-1} \Rightarrow \begin{pmatrix} M_{11}^* & M_{21}^* \\ M_{12}^* & M_{22}^* \end{pmatrix} = \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}.$$

$$\Rightarrow M_{11}^* = M_{22}, \quad M_{21}^* = -M_{12}.$$

There are 4 linearly independent solutions for M ~~disregarding~~

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} & i \\ i & \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

$$\Updownarrow \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow$$

$$(I) \quad (i\sigma_3) \quad (i\sigma_1), \quad (-i\sigma_2).$$

Thus a general solution for M would be a linear combination of these matrices that preserves determinant of 1.

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2. (b). We compute $\exp\left(i\frac{\vec\theta\cdot\vec\sigma}{2}\right)$ explicitly:

$$i\frac{\vec\theta\cdot\vec\sigma}{2} = \frac{i}{2}\begin{pmatrix}\theta_3 & \theta_1 - i\theta_2\\ \theta_1+i\theta_2 & -\theta_3\end{pmatrix}$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^0 = \begin{pmatrix}1 & \\ & 1\end{pmatrix},$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^1 = \frac{i}{2}\begin{pmatrix}\theta_3 & \theta_1 - i\theta_2\\ \theta_1+i\theta_2 & -\theta_3\end{pmatrix},$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^2 = -\frac{1}{4}\begin{pmatrix}\theta_3^2+\theta_2^2+\theta_1^2 & \\ & \theta_3^2+\theta_2^2+\theta_1^2\end{pmatrix}$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^3 = -\frac{i}{8}(\theta_3^2+\theta_2^2+\theta_1^2)\mathbb{1}\begin{pmatrix}\theta_3 & \theta_1 - i\theta_2\\ \theta_1+i\theta_2 & -\theta_3\end{pmatrix}$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^4 = \frac{1}{16}(\theta_3^2+\theta_2^2+\theta_1^2)^2\,\mathbb{1}$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^5 = \frac{i}{32}(\theta_3^2+\theta_2^2+\theta_1^2)^2\begin{pmatrix}\theta_3 & \theta_1 - i\theta_2\\ \theta_1+i\theta_2 & -\theta_3\end{pmatrix}$$

$$\left(\frac{i\vec\theta\cdot\vec\sigma}{2}\right)^6 = -\frac{1}{64}(\theta_3^2+\theta_2^2+\theta_1^2)^3.$$

cos

sin

$$\Rightarrow \boxed{\exp\left(i\frac{\vec\theta\cdot\vec\sigma}{2}\right) = \cos\left(\frac{r}{2}\right)\begin{pmatrix}1 & \\ & 1\end{pmatrix} + i\sin\left(\frac{r}{2}\right)\begin{bmatrix}\theta_3 & \theta_1 - i\theta_2\\ \theta_1+i\theta_2 & -\theta_3\end{bmatrix}\frac{1}{r} \quad \text{where } r=\sqrt{\theta_1^2+\theta_2^2+\theta_3^2}}$$

$$= \cos\left(\frac{r}{2}\right)\begin{pmatrix}1 & \\ & 1\end{pmatrix} + \sin\left(\frac{r}{2}\right)\left[\theta_3 i\sigma_3 + \theta_1 i\sigma_1 + \theta_2 i\sigma_2\right]\frac{1}{r}$$

It's obvious that this is in the form of $aI + bi\sigma_1 + ci\sigma_2 + di\sigma_3$ where $a^2+b^2+c^2+d^2 = 1$. So this is precisely $SU(2)$.