Schwartz 6.1

$$D_F = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{ik(x_1-x_2)} = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{ikx}$$

$$= \int \frac{dk_0}{(2\pi)} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{i}{k_0^2-m^2-|\vec{k}|^2+i\varepsilon} e^{ik_0t} e^{-i\vec{k}\cdot\vec{x}}$$

$$= \int \frac{dk_0}{(2\pi)} e^{ik_0t} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{-i}{-k_0^2+m^2+|\vec{k}|^2-i\varepsilon} e^{-i\vec{k}\cdot\vec{x}}$$

$$= (- - -) \int \frac{k^2\,dk\,d(\cos\theta)\,d\phi}{(2\pi)^3} \frac{-i}{k^2-c^2-i\varepsilon} e^{-ik(\cos\theta)|\vec{x}|}$$

$$c^2 = (k_0^2 - m^2)$$

$$= (- - -) \int \frac{k^2\,dk\,d(\cos\theta)}{(2\pi)^2} \frac{-i}{k^2-c^2-i\varepsilon} e^{-ik(\cos\theta)|\vec{x}|}$$

$$= (- - -) \int \frac{k^2\,dk}{(2\pi)^2} \frac{\cancel{-i}}{k^2-c^2-i\varepsilon} \left[\frac{1}{\cancel{-i}k|\vec{x}|} e^{-ik(\cos\theta)|\vec{x}|}\right]_{-1}^{1}$$

$$= (- - - -) \int \frac{k^{\cancel{2}}\,dk}{(2\pi)^2} \frac{1}{k^2-c^2-i\varepsilon} \left[\frac{1}{\cancel{k}|\vec{x}|}\right]\left[e^{-ik|\vec{x}|} - e^{ik|\vec{x}|}\right]$$

$$= (- - - -) \int \frac{k\,dk}{(2\pi)^2} \frac{1}{k^2-c^2-i\varepsilon} \frac{1}{|\vec{x}|} \left[e^{-ik|\vec{x}|} - e^{ik|\vec{x}|}\right]$$

$$\frac{1}{k^2-c^2-i\varepsilon} = \frac{1}{2c}\left[\frac{1}{k-(c-i\varepsilon)} - \frac{1}{k-(-c+i\varepsilon)}\right]$$

$$= (- - - -) \int \frac{k\,dk}{(2\pi)^2} \frac{1}{2c}\left[\frac{1}{k-(c-i\varepsilon)} - \frac{1}{k-(-c+i\varepsilon)}\right] \frac{1}{|\vec{x}|}\left[e^{-ik|\vec{x}|} - e^{ik|\vec{x}|}\right]$$

$$= (\cdots)\int \frac{k\,dk}{(2\pi)^2 (2c)|\vec{x}|}\left[\frac{1}{k-(c-i\varepsilon)} - \frac{1}{k-(-c+i\varepsilon)}\right]\left[e^{-ik|\vec{x}|} - e^{ik|\vec{x}|}\right]$$

~~By~~ $|\vec{x}| > 0$, so $e^{-ik|\vec{x}|}$ needs to be closed from below, picking up singularity $(c-i\varepsilon)$.

, $e^{ik|\vec{x}|}$ needs to be closed from above, picking up singularity $(-c+i\varepsilon)$.

$$\frac{[k-(c-i\varepsilon)+(c-i\varepsilon)]}{k-(c-i\varepsilon)} = \frac{k}{k-(c-i\varepsilon)} = 1 + \frac{(c-i\varepsilon)}{k-(c-i\varepsilon)}$$

$$\Uparrow$$

$$(c-i\varepsilon)$$

$$e^{-ik|\vec{x}|} = e^{-i\{[k-(c-i\varepsilon)]+(c-i\varepsilon)\}|\vec{x}|}$$

$$= e^{-i[k-(c-i\varepsilon)]}\, e^{-i(c-i\varepsilon)|\vec{x}|}$$

$$\Uparrow \qquad\qquad \Uparrow$$

$$1 \qquad\qquad e^{-i(c-i\varepsilon)|\vec{x}|}$$

$$(-2\pi i)(c-i\varepsilon)e^{-i(c-i\varepsilon)|\vec{x}|} = -2\pi i c\, e^{-ic|\vec{x}|} \quad \text{at} \quad \varepsilon = 0.$$

The other term: $\dfrac{[k-(-c+i\varepsilon)+(-c+i\varepsilon)]}{k-(-c+i\varepsilon)}\, e^{i[(k-(-c+i\varepsilon))+(-c+i\varepsilon)]|\vec{x}|}$

$$\Uparrow \qquad\qquad \Uparrow$$

$$-c \quad (-c+i\varepsilon) \qquad\qquad e^{i[-c+i\varepsilon]|\vec{x}|}$$

$$\Rightarrow (-c+i\varepsilon)(2\pi i)\, e^{i(-c+i\varepsilon)|\vec{x}|} = -2\pi i c\, e^{-ic|\vec{x}|} \quad \text{at} \quad \varepsilon = 0.$$

$$\Rightarrow \; = \cancel{(\cdots)\int} \; (\cdots)\frac{1}{(2\pi)^2 (2c)|\vec{x}|}\left[-4\pi i c\, e^{-ic|\vec{x}|}\right]$$

$$= \int \frac{dk_0}{(2\pi)} e^{ik_0 t} \frac{1}{(2\pi)^2 (\cancel{2}\cancel{i}) |\vec{x}|} \frac{1}{\cancel{2}} \left[-\cancel{4}\pi i \cancel{i}\, e^{-ic|\vec{x}|} \right]$$

$$= \boxed{\int \frac{dk_0}{(2\pi)} e^{ik_0 t} \frac{1}{(2\pi)^2 |\vec{x}|} \left[-\pi i\, e^{-ic|\vec{x}|} \right]}$$

$$c^2 = k_0^2 - m^2, \qquad c = \sqrt{k_0^2 - m^2}$$

$$= \int \frac{dk_0}{(2\pi)^3} \frac{1}{|\vec{x}|} (-\pi i)\, e^{ik_0 t - ic|\vec{x}|}$$

$m \to 0$ limit $\Longleftrightarrow$ $c = k_0$. $\Rightarrow$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dk_0}{(2\pi)^3} \frac{1}{|\vec{x}|} (-\pi i)\, e^{ik_0 (t - |\vec{x}|)}$$

$$\boxed{\text{Schwinger Param: } \frac{i}{A} = \int_0^\infty ds\, e^{isA}}$$

$$= \frac{1}{(2\pi)^3} \frac{1}{|\vec{x}|} (-\pi i) \int_0^\infty dk_0\, e^{ik_0 (t - |\vec{x}|)} \times 2$$

$$= \frac{1}{(2\pi)^3} \frac{1}{|\vec{x}|} (-\pi i) \frac{i}{t - |\vec{x}|} \times 2$$

$$= \frac{2\pi}{(2\pi)^3} \frac{1}{|\vec{x}|} \frac{i(-i)}{t - |\vec{x}|} = \frac{1}{4\pi^2} \frac{1}{|\vec{x}| t - |\vec{x}|^2}$$

$$\boxed{\simeq -\frac{1}{4\pi^2} \frac{1}{(x_1 - x_2)^2 - i\epsilon}}$$

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