Matrices and Basic Operations

Ex 1.

For the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$$

calculate

- $A + B$
- $A - B$
- $2A$
- $3B - 2A$
- $A \cdot B$
- check if $A \cdot B = B \cdot A$.

Ex 2.

For the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 0 \\ 0 & 16 \end{pmatrix}$$

check if

$$A \cdot B \cdot C \cdot D = B \cdot A \cdot D \cdot C = D \cdot C \cdot B \cdot A.$$

Ex 3.

Given the matrix

$$C = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 2 & -1 \end{pmatrix}.$$

Determine the matrix obtained after rearranging rows: swap the 1st and 3rd rows, then add twice the new 1st row to the 2nd row. Write down all steps for each operation.

Ex 4.

For column vectors $u = (1, -2, 3)^\top$ and $v = (2, 0, -1)^\top$, write them as matrices and calculate $u + v$, $u - v$, and the matrix products $u\,v^\top$ and $v\,u^\top$. What is the rank of matrix $u\,v^\top$?

Ex 5.

Show that the diagonal matrix $D = \operatorname{diag}(2, -3, 5)$ commutes with any diagonal matrix $E = \operatorname{diag}(a, b, c)$. Additionally, calculate D^3 and, if it exists, D^{-1}.

Ex 6.

$\star$ For the matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

calculate P^2 and P^3. Does the sequence P^n have a noticeable pattern for $n = 1, 2, 3$?

Ex 7.

$\star$ Rotation coding example

Calculate the product of rotation matrices with angle θ in 2D space:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Check that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$.

Ex 8.

$\star$ Knowing that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

show that the rotation matrix $R(\theta)$ can be written as

$$R(\theta) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

where

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

Ex 9.

$\star\star$ Pauli matrices are defined as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where i is the imaginary unit. Check that:

- $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$ (identity matrix)
- $\sigma_x\sigma_y = i\sigma_z$, $\sigma_y\sigma_z = i\sigma_x$, $\sigma_z\sigma_x = i\sigma_y$
- $\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$ (anticommutator)