

# Calculus

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# Chapter 1

## Sets and functions

### Sets and Functions

#### Cartesian Product and Relations

**Cartesian Product** Given two sets  $A$  and  $B$ , the Cartesian product of  $A$  and  $B$ , denoted as  $A \times B$ , is the set of all ordered pairs  $(a, b)$ , where  $a \in A$  and  $b \in B$ . Formally:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

**Binary Relation** A binary relation  $\mathcal{R}$  (or simply a relation) from a set  $A$  to a set  $B$  is any subset of the Cartesian product  $A \times B$ .

$$\mathcal{R} \subseteq A \times B$$

If  $(a, b) \in \mathcal{R}$ , we say that element  $a$  is in relation  $\mathcal{R}$  with element  $b$ , which is also written as  $a\mathcal{R}b$ . If  $A = B$ , we say that  $\mathcal{R}$  is a relation on the set  $A$ .

**Example** Let  $A = \{1, 2, 3\}$  and  $B = \{x, y\}$ . Then the Cartesian product  $A \times B$  is:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$$

A relation  $\mathcal{R}$  could be, for example, the subset  $\mathcal{R} = \{(1, x), (2, y), (3, x)\}$ .

#### Functions as a Special Type of Relation

**Definition of a Function** A function  $f$  from a set  $X$  (called the domain) to a set  $Y$  (called the codomain) is a binary relation  $f \subseteq X \times Y$  that satisfies the following condition:

$$\forall_{x \in X} \quad \exists!_{y \in Y} \quad (x, y) \in f$$

The symbol  $\exists!$  means "there exists exactly one".

In other words, a function is a relation in which every element of the domain  $X$  is associated with exactly one element of the codomain  $Y$ . Instead of writing  $(x, y) \in f$ , we use the more common notation  $y = f(x)$ .

**Formal Conditions for the Definition of a Function** A relation  $\mathcal{R} \subseteq X \times Y$  is a function if it satisfies two conditions:

1. **Existence Condition:** For every element  $x \in X$ , there exists at least one element  $y \in Y$  such that  $(x, y) \in \mathcal{R}$ .

$$\forall x \in X \exists y \in Y : (x, y) \in \mathcal{R}$$

2. **Uniqueness Condition (Right-Uniqueness):** If an element  $x \in X$  is in relation with  $y_1 \in Y$  and also with  $y_2 \in Y$ , then it must be that  $y_1 = y_2$ .

$$\forall x \in X \forall y_1, y_2 \in Y : ((x, y_1) \in \mathcal{R} \wedge (x, y_2) \in \mathcal{R}) \implies y_1 = y_2$$

**Example** Consider the sets  $X = \{1, 2, 3\}$  and  $Y = \{a, b, c\}$ .

- The relation  $\mathcal{R}_1 = \{(1, a), (2, b), (3, c)\}$  is a function. Every element in  $X$  corresponds to exactly one element in  $Y$ .
- The relation  $\mathcal{R}_2 = \{(1, a), (2, b)\}$  is not a function because the element  $3 \in X$  has no corresponding element in  $Y$  (it fails the existence condition).
- The relation  $\mathcal{R}_3 = \{(1, a), (1, b), (2, c), (3, a)\}$  is not a function because the element  $1 \in X$  is associated with two different elements from  $Y$ :  $a$  and  $b$  (it fails the uniqueness condition).

**Further Examples** Functions can take many forms, depending on their domain and codomain:

- **A polynomial function:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by the formula  $f(x) = x^2 - 4x + 3$ . This is a function where every real number is mapped to another real number. The relation consists of pairs  $(x, x^2 - 4x + 3)$  for all  $x \in \mathbb{R}$ .
- **A function on finite sets:** Let  $A = \{\text{apple}, \text{banana}, \text{cherry}\}$  and  $B = \{\text{red}, \text{yellow}\}$ . A function  $g : A \rightarrow B$  could be defined as  $g(\text{apple}) = \text{red}$ ,  $g(\text{banana}) = \text{yellow}$ , and  $g(\text{cherry}) = \text{red}$ . This is a valid function, even though two different elements from the domain map to the same element in the codomain.
- **The identity function:** For any set  $X$ , the identity function  $id_X : X \rightarrow X$  is defined as  $id_X(x) = x$  for all  $x \in X$ . It maps every element to itself.

## Sequences as Functions

It is worth noting that sequences, which are often treated as a separate topic, are fundamentally a type of function. A sequence of real numbers, for example, is a function whose domain is the set of natural numbers  $\mathbb{N}$  and whose codomain is the set of real numbers  $\mathbb{R}$ .

**Formal Definition of a Sequence** An infinite sequence of elements from a set  $A$  is a function  $f : \mathbb{N} \rightarrow A$ . Instead of the standard function notation  $f(n)$ , we typically use subscript notation, such as  $a_n$ , to denote the value of the function for the argument  $n$ . The entire sequence is then denoted as  $(a_n)_{n=1}^{\infty}$  or simply  $(a_n)$ .

**Example** The sequence defined by the formula  $a_n = 1/n$  for  $n \in \{1, 2, 3, \dots\}$  is actually a function  $f : \mathbb{N} \rightarrow \mathbb{R}$  where  $f(n) = 1/n$ . The sequence is  $(1, 1/2, 1/3, \dots)$ . This perspective is crucial because it allows us to apply concepts from the theory of functions (like limits) directly to sequences.

## Further examples of functions

Many mathematical concepts, which may seem different at first glance, are also examples of functions.

**Determinant of a Matrix** The determinant is a function that takes a square matrix as input and produces a single number (a scalar). For a given dimension  $n$ , the determinant function ‘det’ can be seen as a function:

$$\det : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$$

where  $M_{n \times n}(\mathbb{R})$  is the set of all  $n \times n$  matrices with real coefficients. For each matrix  $A \in M_{n \times n}(\mathbb{R})$ , the function assigns exactly one real number,  $\det(A)$ .

**Operators in Calculus** Operators in calculus, such as differentiation and integration, can also be viewed as functions. These are often called *functionals* or *higher-order functions* because their domain and/or codomain are sets of functions.

- **Differentiation:** The differentiation operator,  $\frac{d}{dx}$ , can be considered a function that maps a differentiable function to another function. For example, if we consider the set  $C^1(\mathbb{R})$  of all continuously differentiable functions on  $\mathbb{R}$  and the set  $C^0(\mathbb{R})$  of all continuous functions on  $\mathbb{R}$ , then:

$$\frac{d}{dx} : C^1(\mathbb{R}) \rightarrow C^0(\mathbb{R})$$

It takes a function, like  $f(x) = x^2$ , and returns its derivative,  $f'(x) = 2x$ .

- **Definite Integration:** A definite integral is a function that maps a function to a number. For instance, the operator that integrates a continuous function over the interval  $[0, 1]$  can be defined as:

$$I : C([0, 1]) \rightarrow \mathbb{R}, \quad \text{where} \quad I(f) = \int_0^1 f(x) dx$$

This function takes a continuous function  $f$  and assigns to it the scalar value representing the area under its curve on the interval  $[0, 1]$ .

## The Importance of Functions in Mathematics

The concept of a function is one of the most fundamental and unifying ideas in all of mathematics. Functions appear in virtually every branch of the discipline, from calculus and algebra to topology and logic. They are the primary tool for describing relationships and transformations between different mathematical objects.

Whether we are modeling physical phenomena, analyzing data, or studying abstract structures, functions provide the language and framework to express how quantities depend on one another. Understanding functions not just as calculation rules, but as versatile mappings between sets, is a key step toward deeper mathematical insight.

# Chapter 2

## Topological Foundations: Metric Spaces

To rigorously define what it means for a variable to "approach" a point, we need to establish a concept of distance and closeness. This is the domain of topology, and its foundational layer for analysis is the theory of metric spaces.

### Metric Spaces

**Definition of a Metric** A **metric** on a non-empty set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  that satisfies the following properties for all  $x, y, z \in X$ :

1. **Non-negativity:**  $d(x, y) \geq 0$
2. **Identity of indiscernibles:**  $d(x, y) = 0 \iff x = y$
3. **Symmetry:**  $d(x, y) = d(y, x)$
4. **Triangle Inequality:**  $d(x, z) \leq d(x, y) + d(y, z)$

The pair  $(X, d)$  is called a **metric space**, and the value  $d(x, y)$  is the **distance** between points  $x$  and  $y$ .

### Examples of Metrics

- **Euclidean Metric on  $\mathbb{R}$ :** This is the standard distance we use on the number line. For  $x, y \in \mathbb{R}$ , the metric is defined as:

$$d(x, y) = |x - y|$$

This is the metric implicitly used in the  $\varepsilon - \delta$  definitions of limits.

- **Manhattan (Taxicab) Metric on  $\mathbb{R}^2$ :** For two points  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$  in the plane, the Manhattan distance is:

$$d_1(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

It measures the distance as if moving along a grid, like a taxi in a city.

- **Discrete Metric:** On any non-empty set  $X$ , we can define the discrete metric as:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

In this space, every two distinct points are "far" from each other.

# Neighborhoods and Open Sets

**Open Ball (Neighborhood)** In a metric space  $(X, d)$ , the **open ball** centered at a point  $c \in X$  with radius  $\varepsilon > 0$  is the set of all points that are at a distance less than  $\varepsilon$  from  $c$ . It is denoted as  $B(c, \varepsilon)$ :

$$B(c, \varepsilon) = \{x \in X \mid d(x, c) < \varepsilon\}$$

An open ball is the most basic form of a **neighborhood** of a point. For instance, in  $\mathbb{R}$  with the standard metric, the open ball  $B(c, \varepsilon)$  is simply the open interval  $(c - \varepsilon, c + \varepsilon)$ .

**Limit Point (Accumulation Point)** Let  $(X, d)$  be a metric space and let  $D$  be a subset of  $X$ . A point  $c \in X$  (which does not have to be in  $D$ ) is called a **limit point** or **accumulation point** of the set  $D$  if every open ball centered at  $c$  contains at least one point of  $D$  that is different from  $c$ . Formally, for every  $\varepsilon > 0$ , the set  $B(c, \varepsilon) \cap (D \setminus \{c\})$  is non-empty.

$$\forall_{\varepsilon > 0} \quad (B(c, \varepsilon) \setminus \{c\}) \cap D \neq \emptyset$$

The set of all limit points of  $D$  is called the derived set, denoted  $D'$ .

**Connecting to Limits** The concept of a limit point is crucial. When we consider  $\lim_{x \rightarrow c} f(x)$ , we are interested in the behavior of  $f$  for values of  $x$  that are "close to" but "not equal to"  $c$ . This is only meaningful if  $c$  is a limit point of the function's domain, ensuring that there are always points in the domain arbitrarily close to  $c$ .

The phrase " $x$  approaches  $c$ " can now be understood formally. It refers to considering values of  $x$  within successively smaller open balls (neighborhoods) around  $c$ . The definition of a function limit,  $\forall \varepsilon > 0, \exists \delta > 0 \dots$ , is a precise way of stating that for any desired closeness in the codomain (an  $\varepsilon$ -neighborhood around the limit  $L$ ), we can find a small enough region in the domain (a  $\delta$ -neighborhood around  $c$ ) that maps into it.

# Chapter 3

## Limits

### Limits

#### Limit of a Sequence

**Formal Definition** A number  $L \in \mathbb{R}$  is the limit of a sequence of real numbers  $(a_n)_{n=1}^{\infty}$  if for every real number  $\varepsilon > 0$ , there exists a natural number  $N$  such that for all  $n > N$ , the inequality  $|a_n - L| < \varepsilon$  holds.

We write this as:

$$\lim_{n \rightarrow \infty} a_n = L$$

which is equivalent to:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |a_n - L| < \varepsilon$$

**Example: Proof of convergence for the sequence  $a_n = 1/n$**  We will prove that the limit of the sequence  $a_n = 1/n$  is 0.

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

**Proof:** According to the formal definition of a limit, we need to show that for any given  $\varepsilon > 0$ , there exists a natural number  $N$  such that for all  $n > N$ , the inequality  $|a_n - L| < \varepsilon$  is satisfied.

In our case,  $a_n = 1/n$  and  $L = 0$ . The inequality becomes:

$$\left| \frac{1}{n} - 0 \right| < \varepsilon$$

Since  $n$  is a natural number,  $n > 0$ , so  $|1/n| = 1/n$ . The inequality simplifies to:

$$\frac{1}{n} < \varepsilon$$

To find an appropriate  $N$ , we can solve this inequality for  $n$ :

$$n > \frac{1}{\varepsilon}$$

By the Archimedean property of real numbers, for any positive real number  $\varepsilon$ , there exists a natural number  $N$  such that  $N > 1/\varepsilon$ . Let's choose  $N = \lceil 1/\varepsilon \rceil$ .

Now, for any  $n > N$ , we have:

$$n > N \geq \frac{1}{\varepsilon}$$

This implies that  $1/n < \varepsilon$ . Therefore, we have shown that for any  $\varepsilon > 0$ , we can find an  $N$  (for example,  $N = \lceil 1/\varepsilon \rceil$ ) such that for all  $n > N$ ,  $|1/n - 0| < \varepsilon$ . This completes the proof that the limit of the sequence  $1/n$  is 0. ■

**Example: Proof of convergence for the sequence  $a_n = \frac{n^2-1}{n^2}$**  We will prove that the limit of the sequence  $a_n = \frac{n^2-1}{n^2}$  is 1.

$$\lim_{n \rightarrow \infty} \frac{n^2-1}{n^2} = 1$$

**Proof:** According to the formal definition of a limit, we need to show that for any given  $\varepsilon > 0$ , there exists a natural number  $N$  such that for all  $n > N$ , the inequality  $|a_n - L| < \varepsilon$  is satisfied.

In our case,  $a_n = \frac{n^2-1}{n^2}$  and  $L = 1$ . The inequality becomes:

$$\left| \frac{n^2-1}{n^2} - 1 \right| < \varepsilon$$

Let's simplify the expression inside the absolute value:

$$\left| \frac{n^2-1-n^2}{n^2} \right| = \left| -\frac{1}{n^2} \right| = \frac{1}{n^2}$$

The inequality simplifies to:

$$\frac{1}{n^2} < \varepsilon$$

To find an appropriate  $N$ , we can solve this inequality for  $n$ :

$$n^2 > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\sqrt{\varepsilon}}$$

Let's choose  $N = \lceil 1/\sqrt{\varepsilon} \rceil$ .

Now, for any  $n > N$ , we have:

$$n > N \geq \frac{1}{\sqrt{\varepsilon}}$$

This implies that  $n^2 > 1/\varepsilon$ , and therefore  $1/n^2 < \varepsilon$ . Thus, we have shown that for any  $\varepsilon > 0$ , we can find an  $N$  such that for all  $n > N$ ,  $|\frac{n^2-1}{n^2} - 1| < \varepsilon$ . This completes the proof that the limit of the sequence is 1. ■

**The Limit of the Sequence**  $a_n = (1 + 1/n)^n$  A fundamental limit in calculus is the limit of the sequence  $a_n = (1 + 1/n)^n$ . This limit exists, and its value is the base of the natural logarithm, the number  $e$ .

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

The proof of the existence of this limit is more involved than the previous examples. It relies on the Monotone Convergence Theorem, which states that a sequence that is both monotonic (either non-decreasing or non-increasing) and bounded must converge.

We will prove that the sequence  $a_n$  is monotonically increasing and bounded above.

**1. Monotonically Increasing** ( $a_n \leq a_{n+1}$ ) Using the binomial theorem, we can expand the term  $a_n$ :

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k \\ &= 1 + \binom{n}{1} \frac{1}{n} + \binom{n}{2} \frac{1}{n^2} + \cdots + \binom{n}{n} \frac{1}{n^n} \\ &= 1 + 1 + \frac{n(n-1)}{2!} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} + \cdots \\ &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots \end{aligned}$$

Similarly, for  $a_{n+1}$ :

$$a_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \frac{1}{3!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) + \cdots$$

By comparing the terms of the expansions for  $a_n$  and  $a_{n+1}$ , we can see that for each  $k \geq 2$ , the term

$$\frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$$

is less than the corresponding term for  $a_{n+1}$ :

$$\frac{1}{k!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right)$$

because for each factor,  $\left(1 - \frac{j}{n}\right) < \left(1 - \frac{j}{n+1}\right)$  for  $j = 1, 2, \dots, k-1$ . Furthermore, the expansion of  $a_{n+1}$  has one additional positive term. Thus, we can conclude that  $a_n < a_{n+1}$  for all  $n \geq 1$ . The sequence is strictly increasing.

**2. Bounded Above** From the binomial expansion of  $a_n$ , we have:

$$a_n = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots$$

Since each term  $\left(1 - \frac{j}{n}\right)$  is less than 1, we can establish an upper bound:

$$a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Now, we can use the inequality  $k! \geq 2^{k-1}$  for  $k \geq 2$ . This gives us  $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$ .

$$a_n < 1 + \left( 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \right)$$

The expression in the parentheses is a finite geometric series with first term 1 and common ratio  $1/2$ . The sum is:

$$S_n = \frac{1 - (1/2)^n}{1 - 1/2} = 2 \left( 1 - \frac{1}{2^n} \right) < 2$$

Therefore, for all  $n$ , we have:

$$a_n < 1 + 2 = 3$$

The sequence is bounded above by 3.

**Conclusion** Since the sequence  $(a_n)_{n=1}^{\infty}$  is monotonically increasing and bounded above, it is convergent. The limit is defined as Euler's number,  $e$ . ■

## Limit of a Function

The limit of a function at a point can be defined in two equivalent ways: the Cauchy definition (using  $\varepsilon$  and  $\delta$ ) and the Heine definition (using sequences).

**Cauchy's Definition ( $\varepsilon$ - $\delta$ )** Let  $f$  be a real-valued function defined on a subset  $D \subseteq \mathbb{R}$ , and let  $c$  be a limit point of  $D$ . A number  $L \in \mathbb{R}$  is the limit of the function  $f$  as  $x$  approaches  $c$  if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in D$ , if  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \varepsilon$ .

We write this as:

$$\lim_{x \rightarrow c} f(x) = L$$

which is equivalent to:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D \quad (0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon)$$

The condition  $0 < |x - c|$  means that we do not consider the value of  $f(c)$  itself when determining the limit.

**Heine's Definition (Sequential)** Let  $f$  be a real-valued function defined on a subset  $D \subseteq \mathbb{R}$ , and let  $c$  be a limit point of  $D$ . A number  $L \in \mathbb{R}$  is the limit of the function  $f$  as  $x$  approaches  $c$  if for every sequence  $(x_n)_{n=1}^{\infty}$  of elements in  $D \setminus \{c\}$  that converges to  $c$ , the sequence of function values  $(f(x_n))_{n=1}^{\infty}$  converges to  $L$ .

Formally:

$$\lim_{x \rightarrow c} f(x) = L$$

is equivalent to:

$$\forall (x_n)_{n=1}^{\infty} \subseteq D \setminus \{c\} \quad \left( \left( \lim_{n \rightarrow \infty} x_n = c \right) \implies \left( \lim_{n \rightarrow \infty} f(x_n) = L \right) \right)$$

**Equivalence** The Cauchy and Heine definitions of the limit of a function are equivalent. This is a fundamental theorem in real analysis.

## Algebra of Limits

In this section, we present the main theorems regarding arithmetic operations on limits. We will focus on sequences, but analogous theorems hold for functions.

**Theorem (Sum of Limits)** Let  $(a_n)$  and  $(b_n)$  be two sequences of real numbers such that  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ . Then the limit of the sum of these sequences exists and is equal to the sum of their limits:

$$\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$$

**Proof** By the definition of a limit, we have:

$$\forall \varepsilon_A > 0 \exists N_A \in \mathbb{N} : \forall n > N_A \quad |a_n - A| < \varepsilon_A$$

$$\forall \varepsilon_B > 0 \exists N_B \in \mathbb{N} : \forall n > N_B \quad |b_n - B| < \varepsilon_B$$

We want to show that for any  $\varepsilon > 0$ , there exists an  $N$  such that for all  $n > N$ ,  $|(a_n + b_n) - (A + B)| < \varepsilon$ .

Let  $\varepsilon > 0$  be given. Let's choose  $\varepsilon_A = \varepsilon/2$  and  $\varepsilon_B = \varepsilon/2$ . From the definitions of the limits of  $(a_n)$  and  $(b_n)$ , there exist  $N_A$  and  $N_B$  corresponding to these choices.

Let  $N = \max(N_A, N_B)$ . Then for any  $n > N$ , both inequalities  $|a_n - A| < \varepsilon/2$  and  $|b_n - B| < \varepsilon/2$  hold.

Using the triangle inequality, we can write:

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, we have shown that for any  $\varepsilon > 0$ , there exists an  $N$  such that for all  $n > N$ ,  $|(a_n + b_n) - (A + B)| < \varepsilon$ . This completes the proof. ■

**Theorem (Product of Limits)** Let  $(a_n)$  and  $(b_n)$  be two sequences such that  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ . Then:

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$$

**Proof** Since the sequence  $(a_n)$  is convergent, it is bounded. This means there exists a real number  $M > 0$  such that  $|a_n| \leq M$  for all  $n \in \mathbb{N}$ .

Let  $\varepsilon > 0$  be given. From the definitions of the limits of  $(a_n)$  and  $(b_n)$ : For  $\frac{\varepsilon}{2M} > 0$ , there exists an  $N_B \in \mathbb{N}$  such that for all  $n > N_B$ :

$$|b_n - B| < \frac{\varepsilon}{2M}$$

For  $\frac{\varepsilon}{2(|B|+1)} > 0$  (we use  $|B|+1$  to avoid division by zero if  $B = 0$ ), there exists an  $N_A \in \mathbb{N}$  such that for all  $n > N_A$ :

$$|a_n - A| < \frac{\varepsilon}{2(|B|+1)}$$

Let  $N = \max(N_A, N_B)$ . For any  $n > N$ , we can estimate the difference  $|a_n b_n - AB|$ :

$$\begin{aligned}
|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\
&= |a_n(b_n - B) + B(a_n - A)| \\
&\leq |a_n(b_n - B)| + |B(a_n - A)| \quad (\text{by triangle inequality}) \\
&= |a_n||b_n - B| + |B||a_n - A| \\
&\leq M|b_n - B| + |B||a_n - A| \\
&< M\left(\frac{\varepsilon}{2M}\right) + |B|\left(\frac{\varepsilon}{2(|B| + 1)}\right) \\
&= \frac{\varepsilon}{2} + \frac{|B|}{|B| + 1} \cdot \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This shows that  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = AB$ . ■

**Theorem (Quotient of Limits)** Let  $(a_n)$  and  $(b_n)$  be two sequences such that  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ . If  $B \neq 0$  and  $b_n \neq 0$  for all  $n$ , then:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$$