Calculus

Damian Chorążkiewicz

December 2, 2025

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Chapter 1

Sets and functions

Sets and Functions

Cartesian Product and Relations

Cartesian Product Given two sets A and B, the Cartesian product of A and B, denoted as $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Formally:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Binary Relation A binary relation $\mathcal{R}$ (or simply a relation) from a set A to a set B is any subset of the Cartesian product $A \times B$.

$$\mathcal{R} \subseteq A \times B$$

If $(a, b) \in \mathcal{R}$, we say that element a is in relation $\mathcal{R}$ with element b, which is also written as $a\mathcal{R}b$. If $A = B$, we say that $\mathcal{R}$ is a relation on the set A.

Example Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Then the Cartesian product $A \times B$ is:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$$

A relation $\mathcal{R}$ could be, for example, the subset $\mathcal{R} = \{(1, x), (2, y), (3, x)\}$.

Functions as a Special Type of Relation

Definition of a Function A function f from a set X (called the domain) to a set Y (called the codomain) is a binary relation $f \subseteq X \times Y$ that satisfies the following condition:

$$\forall_{x \in X} \quad \exists!_{y \in Y} \quad (x, y) \in f$$

The symbol $\exists!$ means "there exists exactly one".

In other words, a function is a relation in which every element of the domain X is associated with exactly one element of the codomain Y. Instead of writing $(x, y) \in f$, we use the more common notation $y = f(x)$.

Formal Conditions for the Definition of a Function A relation $\mathcal{R} \subseteq X \times Y$ is a function if it satisfies two conditions:

1. **Existence Condition:** For every element $x \in X$, there exists at least one element $y \in Y$ such that $(x, y) \in \mathcal{R}$.

$$\forall_{x \in X} \exists_{y \in Y} : (x, y) \in \mathcal{R}$$

2. **Uniqueness Condition (Right-Uniqueness):** If an element $x \in X$ is in relation with $y_1 \in Y$ and also with $y_2 \in Y$, then it must be that $y_1 = y_2$.

$$\forall_{x \in X} \forall_{y_1, y_2 \in Y} : ((x, y_1) \in \mathcal{R} \wedge (x, y_2) \in \mathcal{R}) \implies y_1 = y_2$$

Example Consider the sets $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

- The relation $\mathcal{R}_1 = \{(1, a), (2, b), (3, c)\}$ is a function. Every element in X corresponds to exactly one element in Y.
- The relation $\mathcal{R}_2 = \{(1, a), (2, b)\}$ is not a function because the element $3 \in X$ has no corresponding element in Y (it fails the existence condition).
- The relation $\mathcal{R}_3 = \{(1, a), (1, b), (2, c), (3, a)\}$ is not a function because the element $1 \in X$ is associated with two different elements from Y: a and b (it fails the uniqueness condition).

Chapter 2

Limits

Limits

Limit of a Sequence

Formal Definition A number $L \in \mathbb{R}$ is the limit of a sequence of real numbers $(a_n)_{n=1}^{\infty}$ if for every real number $\varepsilon > 0$, there exists a natural number N such that for all $n > N$, the inequality $|a_n - L| < \varepsilon$ holds.

We write this as:

$$\lim_{n \to \infty} a_n = L$$

which is equivalent to:

$$\forall_{\varepsilon > 0} \quad \exists_{N \in \mathbb{N}} \quad \forall_{n > N} \quad |a_n - L| < \varepsilon$$

Limit of a Function

The limit of a function at a point can be defined in two equivalent ways: the Cauchy definition (using ε and δ) and the Heine definition (using sequences).

Cauchy's Definition (ε-δ) Let f be a real-valued function defined on a subset $D \subseteq \mathbb{R}$, and let c be a limit point of D. A number $L \in \mathbb{R}$ is the limit of the function f as x approaches c if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

We write this as:

$$\lim_{x \to c} f(x) = L$$

which is equivalent to:

$$\forall_{\varepsilon > 0} \quad \exists_{\delta > 0} \quad \forall_{x \in D} \quad (0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon)$$

The condition $0 < |x - c|$ means that we do not consider the value of $f(c)$ itself when determining the limit.

Heine's Definition (Sequential) Let f be a real-valued function defined on a subset $D \subseteq \mathbb{R}$, and let c be a limit point of D. A number $L \in \mathbb{R}$ is the limit of the function f as x approaches c if for every sequence $(x_n)_{n=1}^{\infty}$ of elements in $D \setminus \{c\}$ that converges to c, the sequence of function values $(f(x_n))_{n=1}^{\infty}$ converges to L.

Formally:

$$\lim_{x \to c} f(x) = L$$

is equivalent to:

$$\forall_{(x_n) \subseteq D \setminus \{c\}} \quad \left(\left(\lim_{n \to \infty} x_n = c \right) \implies \left(\lim_{n \to \infty} f(x_n) = L \right) \right)$$

Equivalence The Cauchy and Heine definitions of the limit of a function are equivalent. This is a fundamental theorem in real analysis.

Algebra of Limits

In this section, we present the main theorems regarding arithmetic operations on limits. We will focus on sequences, but analogous theorems hold for functions.

Theorem (Sum of Limits) Let (a_n) and (b_n) be two sequences of real numbers such that $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$. Then the limit of the sum of these sequences exists and is equal to the sum of their limits:

$$\lim_{n \to \infty} (a_n + b_n) = A + B$$

Proof By the definition of a limit, we have:

$$\forall_{\varepsilon_A > 0} \exists_{N_A \in \mathbb{N}} : \forall_{n > N_A} \quad |a_n - A| < \varepsilon_A$$

$$\forall_{\varepsilon_B > 0} \exists_{N_B \in \mathbb{N}} : \forall_{n > N_B} \quad |b_n - B| < \varepsilon_B$$

We want to show that for any $\varepsilon > 0$, there exists an N such that for all $n > N$, $|(a_n + b_n) - (A + B)| < \varepsilon$.

Let $\varepsilon > 0$ be given. Let's choose $\varepsilon_A = \varepsilon/2$ and $\varepsilon_B = \varepsilon/2$. From the definitions of the limits of (a_n) and (b_n), there exist N_A and N_B corresponding to these choices.

Let $N = \max(N_A, N_B)$. Then for any $n > N$, both inequalities $|a_n - A| < \varepsilon/2$ and $|b_n - B| < \varepsilon/2$ hold.

Using the triangle inequality, we can write:

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, we have shown that for any $\varepsilon > 0$, there exists an N such that for all $n > N$, $|(a_n + b_n) - (A + B)| < \varepsilon$. This completes the proof. ■

Theorem (Product of Limits) Let (a_n) and (b_n) be two sequences such that $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$. Then:

$$\lim_{n \to \infty} (a_n \cdot b_n) = A \cdot B$$

Proof Since the sequence (a_n) is convergent, it is bounded. This means there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. From the definitions of the limits of (a_n) and (b_n): For $\frac{\varepsilon}{2M} > 0$, there exists an $N_B \in \mathbb{N}$ such that for all $n > N_B$:

$$|b_n - B| < \frac{\varepsilon}{2M}$$

For $\frac{\varepsilon}{2(|B|+1)} > 0$ (we use $|B|+1$ to avoid division by zero if $B = 0$), there exists an $N_A \in \mathbb{N}$ such that for all $n > N_A$:

$$|a_n - A| < \frac{\varepsilon}{2(|B|+1)}$$

Let $N = \max(N_A, N_B)$. For any $n > N$, we can estimate the difference $|a_n b_n - AB|$:

$$\begin{aligned}
|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\
&= |a_n(b_n - B) + B(a_n - A)| \\
&\leq |a_n(b_n - B)| + |B(a_n - A)| \quad \text{(by triangle inequality)} \\
&= |a_n||b_n - B| + |B||a_n - A| \\
&\leq M|b_n - B| + |B||a_n - A| \\
&< M\left(\frac{\varepsilon}{2M}\right) + |B|\left(\frac{\varepsilon}{2(|B|+1)}\right) \\
&= \frac{\varepsilon}{2} + \frac{|B|}{|B|+1} \cdot \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This shows that $\lim_{n\to\infty}(a_n \cdot b_n) = AB$. ■

Theorem (Quotient of Limits) Let (a_n) and (b_n) be two sequences such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. If $B \neq 0$ and $b_n \neq 0$ for all n, then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$$