Section 2 — Analytic Geometry: Exercises

Vectors

1. For vectors in space

$$\mathbf{u} = [1, 2, -1] \quad \text{and} \quad \mathbf{v} = [2, -1, 3]$$

calculate $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, the dot product $\mathbf{u}\cdot\mathbf{v}$, and the norms $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$. Check if the vectors are orthogonal.

2. For points $A(1, 0, 2)$, $B(3, -1, 1)$, and $C(2, 2, 0)$, calculate the vectors AB and AC and find the angle between them.

3. Calculate the cross product $\mathbf{u}\times\mathbf{v}$ for the vectors from problem 1 and check if it is orthogonal to both vectors.

4. For vectors in the plane: $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [-4, 3]$, calculate their dot product and check if they are perpendicular. Determine the projection of vector $\mathbf{a}$ onto $\mathbf{b}$.

5. Calculate the length of vector $\mathbf{c} = [1, 1]$ and find the unit vector of this vector.

6. Calculate the length of vector $\mathbf{c} = [1, 2, 3]$ and find the unit vector of this vector.

7. Calculate the area of the triangle spanned by the vectors $[2, 1, 2]$ and $[-1, 1, 1]$.

8. Calculate the angle in degrees between the vectors $[4, 2, 1]$ and $[1, 3, 2]$.

9. Find the coordinates of the midpoint of the line segment with endpoints $A(-1, 2)$ and $B(3, -2)$.

10. For three-dimensional vectors: $\mathbf{a} = [a_x, a_y, a_z]$, $\mathbf{b} = [b_x, b_y, b_z]$, $\mathbf{c} = [c_x, c_y, c_z]$, prove that the following identity is satisfied:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

11. Find the most general form of a vector that is simultaneously perpendicular to

$$\mathbf{v} = [-1, 3, 0] \qquad \text{and} \qquad \mathbf{u} = [0, 1, 1]$$

12. For what values of parameters p and q are the vectors $\mathbf{a} = [1-p, 3, -1]$ and $\mathbf{b} = [-2, 4-q, 2]$ parallel?

13. For what values of the parameter s are the vectors $\mathbf{p} = [s, 2, 1-s]$ and $\mathbf{q} = [s, 1, -2]$ perpendicular?

14. Prove that two vectors must have equal lengths if their sum is perpendicular to their difference.

15. We have 2 people (A and B) moving according to the formulas:

    A: $(4, 5) + (1, -2)t$

    B: $(1, -8) + (2, 4)t$

where t denotes time. For what t will the people be closest to each other?

Lines

1. Write the equation of the line passing through points $P(1, 2)$ and $Q(3, -1)$ in slope-intercept and general form.
2. Find the parametric equation of the line perpendicular to the line from problem 1 and passing through point $R(0, 1)$.
3. A line passes through point $A(1, 2)$ and is parallel to the line $y = 2x + 3$. Find the equation of this line.
4. For lines in general form $l_1 : 2x - 3y + 1 = 0$ and $l_2 : 4x - 6y - 5 = 0$, determine if they are parallel, perpendicular, or intersect at one point. If they have a common point, calculate its coordinates.
5. Calculate the angle between the line $y = x + 3$ and the Ox axis.
6. Provide a vector perpendicular to the line $x + y + 1 = 0$.
7. $\star$ Find the distance from point $S(2, 3)$ to the line $l : 3x - 4y + 5 = 0$.
8. $\star$ Write the equation of the line passing through point $T(1, 1)$ and forming an angle of $\pi/6$ with the OX axis. Also, provide the intersection point with the OY axis.

Planes

1. Provide the general and normal equation of the plane passing through point $A(1, 0, 2)$ and with a normal vector $\mathbf{n} = [2, -1, 1]$.
2. Find the equation of the plane passing through points $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$.
3. Determine the angle between the planes: $\pi_1 : x + 2y - 2z + 1 = 0$ and $\pi_2 : 2x - y + z - 3 = 0$.
4. For the plane $\pi : x - 2y + 2z - 4 = 0$, calculate the distance from point $P(3, 0, 1)$ to this plane.
5. Find a vector perpendicular to the plane $x + y + z = 1$.
6. A plane passes through point $A(1, 2, 3)$ and is parallel to the plane $2x + 3y + 4z = 5$. Find the equation of this plane.

7. $\star$ Find the equation of the plane passing through point $D(1,1,1)$ and containing the line passing through points $E(0,0,0)$ and $F(1,2,3)$.

Line and Plane in Space

1. Check if the line given parametrically

$$\ell: \ x = 1 + 2t, \ y = -1 + t, \ z = 3 - t$$

intersects the plane $\pi : 2x - y + z - 4 = 0$. If so, provide the intersection point.

2. $\star$ Calculate the distance from point $G(2,-1,0)$ to the line passing through points $H(0,0,0)$ and $I(1,1,1)$.

3. $\star$ Consider a system of a line and a plane dependent on the parameter λ:

$$\ell(\lambda) : x = \lambda + t, \ y = 1 + 2t, \ z = 2 - t$$

and

$$\pi : x - (\lambda - 1)y + z - 3 = 0$$

Determine the values of λ for which the line is parallel to the plane, contained in the plane, or intersects it at one point.