Oscillations, Chaos, and Waves

1. Objective

This module explores the fundamental concept of periodic motion, starting from simple linear approximations and progressing to complex non-linear dynamics and deterministic chaos. The goal is to understand why harmonic motion is a cornerstone of physics and how simple deterministic rules can lead to unpredictable behavior.

2. Theoretical Scope

Your documentation must be a comprehensive study resource. It should move logically from single-body mechanics to multi-body systems.

2.1. The Simple and Physical Pendulum

Derive the dynamics of a pendulum from first principles (Newton's Laws of Rotation or Lagrangian Mechanics).

- **The Mathematical (Simple) Pendulum**: Derive the equation of motion assuming a point mass and a massless string. Apply the small-angle approximation ($\sin\theta \approx \theta$) to derive the standard Simple Harmonic Motion (SHM) equation.
- **The Physical Pendulum**: Derive the equation of motion for a rigid body.
- **The Non-Linear Reality**: Analyze the equation without the small-angle approximation ($\ddot{\theta} + \frac{g}{L}\sin\theta = 0$). Explain why this cannot be solved with elementary functions and requires elliptical integrals or numerical methods.
- **Significance**: Explain why the harmonic oscillator is such a critical model in physics (hint: Taylor expansion of potential energy near a stable equilibrium).

2.2. The Double Pendulum and Deterministic Chaos

Extend your derivation to a system of two coupled pendulums (one attached to the end of another).

- **Equations of Motion**: Derive the coupled differential equations for the double pendulum (Lagrangian mechanics is recommended here).
- **Deterministic Chaos**: Define what deterministic chaos is. Explain the concept of "sensitivity to initial conditions" (The Butterfly Effect). Discuss how a system can be governed by known laws yet be practically unpredictable in the long term.

2.3. Coupled Oscillators and Wave Propagation

Explore what happens when we couple not just two, but N oscillators (e.g., masses connected by springs).

- **Theory**: Briefly explain how a discrete chain of coupled oscillators transitions into the continuous wave equation as $N \to \infty$.
- **Modes**: Mention the concept of normal modes of vibration.

3. Interactive Simulation Requirements

You must develop a sophisticated, multi-tabbed HTML/JavaScript application to visualize these phenomena. The application should act as a virtual laboratory.

Tab 1: Single Pendulum Analysis (Linear vs. Non-Linear)

- **Visuals**: Display the pendulum animation.
- **Comparison**: Plot the motion of a "perfect" harmonic oscillator (small angle approximation) overlaying the "real" non-linear pendulum (large angle). Show how they diverge as the amplitude increases.
- **Vector Visualization**: Allow the user to toggle real-time vectors for:
  - Velocity ($\vec{v}$).
  - Tangential and Radial Acceleration ($\vec{a}$).
  - Tension and Gravity forces.

Tab 2: The Double Pendulum (Chaos Lab)

- **Simulation**: Simulate the motion of a double pendulum using numerical integration (e.g., Runge-Kutta).
- **Chaos Demonstration**: Implement a feature to spawn "Shadow Pendulums".
  - Allow the user to launch two double pendulums simultaneously with microscopically different initial angles (e.g., $\Delta\theta = 0.0001$ rad).
  - Visualize the "Trace" (trajectory) of both pendulums to demonstrate how they stay together initially and then violently diverge.
- **Phase Space (Optional)**: A view showing the phase space trajectory would be a valuable addition.

Tab 3: Coupled Oscillators (Wave Machine)

- **Simulation**: Create a chain of N masses (e.g., 10-20) connected by springs.
- **Interaction**: Allow the user to perturb the first mass to send a pulse or a continuous wave down the chain.
- **Physics**: Demonstrate reflection of waves and the propagation of energy through the medium.

4. Deliverables and Quality Standards

- **In-Depth Analysis**: Your `problem_solution.md` must link the math to the simulation. When discussing chaos, refer to your specific implementation in the app.
- **Numerical Methods**: Briefly explain the numerical integration method used in your code (e.g., "Why Euler method might fail here and why we used Runge-Kutta or Verlet").
- **Structure**: The note must be self-contained. A student reading it should understand the transition from simple harmonic motion $\rightarrow$ coupled motion $\rightarrow$ waves.