ECE 210 Midterm 1 Worksheet

Note: This worksheet is not guaranteed to be entirely representative of the midterm's contents. Material may appear on this worksheet which will not appear on the midterm, and vice versa.

Complex Number Review

1) Find the roots of the following polynomials:

a) $x^2 + x + 1$

Solution Apply the quadratic formula.

$$\frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \boxed{\frac{-1 \pm j\sqrt{3}}{2}}$$

b) $x^3 + 1$

Solution Isolate the variable term, which gives:

$$x^3 = -1 \rightarrow x = \sqrt[3]{-1} \qquad -1 = e^{\pm(2n+1)j\pi}$$

$$x = (e^{\pm(2n+1)j\pi})^{1/3} = \boxed{e^{j\pi\frac{2n+1}{3}}}$$

c) $x^2 + 4x + 2$

Solution Same process as a)

$$\frac{-4 \pm \sqrt{16 - 8}}{2} = \boxed{-2 \pm \sqrt{2}}$$

d) $x^2 + 2jx + 1$

Solution Same process as a)

$$\frac{-j \pm \sqrt{(2j^2) - 4}}{2} = \frac{-2j \pm -2j\sqrt{2}}{2} = \boxed{-j \pm j\sqrt{2}}$$

2) Evalute the following expressions:

a) $(1+3j)(4-j)$

Solution $(1+3j)(4-j) = 4+3+(12-1)j = \boxed{7+11j}$

b) $(1-j)(2+j)$

Solution $(1-j)(2+j) = 2+1-j = \boxed{3-j}$

c) $(1+3j)(4-j)^{-1}$

Solution $(1+3j)(4-j)^{-1} = \frac{(1+3j)(4+j)}{\|4-j\|} = \frac{4+12j+j-3}{17} = \boxed{\dfrac{1+13j}{17}}$

d) $(1-j)(2+j)^{-1}$

Solution $(1-j)(2+j)^{-1} = \frac{(1-j)(2-j)}{\|2-j\|} = \frac{2-2j-j-1}{5} = \boxed{\dfrac{1-3j}{5}}$

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3) Prove the following:

Triangle Inequality: Show that $|z_1| + |z_2| \geq |z_1 + z_2|$.

Solution Let z_1, z_2 be two arbitrary complex numbers, and $z = a + jb$. We first square the right hand side;

$$\|z_1 + z_2\| = (z_1 + z_2)(z_1^* + z_2^*) = \|z_1\| + z_1 z_2^* + z_1^* z_2 + \|z_2\|$$

Now, we square the left-hand side;

$$(|z_1| + |z_2|)^2 = \|z_1\| + \|z_2\| + 2|z_1||z_2|$$

$$|z_1||z_2| \geq z_1 z_2^* + z_1^* z_2$$

Expanding both sides (and dividing by 2) gives us

$$\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \geq a_1 a_2 + b_1 b_2$$

$$a_1^2 a_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2 + b_1^2 b_2^2 \geq a_1^2 a_2^2 + 2 a_1 a_2 b_1 b_2 + b_1^2 b_2^2$$

$$a_1^2 b_2^2 + a_2^2 b_1^2 \geq a_1 a_2 b_1 b_2$$

$$a_1^2 b_2^2 + a_2^2 b_1^2 - 2 a_1 a_2 b_1 b_2 \geq 0$$

$$(a_1 b_2 - a_2 b_1)^2 \geq 0$$

which is trivially true for all real a, b.

Hyperbolic Functions: Show that $\sin(jx) = j \sinh(x)$.

Solution Expand using Euler's Identity:

$$\frac{e^{j^2 x} - e^{-j^2 x}}{2j} = \frac{e^{-x} - e^x}{2j} = \frac{-1}{j} \frac{e^x - e^{-x}}{2} = j \sinh(x)$$
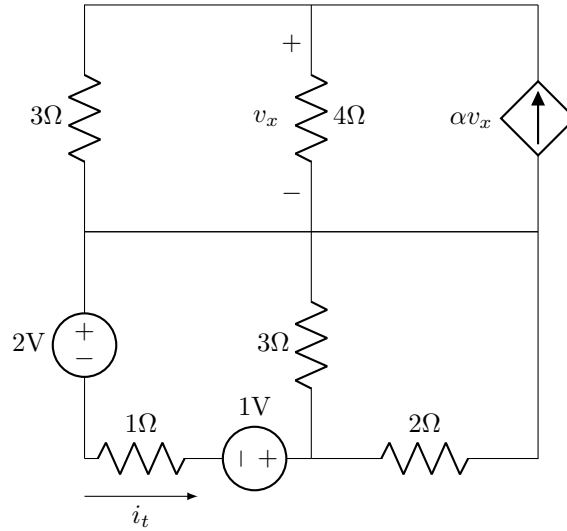
Hyperbolic Functions II: Show that $\sinh(x) = \tan(y) \implies \sin(y) = \pm \tanh(x)$.

Solution We can expand both sides;

$$\frac{e^x - e^{-x}}{2} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Resistive Circuit Analysis

4) Consider the circuit below.

a) There is only one value of α that makes this circuit a valid circuit. Find it.

Solution Use KCL at the top-most node:

$$\alpha * v_x - \tfrac{v_x}{4} - \tfrac{v_x}{3} = 0$$

This equation must sum to zero, so α must satisfy:

$$\alpha = \tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12}$$

$$\boxed{\alpha = \frac{7}{12}}$$

b) Find i_t, the current across the 1 Ω resistor, as indicated.

Solution Suppose the center bottom node is grounded, V_1 is the bottom left node, and the middle node is V_2. Applying KCL at the ground node results in:

$$i_t + \tfrac{V_2}{3} + \tfrac{V_2}{2} = 0$$

Use Ohm's Law to obtain i_t in terms of V_1:

$$i_t = \tfrac{V_1 - (-1)}{1} = V_1 + 1$$

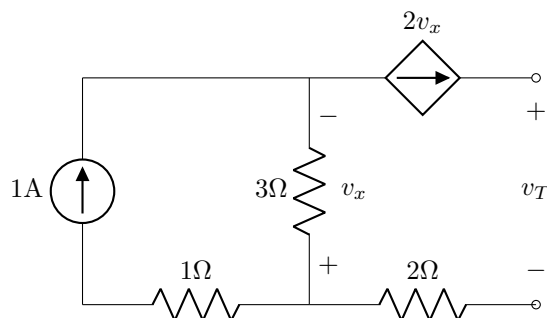Also note that the 2V voltage source creates the following relationship:

$$V_2 = V_1 + 2$$

We can use these relationships to solve for V_1 via substitution in the KCL equation:

$$i_t + \tfrac{5}{6} V_2 = 0$$
$$V_1 + 1 + \tfrac{5}{6} V_1 + \tfrac{5}{3} = 0$$
$$\tfrac{11}{16} V_1 = -\tfrac{8}{3}$$
$$V_1 = -\tfrac{16}{11} \text{V}$$

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Now, we can solve for i_t by simply plugging in our value for V_1:

$$i_t = -\frac{16}{11} + 1 = -\frac{5}{11}A$$

5) Given the circuit below;

a) Find the Thevenin equivalent voltage.

Solution Identify that there is no current through the dependent source because of the open circuit.

$$V_T = -v_x$$

There is only one path for current to flow, so v_x is simply just:

$$v_x = -1(3) = -3\text{V}$$

$$\boxed{V_T = 3V}$$

Note: The 3Ω resistor is using non-standard reference, so an extra negative sign is needed. Watch for this on the test!

b) Find the Norton equivalent current.

Solution Short the two terminals and perform KCL at the top node to solve for v_x:

$$1 + \tfrac{v_x}{3} - 2v_x = 0$$

$$\tfrac{5}{3}v_x = 1$$

$$v_x = \tfrac{3}{5}\text{V}$$

All the current through the short must come from the dependent source:

$$I_N = 2v_x$$

$$\boxed{I_N = \frac{6}{5}A}$$

c) If a 5 Ω load is connected across the terminals of this circuit, how much power is dissipated across the load?

Solution Using the Thevenin equivalent circuit, adding a 5Ω load creates a voltage divider between the load and R_T. First, find R_T using V_T and I_N:

$$R_T = \frac{V_I}{I_N} = \frac{3}{\frac{6}{5}} = \frac{5}{2}\Omega$$

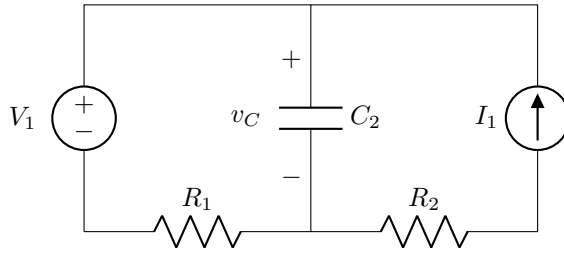Use voltage divider to find the voltage across the 5Ω load:

$$V_L = 3\left(\frac{5}{5 + \frac{5}{2}}\right) = 2V$$

Now, we can solve for the power using $P = \frac{V^2}{R}$:

$$\boxed{P_L = \frac{2^2}{5} = \frac{4}{5}W}$$

N-Order Circuits

6) Initially, the capacitor holds some charge Q_0, and both sources are off, with a value of 0 volts and amps respectively. At time $t = 0$, both the voltage and current source are turned on.

a) Find the zero-input solution for $v_C(t)$, the voltage across the capacitor.

Solution Use the general solution to a first order differential equation for a capacitor's voltage:

$$v_C(t) = v_C(\infty) + (v(0^-) - v_C(\infty))e^{-t/\tau}$$

Note: Knowing this general solution is **very** useful and will save you a lot of time on tests. It also works for inductors! Just replace v_C with i_L and the solution still holds.

We're looking for the zero-input solution, so we can ignore the $v_C(\infty)$ terms (think about why this is). Initially, C_2 holds some charge Q_0. Use the equation

$$Q = CV$$

to determine that

$$v_C(0^-) = \frac{Q_0}{C_2}$$

To find the time constant τ, we suppress sources by shorting V_1 and opening I_1. Therefore, τ will consist of solely R_1 and C_2.

$$\tau = R_1 C_2.$$

Substituting these values into our formula, we get:

$$\boxed{v_{CZI}(t) = \frac{Q_0}{C_2}e^{-t/R_1 C_2}V}$$

b) Find the zero-state solution for $v_C(t)$, the voltage across the capacitor.

Solution Use the same general solution as part a, this time without the $v_C(0^-)$ term (think about what this term represents).

At steady state $(t \to \infty)$, no current enters the capacitor, so the voltage is constant. We can apply KVL to the left loop to find $v_C(\infty)$:

$$v_C(\infty) = V_1 + R_1 I_1$$
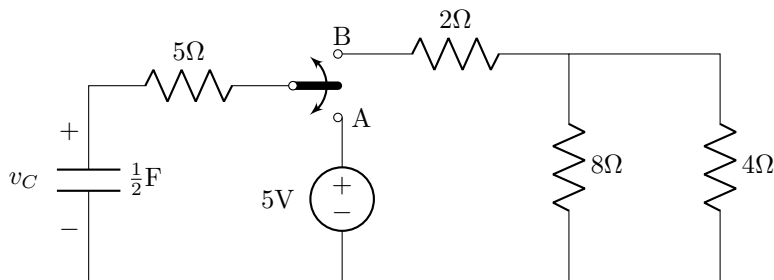
Plug this into the solution to get:

$$\boxed{v_{CZS}(t) = V_1 + I_1 R_1(1 - e^{-t/R_1 C_1})V}$$

c) Find $v_C(t)$.

Solution The full solution is simply the sum of the zero-state and zero-input solutions.

$$v_C(t) = V_1 + I_1 R_1 + (\frac{Q_0}{C_2} - V_1 - I_1 R_1)e^{-t/R_1 C_2} V$$

7) Initially, capacitor C_1 is entirely discharged.

a) At time $t = 0$, the switch is thrown to position A, connecting the left half to the voltage source. Find the expression for $v_C(t)$, the voltage across the capacitor for $t > 0$.

Solution Recall the general solution to a first order differential equation for the voltage across a capacitor

$$v_C(t) = v_C(\infty) + (v_C(0^-) - v_C(\infty))e^{-t/\tau}$$

In this case, we are given that the capacitor starts with no initial charge (i.e., no voltage) so, $v_C(0^-) = 0V$.

At steady state, the capacitor will have the same voltage as the voltage source because it will become completely charged, allowing no current to flow. Hence, $v_C(\infty) = 5V$.

To find the time constant for this circuit, we suppress the voltage source by replacing it with a short:

$$\tau = RC = (5)\frac{1}{2} = \frac{5}{2}s$$

Now we can plug everything into the general solution to arrive at:

$$v_C(t) = 5 - 5e^{-2t/5}V$$

b) Enough time passes such that the capacitor is entirely charged. The switch is then thrown at time $t = t_0$ from position A to position B, disconnecting the left half from the voltage source and connecting it instead to the right half of the circuit, with the voltage source left entirely disconnected. Find the new expression for $v_C(t)$, the voltage across the capacitor for $t > t_0$.

Note: Do not forget the time shift introduced by flipping the switch at time $t = t_0$.

Solution We will use the same general solution from part a:

$$v_C(t) = v_C(\infty) + (v_C(0^-) - v_C(\infty))e^{-t/\tau}$$

This time, we know that the capacitor is charged to 5V in the previous part, so $v_C(0^-) = 5$V.

The circuit is now purely resistive, so the capacitor will completely discharge such that $v_C(\infty) = 0$V.
To find the time constant, we first need to find the equivalent resistance R_{eq} the capacitor sees (note that || means 'in parallel').

$$R_{eq} = 8||4 + 2 + 5$$

$$8||4 = \frac{8 * 4}{8 + 4} = \frac{8}{3}\Omega$$

$$R_{eq} = \frac{8}{3} + 2 + 5 = \frac{29}{3}\Omega$$

Now that we have the equivalent resistance, τ can be found:

$$\tau = R_{eq}C = \frac{29}{3}(\frac{1}{2}) = \frac{29}{6}s$$

Plugging into the general solution (don't forget the time shift):

$$\boxed{v_C(t) = 5e^{-\frac{6(t-t_0)}{29}}V}$$

Notice that this is a zero-input solution, since there are no sources. You might have also noticed that part a was a zero-state solution, as there was no initial condition. This should help build intuition on what zero-state and zero-input solutions represent.
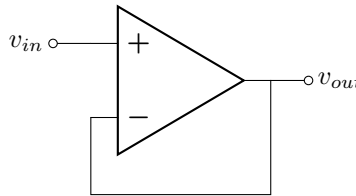
Operational Amplifiers

8) In ECE 210, operational amplifiers are treated as magic triangles with specific input output relations. A more realistic op-amp model would look something like this;

$$i_+ = i_- = 0$$
$$v_{out} = 10^6 \times (v_+ - v_-)$$

a) Using the above equations, show that the ideal op-amp rule $v_+ = v_-$ is approximately true when the output of the chip is connected to the inverting input (v_-) of the chip, as shown in the figure below.

Solution Notice that the output v_{out} is connected directly to the inverting input v_-, so:

$$v_{out} = v_-$$

We can replace v_{out} in the given equation with v_- and solve for v_- in terms of v_+:

$$v_- = 10^6(v_+ - v_-)$$

$$v_- + 10^6 v_- = 10^6 v_+$$
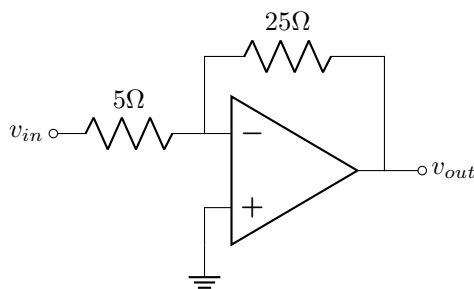
$$v_- = \frac{10^6}{1 + 10^6} v_+$$

However, $\frac{10^6}{1+10^6} = 0.9999 \approx 1$. Therefore:

$$\boxed{v_- \approx v_+}$$

b) What happens when the output of the chip is instead connected back to the non-inverting terminal (v_+)?

Solution In this case, $v_+ = v_{out}$. Notice that if v_+ increases a small amount above v_-, v_{out} will increase by $10^6(v_+ - v_-)$, a very large amount. However, v_+ and v_{out} must be the same voltage because they are connected together, so v_+ will increase again. This creates a positive feedback loop that will quickly saturate the op-amp.

c) Find $v_{out}(t)$ in terms of $v_{in}(t)$ for the figure below using the ideal op-amp approximation. $R_1 = 5\Omega$, $R_2 = 25\Omega$.

Solution Since the non-inverting terminal is grounded:

$$v_+ = v_- = 0V$$

From this, we can conclude that:

$$v_{out} = -v_{R2}$$

No current enters either terminal, so all the current through the 5Ω resistor must pass through the 25Ω resistor. We will call this current i_1. Luckily, because $v_- = 0V$, we can find i_1 using only the 5Ω resistor:

$$i_1 = \frac{v_{in}}{5}$$

The voltage v_{R2} across the 25Ω resistor is simply:

$$v_{R2} = i_1 R_2 = \frac{v_{in}}{5}(25) = 5v_{in}$$

We already know that $v_{out} = -v_{R2}$, so:

$$\boxed{v_{out} = -5v_{in}V}$$

i) What is this op-amp circuit doing? What does it accomplish?

Solution This is a fairly common op-amp circuit called an inverting amplifier. It takes a voltage input and produces an inverted voltage output according to its gain of $-\frac{R_f}{R_{in}}$. In this example, R_2 is R_f and R_1 is R_{in}. Notice that depending on the choices of R_{in} and R_f, an inverting amplifier can either attenuate (weaken) or amplify an input voltage. There are many more interesting uses of op-amps, some of which you will see later in this course.