HKN ECE 310 Review Worksheet 2 Solutions

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1 The very basics

1. What is the relation between the Z-transform and the Discrete-Time Fourier Transform? When is this relation not valid?

$$X_d(\omega) = X(z)|_{z=\exp(j\omega)}$$

2 Devious DTFTs

Use your unbounded knowledge of the DTFT properties and pairs to take the DTFT of these signals.

(i) $\delta[n-1] + (\frac{2}{3})u[n+4]$

(ii) $\cos(n)\sin(n)$

(iii) $n\delta[n-1] * e^{jn}$

Solution:

(i) We can apply the table and DTFT properties to get:

$$\delta[n-1] + \frac{2}{3}u[n+4] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jw} + \frac{2}{3}e^{j4w}(\frac{1}{1-e^{-jw}} + \pi\delta(w))$$

(ii) We could take the DTFT of both $\cos(n)$ and $\sin(n)$ and then convolve them in the frequency domain, but we opt to choose a simpler solution. Recall the folloing trig identity:

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$$

Applying it to our problem, we see that:

$$\sin(n)\cos(n) = \frac{1}{2}(\sin(n+n) + \sin(n-n)) = \frac{1}{2}\sin(2n)$$

The problem is now quite straightforward and we can just apply the table:

$$\sin(n)\cos(n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{2j}(\delta(w-2) - \delta(w+2))$$

(iii) Recall that convolution in the time domain is multiplication in the frequency domain. We first find the DTFT of both input functions:

$$n\delta[n-1] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{dw}(e^{-jw}) = j(-je^{-jw}) = e^{-jw}$$

$$e^{jn} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(w-1)$$

Now it's a straightforward frequency domain multiplication:

$$n\delta[n-1] * e^{jn} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-jw}\delta(w-1)$$

(iii) **Alternate Solution** A slightly quicker way to arrive at the same result is to notice that $n\delta[n-1] = \delta[n]$. It's easy to take the convolution of a function with a delta function, so $n\delta[n-1] * e^{jn} = e^{j(n-1)}$. It's straightforward to apply DTFT properties from here to get the same answer as above.

3 Sampling and DTFTs

(Let the output of a radio be) Consider a signal given by

$$x(t) = 2\cos(10\pi t) + \sin(30\pi t)$$

1. What is the nyquist sampling rate of this signal?

   Since the maximum frequency of any of the components of this signal is 15Hz, the minimum sampling rate must be $2 \cdot 15\text{Hz} = 30\text{Hz}$.

2. Let's say the signal is sampled at twice the nyquist rate. What does the discrete-time signal look like for three samples starting at $n = 0$? What is the n'th sample?

   We are sampling at 60Hz, so the sampled signal is

$$x[n] = 2\cos\left(\frac{1}{6}\pi n\right) + \sin\left(\frac{1}{2}\pi n\right)$$

3. Find the Discrete-Time Fourier Transform of the signal. Plot both the real and imaginary components of the DTFT over the range $(-\pi, \pi)$.

   The DTFT is

$$X_d(\omega) = 2\pi\left[\delta\left(\omega - \frac{\pi}{6}\right) + \delta\left(\omega + \frac{\pi}{6}\right)\right] - j\pi\left[\delta\left(\omega - \frac{\pi}{2}\right) - \delta\left(\omega + \frac{\pi}{2}\right)\right]$$

4. What is the power contained in this signal? Make sure to include units!

   Using Parseval's Relation, the power is 5π watts.

5. If we want to build a low-pass filter to filter out the fastest component of this signal, what is the smallest value of ω at which the filter can start attenuating?

$$\omega_{\text{cutoff}} = \frac{\pi}{6}$$

6. Lets say we have a perfect filter to do said filtering. We then amplify the signal such that the magnitude of each component is doubled. How does the power of the signal change?

   Using Parseval's relation again, the power contained is 16π watts.