# MEASURING GPS SATELLITE ORBITS WITH MOBILE GPS SOFTWARE

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## ABSTRACT

A GPS software installed on a mobile phone can locate our position on the earth (longitude and latitude) as well as GPS satellites' positions (altitude and azimuth) in the sky. In this paper, I provide a method to measure GPS satellites' orbital elements only with mobile GPS softwares. I collected several GPS satellites' position data with local time and calculated their orbital elements, with errors of less than 2.54° compared with the official orbital data from US National Coordination Office (NCO) for Space-Based Positioning, Navigation, and Timing (PNT). Based on the orbital elements I calculated, I forecasted some of the satellites' positions during a specific period. To verify my forecast, I measured these satellites' positions using my mobile phone in this period. The observed values and the theoretical forecast fits well. Using my method, we can measure any satellite's orbit and forecast its position with given observed data.

*Subject headings:* astrometry — ephemerides — methods: data analysis

## 1. INTRODUCTION

The Global Positioning System (GPS) is a U.S.-owned utility that provides users with positioning, navigation, and timing (PNT) services. This system consists of three segments: the space segment, the control segment, and the user segment. (GPS.gov - GPS Overview) The GPS space segment consists of a constellation of satellites transmitting radio signals to users. The United States is committed to maintaining the availability of at least 24 operational GPS satellites, 95% of the time. To ensure this commitment, the Air Force has been flying 31 operational GPS satellites for the past few years. GPS satellites fly in medium Earth orbit (MEO) at an altitude of approximately 20,200 km. Each satellite circles the Earth twice a day. (GPS.gov - Space Segment) So their orbital period is 12 hours in sidereal time.

Assuming the satellites have circular orbits, we can calculate their orbital radius from the orbital period. Given a series of data of a satellite's altitude and azimuth measured by GPS software at a fixed location on the earth, changing over time, we can locate a series of points in the sky, which are in the satellite's orbit. Using least squares method, we can fit an orbit from these observed points and calculate the orbital elements. With the calculated orbital elements and the measured time data, we can make the ephemerides of each satellite and forecast their altitude and azimuth at any time at any location on the earth.

In the next section, I elaborate the method to calculate a satellite's orbital elements from the altitude and azimuth data. In Section 3, I transform the acquired data and manage to plot the orbits as well as the observed points in both equatorial coordinates and geographical coordinates. In Section 4, I provide the method to predict GPS satellite transit. In Section 5, I exhibit my experiment with measured data and results as well as verification of my calculated values.

## 2. CALCULATING ORBITAL ELEMENTS

We first calculate some basic parameters: the local sidereal time, the local earth radius and the satellite's orbital radius. With these parameters, we perform a series of transformations of coordinates. We transform the satellite's horizontal coordinates into local equatorial coordinates, a kind of equatorial coordinates I defined which regard the observer as the origin. Then we transform the observer's geographical coordinates into equatorial coordinates, which regard the centroid of the earth as the origin. Finally with the observer's equatorial coordinates, we can transform the satellite's local equatorial coordinates into equatorial coordinates. We fit these satellite points in equatorial coordinates with a circular equation, whose center of the circle is on the origin (i.e. the centroid of the earth). From the equation, we calculate the orbital elements (RAAN[1] and Inclination) of the satellite.

### 2.1. *Basic parameters*

The local sidereal time is used to contact local coordinates, which are changing over time, with equatorial coordinates, which are fixed relative to the vernal equinox. The earth radius and satellites' orbital radius represent the radial coordinates in these spherical coordinates.

#### 2.1.1. *Sidereal time*

The local sidereal time $t$ can be calculated from the Coordinated Universal Time $UTC$ (Year $Y$, Month $M$, Date $D$, hour $h$, minute $m$ and second $s$). (Wiki - Sidereal time) The Julian Day $JD$ gives

$$JD = \lfloor 365.25Y \rfloor + \left\lfloor \frac{Y}{400} \right\rfloor - \left\lfloor \frac{Y}{100} \right\rfloor + \lfloor 30.59(M-2) \rfloor + D + 1721088.5 + \frac{h}{24} + \frac{m}{1440} + \frac{s}{86400}. \quad (1)$$

The rounding symbol $\lfloor\ \rfloor$ means retaining the integer part of the value. The Truncated Julian Day $TJD$ is

$$TJD = JD - 2440000.5. \quad (2)$$

[1] Right Ascension of the Ascending Node

Then the Greenwich sidereal time $t_G$ can be calculated from

$$t_G = 2\pi \times (0.671262 + 1.0027379094 TJD). \tag{3}$$

The unit of $t_G$ is rad.

Given the observer's longitude $\lambda_0(^\circ)$, we can calculate the local sidereal time

$$t = t_G + \frac{\pi}{180}\lambda_0. \tag{4}$$

The value of $t$ should be transformed within the interval $[0, 2\pi)$.

#### 2.1.2. *Earth radius*

The local earth radius $R$ can be calculated from the local gravitational acceleration $g$ through Newton's equation

$$mg = \frac{GMm}{R^2}, \tag{5}$$

where $G$ is the gravitational constant, $M$ is mass of the earth, and $m$ is mass of any small object at the observer's location. Eq. (5) gives

$$R = \sqrt{\frac{GM}{g}}. \tag{6}$$

We can acquire the values of $G$ and $M$ from an astronomical textbook. We can find the value of local $g$ from a geographical database or conduct an experiment to measure local $g$.

#### 2.1.3. *Orbital radius*

Assume the satellite is flying around the earth in a circular orbit. The orbital radius is $r$, the angular velocity is $\omega$, and the orbital period is $T$. Newton's equation gives

$$\frac{GMm}{r^2} = m\omega^2 r, \tag{7}$$

where $m$ is mass of the satellite. The relation between $T$ and $\omega$ is

$$T = \frac{2\pi}{\omega}. \tag{8}$$

Eq. (7) and eq. (8) gives

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}}. \tag{9}$$

The orbital period $T$ is half of the earth's rotation period, which is

$$T = \frac{23.9344699}{2} \text{ hours}. \tag{10}$$

### 2.2. *Horizontal coordinates → Local equatorial coordinates*

We define local equatorial coordinate system as a coordinate system whose basis vectors have the same directions as basis vectors of equatorial coordinate system and whose origin is at the observer's location. The local equatorial coordinate system is moving relative to the equatorial coordinate system with the earth rotating.

For a satellite's altitude $a$ and azimuth $A$ (measured from the South point, turning positive to the West) at a sidereal time $t$, we can calculate it's local declination $\delta_l$ and hour angle $h$ through basic spherical trigonometry equations (Wiki - Celestial coordinate system) which gives

$$\begin{bmatrix} \cos\delta_l \cos h \\ \cos\delta_l \sin h \\ \sin\delta_l \end{bmatrix} = \begin{bmatrix} \sin\phi_0 & 0 & \cos\phi_0 \\ 0 & 1 & 0 \\ -\cos\phi_0 & 0 & \sin\phi_0 \end{bmatrix} \begin{bmatrix} \cos a \cos A \\ \cos a \sin A \\ \sin a \end{bmatrix}, \tag{11}$$

where $\phi_0$ is the observer's latitude. The above transformation gives

$$\tan h = \frac{\sin A}{\cos A \sin\phi_0 + \tan a \cos\phi_0}, \tag{12}$$

and

$$\sin\delta_l = \sin\phi_0 \sin a - \cos\phi_0 \cos a \cos A. \tag{13}$$

The value of $h$ should be transformed within the interval $[0, 2\pi)$ according to the value of $A$. The value of $\delta_l$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

To transform hour angle $h$ into local right ascension $\alpha_l$, we have

$$\alpha_l = t - h, \tag{14}$$

which should also be transformed within the interval $[0, 2\pi)$.

Now we complete the transformation of the satellite from horizontal coordinates ($a$, $A$) into local equatorial coordinates ($\alpha_l$, $\delta_l$).

### 2.3. *Geographical coordinates → Equatorial coordinates*

With the earth rotating, the right ascension of the observer's location in equatorial coordinates $\alpha_0$ is changing. Since local sidereal time $t$ is the hour angle of the vernal equinox and right ascension is measured from the vernal equinox, the right ascension of the observer's location happens to be the local sidereal time,

$$\alpha_0 = t, \tag{15}$$

and the declination of the observer's location $\delta_0$ equals the local latitude $\phi_0$,

$$\delta_0 = \phi_0. \tag{16}$$

Hence we transform the observer's location from geographical coordinates ($\lambda_0$, $\phi_0$) into equatorial coordinates ($\alpha_0$, $\delta_0$).

### 2.4. *Local equatorial coordinates → Equatorial coordinates*

To simplify the calculation, we transform equatorial coordinates from spherical coordinates into rectangular coordinates. We transform the observer's equatorial coordinates ($R$, $\alpha_0$, $\delta_0$) into rectangular coordinates ($x_0$, $y_0$, $z_0$) through

$$\begin{cases} x_0 = R\cos\delta_0 \cos\alpha_0 \\ y_0 = R\cos\delta_0 \sin\alpha_0 \\ z_0 = R\sin\delta_0 \end{cases}. \tag{17}$$

We transform the satellite's local equatorial coordinates ($d$, $\alpha_l$, $\delta_l$) into rectangular coordinates ($dx$, $dy$, $dz$) through

$$\begin{cases} dx = d\cos\delta_l \cos\alpha_l \\ dy = d\cos\delta_l \sin\alpha_l \\ dz = d\sin\delta_l \end{cases}, \tag{18}$$

where $d$ is the distance between the observer and the satellite, which is to be solved. If we transform the satellite's equatorial coordinates ($r$, $\alpha$, $\delta$) into rectangular coordinates ($x$, $y$, $z$), we have

$$\begin{cases} x = x_0 + dx \\ y = y_0 + dy \\ z = z_0 + dz \end{cases}. \quad (19)$$

The orbital radius gives

$$x^2 + y^2 + z^2 = r^2. \quad (20)$$

Substituting eq. (18), (19) into eq. (20), we have

$$\begin{aligned} &(x_0 + d\cos\delta_l\cos\alpha_l)^2 \\ +\ &(y_0 + d\cos\delta_l\sin\alpha_l)^2 \\ +\ &(z_0 + d\sin\delta_l)^2 = r^2. \end{aligned} \quad (21)$$

In the above equation, only $d$ is unknown quantity. We solve out $d$ that

$$\begin{aligned} d = &\{(y_0^2 - z_0^2)\cos^2\delta_l + r^2 - x_0^2 - y_0^2 \\ &+[(x_0^2 - y_0^2)\cos^2\alpha_l + x_0y_0\sin 2\alpha_l]\cos^2\delta_l \\ &+(x_0\cos\alpha_l + y_0\sin\alpha_l)z_0\sin 2\delta_l\}^{\frac{1}{2}} \\ &-z_0\sin\delta_l - x_0\cos\alpha_l\cos\delta_l - y_0\cos\delta_l\sin\alpha_l. \end{aligned} \quad (22)$$

Substituting $d$ into eq. (18) and (19), we can acquire the satellite's rectangular equatorial coordinates ($x$, $y$, $z$) through

$$\begin{cases} x = x_0 + d\cos\delta_l\cos\alpha_l \\ y = y_0 + d\cos\delta_l\sin\alpha_l \\ z = z_0 + d\sin\delta_l \end{cases}. \quad (23)$$

Finally, we transform the satellite's rectangular equatorial coordinates ($x$, $y$, $z$) into spherical equatorial coordinates ($r$, $\alpha$, $\delta$) through

$$\begin{cases} \tan\alpha = \frac{y}{x} \\ \sin\delta = \frac{z}{r} \end{cases}. \quad (24)$$

The value of $\alpha$ should be transformed within the interval $[0, 2\pi)$ accroding to the values of $x$ and $y$. The value of $\delta$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now we have transformed the satellite's position from local equatorial coordinates ($\alpha_l$, $\delta_l$) into equatorial coordinates ($\alpha$, $\delta$).

### 2.5. *Fitting orbit*

Suppose the normal direction of one satellite's orbital plane is ($\alpha_n$, $\delta_n$), and we can give the orbital equation

$$\cos\delta_n\cos\alpha_n\cos\delta\cos\alpha + \cos\delta_n\sin\alpha_n\cos\delta\sin\alpha + \sin\delta_n\sin\delta = 0. \quad (25)$$

Arrange eq. (25) and we can give a linear relation

$$\tan\alpha = -\frac{\tan\delta_n}{\sin\alpha_n}\frac{\tan\delta}{\cos\alpha} - \cot\alpha_n. \quad (26)$$

Eq. (26) has the same form as the equation

$$y = kx + b, \quad (27)$$

where

$$x = \frac{\tan\delta}{\cos\alpha}, \quad (28)$$

$$y = \tan\alpha, \quad (29)$$

and

$$k = -\frac{\tan\delta_n}{\sin\alpha_n}, \quad (30)$$

$$b = -\cot\alpha_n. \quad (31)$$

Given a series of points ($\frac{\tan\delta}{\cos\alpha}$, $\tan\alpha$) of one satellite, we can fit a line and calculate $k$ and $b$ using least squares method. Then the normal direction of the satellite's orbital plane ($\alpha_n$, $\delta_n$) can be worked out solving eq. (30) and (31), which gives

$$\tan\alpha_n = -\frac{1}{b}, \quad (32)$$

$$\tan\delta_n = \frac{k}{b\sqrt{\frac{b^2+1}{b^2}}}. \quad (33)$$

The value of $\delta_n$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The value of $\alpha_n$ is adjusted according to the values of $\delta_n$, $k$ and $b$. The detailed situations are listed in Table 1. Finally the value of $\alpha_n$ should be transformed within the interval $[0, 2\pi)$.

TABLE 1
DECIDING $\alpha_n$

| Sign of $k\delta_n$ | Sign of $b$ | Range of $\alpha_n$ |
|---|---|---|
| − | − | $(0, \frac{\pi}{2})$ |
| − | + | $(\frac{\pi}{2}, \pi)$ |
| + | − | $(\pi, \frac{3\pi}{2})$ |
| + | + | $(\frac{3\pi}{2}, 2\pi)$ |

We introduce two orbital elements to describe a satellite's orbit, Right Ascension of the Ascending Node (RAAN) and Inclination. RAAN has a difference of $\frac{\pi}{2}$ with the right ascension of the normal direction of the satellite's orbital plane $\alpha_n$, and Inclination is the complementary angle of the declination of the normal direction of the satellite's orbital plane $\delta_n$.

If $\delta_n \geq 0$,

$$\mathrm{RAAN} = \alpha_n + \frac{\pi}{2}, \quad (34)$$

$$\mathrm{Inclination} = \frac{\pi}{2} - \delta_n. \quad (35)$$

If $\delta_n < 0$,

$$\mathrm{RAAN} = \alpha_n + \frac{3\pi}{2}, \quad (36)$$

$$\mathrm{Inclination} = \delta_n + \frac{\pi}{2}. \quad (37)$$

The value of RAAN should be transformed within the interval $[0, 2\pi)$ and the value of Inclination is within $[0, \frac{\pi}{2}]$. Now we complete measuring the GPS satellite's orbit and give the orbital elements (RAAN, Inclination).

## 3. PLOTTING ORBITS

In order to plot the orbit on a map, we create a quantity of points averagely in the orbit. We define orbital coordinate system as a coordinate system which regard the satellite's orbital plane as the fundamental plane. An object's orbital longitude $\alpha_s$ is measured from the ascending node of the satellite, turning positive to the east. The object's orbital latitude $\delta_s$ is measured from the orbital plane, turning positive to the north and negative to the south. The value of $\delta_s$ of the points we created on the orbit is zero, and the values of $\alpha_s$ of these points are averagely distributed in the interval $[0, 2\pi)$.

We transform the points' orbital coordinates into equatorial coordinates to plot them on the celestial sphere. We transform the points' equatorial coordinates into geographical coordinates to plot them on the world map.

### 3.1. *Orbital coordinates → Equatorial coordinates*

Now we transform a point's orbital coordinates ($\alpha_s$, $\delta_s$) into equatorial coordinates ($\alpha$, $\delta$). Suppose the satellite's RAAN is $RA$ and it's Inclination is $I$. We define modified right ascension $\alpha_m$ as the difference between $\alpha$ and $RA$, which is measured from the ascending node of the satellite.

$$\alpha_m = \alpha - RA. \tag{38}$$

Spherical trigonometry gives

$$\begin{bmatrix} \cos\delta\cos\alpha_m \\ \cos\delta\sin\alpha_m \\ \sin\delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \begin{bmatrix} \cos\delta_s\cos\alpha_s \\ \cos\delta_s\sin\alpha_s \\ \sin\delta_s \end{bmatrix}, \tag{39}$$

which gives

$$\tan\alpha_m = \frac{\sin\alpha_s\cos I - \tan\delta_s\sin I}{\cos\alpha_s}, \tag{40}$$

and

$$\sin\delta = \sin\delta_s\cos I + \cos\delta_s\sin I\sin\alpha_s. \tag{41}$$

The value of $\alpha_m$ should be transformed within the interval $[0, 2\pi)$ according to the value of $\alpha_s$. The value of $\delta$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

According to eq. (38), the right ascension of the given point is

$$\alpha = \alpha_m + RA. \tag{42}$$

Finally the value of $\alpha$ should be transformed within the interval $[0, 2\pi)$. Thus we get the given point's equatorial coordinates ($\alpha$, $\delta$) from it's orbital coordinates ($\alpha_s$, $\delta_s$).

### 3.2. *Equatorial coordinates → Geographical coordinates*

The geographical coordinates ($\lambda$, $\phi$) of a point in the satellite's orbit is the geographical coordinates of the radial projection of this point on the surface of the earth. Since Greenwich's longitude $\lambda_G = 0$, the longitude of the given point $\lambda$ is just the difference between the point's right ascension $\alpha$ and Greenwich sidereal time $t_G$, which can be calculated from eq. (1), (2) and (3).

$$\lambda = \alpha - t_G, \tag{43}$$

and the latitude $\phi$ of the given point equals the point's declination $\delta$,

$$\phi = \delta. \tag{44}$$

The value of $\lambda$ should be transformed within the interval $[-\pi, \pi]$ and the value of $\phi$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Now we have transformed the point's equatorial coordinates ($\alpha$, $\delta$) into geographical coordinates ($\lambda$, $\phi$).

### 3.3. *Plotting*

With given points' equatorial coordinates ($\alpha$, $\delta$) and geographical coordinates ($\lambda$, $\phi$), we can plot the satellite's orbit on both the celestial sphere and the world map. The plotted figures of my measurement are given in Section 5.

## 4. FORECAST

Given the two orbital elements, RAAN $RA$ and Inclination $I$, and a initial point in the orbit with time, we can build a satellite's ephemerides and predict it's position at any time. To set the initial point, we transform the original points we measured from equatorial coordinates to orbital coordinates and select the point with the minimum orbital latitude. We change this point's orbital latitude to zero and set this modified point as the initial point. With the initial point we build the ephemerides in orbital coordinates. We transform the ephemerides from orbital coordinates into equatorial coordinates, and then from equatorial coordinates to horizontal coordinates in order to forecast it's position measured by an observer on the earth.

### 4.1. *Equatorial coordinates → Orbital coordinates*

For an observed satellite point transformed into equatorial coordinates ($\alpha$, $\delta$) in Section 2.2 and 2.4, we continue transforming it into orbital coordinates ($\alpha_s$, $\delta_s$) with the measured orbital elements, RAAN $RA$ and Inclination $I$.

The inverse transformation of eq. (39) gives

$$\begin{bmatrix} \cos\delta_s\cos\alpha_s \\ \cos\delta_s\sin\alpha_s \\ \sin\delta_s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{bmatrix} \begin{bmatrix} \cos\delta\cos\alpha_m \\ \cos\delta\sin\alpha_m \\ \sin\delta \end{bmatrix}, \tag{45}$$

which gives

$$\tan\alpha_s = \frac{\sin\alpha_m\cos I + \tan\delta\sin I}{\cos\alpha_m}, \tag{46}$$

and

$$\sin\delta_s = \sin\delta\cos I - \cos\delta\sin I\sin\alpha_m, \tag{47}$$

where $\alpha_m$ can be calculated from eq. (38). The value of $\alpha_s$ should be transformed within the interval $[0, 2\pi)$ according to the value of $\alpha_m$. The value of $\delta_s$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

### 4.2. *Set initial point*

After calculating the points' orbital coordinates, we find out the point whose orbital latitude is the minimum of these points. The point we select is the nearest to the measured orbital plane among the observed points. We set the time (local sidereal time) when this point was recorded as the initial time $t_i$ and set this point's orbital longitude as the initial orbital longitude $\alpha_{si}$. The initial orbital latitude $\delta_{si}$ is 0. Thus we get the initial point in orbital coordinates ($\alpha_{si}$, $\delta_{si}$).

### 4.3. *Satellite ephemerides in Orbital coordinates*

To give the satellite ephemerides in a specific period, we first transform the given time into local sidereal time $t$ using the method in Section 2.1.1. The difference between the given time $t$ and the initial time $t_i$ is

$$\Delta t = t - t_i. \tag{48}$$

Considering the satellite's angular velocity is twice the earth's rotational angular velocity, we have

$$\Delta\alpha_s = 2\Delta t, \tag{49}$$

where

$$\Delta\alpha_s = \alpha_s - \alpha_{si}. \tag{50}$$

Combining eq. (48), (49) and (50), we can work out the satellite's orbital longitude $\alpha_s$ at time $t$ through

$$\alpha_s = \alpha_{si} + 2(t - t_i). \tag{51}$$

The value of $\alpha_s$ should be transformed within the interval $[0, 2\pi)$. The satellite's orbital latitude $\delta_s$ always equals zero.

$$\delta_s = 0. \tag{52}$$

Given a series of time points $t$, we can correspondingly calculate a series of orbital points ($\alpha_s$, $\delta_s$) and thus we build a satellite ephemerides in orbital coordinates.

Using the method in Section 3.1, we can transform the orbital points ($\alpha_s$, $\delta_s$) into equatorial coordinates ($\alpha$, $\delta$) and give the satellite ephemerides in equatorial coordinates.

### 4.4. *Equatorial coordinates → Local equatorial coordinates*

We perform the inverse transformation of Section 2.4. Transform the ephemerides from spherical equatorial coordinates ($r$, $\alpha$, $\delta$) into rectangular equatorial coordinates ($x$, $y$, $z$) through

$$\begin{cases} x = r\cos\delta\cos\alpha \\ y = r\cos\delta\sin\alpha \\ z = r\sin\delta \end{cases}. \tag{53}$$

The observers location in rectangular equatorial coordinates ($x_0$, $y_0$, $z_0$) is given by eq. (15), (16) and (17).

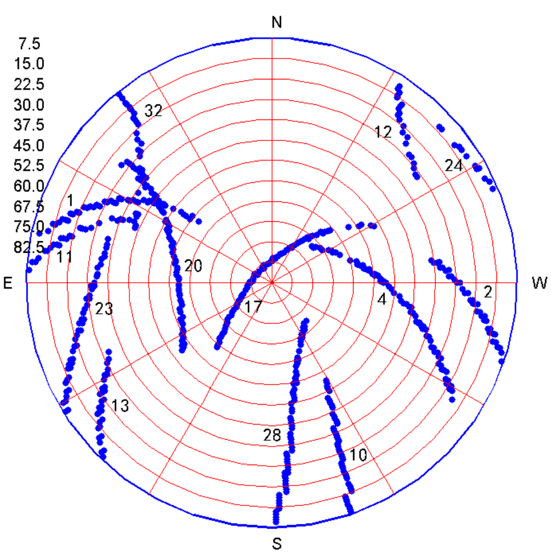


FIG. 1.— GPS satellites' positions (the dots) in horizontal coornidates measured from 8 p.m. to 11 p.m. on May 29, 2014 at (125.2768° E, 43.8253° N). The numbers beside every series of points represent the satellite's number shown in my GPS software.

Eq. (19) gives

$$\begin{cases} dx = x - x_0 \\ dy = y - y_0 \\ dz = z - z_0 \end{cases}. \tag{54}$$

Eq. (18) gives

$$d = \sqrt{dx^2 + dy^2 + dz^2} \tag{55}$$

and

$$\begin{cases} \tan\alpha_l = \frac{dy}{dx} \\ \sin\delta_l = \frac{dz}{d} \end{cases}. \tag{56}$$

Here $dx$, $dy$ and $dz$ are coordinates in local rectangular equatorial coordinate system rather than differentials of $x$, $y$ and $z$. The value of $\alpha_l$ should be transformed within the interval $[0, 2\pi)$ accroding to the values of $dx$ and $dy$. The value of $\delta_l$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus we get the ephemerides in local equatorial coordinates ($\alpha_l$, $\delta_l$).

### 4.5. *Local equatorial coordinates → Horizontal coordinates*

The hour angle $h$ can be calculated from eq. (14), which gives

$$h = t - \alpha_l, \tag{57}$$

and $h$ should be transformed within the interval $[0, 2\pi)$. The inverse transformation of eq. (11) gives

$$\begin{bmatrix} \cos a\cos A \\ \cos a\sin A \\ \sin a \end{bmatrix} = \begin{bmatrix} \sin\phi_0 & 0 & -\cos\phi_0 \\ 0 & 1 & 0 \\ \cos\phi_0 & 0 & \sin\phi_0 \end{bmatrix} \begin{bmatrix} \cos\delta_l\cos h \\ \cos\delta_l\sin h \\ \sin\delta_l \end{bmatrix}, \tag{58}$$

which gives

$$\tan A = \frac{\sin h}{\cos h\sin\phi_0 - \tan\delta_l\cos\phi_0}, \tag{59}$$

and

$$\sin a = \sin\phi_0\sin\delta_l + \cos\phi_0\cos\delta_l\cos h. \tag{60}$$

The value of $A$ should be transformed within the interval $[0, 2\pi)$ according to the value of $h$ and the value of $a$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Note that $A$ is measured from the

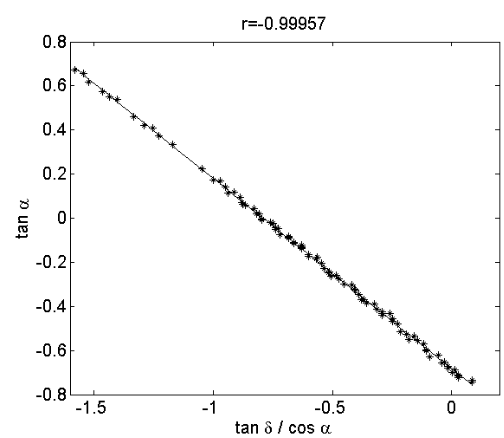


FIG. 2.— An example for using least squares method to fit the orbit. The dots are the observed points ($\frac{\tan\delta}{\cos\alpha}$, $\tan\alpha$) of satellite No. 4 and the line is the fitted line. From the line's slope $k$ and intercept $b$ we can calculate the orbital elements. The parameter $r$ is the correlation coefficient of the points.

TABLE 2
Measured orbital elements

| Number | RAAN(°) | Inclination(°) | $r$ |
|---|---|---|---|
| 1 | 146.67 | 55.64 | -0.999958 |
| 2 | 144.70 | 54.30 | -0.997106 |
| 4 | 145.67 | 54.51 | -0.999574 |
| 10 | 207.23 | 55.13 | -0.997781 |
| 11 | 127.38 | 52.46 | -0.999842 |
| 12 | 27.88 | 54.97 | 0.999964 |
| 13 | 272.87 | 57.23 | 0.994584 |
| 17 | 87.31 | 56.18 | 0.899752 |
| 20 | 203.31 | 53.68 | -0.999994 |
| 23 | 267.25 | 55.04 | -0.996487 |
| 24 | 323.60 | 53.17 | 0.999955 |
| 28 | 28.23 | 55.99 | 0.999403 |
| 32 | 211.73 | 54.31 | -0.999988 |

Note. — $r$ is the correlation coefficient of the observed points $(\frac{\tan\delta}{\cos\alpha}, \tan\alpha)$ calculated to fit the orbit.

South point, turning positive to the West. So far we can forecast a satellite's position ($a$, $A$) at any time with the measured orbital elements RAAN and Inclination.

## 5. MEASUREMENT

I measured 13 satellites for 3 hours from 8 p.m. to 11 p.m. on May 29, 2014 in our campus (125.2768° E, 43.8253° N). I recorded each satellits's altitude $a$ and azimuth $A$ using GPS software on my mobile phone every one or two minutes. The precision of measured time is 1 minute and the precision of $a$ and $A$ is 1°. With the recorded data I plotted the satellite points in horizontal coornidates (Fig. 1).

To calculate the earth radius, I enquired my teacher who gave the local gravitational acceleration measured by former experiments in our laboratory that $g$ = 9.80 m/s$^2$.

After transforming the measured points into equatorial coordinates, I fit the orbit using least squares method (an example is given in Fig. 2). Then I calculated the orbital elements of the 13 satellites, given in Table 2.

The (GPS SPS Performance Standard) gives that all GPS satellites have the Inclination of 55.0°, so the error of Inclination of my measurement is less than 2.54°. Since the precision of my measured time is 1 minute and the precision of $a$ and $A$ is 1°, this scale of error is reasonable.

After measuring the 13 satellites' orbit, I managed to plot them on both the celestial sphere (Fig. 4) and the world map (Fig. 5). In Fig. 4 we find that some of the satellites use the same orbit slot, such as satellite No. 1, 2 and 4, satellite No. 10, 20 and 32, satellite No. 12 and 28, and satellite No. 13 and 23.

To verify my measurement, I build the ephemeris of June 6, 2014 and forcasted a transit of satellite No. 20 and No. 32 in the morning. I measured the two satellites for 25 minutes from 10:20 a.m. to 10:45 a.m. My result is given in Fig. 3. From Fig. 3 we find the error of my measurement is about 2°, which is of the same scale of the error of orbital elements.

## 6. CONCLUSION

In this paper, I give a simple method to measure a satellite's orbit on the earth. Using this method, we can calculate a satellite's orbital elements only by measuring it's altitude and azimuth in a period. With the measured orbit we can forecast the satellite's position, which is testable. By measuring the satellite's position in different periods, we can modify the orbital elements and make the measurement more accurate.

I am grateful to my teacher W. Lv for giving me the gravitational acceleration data. I acknowledge using Astronomy & Astrophysics package for Matlab downloaded at http://www.weizmann.ac.il/home/eofek/matlab/. I acknowledge using the android software *GPS Test* and *Turbo GPS 2* which can be downloaded at http://play.google.com/store.

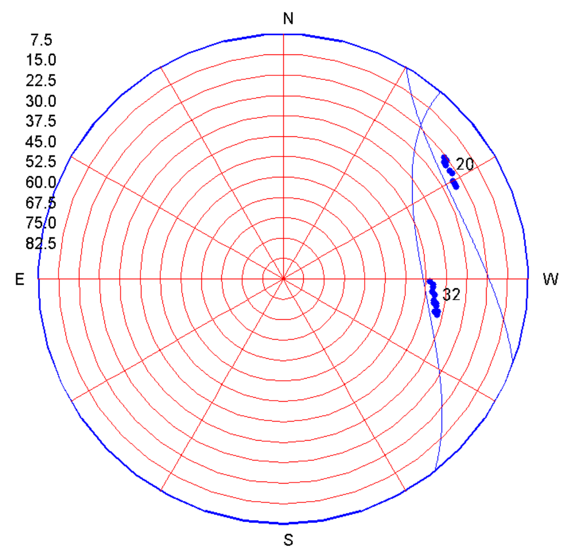


Fig. 3.— A transit of satellite No. 20 and No. 32 in the morning on June 6, 2014 forecasted (the curves) and measured (the dots).

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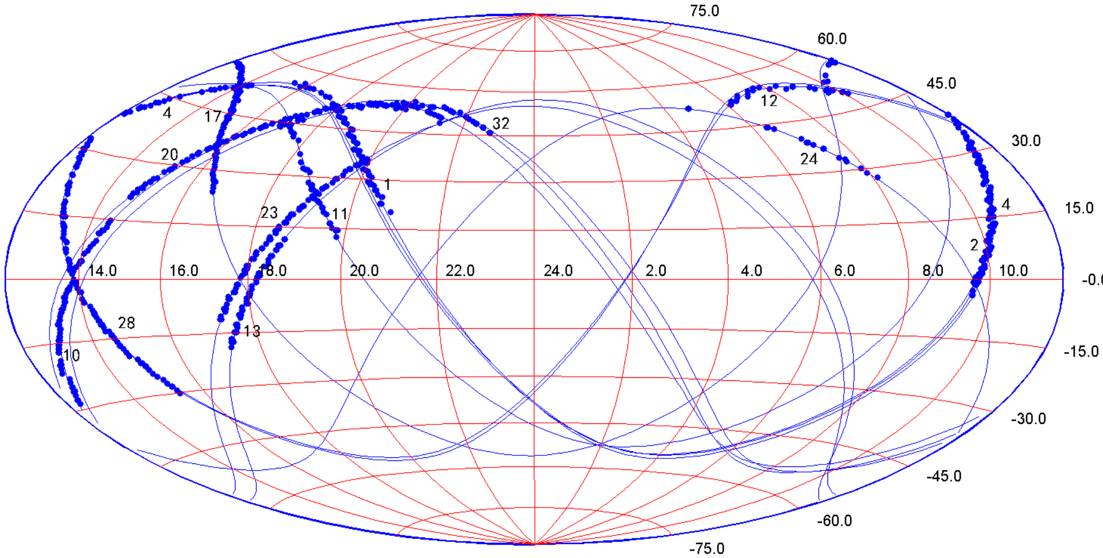


FIG. 4.— Measured points (the dots) and fitted orbit (the curves) in equatorial coordinates. Note that some of the satellites use the same orbit slot, such as satellite No. 1, 2 and 4, satellite No. 10, 20 and 32, satellite No. 12 and 28, and satellite No. 13 and 23.

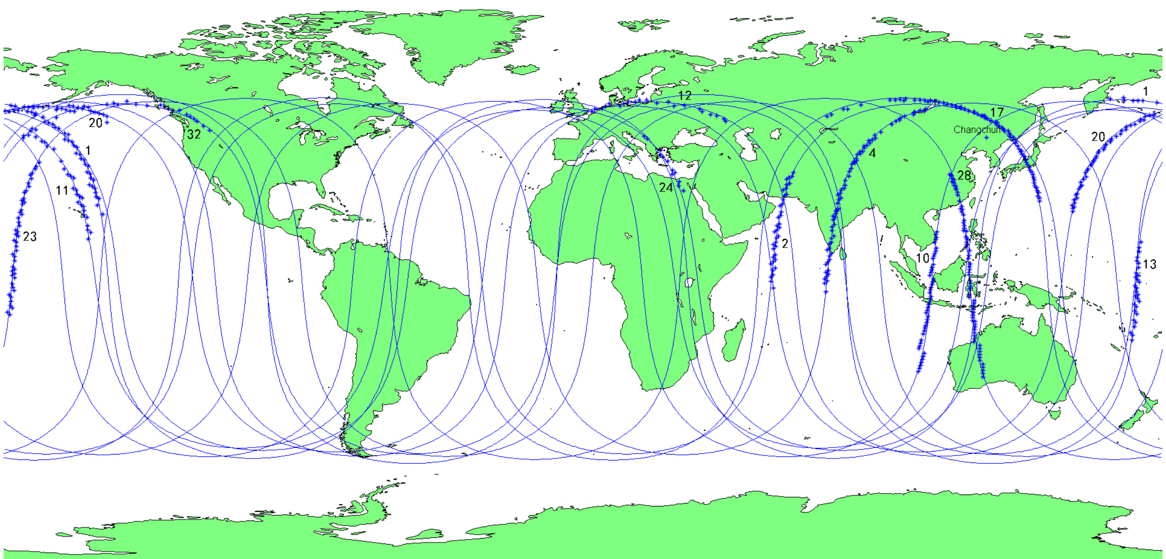


FIG. 5.— Measured points (the dots) and fitted orbit (the curves) plotted on the world map. Note that no satellite above South America and Africa is observed because they are on the opposite side of the earth.