THREE-DIMENSIONAL HORSESHOES AND ORIENTATION REVERSAL IN WILD CHAOS

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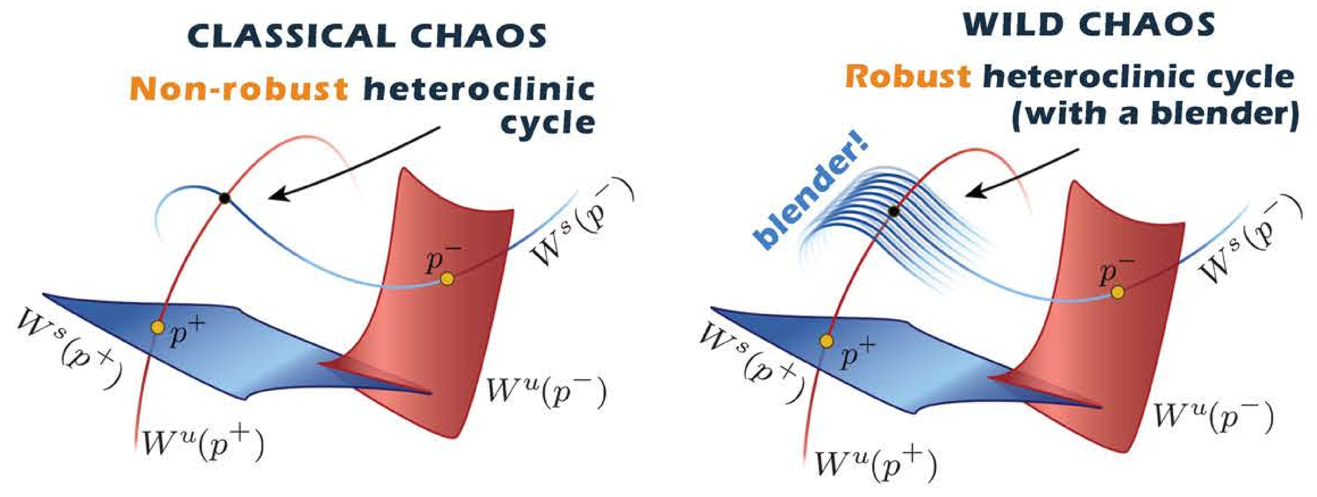
1. BACKGROUND

- **Wild chaos** is a new type of chaotic dynamics; a minimum of three dimensions is required in discrete-time dynamical systems.
- Unlike classical chaos, systems that exhibit wild chaos admit robust heteroclinic cycles leading to **non-hyperbolic robust chaotic dynamics**.

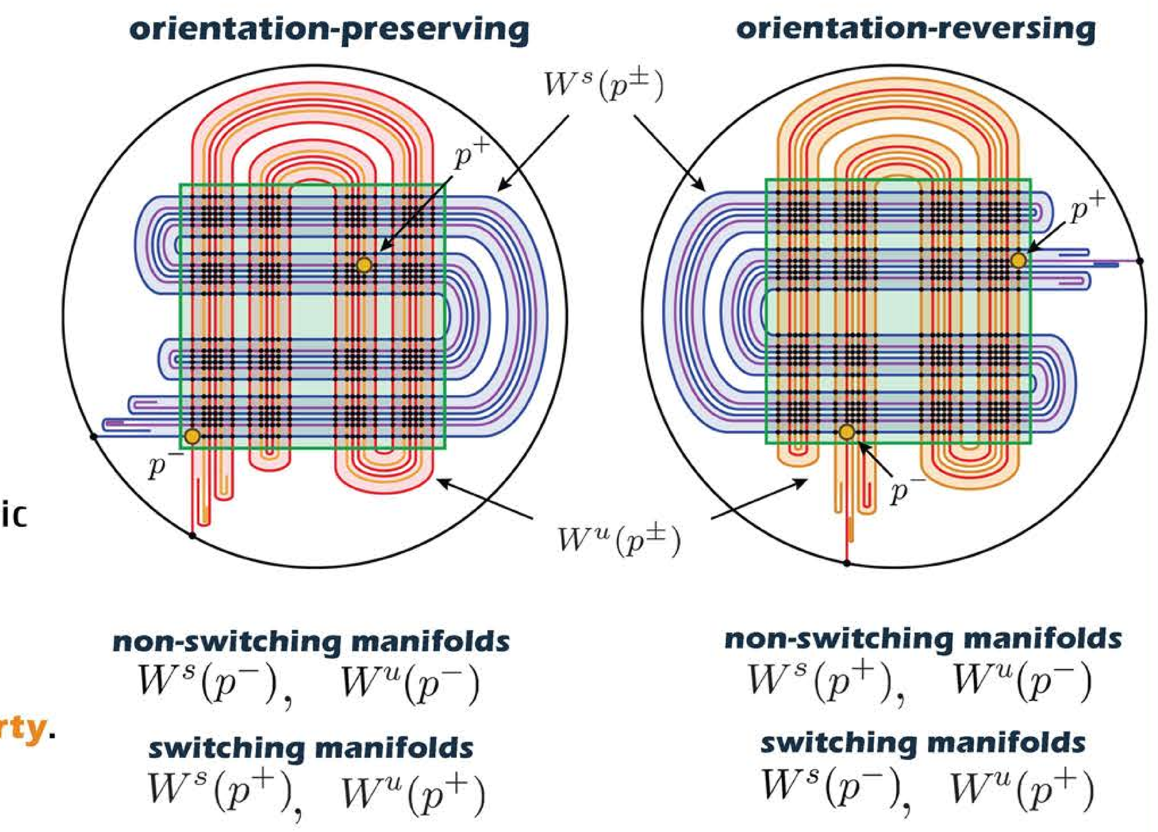


- A mechanism to generate robust heteroclinic cycles and wild chaos is a geometric structure called a **blender**.
- Blenders have invariant manifolds that behave as geometric objects of **higher dimensions** than expected; we say that such a manifold has the **carpet property**.
- Blenders are in some sense a **generalization** of the **Smale's horseshoe** construction to higher dimension.

WHAT IS THE IMPACT OF THE ORIENTATION OF THE HORSESHOE IN THE CREATION OF BLENDERS?

- Sketch of the horseshoe construction of the 2D Hénon map in compactified coordinates:

TWO-DIMENSIONAL HORSESHOES



2. THREE-DIMENSIONAL HÉNON-LIKE MAP

$$\mathcal{H}(x,y,z) = (y, \alpha - y^2 - \beta x, \xi z + y)$$

PROPERTIES OF THE 3D HÉNON-LIKE MAP

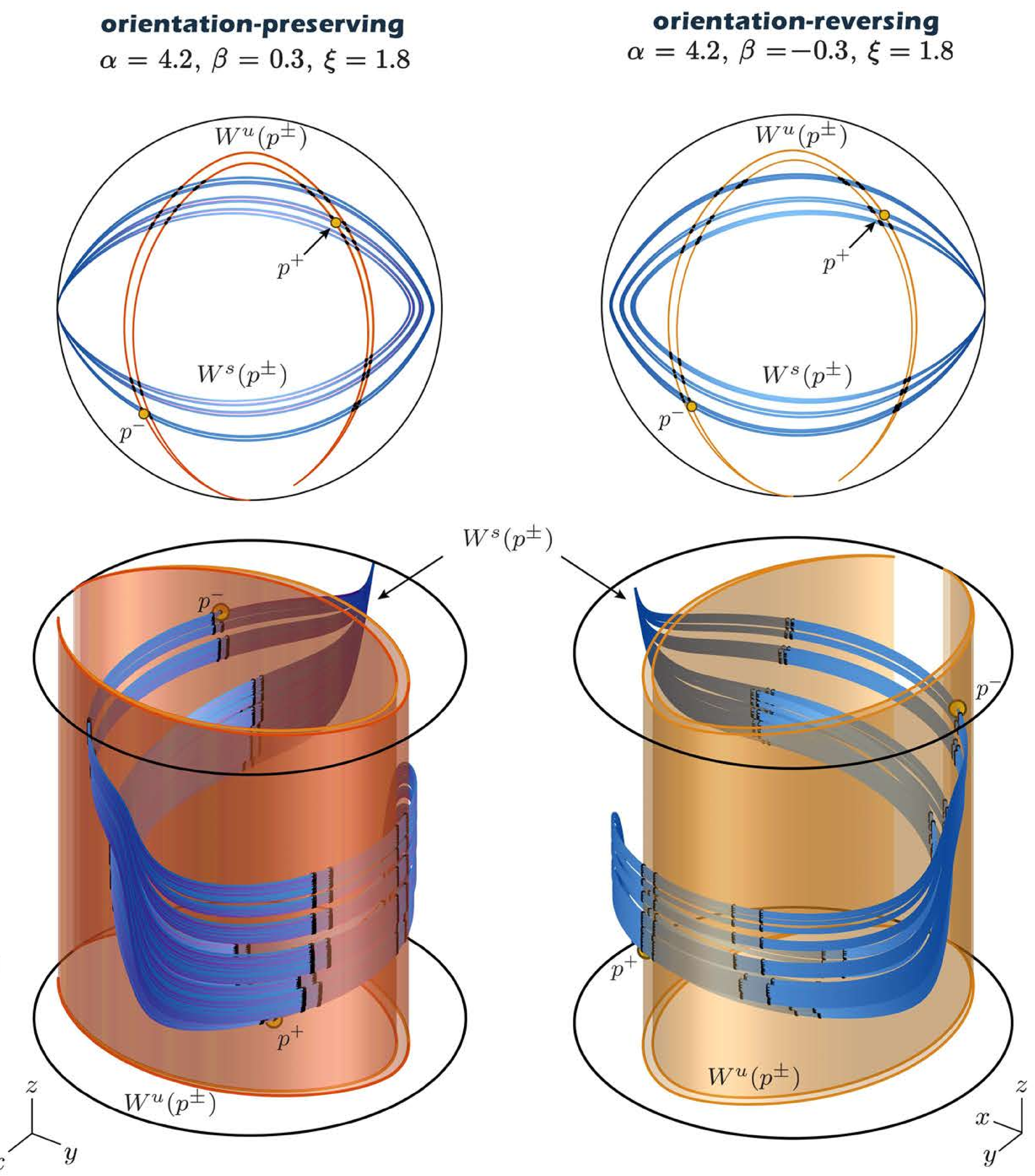
- The first two coordinates corresponds to the 2D Hénon map.
- The determinant of the Jacobian is given by $\beta\xi$; if positive, the map is **orientation-preserving**, if negative, the map is **orientation-reversing**.

WHAT DO WE DO?

- We consider the case of $\xi > 1$: the orientation of the map depends on the **sign** of β.
- If $\beta > 0$ (**orientation-preserving**): $W^s(p^-)$ is a non-switching manifold and $W^s(p^+)$ is a switching manifold.
- If $\beta < 0$ (**orientation-reversing**): $W^s(p^-)$ is a switching manifold and $W^s(p^+)$ is a non-switching manifold.

TO KNOW WHEN THE SYSTEM HAS A BLENDER WE CHECK THE CARPET PROPERTY FOR THE ONE-DIMENSIONAL MANIFOLDS

- Computing long pieces of manifolds to test the carpet property is a major challange since the manifolds make excursions to near infinity.
  SOLUTION: Compactification of the phase space to a cylinder.



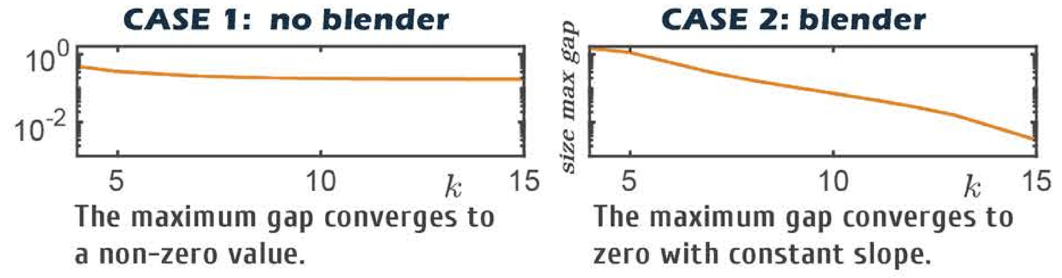
3. IS IT REALLY A BLENDER? THE CARPET PROPERTY (ORIENTATION-PRESERVING)

- We **intersect** the one-dimensional manifolds with a plane Σ and consider the ordered sets of intersection points:

$$\{w^-(\ell)\} := \{(x,y,z) \in W^s(p^-) \cap \Sigma \mid \ell \in \mathbb{Z}/\{0\}\},$$
$$\{w^+(\ell)\} := \{(x,y,z) \in W^s(p^+) \cap \Sigma \mid \ell \in \mathbb{Z}/\{0\}\}.$$

- The manifold has the capet property if the **maximum gap** in the z direction converges to zero as we double the amount of $(\ell = 2^k)$ intersection points $\{w^\pm(\ell)\}$.

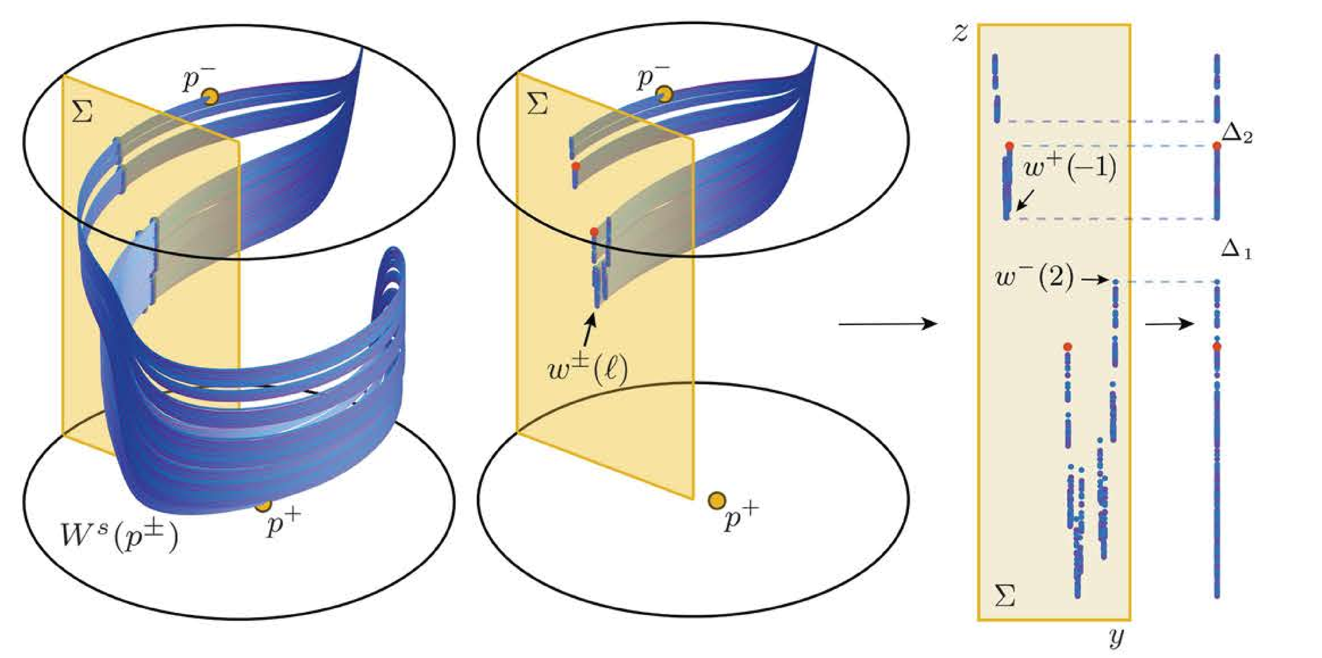
CONVERGENCE OF MAXIMUM GAP



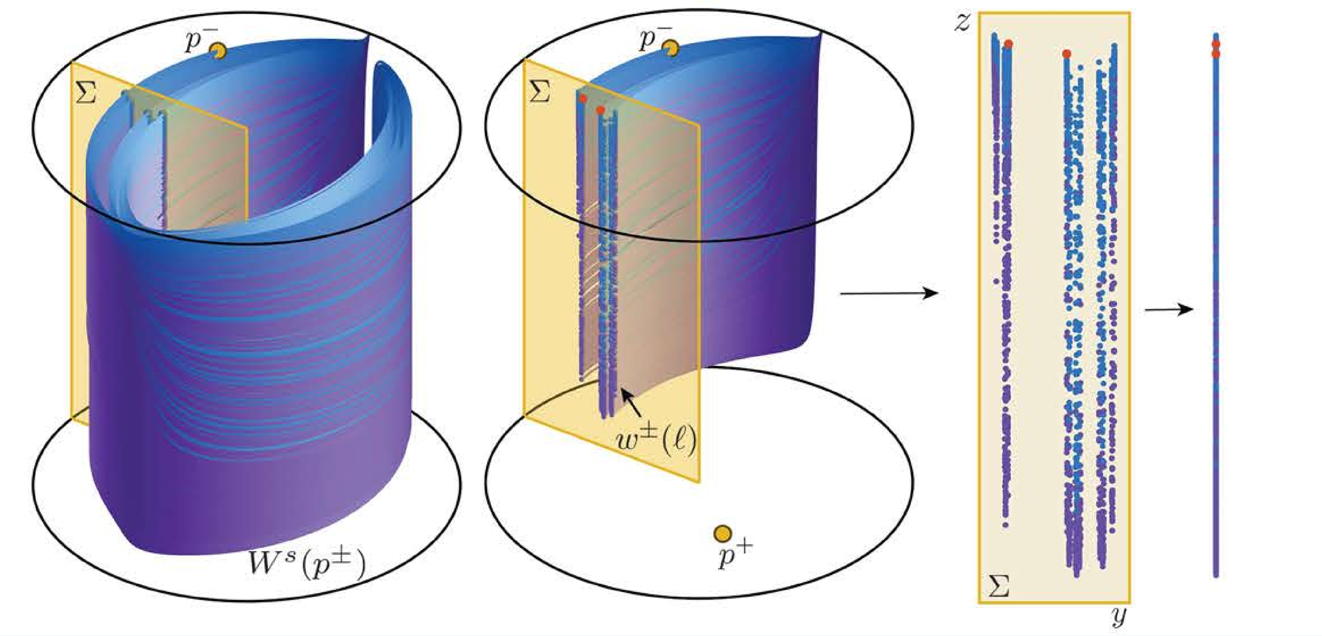
- The generation and desapearence of blenders can be studied as a **function** of ξ.

WHEN ALL THE MAIN GAPS Δ_k CLOSE THEN THE MAXIMUM GAP CONVERGES TO ZERO AND THE SYSTEM EXHIBITS A BLENDER

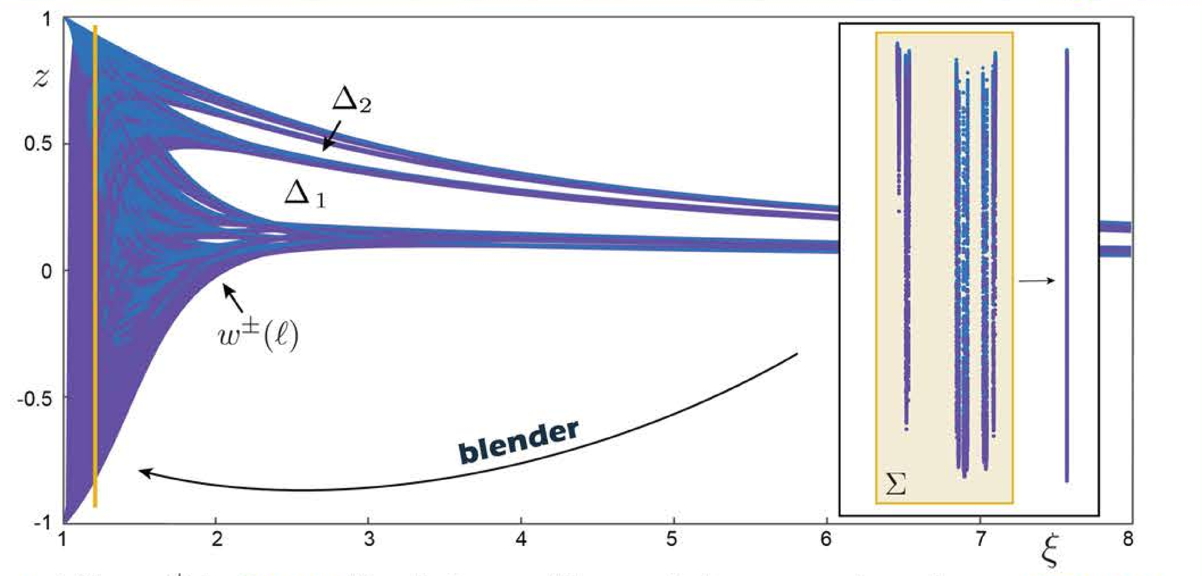
CASE 1: no blender for $\alpha = 4.2$, $\beta = 0.3$, $\xi = 1.8$



CASE 2: blender for $\alpha = 4.2$, $\beta = 0.3$, $\xi = 1.2$



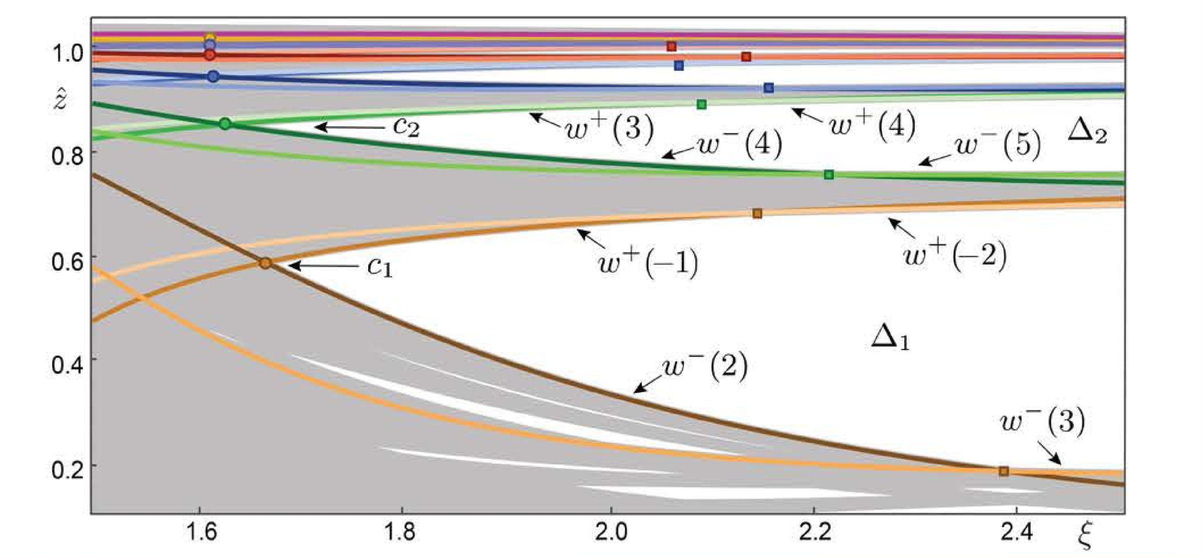
4. CARPET PROPERTY AS A FUNCTION OF ξ



- When ξ is **large** the intersection points are ordered as a **Cantor set** construction, showing clear gaps.
- As ξ **decreases** the intersection points $w^\pm(\ell)$ reorganize following a **recurrent pattern**.

WE DETERMINE WHICH INTERSECTION POINTS OF WHICH MANIFOLD BOUND THE MAIN GAPS FOR WHICH VALUE OF ξ

- The **top boundary** of the gaps Δ_k are defined by the manifold $W^s(p^+)$ and the **bottom boundary** by $W^s(p^-)$; they close at the point C_k, when their boundaries **intersect**.



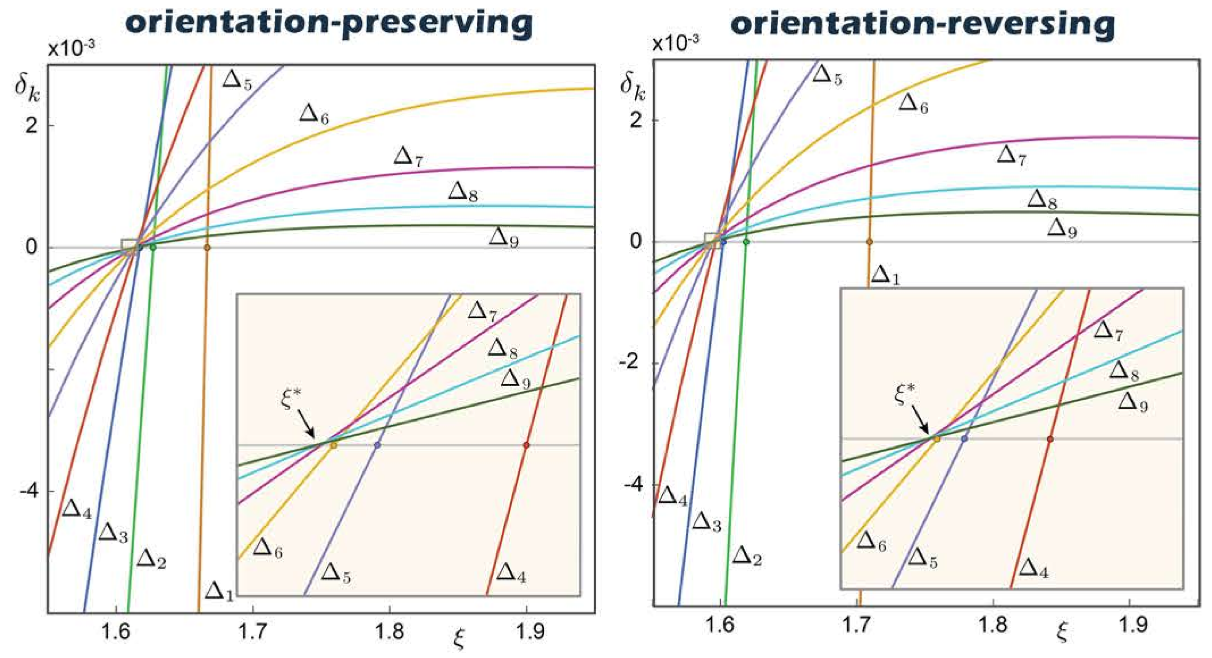
5. PREDICTING WHEN THE GAPS CLOSE

- The previous analysis has been done for both the orientation-preserving and the orientation-reversing case.

WE FIND THAT THERE IS A SIMILAR PATTERN IN THE GAPS FOR BOTH ORIENTATIONS OF THE MAP

- Every gap Δ_k closes in a **sequencial manner**; as ξ decreases, the size δ_k of Δ_k is zero just before δ_{k+1}.
- We **estimate accurately** ξ^* such that there is a blender for all $1 < \xi < \xi^*$ for both orientations of the map.

SIZE δ_k OF GAPS Δ_k AS A FUNCTION OF ξ



Blender for $\xi < \xi^* \approx 1.6135\cdots$ | **Blender for $\xi < \xi^* \approx 1.5935\cdots$**

- It is efficient to check the limit ξ^* as we know exactly the intersection points of the manifold bounding the gaps.

ON GOING WORK: WHAT HAPPENS FOR $\xi < -1$?

- We already have evidence that the manifolds of period-2 points are responsible for bounding the gaps for $\xi < -1$.

[1] C. Bonatti and L.J. Díaz, Persistent nonhyperbolic transitive diffeomorphisms, Annals of Mathematics 143(2): 357–396, 1996.

[2] C. Bonatti, L.J. Díaz and M. Viana, Dynamics beyond Uniform Hyperbolicity, Encyclopaedia of Mathematical Sciences 102, Springer, 2005.

[3] S. Hittmeyer, B. Krauskopf, H.M. Osinga and K. Shinohara, Existence of blenders in a Hénon- like family: geometric insights from invariant manifold computations, Nonlinearity 31(10) , 2018.

[4] B. Krauskopf and H.M. Osinga, Growing 1D and quasi-2D unstable manifolds of maps, Journal of Computational Physics 146(1):404-419, 1998.