

Evolving a Single Puncture Using the Moving Puncture Method*

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In this paper, we investigate the evolution of a single puncture within the framework of the Moving Puncture Method. The Moving Puncture Method has proven to be a powerful technique in numerical relativity for evolving black hole spacetimes. In our study, we focus on its application to the evolution of a single puncture, which can serve as a fundamental building block in simulating more complex spacetime scenarios.

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I. INTRODUCTION

Since the first successful simulations of merging black-hole binaries for computing accurate gravitational waveforms, various formulations and techniques have been employed. These include the BSSN (*Baumgarte-Shapiro-Shibata-Nakamura*) formulation [1–3], conformal decompositions [4, 5], the Bowen-York approach [6], and the moving puncture gauge [7]. These methods have been in use until recently.

Evolving a Schwarzschild black hole serves as a valuable benchmark for assessing the performance of numerical codes, given that we possess its analytical solutions.

In this paper, we present a method for obtaining analytic solutions for a Schwarzschild black hole under time-independent conditions. We provide numerical setups and results for comparison with these analytical solutions.

II. INITIAL DATA

A. Spatial Metric

We need to specify the spatial metric γ_{ij} and the extrinsic curvature K_{ij} which satisfy the Hamiltonian constraint

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \quad (1)$$

and the momentum constraint

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i. \quad (2)$$

To construct a single puncture initial data, we adopt York-Lichnerowicz conformal decompositions [4, 5]. The spatial metric is decomposed into a conformal factor ψ multiplying an conformally related metric:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}. \quad (3)$$

With this, (1) turns into

$$\bar{D}^2\psi - \frac{\psi}{8}\bar{R} + \frac{\psi^5}{8}(K_{ij}K^{ij} - K^2) = -2\pi\psi^5\rho, \quad (4)$$

where we define the conformal Laplace operator

$$\bar{D}^2\psi \equiv \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j\psi. \quad (5)$$

We assume

1) *vacuum spacetime* ($\rho = 0, j^i = 0$),

2) *time symmetry* ($\partial_t\gamma_{ij} = 0, \beta^i = 0$),

so the extrinsic curvature

$$K_{ij} = \frac{1}{2\alpha}(\mathcal{L}_\beta\gamma_{ij} - \partial_t\gamma_{ij}) \quad (6)$$

vanishes and also $K = 0$. Therefore, (2) satisfied identically. If we assume *conformal flatness*, the conformally related metric to be flat,

$$\bar{\gamma}_{ij} = \eta_{ij}, \quad (7)$$

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(4) reduces to the simple Laplace equation

$$\bar{D}^2 \psi = 0, \quad (8)$$

which has the solution

$$\psi = A + \frac{B}{r}. \quad (9)$$

To satisfy asymptotically flat as $r \rightarrow \infty$, and ADM mass to be M , we should choose $A = 1$ and $B = \frac{M}{2}$:

$$\psi = 1 + \frac{M}{2r}. \quad (10)$$

TwoPUNCTURES [8] uses the Bowen-York approach [6] and the method presented in [7]. Since there is a single puncture with neither spin nor momentum, the result should be the same as in (10). We set TP_epsilon to 10^{-6} and TP_Tiny to 0, resulting in

$$\psi = 1 + \frac{M}{2\tilde{r}}, \quad (11)$$

where

$$\tilde{r} = (r^4 + 10^{-24})^{1/4}. \quad (12)$$

The initial metric set by TwoPUNCTURES exactly takes the form

$$\gamma_{ij} = \left(1 + \frac{M}{2\tilde{r}}\right)^4 \eta_{ij}, \quad (13)$$

for all grid points. Fig. 1 shows γ_{xx} along the x -axis at the initial time. Each marker represents a grid point for a respective refinement level. The non-diagonal components of the spatial metric and all components of the extrinsic curvature have explicit values of 0.

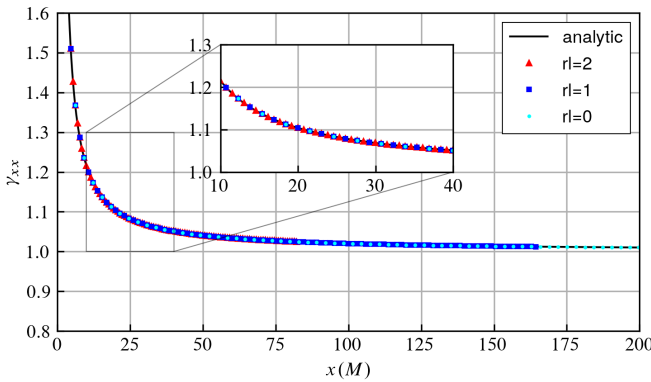


FIG. 1. γ_{xx} along the x -axis at the initial time. Only 3 of the 7 refinement levels are plotted.

B. Constraints

From (1) and (2), we can define

$$\mathcal{H} \equiv R + K^2 - K_{ij} K^{ij} - 16\pi\rho, \quad (14)$$

$$\mathcal{M}^i \equiv D_j (K^{ij} - \gamma^{ij} K) - 8\pi j^i, \quad (15)$$

and it should satisfy that $\mathcal{H} = 0$ and $\mathcal{M}^i = 0$. Fig. 2 shows $\log_{10} |\mathcal{H}|$ on the xy plane at the initial time. Every value at each grid point is less than 10^{-7} . In the case of \mathcal{M}^i , all of the values are explicitly 0.

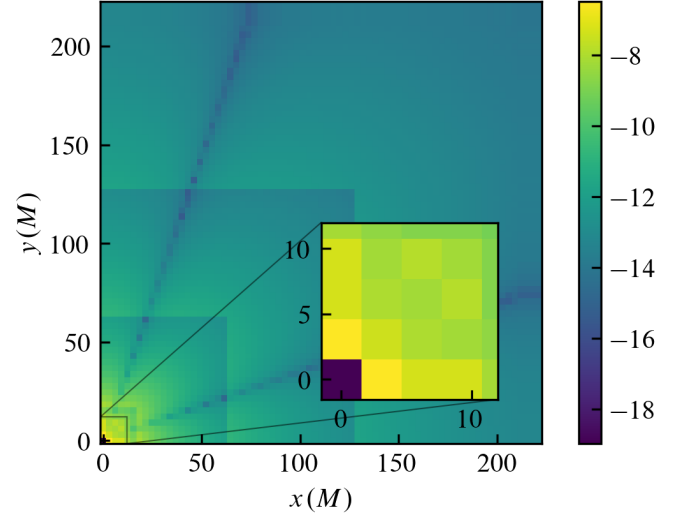


FIG. 2. $\log_{10} |\mathcal{H}|$ on the xy plane at the initial time.

Fig. 3 shows the Hamiltonian constraint violation \mathcal{H} along the x axis. The data at $x = 0$ has been cut since its value is on the order of 1 at the late time. \mathcal{H} increased as time passed, and most peak point's x coordinate also increased. But it still remains on the order of 10^{-6} at $t = 115.2M$.

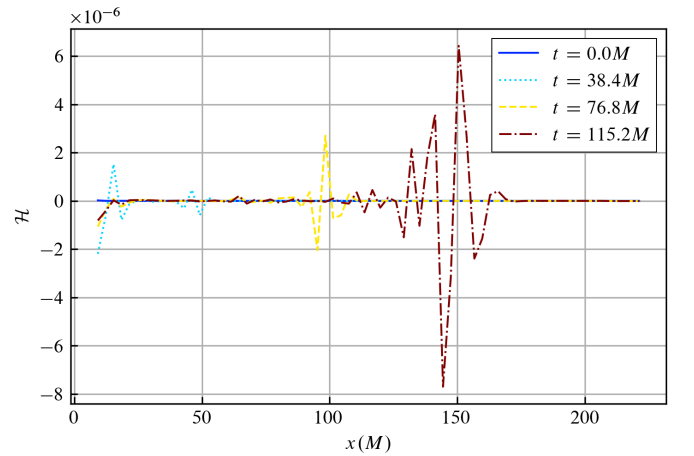


FIG. 3. The Hamiltonian constraint violation along the x axis.

III. EVOLUTION

A. Gauge Condition

The 1+log slicing with advection term is given by

$$\partial_t \alpha = -2\alpha K + \beta^i \partial_i \alpha, \quad (16)$$

and the hyperbolic gamma driver condition for the shift with advection term is defined as

$$\partial_t \beta^i = \frac{3}{4} B^i + \beta^j \partial_j \beta^i, \quad (17)$$

$$\partial_t B^i = \partial_t \bar{\Gamma}^i - B^i + \beta^j \partial_j B^i, \quad (18)$$

which is typically employed in the moving puncture method [9, 10]. Sometimes we can drop the advection term, simplifying (16), (17), and (18) to

$$\partial_t \alpha = -2\alpha K, \quad (19)$$

$$\partial_t \beta^i = \frac{3}{4} B^i, \quad (20)$$

$$\partial_t B^i = \partial_t \bar{\Gamma}^i - B^i, \quad (21)$$

which we adopt in this paper.

We used the twopunctures-averaged initial lapse, which is given by

$$\alpha = \frac{1}{2}(1 + \alpha'), \quad (22)$$

where

$$\alpha' = \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}, \quad (23)$$

ensuring that the lapse satisfies $0 \leq \alpha \leq 1$.

When the simulation progresses sufficiently, and the solution becomes time-independent, we have $\partial_t \alpha = 0$, implying $K = 0$. Since we choose the hyperbolic gamma driver condition, it has a special solution (see [11, 12])

$$\alpha = \sqrt{1 - \frac{2M}{r_s} + \frac{27M^4}{16r_s^4}}, \quad (24)$$

where r_s is an areal radius related to the isotropic radius r as

$$r = \frac{2r_s + M + (4r_s^2 + 4Mr_s + 3M^2)^{1/2}}{4} \times \left(\frac{(4 + 3\sqrt{2})(2r_s - 3M)}{8r_s + 6M + 3(8r_s^2 + 8Mr_s + 6M^2)^{1/2}} \right)^{1/\sqrt{2}}. \quad (25)$$

The conformal factor is given by $\psi = \left(\frac{r_s}{r}\right)^{1/2}$, and when substituted into (25), we obtain

$$\psi = \left(\frac{4r_s}{2r_s + M + (4r_s^2 + 4Mr_s + 3M^2)^{1/2}} \right)^{1/2} \left(\frac{8r_s + 6M + 3(8r_s^2 + 8Mr_s + 6M^2)^{1/2}}{(4 + 3\sqrt{2})(2r_s - 3M)} \right)^{1/2\sqrt{2}}. \quad (26)$$

Fig. 4 shows α along the x axis at several times. After $t = 49.152M$, α remains almost constant. The black solid line represents the analytic solution given in (24).

Fig. 5 shows γ_{xx} along the x axis. The black solid line indicates the analytic solution $\gamma_{ij} = \psi^4 \eta_{ij}$, where ψ is given by (26).

B. BSSN

The 3+1 ADM evolution equations are given as:

$$(\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\alpha K_{ij}, \quad (27)$$

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = -D_i D_j \alpha + \alpha (R_{ij} + K K_{ij} - 2K_{ik} K^k_j) + 4\pi \alpha M_{ij}, \quad (28)$$

However, this set of partial differential equations is only weakly hyperbolic and is therefore not suitable for stable numerical evolution. To address this, we adopt the BSSN (*Baumgarte-Shapiro-Shibata-Nakamura*) formulation [1–3].

In the BSSN formulation, the spatial metric γ_{ij} is decomposed into a conformally related metric as in (3), with $\det(\tilde{\gamma}_{ij}) = 1$. The extrinsic curvature is also decomposed into its trace and traceless parts, and we conformally transform the traceless part as follows:

$$K_{ij} = e^{4\phi} \tilde{A}_{ij} + \frac{1}{3} \gamma_{ij} K. \quad (29)$$

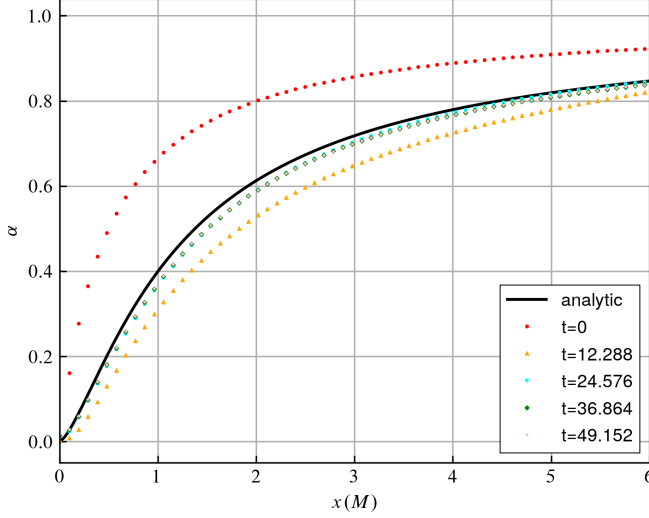


FIG. 4. α along the x axis at several times. Only 5 refinement levels are plotted. The time unit is divided by M .

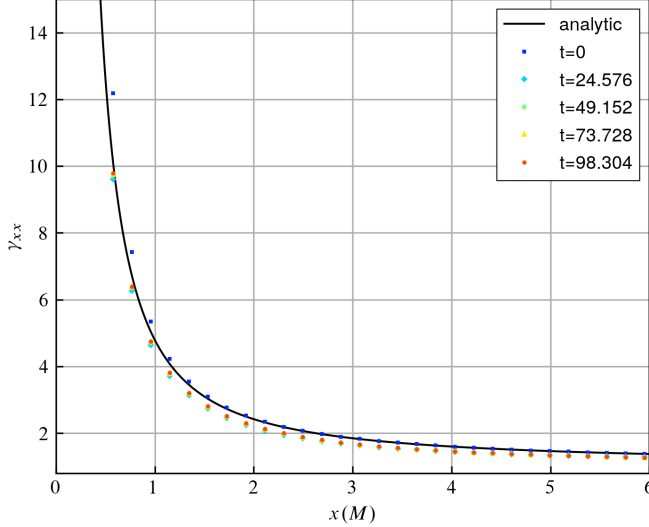


FIG. 5. γ_{xx} along the x axis at several times. Only refinement level 4 is plotted. The time unit is divided by M .

We then promote the following variables to evolution variables:

$$\phi = \ln \psi = \frac{1}{12} \ln \gamma, \quad (30)$$

as well as the conformal connection functions:

$$\bar{\Gamma}^i = \bar{\gamma}^{jk} \bar{\Gamma}^i_{jk} = -\partial_j \bar{\gamma}^{ij}. \quad (31)$$

The evolution equation for γ_{ij} splits into two equations:

$$\partial_t \phi = -\frac{1}{6} \alpha K + \beta^i \partial_i \phi + \frac{1}{6} \partial_i \beta^i, \quad (32)$$

$$\begin{aligned} \partial_t \bar{\gamma}_{ij} = & -2\alpha \bar{A}_{ij} + \beta^k \partial_k \bar{\gamma}_{ij} \\ & + \bar{\gamma}_{ik} \partial_j \beta^k + \bar{\gamma}_{jk} \partial_i \beta^k - \frac{2}{3} \bar{\gamma}_{ij} \partial_k \beta^k. \end{aligned} \quad (33)$$

The evolution equation for K_{ij} also splits into two equations:

$$\begin{aligned} \partial_t K = & -D^i D_i \alpha + \alpha (\tilde{A}_{ij}^{ij} + \frac{1}{3} K^2) \\ & + \beta^i \partial_i K + 4\pi \alpha (\rho + S), \end{aligned} \quad (34)$$

$$\begin{aligned} \partial_t \tilde{A}_{ij} = & e^{-4\phi} [-D_i D_j \alpha + \alpha (R_{ij} - 8\pi S_{ij})]^{TF} \\ & + \alpha (K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}^k_j) + \beta^k \partial_k \tilde{A}_{ij} \\ & + \tilde{A}_{ik} \partial_j \beta^k + \tilde{A}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k, \end{aligned} \quad (35)$$

where the superscript TF denotes the trace-free part of a tensor. The Ricci tensor is also split into:

$$\begin{aligned} R_{ij} = & \bar{R}_{ij} - 2(\bar{D}_i \bar{D}_j \phi + \bar{\gamma}_{ij} \bar{\gamma}^{lm} \bar{D}_l \bar{D}_m \phi) \\ & + 4((\bar{D}_i \phi)(\bar{D}_j \phi) - \bar{\gamma}_{ij} \bar{\gamma}^{lm} (\bar{D}_l \phi)(\bar{D}_m \phi)) \\ \equiv & \bar{R}_{ij} + R_{ij}^\phi. \end{aligned} \quad (36)$$

The $\bar{\Gamma}^i$ are now treated as independent functions that satisfy their own evolution equations:

$$\begin{aligned} \partial_t \bar{\Gamma}^i = & 2\alpha \left(\bar{\Gamma}^i_{jk} \tilde{A}^{kj} - \frac{2}{3} \bar{\gamma}^{ij} \partial_j K - 8\pi \bar{\gamma}^{ij} S_j + 6\tilde{A}^{ij} \partial_j \phi \right) \\ & - 2\tilde{A}^{ij} \partial_j \alpha + \beta^j \partial_j \bar{\Gamma}^i - \bar{\Gamma}^j \partial_j \beta^i \\ & + \frac{2}{3} \bar{\Gamma}^i \partial_j \beta^j + \frac{1}{3} \bar{\gamma}^{il} \partial_l \partial_j \beta^j + \bar{\gamma}^{jl} \partial_j \partial_l \beta^i. \end{aligned} \quad (37)$$

We use the variable:

$$W = \gamma^{-1/6} = e^{-2\phi} \quad (38)$$

instead of ϕ . The evolution equation for W is:

$$\partial_t W = \frac{1}{3} W (\alpha K - \partial_i \beta^i) + \beta^i \partial_i W. \quad (39)$$

The other choice, which evolves ϕ , can lead to crashes due to numerical singularities.

We use McLachlan [13–15] and adopt the set of equations (33), (34), (35), (37), (39) for evolution and the gauge conditions (19), (20).

IV. NUMERICAL SETUP

We utilize the EINSTEIN TOOLKIT [16, 17] to simulate the evolution of a single puncture.

A. Mesh Refinement

We employ mesh refinement [18] for the puncture center with 7 levels. The basic grid resolution is $3.072M$, and the refinement factor is set to 2. Therefore, the minimum resolution is $3.072M \times 2^{-6} = 0.048M$.

B. Boundary Conditions

We used Minkowski boundary condition in ML_BSSN. We used ghost zones size of 3.

C. Symmetry

We used REFLECTIONSYMMETRY to reflection on z plane and CARPETREGRID2 to 180° rotating symmetry.

V. RESULT

A. Gravitational Waves

For wave extraction, we utilized the WEYLSCAL4 and MULTIPOLE thorns [19]. These thorns extract the Weyl scalar and decompose it into the spin-weighted spherical harmonics. To calculate the radiated energy, we considered modes up to $l = 8$, which are considered credible because higher modes are dominated by numerical noise [20].

Since the Schwarzschild metric has spherical symmetry, there should be no gravitational wave radiation. Fig. 6 displays the real and imaginary parts of $r_{\text{ex}}\psi_4$. There is no notable difference between $r_{\text{ex}} = 50M$. Furthermore, all values of $r_{\text{ex}}\psi_4$ are on the order of 10^{-14} , which can be regarded as numerical noise.

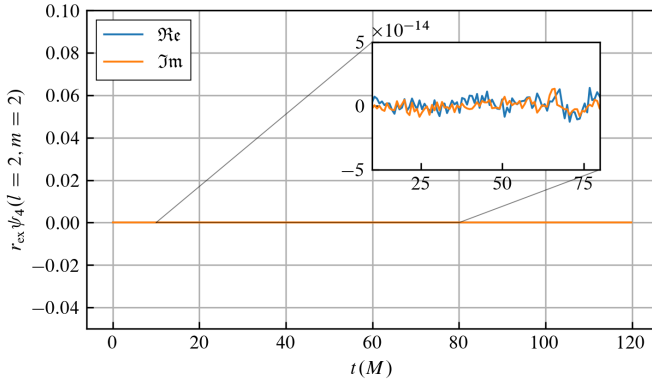


FIG. 6. Real and imaginary parts of $r_{\text{ex}}\psi_4$. The red dashed line indicates $t = r_{\text{ex}} = 50M$.

B. Apparent Horizon

We measured the apparent horizon at each time using AHFINDERDIRECT [21, 22]. This tool identifies an apparent

horizon by numerically solving the equation

$$\Theta \equiv D_i n^i + K_{ij} n^i n^j - K = 0, \quad (40)$$

where n^i represents the outward-pointing unit normal to the apparent horizon, and D_i is the covariant derivative operator associated with the 3-metric in the slice.

We did not fix the puncture location at the origin, but there was no puncture movement, and the puncture remained at the origin. Fig. 7 illustrates the L_∞ norm of expansion Θ of the apparent horizon as a function of time, while Fig. 8 displays the irreducible mass M_{irr} as a function of time.

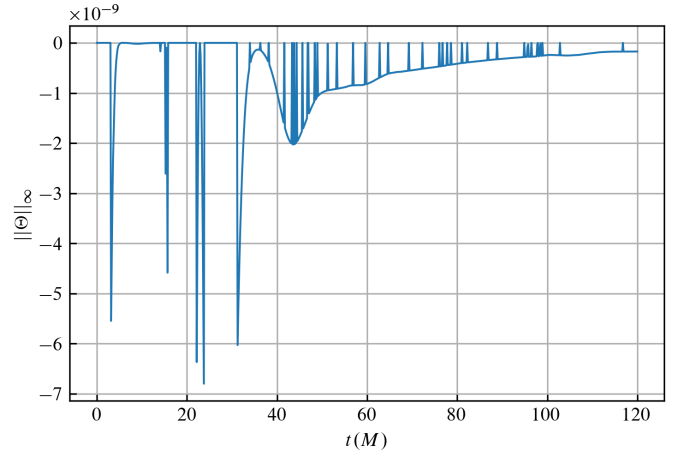


FIG. 7. L_∞ norm of expansion Θ of the apparent horizon as a function of time.

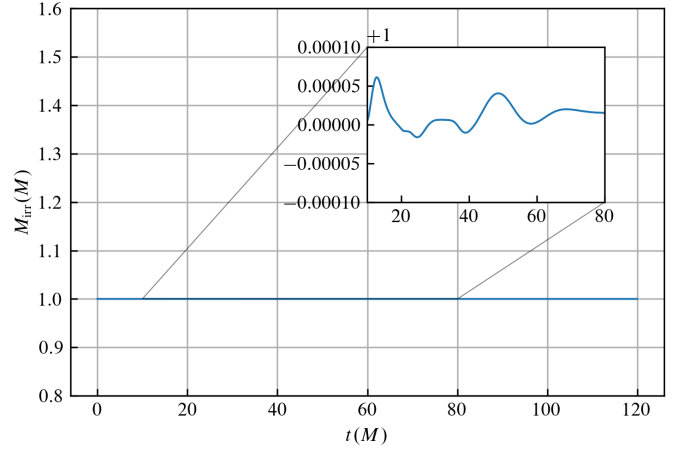


FIG. 8. Irreducible mass M_{irr} as a function of time.

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