Solution to Numerical Relativity

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Contents

2 The 3+1 decomposition of Einstein's equations

Exercise 2.1

Exercise 2.2

From

$$\mathcal{C}_E = D_i E^i - 4\pi\rho, \tag{1}$$

$$\partial_t E_i = D_i D^j A_j - D^j D_j A_i - 4\pi j_i, \tag{2}$$

$$\frac{\partial\rho}{\partial t} + D_i j^i = 0, \tag{3}$$

we can get

$$\begin{aligned}
\partial_t \mathcal{C}_E &= D_i \partial_t E^i - 4\pi \frac{\partial\rho}{\partial t} && (4)\\
&= D_i (D^i D^j A_j - D^j D_j A^i - 4\pi j^i) + 4\pi D_i j^i && (5)\\
&= D_i D^i D^j A_j - D_i D^j D_j A^i && (6)\\
&= 0. && (7)
\end{aligned}$$

Exercise 2.5

$$\omega_{[a} \nabla_b w_{c]} = \frac{1}{6}(\omega_a \nabla_b \omega_c - \omega_a \nabla_c \omega_b + \omega_b \nabla_c \omega_a - \omega_b \nabla_a \omega_c + \omega_c \nabla_a \omega_b - \omega_c \nabla_b \omega_a). \tag{8}$$

Using $\omega_a = \alpha \nabla_a t$, the first two term in parenthesis will be

$$\begin{aligned}
\omega_a \nabla_b \omega_c - \omega_a \nabla_c \omega_b &= \alpha(\nabla_a t)(\nabla_b \alpha \nabla_c t + \alpha \nabla_b \nabla_c t) - \alpha(\nabla_a t)(\nabla_c \alpha \nabla_b t + \alpha \nabla_c \nabla_b t) && (9)\\
&= \alpha(\nabla_a t)(\nabla_b \alpha)(\nabla_c t) - \alpha(\nabla_a t)(\nabla_c \alpha)(\nabla_b t). && (10)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega_{[a} \nabla_b w_{c]} =& \alpha(\nabla_a t)(\nabla_b \alpha)(\nabla_c t) - \alpha(\nabla_a t)(\nabla_c \alpha)(\nabla_b t) && (11)\\
&+ \alpha(\nabla_b t)(\nabla_c \alpha)(\nabla_a t) - \alpha(\nabla_b t)(\nabla_a \alpha)(\nabla_c t) && (12)\\
&+ \alpha(\nabla_c t)(\nabla_a \alpha)(\nabla_b t) - \alpha(\nabla_c t)(\nabla_b \alpha)(\nabla_a t) && (13)\\
&= 0. && (14)
\end{aligned}$$

Exercise 2.6

To show that $\gamma^a_b v^b$ is purely spatial, we should show that $n_a \gamma^a_b v^b = 0$.

$$\begin{aligned}
n_a \gamma^a_b v^b &= n_a(\delta^a_b + n^a n_b) v^b && (15)\\
&= (n_b + n_a n^a n_b) v^b && (16)\\
&= (n_b - n_b) v^b && (17)\\
&= 0. && (18)
\end{aligned}$$

Exercise 2.7

$$
\begin{aligned}
T_{ab} =& \delta_a^c \delta_b^d T_{cd} && (19)\\
=& (N_a^c + \gamma_a^c)(N_b^d + \gamma_b^d) T_{cd} && (20)\\
=& (N_a^c N_b^d + N_a^c \gamma_b^d + \gamma_a^c N_b^d + \gamma_a^c \gamma_b^d) T_{cd} && (21)\\
=& n_a n_b n^c n^d T_{cd} - n_a n^c \perp T_{cb} - n_b n^d \perp T_{ad} + \perp T_{ab}. && (22)
\end{aligned}
$$

Exercise 2.8

It is trivial that $\nabla_a g_{bc} = 0$. Since $\gamma_{bc} = g_{bc} + n_b n_c$, we only need to show that $D_a(n_b n_c) = 0$.

$$
\begin{aligned}
D_a(n_b n_c) =& \gamma_a^d \gamma_b^e \gamma_c^f \nabla_d (n_e n_f) && (23)\\
=& \gamma_a^d (\delta_b^e + n^e n_b)(\delta_c^f + n^f n_c)(n_e \nabla_d n_f + n_f \nabla_d n_e) && (24)\\
=& \gamma_a^d (\delta_b^e \delta_c^f + \delta_b^e n^f n_c + \delta_c^f n^e n_b + n^e n_b n^f n_c)(n_e \nabla_d n_f + n_f \nabla_d n_e) && (25)\\
=& \gamma_a^d (n_b \nabla_d n_c + n_c \nabla_d n_b + n_b n_c n^f \nabla_d n_f + n_c n_f n^f \nabla_d n_b && (26)\\
& + n_b n_e n^e \nabla_d n_c + n_b n_c n^e \nabla_d n_e && (27)\\
& + n_b n_c n^e n_e n^f \nabla_d n_f + n_b n_c n^e n_f n^f \nabla_d n_e && (28)\\
=& n_b \nabla_d n_c + n_c \nabla_d n_b + n_b n_c n^f \nabla_d n_f - n_c \nabla_d n_b && (29)\\
& - n_b \nabla_d n_c + n_b n_c n^e \nabla_d n_e - n_b n_c n^f \nabla_d n_f - n_b n_c n^e \nabla_d n_f && (30)\\
=& 0. && (31)
\end{aligned}
$$

Exercise 2.9

If v^a is purely spatial, $n_a v^a = 0$ and $\gamma_a^b v^a = (\delta_a^b + n^b n_a) v^a = v^b + 0$. Similarly $\gamma_b^a w_a = w_b$. So

$$
\begin{aligned}
D_a(v^b w_b) =& \gamma_a^c \nabla_a (v^b \omega_b) && (32)\\
=& \gamma_a^c (v^b \nabla_c \omega_b + \omega_b \nabla_c v^b) && (33)\\
=& \gamma_a^c (\gamma_d^b v^d \nabla_c \omega_b + \gamma_b^d \omega_d \nabla_c v^b) && (34)\\
=& v^d D_a \omega_d + \omega_d D_a v^d && (35)\\
=& v^b D_a \omega_b + \omega_b D_a v^b. && (36)
\end{aligned}
$$

Exercise 2.12

We need to show that $n^a \nabla_b n_a = 0$.

Bonus

Start from exercise 2.5 and $\omega_a = -n_a$. Contract with n^a,

$$
\begin{aligned}
n^a n_{[a} \nabla_b n_{c]} =& n^a n_a \nabla_b n_c - n^a n_a \nabla_c n_b && (37)\\
&+ \cancelto{0}{n^a n_b \nabla_c n_a} - \cancel{n^a n_b \nabla_a n_c} && (38)\\
&+ \cancel{n^a n_c \nabla_a n_b} - \cancelto{0}{n^a n_c \nabla_b n_a} && (39)\\
=& \nabla_c n_b - \nabla_b n_c && (40)\\
=& 0. && (41)
\end{aligned}
$$

Therefore $\nabla_b n_c = \nabla_c n_b$.

3 Constructing initial data

Exercise 3.1

Starting from connection coefficients for spatial components

$$
\begin{aligned}
\Gamma^i{}_{jk} =& \frac{1}{2}\gamma^{il}(\partial_k \gamma_{lj} + \partial_j \gamma_{lk} - \partial_l \gamma_{jk}) && (42)\\
=& \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}(\partial_k(\psi^4\bar{\gamma}_{lj}) + \partial_j(\psi^4\bar{\gamma}_{lk}) - \partial_l(\psi^4\bar{\gamma}_{jk})) && (43)\\
=& \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}\left\{4\psi^3(\bar{\gamma}_{lj}\partial_k\psi + \bar{\gamma}_{lk}\partial_j\psi + \bar{\gamma}_{jk}\partial_l\psi) + \psi^4(\bar{\gamma}_{lj} + \bar{\gamma}_{lk} + \bar{\gamma}_{jk})\right\} && (44)\\
=& \frac{1}{2}\bar{\gamma}^{il}(\bar{\gamma}_{lj} + \bar{\gamma}_{lk} + \bar{\gamma}_{jk}) + 2\psi^{-1}\bar{\gamma}^{il}(\bar{\gamma}_{lj}\partial_k\psi + \bar{\gamma}_{lk}\partial_j\psi + \bar{\gamma}_{jk}\partial_l\psi) && (45)\\
=& \bar{\Gamma}^i{}_{jk} + 2\bar{\gamma}^{il}(\bar{\gamma}_{lj}\partial_k \ln\psi + \bar{\gamma}_{lk}\partial_j \ln\psi + \bar{\gamma}_{jk}\partial_l \ln\psi) && (46)\\
=& \bar{\Gamma}^i{}_{jk} + 2\Big(\delta^i_j\partial_k \ln\psi + \delta^i_k\partial_j \ln\psi - \bar{\gamma}^{il}\bar{\gamma}_{jk}\partial_l \ln\psi\Big). && (47)
\end{aligned}
$$