ADM equations in isotropic Schwarzschild metric

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This article is based on the p.49 in [2].

What we want to prove is the following theorem.

Theorem 1. *For the isotropic Schwarzschild metric, the numerically computed solutions from the ADM equations are static and stable when lapse and shift are chosen as follows:*

$$\alpha = \frac{1 - M/(2r)}{1 + M/(2r)}, \quad \beta^i = 0. \tag{1}$$

Proof. Assume that Proposition 2 holds. Then $\Sigma_0 = \Sigma_1$ for the initial condition $\Sigma_0 = (\gamma_{ij}, K_{ij})$ and for any t, Σ_t is entirely determined by Σ_{t-1}, so $\Sigma_t = \Sigma_0$. ∎

Here, the spatial metric is given by

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 \mathrm{diag}\big(1, r^2, r^2 \sin^2\theta\big). \tag{2}$$

As initial data, K_{ij} is determined by the equation for the evolution of the spatial metric in the following ADM equation:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i. \tag{3}$$

Since γ_{ij} is time independent as in Eq. (2), and $\beta^i = 0$, we can see that K_{ij} is obviously zero.

Now let's prove the following proposition:

Proposition 2. *Analytically, as t increases, K_{ij} can always have a value of zero, even if the evolution scheme proceeds.*

Proof. The non-trivial three-dimensional Ricci tensor computed from a given spatial metric:

$$R_{rr} = -\frac{8rM}{(2r^2 + Mr)^2}, \quad R_{\theta\theta} = \frac{4r^3 M}{(2r^2 + Mr)^2}, \quad R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{4}$$

By taking the trace from them, we get $R = 0$. Thus, we see that the first equation in the ADM equations, the Hamiltonian constraint

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{5}$$

is satisfied. The second equation in the ADM equations, the momentum constraint

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i, \tag{6}$$

is self-evident. The third equation, the evolution equation for the spatial metric, is a obviously true since we constructed the initial conditions here. Now the non-trivial, fourth equation

$$\partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - D_i D_j \alpha - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)\right) + \beta^k(\text{some terms}), \tag{7}$$

is simplified to

$$\partial_t K_{ij} = \alpha R_{ij} - D_i D_j \alpha. \tag{8}$$

Using $D_a D_b \alpha = \partial_a \partial_b \alpha - \partial_c \alpha \Gamma^c{}_{ba}$, we can see that the second term has a specific form. Significant Christoffel symbol is

$$\Gamma^r{}_{rr} = -\frac{M}{r^2\left(1 + \frac{M}{2r}\right)}, \tag{9}$$

$$\Gamma^r{}_{\theta\theta} = \frac{r(M - 2r)}{M + 2r}, \tag{10}$$

$$\Gamma^r{}_{\phi\phi} = \frac{r(M - 2r)\sin^2\theta}{M + 2r}. \tag{11}$$

And we get

$$D_r D_r \alpha = \frac{8M(M - 2r)}{r(M + 2r)^3}, \tag{12}$$

$$D_\theta D_\theta \alpha = -\frac{4Mr\,(M - 2r)}{(M + 2r)^3}, \tag{13}$$

$$D_\phi D_\phi \alpha = -\frac{4Mr\,(M - 2r)\sin^2\theta}{(M + 2r)^3}. \tag{14}$$

Substituting this into Eq. (8) along with Eq. (4), we see that $\partial_t K_{ij} = 0$. ■

In the numerical calculation, the second term is always constant regardless of t, so a fixed value is used, and the first term is calculated as a simple second-order derivative. Now, in a simulation where $\partial_t K_{ij}$ is computed to be zero with sufficient precision, (γ_{ij}, K_{ij}) for all t will be consistent with the initial value.

If we write $\partial_t K_{ij}$ in the form of a second-order derivative, we have

$$\partial_t K_{ij} = \frac{\alpha}{2}\gamma^{kl}(\partial_i\partial_j\gamma_{kj} + \partial_k\partial_j\gamma_{il} - \partial_i\partial_j\gamma_{kl} - \partial_k\partial_l\gamma_{ij}) + \alpha\gamma^{kl}(\Gamma^m{}_{il}\Gamma_{mkj} - \Gamma^m{}_{ij}\Gamma_{mkl}) - \partial_i\partial_j\alpha + \partial_c\alpha\Gamma^c{}_{ij}. \tag{15}$$

According to the p.168 in [1], 1) the momentum constraints can be guaranteed to be identically satisfied, and 2) either the densitized lapse $\tilde{\alpha} := \alpha/\sqrt{\gamma}$ is assumed to be a known function of spacetime (but not the lapse itself), or we use a slicing condition of the Bona–Masso family, then the ADM system would be strongly hyperbolic.

References

[1] Miguel Alcubierre. *Introduction to 3+ 1 numerical relativity*, volume 140. OUP Oxford, 2008.

[2] Thomas W Baumgarte and Stuart L Shapiro. *Numerical relativity: solving Einstein's equations on the computer*. Cambridge University Press, 2010.