Evolving a Single Puncture Using the Moving Puncture Method*

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In this paper, we investigate the evolution of a single puncture within the framework of the Moving Puncture Method. The Moving Puncture Method has proven to be a powerful technique in numerical relativity for evolving black hole spacetimes. In our study, we focus on its application to the evolution of a single puncture, which can serve as a fundamental building block in simulating more complex spacetime scenarios.

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I. INTRODUCTION

Since the first successful simulations of merging black-hole binaries for computing accurate gravitational waveforms, various formulations and techniques have been employed. These include the BSSN (*Baumgarte-Shapiro-Shibata-Nakamura*) formulation [1–3], conformal decompositions [4, 5], the Bowen-York approach [6], and the moving puncture gauge [7]. These methods have been in use until recently.

Evolving a Schwarzschild black hole serves as a valuable benchmark for assessing the performance of numerical codes, given that we possess its analytical solutions.

In this paper, we present a method for obtaining analytic solutions for a Schwarzschild black hole under time-independent conditions. We provide numerical setups and results for comparison with these analytical solutions.

II. INITIAL DATA

A. Spatial Metric

We need to specify the spatial metric γ_{ij} and the extrinsic curvature K_{ij} which satisfy the Hamiltonian constraint

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{1}$$

and the momentum constraint

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i. \tag{2}$$

To construct a single puncture initial data, we adopt York-Lichnerowicz conformal decompositions [4, 5]. The spatial metric is decomposed into a conformal factor ψ multiplying an conformally related metric:

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij}. \tag{3}$$

With this, (1) turns into

$$\bar{D}^2\psi - \frac{\psi}{8}\bar{R} + \frac{\psi^5}{8}\left(K_{ij}K^{ij} - K^2\right) = -2\pi\psi^5\rho, \tag{4}$$

where we define the conformal Laplace operator

$$\bar{D}^2\psi \equiv \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j\psi. \tag{5}$$

We assume

1) *vacuum spacetime* ($\rho = 0$, $j^i = 0$),

2) *time symmetry* ($\partial_t\gamma_{ij} = 0$, $\beta^i = 0$),

so the extrinsic curvature

$$K_{ij} = \frac{1}{2\alpha}\left(\mathcal{L}_\beta\gamma_{ij} - \partial_t\gamma_{ij}\right) \tag{6}$$

vanishes and also $K = 0$. Therefore, (2) satisfied identically. If we assume *conformal flatness*, the conformally related metric to be flat,

$$\bar{\gamma}_{ij} = \eta_{ij}, \tag{7}$$

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(4) reduces to the simple Laplace equation

$$\bar{D}^2\psi = 0, \tag{8}$$

which has the solution

$$\psi = A + \frac{B}{r}. \tag{9}$$

To satisfy asymptotically flat as $r \to \infty$, and ADM mass to be M, we should choose $A = 1$ and $B = \frac{M}{2}$:

$$\psi = 1 + \frac{M}{2r}. \tag{10}$$

TwoPunctures [8] uses the Bowen-York approach [6] and the method presented in [7]. Since there is a single puncture with neither spin nor momentum, the result should be the same as in (10). We set `TP_epsilon` to 10^{-6} and `TP_Tiny` to 0, resulting in

$$\psi = 1 + \frac{M}{2\tilde{r}}, \tag{11}$$

where

$$\tilde{r} = \left(r^4 + 10^{-24}\right)^{1/4}. \tag{12}$$

The initial metric set by TwoPunctures exactly takes the form

$$\gamma_{ij} = \left(1 + \frac{M}{2\tilde{r}}\right)^4 \eta_{ij}, \tag{13}$$

for all grid points. Fig. 1 shows γ_{xx} along the x-axis at the initial time. Each marker represents a grid point for a respective refinement level. The non-diagonal components of the spatial metric and all components of the extrinsic curvature have explicit values of 0.

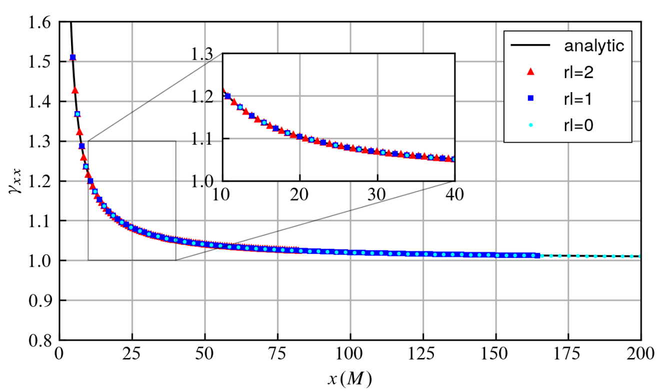

FIG. 1. γ_{xx} along the x-axis at the initial time. Only 3 of the 7 refinement levels are plotted.

B. Constraints

From (1) and (2), we can define

$$\mathcal{H} \equiv R + K^2 - K_{ij}K^{ij} - 16\pi\rho, \tag{14}$$

$$\mathcal{M}^i \equiv D_j(K^{ij} - \gamma^{ij}K) - 8\pi j^i, \tag{15}$$

and it should satisfy that $\mathcal{H} = 0$ and $\mathcal{M}^i = 0$. Fig. 2 shows $\log_{10}|\mathcal{H}|$ on the xy plane at the initial time. Every value at each grid point is less than 10^{-7}. In the case of $\mathcal{M}^i$, all of the values are explicitly 0.

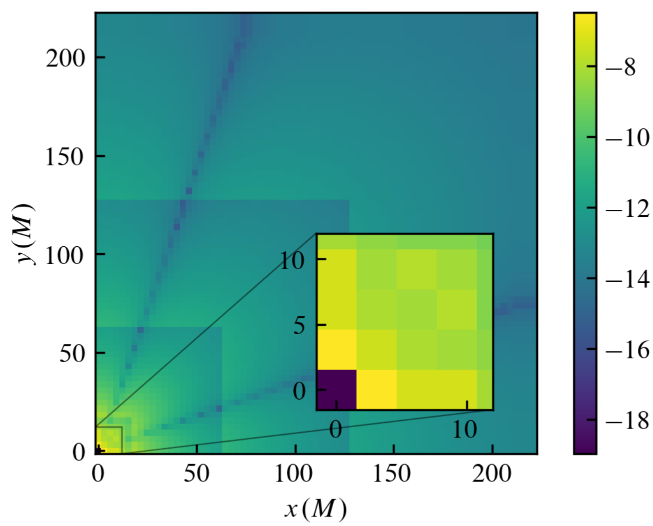

FIG. 2. $\log_{10}|\mathcal{H}|$ on the xy plane at the initial time.

Fig. 3 shows the Hamiltonian constraint violation $\mathcal{H}$ along the x axis. The data at $x = 0$ has been cut since its value is on the order of 1 at the late time. $\mathcal{H}$ increased as time passed, and most peak point's x coordinate also increased. But it still remains on the order of 10^{-6} at $t = 115.2M$.

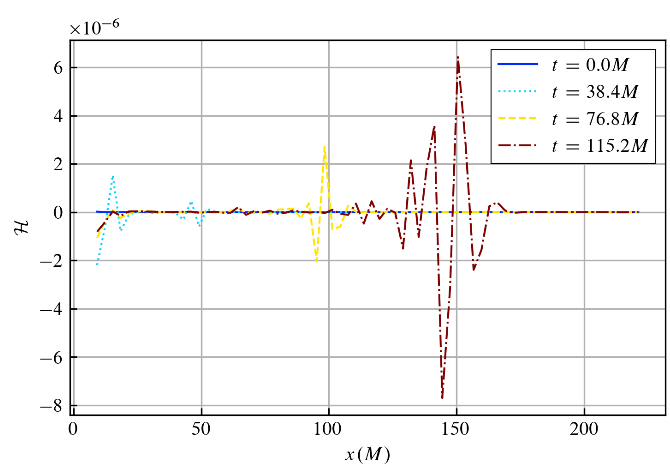

FIG. 3. The Hamiltonian constraint violation along the x axis.

III. EVOLUTION

A. Gauge Condition

The 1+log slicing with advection term is given by

$$\partial_t \alpha = -2\alpha K + \beta^i \partial_i \alpha, \tag{16}$$

and the hyperbolic gamma driver condition for the shift with advection term is defined as

$$\partial_t \beta^i = \frac{3}{4} B^i + \beta^j \partial_j \beta^i, \tag{17}$$

$$\partial_t B^i = \partial_t \bar{\Gamma}^i - B^i + \beta^j \partial_j B^i, \tag{18}$$

which is typically employed in the moving puncture method [9, 10]. Sometimes we can drop the advection term, simplifying (16), (17), and (18) to

$$\partial_t \alpha = -2\alpha K, \tag{19}$$

$$\partial_t \beta^i = \frac{3}{4} B^i, \tag{20}$$

$$\partial_t B^i = \partial_t \bar{\Gamma}^i - B^i, \tag{21}$$

which we adopt in this paper.

We used the `twopunctures-averaged` initial lapse, which is given by

$$\alpha = \frac{1}{2}(1 + \alpha'), \tag{22}$$

where

$$\alpha' = \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}, \tag{23}$$

ensuring that the lapse satisfies $0 \le \alpha \le 1$.

When the simulation progresses sufficiently, and the solution becomes time-independent, we have $\partial_t \alpha = 0$, implying $K = 0$. Since we choose the hyperbolic gamma driver condition, it has a special solution (see [11, 12])

$$\alpha = \sqrt{1 - \frac{2M}{r_s} + \frac{27M^4}{16 r_s^4}}, \tag{24}$$

where r_s is an areal radius related to the isotropic radius r as

$$r = \frac{2r_s + M + (4r_s^2 + 4Mr_s + 3M^2)^{1/2}}{4} \times \left(\frac{(4 + 3\sqrt{2})(2r_s - 3M)}{8r_s + 6M + 3(8r_s^2 + 8Mr_s + 6M^2)^{1/2}} \right)^{1/\sqrt{2}}. \tag{25}$$

The conformal factor is given by $\psi = \left(\frac{r_s}{r}\right)^{1/2}$, and when substituted into (25), we obtain

$$\psi = \left(\frac{4r_s}{2r_s + M + (4r_s^2 + 4Mr_s + 3M^2)^{1/2}} \right)^{1/2} \left(\frac{8r_s + 6M + 3(8r_s^2 + 8Mr_s + 6M^2)^{1/2}}{(4 + 3\sqrt{2})(2r_s - 3M)} \right)^{1/2\sqrt{2}}. \tag{26}$$

Fig. 4 shows α along the x axis at several times. After $t = 49.152M$, α remains almost constant. The black solid line represents the analytic solution given in (24).

Fig. 5 shows γ_{xx} along the x axis. The black solid line indicates the analytic solution $\gamma_{ij} = \psi^4 \eta_{ij}$, where ψ is given by (26).

B. BSSN

The 3+1 ADM evolution equations are given as:

$$(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}, \tag{27}$$

$$(\partial_t - \mathcal{L}_\beta)K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) + 4\pi\alpha M_{ij}, \tag{28}$$

However, this set of partial differential equations is only weakly hyperbolic and is therefore not suitable for stable numerical evolution. To address this, we adopt the BSSN (*Baumgarte-Shapiro-Shibata-Nakmura*) formulation [1–3].

In the BSSN formulation, the spatial metric γ_{ij} is decomposed into a conformally related metric as in (3), with $\det(\bar{\gamma}_{ij}) = 1$. The extrinsic curvature is also decomposed into its trace and traceless parts, and we conformally transform the traceless part as follows:

$$K_{ij} = e^{4\phi} \tilde{A}_{ij} + \frac{1}{3}\gamma_{ij} K. \tag{29}$$

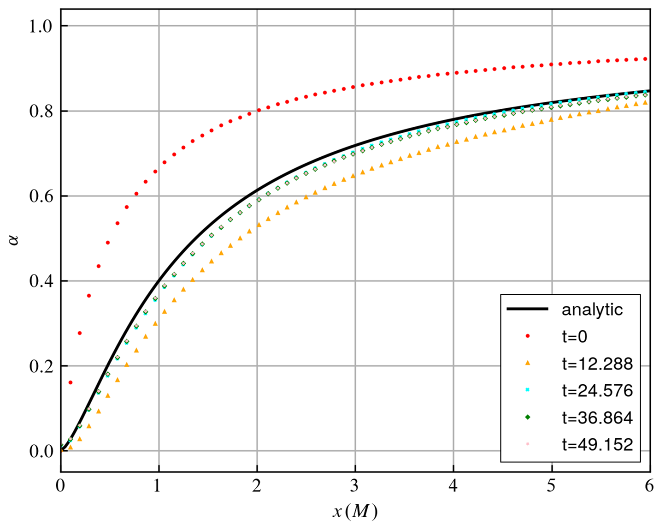

FIG. 4. α along the x axis at several times. Only 5 refinement levels are plotted. The time unit is divided by M.

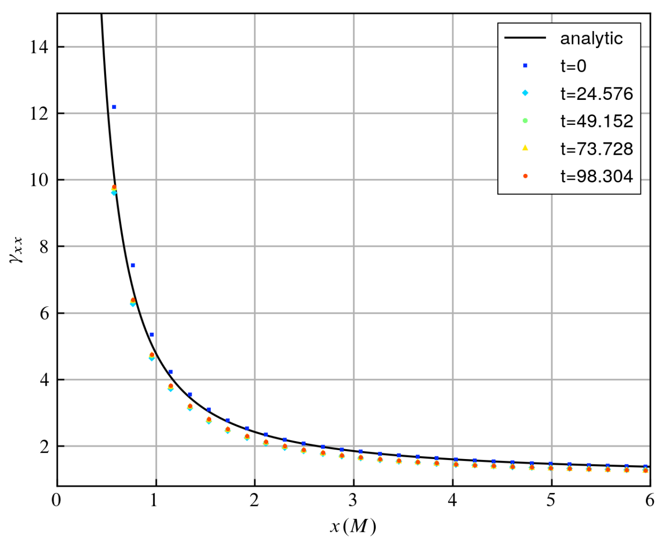

FIG. 5. γ_{xx} along the x axis at several times. Only refinement level 4 is plotted. The time unit is divided by M.

We then promote the following variables to evolution variables:

$$\phi = \ln\psi = \frac{1}{12}\ln\gamma, \tag{30}$$

as well as the conformal connection functions:

$$\bar{\Gamma}^i = \bar{\gamma}^{jk}\bar{\Gamma}^i{}_{jk} = -\partial_j\bar{\gamma}^{ij}. \tag{31}$$

The evolution equation for γ_{ij} splits into two equations:

$$\partial_t\phi = -\frac{1}{6}\alpha K + \beta^i\partial_i\phi + \frac{1}{6}\partial_i\beta^i, \tag{32}$$

$$\begin{aligned}\partial_t\bar{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij}\\ &+\bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k.\end{aligned} \tag{33}$$

The evolution equation for K_{ij} also splits into two equations:

$$\begin{aligned}\partial_t K &= -D^i D_i\alpha + \alpha(\tilde{A}^{ij}_{ij} + \frac{1}{3}K^2)\\ &+\beta^i\partial_i K + 4\pi\alpha(\rho + S),\end{aligned} \tag{34}$$

$$\begin{aligned}\partial_t\tilde{A}_{ij} &= e^{-4\phi}\left[-D_iD_j\alpha + \alpha(R_{ij} - 8\pi S_{ij})\right]^{TF}\\ &+\alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) + \beta^k\partial_k\tilde{A}_{ij}\\ &+\tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k,\end{aligned} \tag{35}$$

where the superscript TF denotes the trace-free part of a tensor. The Ricci tensor is also split into:

$$\begin{aligned}R_{ij} &= \bar{R}_{ij} - 2\left(\bar{D}_i\bar{D}_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\phi\right)\\ &+4\left((\bar{D}_i\phi)(\bar{D}_j\phi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l\phi)(\bar{D}_m\phi)\right)\\ &\equiv \bar{R}_{ij} + R^{\phi}_{ij}.\end{aligned} \tag{36}$$

The $\bar{\Gamma}^i$ are now treated as independent functions that satisfy their own evolution equations:

$$\begin{aligned}\partial_t\bar{\Gamma}^i &= 2\alpha\left(\bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right)\\ &-2\tilde{A}^{ij}\partial_j\alpha + \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i\\ &+\frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{il}\partial_l\partial_j\beta^j + \bar{\gamma}^{jl}\partial_j\partial_l\beta^i.\end{aligned} \tag{37}$$

We use the variable:

$$W = \gamma^{-1/6} = e^{-2\phi} \tag{38}$$

instead of ϕ. The evolution equation for W is:

$$\partial_t W = \frac{1}{3}W(\alpha K - \partial_i\beta^i) + \beta^i\partial_i W. \tag{39}$$

The other choice, which evolves ϕ, can lead to crashes due to numerical singularities.

We use `McLachlan` [13–15] and adopt the set of equations (33), (34), (35), (37), (39) for evolution and the gauge conditions (19), (20).

IV. NUMERICAL SETUP

A. Mesh Refinement

We employ mesh refinement [16] for the puncture center with 7 levels. The basic grid resolution is $3.072M$, and the refinement factor is set to 2. Therefore, the minimum resolution is $3.072M \times 2^{-6} = 0.048M$.

We utilize the Einstein Toolkit [17] to simulate the evolution of a single puncture.

V. RESULT

A. Gravitational Waves

For wave extraction, we utilized the WeylScal4 and Multipole thorns [18]. These thorns extract the Weyl scalar and decompose it into the spin-weighted spherical harmonics. To calculate the radiated energy, we considered modes up to $l = 8$, which are considered credible because higher modes are dominated by numerical noise [19].

Since the Schwarzschild metric has spherical symmetry, there should be no gravitational wave radiation. Figure 6 displays the real and imaginary parts of $r_{\rm ex}\psi_4$. There is no notable difference between $r_{\rm ex} = 50M$. Furthermore, all values of $r_{\rm ex}\psi_4$ are on the order of 10^{-14}, which can be regarded as numerical noise.

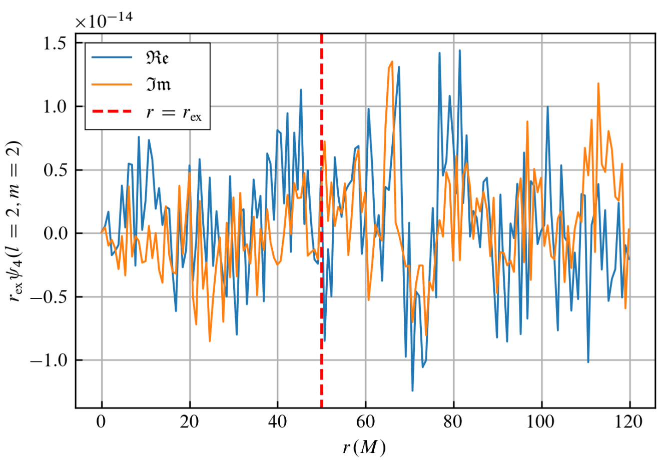

FIG. 6. Real and imaginary parts of $r_{\rm ex}\psi_4$. The red dashed line indicates $r = r_{\rm ex} = 50M$.

B. Apparent Horizon

We measured apparent horizon at each time using AHFinderDirect [20, 21]. It finds an apparent horizon by numerically solving equation

$$\Theta \equiv D_i n^i + K_{ij} n^i n^j - K = 0, \tag{40}$$

where n^i is the outward-pointing unit normal to the apparent horizon, and D_i is the covariant derivative operator associated with the 3-metric in the slice.

We don't fixed puncture location at the origin, but there was no puncture moving and the puncture still at the origin. Fig. 7 shows expansion of apparent horizon as function of time. Fig. 8 shows irreducible mass as function of time.

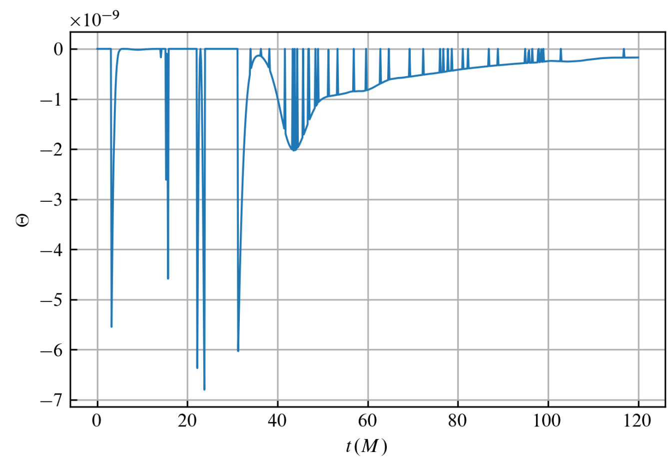

FIG. 7. Expansion Θ of apparent horizon as function of time.

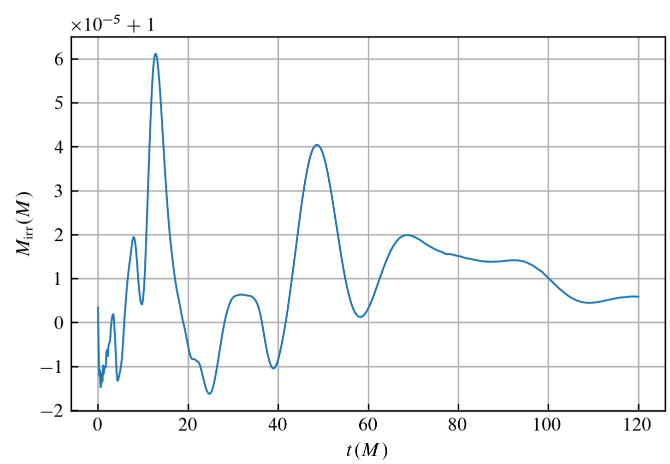

FIG. 8. Irreducible mass $M_{\rm irr}$ as function of time.

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