

Solution to Numerical Relativity

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2 The 3+1 decomposition of Einstein's equations

Exercise 2.1

Exercise 2.2

From

$$\mathcal{C}_E = D_i E^i - 4\pi\rho, \quad (1)$$

$$\partial_t E_i = D_i D^j A_j - D^j D_j A_i - 4\pi j_i, \quad (2)$$

$$\frac{\partial\rho}{\partial t} + D_i j^i = 0, \quad (3)$$

we can get

$$\partial_t \mathcal{C}_E = D_i \partial_t E^i - 4\pi \frac{\partial\rho}{\partial t} \quad (4)$$

$$= D_i (D^i D^j A_j - D^j D_j A^i - 4\pi j^i) + 4\pi D_i j^i \quad (5)$$

$$= D_i D^i D^j A_j - D_i D^j D_j A^i \quad (6)$$

$$= 0. \quad (7)$$

Exercise 2.5

$$\omega_{[a} \nabla_b w_{c]} = \frac{1}{6} (\omega_a \nabla_b w_c - \omega_a \nabla_c w_b + \omega_b \nabla_c w_a - \omega_b \nabla_a w_c + \omega_c \nabla_a w_b - \omega_c \nabla_b w_a). \quad (8)$$

Using $\omega_a = \alpha \nabla_a t$, the first two term in parenthesis will be

$$\omega_a \nabla_b w_c - \omega_a \nabla_c w_b = \alpha (\nabla_a t) (\nabla_b \alpha \nabla_c t + \alpha \nabla_b \nabla_c t) - \alpha (\nabla_a t) (\nabla_c \alpha \nabla_b t + \alpha \nabla_c \nabla_b t) \quad (9)$$

$$= \alpha (\nabla_a t) (\nabla_b \alpha) (\nabla_c t) - \alpha (\nabla_a t) (\nabla_c \alpha) (\nabla_b t). \quad (10)$$

Therefore,

$$\omega_{[a} \nabla_b w_{c]} = \alpha (\nabla_a t) (\nabla_b \alpha) (\nabla_c t) - \alpha (\nabla_a t) (\nabla_c \alpha) (\nabla_b t) \quad (11)$$

$$+ \alpha (\nabla_b t) (\nabla_c \alpha) (\nabla_a t) - \alpha (\nabla_b t) (\nabla_a \alpha) (\nabla_c t) \quad (12)$$

$$+ \alpha (\nabla_c t) (\nabla_a \alpha) (\nabla_b t) - \alpha (\nabla_c t) (\nabla_b \alpha) (\nabla_a t) \quad (13)$$

$$= 0. \quad (14)$$

Exercise 2.6

To show that $\gamma_b^a v^b$ is purely spatial, we should show that $n_a \gamma_b^a v^b = 0$.

$$n_a \gamma_b^a v^b = n_a (\delta_b^a + n^a n_b) v^b \quad (15)$$

$$= (n_b + n_a n^a n_b) v^b \quad (16)$$

$$= (n_b - n_b) v^b \quad (17)$$

$$= 0. \quad (18)$$

Exercise 2.7

$$T_{ab} = \delta_a^c \delta_b^d T_{cd} \quad (19)$$

$$= (N_a^c + \gamma_a^c)(N_b^d + \gamma_b^d) T_{cd} \quad (20)$$

$$= (N_a^c N_b^d + N_a^c \gamma_b^d + \gamma_a^c N_b^d + \gamma_a^c \gamma_b^d) T_{cd} \quad (21)$$

$$= n_a n_b n^c n^d T_{cd} - n_a n^c \perp T_{cb} - n_b n^d \perp T_{ad} + \perp T_{ab}. \quad (22)$$

Exercise 2.8

It is trivial that $\nabla_a g_{bc} = 0$. Since $\gamma_{bc} = g_{bc} + n_b n_c$, we only need to show that $D_a(n_b n_c) = 0$.

$$D_a(n_b n_c) = \gamma_a^d \gamma_b^e \gamma_c^f \nabla_d (n_e n_f) \quad (23)$$

$$= \gamma_a^d (\delta_b^e + n^e n_b) (\delta_c^f + n^f n_c) (n_e \nabla_d n_f + n_f \nabla_d n_e) \quad (24)$$

$$= \gamma_a^d (\delta_b^e \delta_c^f + \delta_b^e n^f n_c + \delta_b^e n^f n_c + n^e n_b n^f n_c) (n_e \nabla_d n_f + n_f \nabla_d n_e) \quad (25)$$

$$= \gamma_a^d (n_b \nabla_d n_c + n_c \nabla_d n_b + n_b n_c n^f \nabla_d n_f + n_c n_f n^f \nabla_d n_b \quad (26)$$

$$+ n_b n_e n^e \nabla_d n_c + n_b n_c n^e \nabla_d n_e \quad (27)$$

$$+ n_b n_c n^e n_e n^f \nabla_d n_f + n_b n_c n^e n_f n^f \nabla_d n_e \quad (28)$$

$$= n_b \nabla_d n_c + n_c \nabla_d n_b + n_b n_c n^f \nabla_d n_f - n_c \nabla_d n_b \quad (29)$$

$$- n_b \nabla_d n_c + n_b n_c n^e \nabla_d n_e - n_b n_c n^f \nabla_d n_f - n_b n_c n^e \nabla_d n_f \quad (30)$$

$$= 0. \quad (31)$$

Exercise 2.9

If v^a is purely spatial, $n_a v^a = 0$ and $\gamma_a^b v^a = (\delta_a^b + n^b n_a) v^a = v^b + 0$. Similarly $\gamma_b^a w_a = w_b$. So

$$D_a(v^b w_b) = \gamma_a^c \nabla_a (v^b w_b) \quad (32)$$

$$= \gamma_a^c (v^b \nabla_c w_b + w_b \nabla_c v^b) \quad (33)$$

$$= \gamma_a^c (\gamma_d^b v^d \nabla_c w_b + \gamma_b^d w_d \nabla_c v^b) \quad (34)$$

$$= v^d D_a w_d + w_d D_a v^d \quad (35)$$

$$= v^b D_a w_b + w_b D_a v^b. \quad (36)$$

Exercise 2.12

We need to show that $n^a \nabla_b n_a = 0$.

Bonus

Start from exercise 2.5 and $\omega_a = -n_a$. Contract with n^a ,

$$n^a n_{[a} \nabla_b n_{c]} = n^a n_a \nabla_b n_c - n^a n_a \nabla_c n_b \quad (37)$$

$$+ \cancel{n^a n_b \nabla_c n_a} \xrightarrow{0} - \cancel{n^a n_b \nabla_a n_c} \quad (38)$$

$$+ \cancel{n^a n_c \nabla_a n_b} - \cancel{n^a n_c \nabla_b n_a} \xrightarrow{0} \quad (39)$$

$$= \nabla_c n_b - \nabla_b n_c \quad (40)$$

$$= 0. \quad (41)$$

Therefore $\nabla_b n_c = \nabla_c n_b$.

3 Constructing initial data

Exercise 3.1

Starting from connection coefficients for spatial components

$$\Gamma^i_{jk} = \frac{1}{2} \gamma^{il} (\partial_k \gamma_{lj} + \partial_j \gamma_{lk} - \partial_l \gamma_{jk}) \quad (42)$$

$$= \frac{1}{2} \psi^{-4} \bar{\gamma}^{il} (\partial_k (\psi^4 \bar{\gamma}_{lj}) + \partial_j (\psi^4 \bar{\gamma}_{lk}) - \partial_l (\psi^4 \bar{\gamma}_{jk})) \quad (43)$$

$$= \frac{1}{2} \psi^{-4} \bar{\gamma}^{il} \{ 4\psi^3 (\bar{\gamma}_{lj} \partial_k \psi + \bar{\gamma}_{lk} \partial_j \psi + \bar{\gamma}_{jk} \partial_l \psi) + \psi^4 (\bar{\gamma}_{lj} + \bar{\gamma}_{lk} + \bar{\gamma}_{jk}) \} \quad (44)$$

$$= \frac{1}{2} \bar{\gamma}^{il} (\bar{\gamma}_{lj} + \bar{\gamma}_{lk} + \bar{\gamma}_{jk}) + 2\psi^{-1} \bar{\gamma}^{il} (\bar{\gamma}_{lj} \partial_k \psi + \bar{\gamma}_{lk} \partial_j \psi + \bar{\gamma}_{jk} \partial_l \psi) \quad (45)$$

$$= \bar{\Gamma}^i_{jk} + 2\bar{\gamma}^{il} (\bar{\gamma}_{lj} \partial_k \ln \psi + \bar{\gamma}_{lk} \partial_j \ln \psi + \bar{\gamma}_{jk} \partial_l \ln \psi) \quad (46)$$

$$= \bar{\Gamma}^i_{jk} + 2 \left(\delta^i_j \partial_k \ln \psi + \delta^i_k \partial_j \ln \psi - \bar{\gamma}^{il} \bar{\gamma}_{jk} \partial_l \ln \psi \right). \quad (47)$$