Solid State Basics

2. Specific Heat of Solids

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Why numerical relativity?

- How to determine the dynamical evolution of a physical system governed by Einstein's equations of general relativity?
- Analytic solutions for the evolution of such systems do not exist.
- We have to recast Einstein's 4-dimensional field equations into a form that is suitable for numerical integration.

3+1 decompostion

- The problem of evolving the gravitational field in GR can be posed in terms of a traditional initial value problem or "Cauchy" problem.
- The evolution of a general relativistic gravitational field is determined by specifying the metric quantities g_{ab} and $\partial_t g_{ab}$ at a given (initial) instant of time t.
- In particular, we need to specify the metric field components and their first time derivatives everywhere on some 3-dimensional spacelike hypersurface labeled by coordinate $x^0 = t =$ constant.

3+1 decompostion

1 Introduction

- The different points on this surface are distinguished by their spatial coordinates x^i.
- Now these metric quantities can be integrated forward in time provided we can obtain from the Einstein field equations expressions for $\partial_t^2 g_{ab}$ at all points on the hypersurface.
- That way we can integrate these expressions to compute g_{ab} and $\partial_t g_{ab}$ on a new spacelike hypersurface at some new time $t + \delta t$, and then, by repeating the process, obtain g_{ab} for all other points x^0 and x^i in the (future) spacetime.

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Introduction

- Lagrangian formulation of a field theory is "spacetime covariant."
- Hamiltonian formulation of a field theory requires a breakup of spacetime into space and time.
- Indeed, the first step in producing a Hamiltonian formulation of a field theory consists of choosing a time function t and a vector field t^a on a spacetime such that the surfaces, Σ_t, of constant t are spacelike Cauchy surfaces and such that $t^a \nabla_a t = 1$.
- The vector field t^a may be interpreted as describing the "flow of time" in the spacetime and can be used to identify each Σ_t with the initial surface Σ_0.

Introduction

- In performing integrals over Σ_t, it would be natural in most cases to use the volume element

$$^{(3)}\epsilon_{abc} = \epsilon_{dabc} n^d \tag{1}$$

, where n^d is the unit normal to Σ_t.

- However, volume elements will be "time dependent" in the sense that

$$\mathcal{L}_t \epsilon_{abcd} \neq 0 \quad \text{and} \quad \mathcal{L}_t^{(3)} \epsilon_{abc} \neq 0 \tag{2}$$

, which inconvenient.

- Therefore, we shall introduce a fixed volume element e_{abcd} on M satisfying $\mathcal{L}_t e_{abcd} = 0$.

Introduction

- The next step in giving a Hamiltonian formulation is to define a configuration space for the field by specifying what tensor field (or fields) q on Σ_t physically describes the instantaneous configuration of the field ψ.
- The space of possible momenta of the field at a given configuration q then is taken to be the "cotangent space," V_q^*, of the configuration space at q.
- Since the set of possible configurations of the field is infinite-dimensional, we shall not attempt here to give a precise definition of V_q^*.

- The final and most nontrivial step required for a Hamiltonian formulation of a field theory is the specification of a functional $H[q,\pi]$ on Σ_t, called the *Hamiltonian*, which is of the form

$$H = \int_{\Sigma_t} \mathscr{H}, \tag{3}$$

where the *Hamiltonian density* $\mathscr{H}$ is the local function of q, π and of their spatial derivatives up to a finite order, such that the pair of equations,

$$\dot{q} \equiv \pounds_t q = \frac{\delta H}{\delta \pi}, \tag{4}$$

$$\dot{\pi} \equiv \pounds_t \pi = -\frac{\delta H}{\delta q}, \tag{5}$$

is equivalent to the field equation satisfied by ψ.

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- Consider a spacetime $(\mathcal{M}, \boldsymbol{g})$ such that $\boldsymbol{g}$ obeys the Einstein equation:

$$ {}^4\boldsymbol{R} - \frac{1}{2}\,{}^4R\,\boldsymbol{g} = 8\pi \boldsymbol{T}, \tag{6} $$

  where ${}^4\boldsymbol{R}$ is the Ricci tensor associated with $\boldsymbol{g}$, ${}^4R$ the corresponding Ricci scalar, and $\boldsymbol{T}$ the matter stress-energy tensor.

- We are using units in which Newton's gravitational constant G is set to unity, otherwise, the coefficient in front of $\boldsymbol{T}$ in Eq.(6) should read $8\pi G$, and even $8\pi G/c^4$ if we are relaxing $c = 1$.

The Einstein Equation

- We shall also use the equivalent form

$$ {}^4\boldsymbol{R} = 8\pi\left(\boldsymbol{T} - \frac{1}{2}T\,\boldsymbol{g}\right), \tag{7} $$

  where $T := g^{\mu\nu}T_{\mu\nu}$ stands for the trace (with respect to $\boldsymbol{g}$) of $\boldsymbol{T}$.
- Let us assume that the spacetime $(\mathcal{M}, \boldsymbol{g})$ is globally hyperbolic and let $(\Sigma_t)_{t\in\mathbb{R}}$ be a foliation of $\mathcal{M}$ by a family of spacelike hypersurfaces.
- The 3+1 formalism for GR amounts to projecting the Einstein equation onto Σ_t and perpendicularly to Σ_t.
- To this aim let us first consider the 3+1 decomposition of the stress-energy tensor.

3+1 Decomposition of the Stress-Energy Tensor

- From the very definition of a stress-energy tensor, the *matter energy density* as measured by the Eulerian observer introduced is

$$E := \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n}). \tag{8}$$

- This follows from the fact that the 4-velocity of the Eulerian observer is the unit normal vector $\boldsymbol{n}$ of the hypersurfaces Σ_t.