Given a permutation of 1-n, and let random variable X_n be be the number of integers that have the same position as their original position, e.g. if $n = 3$ then 321 has only 1 digit (which is 2) that has the same position as its original position. What's the expectation of X_n?

As what we've done before, we might try to use method of conditioning to solve this problem. However, this is not a good direction because there's no obvious random variable to condition on to so that the conditional distribution is simple. There's a classic way to solve this problem, which is to decompose random variable X_n into a sum of n random variables.

Let $Y_i = I(\text{integer } i \text{ has the same position as its original position}), i = 1, \ldots, n$, where I is a indicator function, then $X_n = \sum_{i=1}^n Y_i$. Since all the five integers $1, \ldots, n$ plays a equal role in the problem (the only difference is the difference symbol we use), random variables $Y_1, \ldots, Y_n$ are exchangeable, i.e. they have a same distribution, and any two of them has a same joint distribution, and so on. It's obvious that $Y_1 \sim Bernoulli(\frac{1}{n})$ and $Y_1Y_2 \sim Bernoulli(\frac{1}{n(n-1)})$. So $EY_1 = \frac{1}{n}$ and $EY_1Y_2 = \frac{1}{n(n-1)}$ which yields that $Var(Y_1) = \frac{n-1}{n^2}$ and $Cov(Y_1, Y_2) = EY_1Y_2 - EY_1EY_2 = \frac{1}{n^2(n-1)}$. Thus we know that

$$E(X_n) = E\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n EY_i = nEY_1 = n\frac{1}{n} = 1$$

$$\begin{aligned} Var\left(X_n\right) &= Var\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n Var(Y_i) + \sum_{i \neq j} Cov(Y_i, Y_j) \\ &= nVar(Y_1) + n(n-1)Cov(Y_1, Y_2) \\ &= n\frac{n-1}{n^2} + n(n-1)\frac{1}{n^2(n-1)} = 1 \end{aligned} \tag{1}$$

Surprisingly, both the mean and the variance of X_n is 1, which suggests that we can predict X_n, i.e. the number of integers that have the same position as their original position very well.