MAXIMAL INDEPENDENCE AND SINGULAR CARDINALS

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Abstract. In this paper, we study the concept of a *maximal δ-independent family* of subsets of λ, when λ is a singular cardinal of cofinality κ and δ is a regular cardinal $\leq \kappa$. We first show that if λ is a singular cardinal which is a limit of a sequence of regular cardinals $(\lambda_\alpha : \alpha < \kappa)$ and there are maximal δ-independent families at each cardinal λ_α; then it is possible to build a maximal δ-independent family at the singular λ. Afterward, we use this fact together with the results of Kunen in [Kun83] regarding the existence of maximal independent families at regular cardinals to prove our main result: If λ is a singular cardinal which is a limit of supercompact cardinals $(\lambda_\alpha : \alpha < \kappa)$ and $\mathrm{cf}(\lambda) = \kappa$, then consistently there exists a maximal κ-independent family of subsets of λ. Finally, we add a discussion on the possible sizes of these families.

1. Introduction

In the last years, particular interest has been given to the study of higher analogs of classical cardinal characteristics of the continuum. By this time, there are several results involving higher analogs of characteristics on the generalized Baire spaces κ^κ, when κ is a ***regular*** uncountable cardinal. Some references which are relevant for this paper are [EF21; Bro+17; BG20; BG15]. Additionally, several results for the specific case when κ is a *singular* cardinal have also appeared. It is remarkable that in many of these results the classical theory of *possible cofinalities (pcf)* plays an important role. Examples of this behavior can be found in [GGS20; Hec72]. This particular paper deals with the latter case for the specific case of independent and maximal independent families.

A family of subsets of ω is said to be *independent* if for every two finite disjoint subfamilies $\mathcal{B}$ and $\mathcal{C}$ the set $\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}$ is infinite. It is additionally called *maximal independent* if it is maximal with respect to the inclusion order on $\mathcal{P}(\omega)$. The study of the concept of maximal independence at subsets of ω, as well as the possible sizes of these families has contributed to the development of many forcing techniques and to the understanding of the classical Baire space.

Kunen was the first person interested in generalizing the concept of maximality and introduced the following definition of independence within the framework of uncountable cardinals:

Notation 1. Assume that κ is a regular cardinal and χ is an infinite cardinal. Let $\mathcal{A}$ be a family of subsets of χ such that $|\mathcal{A}| \geq \kappa$:

- We denote by $\mathrm{BF}_\kappa(\mathcal{A})$ the family of partial functions $\{h : \mathcal{A} \to 2 : |\mathrm{dom}(h)| < \kappa\}$ and call it *the family of bounded functions on $\mathcal{A}$*.

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- Given $h \in \mathrm{BF}_\kappa(\mathcal{A})$, we define $\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathcal{A} \cap \mathrm{dom}(h)\}$, where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \chi \setminus A$ otherwise. We call $\mathcal{A}^h$ the Boolean combination of $\mathcal{A}$ associated to h and we refer to $\{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\}$ as the family of *generalized boolean combinations* of the family $\mathcal{A}$.

Definition 1 (Independence and maximal independence)**.** [Kun83] Let κ be a regular cardinal. A family $\mathcal{A} \subseteq \mathcal{P}(\chi)$ such that $|\mathcal{A}| \geq \kappa$ is called *κ-independent* if for for every $h \in \mathrm{BF}_\kappa(\mathcal{A})$, the set $\mathcal{A}^h$ has size χ. A κ-independent family $\mathcal{A}$ is said to be *maximal κ-independent* if it is *not* properly contained in another κ-independent family. We call the cardinal κ the *degree of independence* of the family $\mathcal{A}$.

Kunen noticed that, unlike in the classical case, the existence of a maximal κ-independent family of subsets of some χ does not follow from the axiom of choice. Moreover, he provided necessary and sufficient conditions to guarantee the existence of such families and pointed out that large cardinals are required (See Theorem 3).

This paper deals with the study of the existence of maximal δ-independent families of subsets of λ, when λ is a singular cardinal of cofinality $\kappa \geq \delta$ and it is organized as follows: Section 3 presents the preliminary results of Kunen which will be necessary for the main results of this paper. They deal with the existence of maximal κ-independent families at some regular cardinal $\chi \geq \kappa$.

Section 4 presents the main results of this paper. First, we prove that given a singular cardinal λ of $\mathrm{cf}(\lambda) = \kappa$ which is a limit of a sequence $(\lambda_\alpha : \alpha < \kappa)$ of regular cardinals, such that for each $\alpha < \kappa$ there exists a maximal δ-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ for δ a successor cardinal $\delta \leq \kappa$, then there is a maximal δ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$ (Theorem 15). The rest of the section is devoted to a consistency result realizing the conditions of the Theorem above. The main result of this paper states the following:

Theorem 2. *Start with a ground model V in which* GCH *holds. Suppose that λ is a singular cardinal of cofinality $\kappa > \omega$ which is a limit of supercompact cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \kappa)$ be a sequence of regular successor cardinals converging to κ so that $\delta_\alpha^{<\delta_\alpha} = \delta_\alpha$. Then there is a generic extension of V in which:*

$$V^{\mathbb{P}} \models \textit{ There is a maximal } \kappa\textit{-independent family of subsets of } \lambda.$$

Section 6 deals with the sizes of maximal independent families at singular cardinals. We define the *independence number* for a singular cardinal λ of cofinality κ as follows:

$$\mathfrak{i}_\kappa(\lambda) = \{|\mathcal{A}| : \mathcal{A} \text{ is a maximal } \kappa - \text{independent family of subsets of } \lambda\}.$$

First we prove that for λ a singular cardinal of uncountable cofinality, if for all $\gamma < \mathfrak{d}(\lambda)$ we get $\gamma^{<\kappa} < \mathfrak{d}(\lambda)$ then $\mathfrak{i}_\kappa(\lambda) \geq \mathfrak{d}(\lambda)$. Here, $\mathfrak{d}(\lambda)$ denotes the dominating number for singulars (see Definition 26). Also, by using Shelah's results in [She94] on $\mathfrak{d}(\lambda)$ for λ a strong limit singular cardinal we get that in some instances $\mathfrak{i}_\kappa(\lambda) = 2^\lambda$. Finally, Section 7 presents some open questions and possible lines of further research.

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3. Independent families for uncountable cardinals.

In this section, we introduce some preliminaries in the study of the existence of generalized maximal independent families at uncountable cardinals. We focus on the results of Kunen on this matter and we outline some of their proofs because they are crucial to the main results of this paper. Unlike other generalizations of combinatorial notions to the context of uncountable cardinals (splitting families, maximal almost disjoint families), the concept of maximal independence seems to be much harder to study. Specifically, it is not clear beforehand whether maximal κ-independent families do exist.

In the classical case ($\chi = \kappa = \omega$) the existence of maximal objects can be assured by using Zorn's lemma. On the other hand, if both χ and κ are uncountable it is possible to build κ-independent families of size 2^χ but one cannot use Zorn's lemma to prove the existence of maximal κ-independent families (just notice that the union of a countable chain of κ-independent families is not necessarily κ-independent). The following result of Kunen provides necessary conditions for the existence of maximal κ-independent families in the general context when κ is a regular uncountable cardinal.

Theorem 3 (See Theorem 1 in [Kun83])**.** *Suppose that κ is regular and uncountable and χ is any infinite cardinal. Also assume that there is a maximal κ-independent family $\mathcal{A} \subseteq \mathcal{P}(\chi)$, with $|\mathcal{A}| \geq \kappa$. Then:*

(1) $2^{<\kappa} = \kappa$ and,
(2) there is a Γ with $\sup\{(2^\alpha)^+ : \alpha < \kappa\} \leq \Gamma \leq \min\{\chi, 2^\kappa\}$ such that there is a non-trivial κ^+-saturated Γ-complete ideal over Γ.

In particular, if $\chi = \kappa$ and κ is strongly inaccessible the existence of a κ-maximal independent family of subsets of κ implies that $2^{<\kappa} = \kappa$ and there is a κ-complete κ^+-saturated ideal on κ.

Notice that the former theorem implies that proving the existence of maximal κ-independent families requires the existence of large cardinals. In particular, since $\Gamma \geq \kappa$, then the ideal given by the theorem must be Γ^+-saturated, which yields an inner model with a measurable cardinal.

An outline of Kunen's proof. We review a few details of the proof of Theorem 3 which will be relevant for the results to come: Suppose that κ is a regular cardinal and let $\mathcal{A}$ be a *κ-maximal* independent family of subsets of χ, where $\chi \geq \kappa$. Define the map $\varphi : \mathrm{Fn}_\kappa(\mathcal{A}, 2) \to \mathcal{P}(\chi)$ by $\varphi(p) = \mathcal{A}^p$ where $\mathrm{Fn}_\kappa(\mathcal{A}, 2)$ is the classical poset of partial functions $p : \mathcal{A} \to 2$ with $|\mathrm{dom}(p)| < \kappa$ ordered by reverse inclusion. [1]

[1] As sets $\mathrm{Fn}_\kappa(\mathcal{A}, 2) = \mathrm{BF}_{<\kappa}(\mathcal{A})$.

Note that φ is an isomorphism from $\mathrm{Fn}_\kappa(\mathcal{A},2)$ into $[\chi]^\chi$: Clearly $p \leq q$ implies $\varphi(p) \subseteq \varphi(q)$ and two conditions p,q are compatible in $\mathbb{P} = \mathrm{Fn}_\kappa(\mathcal{A},2)$ if and only if $\varphi(p) \cap \varphi(q) \neq \emptyset$.

The family $\mathcal{A}$ is maximal if and only if for all $X \subseteq \chi$ there is a $p \in \mathbb{P}$ such that $\varphi(p) \subseteq^* X$ or $\varphi(p) \subseteq^* \chi \backslash X$ [2]. Moreover, we can assume that $\mathcal{A}$ has a stronger property, i.e. that it is *densely maximal*.

Definition 4. A family $\mathcal{A}$ is said to be *densely maximal* if for all $X \subseteq \chi$ and all $h \in \mathrm{BF}_\kappa(\mathcal{A})$, there is a $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq^* X$ or $\mathcal{A}^{h'} \subseteq^* \chi \backslash X$.

Proposition 5. It is possible to assume, without loss of generality that $\mathcal{A}$ is densely maximal.

Proof. This is due to Glazer (see [Kun83]): Given $p \in \mathbb{P}$, define $\mathcal{A}_p$ to be the collection of sets $\mathcal{A}_p := \{A \cap \varphi(p) : A \in \mathcal{A}\}$. Note that if for no $p \in \mathbb{P}$ the set $\mathcal{A}_p$ is densely maximal, then the set

$$\mathcal{D} = \{q \in \mathbb{P} : \text{ There exists } A_q \subseteq \varphi(q) \text{ such that } \forall r \leq q \ (\varphi(r) \nsubseteq^* A_q \text{ or } \varphi(r) \nsubseteq^* \varphi(r) \backslash A_q)\}$$

is dense in $\mathbb{P}$ and we can take $\mathcal{B}$ to be a maximal antichain contained in $\mathcal{D}$. Put $A = \bigcup_{q \in \mathcal{B}} A_q$ and use that $\mathcal{A}$ is maximal κ-independent to find $p^* \in \mathbb{P}$ such that either $\varphi(p^*) \subseteq^* A$ or $\varphi(p^*) \subseteq^* \chi \backslash A$, but then p^* has to be incompatible with all conditions $q \in \mathcal{B}$, a contradiction. Thus, there is a $p \in \mathbb{P}$ such that $\mathcal{A}_p \subseteq \mathcal{P}(\varphi(P))$ is densely maximal κ-independent. $\square$

Define the ideal associated to the dense maximal κ-independent family $\mathcal{A}$, $\mathcal{I}_\mathcal{A} := \{X \subseteq \chi : \forall p \in \mathbb{P} \, (\varphi(p) \nsubseteq^* X)\}$. To finish the proof of Theorem 3, Kunen showed that the ideal $\mathcal{I}_\mathcal{A}$ is $(2^\alpha)^+$-complete for all $\alpha < \kappa$, that it is $(2^{<\kappa})^+$-saturated and also that $2^{<\kappa} = \kappa$. This implies that $\mathcal{I}_\mathcal{A}$ is in fact, κ^+-saturated. Hence if Γ is the minimum cardinal such that $\mathcal{I}_\mathcal{A}$ is not Γ-complete, one gets the desired result. $\square$

Moreover, Kunen also gave sufficient conditions for the existence of maximal λ-independent families at κ.

Lemma 6. Suppose κ is regular, $2^{<\kappa} = \kappa$, $\kappa \leq \chi$ and $\mathcal{I}$ is a κ^+-saturated χ-complete ideal over χ with $\mathcal{B}(\mathrm{Fn}_\kappa(2^\chi,2))$ isomorphic to $\mathcal{P}(\chi)/\mathcal{I}$. Then, there is a maximal κ-independent family of subsets of χ.

Proof sketch. From the isomorphism $\varphi : \mathcal{B}(\mathrm{Fn}_\kappa(2^\chi,2)) \to \mathcal{P}(\chi)/\mathcal{I}$, one can define for each $\delta < 2^\chi$, the set $A_\delta = \varphi(\{(\delta,1)\})$. Then a small modification of the family $\mathcal{A} = \{A_\delta : \delta < 2^\chi\}$ is maximal κ-independent.

$\square$

3.1. **Consistency results:** Kunen additionally provided a list of consistency results by showing how to build models in which there are maximal independent families from some large cardinal assumptions. We include its results and prove that they can be generalized to larger cardinals. These generalizations will be crucial for the theorems in Section 4.

Theorem 7 (Kunen)**.** *If there is a measurable cardinal, then there is a maximal σ-independent family $\mathcal{A} \subseteq \mathcal{P}(2^{\omega_1})$.*

[2] Here $A \subseteq^* B$ if and only if $|A \backslash B| < \chi$.

Proof of Theorem 7. Start with a measurable cardinal κ in a ground model V where CH holds. Let $\mathcal{U}$ be a normal measure witnessing the measurability of κ. We shall construct a model in which CH still holds and if $\kappa = 2^{\aleph_1}$, there is an ω_2-saturated, κ-complete ideal $\mathcal{I}$ over κ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{I}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(2^\kappa, 2))$ are isomorphic.

Let $\mathbb{P}$ be $\mathrm{Fn}_{\omega_1}(\kappa, 2)$ and let G be a $\mathbb{P}$-generic filter over V. In $V[G]$, $\kappa = 2^{\aleph_1}$ and we can define the following collection of subsets of κ:

$$\mathcal{J} = \{X \subseteq \kappa : \exists Y \in \mathcal{U}(X \cap Y = \emptyset)\}$$

In other words, $\mathcal{J}$ is the ideal generated by the dual ideal of $\mathcal{U}$ in the generic extension $V[G]$. The ideal $\mathcal{J}$ is, in turn, a κ-complete ω_2-saturated ideal, by applying Theorem 17.1 in [Kan03] to the forcing notion $\mathbb{P}$ which has the ω_2-cc. In a few words, this Theorem ensures that the saturation of an ideal can be preserved after forcing with a partial order with a nice chain condition.

The rest of the argument aims to build an isomorphism between the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{I}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(2^\kappa, 2))$ in $V[G]$. First, let $j : V \to M = \mathrm{Ult}(V,\mathcal{U})$ be the ultrapower embedding associated to $\mathcal{U}$, i.e. j is elementary, $\mathrm{crit}(j) = \kappa$. Let $\kappa^* = j(\kappa) > \kappa$, then $2^\kappa < \kappa^* < (2^\kappa)^+$ and the posets $\mathrm{Fn}_{\omega_1}(2^\kappa, 2)$ and $\mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2)$ are isomorphic.

Let's define the isomorphism $\Gamma : \mathcal{P}(\kappa)/\mathcal{I} \to \mathcal{B}(\mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2))$ in $V[G]$ as follows: Given $[X] \in (\mathcal{P}(\kappa)/\mathcal{I})^{V[G]}$, and let $\dot{X}$ be a $\mathbb{P}$-name for the set X. We define the function as follows:

$$\Gamma([X]) := \bigvee\{q \in \mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2) : \exists p \in G(p \cup q \Vdash \check{\kappa} \in j(\dot{X}))\}.$$

Recall that $j(\mathbb{P}) = j(\mathrm{Fn}_{\omega_1}(\kappa, 2)) = \mathrm{Fn}_{\omega_1}(\kappa^*, 2) \simeq \mathbb{P} \times \mathbb{Q}$, where $\mathbb{Q} = \mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2)$. Also, every element of the poset $\mathbb{Q}$ is represented in $\mathrm{Ult}(V,\mathcal{U})$ by a sequence $(q_\alpha : \alpha < \kappa)$ such that $q_\alpha \in \mathbb{Q}$ for all $\alpha < \kappa$.

Let H be $\mathbb{Q}$-generic over $V[G]$, then $G \times H$ is $j(\mathbb{P})$-generic over V and we can define a map j to $j^* : V[G] \to M[G \times H]$ as $j^*(X) = (j(\dot{X}))^{G \times H}$ in $V[G \times H]$. So, we can ask for a given set $Y \in V[G]$ whether or not $\check{\kappa} \in (j(\dot{Y}))^{G \times H}$. Hence, for all $p \in \mathbb{P}$, $q \in \mathbb{Q}$ and a $\dot{Y} \in V[G]$ ($j(\mathbb{P})$-name):

$$p \cup q \Vdash \check{\kappa} \in (j(\dot{Y})) \text{ if and only if } \{\alpha < \kappa : p \cup q_\alpha \Vdash \check{\alpha} \in \dot{Y}\} \in \mathcal{U}.$$

□

As a Corollary, Kunen concluded the following:

Corollary 8 (Kunen)**.** Assume κ is strongly compact in V. Then in $V[G]$, where G is $\mathbb{P}$-generic (for $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\kappa, 2)$ like in the theorem above) for every cardinal $\chi \geq \kappa$ such that $\mathrm{cf}(\chi) \geq \kappa$ there is a maximal σ-independent family of subsets of χ.

Proof. Since κ is strongly compact, for each $\chi \geq \kappa$ there is a fine ultrafilter on $\mathcal{P}_\kappa(\chi)$. Take $\mathcal{U}_\chi$ to be a fine measure witnessing this property and $j_\chi : V \to M_\chi$ be an embedding with $\mathrm{crit}(j_\chi) = \kappa$ such that: for any $X \subseteq M_\chi$ with $|X| \leq \chi$, there is $Y \in M_\chi$ such that $Y \supseteq X$ and $M_\chi \models |Y| < j_\chi(\kappa)$.

Define $\mathcal{J}_\chi$ for all χ likewise and notice that if $\kappa^*_\chi = j_\chi(\kappa)$, $(2^\kappa)^{<\chi} \leq ((2^\kappa)^{<\chi})^M < j_\chi(\kappa) < (2^\kappa)^{<\chi})^+$ and since $\mathrm{cf}(\chi) \geq \kappa$, $(2^\kappa)^{<\chi} = 2^\chi$ and we get $\mathrm{Fn}_{\omega_1}(2^\chi, 2) \simeq \mathrm{Fn}_{\omega_1}(\kappa^*_\chi \setminus \kappa, 2)$.

Hence, we can define isomorphisms Γ_χ between the Boolean algebras $\mathcal{P}(\chi)/\mathcal{J}_\chi$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(\kappa^*_\chi \setminus \kappa, 2))$ as in the theorem above and Lemma 6 gives us the result. □

Now, we want to point out that the results of Kunen in Theorem 7 and Corollary 8 can be generalized to cardinals different than ω_1. Indeed, similar proofs give us the following:

Corollary 9. Let δ be a regular cardinal such that $2^{<\delta} = \delta$ and κ be a measurable cardinal above it. Then there is a maximal δ-independent family $\mathcal{A} \subseteq \mathcal{P}(2^\delta)$.

Proof. Start with a measurable cardinal κ in a model of GCH and $\mathcal{U}$ be a normal measure witnessing this fact. As mentioned the proof follows the same lines as the one of Theorem 7, the main difference is the choice of the poset $\mathbb{P}$ which in this case should be $\mathbb{P} = \mathrm{Fn}_\delta(\kappa, 2)$. This poset is $(2^{<\delta})^+ = \delta^+$-cc, so the ideal $\mathcal{J}$ generated by the dual ideal of $\mathcal{U}$ in the generic extension will turn out to be δ^+-saturated.

In a similar fashion one can express the poset $j(\mathbb{P}) = j(\mathrm{Fn}_\delta(\kappa, 2)) = \mathrm{Fn}_\delta(\kappa^*, 2) \simeq \mathbb{P} \times \mathbb{Q}$, where $\mathbb{Q} = \mathrm{Fn}_\delta(\kappa^* \setminus \kappa, 2)$ and define the final isomorphism likewise. □

The analog of Corollary 8 has then the following form:

Corollary 10. Assume κ is strongly compact in V and $\delta < \kappa$ is regular such that $2^{<\delta} = \delta$. Then in $V[G]$, where G is $\mathbb{P}$-generic (for $\mathbb{P} = \mathrm{Fn}_\delta(\kappa, 2)$ like in the Theorem above) for every cardinal $\chi \geq \kappa$ such that $\mathrm{cf}(\chi) \geq \kappa$ there is a maximal δ-independent family of subsets of χ.

A comment on countable independence degree and the regular uncountable case: As mentioned above, if we assume $\kappa = \omega$ the existence of maximal κ-independent families at a cardinal χ is a straightforward consequence of Zorn's lemma. In [FM20], the notion of ω-independence (or simply *independence*) at χ when χ is an uncountable regular cardinal has been studied. Specifically, it has been proven that for this case, if χ is a measurable cardinal in a model V of GCH and $\delta > \chi$, there is a maximal ω-independent family $\mathcal{A}$, which is moreover preserved after forcing with δ-many copies of χ-Sacks forcing, and hence there is a generic extension in which the minimum size of a maximal ω-independent family of subsets of χ is χ^+ while $2^\chi = \delta$.

Additionally, Eskew and Fischer had studied in [EF21] the concept of maximal κ-independence for uncountable cardinals and the spectrum of independence, i.e., the set of possible cardinalities which can be realized as possible sizes of maximal κ-independent families.

4. The singular case

Now, we want to study the concept of independence in the case when λ is a singular cardinal of cofinality $\kappa < \lambda$. Note that in the Definition 1 above, there is no restriction when considering the case $\chi = \lambda$. Moreover, Kunen's result (Theorem 3) holds as long as κ, the degree of independence, is a regular cardinal.

We first introduce an example at the first singular cardinal ($\aleph_\omega$) of a ω-independent family.

Example 11. Hausdorff's example at $\chi = \lambda = \aleph_\omega$ of a ω-independent family at $\aleph_\omega$.

Proof. Let $\mathcal{C} = \{(a, A) : a \in [\lambda]^{<\omega}, A \subseteq \mathcal{P}(a)\}$ and note $|\mathcal{C}| = \aleph_\omega^{<\omega} = \aleph_\omega$. For $X \subseteq \lambda$ define $\mathcal{Y}_X = \{(a, A) \in \mathcal{C} : X \cap a \in A\}$. Then, $\mathcal{A} = \{\mathcal{Y}_X : X \subseteq \lambda\} \subseteq \mathcal{P}(\mathcal{C}) \simeq \mathcal{P}(\aleph_\omega)$ is ω-independent (or σ-independent): Given $X_0, X_1, \dots X_i$ and $Z_0, Z_1, \dots Z_j$ for $i, j < \omega$ the set $\bigcap_{l \leq i} \mathcal{Y}_{X_l} \cap \bigcap_{l \leq j} \lambda \backslash \mathcal{Y}_{Z_j}$ is non-empty; simply fix $a \in [\lambda]^{<\omega}$ such that $X_l \cap a \neq X_{l'} \cap a \neq Z_n \cap a \neq Z_{n'} \cap a$ for all $l, l' \leq i$ and $n, n' \leq j$, then $(a, \{X_l : l < i\})$ belongs to the corresponding intersection. □

Note that $\mathcal{A}$ is not ω_1-independent: Suppose $X_0 \subseteq X_1 \subseteq \dots X_n \subseteq \dots$ is an increasing cofinal sequence of subsets of λ and assume towards a contradiction that $\mathcal{A}$ is ω_1-independent. Let $(a, A) \in \bigcap_{i \text{ even}} \mathcal{Y}_{X_i} \cap \bigcap_{i \text{ odd}} \lambda \backslash \mathcal{Y}_{X_i}$, since the sequence of the X_n's is cofinal there is $n \in \omega$ (we can take it minimal) such that $a \subseteq X_n$, but then for all $i \geq n$, $a \cap X_i = a$ which is a contradiction.

Hausdorff's example can also be generalized to a more general situation:

Proposition 12. Suppose λ is a strong limit singular cardinal with $\mathrm{cf}(\lambda) = \kappa$. Then there is a κ-independent family of subsets of λ of size 2^λ.

Proof. Since λ is a strong limit, if we define $\mathcal{C} = \{(\gamma, A) : \gamma < \lambda \wedge A \subseteq \gamma\} \subseteq \lambda \times 2^{<\lambda}$ we can put this set in bijective correspondence with λ. Now, given $X \subseteq \lambda$ define the set $\mathcal{Y}_X = \{(\gamma, A) \in \mathcal{C} : X \cap \gamma \in A\}$ we claim that $\{\mathcal{Y}_X : X \subseteq \lambda\}$ is κ-independent.

Let $\{X_\alpha : \alpha < \delta_1\}$ and $\{Z_\beta : \beta < \delta_2\}$ be two disjoint families of different subsets of λ and $\delta_1, \delta_2 < \kappa$. Note that $(\gamma, A) \in \mathcal{X} = \bigcap_{\alpha < \delta_1} \mathcal{Y}_{X_\alpha} \cap \bigcap_{\beta < \delta_2} (\mathcal{C} \setminus \mathcal{Y}_{Z_\beta})$ if for all $\alpha < \delta_1$, $X_\alpha \cap \gamma \in A$ and for all $\beta < \delta_2$, $Z_\beta \cap \gamma \notin A$. Then it is enough to notice that there are unboundedly many ordinals $\gamma < \lambda$ for which $X_\alpha \cap \gamma \neq X_{\alpha'} \cap \gamma$ when $\alpha \neq \alpha' < \delta_1$, $Z_\beta \cap \gamma \neq Z_{\beta'} \cap \gamma$ when $\beta \neq \beta' < \delta_2$ and $X_\alpha \cap \gamma \neq Z_{\beta'} \cap \gamma$ for all $\alpha < \delta_1$ and $\beta < \delta_2$. Then, for such indexes γ putting $A_\gamma = \{X_\alpha \cap \gamma : \alpha < \delta_1\}$ we get that $(\gamma, A_\gamma) \in \mathcal{X}$ and so that $\mathcal{X}$ has size λ. □

The phenomenon in the example above is much more general, we have the following property:

Proposition 13. Let λ be a singular cardinal of cofinality $\kappa < \lambda$. Suppose that $\mathcal{A}$ is a κ-independent family of subsets of λ, then $\mathcal{A}$ is <u>not</u> κ^+-independent.

Proof. Let $\mathcal{A}$ be a κ-independent family of subsets of λ and let $(\lambda_\beta : \beta < \kappa)$ be a cofinal sequence of regular cardinals converging to λ. Put $X_\beta = [\lambda_\beta, \lambda)$ for all $\beta < \kappa$. We shall construct a strictly increasing sequence of partial functions $\{h_\alpha : \alpha < \kappa\} \subseteq \mathrm{BF}_\kappa(\mathcal{A})$ and a sequence $(\gamma_\alpha : \alpha < \kappa)$ such that $\mathcal{A}^{h_\alpha} \subseteq X_{\gamma_\alpha}$ for some $\beta_\alpha < \kappa$.

Start with any $h_0 \in \mathrm{BF}_\kappa(\mathcal{A})$ and put γ_0 to be the minimum ordinal such that $\mathcal{A}^{h_0} \cap X_{\gamma_0} \neq \emptyset$. If we have already constructed both h_α and γ_α take $h_{\alpha+1} = h_\alpha \cup \{(A, 0)\}$ to be a proper extension of h_α, so that A is an arbitrary element of $\mathcal{A} \backslash \mathrm{dom}(h_\alpha)$ such that $A \cap \lambda_{\gamma_\alpha} = \emptyset$. This is always possible because we can modify elements of $\mathcal{A}$ in boundedly many coordinates.

Then, define $\gamma_{\alpha+1}$ again to be the minimum ordinal for which $X_{\gamma_{\alpha+1}} \cap \mathcal{A}^{h_{\alpha+1}} \neq \emptyset$, because of our construction $\gamma_{\alpha+1} > \gamma_\alpha$.

Finally, if we put $h = \bigcup_{\alpha < \kappa} h_\alpha$, this is a bounded function in $\mathrm{BF}_{\kappa^+}(\mathcal{A})$ and clearly $\mathcal{A}^h \subseteq \bigcap_{\alpha < \kappa} (\mathcal{A}^{h_\alpha} \cap X_{\beta_\alpha}) = \emptyset$, so $\mathcal{A}$ is not κ^+-independent. □

Now we turn to maximality and the issue of the existence of these families at singular cardinals.

Definition 14. From now on, if λ is a singular cardinal such that $\mathrm{cf}(\lambda) = \kappa$ when we talk about ***independent families at*** λ, we mean a family $\mathcal{A} \subseteq [\lambda]^\lambda$ that is κ-independent, i.e. such that for all $h \in \mathrm{BF}_\kappa(\mathcal{A})$, $\mathcal{A}^h$ is unbounded in λ.

First of all, let's look at the case where λ is singular of countable cofinality. In this case the existence of a maximal ω-independent family (or just *independent*) of subsets of λ can be proven using Zorn's lemma. Indeed, if $(\mathcal{A}_i : i \in I)$ is a chain of ω-independent families, its union is also ω-independent (boolean combinations have always a finite domain).

In the case of λ singular of cofinality $\kappa > \omega$ we have that Kunen's result still applies: if there exists $\mathcal{A} \subseteq [\lambda]^\lambda$ a maximal κ-independent family, then Theorem 3 implies that $2^{<\kappa} = \kappa$ and that there is an ordinal Γ with $\sup\{(2^\alpha)^+ : \alpha < \lambda\} \leq \Gamma \leq \min\{\lambda, 2^\kappa\}$ such that there is a non-trivial κ^+-saturated Γ-complete ideal over Γ.

4.1. **The main Lemma.** The next result guarantees the existence of a maximal κ-independent family at a singular limit cardinal λ of cofinality κ, when we assume the existence of maximal κ-independent families at cardinals $(\lambda_\alpha : \alpha < \kappa)$ converging to λ.

Lemma 15. Assume that λ is a singular cardinal of cofinality $\kappa > \omega$ which is a limit of the discrete sequence of regular cardinals $(\lambda_\alpha : \alpha < \kappa)$ and that $\delta \leq \kappa < \lambda_0$ is a successor cardinal $\delta = \mu^+$. If for each $\alpha < \kappa$ there is a dense maximal δ-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ and also there is a maximal δ-independent family $\mathcal{C}$ of subsets of κ, then, there is a maximal δ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$. [3]

Proof. Let $\mathcal{A}_\alpha$ be a dense maximal δ-independent family at λ_α; without loss of generality let us assume that for each $\alpha < \kappa$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\lambda_\alpha \backslash \lambda_{<\alpha})$ where $\lambda_{<\alpha} = \sup_{\beta<\alpha} \lambda_\beta$. Also, let $\mathcal{C} \subseteq [\kappa]^\kappa$ be a maximal δ-independent family of subsets of κ. We first prove that we can make a stronger assumption on the maximality of the families $\mathcal{A}_\alpha$'s:

Claim 16. We may assume that the families $\mathcal{A}_\alpha$ are densely maximal in the following stronger sense: For all $\alpha < \kappa$:

$$\forall X \in [\lambda_\alpha]^{\lambda_\alpha} \forall g \in \mathrm{BF}_\delta(\mathcal{A}_\alpha) \exists h \supseteq g \text{ such that either } \mathcal{A}^h \cap X \text{ or } \mathcal{A}^h \backslash X \text{ is empty.}$$

Proof. Given an enumeration $\{A^\alpha_\gamma : \gamma < 2^{\lambda_\alpha}\}$ of $\mathcal{A}_\alpha$ and an enumeration $\{I_\gamma : \gamma < 2^{\lambda_\alpha}\}$ of the ideal $\mathcal{I}_{\mathcal{A}_\alpha}$ such that each element of $\mathcal{I}_{\mathcal{A}_\alpha}$ appears δ-many times and define $\mathcal{D}_\alpha = \{D^\alpha_\gamma : \gamma < 2^{\lambda_\alpha}\}$ where $D^\alpha_\gamma = A^\alpha_\gamma \backslash I_\gamma$. We claim that $\mathcal{D}_\alpha$ is maximal in the above sense.

Let $X \in [\lambda_\alpha]^{\lambda_\alpha}$ and $g \in \mathrm{BF}_\delta(\mathcal{D}_\alpha)$, then g induces a Boolean combination $x_g \in \mathrm{BF}_\delta(\mathcal{A}_\alpha)$ as follows: put $\mathrm{dom}(g) = \{D^\alpha_{i_1}, \dots, D^\alpha_{i_l}\}$ where $l < \delta$ and define $\mathrm{dom}(x_g) = \{A^\alpha_{i_1}, \dots, A^\alpha_{i_l}\}$ so that $x_g(A^\alpha_{i_j}) = g(D^\alpha_{i_j})$. Since $\mathcal{A}_\alpha$ is densely maximal there is $x_{g'} \supseteq x_g$ such that either $(\mathcal{A}_\alpha)^{x_{g'}} \cap X$ or $(\mathcal{A}_\alpha)^{x_{g'}} \backslash X$ is bounded in λ_α.

[3]One can prove an analog of this theorem in which instead of densely maximal, we just have *maximal*.

Suppose without loss of generality that $|(\mathcal{A}_\alpha)^{x_{g'}} \cap X| < \lambda_\alpha$ and choose γ^* such that $A^\alpha_{\gamma^*} \notin \mathrm{dom}(x_{g'})$ and $I_{\gamma^*} = (\mathcal{A}_\alpha)^{x_{g'}} \cap X$. Finally, define $y' = x_{g'} \cup \{(A^\alpha_{\gamma^*}, 1)\}$ and take the associated $h_y \in \mathrm{BF}_\delta(\mathcal{D}_\alpha)$. Thus we get $(\mathcal{D}_\alpha)^{h_y} \subseteq A^\alpha_{\gamma^*}$ and so $(\mathcal{D}_\alpha)^{h_y} \cap X = \emptyset$. $\square$

Now, consider a function $f \in \Pi_{\alpha\in C}\mathcal{A}_\alpha$ for some $C \in \mathcal{C}$ and define the set $B^{f,C} = \bigcup_{\alpha\in C} f(\alpha)$. We claim that the family

$$\mathcal{B} = \{B^{f,C} : C \in \mathcal{C} \wedge f \in \Pi_{\alpha\in C}\mathcal{A}_\alpha\}$$

is maximal δ-independent at λ. First, we prove that $\mathcal{B}$ is δ-independent: To start, note that given $B^{f,C} \in \mathcal{B}$,

$$\lambda\backslash B^{f,C} = \lambda\backslash \bigcup_{\alpha\in C} f(\alpha) = \bigcap_{\alpha\in C} (\lambda\backslash f(\alpha)) \supseteq \bigcup_{\alpha\in\delta\backslash C} \bar{f}(\alpha) = B^{\bar{h},\delta\backslash C}. \ (*)$$

where $\bar{f}(\alpha)$ chooses an arbitrary element in $\mathcal{A}_\alpha$ for $\alpha \in \delta\backslash C$.

For the general case, let $\varphi \in \mathrm{BF}_\delta(\mathcal{B})$ and notice that φ reflects into a Boolean combination in $\mathcal{A}_\alpha$ for each $\alpha \in C$ as follows: Put $\mathrm{dom}(\varphi) = \{B^{f_i,C_i} : i < l_1\} \cup \{B^{f'_i,C'_i} : i < l_2\}$ for $l_1, l_2 < \delta$ and $C = \bigcap_{i<l_1} C_i \cap \bigcap_{i<l_2} \delta\backslash C'_i$.

For fixed $\alpha \in C$, φ induces $\mathcal{A}^\varphi_\alpha = \bigcap_{i<l_1} f_i(\alpha) \cap \bigcap_{i<l_2} \bar{f}'_i(\alpha)$ where $\bar{f}'_i$ is like in $(*)$. This is a Boolean combination within the maximal δ-independent family $\mathcal{A}_\alpha$, and so unbounded on λ_α. Then clearly $\mathcal{B}^\varphi = \bigcap_{i<l_1} B^{f_i,C_i} \cap \bigcap_{i<l_2}(\lambda \setminus B^{f'_i,C'_i})$ is unbounded in λ because:

$$\mathcal{B}^\varphi \supseteq \bigcap_{i<l_1} B^{f_i,C_i} \cap \bigcap_{i<l_2} B^{\bar{f}'_i,\delta\backslash C_i} \supseteq \bigcup_{\alpha\in C} \mathcal{A}^\varphi_\alpha$$

which is clearly an unbounded set in λ.

It is just left to prove that $\mathcal{B}$ is maximal: Let $Y \in [\lambda]^\lambda$ and put $Y_\alpha = Y \cap (\lambda_\alpha\backslash\lambda_{<\alpha})$, we split into two cases: Case 1 is when there are unboundedly many $\alpha' s$ (in κ) so that Y_α is unbounded in λ_α. Case 2 is when Y_α is bounded in λ_α for all but boundedly many $\alpha < \kappa$. The argument for the second case is simpler and easily derived from the argument for case 1. We therefore focus on case 1: Let $N = \{\alpha < \kappa : |Y_\alpha| = \lambda_\alpha\}$.

Using that $\mathcal{A}_\alpha$ is maximal dense δ-independent in the strong sense described above, we have that for each $\alpha \in N$ there is $h_\alpha \in \mathrm{BF}_\delta(\mathcal{A}_\alpha)$ so that either $(\mathcal{A}_\alpha)^{h_\alpha} \cap Y_\alpha$ or $(\mathcal{A}_\alpha)^{h_\alpha}\backslash Y_\alpha$ is empty.

Hence, either one of the sets $Z_0 = \{\alpha \in N : (\mathcal{A}_\alpha)^{h_\alpha} \cap Y_\alpha = \emptyset\}$ or $Z_1 = \{\alpha \in N : (\mathcal{A}_\alpha)^{h_\alpha}\backslash Y_\alpha = \emptyset\}$ must be unbounded in κ, without loss of generality $|Z_0| = \kappa$. On the other hand, by the maximality of the family $\mathcal{C}$ there is $p \in \mathrm{BF}_\delta(\mathcal{C})$ such that either $\mathcal{C}^p \cap Z_0$ or $\mathcal{C}^p\backslash Z_0$ is empty.

Suppose again without loss of generality that $\mathcal{C}^p \cap Z_0 = \emptyset$ and recall that $N = Z_0 \cup Z_1$ and $Z_0 \cap Z_1 = \emptyset$. In order to finish the proof, we want to build a Boolean combination of the family $\mathcal{B}$ of elements of the form $B^{h,C}$ as follows:

First, for all $\alpha \in N$, let's put $\mathrm{dom}(h_\alpha) = \{E^\alpha_j : j < l_\alpha\} \cup \{F^\alpha_i : j < m_\alpha\}$ where for each $j < l_\alpha$, $h_\alpha(E^\alpha_j) = 0$ and for each $i < m_\alpha$, $h_\alpha(F^\alpha_m) = 1$. The first thing we would like to argue is that we can uniformize the lengths of the domains, i.e. we want to assume that $l_\alpha = l_\beta$ for all α, $\beta < \kappa$ and likewise for the m_α's.

Let $l = \sup\{l_\alpha : \alpha < \kappa\}$ and $m = \sup\{m_\alpha : \alpha < \kappa\}$, since δ is a successor cardinal it is clear that both $m, l < \delta$. In order to extend the domains of the h_α's we just have to notice that, for each $\alpha \in N$ if we add arbitrary elements to the $\mathrm{dom}(h_\alpha)$ and build an extension h'_α of h_α we get $(\mathcal{A}_\alpha)^{h'_\alpha} \subseteq (\mathcal{A}_\alpha)^{h_\alpha}$ and so $(\mathcal{A}_\alpha)^{h'_\alpha} \cap Y_\alpha = \emptyset$.

Now, we define functions f_i, g_j in $\Pi_{\alpha\in\mathcal{C}^p}\mathcal{A}_\alpha$ for $i < l$ and $j < m$ as follows: $f_j(\alpha) = E_\alpha$ for all $\alpha \in N$ and $g_j(\alpha) = \lambda_\alpha \backslash F_\alpha$ for all $\alpha \in N$. We claim that

$$(\bigcap_{i<l} B^{f_i,\mathcal{C}^p} \cap \bigcap_{j<m} B^{g_j,\mathcal{C}^p})\backslash Y = \emptyset$$

Note first that if $\mathrm{dom}(p) = \{P_i : i < l\} \cup \{Q_j : j < m\}$, then for an arbitrary $f \in \Pi_{\alpha\in\mathcal{C}^p}\mathcal{A}_\alpha$, then $B^{f,\mathcal{C}^p} = \bigcup_{\alpha\in\mathcal{C}^p} f(\alpha) = \bigcap_{i<l} B^{f_i^*,P_i} \cap \bigcap_{j<m} B^{f_j^*,Q_j^c}$ and the latter is a Boolean combination of elements of $\mathcal{B}$. To finish the proof, notice that:

$$(\bigcap_{i<l} B^{f_i,\mathcal{C}^p} \cap \bigcap_{i<l} B^{g_j,\mathcal{C}^p})\backslash Y \subseteq \bigcup_{\alpha\in\mathcal{C}^p} \mathcal{A}^{h'_\alpha} \backslash Y_\alpha = \emptyset$$

as we wanted. The other cases are analogous. □

We can have an even stronger version of the result above, one in which the degrees of independence increase. First, notice that the part in the proof above, in which we uniformize the lengths of the Boolean combinations h_α can be achieved by adding sort of "dummy" coordinates to the domains of the h_α's.

Theorem 17. *Assume that λ is a singular cardinal of cofinality κ which is a limit of a discrete sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \mu)$ be a sequence of cardinals with limit $\mu \leq \kappa$ where μ is a regular cardinal. Suppose also that for each $\alpha < \kappa$, there is a maximal δ_α^+-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ and $\kappa < \lambda_0$ is regular such that there is a maximal μ-independent family of subsets of κ. Then, there is a maximal μ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$.*

Proof. The proof of this is similar to the proof of the result above, so we will focus on the main differences of this argument in comparison with the one above. Let $\mathcal{A}_\alpha$ be a maximal δ_α^+-independent family at λ_α, without loss of generality $\mathcal{A}_\alpha \subseteq \mathcal{P}(\lambda_\alpha \backslash \lambda_{<\alpha})$ where $\lambda_{<\alpha} = \sup_{\beta<\alpha} \lambda_\beta$. Also, let $\mathcal{C} \subseteq [\kappa]^\kappa$ be a maximal μ-independent family of subsets of κ; consider $f \in \Pi_{\alpha\in C}\mathcal{A}_\alpha$ for some $C \in \mathcal{C}$ and define likewise the set $B^{f,C} = \bigcup_{\alpha\in C} f(\alpha)$.

Again, we claim that $\mathcal{B} = \{B^{f,C} : C \in \mathcal{C} \wedge f \in \Pi_{\alpha\in C}\mathcal{A}_\alpha\}$ is maximal μ-independent at λ.

First, we prove that $\mathcal{B}$ is μ-independent:

Let $\varphi \in \mathrm{BF}_\mu(\mathcal{B})$ and put $\mathrm{dom}(\varphi) = \{B^{f_i,C_i} : i < l_1\} \cup \{B^{f'_i,C'_i} : i < l_2\}$ for $l_1, l_2 < \mu$. Also, define $C = \bigcap_{i<l_1} C_i \cap \bigcap_{i<l_2} \delta \backslash C'_i$. Note that C is unbounded in λ because the family $\mathcal{C}$ is μ-independent.

Since l_1 and $l_2 < \mu$ there is a $\beta < \kappa$ such that $l_1, l_2 < \delta_\beta^+$. Thus for $\alpha \in C$, $\alpha \geq \beta$ the function φ induces a Boolean combination in the corresponding $\mathcal{A}_\alpha$-family. Namely, $\mathcal{A}_{\varphi,\alpha} = \bigcap_{i<l_1} f_i(\alpha) \cap \bigcap_{i<l_2} \bar{f}'_i(\alpha)$ where $\bar{f'}_i$ is like in $(*)$ in the Theorem above.

The Boolean combination $\mathcal{A}_{\varphi,\alpha}$ can be expressed as $\mathcal{A}_{\psi,\alpha}$ where $\psi \in \mathrm{BF}_{\delta_\beta}(\mathcal{A}_\alpha)$. Thus, for every $\alpha \geq \beta$ we can use that the family $\mathcal{A}_\alpha$ is δ_α^+-independent to conclude that $\mathcal{A}_{\psi,\alpha}$ is unbounded on λ_α.

Then clearly $\mathcal{B}^\varphi = \bigcap_{i<l_1} B^{f_i,C_i} \cap \bigcap_{i<l_2}(\lambda \setminus B^{f'_i,C'_i})$ is unbounded in λ because:

$$\mathcal{B}^\varphi \supseteq \bigcap_{i<l_1} B^{f_i,C_i} \cap \bigcap_{i<l_2} B^{\bar{f}'_i,\delta\setminus C_i} \supseteq \bigcup_{\alpha\in C\cap[\beta,\kappa)} \mathcal{A}_{\varphi,\alpha}$$

which is clearly an unbounded set in λ.

Maximality: Let $Y \in [\lambda]^\lambda$ and put $Y_\alpha = Y \cap (\lambda_\alpha \backslash \lambda_{<\alpha})$, as before we split into two cases and we give a proof in the case were there are unboundedly many $\alpha's$ (in κ) so that Y_α is unbounded in λ_α. Let $N = \{\alpha < \kappa : |Y_\alpha| = \lambda_\alpha\}$.

Using that $\mathcal{A}_\alpha$ is maximal dense δ_α^+-independent in the strong sense described above, we have that for each $\alpha \in N$ there is $h_\alpha \in \mathrm{BF}_{\delta_\alpha}(\mathcal{A}_\alpha^+)$ so that either $(\mathcal{A}_\alpha)^{h_\alpha} \cap Y_\alpha$ or $(\mathcal{A}_\alpha)^{h_\alpha} \backslash Y_\alpha$ is empty.

Suppose again without loss of generality $|Z_0| = |\{\alpha \in N : (\mathcal{A}_\alpha)^{h_\alpha} \cap Y_\alpha\}| = \kappa$. On the other hand, by the maximality of the family $\mathcal{C}$ there is $p \in \mathrm{BF}_\delta(\mathcal{C})$ such that either $\mathcal{C}^p \cap Z_0$ or $\mathcal{C}^p \backslash Z_0$ is empty.

Suppose again without loss of generality that $\mathcal{C}^p \cap Z_0 = \emptyset$. We build now a Boolean combination of the family $\mathcal{B}$ of elements of the form $B^{h,C}$ as follows:

First, for all $\alpha \in N$, let's put $\mathrm{dom}(h_\alpha) = \{E_j^\alpha : j < l_\alpha\} \cup \{F_i^\alpha : j < m_\alpha\}$ where for each $j < l_\alpha$, $h_\alpha(E_j^\alpha) = 0$ and for each $i < m_\alpha$, $h_\alpha(F_m^\alpha) = 1$. Clearly l_α and m_α are both $< \delta_\alpha^+$. Now, we want to uniformize the domains but slightly different from the argument in the last Theorem:

Let $l = \sup\{l_\alpha : \alpha < \kappa\}$ and $m = \sup\{m_\alpha : \alpha < \kappa\}$, since μ^+ is a successor cardinal it is clear that both $m, l < \mu^+$. We define sequences $(G_i^\alpha : i < l)$ and $(H_j^\alpha : j < m)$ of elements of the family $\mathcal{A}_\alpha$ as follows:

$$G_i^\alpha = \begin{cases} E_i^\alpha, & \text{if } i < l_\alpha \\ E_0^\alpha, & \text{otherwise} \end{cases}$$

Likewise:

$$H_j^\alpha = \begin{cases} F_j^\alpha, & \text{if } j < m_\alpha \\ F_0^\alpha, & \text{otherwise} \end{cases}$$

Now, we define functions f_i, g_j in $\Pi_{\alpha\in\mathcal{C}^p}\mathcal{A}_\alpha$ for $i < l$ and $j < m$ as follows: $f_j(\alpha) = G_j^\alpha$ for all $\alpha \in N$ and $g_j(\alpha) = \lambda_\alpha \backslash F_j^\alpha$ for all $\alpha \in N$. We claim that

$$(\bigcap_{i<l} B^{f_i,\mathcal{C}^p} \cap \bigcap_{j<m} B^{g_j,\mathcal{C}^p}) \backslash Y = \emptyset$$

To this end, notice that:

$$(\bigcap_{i<l} B^{f_i,\mathcal{C}^p} \cap \bigcap_{i<l} B^{g_j,\mathcal{C}^p}) \backslash Y \subseteq \bigcup_{\alpha\in\mathcal{C}^p} \mathcal{A}^{h'_\alpha} \backslash Y_\alpha = \emptyset$$

as we intended. $\square$

5. A model with a maximal κ-independent family at λ

Recall that a cardinal κ is *supercompact*, if for all $\lambda > \kappa$ there are elementary embeddings $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, the model M is closed under λ-sequences (write $M^\lambda \subseteq M$) and $j(\kappa) > \lambda$. If κ is supercompact, Laver preparation gives us a tool, to preserve the supercompactness of κ after forcing with $<\kappa$-directed closed forcings.

Theorem 18 (Laver [Lav78])**.** *If κ is supercompact, then there is a κ^+-cc forcing notion $\mathbb{Q}$ of size κ, such that in $V^{\mathbb{Q}}$, κ is supercompact and remains supercompact upon forcing with any κ-directed closed partial ordering.*

The proof of the theorem needs the following lemma, which assures the existence of the well-known Laver diamonds.

Lemma 19. Let κ be supercompact. Then there is a function $h : \kappa \to V_\kappa$ such that for every x and every $\lambda \geq \kappa$ such that $x \in H_{\lambda^+}$, there is a $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$. h is called a *Laver diamond* for κ.

In the following proposition, we explore the idea of Laver preparing a set of κ-many supercompact cardinals simultaneously, so let us assume that $(\kappa_\alpha : \alpha < \kappa)$ is a fixed increasing sequence of supercompact cardinals.

Proposition 20. Suppose that $(\kappa_\alpha : \alpha < \kappa)$ is a sequence of supercompact cardinals, then there is a poset $\mathbb{P}$, such that in the generic extension by $\mathbb{P}$ one has that for every $\alpha < \kappa$, κ_α is supercompact and remains supercompact upon forcing with any κ_α-directed closed forcing.

Proof. We modify Laver's proof in order to include the whole set of κ_α's simultaneously. Let $h_\alpha : \kappa_\alpha \to V_{\kappa_\alpha}$ be a Laver diamond for κ_α. Consider also bijections $\varphi : \kappa_\alpha \backslash \kappa_{<\alpha} \to \kappa_\alpha$.

We define a forcing notion $\mathbb{Q}$ which is obtained as an Easton reverse iteration of length $\kappa = \sup_{\alpha<\kappa} \kappa_\alpha$. We define inductively posets $\mathbb{Q}_\beta$ and ordinals λ_β for $\beta < \kappa$ as follows:

At limit stages, we follow the same recipe as in Laver's Theorem above. For the successor step, let's assume that we have already defined $\mathbb{Q}_\beta$ and consider α_β be the unique ordinal $< \kappa$ such that $\beta \in [\kappa_{<\alpha_\beta}, \kappa_{\alpha_\beta})$, then $\dot{\mathbb{P}}_\beta$ is trivial unless $\lambda_\gamma < \beta$ for all $\gamma < \beta$ and $h_{\alpha_\beta}(\varphi_\alpha(\beta)) = (\dot{P}, \lambda)$ where λ is an ordinal and $\dot{P}$ is a $\mathbb{Q}_\beta$-name for a β-directed closed forcing. In this case $\mathbb{Q}_{\beta+1} = \mathbb{Q}_\beta * \dot{P}$ and $\lambda_{\beta+1} = \lambda$.

Let G be a generic filter, then it is enough to prove that for all $\alpha < \kappa$ and $\mathbb{P} \in V[G]$ is so that:

$$\Vdash_{\mathbb{Q}} \dot{\mathbb{P}} \text{ is } \kappa_\alpha - \text{directed closed}$$

Then, κ_α stays supercompact after forcing with $\mathbb{P}$. The proof follows exactly the same arguments as Laver's above. We run the argument relative to a specific κ_α we are considering, i.e. we use the Laver diamond h_α in order to find an elementary embedding $j_\alpha : V \to M_\alpha$ with $\mathrm{crit}(j_\alpha) = \kappa_\alpha$, $j(\kappa_\alpha) > \lambda$ and $M_\alpha^\lambda \subseteq M_\alpha$ and $j_\alpha(h\alpha)(\kappa_\alpha) = \dot{\mathbb{P}}$.

Finally, just notice that the intervals $[\kappa_{<\alpha}, \kappa_\alpha)$ translate into intervals in M_α via j_α of the form $[j_\alpha(\kappa_{<\alpha}), j_\alpha(\kappa_\alpha))$ taking into account that $j_\alpha(\kappa_\delta) = \kappa_\delta$ for all $\delta < \alpha$. Thus, for instance the interval's $[\kappa_{<\alpha}, \kappa_\alpha)$ image on the M_α side corresponds to the interval $[\kappa_{<\alpha}, j_\alpha(\kappa_\alpha))$.

In M_α, $j_\alpha(\mathbb{Q}_\alpha : \alpha \leq \kappa)$ is an iteration of length $j_\alpha(\kappa)$ which, by elementarity and the fact that $\mathrm{crit}(j_\alpha) = \kappa_\alpha$ has to have as initial segment the iteration $\mathbb{Q} \restriction \kappa_\alpha$. Write then $j_\alpha(\mathbb{Q}_\alpha : \alpha \leq \kappa) = (\mathbb{Q}_\alpha : \alpha \leq j_\alpha(\kappa))$ (we refer to this iteration as $\mathbb{Q}_{j_\alpha(\kappa)}$). Moreover, note that by definition of the forcing $\mathbb{Q}$ and elementarity $\mathbb{Q}_{\kappa_\alpha+1} = \mathbb{Q} * \dot{\mathbb{R}}$ and for $\kappa_\alpha < \delta < \lambda$, $\mathbb{Q}_{\delta+1} = \mathbb{Q}_\delta * \mathbb{1}$.

Likewise, the part of the iteration $(\mathbb{Q}_\alpha : \lambda \leq \alpha \leq j(\kappa))$ is a $\geq \lambda$-closed forcing notion in M_α (and hence in V because $M^\lambda \subseteq M$). Also $j_\alpha(\dot{\mathbb{P}})$ is forced to be a λ-directed closed forcing notion, thus $\mathbb{Q}_{j_\alpha(\kappa)} * j(\dot{\mathbb{P}}) = \mathbb{Q} * \dot{\mathbb{P}} * \dot{\mathbb{R}}$ where $\mathbb{R}$ is a λ-directed closed forcing living in the extension $V^{\mathbb{Q}*\dot{\mathbb{P}}}$. The rest of the argument works analogously. $\square$

Now, we are ready to state our main result:

Theorem 21. *Start with a ground model V in which* GCH *holds. Suppose that λ is a singular cardinal of cofinality $\kappa > \omega$ which is a limit of supercompact cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \kappa)$ be a sequence of regular successor cardinals converging to κ so that $\delta_\alpha^{<\delta_\alpha} = \delta_\alpha$. Then there is a generic extension of V such that*

$$V^{\mathbb{P}} \models \textit{There is a maximal } \kappa\textit{-independent family of subsets of } \lambda.$$

Proof. Start with $(\lambda_\alpha : \alpha < \kappa)$ and $(\delta_\alpha : \alpha < \kappa)$ two sequences cardinals as in the statement of the Theorem such that $\sup_{\alpha<\kappa} \lambda = \lambda_\alpha$ and $\sup_{\alpha<\kappa} \delta_\alpha = \delta \leq \kappa$.

Also, assume that our base model is Laver prepared for all λ_α's, i.e. $V = V^{\mathbb{P}}$ where $\mathbb{P}$ is the forcing from Proposition 20. This will ensure that after forcing with $< \lambda_\alpha$-directed closed forcings, not just the cardinal λ_α stays supercompact, but all cardinals $\lambda_\beta > \lambda_\alpha$ for $\beta > \alpha$ likewise.

We use the results above regarding the existence of maximal independent families for regular cardinals in Theorem 9 and Corollary 10. Let's consider the poset $\mathbb{Q}$ constructed as the Easton support product of the posets $\mathbb{P}_\alpha = \mathrm{Fn}_{\delta_\alpha}(\lambda_\alpha, 2)$ for each $\alpha < \kappa$.

Let K be a $\mathbb{Q}$-generic filter over V and fix $\alpha < \lambda$. Then, by Easton's Theorem, we have that in the generic extension $V[K] \models \forall\alpha(2^{\delta_\alpha} = \lambda_\alpha)$.

Now, we claim that additionally the conditions in Corollary 10 are fulfilled for all $\alpha < \kappa$ in $V[K]$, and so, in the generic extension the following holds:

$$V[K] \models \text{For all } \alpha < \kappa \text{ there is a maximal } \delta_\alpha\text{-independent family of subsets of } \lambda_\alpha \ (*)$$

Recall that $\mathbb{Q} \simeq \mathbb{Q}_\alpha^- \times \mathbb{Q}_\alpha^+$ where $\mathbb{Q}_\alpha^-$ is the Easton poset restricted to the cardinals $(\lambda_\beta : \beta \leq \alpha)$ and $\mathbb{Q}_\alpha^+$ is the Easton poset restricted to the set of cardinals $(\lambda_\beta : \beta > \alpha)$. Moreover, by using the product lemma: If K is $\mathbb{Q}$-generic over the ground model V, then we can factorize $V[K] = V[H][G]$ where H is $\mathbb{Q}_\alpha^+$ generic over V and G is $\mathbb{Q}_\alpha^-$-generic over $V[H]$.

First, fix $\alpha < \kappa$ and look at the intermediate extension $V[G]$ (recall this is the extension given by $\mathbb{Q}_\alpha^+$). Note that the cardinal λ_α is still supercompact $V[G]$ because the iteration product is λ_α-directed closed.

Since λ_α is supercompact in $V[G]$, there is a fine ultrafilter on $\mathcal{P}_{\lambda_\alpha}(\lambda_\alpha)$. Take $\mathcal{U}_\alpha$ to be a fine measure witnessing this property and $j_\alpha : V \to M_\alpha$ to be an elementary embedding with $\mathrm{crit}(j_\alpha) = \lambda_\alpha$ such that: for any $X \subseteq M_\alpha$ with $|X| \leq \alpha$, there is $Y \in M_\alpha$ such that $Y \supseteq X$ and $M_\alpha \models |Y| < j_\alpha(\lambda_\alpha)$.

Note also that the poset $\mathbb{Q}_\alpha^-$ is α^+-cc, so for the normal measure $\mathcal{U}_\alpha$ we get that after forcing with $\mathbb{Q}_\alpha^-$, the ideal generated by $\mathcal{U}$ in $V[K]$ is δ_α^+-saturated and λ_α-complete.

Then it is left to define an isomorphism $\Gamma_\alpha : \mathcal{P}(\kappa_\alpha)/\mathcal{I} \to \mathcal{B}(\mathrm{Fn}_{\delta_\alpha}(\kappa_\alpha^* \backslash \kappa_\alpha, 2))$ (here $\kappa_\alpha^* = j_\alpha(\kappa_\alpha)$) in $V[K]$ as follows: Given $[X] \in (\mathcal{P}(\kappa_\alpha)/\mathcal{I})^{V[K/G]}$, and let $\dot{X}$ be a $\mathbb{Q}_\alpha^+$-name for the set X. We define the function as follows:

$$\Gamma([X]) := \bigvee \{q(\alpha) \in \mathrm{Fn}_{\delta_\alpha}(\kappa_\alpha^* \backslash \kappa_\alpha, 2) : q \in \mathbb{Q}_\alpha^+ \wedge \exists p \in G(p \cup q \Vdash \check{\kappa_\alpha} \in j(\dot{X}))\}.$$

Following the same arguments as above, one can easily prove that Γ is the desired isomorphism. The proof of the result is then finished by using Theorem 17. □

6. Sizes of maximal independent families

In this section, we are interested in the study of the possible sizes of maximal κ-independent families at a singular cardinal λ of cofinality κ. First of all, we provide all the necessary definitions regarding relevant cardinal characteristics which will be used throughout this section. Let λ be a singular cardinal so that $\mathrm{cf}(\lambda) = \kappa$:

Definition 22 (The independence number)**.**

$$\mathfrak{i}_\kappa(\lambda) = \min\{|\mathcal{A}| \colon \mathcal{A} \text{ is a maximal } \kappa\text{-independent family of subsets of } \lambda\}.$$

Definition 23 (The dominating and eventually different numbers)**.**

- For two functions $f, g \in \kappa^\lambda$, we say $f \le g \,(\mathrm{mod}(\lambda))$ if and only if there exists $\alpha < \lambda$ such that for all $\beta > \alpha$, $f(\beta) \le g(\beta)$.
- $\mathfrak{d}(\lambda) = \min\{|\mathfrak{F}| \colon \mathfrak{F} \subseteq {}^\lambda \mathrm{cf}(\lambda)$ and $\forall g \in {}^\lambda \mathrm{cf}(\lambda) \exists f \in \mathcal{F}$ such that $f < g$ $(\mathrm{mod}(\lambda))\}$.
- $\mathfrak{e}(\lambda) = \min\{|\mathfrak{F}| \colon \mathfrak{F} \subseteq {}^\lambda \mathrm{cf}(\lambda)$ and $\forall g \in {}^\lambda \mathrm{cf}(\lambda)$ $\exists f \in \mathcal{F}$ such that $f \neq g \,(\mathrm{mod}[\lambda]^{<\lambda})\}$.

Definition 24 (The reaping number)**.**

- For A and $B \in \mathcal{P}(\lambda)$, say $A \subseteq^* B$ (A is almost contained in B) if $A \setminus B$ has size $< \lambda$. We also say that A splits B if both $A \cap B$ and $B \setminus A$ have size λ. A family $\mathcal{A}$ is called a splitting family if every unbounded (with supremum λ) subset of λ is split by a member of $\mathcal{A}$. Finally, $\mathcal{A}$ is unsplit if no single set splits all members of $\mathcal{A}$.
- $\mathfrak{r}(\lambda) = \min\{|\mathcal{A}| \colon \mathcal{A}$ is an unsplit family of subsets of $\lambda\}$.

Lemma 25. The following inequalities hold:

(1) $\mathfrak{i}_\kappa(\lambda)^{<\kappa} \ge \lambda^+$.
(2) $\mathfrak{i}_\kappa(\lambda)^{<\kappa} \ge \mathfrak{r}(\lambda)$.

Proof. (1) Let $\mathcal{B}$ be a Boolean algebra and let $\mathcal{A} \subseteq \mathcal{B}^+$. We say that $\mathcal{A}$ is κ-indecomposable if there is a disjoint collection $P \subseteq \mathcal{A}^+$ such that $|P| = |P \wedge \wedge a| = |\{p \wedge a : p \in P\}|$ for all $a \in \mathcal{A}$. Balçar, Simon, and Vojtaš proved that if $\mathcal{A}$ is κ-decomposable, then it has a disjoint refinement. The proof of the desired inequality is due to Eskew and Fischer in [EF21] and uses the fact above: Let $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ be κ-independent such that $\delta = |\mathcal{A}|^{<\kappa} \le \lambda$. Let $\mathcal{A} = \{A_\alpha : \alpha < \delta\}$ be an enumeration of $\mathcal{A}$, then there is a disjoint subfamily $\{B_\alpha : \alpha < \delta\}$ where each $B_\alpha \subseteq A_\alpha$ and $|B_\alpha| = \lambda$. Note that this is possible because $\mathcal{A}$ is

κ-decomposable as a subset of the Boolean algebra $\mathcal{P}(\kappa)/\text{fin}$, just take C be a maximal antichain in the poset $\text{Fn}_\kappa(\mathcal{A}, 2)$, so $|C| \leq \kappa$ and so $P = \{\mathcal{A}^h : h \in C\}$.

Then we can partition each B_α into two disjoint sets of the same cardinality, B^0_α and B^1_α. Let $C = \bigcup\{B^0_\alpha : \alpha < \delta\}$. Then $\mathcal{A} \cup \{C\}$ is κ-independent.

(2) Let $\mathcal{A}$ be maximal κ-independent and put $\mathfrak{r}(\lambda) = \delta$ and $\mathcal{X} = \{\mathcal{A}^h : h \in \text{BF}_{<\kappa}(\mathcal{A})\}$. Then $|\mathcal{X}| = \delta^{<\kappa}$ and this is an unsplit family. Otherwise, there is a set $Z \in [\lambda]^\lambda$ such that both sets $\mathcal{A}^h \cap Z$ and $\mathcal{A}^h \backslash Z$ are unbounded for all $h \in \text{BF}_{<\kappa}(\mathcal{A})\}$. Then $\mathcal{A}$ is not maximal.

□

Proposition 26. If for all $\gamma < \mathfrak{d}(\lambda)$ we get $\gamma^{<\kappa} < \mathfrak{d}(\lambda)$ then $\mathfrak{i}_\kappa(\lambda) \geq \mathfrak{d}(\lambda)$.

Proof. Let $\mathcal{A}$ be a κ-independent family so that $|\mathcal{A}| < \mathfrak{d}(\lambda)$. First, we prove the following lemma:

Lemma 27. Suppose $\mathcal{C} = (C_\alpha : \alpha < \kappa)$ is a $\subseteq^*$-decreasing sequence of unbounded subsets of λ and $\mathcal{A}$ is a family of less than $\mathfrak{d}(\lambda)$ many subsets of λ such that each set in $\mathcal{A}$ intersects every C_α in a set of size λ. Then $\mathcal{C}$ has a pseudointersection B that also has an unbounded intersection with each member of $\mathcal{A}$.

Proof. Without loss of generality, we assume that the sequence $\mathcal{C}$ is $\subseteq$-decreasing. Fix an increasing sequence of ordinals $(\lambda_\alpha : \alpha < \kappa)$, for any $h \in \kappa^\lambda$ define $B_h = \bigcup_{\alpha<\kappa}(C_\alpha \cap h(\alpha))$, clearly B_h is a pseudointersection of $\mathcal{C}$ (i.e. $|B_h \setminus C_\alpha| < \lambda$). Thus, it is enough to find $h \in \kappa^\lambda$ such that $|B_h \cap A| = \lambda$ for each $A \in \mathcal{A}$.

For each $A \in \mathcal{A}$ and $\alpha < \kappa$ define $f \in \kappa^\lambda$ by letting $f_A(\alpha)$ be the α-th element of the set $C_\alpha \cap A$. The set $\{f_A : A \in \mathcal{A}\}$ has cardinality $< \mathfrak{d}(\lambda)$, then we can find $h \in \kappa^\lambda$ such that for all $A \in \mathcal{A}$, $h \nleq f_A(\text{mod}(\lambda))$ (i.e. $X_A = \{\alpha < \kappa : f_A(\delta) < h(\delta)\}$ is unbounded in λ). We claim that B_h is the pseudointersection we need. For this notice that

$$B_h \cap A = (\bigcup_{\alpha<\kappa} C_\alpha \cap h(\alpha)) \cap A = \bigcup_{\alpha<\kappa} (C_\alpha \cap A) \cap h(\alpha) \supseteq \bigcup_{\alpha \in X_A} (C_\alpha \cap A) \cap f_A(\alpha)$$

and the latter set is unbounded in λ. □

Now, for the proof of the proposition put $\mathcal{D} = (A_\alpha : \alpha < \kappa) \subseteq \mathcal{A}$ and $\mathcal{A}' = \mathcal{A} \backslash \mathcal{D}$. Consider the set of Boolean combinations of the family $\mathcal{A}$, $\mathcal{X} = \{\mathcal{A}^h : h \in \text{BF}_\kappa(\mathcal{A})\}$ and notice again that $|\mathcal{X}| = |\mathcal{A}|^{<\kappa} < \mathfrak{d}(\lambda)$. For each $f : \kappa \to 2$ consider the set $C^f_\alpha = \bigcap_{\beta<\alpha} A^{f(\beta)}_\beta$ where $A^0 = A$ and $A^1 = \lambda \backslash A$. The sequence of sets $(C^f_\alpha : \alpha < \kappa)$ is $\subseteq$-decreasing and moreover, every set C^f_α intersects all elements of the family $\mathcal{A}$ in an unbounded (in λ) set.

Thus, using the lemma before there exists a pseudointersection B_f of the family $(C^f_\alpha : \alpha < \kappa)$ that intersects in an unbounded set all members of $\mathcal{A}$. Moreover, if $f \neq g$ we have that putting $\gamma = \min\{\delta < \kappa : f(\delta) \neq g(\delta)\}$, $C^f_{\gamma+1} \cap C^g_{\gamma+1} = \emptyset$ and so $|B_f \cap B_g| < \lambda$.

Now, fix two disjoint dense subsets X and X' of 2^κ. Take $Y = \bigcup_{f \in X} B_f$ and $Y' = \bigcup_{f \in X'} B_f$, note that $Y \cap Y' = \emptyset$. Finally, we claim that $\mathcal{A} \cup \{Y\}$ is κ-independent: for this, it is enough to show that both Y and Y' have unbounded intersection (on λ) with each member of $\mathcal{X}$. We write the argument for Y (for Y' the argument is analogous).

Take $h \in \mathrm{BF}_\kappa(\mathcal{A})$ and put $J_i = \{A \in \mathrm{dom}(h) : h(A) = i\}$ for $i \in \{0,1\}$. By density of X we can find $h' \supseteq h$ such that $h' \in X$, then we have the following:

$$\mathcal{A}^h \cap Y \supseteq \mathcal{A}^{h'} \cap Y = (\mathcal{D}^g \cap (\mathcal{A}')^{g'}) \cap Y = (\mathcal{D}^g \cap (\mathcal{A}')^{g'}) \cap \bigcup_{f \in X} B_f$$

where $g = h' \restriction_{\mathcal{D}}$ and $g' = h' \restriction_{\mathcal{A}'}$, also notice that $\mathcal{D}^g \supseteq C^F_\alpha$ for $\alpha = \min\{\delta : A_\delta \notin \mathrm{dom}(g)\}$ and $F = g \cup 0 \restriction_{\kappa \setminus \mathrm{dom}(g)}$. Hence:

$$\mathcal{A}^h \cap Y \supseteq (C^F_\alpha \cap (\mathcal{A}')^{g'}) \cap \bigcup_{f \in X} B_f \mathbin{^*\supseteq} (B_F \cap (\mathcal{A}')^{g'}) \cap \bigcup_{f \in X} B_f = (B_F \cap \bigcup_{f \in X} B_f) \cap (\mathcal{A}')^{g'}.$$

and the latter set satisfies:

$$\begin{aligned}(B_F \cap \bigcup_{f \in X} B_f) \cap (\mathcal{A}')^{g'} &= (B_F \cup \bigcup_{f \in X, f \neq F} (B_f \cap B_F)) \cap (\mathcal{A}')^{g'} \\ &= (B_F \cap (\mathcal{A}')^{g'}) \cup (\bigcup_{f \in X, f \neq F} (B_f \cap B_F)) \cap (\mathcal{A}')^{g'})\end{aligned}$$

To end, just notice that $B_F \cap (\mathcal{A}')^{g'}$ is unbounded by hypothesis. $\square$

An important result regarding the dominating number at singulars due to Saharon Shelah is presented now:

Theorem 28 (Shelah, see [GGS20])**.**

- *If λ is strong limit singular, then $\mathfrak{d}(\lambda) = 2^\lambda$.*
- *If λ is singular and for each $\alpha < \lambda$ we have that $|\alpha|^{\mathrm{cf}(\lambda)} < \lambda$, then $\mathfrak{d}(\lambda) = 2^\lambda = \mathfrak{e}(\lambda)$.*

The results above imply the following:

Corollary 29. Let λ be a strong limit cardinal for which $\gamma < 2^\lambda$ implies $\gamma^{<\kappa} < 2^\lambda$, then $\mathfrak{i}(\lambda) = 2^\lambda$.

We conjecture that for all λ singular strong limit it is the case that $\mathfrak{i}(\lambda) = 2^\lambda$. Unfortunately, we do not have full proof of this fact yet.

The following section aims to go for an approximation of this problem. The concept of regular ultrafilters was introduced by Ketonen in [Ket72] and it turns out that the existence of such ultrafilters is crucial when proving that if κ is a strongly compact cardinal, then the singular cardinal hypothesis (SCH) holds for every singular strong limit of cofinality $\geq \kappa$ above κ. We introduce the concept of regular maximal independent families and prove that for these it is the case that their sizes can be maximal (i.e. 2^λ).

6.1. **Regular ultrafilters and maximal independent families.** First, we introduce some preliminaries on the theory of regular ultrafilters, we refer the reader to [Ket72] for an extensive reference on this subject.

Definition 30. Let κ, λ, μ be cardinals so that $\kappa \leq \mu$. An ultrafilter $\mathcal{D}$ over λ is (κ, μ)*-regular* if there is a family $\{X_\alpha : \alpha < \mu\}$ of elements of $\mathcal{D}$ so that every subset S of μ of cardinality κ, $\bigcap_{\alpha\in S} X_\alpha = \emptyset$. We call the sequence $\{X_\alpha : \alpha < \mu\}$ a *regularizing sequence for* $\mathcal{U}$.

Classical results:

Theorem 31. *If $\kappa > \omega$ is a regular cardinal, then κ is strongly compact if and only if for every $\lambda > \kappa$ there is a κ-complete, (κ, λ)-regular ultrafilter.*

Theorem 32 (Ketonen)**.** *If $\kappa > \omega$ is strongly compact, then for every regular $\lambda > \kappa$ there is a uniform κ-complete, (κ, λ)-regular ultrafilter over λ.*

Proposition 33. Any κ-complete, $(\kappa, \lambda)-$regular ultrafilter is $(\kappa, \lambda^{<\kappa})$-regular.

Proof. Just notice that if $\{X_\alpha : \alpha < \lambda\}$ is a family witnessing that an ultrafilter $\mathcal{D}$ is (κ, λ)-regular, then the family $\{\bigcap_{\beta\in T} X_\beta : T \subseteq \lambda \wedge |T|< \kappa\}$ witnesses that $\mathcal{D}$ is $(\kappa, \lambda^{<\kappa})$-regular. □

Theorem 34 (Solovay)**.** *If $\lambda > \kappa$ is a regular and κ is strongly compact then $\lambda^{<\kappa} = \lambda$. Hence, the singular cardinal hypothesis holds at any singular strong limit cardinal $> \kappa$ of cofinality κ.*

Proof sketch: Assume that $\lambda \geq \kappa$ is a strong limit singular cardinal. If $\mathrm{cf}(\lambda) < \kappa$, $2^\lambda = \lambda^{\mathrm{cf}(\lambda)} \leq (\lambda^+)^{\mathrm{cf}(\lambda)} = \lambda^+$.

If $\mathrm{cf}(\lambda) \geq \kappa$ the set $S = \{\alpha < \lambda : \alpha$ is a singular strong limit cardinal of cofinality $< \kappa\}$ is a stationary subset of λ and $\alpha \in S$ implies that $2^\alpha = \alpha^+$. Now, Silver's result states that if μ is a singular cardinal of uncountable cofinality so that those $\alpha < \mu$ with $2^\mu = \mu^+$ forms a stationary subset of μ, then $2^\mu = \mu^+$. Thus $2^\lambda = \lambda^+$. □

Now, we want to use both Theorems above together with the theory of maximal κ-independent families. We first define a special type of a maximal independent family at a given cardinal χ.

Definition 35. Let $\mathcal{A}$ be a maximal κ-independent family at some cardinal χ, we say that $\mathcal{A}$ is regular if for every $g \in \mathrm{BF}_{\kappa^+}(\mathcal{A})$ with $|\mathrm{dom}(g)|= \kappa$, the corresponding Boolean combination $\mathcal{A}^g$ is empty.

Proposition 36. Suppose that $\mathcal{A}$ is a maximal κ-independent family of subsets of χ, then the filter associated to $\mathcal{A}$, $\mathcal{F}_\mathcal{A}$ is a (κ, κ^+)-regular filter.

Proof. Recall that $\mathcal{F}(\mathcal{A}) = \{X \subseteq \chi : \forall h \exists h' \supseteq h(\mathcal{A}^{h'} \subseteq^* X)\}$. Giving $g \in \mathrm{BF}_{\kappa^+}(\mathcal{A})$ with $|\mathrm{dom}(g)|= \kappa$ then, once we eliminate possible repetitions the sets $B_\alpha = \{\mathcal{A}^{g\restriction\alpha} : g \in \mathrm{BF}_{\kappa^+}(\mathcal{A}) \wedge |\mathrm{dom}(g)|= \kappa\}$ correspond to maximal antichains in the set $\mathrm{BF}_\kappa(\mathcal{A})$ for each $\alpha < \kappa$.

Put $X_\alpha = \bigcup\{\mathcal{A}^h : h \in B_\alpha\}$, then every intersection of κ-many sets in the family $\{X_\alpha : \alpha < \kappa\}$ is empty. Indeed, if $Z \subseteq \kappa$ is unbounded in κ then:

$$\begin{aligned}\bigcap_{\alpha\in Z} X_\alpha = \bigcap_{\alpha\in Z} X_\alpha = \bigcap_{\alpha\in Z} (\bigcup\{\mathcal{A}^h : h \in B_\alpha\}) \\ \subseteq \bigcap\{\mathcal{A}^g : g \in \mathrm{BF}_{\kappa^+}(\mathcal{A}) \wedge |\mathrm{dom}(g)|= \kappa \wedge g \restriction \alpha \in B_\alpha \text{ for some } \alpha \in Z\}\end{aligned}$$

which is an empty set, by the regularity of the family $\mathcal{A}$. □

Lemma 37. Suppose κ is a regular cardinal, $2^{<\kappa} = \kappa$, $\kappa \leq \chi$ and $\mathcal{I}$ is a κ^+-saturated χ-complete ideal over χ such that:

(1) $\mathcal{B}(\mathrm{Fn}_\kappa(2^\chi, 2))$ is isomorphic to $\mathcal{P}(\chi)/\mathcal{I}$.
(2) The dual filter $\mathcal{F}_\mathcal{I}$ associated to $\mathcal{I}$ is (κ, κ^+)-regular.

Then, there is a maximal κ-independent family of subsets of χ, that is regular.

Proof. This is a refinement of Lemma 6. Recall that from the isomorphism $\varphi : \mathcal{B}(\mathrm{Fn}_\kappa(2^\chi, 2)) \to \mathcal{P}(\chi)/\mathcal{I}$, one can define for each $\delta < 2^\chi$, the set $A_\delta = \varphi(\{(\delta, 0)\})$. Then if we enumerate the ideal $\mathcal{I}$ as $\{I_\alpha : \alpha < 2^\chi\}$ so that each element appears at least κ-many times then the family $\mathcal{A} = \{A'_\delta : \delta < 2^\chi\}$ is maximal κ-independent where $A'_\delta = A_\delta \setminus I_\delta$.

Now, it is left to prove that $\mathcal{A}$ is regular: Let $g \in \mathrm{BF}_{\kappa^+}(\mathcal{A})$ such that $|\mathrm{dom}(g)| = \kappa$. First, since $g \in \mathcal{B}(\mathrm{Fn}_\kappa(2^\chi, 2))$ we get $\varphi(g) = \bigcap_{\alpha<\kappa} \varphi(g \restriction \alpha)$. Let $\{X_\alpha : \alpha < \kappa^+\}$ be a family of subsets of χ witnessing the regularity of the associated filter $\mathcal{F}_\mathcal{I}$.

Put $\mathrm{dom}(g) = \{B_\alpha : \alpha < \kappa\} \cup \{C_\alpha : \alpha < \kappa\}$ where $g(B_\alpha) = 0$ and $g(C_\alpha) = 1$ for all $\alpha < \kappa$. Note that, from the construction of the family $\mathcal{A}$, all its elements are $\mathcal{I}$ positive.

We consider the following cases: The easiest corresponds to the situation in which there is a set $Z \subseteq \kappa$ unbounded on κ such that for all $\beta \in Z$, $\mathcal{A}^{g\restriction\beta} \subseteq^* X_{\gamma(\beta)}$. Then, we can use the regularity of $\mathcal{F}_\mathcal{I}$ to get that

$$\mathcal{A}^g \subseteq^* \bigcap_{\beta \in Z} X_\beta = \emptyset$$

So, let's suppose that there is a $\beta^* < \kappa$ such that for all $\alpha \geq \beta$ such that $|\mathcal{A}^{g\restriction\alpha} \setminus X_\delta| = \chi$ for all $\delta \in W \subseteq \kappa$ with $|W| = \kappa$. Now, since $X_\delta \in \mathcal{F}_\mathcal{I}$ for each δ, given the partial function $g \restriction_{\beta^*} \in \mathrm{BF}_\kappa(\mathcal{A})$ there exists $h \supseteq g \restriction_{\beta^*}$ such that $\mathcal{A}^{g\restriction_{\beta^*}} \subseteq^* X_\delta$, this implies that $h \perp g \restriction_{\beta^*}$ for all $\alpha \geq \beta^*$.

Also, if $Y = \mathcal{A}^g \notin \mathcal{I}$ recall that we may assume that the family $\mathcal{A}$ is dense maximal and so, for the pair $\mathcal{A}^g$ and $h \in \mathrm{BF}_\kappa(\mathcal{A})$ there exists $h' \supseteq h$ such that either $\mathcal{A}^{h'} \cap Y$ or $\mathcal{A}^{h'} \setminus Y$ is bounded. Note that the latter must happen because of our assumption $Y \in \mathcal{A}$. Thus, $\mathcal{A}^{h'} \setminus Y$ is bounded, so $\mathcal{A}^{h'} \subseteq^* \mathcal{A}^g$ and so g and h' are compatible and so $h' \subseteq g \restriction_\delta$ for some $\delta \geq \beta^*$.

□

Let κ be a strongly compact cardinal in a model where CH holds. Let $\mathcal{U}$ be a κ-complete ultrafilter on κ which is (κ, κ^+)-regular. Similarly as in the proof of Theorem 7 consider the poset $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\kappa, 2)$, then in the generic extension $V[G]$ there is a maximal independent family $\mathcal{A}_G$ of subsets of ω_1 Also, the V-ultrafilter $\mathcal{U}^*$ generated by $\mathcal{U}$ in the generic extension $V[G]$ is still (κ, κ^+)-regular (the same regularizing sequence works). Let $\{Y_\alpha : \alpha < \kappa^+\}$ be such a sequence.

Corollary 38. The maximal κ-independent family $\mathcal{A}_G$ in the extension $V[G]$ is regular.

Proof. Let $g \in \mathrm{BF}_{\kappa^+}(\mathcal{A})$ and recall that we have an isomorphism Γ between the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{I}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2))$. Hence, given that $g \in \mathcal{B}(\mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2))$ because $g = \bigcup_{\alpha<\kappa} g \restriction \alpha$, we have that $\Gamma(g) = \bigwedge \Gamma(g \restriction \alpha)$.

Suppose first that $\mathcal{A}^g \notin \mathcal{J}$, then for all $Y \in \mathcal{U}^*$, $\mathcal{A}^g \cap Y \neq \emptyset$. In particular, for the set of Y_α's witnessing regularity, we have $\mathcal{A}^g \cap Y_\alpha \neq \emptyset$. Now, for a particular $\alpha < \kappa^+$, $\mathcal{A}^g = (\mathcal{A}^g \cap Y_\alpha) \cup (\mathcal{A}^g \setminus Y_\alpha)$ and the latter set belongs to $\mathcal{J}$ and so $[\mathcal{A}^g] = [\mathcal{A}^g \cap Y_\alpha]$, but since $\bigcap_{\alpha<\kappa} Y_\alpha = \emptyset$, so $\mathcal{A}^g \in \mathcal{I}$.

□

From the arguments above we can conclude that regular maximal independent families (in the presence of a strongly compact cardinal) have maximal size. So, the following question arises:

Question 39. Are all maximal κ-independent families regular, or in other words: Given an uncountable cardinal κ, is there a cardinal χ such that there are no regular maximal κ-independent families of subsets of χ?

The existence of a strongly compact cardinal implies that the SCH holds above κ for all cardinals of cofinality at most κ. So the following question follows:

Question 40. Let κ be an uncountable cardinal and assume there exists a maximal κ independent family of subsets of κ. Does this imply SCH above κ in an analogous way as above?

7. More questions and final discussion

We finish this paper with a last discussion on the possible future lines of research in this topic by mentioning some open questions left after the research done in this article. We mention some open questions regarding the regular case:

Question 41. Let χ and κ be regular cardinals, is there a full characterization of the spectrum of maximal κ-independent families at χ? The work of Eskew and Fischer in [EF21] provides many results in this area but a full characterization is still missing.

More specifically, if A is a set of cardinals, is there a generic extension of the universe for which there are maximal κ-independent families at χ for all $\chi \in A$ and none of them for all $\chi \notin A$?

Question 42. Given a singular strong limit cardinal λ, is it the case that $\mathfrak{i}(\lambda) = 2^\lambda$?

Question 43. Given a maximal κ-independent family $\mathcal{A}$ at a singular strong limit cardinal, is there a family $\mathcal{A}'$ (most probably depending on $\mathcal{A}$) that is regular?

Question 44. Looking at the construction of the maximal δ-independent family at a singular from Theorem 15 one has that, regardless of the sizes of the intermediate families $\mathcal{A}_\alpha$ for $\alpha < \kappa$, the size of the final family $\mathcal{B}$ is always 2^λ. Is it possible to refine the family $\mathcal{B}$ in such a way that its size is dependent on the size of the intermediate families?

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