Noise Parameter Estimation From Quantized Data

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***Abstract*—In this paper, the parametric estimation of the variance of white Gaussian noise is considered when available data are obtained from a quantized noisy stimulus. The Cramér–Rao lower bound is derived, and the statistical efficiency of a maximum-likelihood parametric estimator is discussed, along with the estimation algorithm proposed in IEEE Standard 1241.**

***Index Terms*—Analog-to-digital conversion (ADC), Cramér–Rao lower bound (CRLB), maximum likelihood, noise parametric estimation, statistical efficiency.**

I. Introduction

TESTING activities are currently an increasingly important part of modern manufacturing processes, taking up as much as 50% of the cost of electronic device production. Thus, several research and standardization activities have been dedicated to this issue. Particularly important is the test of analog-to-digital converters (ADCs) because of their wide usage in various technical fields [1], [2]. Various techniques are available based on the usage of different stimuli for both static and dynamic ADC characterizations. It is worthy of notice that the used testing signals are usually not fully known in advance, and their parameters are estimated using the ADC output codes before applying the ADC testing algorithm. Moreover, the stimuli are often affected by random noise, which is frequently modeled as additive white Gaussian noise (AWGN) [3]. Consequently, ADC testing may be strongly affected by noise, which can reduce the accuracy of both the preliminary stimulus parametric estimation and the following ADC characterization. In previous works, the authors have analyzed the statistical efficiency of the four-parameter sine-wave fitting, modeling and describing the Cramér–Rao lower bound (CRLB) on the sine-wave parametric estimation from quantized data [4]–[7]. In this paper, a similar approach is initially followed, and the CRLB that is related to the estimation of noise parameters is evaluated. Notice that when a measurement is carried out, a model is implicitly assumed. Within this model, the CRLB, as a lower bound on the estimator variance, implies the existence of a lower bound on measurement uncertainty as well, which is defined as the standard deviation associated with measurement results. The results are then used to analyze and compare the statistical efficiency of both the algorithm proposed in [2] to assess the noise variance and of a new algorithm implementing the maximum-likelihood estimator (MLE). By extending the results presented in [8], a procedure has been considered based on the acquisition of a data record, whose quantized codes are used to estimate the stimulus parameters, the ADC transition levels, and, finally, the power of the noise superimposed to the stimulus.

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II. CRLB on Noise Estimators

To analyze the statistical efficiency of the considered estimation algorithms, the CRLB on the estimator of the noise standard deviation has been evaluated using a direct discrete analysis [6], [7]. Notice that both noise variance estimators considered in this paper require the collection of two records $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$ of ADC output codes, which are obtained by feeding twice the ADC with a period of a triangular signal $s[\cdot]$, under the same trigger conditions, according to the following:

$$\begin{cases} s_{q1}[n] = Q\left(s[n] + w_1[n]\right) \\ s_{q2}[n] = Q\left(s[n] + w_2[n]\right) \end{cases}, \qquad n = 0, \ldots, N-1 \tag{1}$$

where $w_1[\cdot]$ and $w_2[\cdot]$ are two realizations of an AWGN process with variance σ^2, $Q(\cdot)$ is the quantization law corresponding to the adopted ADC, and N is the record length.

Then, the sequence $x_q[\cdot] = s_{q2}[\cdot] - s_{q1}[\cdot]$ is used to estimate the AWGN variance σ^2. Accordingly, any unknown and unwanted deterministic component in $s[\cdot]$ tends to be canceled out, and the distortion it may introduce on the estimates is reduced. As all of the collected ADC samples of both $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$ are statistically independent, the Fisher information $I_\sigma(\sigma)$ on the AWGN standard deviation σ can be obtained as the summation of Fisher information $I_\sigma(n,\sigma)$ that is associated to each sample $x_q[n]$, i.e.,

$$\begin{aligned} I_\sigma(\sigma) &= \sum_{n=0}^{N-1} I_n(n,\sigma) \\ I_\sigma(n,\sigma) &= \sum_{m=-M+1}^{M-1} \frac{1}{P(x_q[n]=m;\sigma)} \left(\frac{\partial P(x_q[n]=m;\sigma)}{\partial \sigma} \right)^2 \end{aligned} \tag{2}$$

where $M = 2^b$ is the number of output codes of a b-bit ADC converter, and $P(x_q[n]=m;\sigma)$ is the probability that the nth sample of the sequence $x_q[\cdot]$ equals m. Then, by properly expressing $P(x_q[n]=m;\sigma)$ as a function of the probability of occurrence of the ADC output codes $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$, the CRLB on estimators of σ is obtained (see the Appendix) as

$$\mathrm{CRLB}_\sigma(\sigma) = \frac{1}{I_\sigma(\sigma)}. \tag{3}$$

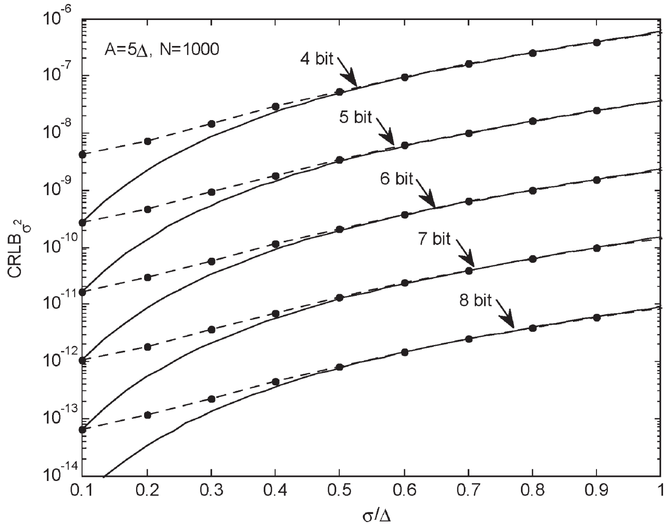

Fig. 1. CRLB on the estimator variance σ^2 as a function of σ/Δ. The continuous lines have been obtained using the direct discrete analysis, whereas the dotted lines have been obtained using (7).

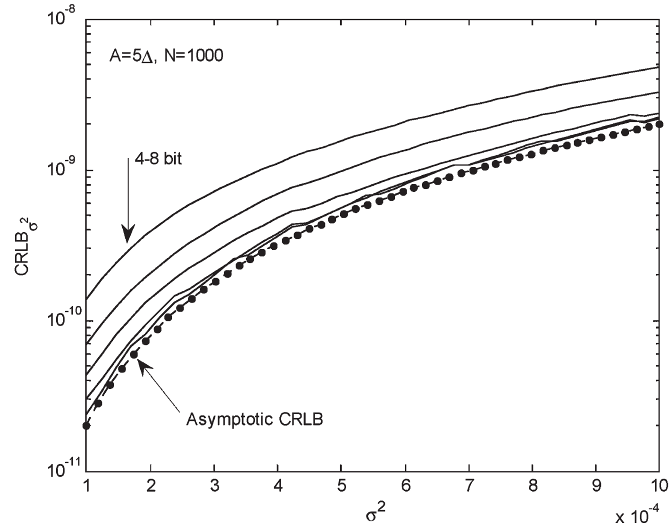

Fig. 2. CRLB on the estimator variance σ^2 as a function of σ^2. The continuous lines have been obtained using the direct discrete analysis, whereas the dotted line is the asymptotic model obtained for an infinite resolution ADC.

Notice that the CRLB on the noise variance σ^2 can be obtained from the CRLB on σ using

$$\mathrm{CRLB}_{\sigma^2}(\sigma) = \left(\frac{dg(\sigma)}{d\sigma}\right)^2 \mathrm{CRLB}_{\sigma}(\sigma), \qquad \sigma > 0 \quad (4)$$

where $g(\sigma) = \sigma^2$ [4]. It should be observed that according to [9], measurement uncertainty is expressed as the standard deviation of the estimator, which is the square root of the estimator variance. Consequently, when a parameter α is to be measured, a lower bound on the estimator uncertainty $u_{C\alpha}(\alpha)$ may be obtained as follows:

$$u_{C\alpha}(\alpha) \geq \sqrt{\mathrm{CRLB}_{\alpha}}. \quad (5)$$

Fig. 1 reports the CRLB on estimators of the noise variance that is obtained for various resolutions, assuming an ideal uniform ADC with unitary full scale FS as a function of the ratio σ/Δ, where $\Delta = 2FS/2^b$ is the ADC quantization step. All of the reported curves have been obtained by considering as a stimulus one period of a zero-mean triangular wave of peak amplitude $A = 5\Delta$ and unitary period T, which is acquired with a sampling period $T_C = T/N$, where $N = 1000$. It can be observed that the CRLB grows with σ and reduces when the ADC resolution is increased. Notice that an asymptotic bound to the CRLB when ADC resolution is increased is given by the CRLB obtained for a nonquantized signal affected by AWGN alone, because in such a case, the effect of quantization tends to be negligible, and (4) becomes [4]

$$\mathrm{CRLB}_{\sigma^2}(\sigma) = \frac{2\sigma^4}{N}. \quad (6)$$

Moreover, provided that the AWGN power is much higher than the quantization noise power, a simplified formula describing the effect of quantization on the CRLB for high ADC resolutions may be obtained by replacing σ^2 with $\sigma^2 + \Delta^2/12$ so that (4) is almost equal to [7], [10]

$$\mathrm{CRLB}_{\sigma^2}(\sigma) \cong \frac{2\left(\sigma^2 + \frac{\Delta^2}{12}\right)^2}{N}, \qquad \sigma > 0.3\Delta \quad (7)$$

and the corresponding lower bound on the estimator uncertainty may be obtained as the square root of (7). Fig. 1 also reports, for each considered resolution, the corresponding curves obtained using (7). It can be observed that provided that σ is not significantly lower than Δ, (7) accurately describes the CRLB. In particular, for $\sigma > 0.3\Delta$, (7) is very close to (2), and for $\sigma > 0.5\Delta$, the two models give nearly identical results.

Fig. 2 reports the CRLB on σ^2 as a function of σ^2 for various ADC resolutions. A curve obtained using (2) and (4) is reported for each resolution. It can be observed that for high ADC resolutions, the CRLB tends to the asymptotic curve (6), as also reported in Fig. 2. Equation (7) also shows that the CRLB tends to zero like $1/N$. This may be justified by observing that with the triangular-wave stimulus, the number of samples in $s[\cdot]$ belonging to each ADC code bin is proportional to N. Furthermore, (7) may be rewritten as a function of σ/Δ, obtaining

$$\mathrm{CRLB}_{\sigma^2}\left(\frac{\sigma}{\Delta}\right) \cong \frac{2FS^4}{9N2^{4b}}\left(12\left(\frac{\sigma}{\Delta}\right)^2 + 1\right)^2, \qquad \sigma > 0.3\Delta \quad (8)$$

which shows that provided that Gaussian noise power does not dominate quantization error power, the curves in Fig. 2 tend to be scaled replicas of each other and that the CRLB is reduced by a factor of 2^4 for each bit of increased resolution. Such a behavior has also been verified using the direct discrete analysis for the considered ADC resolution and σ/Δ range.

Finally, it is worthy of notice that if an estimation were carried out using directly the ADC output codes, the CRLB would be lower than that evaluated using $x_q[\cdot]$ as a data source

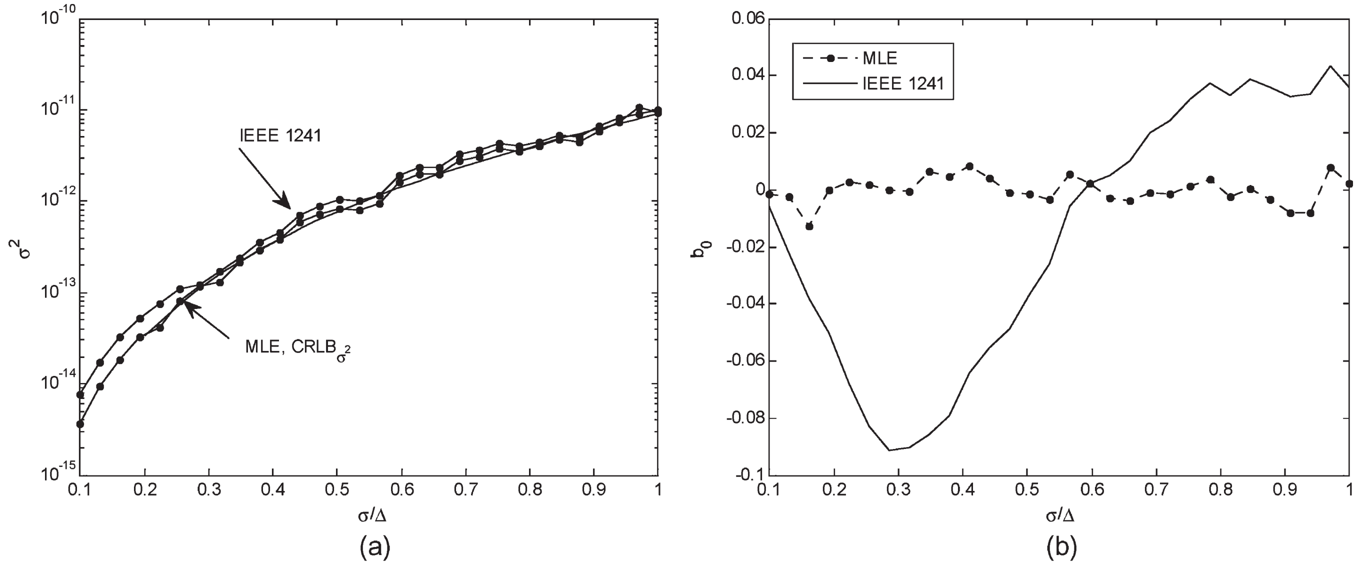

Fig. 3. (a) Variance of (solid curve) IEEE 1241 estimator and (dotted curve) MLE. The continuous curve reports the corresponding CRLB. (b) Normalized bias of (solid curve) IEEE 1241 estimator and (dotted curve) MLE.

because $2N$ samples would be available rather than N. In particular, it can be shown that for high ADC resolutions, a halved asymptotic CRLB holds with respect to (6) and (7) [4]. Thus, using $x_q[\cdot]$ as a data source may reduce the amount of available information and the achievable statistical efficiency but at a price of a reduced estimator accuracy when the stimulus is affected by unknown systematical effects.

III. IEEE 1241 and Maximum-Likelihood Noise Variance Estimators

A. IEEE 1241 Estimator

The IEEE Standard 1241 proposes three techniques to estimate the variance of the overall AWGN adding to the desired stimulus; two of them are reported here for the sake of readability. The first one requires feeding of the ADC with a constant signal and the collection of two records of ADC output codes, both of size N. Then, the AWGN variance is estimated as

$$\hat{\sigma}_1^2 = \frac{1}{2N}\sum_{n=0}^{N-1}\left(s_{q2}[n]-s_{q1}[n]\right)^2 = \frac{1}{2N}\sum_{n=0}^{N-1}\left(x_q[n]\right)^2. \quad (9)$$

The second technique requires feeding the ADC twice with a period of a triangular wave, spanning over a few ADC bins, and keeping the same synchronization conditions. Then, the ADC mse and variance are obtained from

$$\hat{\sigma}_2^2 = \left[\left(\frac{mse}{2}\right)^{-2} + \left(\frac{0.886 mse}{\Delta}\right)^{-4}\right]^{-\frac{1}{2}} \quad (10)$$

where

$$mse = \frac{1}{N}\sum_{n=0}^{N-1}\left(s_{q2}[n]-s_{q1}[n]\right)^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left(x_q[n]\right)^2.$$

Both techniques rely on subtraction of corresponding ADC output codes to remove deterministic effects. Fig. 3(a) and (b), which is obtained throughout meaningful Monte Carlo simulations for an 8-bit uniform ADC with $FS = 1$ and fed with a triangular wave with $A = 5\Delta$, reports the estimator variance and bias, respectively, as a function of the ratio σ/Δ, where the normalized bias b_0 is defined as

$$b_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\sigma^2}. \quad (11)$$

It can be observed that the estimator variance grows with σ/Δ and that, at least in the observed σ/Δ interval, the estimator appears to be biased.

B. MLE

The MLE proposed in this paper is obtained by maximizing the likelihood function with respect to the noise variance, which is the probability density function of the difference between each code pair and seen as a function of the considered parameter, which, in this case, is given by

$$P(x_q = x_{0q};\sigma) = \prod_{n=0}^{N-1} P\left(x_q[n] = x_{0q}[n];\sigma\right)$$
$$x_q = \{x_q[0],\ldots,x_q[N-1]\}$$
$$x_{0q} = \{x_{0q}[0],\ldots,x_{0q}[N-1]\} \quad (12)$$

where $x_{0q}[\cdot]$ is the collected record of code bin differences. This estimator, when existing, is guaranteed to be asymptotically unbiased and efficient [4], [5]. Fig. 3(a) and (b), which is obtained by assuming an 8-bit ADC and $N = 1000$, also reports the MLE variance and bias for a test based on the same triangular input signal, where the likelihood function has been maximized using numerical techniques. It can be observed that this estimator, in the considered σ/Δ range, appears to be slightly superior to the IEEE 1241 one, showing smaller variance and bias at the price of a higher algorithm convergence time.

To assess the statistical efficiency of both considered estimators, Fig. 3(a) also reports the corresponding CRLB. Notice that while the IEEE 1241 estimator variance is reasonably close to the CRLB, the MLE appears to be superior, as its variance coincides with the corresponding CRLB. Notice that the accuracy of the minimization of (12) may be strongly affected by both the adopted numerical technique and the algorithm initial conditions. In particular, Fig. 3(a) and (b) has been obtained by using the Nelder–Mead algorithm and, to improve the convergence properties, the results of the IEEE 1241 estimator as the initial condition. Thus, the proposed approach appears to be capable of removing the bias and reducing the variance of the IEEE 1241 estimator. Notice that removing the noise variance estimator bias is of great importance for testing ADC linearity, because the knowledge of such parameter is required to chose the proper ADC overdrive [2], [11]. In fact, a negative estimator bias may lead to insufficient overdriving, resulting in a biased ADC transition-level estimation. Conversely, a positive bias may lead to an excessively high overdriving, leading to deep ADC overloading and an inadequate estimation of the stimulus amplitude and offset [7].

It is worthy of notice that the MLE requires the knowledge of both the parameters of the deterministic stimulus and the ADC transition levels (see the Appendix). As these parameters are usually not known in advance, they have to be measured. Consequently, the uncertainty associated with the measurement results may potentially reduce the MLE accuracy. Thus, to further develop the analysis, a more realistic procedure has been considered, where the acquired ADC codes of the noisy stimulus are sequentially used to estimate the triangular-wave peak value, the ADC transition levels, and, finally, the noise variance by using both the MLE and the IEEE 1241 estimator.

In particular, the considered ADC stimulus can be expressed as

$$\begin{aligned} s[n] &= As_0[n] + d \\ s_0[n] &= s_0(nT_c) \\ s_0(t) &= 1 - \left|\frac{4}{T}\left(t - \frac{T}{2}\right)\right|, \qquad t \in [0, \ldots, T] \end{aligned} \tag{13}$$

where $s_0(\cdot)$ is a noiseless zero-mean triangular signal with the same period T and initial phase of the acquired signal and unitary peak value. Thus, $s[\cdot]$ can be rewritten as a linear model, relating the peak amplitude A and the offset d to $s_0[\cdot]$, i.e.,

$$\begin{aligned} s &= H\theta \\ \theta &= [A, d]^T \\ H &= \left[s_0, u(N)^T\right] \\ s &= \{s[0], s[1], \ldots, s[N-1]\}^T \\ s_0 &= \{s_0[0], s_0[1], \ldots, s_0[N-1]\}^T \end{aligned} \tag{14}$$

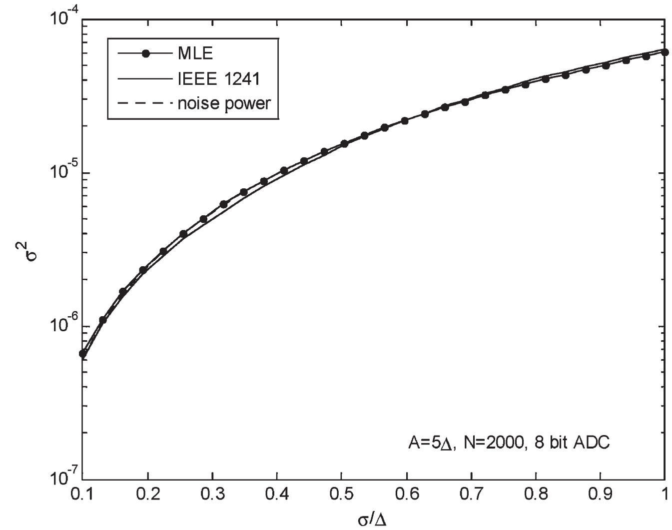

Fig. 4. Noise variance that is measured using both the MLE and the IEEE 1241 estimator for records of $N = 2000$ samples when the MLE is evaluated after a preliminary estimation of both the noiseless stimulus and the ADC transition levels.

where H is an $N \times 2$ matrix, θ is a 2×1 matrix, $u(N)$ is a unit row vector of length N, and s is an N-length column vector containing the noiseless triangular signal samples. Thus, the acquired signal $s_q[\cdot]$ is obtained as

$$\begin{aligned} s_q &= H\theta + \eta \\ s_q &= \{s_q[0], s_q[1], \ldots, s_q[N-1]\}^T \\ \eta &= \{\eta[0], \eta[1], \ldots, \eta[N-1]\}^T \end{aligned} \tag{15}$$

where η is a noise vector, keeping in account both the AWGN and the quantization error. Under such conditions, and by ignoring the correlation between the noise samples, A and d can be obtained using a linear estimator, which is expressed as [4]

$$\begin{aligned} \hat{\theta} &= (H^T H)^{-1} H^T s_q \\ \hat{\theta} &= [\hat{A}, \hat{d}]^T \end{aligned} \tag{16}$$

where H^T is the transpose of the matrix H, and the vector parameter to be estimated, i.e., $\hat{\theta}$, comprises the estimated peak amplitude $\hat{A}$ and offset $\hat{d}$ of the triangular stimulus. Notice that when η can be modeled as an AWGN, the linear estimator is guaranteed to be both unbiased and efficient [4]. As shown in Section II, such a condition is satisfied when $\sigma/\Delta > 0.3$. Finally, the ADC thresholds have been estimated using a triangular histogram test, and the obtained results have been used to feed the MLE.

Figs. 4 and 5 report simulation results that are obtained from two records of $N = 2000$ samples, each of a triangular wave with $A = 2.5\Delta$ and $T = 1$, affected by AWGN, and coherently sampled with period $T_C = T/N$ by an 8-bit ADC. Two curves report the estimated noise variance as a function of σ/Δ: one for the MLE and the other for the IEEE 1241 estimator. For comparison purposes, the actual noise variance is also reported. It can be observed that the MLE is still less biased than the IEEE 1241 estimator, even if it relies on a preliminary estimation of

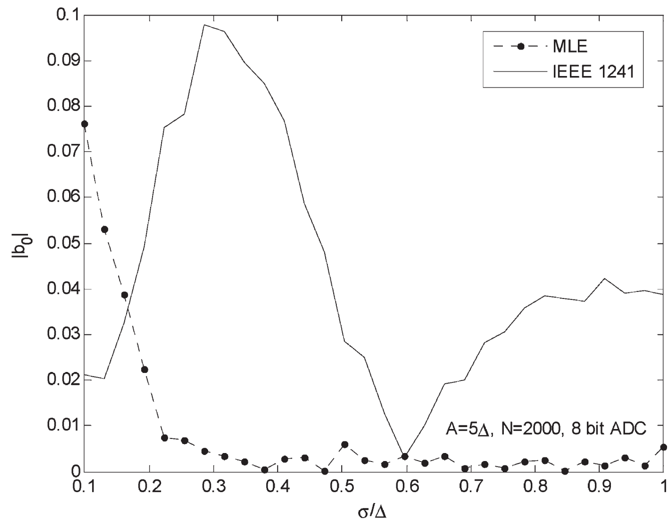

Fig. 5. Normalized bias of (solid curve) IEEE 1241 estimator and (dotted curve) MLE for records of $N = 2000$ samples when the MLE is evaluated after a preliminary estimation of both the noiseless stimulus and the ADC transition levels.

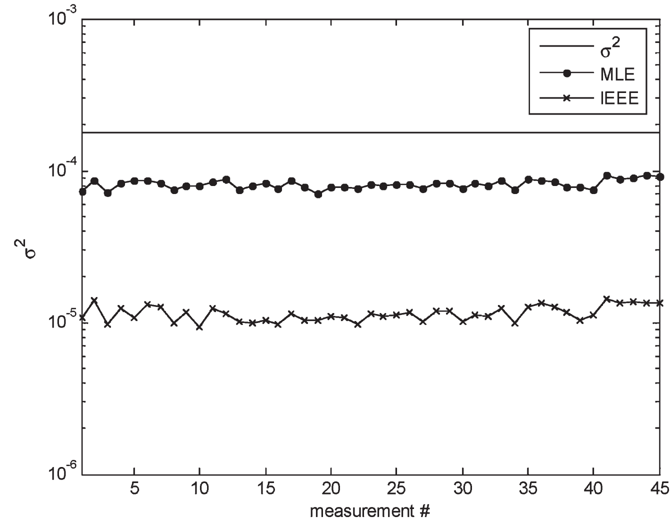

Fig. 6. Noise variance that is measured using both the MLE and the IEEE 1241 estimator.

both A and the ADC quantizer thresholds. This can be better seen in Fig. 5, which shows that the relative bias (11) obtained for the MLE, which, for $\sigma/\Delta > 0.3$, is much lower than that obtained from the IEEE 1241 test. Moreover, it can be observed that the MLE is scarcely affected by the inaccuracies of the joint estimation of both the stimulus parameters and ADC transition levels, except for very low values of σ/Δ. In this case, the MLE accuracy may be reduced because of the inaccuracy introduced by the estimation of both the stimulus and the ADC transition levels.

IV. Experimental Verification

To validate the proposed methodology, the performances of both the MLE and the IEEE 1241 estimator have been compared. In particular, the noisy stimulus has been obtained using two function generators: One is used to generate the triangular wave, and the other is used to inject arbitrary levels of white noise by means of an analog adder. The obtained signal was acquired using a Texas Instruments DSK6713 digital signal processor board, which is programmed to drive an 8-bit ADC with $FS = 4.098$ V. A set of 45 measurements was carried out. For each measurement, two records of 1000 samples of a noisy triangular wave of amplitude $A = 100$ mV and frequency $F = 1/T = 2.809$ Hz were acquired at a sampling period $T_c = T/1000$. The injected noise standard deviation was set to $\sigma = 0.0134$ V, which is approximately equal to 0.5Δ, and the collected data were used to estimate σ^2, both using (10) and the MLE. Fig. 6 reports the measurement results, together with the actual noise, independently measured in the absence of the triangular stimulus. It can be observed that MLE tends to provide more correct estimates of the noise variance. The results obtained from the 45 measurements have been used to evaluate bias and variance of both estimators. The MLE provides a normalized bias of -0.54 and a variance of $3 \cdot 10^{-11}$, whereas the IEEE 1241 test gives a normalized bias of -0.93

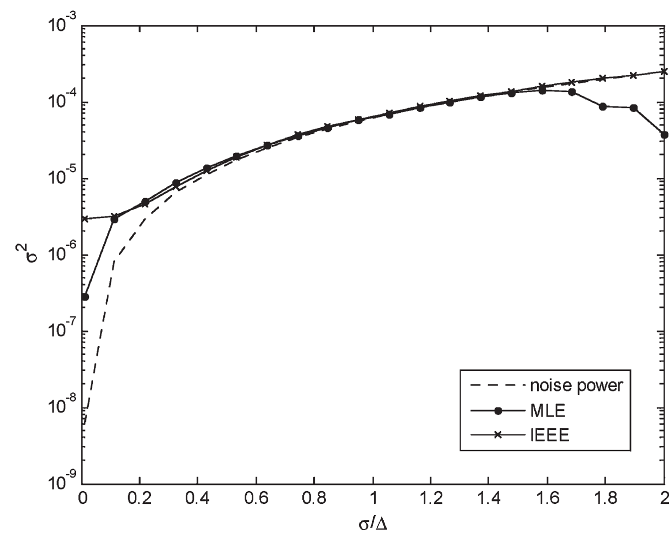

Fig. 7. Noise variance that is estimated with simulations using both the MLE and the IEEE 1241 estimator when the record $s_{q2}[n]$ is acquired after a delay of $0.5T_c$ with respect to the ideal trigger condition.

and a variance of $1.7 \cdot 10^{-12}$. In both cases, the estimator variance is higher than the corresponding CRLB. The MLE bias is possibly due to inaccuracies in the estimation of both the ADC transition levels and the nonquantized values of the input signal acquired samples that are needed to evaluate the likelihood function. Moreover, a further contribution affecting both estimators is the synchronization between the two acquired records, which, in the adopted setup, is affected by a jitter on the sampling trigger upper bounded in magnitude by $T_c/2$.

Notice that the MLE test is potentially more sensitive than the IEEE one to synchronization problems, particularly when the sampling jitter is such that the acquired signal differs from the expected one by multiples of the ADC quantization step Δ. In fact, the code pair probabilities in (12) are evaluated by assuming that the collected records $s_{q2}[\cdot]$ and $s_{q1}[\cdot]$ are perfectly synchronized. Such behavior has been further investigated through meaningful simulations. In particular, Fig. 7

reports simulation results obtained by assuming that the record $s_{q2}[\cdot]$ is acquired with a delay of 0.5 T_c with respect to $s_{q1}[\cdot]$, where the ideal signal $s[\cdot]$ is a triangular wave with unitary peak value and period $T = 1000\ T_c$. Simulations show that the MLE accuracy is still comparable to the one of the IEEE test and is slightly better for low noise levels. Finally, the MLE test requires an accurate characterization of the ADC transition levels, which are required to evaluate the probability of occurrence of the ADC output codes. Thus, provided that a good level of synchronization is ensured and that the ADC is properly characterized, the MLE test is potentially capable of higher performances than the IEEE test.

V. Conclusion

An MLE for the variance of an AWGN has been proposed and compared to the corresponding IEEE 1241 estimator. The statistical efficiency of both noise estimators has been assessed by evaluating the corresponding CRLB both by means of direct discrete analysis and deriving a simple formula that is useful when high-resolution ADCs are used. It is shown that the MLE statistical efficiency may exceed that of the corresponding IEEE-1241-based estimator.

Appendix
CRLB Direct Discrete Analysis

As reported in Section II, the CRLB on the AWGN standard deviation σ may be evaluated by expressing $P(x_q[n] = m; \sigma)$ in terms of $P(s_{q1}[n] = h, s_{q2}[n] = k; \sigma)$, which is the joint probability that $s_{q1}[n] = h$ and $s_{q2}[n] = k$. As the sequences $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$ are statistically independent, $P(s_{q1}[n] = h, s_{q2}[n] = k; \sigma)$ can be factorized into

$$P\left(s_{q1}[n] = h, s_{q2}[n] = k; \sigma\right) = P\left(s_{q1}[n] = h; \sigma\right) P\left(s_{q2}[n] = k; \sigma\right) \quad \text{(A.1)}$$

where $P(s_{q1}[n] = h; \sigma)$ and $P(s_{q2}[n] = k; \sigma)$ are the corresponding marginal probabilities, and

$$P\left\{s_q[n] = i; \sigma\right\} = \begin{cases} \Phi\left(\frac{T_i - s[n]}{\sigma}\right), & i = 0 \\ \Phi\left(\frac{T_i - s[n]}{\sigma}\right) - \Phi\left(\frac{T_{i-1} - s[n]}{\sigma}\right), & 0 < i < M-1 \\ 1 - \Phi\left(\frac{T_{i-1} - s[n]}{\sigma}\right), & i = M-1 \end{cases} \quad \text{(A.2)}$$

where $\Phi(\cdot)$ is the error function [12], which is given by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt. \quad \text{(A.3)}$$

As for different pairs of ADC output codes (h_1, k_1) and (h_2, k_2), the corresponding events $(s_{q1}[n] = h_1, s_{q2}[n] = k_1)$ and $(s_{q1}[n] = h_2, s_{q2}[n] = k_2)$ are mutually exclusive, and $P(x_q[n] = m; \sigma)$ can be written as

$$\begin{aligned} P\left(x_q[n] = m; \sigma\right) &= \sum_{h,k|k-h=m} P\left(s_{q1}[n] = h; \sigma\right) P\left(s_{q2}[n] = k; \sigma\right) \\ & h \in [0, \ldots, M-1]; \quad k \in [0, \ldots, M-1] \\ & -M+1 \le m \le M-1 \end{aligned} \quad \text{(A.4)}$$

where the adopted notation indicates that the summation (A.4) is restricted to the values of h and k such that $k - h = m$, and its derivative with respect to σ is given by

$$\begin{aligned} \frac{\partial P\left(x_q[n] = m; \sigma\right)}{\partial \sigma} &= \sum_{h,k|k-h=m} \frac{\partial\left(P\left(s_{q1}[n] = h; \sigma\right) P\left(s_{q2}[n] = k; \sigma\right)\right)}{\partial \sigma} \\ &= \sum_{h,k|k-h=m} \left[\frac{\partial P\left(s_{q1}[n] = h; \sigma\right)}{\partial \sigma} P\left(s_{q2}[n] = k; \sigma\right) \right. \\ &\quad \left. + \frac{\partial P\left(s_{q2}[n] = k; \sigma\right)}{\partial \sigma} P\left(s_{q1}[n] = h; \sigma\right)\right] \\ & h \in [0, \ldots, M-1]; \quad k \in [0, \ldots, M-1] \\ & -M+1 \le m \le M-1 \end{aligned} \quad \text{(A.5)}$$

where again, the summation (A.5) is restricted to the values of h and k corresponding to code pairs that exhibit a difference equal to m.

Finally, by using (A.4) and (A.5) in (2), the CRLB on σ is obtained.

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