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Noise Parameter Estimation from Quantized Data

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Abstract – *In this paper, the parametric estimation of Additive White Gaussian Noise is considered, when available data are obtained from a quantized noisy stimulus. The Cramér-Rao Lower Bound is derived, and the statistically efficiency of a maximum likelihood parametric estimator is discussed, along with the estimation algorithm proposed in IEEE standard IEEE 1241.*

Keywords – *Noise parametric estimation, statistical efficiency, maximum likelihood*

I. INTRODUCTION

Testing activities are nowadays an increasingly important part of modern manufacturing processes, weighting for as much as 50% of the cost of electronic devices production. Thus, several research and standardization activities have been dedicated to this issue. Particularly important is the test of A/D converters (ADC), because of their wide appliance in various technical fields [1][2]. Various techniques are available, based on the usage of different stimuli, for both static and dynamic ADC characterization. It is worthy of notice that the used testing signals are usually not fully known in advance, and their parameters are usually estimated using the ADC output codes before applying the ADC testing algorithm. Moreover, the stimuli are often affected by random noise, usually modeled as Additive White Gaussian Noise (AWGN) [3]. Consequently, ADC testing may be strongly affected by noise, which can reduce both the accuracy of the preliminary stimulus parametric estimation and the following ADC characterization. In previous works, the authors have analyzed the statistical efficiency of the four-parameter sinewave fitting, modeling and describing the Cramér-Rao Lower Bound (CRLB) on the sinewave parametric estimation from quantized data [4]-[7]. In this paper, a similar approach is initially followed, and the CRLB on the estimators of noise parameters is evaluated. Notice that, when a measurement is carried out, a model is implicitly assumed. Within this model, the CRLB, being a lower bound on the estimator variance, implies the existence of a lower bound on measurement uncertainty as well, defined as the standard deviation associated to measurement results. The results are then used to analyze and compare the statistical efficiency of both the algorithm proposed in [2] to assess the noise variance, and one based on the Maximum Likelihood Estimator (MLE) procedure.

II. CRLB ON NOISE ESTIMATORS

In order to analyze the statistical efficiency of the considered estimation algorithms, the CRLB on the estimator of the noise standard deviation has been carried out, using a direct discrete analysis [6][7]. Notice that both the noise variance estimators considered in this paper require the collection of two records $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$ of ADC output codes, obtained by feeding twice the ADC with a period of a triangular signal $s[\cdot]$, under the same trigger conditions, according to the following.

$$\begin{cases} s_{q1}[n] = Q\big(s[n] + w_1[n]\big) \\ s_{q2}[n] = Q\big(s[n] + w_2[n]\big) \end{cases}, \qquad n = 0,..,N-1 \tag{1}$$

where $w_1[\cdot]$ and $w_2[\cdot]$ are two realizations of an AWGN process with variance σ^2, $Q(\cdot)$ is the quantization law corresponding to the adopted ADC, and N is the record length.

Then, the sequence $x_q[\cdot]$=$s_{q2}[\cdot]$-$s_{q1}[\cdot]$ is used to estimate the AWGN variance σ^2. Accordingly, any unknown deterministic component in $s[\cdot]$ tends to be canceled out, and the distortion it may introduce on the estimates is reduced. As all of the collected ADC samples of both $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$ are statistically independent, the Fisher information $I_\sigma(\sigma)$ on the AWGN standard deviation σ can be obtained as the summation of Fisher information $I_\sigma(n,\sigma)$ associated to each sample $x_q[n]$, that is

$$\begin{aligned} I_\sigma(\sigma) &= \sum_{n=0}^{N-1} I_n(n,\sigma), \\ I_\sigma(n,\sigma) &= \sum_{m=-M+1}^{M-1} \frac{1}{P\big(x_q[n]=m;\sigma\big)} \left(\frac{\partial P\big(x_q[n]=m;\sigma\big)}{\partial \sigma} \right)^2 \end{aligned}, \tag{2}$$

where M=2^b is the number of output codes of a b bit ADC converter and $P(x_q[n]=m;\sigma)$ is the probability that the n-th sample of the sequence $x_q[\cdot]$ equals m. Then, by properly expressing $P(x_q[n]=m;\sigma)$ as a function of the probability of occurrence of the ADC output codes $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$, the CRLB on estimators of σ is obtained (see appendix A for the details) as

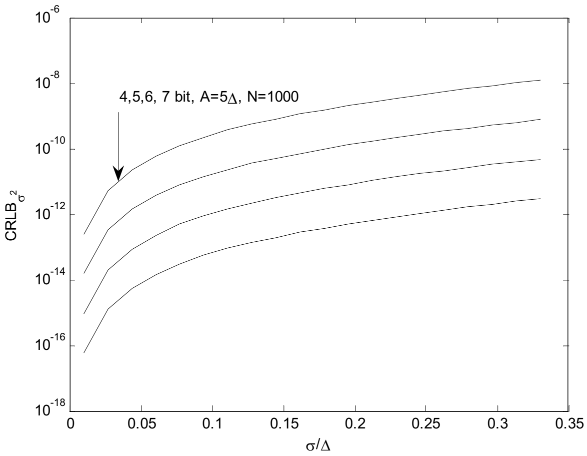

Fig. 1: CRLB on the estimator variance σ^2, as a function of σ/Δ

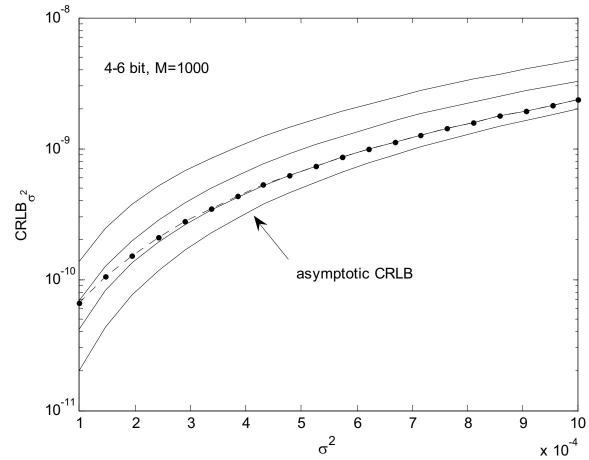

Fig. 2: CRLB on the estimator variance σ^2, as a function of σ/Δ. The continuous lines have been obtained using the Direct discrete Analysis, while the dotted lines have been obtained using (7)

$$CRLB_{\sigma}(\sigma) = \frac{1}{I_{\sigma}(\sigma)} . \tag{3}$$

Notice that the CRLB on the noise variance σ^2 can be obtained from the CRLB on σ, using the following

$$CRLB_{\sigma^2}(\sigma) = \left(\frac{\partial g(\sigma)}{\partial \sigma}\right)^2 CRLB_{\sigma}(\sigma) \tag{4}$$

where $g(\sigma)=\sigma^2$ [4]. Consequently, as the uncertainty is expressed as a standard deviation, a lower bound on the estimator uncertainty $u_C(\sigma)$ may be obtained as

$$u_C(\sigma) \geq \sqrt{CRLB_{\sigma^2}(\sigma)} \tag{5}$$

Fig. 1 reports the CRLB on estimators of the noise variance, obtained for various resolutions, assuming an ideal uniform ADC with unitary Full Scale *FS*, as a function of the ratio σ/Δ, where $\Delta=2FS/2^b$ is the ADC quantization step. All of the reported curves have been obtained by considering as a stimulus one period of a zero mean triangular wave of peak amplitude $A=5\Delta$, and unitary period T, acquired with a sampling period $T_C=T/N$, where $N=1000$. It can be observed that the CRLB grows with σ, and reduces when the ADC resolution is increased. Notice that an asymptotic bound to the CRLB when ADC resolution is increased is given by the CRLB obtained for a non-quantized signal affected by AWGN alone, because in such a case the effect of quantization tends to be negligible, obtaining [4]

$$CRLB_{\sigma^2}(\sigma) = \frac{2\sigma^4}{N} \tag{6}$$

Moreover, provided that the AWGN power is much higher than the quantization noise power, a simplified formula, describing the effect of quantization on the CRLB for high ADC resolutions, may be obtained by replacing σ^2 with $\sigma^2+\Delta^2/12$, obtaining [7]

$$CRLB_{\sigma^2}(\sigma) \cong \frac{2\left(\sigma^2 + \frac{\Delta^2}{12}\right)^2}{N} , \tag{7}$$

and the corresponding lower bound on the estimator uncertainty may be obtained as

$$u_{\sigma^2}(\sigma) \geq \left(\sigma^2 + \frac{\Delta^2}{12}\right)\sqrt{\frac{2}{N}} \tag{8}$$

Fig. 2 reports the CRLB on σ^2, as a function of σ^2, for various ADC resolutions. For each resolution, two curves are reported, one obtained using (2) and (4), the other obtained using (7). Eq. (7) also shows that the CRLB tends to zero like $1/N$. This may be justified by observing that, being the stimulus a triangular wave, the number of samples in $s[\cdot]$ belonging to each ADC code bin is proportional to N. Furthermore, (7) may be rewritten as a function of σ/Δ, obtaining

$$CRLB_{\sigma^2}\left(\frac{\sigma}{\Delta}\right) \cong \frac{2FS^4}{9N2^{4b}}\left(12\left(\frac{\sigma}{\Delta}\right)^2 + 1\right)^2 , \tag{9}$$

which shows that the curves of Fig. 2 tend to be scaled replicas of each other, and that the CRLB is reduced by a factor 2^4 for each bit of resolution. Such a behavior has been verified also using the direct discrete analysis, for the considered ADC resolution and σ/Δ range.

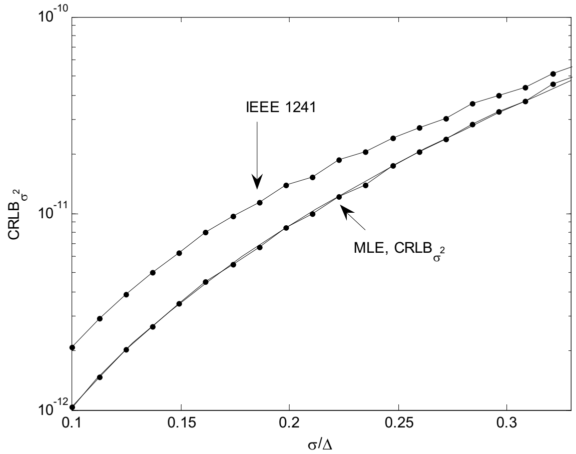

Fig. 3(a): Variance of IEEE 1241 and MLE estimators (dotted curves). The continuous curve reports the corresponding CRLB

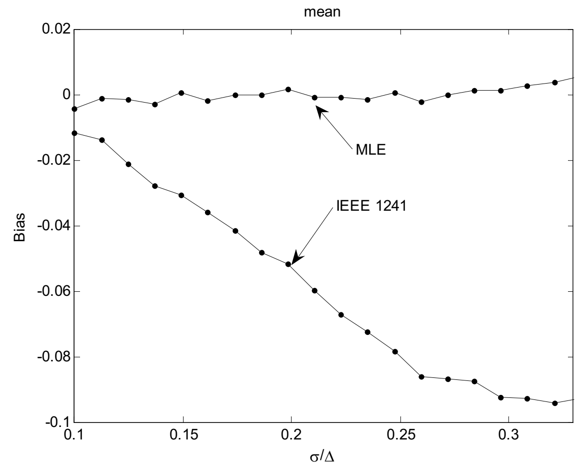

Fig. 3(b): Bias of IEEE 1241 and MLE estimators (dotted curves).

It can be observed that, provided that σ is not significantly lower than Δ, (7) accurately describes the CRLB, and that for high ADC resolutions the CRLB tends to the asymptotic curve (4), also reported in Fig. 2. It worthy of notice that, if an estimation were carried out using directly the ADC output codes, the CRLB would be lower than the one evaluated using $x_q[\cdot]$ as a data source, because $2N$ samples would be available rather than N. In particular, it can be shown that, for high ADC resolutions, an halved asymptotic CRLB holds with respect to (6) and (7) [4]. Thus, using $x_q[\cdot]$ as a data source may reduce the amount of available information and the achievable statistical efficiency.

III. IEEE 1241 AND MAXIMUM LIKELIHOOD NOISE VARIANCE ESTIMATORS

A. IEEE 1241 estimator

The IEEE standard 1241 proposes three techniques to estimate the variance of the overall AWGN adding to the desired stimulus, two of them reported here for the sake of readability. The first one requires to feed the ADC with a constant signal, and to collect two records of ADC output codes, both of size M. Then, the AWGN variance is estimated as

$$\hat{\sigma}^2 = \frac{1}{2N}\sum_{n=0}^{N-1}\left(s_{q2}[n] - s_{q1}[n]\right)^2 . \tag{10}$$

The second technique requires to feed the ADC twice with a period of a triangular wave, spanning over a few ADC bins and keeping the same synchronization conditions. Then the ADC *mse* and variance are obtained from

$$\begin{aligned}\sigma^2 &= \left[\left(\frac{mse}{2}\right)^{-2} + \left(\frac{0.886 mse}{\Delta}\right)^{-4}\right]^{-\frac{1}{2}},\\ mse &= \frac{1}{N}\sum_{n=0}^{N-1}\left(s_{q2}[n] - s_{q1}[n]\right)^2\end{aligned} \tag{11}$$

Both techniques rely on subtraction of corresponding ADC output codes to remove deterministic effects. Fig. 3(a) and 3(b), obtained throughout meaningful Monte Carlo simulations for a 6 bit uniform ADC with FS=1, fed with a triangular wave with A=5Δ, report the estimator variance and bias respectively, as a function of the ratio σ/Δ, where the normalized bias is defined as

$$bias = \frac{\hat{\sigma}^2 - \sigma^2}{\sigma^2} . \tag{12}$$

It can be observed that the estimator variance grows with σ/Δ, and that, at least in the observed σ/Δ interval, the estimator appears to be biased.

B. Maximum Likelihood Estimator

The Maximum Likelihood Estimator (MLE) is obtained by using numerical techniques to maximize with respect to the noise variance the likelihood function, that is the probability density function of the difference between each code pair, seen as a function of the considered parameter, which in this case is given by

$$\begin{aligned}&P(\overline{x_q} = \overline{x_{0q}};\sigma) = \prod_{n=0}^{N-1} P(x_q[n] = x_{0q}[n];\sigma)\\ &\overline{x_q} = \{x_q[0],\ldots x_q[N-1]\},\ \ \overline{x_{0q}} = \{x_{q0}[0],\ldots x_{q0}[N-1]\}\end{aligned}, \tag{13}$$

where $x_{q0}[\cdot]$ is the collected record of the signal $x_q[\cdot]$. This estimator, when existing, is guaranteed to be asymptotically unbiased and efficient [4][5]. Fig. 3(a) and 3(b) also report the MLE variance and bias, for a test based on the same triangular input signal. It can be observed that this estimator, in the considered σ/Δ range, appears to be slightly superior to the IEEE 1241 one, showing smaller variance and bias at the price of a higher algorithm convergence time.

In order to assess the statistical efficiency of both considered estimators, fig 3(a) reports also the corresponding CRLB, Notice that, while the IEEE 1241 estimator variance is reasonably close to the CRLB, the MLE appears to be superior, as its variance coincides with the corresponding CRLB.

IV. EXPERIMENTAL VERIFICATION

In order to validate the proposed methodology, the performances of both the MLE and the IEEE 1241 estimator have been compared. In particular, the noisy stimulus has been obtained using two function generators, one used to generate the triangular wave, the other used to inject arbitrary levels of white noise by means of an analog adder. The obtained signal was acquired using a Texas Instruments DSK6713 DSP Board, programmed to drive a 8 bit A/D converter with FS=4.098V. A set of 30 measurements was carried out. For each measurement, two records of 1000 samples of a noisy triangular wave of amplitude A=100mV and period T=2.809Hz were acquired, at a sampling period $T_c=T/1000$. The injected noise standard deviation was set to σ=0.0134V, approximately equal to 0.5Δ, and the collected data were used to estimate σ^2, both using (10) and the MLE. Figure (4) reports the measurement results, together with the actual noise, independently measured in absence of the triangular stimulus. It can be observed that MLE tends to provide more correct estimates of the noise variance. The results obtained from the 30 measurements have been used to evaluate bias and variance of both estimators. The MLE provides a normalized bias of 0.27 and a variance of $1.1\cdot10^{-9}$, while the IEEE1241 test gives a normalized bias of –0.67 and a variance of $0.1\cdot10^{-9}$. In both cases the estimator variance is higher than the corresponding CRLB. The MLE bias is possibly due to inaccuracies in the estimation of both the ADC transition levels and the non quantized values of the acquired samples prior to quantization, needed to evaluate the likelyhood function. Moreover, a further contribution affecting both estimators is the synchronization between the two acquired records, which in the adopted setup is affected by a jitter on the sampling trigger upper bounded in magnitude by $T_c/2$. Thus, while the MLE test is potentially capable of higher performances, it requires highly accurate synchronization and characterization of the noiseless stimulus.

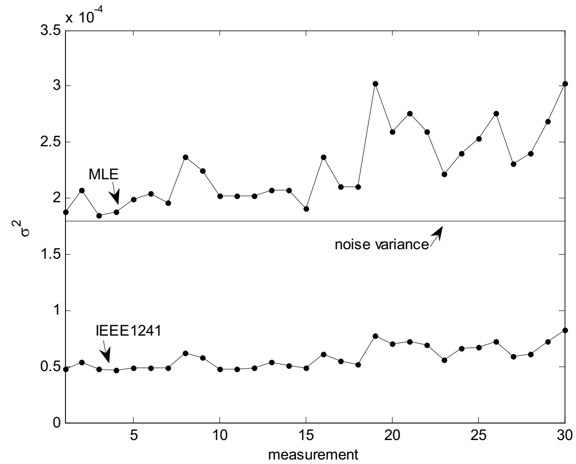

Fig. 4: Noise variance, measured using both the MLE and the IEEE1241estimator.

V. CONCLUSIONS

A MLE estimator for the variance of an AWGN has been proposed and compared to the corresponding IEEE 1241 estimator. The statistical efficiency of both noise estimators has been assessed, by evaluating the corresponding CRLB both by means of direct discrete analysis and deriving a simple formula, useful when high resolution A/D converters are used. It is shown that the MLE statistical efficiency may exceed that of the corresponding IEEE 1241 based estimator.

APPENDIX A: CRLB DIRECT DISCRETE ANALYSIS

As reported in section II, the CRLB on σ may be evaluated by expressing P($x_q[n]=m;\sigma$) in terms of P($s_{q1}[n]=h$, $s_{q2}[n]=k;\sigma$), that is the joint probability that $s_{q1}[n]=h$ and $s_{q2}[n]=k$. As the sequences $s_{q1}[\cdot]$ and $s_{q2}[\cdot]$ are statistically independent, P($s_{q1}[n]=h$, $s_{q2}[n]=k;\sigma$) can be factorized into

$$P\left(s_{q1}[n]=h, s_{q2}[n]=k;\sigma\right)=P\left(s_{q1}[n]=h;\sigma\right)P\left(s_{q2}[n]=k;\sigma\right) \tag{A.1}$$

where P($s_{q1}[n]=h;\sigma$) and P($s_{q2}[n]=k;\sigma$) are the corresponding marginal probabilities, and

$$P\left\{s_q[n]=i;\sigma\right\}=\begin{cases}\Phi\left(\dfrac{T_i-s[n]}{\sigma}\right), & i=0\\ \Phi\left(\dfrac{T_i-s[n]}{\sigma}\right)-\Phi\left(\dfrac{T_{i-1}-s[n]}{\sigma}\right), & 0<i<M-1\\ 1-\Phi\left(\dfrac{T_{i-1}-s[n]}{\sigma}\right), & i=M-1\end{cases} \tag{A.2}$$

where $\Phi(\cdot)$ is the error function [9], given by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt . \qquad \text{(A.3)}$$

As for different pairs of ADC output codes (h_1,k_1), (h_2,k_2) the corresponding events $(s_{q1}[n]=h_1,s_{q2}[n]=k_1)$ and $(s_{q1}[n]=h_2,s_{q2}[n]=k_2)$ are mutually exclusive, $P(x_q[n]=m;\sigma)$ can be written as

$$P\left(x_q[n]=m;\sigma\right) = \sum_{h,k|k=h+m} P\left(s_{q1}[n]=h;\sigma\right) P\left(s_{q2}[n]=k;\sigma\right) \qquad \text{(A.4)}$$

and its derivative with respect to σ is given by

$$\frac{\partial P\left(x_q[n]=m;\sigma\right)}{\partial\sigma} = \sum_{h,k|k=h+m} \frac{\partial\left(P\left(s_{q1}[n]=h;\sigma\right)P\left(s_{q2}[n]=k;\sigma\right)\right)}{\partial\sigma} =$$
$$= \sum_{h,k|k=h+m}\left[\frac{\partial P\left(s_{q1}[n]=h;\sigma\right)}{\partial\sigma} P\left(s_{q2}[n]=k;\sigma\right) + \frac{\partial P\left(s_{q2}[n]=k;\sigma\right)}{\partial\sigma} P\left(s_{q1}[n]=h;\sigma\right)\right] \qquad \text{(A.5)}$$

By using (A.4) and (A.5) in (2), the CRLB on σ is obtained.

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