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ADC Testing with Verification

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Abstract *– An important method of Analog-to-Digital Converter (ADC) testing is sine wave fitting. By this, the device is excited with a sine wave, and another sine wave is fitted to the samples at the output of the ADC. The acquisition device can be analyzed by looking at the differences between the fitted signal and the samples. The fit happens with the Least Squares (LS) method. If the samples of the error (the difference of the fitted signal and the samples) were random and independent of each other and of the signal, the LS fit would have very good properties. However, when the error is dominated by the quantization error, especially when low bit number is used, these conditions are not fulfilled. The estimation will be biased, the estimation must be corrected.*

The independence of the error samples is more or less true if the sine wave is noisy, or dither is used. In these cases the correction is not necessary. Therefore, it is reasonable to analyze the effect of the maybe unnecessary correction to noisy data, and it is desirable to know the magnitude of the noise. In this paper, these two questions are investigated. A new noise estimation method is developed and analyzed.

Keywords *– IEEE standard 1241-2000, ADC test, analog-to-digital converter, Least Squares fit, sine wave fitting, sine wave test, elimination of samples, effective number of bits, ENOB, noise estimation.*

I. INTRODUCTION

Sine fitting is maybe the most important method of ADC (Analog-to-Digital Converter) testing in the IEEE 1241-2000 standard. The essence of this method is fitting of a sine wave to the samples which appear at the ADC output. The errors of the converter can be analyzed by looking at the difference between the fitted signal and the samples.

Fitting is executed using the Least Squares (LS) method. The error (e) is defined as the difference between the observations (y) and the model (m). The observations are the samples, the model is the test signal whose parameters are unknown. Minimizing the sum of e_i^2 the LS fit is done:

$$\min_m \sum_n (y_n - m_n)^2 = \min_m \sum_n e_n^2 \qquad (1)$$

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The LS method has very good properties, especially when the error is random, zero-mean Gaussian and the samples are independent. However, when the error (e) is dominated by the quantization noise, neither of these is true. The main problem is that even by ideal quantization the quantization error depends strongly on the input signal (see Fig. 1). Therefore, the estimated parameters, especially the estimated amplitude, will usually be biased.

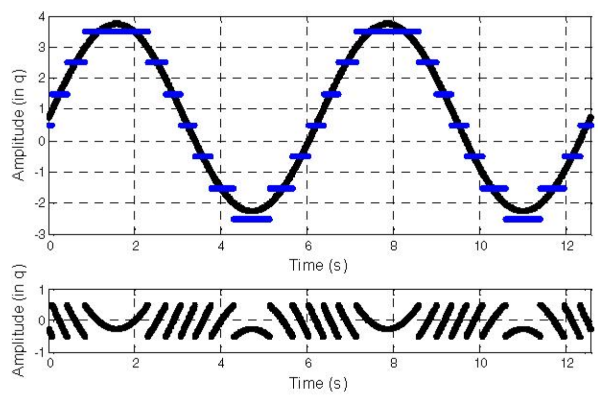

Figure 1. The original sine wave, the quantized samples, and the quantization error (B=3 bits, DC≠0, q=1)

An easy-to-see example is the case when the DC level of the input signal is not zero. Then the peak in the PDF of the quantization error (the part related to the peaks of the sine wave) is not any more in the middle, and the mean value of the error will not be zero (see Fig. 2). This results an error in the amplitude estimation. But, as we are going to see, amplitude estimation can be erroneous even if the error distribution is symmetrical to zero.

A possible solution to avoid this problem is to eliminate the samples belonging to the almost constant curves in the quantization error [2] [3]. This indeed decreases bias, but adds to the variance of the estimator, since information is lost by elimination of some samples. Therefore, it is reasonable to analyze how the variance changes as a consequence of eliminating samples.

When the sine wave is noisy, or dither is used, the samples of the error will be more or less independent of the input signal, and the estimation will be unbiased.

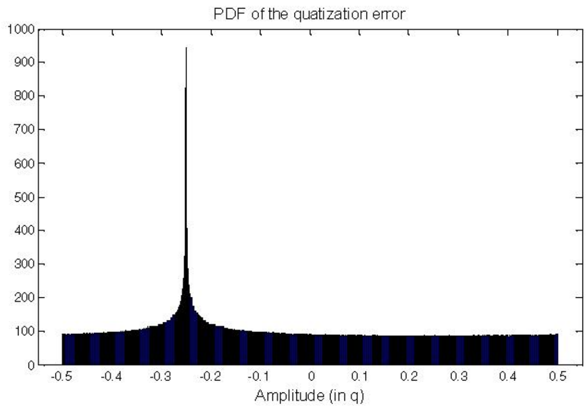

Figure 2. PDF of the quantization error (B=3bit, DC≠0, q is the quantum size)

If sample elimination is applied blindly, in the noisy case the otherwise zero bias is not destroyed, but the variance of the estimation grows. Therefore, it is necessary to know how noisy the sine wave was. In the second part of the paper, a new signal-to-noise estimation method will be shown. When the signal-to-noise ratio is known, one can decide if the sample elimination is necessary or not.

The basis of the noise estimation method is the measurement of the granularity in the output samples. The length of this granularity depends on the magnitude (standard deviation) of the noise. This is the best way of estimation, because the available information on the noise is considered.

II. THE LEAST SQUARES FIT

According to the standard, the following expression has to be minimized:

$$\min_{A_0,B_0,C_0,\omega} \sum_{n=1}^{M} \left[y_n(n) - A_0 \cos(\omega t_n) - B_0 \sin(\omega t_n) - C_0 \right]^2 =$$

$$= \min_{A_0,B_0,C_0,\omega} \sum_{n=1}^{M} e_n^2 = \min_{A_0,B_0,C_0,\omega} (\mathbf{y} - \mathbf{Dx})^T (\mathbf{y} - \mathbf{Dx}) \qquad (2)$$

where $\mathbf{y}$ is the vector of samples, M is the length of $\mathbf{y}$. If ω is known, 3-parameter sine fitting can be applied. Minimizing the expression, the estimator of the input signal is obtained:

$$\hat{\mathbf{x}} = \left(\mathbf{D}^{\mathrm{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} \hat{A}_0 \\ \hat{B}_0 \\ \hat{C}_0 \end{bmatrix} \qquad (3)$$

Assuming independent, identically distributed error, the covariance matrix of the estimator is the following:

$$\mathrm{cov}\{\hat{\mathbf{x}}, \hat{\mathbf{x}}\} = E\left\{(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\right\} = \sigma^2 \left(\mathbf{D}^{\mathrm{T}}\mathbf{D}\right)^{-1}, \qquad (4)$$

where σ^2 is the variance of the noise (e). The elements of matrix $(\mathbf{D}^{\mathbf{T}}\mathbf{D})^{-1}$ are the following:

$$\mathrm{cov}\{\hat{\mathbf{x}}, \hat{\mathbf{x}}\} = \sigma^2 \begin{bmatrix} \mathrm{var}\{\hat{A}_0\} & \mathrm{cov}\{\hat{A}_0, \hat{B}_0\} & \mathrm{cov}\{\hat{A}_0, \hat{C}_0\} \\ \mathrm{cov}\{\hat{B}_0, \hat{A}_0\} & \mathrm{var}\{\hat{B}_0\} & \mathrm{cov}\{\hat{B}_0, \hat{C}_0\} \\ \mathrm{cov}\{\hat{C}_0, \hat{A}_0\} & \mathrm{cov}\{\hat{C}_0, \hat{B}\} & \mathrm{var}\{\hat{C}_0\} \end{bmatrix} \qquad (5)$$

If sampling is coherent, that is, the sampled record contains an integer number of periods, the columns of $\mathbf{D}$ are orthogonal, thus the entries outside the main diagonal in the covariance matrix become zero.

The calculated main diagonal entries are the following:

$$\mathrm{var}\{\hat{A}_0\} = \frac{1}{\sum_{n=1}^{M} \cos^2(\omega t_n)} = \frac{2}{M} \qquad (6)$$

$$\mathrm{var}\{\hat{B}_0\} = \frac{1}{\sum_{n=1}^{M} \sin^2(\omega t_n)} = \frac{2}{M} \qquad (7)$$

$$\mathrm{var}\{\hat{C}_0\} = \frac{1}{\sum_{n=1}^{M} 1} = \frac{1}{M} \qquad (8)$$

For non-coherent sampling, or when samples are eliminated, orthogonality is not true. This means that the estimated parameters become correlated.

Thus, in the general case, when eliminating samples, the sums decrease and the variances grow.

III. THE INCREASE OF THE VARIANCE WHEN ELIMINATING SOME SAMPLES

When the phase (φ) of the signal to be sampled is zero (a simplifying assumption without the loss of generality), $\hat{A}_0$ becomes the amplitude estimator:

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{A}_0 \\ \hat{B}_0 \\ \hat{C}_0 \end{bmatrix} = \begin{bmatrix} \hat{A} \cdot \cos(0) \\ -\hat{A} \cdot \sin(0) \\ \hat{C} \end{bmatrix} = \begin{bmatrix} \hat{A} \\ 0 \\ \hat{C} \end{bmatrix} \qquad (9)$$

In this case it is enough to analyze $\mathrm{var}\{\hat{A}_0\}$ and $\mathrm{var}\{\hat{C}_0\}$. For simplicity, we reduce the number of periods to one. The effect of the elimination of samples can be seen in Fig. 3.

The quantization error is not zero-mean (asymmetric) when the input signal has a DC component. When eliminating the samples, the distribution of the quantization error becomes more or less uniform (see Fig. 4). This illustrates that we got close to the quantization noise model.

The elimination of samples influences the matrix $\mathbf{D}$. The number of rows of $\mathbf{D}$ matches the number of rows of $\mathbf{y}$. When we eliminate samples from $\mathbf{y}$, we also have to eliminate the corresponding samples from $\mathbf{D}$. This is illustrated in Fig. 5.

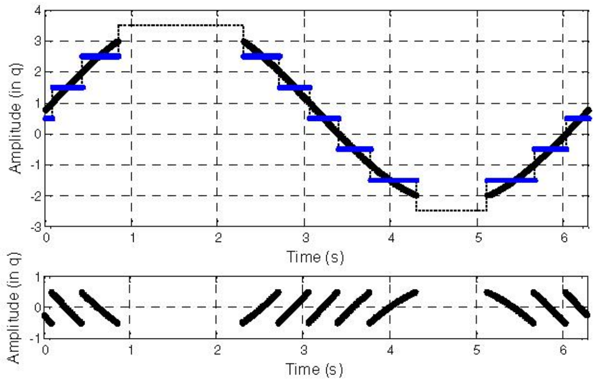

Figure 3. Effect of the sample elimination from y illustrated in the time domain quantization error

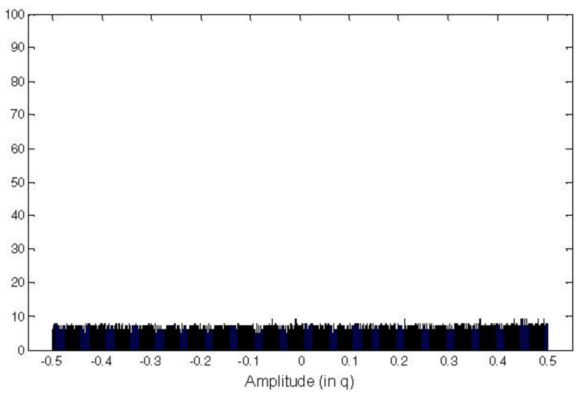

Figure 4. PDF of the quantization error after the elimination

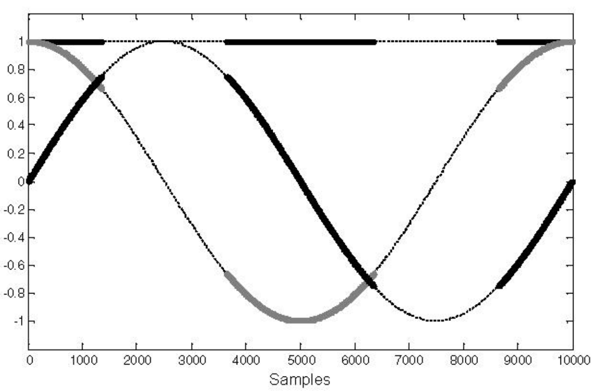

Figure 5. Eliminating rows (cos, sin, 1) from the **D** matrix ($\varphi = 0$)
Dotted line: eliminated samples, solid line: left samples

When we eliminate rows from matrix **D**, than the values of the main diagonals of $\mathbf{D}^\mathbf{T}\mathbf{D}$ change (see Fig. 6).

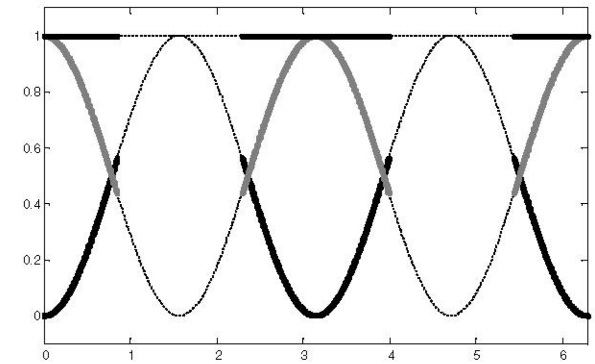

Figure 6. Changing of the elements ($\Sigma\cos^2$, $\Sigma\sin^2$, $\Sigma 1$) in the main diagonal of $\mathrm{D}^\mathrm{T}\mathrm{D}$ ($\varphi = 0$)
Dotted line: eliminated samples, solid line: remaining samples

The change in var$\{\hat{A}_0\}$

Let us assume that we eliminate $k/2$ samples from the positive and negative half-periods each. When $\varphi = 0$ and just one period was measured, samples are eliminated in a symmetrical manner around positions $M/4$ and $3M/4$. Therefore, the following can be written:

$$\operatorname{var}\left\{\hat{A}_{0\mathrm{elim}}\right\} = \frac{1}{\sum_{n=1}^{M} \sin^2(\omega t_n) - 2 \cdot \sum_{i=M/4-k/4}^{M/4+k/4} \sin^2(\omega t_i)} \tag{10}$$

This causes a significant modification (growth of the variances), because the samples are left out from near the maximum of the $\sin^2$ function. After some manipulation, the new variance is obtained:

$$\operatorname{var}\left\{\hat{A}_{0\mathrm{elim}}\right\} = \frac{2}{M - k - 2\sum_{n=-k/4}^{k/4} \cos(2\omega t_n)} \tag{11}$$

To see how much the variance of the amplitude estimator grows, k samples are left out from M (all close to $M/4$ and $3M/4$):

$$\frac{\operatorname{var}\left\{\hat{A}_{0\mathrm{elim}}\right\}}{\operatorname{var}\left\{\hat{A}_0\right\}} = \frac{1}{1 - \frac{k}{M} - \frac{2}{M}\sum_{n=-k/4}^{k/4} \cos(2\omega t_n)} \tag{12}$$

This can be approximated as:

$$\frac{\operatorname{var}\left\{\hat{A}_{0\mathrm{elim}}\right\}'}{\operatorname{var}\left\{\hat{A}_0\right\}} \cong \frac{1}{\left(1 - 1.15 \cdot \frac{k}{M}\right)^2} \tag{13}$$

In Fig. 7 the result of both formulas can be seen.

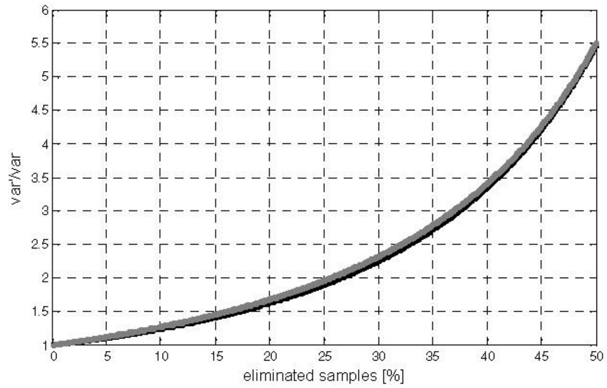

Figure 7. Ratio of the variance vs. proportion of the eliminated samples (grey curve: approximate formula, black curve: theoretical formula)

According to this plot, for 1,5-fold growth of the variance approximately 17% of the samples left out, and for doubling of the variance, about 26%. These values are not reached with practical bit numbers. The worst case values according to the elimination algorithm [2-3] are:

ADC bit number	eliminated samples
6	18.65%
8	9.22%
10	4.59%
12	2.29%
16	0.57%
24	0.04%

The change in var$\{\hat{C}_0\}$

The new variance of $\hat{C}_0$ after eliminating k samples is:

$$\operatorname{var}\left\{\hat{C}_{0\mathrm{elim}}\right\} = \frac{1}{\sum_{n=1}^{M} 1 - \sum_{i=1}^{k} 1} = \frac{1}{M-k} \tag{14}$$

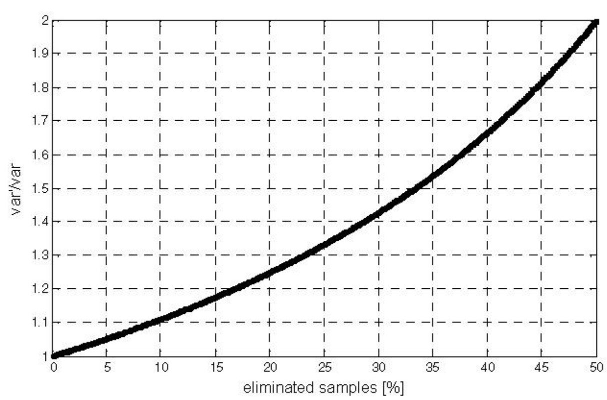

Figure 8. Ratio of the variances vs. proportion of eliminated samples

For the ratio of the variances it can be written:

$$\frac{\operatorname{var}\left\{\hat{C}_{0\mathrm{elim}}\right\}}{\operatorname{var}\left\{\hat{C}_0\right\}} = \frac{M}{M-k} = \frac{1}{1-\dfrac{k}{M}} \tag{15}$$

In Fig. 8 it is illustrated how the ratio of the variances grow when we eliminate samples.

IV. SIMULATION

The previous calculations have been verified by simulations, using MATLAB. During simulation, we simulated the worst case, that is, the case when the samples around the peaks are eliminated by elimination the samples falling into the maximum histogram bins and the ones larger outside these.

Measurements were somewhat noisy. This noise was approximately Gaussian. The standard deviation (σ) of this noise was approximately equal to q. The condition $\sigma > 0.5q$ assures to satisfy Quantization Theorem II [5]. The quantization error was therefore approximately uniformly distributed. As expected, the input signal was uncorrelated with **e**.

Properties of the input signal and of the quantizer were as follows during the simulation:

- $M = 1000$ (number of samples)
- $A = 1$, $\varphi = 0$, $C = 0$
- quantizer: 6 bit, linear, ideal

The simulation routine was repeated 100 times, with elimination of samples, and 100 without elimination. The simulation results are the following.

Simulation results with all samples:

$$mean\{\hat{\mathbf{x}}\} = \begin{bmatrix} 0.9771 \\ -0.0001 \\ 0.0002 \end{bmatrix} \rightarrow \begin{matrix} A = 0.9771 \\ \varphi = 0.0054° \\ C = 0.0002 \end{matrix} \tag{16}$$

$$\operatorname{cov}\{\hat{\mathbf{x}}, \hat{\mathbf{x}}\} = \sigma^2 \begin{bmatrix} 2\cdot 10^{-3} & 0 & 0 \\ 0 & 2\cdot 10^{-3} & 0 \\ 0 & 0 & 1\cdot 10^{-3} \end{bmatrix} \tag{17}$$

Simulation results with elimination of samples:

Number of ADC bits: see above
Mean number of the eliminated samples: ≈ 173
Mean number of the remaining samples: $M_{\mathrm{elim}} \approx 827$

$$\operatorname{mean}\{\hat{\mathbf{x}}\} = \begin{bmatrix} 0.9718 \\ 0.006 \\ 0.0002 \end{bmatrix} \rightarrow \begin{matrix} A = 0.9718 \\ \varphi = -0.3525° \\ C = 0.0002 \end{matrix} \tag{18}$$

$$\operatorname{cov}(\hat{\mathbf{x}},\hat{\mathbf{x}}) = \sigma^2 \begin{bmatrix} 3.0024\cdot 10^{-3} & 7.7\cdot 10^{-6} & 4.8\cdot 10^{-6} \\ 7.7\cdot 10^{-6} & 2.026\cdot 10^{-3} & 5\cdot 10^{-7} \\ 4.8\cdot 10^{-6} & 5\cdot 10^{-7} & 1.2097\cdot 10^{-3} \end{bmatrix} \quad (19)$$

The variances were increased. Using (12),

$$\sigma^2 \operatorname{var}\left\{\hat{A}_{0\text{elim}}\right\} = \sigma^2 \frac{2}{M - k - 2\sum_{n=-k/4}^{k/4} \cos(2\omega t_n)} = 0.003\sigma^2 \quad (20)$$

Comparing this to (19) it can seen that the formula works well.

Comparison of the two cases

The ratio of the two variances is following:

$$\frac{\sigma^2 \operatorname{var}\left\{\hat{A}_{0\text{elim}}\right\}}{\sigma^2 \operatorname{var}\left\{\hat{A}_0\right\}} = \frac{0.0030024\sigma^2}{0.002\sigma^2} \cong 1.5 \quad (21)$$

This means that about 50% more samples are needed to reach the same variance:

$$M_{\text{needed}} = \frac{\operatorname{var}_{\text{elim}}}{\operatorname{var}} M = 1.5M \quad (22)$$

V. ESTIMATION OF THE NOISE LEVEL FROM THE OUTPUT SAMPLES

The above analysis illustrates that in the presence of noise, elimination of samples increases variance. On the other hand, noise removes bias, thus with noise, elimination of samples is not necessary. Therefore, it is of interest to determine the input noise level from the output samples.

The model is that the input is a sine wave, corrupted by Gaussian white noise, and this is quantized by an ADC. In the examples below, the ADC is handled as ideal, but we will see that this is not a requirement. A typical output signal is illustrated in Fig. 9.

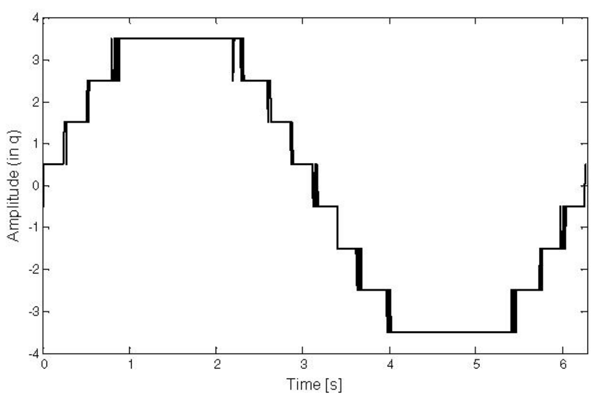

Figure 9. Output samples of an ADC

The noise is exhibited by the oscillating parts around the quantization levels. It makes sense to analyze the length of the oscillations, because their length directly depends on the standard deviation of the noise.

In Fig. 10 one oscillation can be seen. The local slope of the sine is more or less constant, and the change of the sine is much slower than the change of the noise. In this case we can say that the sine is a "drift" component of the noise.

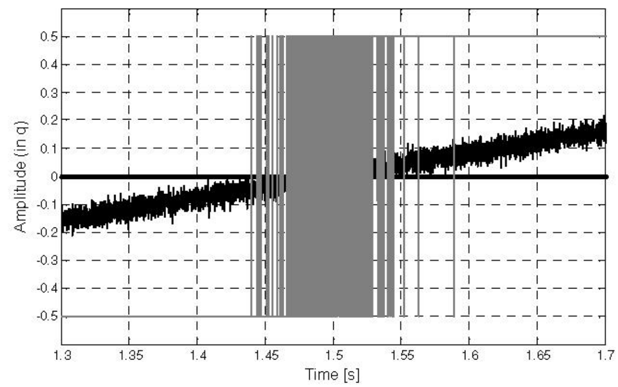

Figure 10. One transient in the output code because of noise

The centers of the oscillations can be considered like the "center of mass" in mechanics, and the length of the oscillations can be measured as deviations from these center points.

The central point of one oscillation can be calculated as follows:

$$c = \frac{\sum_{n=1}^{M} n \cdot x_n}{\sum_{n=1}^{M} x_n} \quad (25)$$

where c is the "center of mass", x_n is an element of the oscillating part (points around the given comparison level where the absolute value of the derivative is non-zero). The length of an oscillation can be determined as follows:

$$\sigma = \sqrt{\frac{1}{M-1}\sum_{n=1}^{M}(c - x_n)^2} \quad (26)$$

The obtained oscillation lengths must be weighted, because the lengths of the oscillations depend on the local slope of the sine. This weighting can be done using the local slope of the sine. The weighted values can be averaged. The dependence of this average on the standard deviation of the noise can be seen in Fig. 11.

The nature of this curve is similar for all investigated bit numbers.

In the next step the fitted curve can be determined. With help of the fitted curve and the measured length of the oscillations the standard deviation of the noise can be computed.

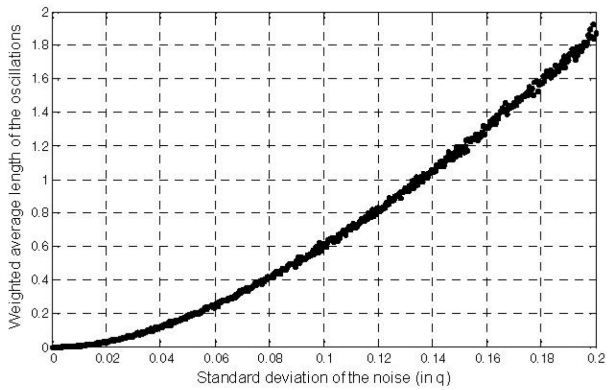

Figure 11. Standard deviation of the noise vs. average length of the oscillations (M = 1000 points, B = 3 bit)

A possible approximation of the curve in Fig. 11 can be done by making an LS fit with a cosine function. The parameters of the cosine can be finding in Table 1. The phase of the cosine is in all cases zero.

	Amplitude (sample) A	Period (in q) P	Offset (sample) C
3 bit	-10.432	0.32752	10.873
4 bit	-3.8301	0.32563	3.9769
5 bit	-1.4071	0.32087	1.4511
6 bit	-0.52826	0.30706	0.53915

Table 1. Parameters of the fitted cosine (M = 8000 samples)

The fitted curve can be determinate as follows:

$$f_i = A \cdot \cos\left(2 \cdot \pi \cdot p \cdot \frac{i}{M}\right) + C \tag{27}$$

The LS method causes an apparent relative error at the beginning of the fitted curve. The relative error around zero can be forced to be small in the fit. The modified fitting can be seen in Fig. 12.

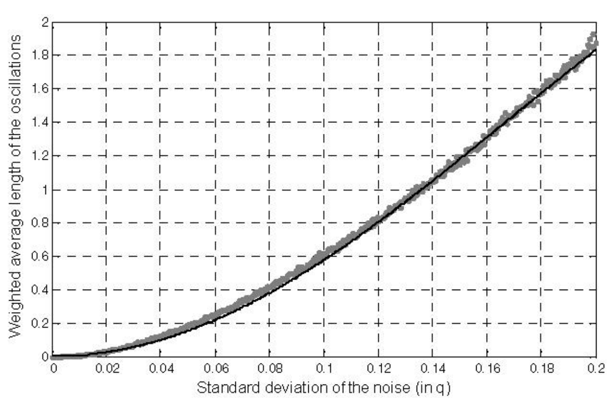

Figure 12. Comparison of the simulation results and the modified fitted curve (M = 1000 points, B = 3 bit)

The modified fit is less accurate. One would like to approximate the simulation results more closely. In the future an analytical analysis will be done to study the nature of the curve.

An approximate formula for the amplitude of the cosine is as follows:

$$\hat{A} \cong 1.246 \cdot 10^{-4} \cdot N \cdot \left(-210.2 \cdot e^{-b}\right) \tag{28}$$

where N is the number of samples, b is the number of bits.

To calculate the signal-to-noise ratio using the observations (the output samples of an ADC), the following steps can be suggested:

- measure the oscillation lengths at the output of the ADC,
- form a weighted average: the weights are the local slope of the sine,
- determine the cosine curve using Table 1 or (28), (27),
- determine the standard deviation of the noise with help of the inverse cosine function.

VI. CONCLUSIONS

The sample elimination method [2-3] can be well used for the enhancement of the sine wave fit. The samples around sine peaks have to be eliminated, and to have estimates with the same variance, more samples have to be taken. The necessary number of samples can be determined from the increase of the variance.

To verify the results without sample elimination, the standard deviation of the noise has to be measured. For this, a usable algorithm was given.

VII. REFERENCES

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