Preliminaries

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Geometric Series

A ***geometric series*** is the sum of a(n) (in)finite number of terms that have a constant ratio r between successive terms as

$$a + ar + ar^2 + \cdots + ar^U = \sum_{n=0}^{U} ar^n$$

For $r \neq 1$, the sum S of $U - L + 1$ terms of a geometric series is

$$\begin{array}{rcc} S & = & r^L + r^{L+1} + r^{L+2} \cdots + r^U \\ rS & = & r^{L+1} + r^{L+2} \cdots + r^U + r^{U+1} \\ S - rS & = & r^L - r^{U+1} \\ (1-r)\,S & = & r^L - r^{U+1} \\ S & = & \frac{r^L - r^{U+1}}{1-r} \end{array}.$$

The general formula is

$$\sum_{n=L}^{U} ar^n = a\left(\frac{r^L - r^{U+1}}{1-r}\right).$$

In particular, when $L = 0$, the formula reduces to

$$\sum_{n=0}^{U} ar^n = a\left(\frac{1 - r^{U+1}}{1-r}\right).$$

As U approaches infinity, the absolute value of r must be less than one for the series to converge. The sum then becomes

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

Symmetric vs Periodic Windows

In the symmetric case, the second half of the window, $M \leq n \leq N{-}1$, is obtained by reflecting the first half around the midpoint. The symmetric option is the preferred method when using a window in FIR filter design.

The periodic window is constructed by extending the desired window length L by one sample to $L = N + 1$, constructing a symmetric window, and removing the last sample. The periodic version is the preferred method when using a window in spectral analysis *without zero-padding* because the discrete Fourier transform assumes periodic extension of the input vector. Zero-padding removes the advantage of symmetry.

Fourier Transforms

Discrete-Time Fourier Transform

The discrete-time Fourier transform (DTFT) of the sequence $x(n)$ is

$$X(\tilde{\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\tilde{\omega}n},$$

where $\tilde{\omega} \triangleq \omega T_s \in [-\pi, \pi)$ is the *continuous* normalized radian frequency variable and $T_s = \frac{1}{f_s}$ is the sampling period, the inverse of the sampling frequency f_s. Note that $\omega = 2\pi f$ is the radian frequency in rad/s and f is the frequency in Hertz.

Therefore, the DTFT uses the continuous angular frequency variable $\tilde{\omega}$, whereas the discrete Fourier transform (DFT) is obtained from the DTFT by discretization of $\tilde{\omega}$ as $\omega T_s = \omega_k T_s$, where ω_k are the normalized discrete radian frequency given by

$$\omega_k = \frac{2\pi}{N}k, \text{ where } k = 0, 1, \ldots, N-1,$$

where N is the size of the DFT.

Useful Relationships

Variable	Name	Unit
T	Period	s
$f = \frac{1}{T}$	Frequency	Hz (cycle/s)
$\omega = 2\pi f$	Angular frequency	rad/s
f_s	Sampling frequency	samples/s
$\tilde{T} = \frac{1}{\nu} = \frac{f_s}{f}$	Normalized period	samples
$\nu = \frac{f}{f_s}$	Normalized frequency	cycles/sample
$\tilde{\omega} = 2\pi\nu$	Normalized angular frequency	rad/sample
k	Frequency bin	samples
$\frac{k}{N}$	Discrete frequency	-

Variable	Name	Unit
$\omega_k = 2\pi\frac{k}{N}$	Discrete angular frequency	radians

Window size $M = m\frac{f_s}{f_0}$ samples, where m is an integer number of periods.

Angular frequency $\tilde{\omega} f_s = 2\pi f = \omega$ radians/s

Normalized frequency $\frac{k}{N} = \frac{f}{f_s}$