Project: “Multigrid Convergence of Geometrical moments”

Introduction

The objective of this project is to implement and evaluate geometrical moments computation on digital data.

We expect from you:

- A short report with answers to the ”formal” questions and a description of our implementation choices and results.
- A C++ project (`CMakeLists.txt` plus several **commented** cpp program files).

1 Properties of moments

We are interest in the analysis of geometrical moment of order $m_{p,q}$ defined as follows for $X \subset \mathbb{R}^2$:

$$m_{p,q}(X) = \int\int_X x^p y^q \, dxdy$$

On a digital object $Z \in \mathbb{Z}^2$, we will use the following approximation:

$$\hat{m}_{p,q}(Z) = \sum_{(i,j)\in Z} i^p j^q$$

Similarly, we will also consider **central geometrical moments** defined by:

$$\mu_{p,q}(X) = \int\int_X (x-\mu_x)^p (y-\mu_y)^q \, dxdy$$

$$\hat{\mu}_{p,q}(Z) = \sum_{(i,j)\in Z} (i-\mu_i)^p (p-\mu_j)^q$$

where (μ_x, μ_y) (resp. (μ_i, μ_j)) is the centroid of X (resp. Z).

Question 1 *Express (μ_x, μ_y) coordinates as function of $m_{a,b}$ for some $a, b \in \mathbb{Z}$*

Question 2 *Express first $\mu_{p,q}$ central moments ($p + q \leq 2$) as functions of $m_{p,q}$.*

Question 3 *Geometrical moments are not scale-invariant. We consider a scaling of X by a factor k (denoted $k \cdot X$), express $m_{p,q}(k \cdot X)$ as a function of $m_{p,q}(X)$*

Similarly, you also have $\mu_{p,q}(k \cdot X)$ as a function of $\mu_{p,q}(X)$. We can use the previous result to design moments $\eta_{p,q}$ with scale invariant properties.

Question 4
- *Express $m_{0,0}(k \cdot X)$ from $m_{0,0}(X)$.*
- *Find the α such that*
$$\frac{m_{p,q}(k \cdot X)}{m_{0,0}(k \cdot X)^\alpha} = \frac{m_{p,q}(X)}{m_{0,0}(X)^\alpha}$$
- *Define $\eta_{p,q}$ as a function of $\mu_{p,q}$ and $\mu_{0,0}$. Conclude on the fact that $\eta_{p,q}$ are translation and scale invariant.*

2 Multigrid Analysis

Here, we evaluate the multigrid convergence of $\hat{m}_{pq}$ estimators. First, remember that we consider here Gauss digitization at gridstep h of $X \subset \mathbb{R}^2$:

$$Z = Dig(X,h) = \left(\frac{1}{h} \cdot X\right) \cap \mathbb{Z}^2 = X \cap (h \cdot \mathbb{Z}^2)$$

As discussed above, we denote

$$\hat{m}_{pq}(Z) = \sum_{(i,j)\in Z} i^p j^q$$

and define

$$\hat{m}_{pq}(Z,h) = \hat{m}_{pq}(h \cdot Z)$$

Question 5 *From the definition and using the result of Question 4, express $\hat{m}_{pq}(Z,h)$ as a function of $\hat{m}_{pq}(Z)$ and h.*

Question 6 *Implement in* DGtal *geometrical moments $\hat{m}_{pq}(Z,h)$ computations in 2D. In this multigrid context, please consider the digitization of a parametric shape (* e.g. *an ellipse).*

Question 7 *Perform a multigrid analysis of $\hat{m}_{pq}(Dig(X,h),h)$*

- *Implement function which constructs the digitization of an Euclidean ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at grid step h*
- *For the first moments ($p+q \leq 3$), output $|\hat{m}_{pq}(Dig(X,h),h) - m_{pq}(E)|$ values for h tending to 0.*
- *Plot error graphs $h \times Error$ in logscale using `gnuplot`*

See Appendix A for first geometrical moments of the ellipse.

Question 8 *We guess that $\hat{m}_{pq}$ has an error in $O(h^{\alpha_{pq}})$ for some $\alpha_{pq} \in \mathbb{R}$ depending only on p and q. Experimentally, can you estimate such α_{pq} for first moments ? (Hint: the slope β of linear fitting in logscale gives you the exponent of something on x^β). Can you guess the general form for the error of m_{pq}.*

So far, we saw in the lectures that $\hat{m}_{00}$ is convergent with speed at least $O(h)$ ([Gauss] with general convex hypothesis on X). Hence, we have $\alpha_{00} = 1$. Adding hypothesis on ∂X (e.g. being C^3), we have $\alpha_{00} = \frac{15}{11} - \epsilon$ [Huxley]

3 Extensions

Question 9 *In dimension 3, are the convergence speeds different ?*

Question 10 *Are moments m_{pq} or $\hat{m}_{pq}$ rotational invariant ? Can you construct rotational invariant shape descriptor from m_{pq} or $\hat{m}_{pq}$?*

A Geometrical moments of a General Ellipse

Notations:

- a length of the semi-major axis
- b length of the semi-minor axis
- x_0, y_0 coordinate of the center of the ellipse
- λ angle of the major axis with the $x-$axis

$$m_{00} = \pi ab \tag{1}$$
$$m_{10} = \pi ab x_0 \tag{2}$$
$$m_{01} = \pi ab y_0 \tag{3}$$
$$m_{20} = \pi ab \left(\frac{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}{4} + x_0^2 \right) \tag{4}$$
$$m_{02} = \pi ab \left(\frac{a^2 \sin^2 \lambda + b^2 \cos^2 \lambda}{4} + y_0^2 \right) \tag{5}$$
$$m_{11} = \pi ab \left(\frac{(a^2 - b^2) \cos \lambda \sin \lambda}{4} + x_0 y_0 \right) \tag{6}$$
$$m_{30} = \pi ab \left(\frac{3x_0(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)}{4} + x_0^3 \right) \tag{7}$$
$$m_{03} = \pi ab \left(\frac{3y_0(a^2 \sin^2 \lambda + b^2 \cos^2 \lambda)}{4} + y_0^3 \right) \tag{8}$$
$$m_{21} = \pi ab \left(\frac{y_0(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)}{4} + \frac{x_0(a^2 - b^2) \sin \lambda \cos \lambda}{2} + x_0^2 y_0 \right) \tag{9}$$
$$m_{12} = \pi ab \left(\frac{x_0(a^2 \sin^2 \lambda + b^2 \cos^2 \lambda)}{4} + \frac{y_0(a^2 - b^2) \sin \lambda \cos \lambda}{2} + x_0 y_0^2 \right) \tag{10}$$