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A Comparative Evaluation of Length Estimators

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A Comparative Evaluation of Length Estimators

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Abstract

The paper compares previously published length estimators in image analysis having digitized curves as input. The evaluation uses multigrid convergence (theoretical results and measured speed of convergence) and further measures as criteria. The paper also suggests a new gradient-based method for length estimation.

Keywords

Length estimator, digital geometry, curve length, multigrid convergence

I. Introduction

The digitization of curves or boundaries has been studied in image analysis for about 40 years [1]. Since these first studies, many algorithms have been proposed to estimate the length of a digitized curve. Some approaches are based on local metrics such as the chamfers metrics, other approaches are based on polygonalizations of digital curves, e.g. directed on subsequent calculations of maximum-length digital straight segments (DSSs) or of minimum length polygons (MLPs), and we also propose a new length estimator based on calculated gradients.

The computational problem is as follows : the input is a sequence of chain codes $i(0), i(1), \ldots$ with $i(k) \in A = \{0, \ldots, 7\}$, $k \geq 0$. An *off-line algorithm* takes finite words $u \in A^\star$ as input, and performs the required calculations, e.g. whether u is a DSS or not. An *on-line algorithm* reads successive chain codes $i(0), i(1), \ldots$ and provides always a correct result up to the most recent input value $i(n)$, e.g. it decides whether $i(0), i(1), \ldots, i(n)$ is still a DSS, and if not, then it initializes a new DSS with $i(n-1)i(n)$. An off-line algorithm is *linear* iff it runs in $\mathcal{O}(n)$ time, i.e. it performs at most $\mathcal{O}(|u|)$ basic computation steps for any input word $u \in A^\star$. An on-line algorithm is linear iff it uses *on the average* a constant number of operations for any incoming chain code symbol.

An obvious benefit of local metrics based approaches is that they support linear on-line implementations. This method has frequently been suggested in the image analysis literature for length estimation, combined with proposals of local weights to improve these estimations. However, it is known that these methods are only of limited use if multigrid convergence is applied as a selection criteria.

DSS-based polygonalization is a popular method in image analysis, allowing to transform digital boundaries into polygonal objects. Linear off-line algorithms for DSS recognition were published in 1981 in [5] and in 1982 in [8]. A linear off-line algorithm for cellular straight segment recognition, based on convex hull construction, is briefly sketched in [7]. Two linear on-line algorithms for DSS recognition were published in 1982 in [6]; one of them is an on-line version of the off-line algorithm published in [5].

The general problem of decomposing a digital curve into a sequence of DSSs, which includes DSS recognition as a subproblem, is discussed in, e.g. [14], [15], [19], [26]. Obviously, linear on-line DSS recognition algorithms will support linear decomposition algorithms, but the application of a linear off-line algorithm for input-sequences of increasing length, i.e. first for $i(0)$, then for $i(0)i(1)$, then for $i(0)i(1)i(2)$ etc., leads to quadratic run-time behavior.

MLP-based polygonalization provides a third method, which is not yet of widespread use in image analysis. An MLP-approximation [21], [27] calculates a minimum-length polygon circumscribing a given (closed) inner boundary (given by a sequence of chain codes), and being in the interior of an outer boundary (typically in Hausdorff-Chebyshev distance 1 to the inner boundary). This polygon is also known as relative convex hull [4], and calculations of relative convex hulls have a history in computational geometry and robotics.

As a fourth method we also introduce gradient-based length estimation in this paper which may be seen as an extension of the DSS-based approach. The notions on-line, off-line, or linear time apply for algorithms following any of these four design strategies.

In this article, we present a comparative evaluation of length estimators covering these four types of strategies. We are especially interested in evaluating these algorithms (and underlying methodologies) with respect to the accuracy of length estimation. Multigrid convergence is one option of characterizing this accuracy, and experimental studies provide another way for performance testing. Our experiments are directed on illustrating accuracy and stability of the chosen algorithms on convex and non-convex curves (see Fig. 1 for the used test data which had been proposed in [26]). These given Jordan curves of known length are digitized for increases in grid resolution, allowing to study and illustrate

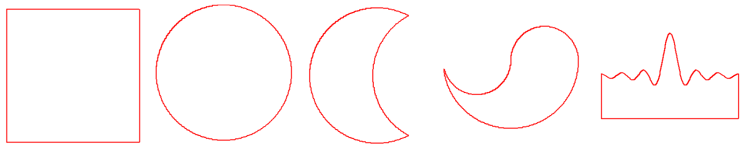

Fig. 1. Test data set.

experimental multigrid convergence.

In our experiments we digitize these curves in grids varying between 30 times 30, and 1000 times 1000. For the digitization of a planar Jordan curve (up to a given grid resolution) we may adopt any of the digitization models known in the image analysis literature.

We assume an orthogonal grid with grid constant $0 < \theta \leq 1$ in the Euclidean plane $\mathbb{R}^2$, *i.e.* θ is the uniform spacing between grid points parallel to one of the coordinate axes. Let $r = 1/\theta$ be the *grid resolution*, and the *r-grid* $\mathbb{Z}_r^2$ has resolution r, defined by *r-points* whose coordinates are $(\theta \cdot i, \theta \cdot j)$, with $i, j \in \mathbb{Z}$. Now, we consider a Jordan curve $\gamma : [0,1] \rightarrow \mathbb{R}^2$, being the topological frontier of a set S in the Euclidean plane. Let $D_r(\gamma)$ be an *r-digitization* of γ in $\mathbb{Z}_r^2$:

Definition 1: In this paper, an *r-digitization* of curve γ is one of the following:

(i) the cyclic 4- or 8-path of r-grid points following an r-grid-intersection digitization of γ, see [2] for the original definition of grid-intersection digitization with $r = 1$;

(ii) the cyclic 4- or 8-path following vertices of r-grid squares in the frontier of the *Gauss digitization* $\mathbf{G}_r(S)$ of set S in the r-grid, where $\mathbf{G}_r(S)$ is the union of all r-grid squares having their centroids in the given set S, or

(iii) the closed difference set between *outer and inner Jordan digitization* $\mathbf{J}_r^+(S)$ and $\mathbf{J}_r^-(S)$, i.e. $\mathrm{cl}(\mathbf{J}_r^+(S) \setminus \mathbf{J}_r^+(S))$, where $\mathbf{J}_r^+(S)$ is the union of all r-grid squares having a non-empty intersection with the given set S, and $\mathbf{J}_r^-(S)$ is the union of all r-grid squares contained in the topological interior of the given set S.

See, e.g., [29] for more details and historic citations for these digitization methods. We denote by $\mathcal{P}(\gamma) \in \mathbb{R}$ a property of a curve γ, which is the length $\mathcal{L}(\gamma)$ of γ in this article. We denote by $\mathcal{E}$ an estimated property. Assume that $\mathcal{E}$ is defined for digitizations $D_r(\gamma)$, for $r > 0$ and all curves γ in a class Γ of curves. In this paper we only consider the class

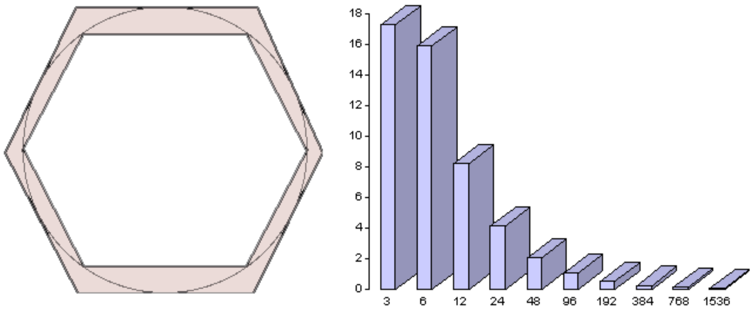

Fig. 2. Inner and outer hexagon approximating a circle (left), and percentage errors for perimeter estimation using inner n-gons.

Γ of all Jordan curves in the Euclidean plane.

Definition 2: The estimated property $\mathcal{E}$ is said to be *multigrid convergent* towards $\mathcal{P}$ with respect to digitization model D_r and class Γ iff for any curve $\gamma \in \Gamma$ we have that $\mathcal{E}(D_r(\gamma))$ converges to $\mathcal{P}(\gamma)$, for $r \to \infty$. More formally:

$$|\mathcal{E}(D_r(\gamma)) - \mathcal{P}(\gamma)| \leq \kappa(r) \tag{1}$$

with $\lim_{r\to\infty} \kappa(r) = 0$. The order $O(1/\kappa(r))$ denotes the *speed* of this convergence.

Multigrid convergency of estimated properties is a standard constraint in numerical mathematics for discrete versions of 'continuous' properties. We conclude this introduction with a citation of a historic example in [33]:

Example 1: In ancient mathematics, Archimedes and Liu Hui [9] estimated the length $\mathcal{L}(\gamma)$ of a circular curve γ. Liu Hui used regular n-gon approximations, with $n = 3, 6, 12, 24, 48, 96, ...$, see left of Fig. 2. In case of $n = 6$ it follows $3 \cdot d < \mathcal{L}(\gamma) < 3.46 \cdot d$ for diameter d, and for $n = 96$ it follows that

$$223/71 < \pi < 220/70 \ , \quad \text{i.e.} \quad \pi \approx 3.14 \ . \tag{2}$$

The used method is mathematically correct because the perimeters of inner and outer regular n-gons converge towards the circle's perimeter for $n \to \infty$. For example, for the inner 3×2^n-gons, having perimeters

$$p_{2n} = 2n \cdot r\sqrt{2r^2 - r\sqrt{4r^2 - p_n^2}} \ , \tag{3}$$

it follows

$$\kappa(n) = |p_n - 2\pi r| \approx 2\pi r/n \ , \quad \text{for} \quad n \geq 6 \ . \tag{4}$$

The function $\kappa(n)$ defines the speed of convergence, which is linear in this case. Altogether, the estimated length converges towards the true length with respect to regular n-gon approximation and the class of all circular curves.

II. Local metrics

Local metrics were historically the first attempts towards a solution of the length estimation problem in image analysis. These algorithms apply to digital curves defined by options (i) or (ii) in Definition 1, and can be viewed as shortest path calculations in weighted adjacency graphs of pixel locations. Weights have been designed with the intention of approximating the Euclidean distance. For example, horizontal and vertical moves in the orthogonal r-grid may be weighted by $1/r$, and diagonal moves may be weighted by $\sqrt{2}/r$. More generally, a chamfer metrics definition first lists elementary moves and then associates weights to each move, see [11], [12], [17].

In order to make length estimation as accurate as possible, the use of statistical analysis has been suggested to find those weights which minimize the mean square error between estimated and true length of a straight segment. For example, [12] presents a best linear unbiased estimator (BLUE for short) for straight lines, and defines the following length estimator:

$$\mathcal{E}_{BLUE}(D_r(\gamma)) = \frac{1}{r} \cdot (1.059 \cdot n_i + 1.183 \cdot n_d) \ , \tag{5}$$

where n_i is the number of isothetic steps and n_d of diagonal steps in the r-grid. We include this estimator into our comparative study. It ensures a superlinear convergence $O(r^{-1.5})$ of asymptotic length estimation [15] in case of digitized straight lines, i.e. it is multigrid convergent for r-grid-intersection digitization and the class Γ of all straight segments. However, it fails to be multigrid convergent for more general situations of digitized arcs or curves.

III. Polygonal DSS-approaches

Many algorithms have been published for the DSS recognition problem. Digital curves are again defined by options (i) or (ii) in Definition 1. DSS-approaches are based on characterizations of digital lines, such as syntactic chain code properties [5], [8], arithmetical properties defining tangential lines [19], properties of feasible regions in the (dual) parameter space [13], [15], or use linear programming tools such as the Fourier-Motzkin algorithm [20]. All these algorithms present a solution for deciding whether a given sequence of r-grid points is a DSS, or even for segmenting a digital curve into a sequence of maximum-length DSSs. The length estimator $\mathcal{E}_{DSS}$ is then defined by the length of the obtained polyline. Note that DSS-approximations are not uniquely defined, they vary from method to method, and depend in general upon the chosen start point and the orientation of curve tracing.

In our comparative study we include two representative implementations of DSS-based length estimators: if the digital curve is defined as an 8-curve, we use the Debled-Reveillès algorithm [19] and call it the $\mathcal{E}_{DR-DSS}$ estimator. In our implementation we strictly follow the algorithm as described in [19]. If the digital curve is defined as a 4-curve, we consider a length estimator based on Kovalevsky's algorithm [14] and call it the $\mathcal{E}_{VK-DSS}$ estimator. We use the algorithm as implemented for, and detailed in [26].

These two DSS-based length estimators are known to be multigrid-convergent for convex Jordan curves γ [18], [29], [31]. Given a simply-connected compact set S in the Euclidean plane and a grid resolution r, the *r-frontier* $\partial\mathbf{G}_r(S)$ of S is uniquely determined (i.e. the frontier of the Gauss digitization $\mathbf{G}_r(S)$ with respect to the topology of the Euclidean plane). Note that an r-frontier may consists of several non-connected curves even in the case of a bounded convex set S. A set S is *r-compact* iff there is a number $r_S > 0$ such that $\partial\mathbf{G}_r(S)$ is just one (connected) curve, for any $r \geq r_0$.

Theorem 1: (Kovalevsky/Fuchs 1992, Klette/Zunic 2000) Let S be a convex, r-compact polygonal set in $\Re^2$. Then there exists a grid resolution r_0 such that for all $r \geq r_0$, any DSS approximation of the r-frontier $\partial\mathbf{G}_r(S)$ is a connected polygon with perimeter p_r satisfying the inequality

$$|\mathcal{L}(\partial(S)) - p_r| \leq \frac{2\pi}{r}\left(\varepsilon_{DSS}(r) + \frac{1}{\sqrt{2}}\right) . \qquad (6)$$

This theorem and its proof can be found in [29]. The proof is based to a large extent on material given in [18]. The value of r_0 depends on the given set, and $\varepsilon_{DSS}(r) \geq 0$ is an algorithm-dependent approximation threshold specifying the maximum Hausdorff-Chebyshev distance (generalizing the Euclidean distance between points to a distance between sets of points) between the r-frontier $\partial \mathbf{G}_r(S)$ and the constructed (not uniquely specified - see comments above) DSS approximation polygon. Assuming $\varepsilon_{DSS}(r) = 1/r$, it follows that the upper error bound for DSS approximations is characterized by[1]

$$\frac{2\pi}{r^2} + \frac{2\pi}{r \cdot \sqrt{2}} \approx \frac{4.5}{r} \quad \text{if} \quad r \gg 1 \quad \text{(i.e. } r \text{ is large)} \tag{7}$$

Grid resolution $1/r$ is assumed in the chord property defined in [2], where a DSS is assumed to be a finite 8-path. In the case of using cell complexes it is appropriate to consider a finite 4-path as a DSS iff its main diagonal width is less than $\sqrt{2}$, see [10], [16], [18].

IV. Polygonal MLP-approaches

MLP-based length estimators consider a situation where a given simple digital curve C is described by two discrete curves γ_1 and γ_2, bounding sets S_1 and S_2 respectively, such that S_2 is contained in the interior S_1° of set S_1, and C is contained in $B = S_1 \setminus S_2^\circ$. Digital curves are defined in this case by option (iii) in Definition 1. The task consists in calculating that MLP which is contained in B and circumscribes γ_2. The length estimator $\mathcal{E}_{MLP}$ is then defined by the length of this (uniquely defined [21], [24]) MLP.

In our comparative study we include two representative implementations of MLP-based length estimators: the *grid-continua MLP* approach of [21], [24] has been derived for the model of using inner and outer Jordan digitization, and defining B to be the difference set between outer and inner Jordan digitization. We use the MLP algorithm as reported in [26]. We call it the $\mathcal{E}_{SZ-MLP}$ estimator.

As another MLP-method we include the *approximation-sausage MLP* approach of [27], [30] defining the $\mathcal{E}_{AS-MLP}$ estimator, which actually involves a parameter δ, with $0 < \delta \leq .5/r$. In our experimental evaluation we use the algorithm provided by the authors of this approach with $\delta = .5/r$

We briefly describe the approximation-sausage MLP method following [27], [30]: assume an r-frontier of S which can be represented in the form $P = (v_0, v_1, \ldots, v_{n-1})$ where vertices are clockwise ordered and the interior of S lies to the right. For each vertex of P we define forward and backward shifts: The *forward shift* $f(v_i)$ of v_i is the point on

[1] Let $\kappa(r) = 2\pi/r^2 + 2\pi/r \cdot \sqrt{2}$. Then it follows that $\kappa(r) \to \pi\sqrt{2}$ as $r \to \infty$.

the edge (v_i, v_{i+1}) at the distance δ from v_i. The *backward shift* $b(v_i)$ is that on the edge (v_{i-1}, v_i) at the distance δ from v_i.

For curve approximation, we replace an edge (v_i, v_{i+1}) by a line segment $(v_i, f(v_{i+1}))$ interconnecting v_i and the forward shift of v_{i+1}, which is referred to as the *forward approximating segment* and denoted by $L_f(v_i)$. The *backward approximating segment* $(v_i, b(v_{i-1}))$ is defined similarly and denoted by $L_b(v_i)$. Now we have three sets of edges, original edges of the r-frontier, forward and backward approximating segments. Based on these edges we define a connected region $A_r^\delta(S)$, which is homeomorphic to the annulus, as follows:

Given a polygonal circuit P describing an r-frontier in clockwise orientation, by reversing P we obtain a polygonal circuit Q in counterclockwise order. In the initialization step of our approximation procedure we consider P and Q as the *external* and *internal* bounding polygons of a polygon P_B homeomorphic to the annulus. It follows that this initial polygon P_B has area contents zero, and as a set of points it coincides with $\partial \mathbf{G}_r(S)$.

Now we 'move' the external polygon P 'away' from $\mathbf{G}_r(S)$, and the internal polygon Q 'into' $\mathbf{G}_r(S)$ as specified below. This process will expand P_B step by step into a final polygon which contains $\partial \mathbf{G}_r(S)$, and where the Hausdorff-Chebyshev distance between P and Q becomes non-zero. For this purpose, we add forward and backward approximating segments to P and Q in order to increase the area contents of the polygon P_B.

To be precise, for any forward or backward approximating segment $L_f(v_i)$ or $L_b(v_i)$ we first remove the part lying in the interior of the current polygon P_B and updating the polygon P_B by adding the remaining part of the segment as a new boundary edge. The direction of the edge is determined so that the interior of P_B lies to the right of it. The resulting polygon P_B^δ is referred to as the *approximating sausage* of the r-frontier and denoted by $A_r^\delta(S)$. The width of such an approximating sausage depends on the value of δ. An *AS-MLP curve* for approximating the boundary of S is defined as being a shortest closed curve $\gamma_r^\delta(S)$ lying entirely in the interior of the approximating sausage $A_r^\delta(S)$, and encircling the internal boundary of $A_r^\delta(S)$. It follows that such an AS-MLP curve $\gamma_r^\delta(S)$ is uniquely defined, and that it is a polygonal curve defined by finitely many straight segments. Note that this curve depends upon the choice of the approximation constant δ, and we have taken $\delta = .5/r$ for the experimental evaluation.

Both cited MLP-based length estimators are known to be multigrid-convergent for convex Jordan curves γ [21], [27], [30].

Theorem 2: (Asano/Kawamura/Klette/Obokkata 2000) Let S be a bounded, r-compact convex polygonal set. Then, there exists a grid resolution r_0 such that for all $r \geq r_0$ it holds that any AS-MLP approximation of the r-frontier $\partial\mathbf{G}_r(S)$, with $0 < \delta \leq .5/r$, is a connected polygon with a perimeter l_r and

$$|\mathcal{L}(\partial S) - l_r| \leq (4\sqrt{2} + 8 * 0.0234)/r = 5.844/r. \tag{8}$$

For the case of SZ-MLP approximations there are several convergence theorems in [24], showing that the perimeter of the SZ-MLP approximation is a convergent estimator of the perimeter of a bounded, convex, smooth or polygonal set in the Euclidean plane. The following theorem is basically a quotation from [24]; it specifies the asymptotic constant for SZ-MLP perimeter estimates. A sequence of r-squares, where any r-square has exactly two edge-neighboring r-squares in the sequence, is called a *one-dimensional grid continuum.*

Theorem 3: (Sloboda/Zaťko/Stoer 1998) Let γ be a (closed) convex curve in the Euclidean plane which is contained in a one-dimensional grid continuum of r-squares, for $r \geq 1$. Then the SZ-MLP approximation of this one-dimensional grid continuum is a connected polygonal curve with length l_r satisfying the inequality:

$$l_r \leq \mathcal{L}(\gamma) < l_r + \frac{8}{r} \tag{9}$$

These three Theorems 1, 2 and 3 show that the DSS error bound of $4.5/r$ is smaller than the AS-MLP bound $5.844/r$, and the AS-MLP is smaller than the SZ-MLP bound $8/r$.

V. Gradient-based approach

We use a gradient integration process to estimate the length of a curve. Let $\vec{n} : [0,1] \rightarrow \mathbb{R}^2$ denote the gradient or normal vector field associated with curve γ. The length of γ can be expressed as

$$l(\gamma) = \int_0^1 \vec{n}(s)ds \tag{10}$$

The main idea of the gradient-based approach consists in using discrete estimates of products $\vec{n}(s)ds$. This method was originally proposed by Ellis *et al.* in [3]. We present here both a linear in time algorithm and a multigrid convergence proof of such an estimator.

The discrete tangent on a digital curve was proposed in [22], and it is based on a chosen DSS definition (see Fig. 3): the discrete tangent at point p of a digital curve is the longest DSS centered at point p. Note that this definition is applicable whatever DSS definition (and related recognition algorithm) is chosen. A straightforward application of a linear on-line DSS recognition algorithms (at any point p) leads to an $O(n^2)$ solution. However, the optimization proposed in [25] allows to compute all tangents in linear time, assuming a cellular approach (i.e. a grid square has four $0-$cells as its vertices etc.) for defining pixel locations. The Vialard algorithm computes the discrete tangent and thus the discrete normal vector at each $0-$cell of the curve.

We define the normal vector $\tilde{n}$ associated to a $1-$cell as the mean vector of both vectors calculated at its two bounding $0-$cells. We also define an elementary normal vector n_{el} to a $1-$cell as the unit vector orthogonal to this $1-$cell (see Fig. 3). Hence, the discrete version of eq. (10) is:

$$\mathcal{E}_{TAN}(D_r(\gamma)) = \sum_{s \in \mathcal{S}} \tilde{n}(s) \cdot n_{el}(s) \tag{11}$$

where '$\cdot$' denotes the scalar product and $\mathcal{S}$ is the set of all $1-$cells of $D_r(\gamma)$, which is assumed to be an alternating sequence of $0-$cells and $1-$cells. The main idea of this approach is to compute the contribution of each $1-$cell to the global length estimation by projecting the $1-$cell according to the direction of the normal. In [32] it is shown that length estimator $\mathcal{E}_{TAN}$ is multigrid convergent for convex Jordan curves.

Theorem 4: (Coeurjolly/Teytaud 2001) Let γ be a C^2 curve with bounded curvature, then both the discrete normal vector direction and the gradient-based length estimator are multigrid convergent.

The speed of multigrid convergence, and the maximum error bound of the gradient-based length estimator require further studies to be determined.

VI. Evaluation

We compare the chosen length estimator algorithms based on practical experiments and available theoretical results. Regarding algorithmic complexity note that all are linear on-line algorithms. Theoretical studies should answer the following questions:

multigrid convergence : Is the estimator multigrid convergent for convex curves (if

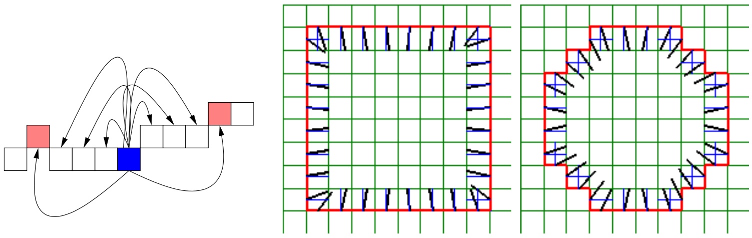

Fig. 3. Illustration of the Discrete tangent calculus and examples of discrete normal vector computations for a square and a circle: black (bold) vectors are estimated normals and blue (thin) vectors are elementary normals.

Method	multigrid	discrete	unique	3D extension	References
DR-DSS	yes	yes	no	[31]	[19]
VK-DSS	yes	yes	no	-	[14]
SZ-MLP	yes	yes	yes	[28]	[21], [27]
AS-MLP	yes	no	no	-	[30]
TAN	yes	yes	yes	-	[3], this article
Local metrics	no	depends	yes	[23]	[11], [12]

TABLE I

LENGTH ESTIMATORS USED IN THIS COMPARISON.

'yes', we are also interested in an analysis of convergence speed)?

discrete : Does the core of the algorithm only deal with integers?

unique : Does the result depend on initialization?

3D extension : May the approach be extended to length estimation of digital curves in 3D space? This point will discussed in the next section.

Table 1 informs on the situation. The convergence speed is known to be linear for DR-DSS, VK-DSS, SZ-MLP and AS-MLP.

We consider two measures: (i) the relative error in percent between estimated and true curve length, and (ii) for DSS- and MLP-approaches, also a *trade-off measure* defined as the product of relative error times the number of generated segments (in [24], [26] it has been called the *efficiency of convergence*).

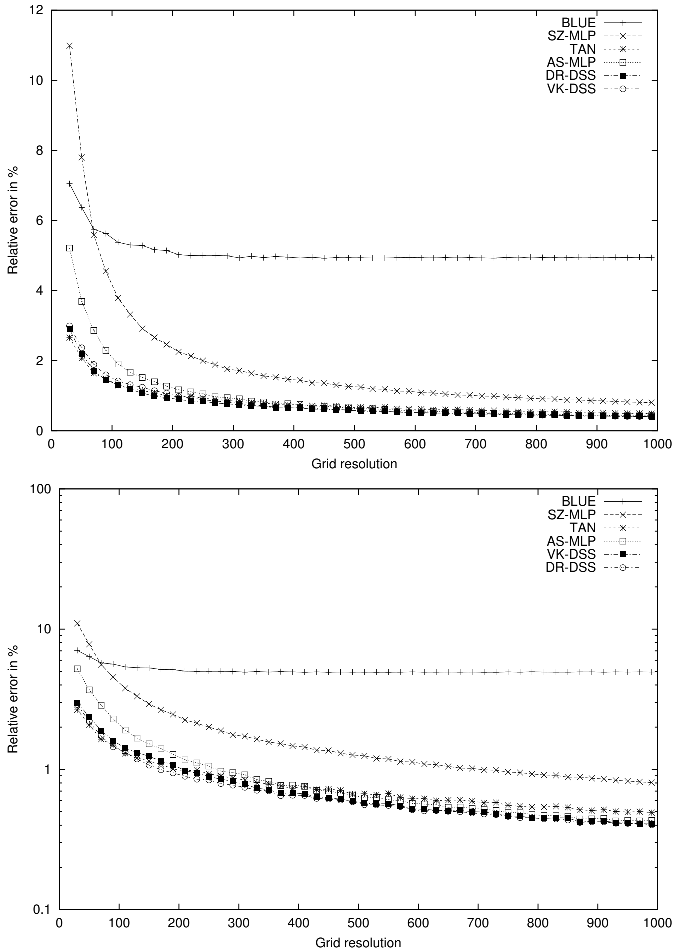

Fig. 4. Evident multigrid convergence of length estimators: sliding means of relative errors (above), also shown in a logarithmic scale (below).

In Fig. 4, experimental convergence is evident for all methods. However, in case of method BLUE we have convergence to a false value! Errors are calculated for all curves, transformed into a mean value for a given grid size, and curves are generated by sliding means (of 30 values) along different grid sizes. We show a decimal and a logarithmic scale of errors.

The trade-off measure is presented in Fig. 5 for polygonal approaches only. Again, values are calculated for all curves, transformed into a mean value for a given grid size, and curves are generated by sliding means (of 30 values) along different grid sizes.

As an additional test we also rotated a square of fixed size in a grid of resolution 128. Figure 6 shows the behavior of estimated perimeters for such a rotated square curve, for

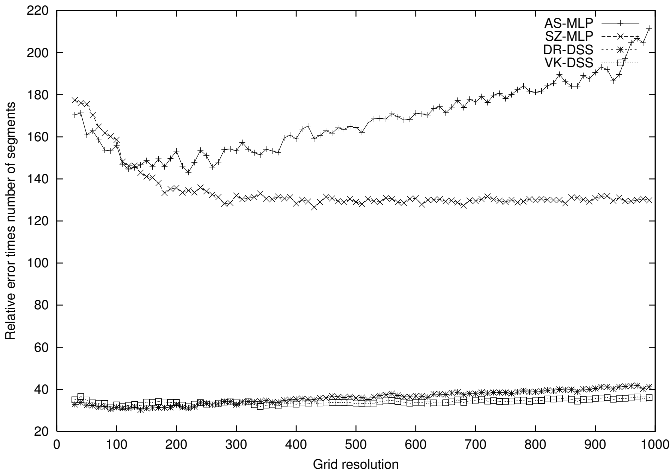

Fig. 5. Trade-off value diagrams for polygonal approaches.

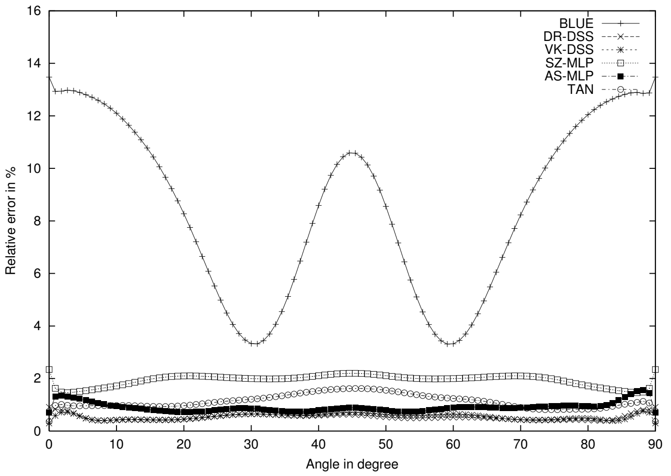

Fig. 6. Relative errors for a rotating square.

different estimation algorithms. Ideal would be a totally orientation-independent estimator, and all methods besides BLUE approximate this ideal situation reasonably well.

Figure 7 summarizes the run-times of polygonalization-based estimators compared to a local-metric estimator. Obviously, the local algorithm BLUE is the fastest. Method SZ-MLP in the implementation of [26] provides the fastest global estimator, but VK-DSS and DR-DSS are close. The AS-MLP algorithm has not yet been optimized, and faster implementations might be possible. Theoretically it is known that the TAN method allows a linear asymptotic run-time implementation [25]. However, the used program showed quadratic run-time. Further algorithmic optimization is needed.

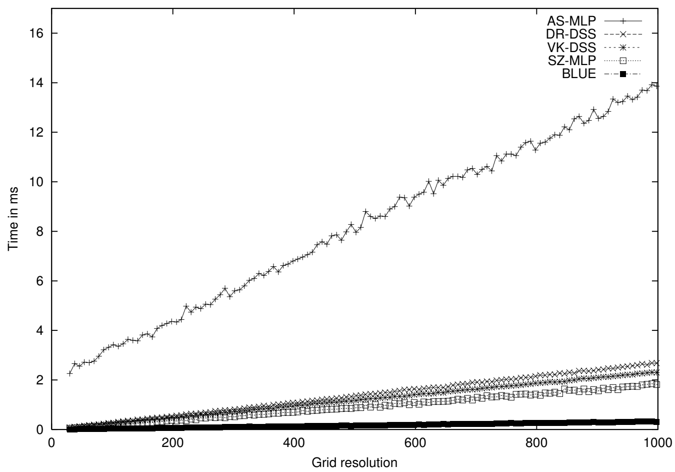

Fig. 7. Run-times of polygonalization-based estimators on an Ultra 10 Sparc Workstation.

VII. Conclusions

The experiments show that the local-metrics approach is not multigrid convergent for the test data set, but all five global approximation methods confirm known theoretical convergence results by experimental evidence. Furthermore, the increase in run-times of the studied polygonal methods is only minor compared to that of local-metric algorithms. Hence, the use of an (incorrect) local-metric algorithm is also not justified by a run-time argument. The choice of a global method may depend on preferences defined by the context of an image processing software package, and the authors can recommend any of the five studied global methods. Studies on test data might be useful for selecting the most efficient implementation for a given application context.

Originally the authors also indented to use a third measure for comparison, the minimum value r_0 of grid resolution r such that a method estimates π first time within the error interval defined by Equ. 2 (i.e. where a circular region is digitized in a grid with edge length $1/r$), what might be called the *Archimedes-Hui constant* of the algorithm implementing a method. However, due to oscillations of calculated estimations, results of an algorithm may be outside of this interval again for $r > r_0$ for an Archimedes-Hui constant r_0 of this algorithm, and just using a sliding mean can also not be recommended because of unsecured knowledge on the general behaviour of the measured error sequence. A methodically correct introduction of such an Archimedes-Hui constant remains an open problem.

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