TP4: Curvature estimation of digital curves

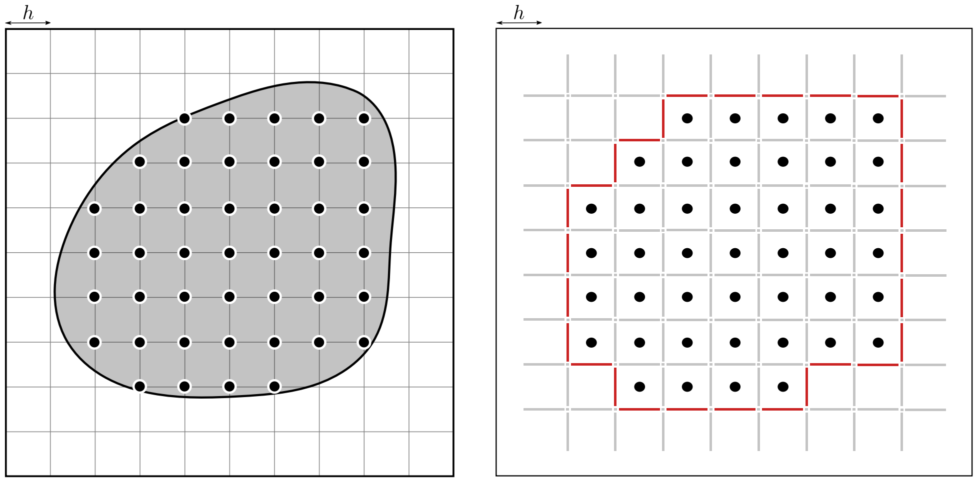

Figure 1: The Gauss Digitization $\mathtt{D}_h(X)$ of and Euclidean shape X (in gray) is the set of black points that lies within X (that are the embedded centers of 2-cells). The set of surfels of the Gauss Digitization is pictured in red.

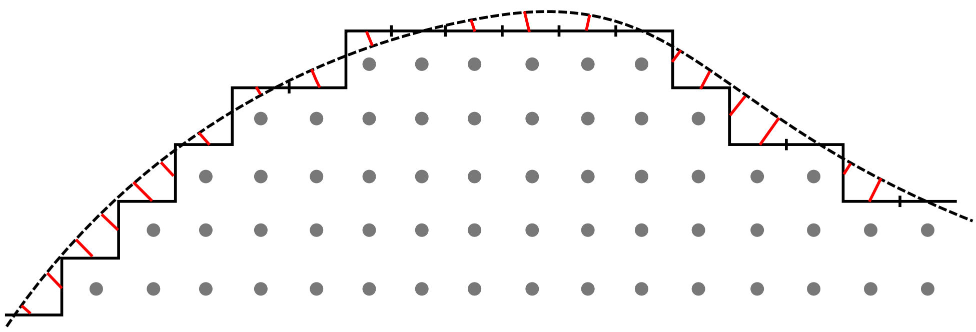

Figure 2: For a given linel l, the projection map ξ associates the centroid of l to the closest point on X.

We consider a compact shape $X \in \mathbb{R}^2$. The Gauss Digitization of X, namely $\mathtt{D}_h(X)$, at a grid step h is:

$$\mathtt{D}_h(X) := X \cap (h\mathbb{Z})^2,$$

which is simply the set of points of the infinite regular grid of size h that are inside X (see Fig.1). The discrete border of $\mathtt{D}_h(X)$ is roughly defined as the border of the shifted $h + h/2$-grid. In 2D, elements of dimensions 1 are called linels and elements of dimension 0 are called pointels (they correspond to the center of the original h-grid). The set of linels is denoted $\mathbb{E}^1$ and the set of pointels $\mathbb{E}^0$ for convenience.

For each elements of $l \in \mathbb{E}^1$ we call the projection of l the closest point of the centroid of l on X (see Fig.2). The associated map is denoted ξ and is called the projection map between X and $\mathtt{D}_h(X)$.

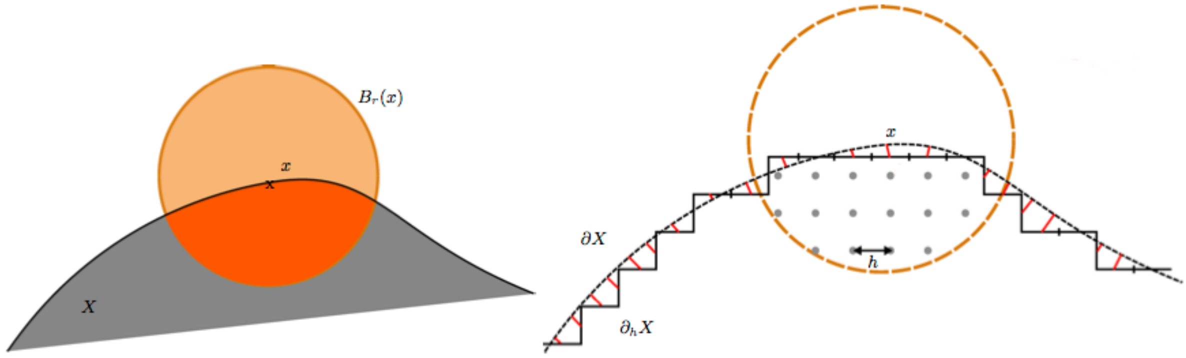

Figure 3: The volumetric integral (on the left) $V_R(x)$ is the area of the intersection between an euclidean ball $B_r(x)$ centered in x and the shape X.

Goal Compute a convergent estimator $\tilde{\kappa}$ of the real curvature κ at a point $x \in X$ using only the discrete structure. More precisely we want that $\forall \mathbf{x} \in \mathbb{E}^{\mathbf{1}}$

$$||\kappa(\xi(\dot{\mathbf{x}})) - \tilde{\kappa}(\dot{\mathbf{x}})||_\infty \leq \sigma(h)$$

where the limit of σ is zero as h tends to zero.

Integral Invariant Estimator

Given an euclidean ball $B_r(x)$ of radius r centered in x, we define the volumetric integral $V_r(x)$ (see Fig.3) as

$$V_r(x) := \int_{B_r(x)} \mathcal{X}(p)dp$$

where $\mathcal{X}$ is the characteristic function of X (it is equal to one if x is in X, zero otherwise).

We have

$$\kappa(x) \approx \frac{3\pi}{2r} - \frac{3V_r(x)}{r^3}$$

that links the curvature κ and the volumetric integral V_r at a point x. If we are able to compute the volumetric integral at a point x, we can estimate the curvature at the same point.

An existing result for digital area approximation is

$$\text{Area}(\mathtt{D}_h(X)) := h^2\text{Card}(\mathtt{D}_h(X)) = \text{Area}(X) + O(h)$$

where Card is simply the number of digital points within $\mathtt{D}_h(X)$. Therefore, to estimate $V_r(x)$ we can count the number of digital points in the intersection between $B_r(x)$ and $\mathtt{D}_h(X)$.

In order to achieve convergence of the estimator $\tilde{\kappa}$, the ball radius r must be set to $kh^{\frac{1}{3}}$ where k depends on the shape.

Assignment

You have to implement the integral invariant curvature estimator $\tilde{\kappa}$ on 2D digital curves. To do so, have a look at the file flowers.h. This header file provides helpers to compute the Gauss Digitization $\mathtt{D}_h(X)$ (where X can be here either a Flower or an Accelerated Flower), and to compute the real curvature.

Question Given a 2-cell $\mathbf{s}$, implement the estimator of the volumetric integral $V_r(\mathbf{s})$ (it must be parametrized by the ball radius r)

Question Implement the curvature estimator $\tilde{\kappa}$. For each linels l, compute the mean of the volumetric integral estimator of the two adjacent 2-cell.

Question Plot (using gnuplot for example) the maximum error between the real curvature κ and the estimated one $\tilde{\kappa}$ with a decreasing grid-step h (in logscale).