The ray theory of von Nardroff gives very nearly the same answer, Eq. (1).

The discrepancy between the two formulas apparently arises from the difference in treatment of intensity of rays near the edge of a particle. The von Nardroff calculation also errs in this respect, since it assumes perfect transmission of all rays except those suffering total external reflection. The difference in the formulas is actually very minor: for $\delta=10^{-5}$, the von Nardroff value of w is only about 3 percent larger than the width calculated here.

Confirmation of the supposition that "edge" effects are responsible for the above discrepancies is afforded rather easily by considering an artificial case in which such effects would be absent, namely, the scattering from right circular cones. We suppose the cones to have altitude h, radius of base R, and to be oriented always with their axes parallel to the incident beam. Every ray striking a cone will then be deviated, according to geometrical optics, by an angle $h|\delta|/R$ (to first order in δ). The root mean square deviation of rays will then also be $h|\delta|/R$, and von Nardroff's considerations of multiple scattering by an average of p scatterers per ray would lead to a Gaussian distribution of scattered rays with a width $w_{VN}'=p^{\frac{1}{2}}h|\delta|/R$. If, instead, the Huygen's principle calculation be applied to this case, the phase shift function $f(\mathbf{x},\mathbf{x}_n)$ of Eq. (2) is replaced by $f'(\mathbf{x},\mathbf{x}_n)$, where

$$f'(\mathbf{x},\mathbf{x}_n)=(2\pi\delta/\lambda)(h/R)(R-|\mathbf{x}-\mathbf{x}_n|),\quad |\mathbf{x}-\mathbf{x}_n|\leq R,$$
$$f'(\mathbf{x},\mathbf{x}_n)=0,\quad |\mathbf{x}-\mathbf{x}_n|>R.$$

All of the previous calculation for spheres now applies, and it is only necessary to work out the new value for the coefficient a, say a', corresponding to the phase shift f'. This proves to be $a'=(\pi^2h^2\delta^2)/(\lambda^2R^2)$. Inserting this in Eq. (7), the width becomes $w'=\lambda\pi^{-1}(pa')^{\frac{1}{2}}=p^{\frac{1}{2}}h|\delta|/R$. This width agrees precisely with the ray theory width given above, and, of course the distribution of scattered intensity is also Gaussian.

We conclude that the von Nardroff formulation is indeed justified for the case where (a), the scattering is highly multiple and, (b), the particles are sufficiently large that $R|\delta|/\lambda\gg 1$, but that it is not justified for scattering by one, or only a few isolated particles. It is not clear whether formula (9) or formula (1) is the better approximation, but for δ as ordinarily encountered, the discrepancy between them is not important.

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The Impulse Approximation*

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An analysis is made of the assumption that the scattering of an elementary particle by a complex nucleus is well approximated by the superposition of outgoing waves generated by the individual nucleons acting independently. The special case of nucleon-nucleon scattering is discussed in detail and an estimate given of the order of magnitude of the error in the approximation. Application of the method to meson problems is also discussed.

INTRODUCTION

IN an earlier paper[1] one of us proposed an "impulse approximation" for the study of collisions between fast nucleons and deuterons. This method, which can also be applied to other collision problems,[2] was advanced on the basis of plausibility arguments, while a more careful analysis of its validity by the present authors was promised. Some crude estimates of the errors involved in the method were quickly obtained for a typical case; further efforts led to some formal improvements, but in trying to explore the validity of the approximations under much more general conditions, we encountered serious difficulties. In view of the improbability of further rapid progress and because of the appearance of new applications,[2] we have decided to publish our present crude and incomplete results as a partial answer to the need for a better justification of the method.

Owing to the well-known inadequacy of the Born approximation in the case of nuclear forces,[3] one has no really good and simple method for the study of nuclear

* Work partially supported by the AEC.

[1] G. F. Chew, Phys. Rev. **80**, 196 (1950).

[2] Fernbach, Green, and Watson, Phys. Rev. **82**, 980 (1951); B. Segall, Phys. Rev. **83**, 1247 (1951); W. Cheston, Phys. Rev. (to be published); Y. Fujimoto and Y. Yamaguchi, Prog. Theor. Phys. **6**, 166 (1951); G. F. Chew and H. W. Lewis, Phys. Rev. **84**, 779 (1951).

[3] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

collisions involving more than two nucleons. There are many circumstances, however, under which the many-body character of a problem is only a secondary feature, and the collision may be decomposed, as it were, into a superposition of simple two-body collisions. This is a familiar thought, which finds expression in a variety of approximations, ranging from the crude additivity rule for cross sections, which is seldom applicable, to more sophisticated procedures, such as the Fermi theory of the effect of the chemical bond on slow neutron scattering.[4] The impulse approximation is another expression of the same thought.

Mention should be made of a modification of the Born approximation[5] which leads to expressions essentially equivalent to those obtained with the impulse approximation. The difference between the two methods is then mainly one of emphasis and physical picture. The assumption on which the modified Born method is based can perhaps be expressed as follows. One calculates in the first approximation the scattering amplitudes for (a) the nucleon-nucleon and (b) the nucleon-deuteron collision problem, for example; one then assumes that the relationship that one obtains between the two, via the elimination of the Fourier components of the nucleon-nucleon interaction, is valid even though the Born approximation is not.

The assumptions underlying the impulse approximation are somewhat different and perhaps a little more physically plausible. Consider as a typical case a simple particle (say a nucleon) striking a complex system (deuteron, light nucleus). The assumptions are then the following: (I) The incident particle never interacts strongly with two constituents of the system at the same time; (II) the amplitude of the incident wave falling on each constituent (nucleon) is nearly the same as if that constituent were alone; and (III) the binding forces between the constituents of the system are negligible during the decisive phase of the collision, when the incident particle interacts strongly with the system.

Of these assumptions, I and III are seen to be vital, if one wants to avoid essential n-body features in the problem; II, which may be called the transparency requirements, is less vital, but simplifies matters considerably. Condition I is the one that perhaps needs least justification in most applications, for instance in the case of a loose structure like the deuteron, in which the average distance between the constituents is noticeably larger than the range of the interaction. I and II will be discussed in the third section of the paper.

The main part of the paper is thus devoted to a study of the consequences and validity of condition III, which is in fact the most interesting one. It is from this condition that the name of the approximation is derived, in analogy with those problems in classical dynamics in which one considers the effect on a system of a force acting for a time so short, that all other forces may be neglected during that time.[6] In a quantum-mechanical problem, of course, such an assumption involves some well-known paradoxical features. For example, the binding forces determine the energy levels of the struck system before and after the collision, so that in the case of discrete energy levels it is obvious that the change in energy suffered by the incident particle in the collision is directly affected by the binding forces, in a manner which at first sight is incompatible with our assumption. A proper formulation of the "impulse" assumption in quantum mechanics is possible, however, as soon as due attention is paid to some perfectly familiar points. Thus the energy paradox is removed when one remembers that the very concept of a collision of short duration τ implies the description of the incident particle by means of a short train of waves, whereby any attempt to control the energy E of the system to better than $\Delta E \sim \hbar/\tau$ becomes impossible. Clearly, the process under study must not be too strongly energy dependent, or the whole method is meaningless.

The basic assumption is then that τ can be chosen so small that the change of the many-body wave function during the collision is not noticeably affected by the binding forces. The role of the binding forces is then merely to determine the eigenstates in terms of which the wave function is expanded before and after the collision.

Now to put these ideas into formulas.[7] Let K be the total kinetic energy operator, V the interaction between the incident particle and the system, and U the potential of the binding forces, so that the total Hamiltonian is

$$H = K + V + U, \tag{1}$$

while

$$H_0 = K + U \tag{2}$$

is the "unperturbed" Hamiltonian.

Let Φ_a be the wave function for some initial state, say the product of the eigenfunction of some definite state of the struck system by a wave packet for the motion of the incident particle relative to the system. We write the time dependence of the wave function explicitly as a factor multiplying Φ_a (for simplicity in the rest of this discussion we set $\hbar = 1$). This will hold until some time t_1 when the collision begins. At that time, therefore, the wave function is

$$\exp(-iH_0 t_1)\Phi_a. \tag{3}$$

From then on the wave function evolves according to the complete Hamiltonian, so that at the end of the collision, at a time

$$t_2 = t_1 + \tau,$$

[4] E. Fermi, Ricerca Sci. **VII–II**, 13 (1936).
[5] R. L. Gluckstern and H. A. Bethe, Phys. Rev. **81**, 761 (1951).

[6] The further assumption, often made, that the displacement of the struck system can be neglected during the collision, will not be made here.
[7] See Ashkin and Wick, this issue of *The Physical Review*, for a formulation much improved over the one presented here.

the wave function is

$$e^{-iH\tau}e^{-iH_0t_1}\Phi_a. \tag{4}$$

To find the probability of transition to some final state Φ_b, defined analogously to Φ_a, and which is, therefore, given at the time t_2 by

$$\exp(-iH_0t_2)\Phi_b, \tag{5}$$

one must take the scalar product of (4) and (5), which is equal to the matrix element $(\Phi_b, S\Phi_a)$, where

$$S=e^{iH_0t_2}e^{-iH\tau}e^{-iH_0t_1}. \tag{6}$$

It is easy to see that (6) is the transformation function which Schwinger, Dyson, and others[8] use to define the Heisenberg S-matrix. The exact S-matrix is obtained in the limit $t_1\rightarrow-\infty$, $t_2\rightarrow+\infty$, in which case the energy of the wave packets Φ_a and Φ_b can be defined with arbitrary accuracy. We here use (6) as an approximate expression for S, applicable only to suitably chosen wave packets, such that $t_2-t_1=\tau$ can be assumed very small.

Our basic assumption is now that in the middle exponential in (6) one can neglect U, i.e., that

$$|U|\tau\ll1 \tag{7}$$

(in ordinary units: $|U|\tau\ll\hbar$). For the same reason one can neglect U in the lateral exponentials; to see this most simply consider that, for the wave packets to which it is to be applied, S is obviously independent of how we choose the origin of time. Let then $t_1+t_2=0$, so that $t_1=-\tau/2$, $t_2=+\tau/2$. Neglecting then $\pm U\tau/2$ in the side exponentials, one has approximately $S\approx s$ where

$$s=e^{iKt_2}e^{-i(K+V)(t_2-t_1)}e^{-iKt_1}. \tag{8}$$

Now s is just the transformation which in the limit $t_1\rightarrow-\infty$, $t_2\rightarrow+\infty$ defines the S-matrix for the collision between the incident particle and a system of unbound particles. In order to arrive at the simple expression of the impulse approximation one has still to argue that owing to the high velocity of the incident particle and short range of the force, the expression (8) is already practically equal to the limit, $t_1=-\infty$, $t_2=+\infty$, when the collision time, $\tau=t_2-t_1$ satisfies (7); furthermore, one has to use assumptions (I) and (II) (we have only used III so far) to reduce s to an expression involving only two particles at a time. These assumptions clearly imply that the initial spatial distribution of the scattering particles is such that, once these particles are treated as free, the scatterings of the incident particle by particles number 1, 2, $\cdots$ are independent.

Thus the transition operator for the complex system, when our three assumptions are valid, will be simply a linear superposition of the two particle transition operators which describe the scattering of the incident particle by the individual target particles. This result is given its most useful expression in terms of that form of the T matrix[8] which does not contain as a factor the energy delta-function

$$T\approx t=t_1+t_2+\cdots, \tag{9}$$

where t_i is the t-matrix for the scattering of the incident particle by particle i. The probability amplitude for the transition from state a to state b is finally,

$$(\Phi_b, [t_1+t_2+\cdots t_n]\Phi_a). \tag{10}$$

This is the basic formula that has been used so far under the name of impulse approximation. The ambiguities about treatment of energy conservation which may trouble the reader at this point will be completely removed in the next section, where a systematic study is to be made of the error arising from assumption III. For that purpose we shall consider a simplified problem in which only that assumption is involved, namely, the case in which the incident particle interacts with a single target particle bound to a fixed field of force.

THE IMPULSE ASSUMPTION

Using the formalism of Lippmann and Schwinger,[8] the exact solution of the simplified problem is a wave function Ψ_a, which satisfies the integral equation,

$$\Psi_a=\Phi_a+\frac{1}{E_a+i\eta-K-U}V\Psi_a. \tag{11}$$

The quantity η is a real positive number which is to be allowed to go to zero after one has performed the sum over states required to give meaning to the operator $(E_a+i\eta-K-U)^{-1}$. The function Φ_a is a product of an incident plane wave in the first particle's coordinates and a bound state function in the coordinates of the second. That is to say, Φ_a is an eigenfunction of the operator, $K+U$, with the eigenvalue, E_a.

If we know Ψ_a, then the exact T-matrix is given by

$$T_{ba}=(\Phi_b, V\Psi_a). \tag{12}$$

The impulse approximation (10) amounts in this simple case of a single scatterer to the statement that $T\approx t$ or

$$T_{ba}\approx(\Phi_b, t\Phi_a), \tag{13}$$

where t is the operator which describes the free, two particle scattering. We can define t in terms of the wave function, ψ_n, which describes the free scattering of the pair of particles, starting from an eigenstate χ_n of the kinetic energy operator, K. That is, the function ψ_n satisfies the integral equation

$$\psi_n=\chi_n+\frac{1}{E_n+i\eta-K}V\psi_n, \tag{14}$$

if E_n is the eigenvalue of K belonging to χ_n. In the χ_n representation, the matrix elements of t are defined by

$$t_{mn}=(\chi_m, V\psi_n). \tag{15}$$

[8] See, for instance, B. A. Lippmann and J. S. Schwinger, Phys. Rev. **79**, 469 (1950).

The function $t\Phi_a$ is, therefore, to be understood in terms of the expansion of Φ_a in the set χ_n,

$$t\Phi_a = \sum_{m,n} \chi_m t_{mn}(\chi_n, \Phi_a) = V \sum_n \psi_n(\chi_n, \Phi_a). \quad (16)$$

Comparing (16) with (13) and (12) we see that an alternative way of expressing the impulse approximation is to say that where V is important (has large matrix elements), one assumes the correct wave function, Ψ_a, to be well represented by the function,

$$\Psi_a{}^0 = \sum_n \psi_n(\chi_n, \Phi_a). \quad (17)$$

As already discussed by Chew,[1] this function represents the scattering of the incident particle by a wave packet of the free target particle, a packet which contains the same momentum distribution as the actual bound state.

Since there is a one to one correspondence between the two particle scattering functions ψ_n and the basis functions χ_n, one may define a matrix,

$$\Omega_{mn} = (\chi_m, \psi_n), \quad (18)$$

and write (17) in the shorthand form[9]

$$\Psi_a{}^0 = \sum_{m,n} \chi_m \Omega_{mn}(\chi_n, \Phi_a) = \Omega\Phi_a. \quad (19)$$

It is now proposed to investigate the accuracy of the impulse approximation (13) by employing $\Psi_a{}^0$ as the trial function in a variational expression for T_{ba}. Such an expression has been given by Lippmann and Schwinger,[8]

$$\mathcal{T}_{ba} = (\Phi_b, V\Psi_a{}^+) + \left(\Psi_b{}^-, V\left[\Phi_a + \frac{1}{E_a + i\eta - K - U} V\Psi_a{}^+ - \Psi_a{}^+\right]\right). \quad (20)$$

This expression is stationary with respect to independent variations of the functions $\Psi_a{}^+$ and $\Psi_b{}^-$ about the solutions of the following equations:

$$\Psi_a{}^+ = \Phi_a + \frac{1}{E_a + i\eta - K - U} V\Psi_a{}^+, \quad (21)$$

$$\Psi_b{}^- = \Phi_b + \frac{1}{E_b - i\eta - K - U} V\Psi_b{}^-. \quad (21')$$

Thus $\mathcal{T}_{ba}$ is stationary about $\Psi_a{}^+ = \Psi_a$, since Eq. (11) is identical with (21). The stationary value of $\mathcal{T}_{ba}$ is T_{ba}, the matrix element we are seeking.

We argue now that to replace $\Psi_a{}^+$ in (20) by $\Psi_a{}^0$ will give a better value for T_{ba} than simply substituting into (12), the latter procedure being the one which leads to the impulse approximation. The reason, of course, is that (20) is stationary, whereas (12) is not. Thus we can expect to find a correction to the impulse approximation by employing (20).

Using $\Psi_a{}^0$ as a trial function in this way, one finds

$$T_{ba} = (\Phi_b, V\Psi_a{}^0) + (\Psi_b{}^-, V\Psi_a{}'), \quad (22)$$

where

$$\Psi_a{}' = \Phi_a + \frac{1}{E_a + i\eta - K - U} V\Psi_a{}^0 - \Psi_a{}^0. \quad (23)$$

Equation (22) is an identity so long as $\Psi_b{}^-$ is exact. It is clear even from (23) that the smallness of $\Psi_a{}'$ is somehow a measure of the accuracy of the function $\Psi_a{}^0$, for if the latter were exact, then $\Psi_a{}'$ would vanish identically.

Using the fact that from (14) and (17),

$$\Psi_a{}^0 = \Phi_a + \sum_n \frac{1}{E_n + i\eta - K} V\psi_n(\chi_n, \Phi_a), \quad (24)$$

one finds

$$\Psi_a{}' = \sum_n \left[\frac{1}{E_a + i\eta - K - U} - \frac{1}{E_n + i\eta - K}\right] V\psi_n(\chi_n, \Phi_a), \quad (25)$$

which may be contracted by using the operator relation,

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}. \quad (26)$$

In our case, $B - A = (E_n - E_a) + U$, so that a useful further relation is

$$(E_a - E_n)(\chi_n, \Phi_a) = (\chi_n, U\Phi_a). \quad (27)$$

This last equality is simply the Schrödinger equation for the state Φ_a, expressed in the χ_n representation. Putting together (26) and (27) and once again making use of (14), one finds

$$\Psi_a{}' = \frac{1}{E_a + i\eta - K - U} \sum_n \{U\psi_n(\chi_n, \Phi_a) - \psi_n(\chi_n, U\Phi_a)\}. \quad (28)$$

Using the shorthand notation as in (19), this may be rewritten as

$$\Psi_a{}' = \frac{1}{E_a + i\eta - K - U} [U, \Omega]\Phi_a. \quad (28')$$

Returning to (22) one may consistently replace the function $\Psi_b{}^-$ by

$$\sum_n \psi_n{}^-(\chi_n, \Phi_b), \quad (29)$$

where $\psi_n{}^-$ satisfies the integral equation

$$\psi_n{}^- = \chi_n + \frac{1}{E_n - i\eta - K} V\psi_n{}^-. \quad (30)$$

This is analogous to the replacement of $\Psi_a{}^+$ by $\Psi_a{}^0$ and has exactly the same justification. Corresponding to the matrix t, one can define a matrix t^- with elements

$$t_{mn}{}^- = (\psi_m{}^-, V\chi_n), \quad (31)$$

and it can be shown[8] that for $E_m = E_n$, $t_{mn}{}^- = t_{mn}$. In

[9] We are indebted to M. Gell Mann for calling to our attention the appropriate nature of the matrix notation in this problem. Ω is essentially the matrix wave function of C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 23, No. 1 (1945).

fact, since T_{ba} can just as well be written $(\Psi_b^-, V\Phi_a)$ as $(\Phi_b, V\Psi_a^+)$, one may expect t^- to be equally as accurate as t as an approximation to T. Evidently the validity of the impulse approximation implies that the relevant matrix elements of t^- are not much different from those of t.

Making the approximation (29) for Ψ_b^- as it appears in (22), one finds for the second term of (22)

$$\sum_n (\Phi_b, \chi_n)(\psi_n^-, V\Psi_a'),$$

which by virtue of (31) can be expressed as

$$(\Phi_b, t^-\Psi_a'). \tag{32}$$

Now we have argued that if the second term of (22) is small compared to the first, then the difference between t^- and t cannot be important. Thus it is consistent to replace t^- by t in (32), so long as we regard the latter as only a correction term and rewrite (22) in the form

$$\begin{aligned} T_{ba} &\approx (\Phi_b, t\Phi_a)+(\Phi_b, t\Psi_a') \\ &= (\Phi_b, t\{1+(E_a+i\eta-K-U)^{-1}[U, \Omega]\}\Phi_a). \end{aligned} \tag{22'}$$

There exists a strong suspicion that the derivation of (22′) presented here is unnecessarily complicated. Evidently an expansion of the T matrix in powers of U is the desired goal, and we have here obtained the zeroth- and first-order terms, *viz.*,

$$T = t\left\{1 - i\int_{-\infty}^{0} dx e^{\eta x} e^{iKx}[U, \Omega]e^{-iKx} + 0(U^2)\right\} \tag{33}$$

using a manner of expression recommended by Snyder.[10] Since the corresponding exact integral equation for T is

$$T = V\left\{1 - i\int_{-\infty}^{0} dx e^{\eta x} e^{i(K+U)x} T e^{-i(K+U)x}\right\}, \tag{34}$$

while that for t is

$$t = V\left\{1 - i\int_{-\infty}^{0} dx e^{\eta x} e^{iKx} t e^{-iKx}\right\}, \tag{35}$$

the suggestion is unavoidable that (33) can be obtained directly by expanding (34) with reference to (35) in powers of U. The technique of carrying out this expansion has eluded us so far, however, and we have resorted to the variational approach as a temporary "crutch."

The form (22′) shows that what was meant in the introduction, when speaking of the "collision time" τ, was $1/\langle\Delta E\rangle_{\mathrm{Av}}$, where $\langle\Delta E\rangle_{\mathrm{Av}}$ is some kind of average of the extent to which energy conservation is "violated" during the collision. This average is represented formally by the denominator in (28′). On the other hand, the assumption (7) was unnecessarily strong. We have found that (7) should be replaced by

$$\langle|\tau[U, \Omega]|\rangle_{\mathrm{Av}} \ll 1, \tag{7'}$$

[10] H. Snyder, private communication.

so that strong binding potentials, if they depend "weakly" on the nucleon variables, may be tractable with the impulse approximation. Actually, nuclear forces are strongly spin dependent, so the difference between (7) and (7′) is not important when the target is a nucleus. The difference would be important if the binding were molecular.

ASSUMPTIONS I AND II

A quantitative criterion has been achieved for the validity of assumption III, the impulse assumption, which can be used for any type of scattering problem. We have not yet succeeded in formulating so generally the criterion for the validity of assumption II, the transparency assumption.† One can, however, make two qualitative statements.

(1) When the target system contains many particles, the "mean free path" for the incident particle in the scattering material must certainly be large compared to the over-all dimensions of the scatterer. Otherwise the target particles on the "far" side of the scattering region will be shielded and will not contribute as much as if they were free and alone. It is well known that for special cases a correction for this attenuation of the incident beam can be made without difficulty, and it seems not impossible that a general formulation of the correction procedure may yet be achieved. However, for the time being, applications of the impulse approximation must be restricted to light nuclei. The exact value of the maximum permissible radius will depend on the value of the single particle scattering cross section. If we assume constant nuclear density, so that the nuclear radius is given by $r_0 A^{\frac{1}{3}}$ ($r_0 = 1.4\times 10^{-13}$ cm), then the mean free path within the nucleus is $(4\pi/3)r_0^3/\sigma$ (σ is the single nucleon cross section), and the transparency requirement becomes

$$(3/4\pi)\sigma A^{\frac{1}{3}}/r_0^2 \ll 1. \tag{36}$$

(2) For target systems with only a few particles, the transparency requirement has a more subtle aspect. The amplitude of the scattered wave sent out by a particular nucleon must, by the time it reaches another nucleon have become negligible in comparison to the amplitude of the incident wave. Otherwise "multiple scattering" effects will be significant. Naively, one would expect the multiple scattering error to be of the order of the amplitude of the scattered wave from a single nucleon evaluated, for unit incident amplitude, at a distance $\bar{R}$, if $\bar{R}$ is the average internucleon distance. In terms of the cross section, σ, this becomes

$$(\sigma/4\pi)^{\frac{1}{2}}/\bar{R} \ll 1. \tag{37}$$

Actually, it will be shown that when the wavelength of the scattered particle is small compared to the average nucleon separation distance R, interference reduces the average amplitude of the scattered wave by a factor

† This formulation will be presented in a forthcoming paper by G. F. Chew and M. L. Goldberger.

$\sim\lambda/\bar{R}$, so that (37) is replaced by

$$(\lambda/\bar{R})(\sigma/4\pi)^{\frac{1}{2}}/\bar{R}\ll 1. \qquad (37')$$

It is plausible that (36) and (37) or (37′) together form the general criteria for the validity of assumption II. However, no systematic quantum-mechanical treatment comparable to that of the last section for assumption III has yet been achieved.† The situation with regard to assumption I is even worse. One might say that the "range" of the interaction of the incident particle with a target nucleon must be much less than $\bar{R}$; but in some cases the range of the interaction may be difficult to define. Each problem requires individual study. The next section will discuss in some detail the special case of nucleon-nucleon scattering. The following one will treat, in a more qualitative way, meson-nucleon scattering. These two examples will fairly well illustrate the meaning of requirement I with respect to that of II and III.

THE SCATTERING OF HIGH ENERGY NUCLEONS BY COMPLEX NUCLEI

With the usual assumption that the interaction between two nucleons is describable in terms of a velocity independent potential, the "range" of the fundamental interaction would seem to have a fairly definite meaning. Nevertheless this "range" decreases with increasing particle energy as the wavelength becomes shorter, and it would be incorrect to use the effective range as defined by low energy scattering experiments to estimate the validity of assumption I in the hundred million-volt region. Probably a more reasonable way to make the estimate at high energies is to say the total cross section is "geometrical," that is define the range, ρ, by $\pi\rho^2=\sigma^{\text{total}}$. Using $\sigma^{\text{total}}=4\times 10^{-26}$ cm^2 as an average value, this estimate leads to $\rho\approx 1.1\times 10^{-13}$ cm. Assumption I is thus marginal for an average nucleus where $\bar{R}=2.2\times 10^{-13}$ cm but valid for the deuteron. For the deuteron, where $\bar{R}=3.2\times 10^{-13}$ cm, one can make a simple geometrical estimate of the probability that a particle which is less than 1.1×10^{-13} cm from the proton should at the same time be less than 1.1×10^{-13} cm from the neutron. The result is about 10 percent. Note that it does not necessarily follow that the error in the impulse approximation due to this source is as much as 10 percent. (For example, if the Born approximation were valid, there would be no error at all!)

The analysis of the transparency requirement (II) can be carried out quite simply if the target nucleons are replaced by fixed scattering centers of zero range force whose individual total cross sections are equal to those of the actual nucleons. It is evident that the essence of the transparency question is unaltered by this simplification. Consider, then, two fixed centers, separated by a displacement $\mathbf{R}$, which would individually have scattering lengths a_1 and a_2 (e.g., for unit incident amplitude, the scattered wave from the first center alone would have amplitude $(a_1/r_1)\exp(ir_1/\lambda)$, where r_1 is the distance from the center). A short calculation shows that in the presence of the second center the effective scattering length of the first is altered to

$$a_1'=a_1\frac{1+\exp(i\mathbf{k}_a\cdot\mathbf{R})(a_2/R)\exp(ik_aR)}{1-(a_1/R)\exp(ik_aR)(a_2/R)\exp(ik_aR)}, \qquad (38)$$

where $\mathbf{k}_a$ is the wave number of the incident particle ($k_a=1/\lambda$). The modification in the numerator is obviously due to the initial outgoing wave from the second center which impinges upon the first. The denominator represents waves which bounce back and forth many times. Keeping only linear and quadratic terms in the scattering lengths, the denominator of (38) may be neglected; and if one averages the displacement vector R over a spherical distribution, one finds for $k_a\bar{R}\gg 1$,

$$a_1'\approx a_1(1+ia_2/2k_a\bar{R}^2), \qquad (39)$$

so that the error involved in approximating a_1' by a_1 is $\sim(a_2/\bar{R})(\lambda/\bar{R})$. This is just the result (37′) with $\sigma=4\pi a_2^2$. For 200-Mev nucleon-deuteron scattering, $\lambda/\bar{R}\approx\frac{1}{6}$ and $a/\bar{R}\approx\frac{1}{6}$. This makes the multiple scattering error $\sim$3 percent. The wavelength varies inversely with the square root of the energy, and below 100 Mev the nucleon-nucleon cross sections increase sharply. Thus at 50 Mev, $\lambda/\bar{R}\approx\frac{1}{2}$ and $a/\bar{R}\approx\frac{1}{3}$, pushing the multiple scattering error to 16 percent.

It is doubtful that one can or ought to talk of "attenuation" in a structure as simple as the deuteron. In a nucleus heavier than the alpha-particle, however, an estimate should be made of the requirement (36). We find, with $\sigma=4\times 10^{-26}$ cm^2, that

$$(3/4\pi)\sigma A^{\frac{1}{3}}/r_0^2\approx\tfrac{1}{2}A^{\frac{1}{3}}. \qquad (40)$$

Thus the attenuation error can be very serious in complex nuclei.

We come finally to assumption III, the impulse assumption. The problem is to estimate the importance of the second term in (22′) with respect to the first. As stated above, the order of magnitude is $\tau[U,\Omega]$, where $1/\tau$ is the "average" violation of energy conservation "during" the collision. The value of this ratio depends considerably on the model adopted to describe the nucleon-nucleon interaction, i.e., on Ω. For a zero range force with strong spin dependence, for example, one finds, upon evaluation of (22′) in the limit $\lambda\ll\bar{R}$,

$$\tau[U,\Omega]\sim(\langle\Delta U\rangle_{\text{Av}}/E_a)(a/\lambda), \qquad (41)$$

where $\langle\Delta U\rangle_{\text{Av}}$ is the average change of potential energy of the target nucleon in its ground state when its spin is flipped, and a is the scattering length for spin flip. E_a is the kinetic energy of the incident nucleon. One may interpret this result by saying that a zero range force is intrinsically capable of an arbitrarily large "violation" of energy conservation. The average value of $1/\tau$ then is determined by the incident energy itself.

A simpler interpretation is that the collision time is the time required for one wavelength of the incident particle to pass a fixed point. This is $\sim 1/E_a$.

The occurrence of the factor a/λ in (41) is surprising until one remembers that when the first Born approximation is valid there is no error in the impulse approximation. The error in the Born approximation decreases as the scattering becomes weaker. Therefore, so must the error in the impulse approximation.

The situation is quite different when field theory is used to describe the scattering as occurring through the exchange of a meson. Here the scattering cannot proceed in first Born approximation, so the impulse error is not proportional to the scattering length. On the other hand the average violation of energy conservation is greater, being of the order of the total energy of the exchanged meson. Thus (41) is replaced by

$$\langle \Delta U \rangle_{\mathrm{Av}} / (\mu^2 c^4 + c^2 k_a^2)^{\frac{1}{2}}, \tag{42}$$

if μ is the meson mass and $\langle \Delta U \rangle_{\mathrm{Av}}$ is now the change in potential energy which occurs when a meson is emitted by the target nucleon. This may be due to a flip of either the spin or the isotopic spin or both.

Taking $\langle \Delta U \rangle_{\mathrm{Av}} \approx 10$ Mev, one finds for $E_a = 200$ Mev and $a = 0.6 \times 10^{-13}$ cm that (41) gives about 5 percent for the impulse error. Formula (42) gives an error of 2 percent, so there is no significant difference between (41) and (42) at high energies. However, below 100 Mev (41) becomes very much larger with decreasing energy, whereas (42) approaches a constant value of only a few percent. At the lower energies, however, the probability of exchanging more than one meson in a collision becomes large and (42) cannot be expected to hold. That is to say, account must be taken of important higher order intermediate states in which no mesons are present and which violate energy conservation by much less than μc^2.

Neither of the models upon which (41) and (42) are based can be taken literally. The fact that both predict only a small error in the impulse assumption at high energies, however, is certainly significant. It would be very surprising if the correct model of the nuclear force, when and if it is discovered, should lead to a large error.

We see in retrospect that the most serious limitations of the "impulse" approximation, as applied to the collisions of nucleons with nuclei, are not associated with the impulse assumption itself but rather with the "secondary" assumptions I and II. All three assumptions are valid to within a few percent for high energy nucleon-deuteron scattering. In more tightly packed nuclei, however, assumption I becomes questionable. Assumption II definitely will lead to a large error for any target heavier than the alpha-particle.

THE SCATTERING OF MESONS BY COMPLEX NUCLEI

The scattering of mesons by nucleons is even less understood theoretically, if that is possible, than the scattering of nucleons by nucleons. Nevertheless, the validity of the impulse approximation is highly probable. In the hundred million-volt region, the meson-nucleon cross section is of about the same size[11] as the nucleon-nucleon cross section. All statements made in the previous section about assumptions I and II, therefore, apply equally well to the scattering of high energy mesons by complex nuclei. The meson cross section decreases sharply with decreasing meson momentum, however, so the transparency assumption (II) gets better instead of worse at low energies, and the method can be applied there to correspondingly heavier target nuclei.

With respect to the impulse assumption one can argue that formula (42) should give the order of magnitude of the error, if $\mathbf{k}_a$ is now the incident meson momentum. That is, if the scattering process takes place via an intermediate state in which there are either no mesons or two mesons present, the violation of energy conservation is simply the energy of the one meson which has been created or destroyed. Segall[2] demonstrates this result in detail for meson-deuteron scattering, showing that the impulse error, even for slow mesons, is unlikely to be more than about 10 percent.

One may worry that, as in the case of nucleon-nucleon scattering, formula (42) is completely untrustworthy at low energies. Empirically, however, there does not seem to be a strong resonance in meson-nucleon scattering comparable to that which occurs at low energies in nucleon-nucleon scattering.‡ Therefore, while the weak coupling theory is certainly quantitatively inadequate, one may hope that the estimate of the collision time which it yields is not wrong by an order of magnitude.

The above considerations apply almost word for word to the problem of photomeson production and its inverse, meson capture with gamma-ray emission. The application to processes involving gamma-rays alone is trivial because the Born approximation is genuinely applicable there. An application has also been made to the production of mesons in nucleonic collisions.[12] The criterion of validity is here obscured by the great loss of energy suffered by the incident nucleon in creating the meson. It thus interacts much more strongly with the nucleus after emitting the meson than it did before. This case merits independent study.

Note added in proof.—The formal deficiencies of this paper have been remedied, partly by Ashkin and Wick (this issue of the Phys. Rev., p. 686), who use a time dependent formalism to make a systematic expansion of T in powers of U, and partly by Chew and Goldberger (forthcoming publication in the Phys. Rev.) who use a stationary state formalism. The latter authors have placed assumptions I and II and the corrections thereto on a basis comparable to that for assumption III.

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[11] Chedester, Isaacs, Sachs, and Steinberger, Phys. Rev. **82**, 958 (1951). H. L. Anderson, Phys. Rev. this issue, p. 729.

‡ There may be a P-state resonance, but the width is so large ($\sim$100 Mev) that the collision time is still short compared to nuclear periods.

[12] H. P. Noyes, Phys. Rev. **81**, 924 (1951).