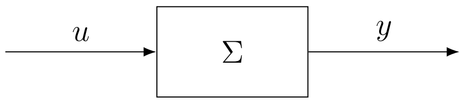
# Exercise 3 - Systems and Linearization

## 1 System Classification

After today we should be able to classify systems in different groups. This classification is important because in this course we will focus on a specific group of systems: the Linear Time Invariant (LTI) systems.
Let's watch at an input-output system:




Depending on the properties of the system $\Sigma$, we can classify it in different ways:

- Linear vs Non-Linear
- Causal vs Non-Causal
- Static(Memoryless) vs Dynamic
- Time-Invariant vs Time-Variant

### 1.1 Linearity

An input-output system is linear if it satisfies the principle of superposition, which is composed of two properties: **Additivity** and **Homogeneity**.

- **Additivity**: $\Sigma(u_1 + u_2) = \Sigma u_1 + \Sigma u_2$
- **Homogeneity**: $\Sigma(\mathbf{k}u) = \mathbf{k}\Sigma u$ ,where $\mathbf{k} \in \mathbb{R}$

In other words, a system is linear if, for all input signals $u_1$ and $u_2$, and for all scalars $\alpha, \beta \in \mathbb{R}$, the following condition holds:

$$\Sigma(\alpha u_1 + \beta u_2) = \alpha\Sigma(u_1) + \beta\Sigma(u_2) = \alpha y_1 + \beta y_2$$

Important note: Differentiation and Integration are linear operations!
As we said earlier, the key idea is superposition. This means that if a system is linear, we can:

- Break down "complicated" inputs into simpler components
- Compute the output for each component
- Add the outputs to get the overall output

If a system does not satisfy the principle of superposition, it is called Non-Linear.

**Exam Problem: HS23**

| | Linear | Non-Linear |
|---|---|---|
| $y(t) = 3cos(t)u(t) + \int_0^t u(\tau - 4)d\tau$ | X | |
| $y(t) = cos(u(t)) \int_{-\infty}^{t-2} u(\tau)d\tau$ | | X |
| $y(t) = \sqrt{cos(\frac{3\pi}{2})}u(3t+1)$ | X | |

## 1.2 Causality

An input-output system is causal if the output at any time $t$ depends only on the values of the input on $(-\infty, t]$.
In other words, a system is **causal** if the output at time $t$ **does not depend** on future values of the input.
All practical systems are causal. (Otherwise we would have to predict the future...)

**Exam Problem: HS24**

| | Causal | Non-Causal |
|---|---|---|
| $y(t) = 2\sqrt{5}t^2u(t+2)$ | | X |
| $y(t) = 4\int_{-\infty}^{t-2} e^{-t}u(\tau+2)d\tau$ | X | |
| $y(t) = \int_{-\infty}^{t} u(t-2)u(\tau)d\tau$ | X | |

## 1.3 Static vs Dynamic

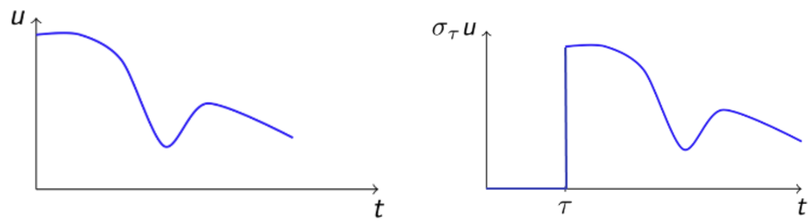
An input-output system is static (memoryless) if the output at any time $t$ depends only on the value of the input at that same time $t$.
In other words, a system is **static** if the output at the **present time** depends only on the input at the **present time**; not on past or future inputs.
In all other cases the system is called **dynamic**.
Important note: All static systems are causal, but not all causal systems are static!

**Exam Problem: FS23**

| | Static | Dynamic |
|---|---|---|
| $y(t) = 1 + u(t-1) + 2u(t)$ | | X |
| $y(t) = u(t)((sin(t) - 5)^2 + tcos(t))$ | X | |
| $y(t) = 2\int_{-\infty}^{t} u^2(\tau)d\tau$ | | X |

## 1.4 Time-Invariant vs Time-Variant

A time-invariant system is a system whose behavior and characteristics are unchanging over time.
It is a time-independent map between input and output that is the same at all times.
In other words, this means that if the input signal is shifted in time, the output signal will be shifted by the same amount, without any change in shape or form.




**Exam Problem: FS23**

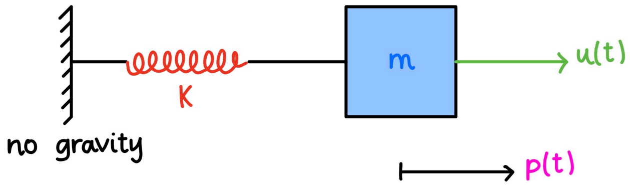
| | Time-Invariant | Time-Variant |
|---|---|---|
| $y(t) = 1 + u(t-1) + 2u(t)$ | X | |
| $y(t) = u(t)((sin(t) - 5)^2 + tcos(t))$ | | X |
| $y(t) = 2\int_{-\infty}^{t} u^2(\tau)d\tau$ | X | |

But which systems do we care about in this course?
We will focus on systems that are Linear, Time Invariant, Causal, Single Input Single Output **(LTI SISO Systems)**.
This is a very restrictive class of systems, in fact most sysems are NOT LTI. On the other hand, many systems can be **well approximated** by LTI SISO models.

The main characteristic of LTI systems is that they can be represented by a **state-space model** of the form:

$$\boxed{\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)\end{aligned}}$$

where **A**, **B**, **C** and **D** are constant matrices or vectors.
One example of a LTI SISO system is the mass-spring system that we saw last week.




The state-space model is given by:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

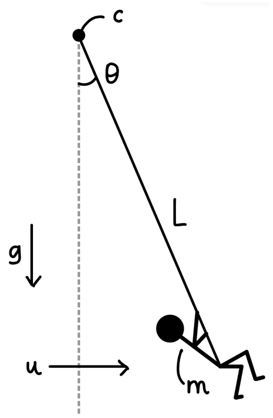
In this example **A**, **B**, **C** and **D** are defined as:

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & 0\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix}0 & 1\end{bmatrix}, \quad \mathbf{D} = 0$$

The vector $x(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$ is the **state** of the system. It contains all the information about the past behavior of the system, together with the current input $u$, that is necessary to predict its future behavior.
The choice of the state for a system is not unique. Every different choice of state is called **realization** of the system (there are an infinite number of realizations).
However, there are some states that are more convenient than others. In particular, we can look for a **minimal realization**, which is the one with the smallest number of states.
For this we define the **dimension n** of a causal system as the minimal number of variables sufficient to describe the system's state.
The dimension **n** of the state vector $x(t) \in \mathbb{R}^n$ is equal to the order of the system.
Important note: A static system has dimension $\mathbf{n} = \mathbf{0}$.
Another important note: A system is **strictly causal** if the "feedthrough" term **D** is

zero. This means that the output does not depend directly on the input.

Now let's take a look at the other example we saw last week, that is the pendulum/swing:




$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{1}{mL^2}\left(-mgL\sin(x_1(t)) - cx_2(t) + L\cos(x_1(t))u(t)\right) \\ y(t) = x_1(t) \end{cases}$$

How do we write this system in the form of a state-space model?
This system is **non-linear** because of the $\sin(x_1(t))$ and $\cos(x_1(t))$ terms. Therefore we can't directly write it in the form of a LTI System.
However, we can **linearize** the system and approximate it as a LTI System.
This is an important concept because many systems are non-linear, but can be approximated as LTI systems around a certain operating point.

## 2 Linearization

The key idea of linearization is to approximate a non-linear system by a linear one around a certain operating point, also called equilibrium point.
This approximation works well and lets us use the tools we have for LTI systems even on nonlinear ones.

For a system described by an ODE $\dot{x}(t) = f(x(t), u(t))$ the equilibrium point is the point $(x_e, u_e)$ where the state does not change, that is:

$$\boxed{f(x_e, u_e) = 0}$$

Let's take a look at the swing example again. We will look at equilibrium where the input is zero, that is $u_e = 0$.
Intuitevely, we can see that the equilibrium points are when the swing is at rest, that is $x_2 = 0$, and the angle is either $x_1 = 0$ (hanging down) or $x_1 = \pi$ (inverted).

Let's verify this mathematically:

$$f(x_e, 0) = 0$$
$$\Downarrow$$
$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} x_{2e} \\ \frac{1}{mL^2}(-mgL\sin(x_{1e}) - cx_{2e} + L\cos(x_{1e}) \cdot 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} x_{2e} \\ -\frac{g}{L}\sin(x_{1e}) - \frac{c}{mL^2} \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this we get that:

$$\dot{x_2}(t) = sin(x_1(t)) = 0 \Rightarrow x_{1e} = 0 \text{ or } x_{1e} = \pi$$

So for $u_e = 0$ there are two equilibirum points:

$$x_e = (0, 0), u_e = 0 \Rightarrow \text{stable}$$
$$x_e = (\pi, 0), u_e = 0 \Rightarrow \text{unstable}$$

In order to linearize around an equilibrium point $(x_e, u_e)$ we compute the Taylor series expansion of the nonlinear system's dynamics around $(x_e, u_e)$. The linearized LTI matrices are given by:

$$A = \frac{\partial f(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(x_e,u_e)} \in \mathbb{R}^{n \times n}$$

$$B = \frac{\partial f(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{(x_e,u_e)} \in \mathbb{R}^{n \times m}$$

Similarly:

$$C = \frac{\partial h(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}_{(x_e,u_e)} \in \mathbb{R}^{p \times n}$$

$$D = \frac{\partial h(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}_{(x_e,u_e)} \in \mathbb{R}^{p \times m}$$

For the swing example, the nonlinear dynamics are:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{1}{mL^2}\left(-mgL\sin(x_1(t)) - cx_2(t) + L\cos(x_1(t))u(t)\right) \\ y(t) = x_1(t) \end{cases}$$

With equilibrium points at: $u_e = 0$ and $x_e = (0,0)$ or $x_e = (\pi, 0)$.
To linearize around $x_e = (0,0)$ and $u_e = 0$, we compute the Jacobians:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(0,0)}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(0,0)}, \qquad C = \left.\frac{\partial h}{\partial x}\right|_{(0,0)}, \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(0,0)}$$

Calculating the partial derivatives for A:

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_2}{\partial x_1} = \frac{1}{mL^2}\left(-mgL\cos(x_1)\right)\Big|_{x_1=0} = -\frac{g}{L}$$

$$\frac{\partial f_2}{\partial x_2} = -\frac{c}{mL^2}$$

So the linearized system matrix A :

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{c}{mL^2} \end{bmatrix}$$

If we do the same for B, C and D we get:

$$B = \begin{bmatrix} 0 \\ \frac{1}{mL} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Important: we linearized the system around $x_e = (0,0), u_e = 0$. If we linearize near the other eq. point $x_e = (\pi, 0), u_e = 0$ we will get another state space representation:

$$A = \begin{bmatrix} 0 & 1 \\ +\frac{g}{L} & -\frac{c}{mL^2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{mL} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

In conclusion, this LTI approximations work well if we stay close to their equilibrium point. By choosing the right controller we can also ensure it works well with the real nonlinear system.

## Exam Problem: FS24

**Problem:** Consider the following system in nonlinear state-space form, where $x(t) \in \mathbb{R}^3$ represents the state vector and where $u(t) \in \mathbb{R}$ represents the input to the system,

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \cos(x_1(t)) + u(t) \\ 9x_2^2(t) + 6x_1(t)x_2(t) + x_1^2(t) \\ e^{x_3(t)} - u(t) \end{bmatrix}$$

**Q6** **(1 Points)** Find the equilibrium ($x_e = (x_{1,e}, x_{2,e}, x_{3,e}), u_e$) such that $x_{2,e} = -\frac{\pi}{3}$.

$$\dot{x}(t) = 0 \Rightarrow \begin{cases} \cos(x_1(t)) + u(t) = 0 \\ 9x_2^2(t) + 6x_1(t)x_2(t) + x_1^2(t) = 0 \quad (*) \\ e^{x_3(t)} - u(t) = 0 \end{cases}$$

$$(*)\quad 9\left(-\frac{\pi}{3}\right)^2 + 6\left(-\frac{\pi}{3}\right)x_1(t) + x_1^2(t) = 0 \Leftrightarrow \pi^2 - 2\pi x_1(t) + x_1^2(t) = 0$$

$$(x_1(t) - \pi)^2 = 0$$

$$\Rightarrow x_{1,e} = \pi$$

- $\cos(x_{1,e}) + u(t) \Leftrightarrow \cos(\pi) + u(t) = 0$

  $u_e = 1$

- $e^{x_3(t)} - u_e = 0 \Leftrightarrow e^{x_3(t)} = 1$

  $x_{3,e} = \ln(1) = 0$

## Exam Problem: FS24

**Problem:** Consider the nonlinear system with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}_{>0}$, and output $y(t) \in \mathbb{R}$ described by,

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1 + 3\ln(u(t)) \\ \cos(x_1(t)) - 5\cos(2x_1(t))u(t) + 3x_2^2(t) + u(t) \end{bmatrix}, \\ y(t) &= x_1(t)x_2^2(t) + x_2(t)u^2(t). \end{aligned}$$

Derive the $(A, B, C, D)$ matrices of the linearized state-space representation of the above system around the equilibrium point ($x_{1,e} = 0$, $x_{2,e} = -1$, $u_e = 1$).

**Q7** **(1 Points)** What is the $A$ matrix of the above linearized system around the equilibrium ($x_{1,e}$, $x_{2,e}$, $u_e$)? The resulting $A$ matrix must only contain numerical values.

**Q8** **(0.5 Points)** What is the $B$ matrix of the above linearized system around the equilibrium ($x_{1,e}$, $x_{2,e}$, $u_e$)? The resulting $B$ matrix must only contain numerical values.

**Q9** **(0.5 Points)** What is the $C$ matrix of the above linearized system around the equilibrium ($x_{1,e}$, $x_{2,e}$, $u_e$)? The resulting $C$ matrix must only contain numerical values.

**Q10** **(0.25 Points)** What is the $D$ matrix of the above linearized system around the equilibrium ($x_{1,e}$, $x_{2,e}$, $u_e$)? The resulting $D$ matrix must only contain numerical values.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} 1 & 0 \\ -\sin(x_1(t)) + 10\sin(2x_1(t))u(t) & 6x_2(t) \end{pmatrix}_{(x_e, u_e)}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} \frac{3}{u(t)} \\ 1 - 5\cos(2x_1(t)) \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} 3 \\ 1-5 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} x_2^2(t) & 2x_1(t)x_2(t) + u^2(t) \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{\partial h}{\partial u} \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} 2u(t)\,x_2(t) \end{pmatrix}_{(x_e, u_e)} = \begin{pmatrix} -2 \end{pmatrix}$$