Exercise 4 - Time Response

1 Time Response

From last week, we know how to formulate physical systems in the LTI state-space form:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

But what is it's solution?
In other words, how can we express $x(t)$ and $y(t)$ for $t \geq t_0$?
First, we recall the Linearity of our equations. Since the system is linear, we can decompose the solution into the response due to the initial conditions and the response due to the input:

- Initial-conditions response:

$$\begin{cases} x_{IC}(0) = x_0 \\ u_{IC}(t) = 0, \ t \geq 0 \end{cases} \implies y_{IC}$$

- Forced response:

$$\begin{cases} x_F(0) = 0 \\ u_F(t) = u(t), \ t \geq 0 \end{cases} \implies y_F$$

Due to Linearity, after solving each case separately, we can simply add y_{IC} and y_F to get the complete output. This allows us to **study separately the effects of non-zero inputs and of non-zero initial conditions**.

1.1 Initial condition (homogeneous) response

In this case we consider an initial condition $x(0) = x_0$ and zero input, i.e $u(t) = 0$.
Thus, we need to solve:

$$\begin{cases} \dot{x}(t) = ax(t) \\ y(t) = cx(t) \end{cases}$$

And the solution of this ODE is given by:

$$x_{IC}(t) = e^{at}x_0 = \phi(t)x_0$$
$$y_{IC}(t) = ce^{at}x_0 = c\phi(t)x_0$$

Where $\phi(t) = e^{at}$ is the **state-transition** function.

1.2 Forced response

For this case we consider instead that the initial condition is $x(0) = 0$ and the input $u(t) \neq 0$. We then have to solve:

$$\begin{cases} \dot{x}(t) = ax(t) + bu(t) \\ y(t) = cx(t) + du(t) \end{cases}$$

And in this case we get (the derivation is shown in the slides of the Lecture or in the Notes):

$$x_F(t) = \int_0^t e^{a(t-\tau)}bu(\tau)d\tau = \int_0^t \phi(t-\tau)bu(\tau)d\tau$$
$$y_F(t) = c\int_0^t e^{a(t-\tau)}bu(\tau)d\tau + du(t) = c\int_0^t \phi(t-\tau)bu(\tau)d\tau + du(t)$$

1.3 Complete response

As we said earlier, to get the complete response we just need to sum up the two cases. Therefore $x = x_{IC} + x_F$, $u = u_{IC} = u_F$ and $y = y_{IC} + y_F$.
We finally get:

$$\boxed{\begin{aligned} x(t) &= e^{at}x_0 + \textstyle\int_0^t e^{a(t-\tau)}bu(\tau)d\tau \\ y(t) &= ce^{at}x_0 + c\textstyle\int_0^t e^{a(t-\tau)}bu(\tau)d\tau + du(t) \end{aligned}}$$

where the first term of $y(t)$ represents the *natural response*, the second term represents the *forced response*, and the last term corresponds to the *feedthrough*.

Important note: Here a, b, c, d are scalars.

1.4 Complete response for higher-order system

However, in general A, B, C, and D are NOT scalars like above. Fortunately, the solution looks very similar :D. Keep in mind that whenever A, B, C and D are matrices you have to maintain the order of multiplication.
Here, using the matrix exponential, we get:

$$\boxed{\begin{aligned} x(t) &= e^{\mathrm{A}t}x_0 + \textstyle\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau \\ y(t) &= \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}\textstyle\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau + \mathrm{D}u(t) \end{aligned}}$$

We see that, as before, some terms contain an exponential. But how do we compute a matrix exponential like $e^{\mathrm{A}t}$?

- Brute force (compute many terms of the Taylor expansion)
- Matlab (with expm(A*t))
- Find a realization of the system such that the matrix A is either **Diagonal** or in **Jordan form**:

$$\exp\left(\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} t\right) = \begin{bmatrix}\exp(\lambda_1 t) & 0\\ 0 & \exp(\lambda_2 t)\end{bmatrix} \qquad \text{(Diagonal form)}$$

$$\exp\left(\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} t\right) = \begin{bmatrix}\exp(\lambda t) & t\exp(\lambda t)\\ 0 & \exp(\lambda t)\end{bmatrix} \qquad \text{(Jordan form)}$$

To get the matrix in one of this two forms we need to do a coordinate transformation, also called **similarity transformation** (see Chapter 2.3.1 in Lecture Notes). As you saw in class, through the transformation the time response doesn't change!
P.S. For Diagonalization see Linear Algebra II.

Initial condition (homogeneous) response:

Let us now take a look at a system where A is diagonal. We will look at the initial condition repsonse, that means $u(t) = 0$.

- For a diagonal, real matrix:

$$A = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \qquad ,\lambda_i \in \mathbb{R}$$

$$y(t) = Ce^{At}x_0$$

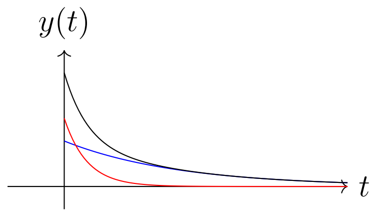
we can write all terms for A being diagonal and get:

$$y(t) = \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

$$\Downarrow$$

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$

So we can say that for real, diagonal matrices the initial condition response is given by the linear combination of exponentials of the form $\exp(\lambda_i t)$.



- For a diagonal, complex matrix:

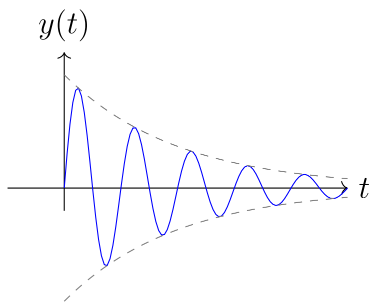
$$A = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}$$

$$y(t) = Ce^{At}x_0$$

we get:

$$y(t) = c_1 e^{\sigma t} e^{j\omega t} x_1(0) + c_2 e^{\sigma t} e^{j\omega t} x_2(0) = e^{\sigma t}(\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t))$$
$$\Downarrow$$
$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \phi)$$

In this case, when there are two complex conjugate eigenvalues, the homogeneous response is given by a sinusoid of frequency ω, with amplitude increasing/decreasing as $\exp(\sigma t)$.



In the general case, the response of a linear system will always be a combination of terms!

Exam Problem: FS24

Problem: Consider the following linear time-invariant system in state-space representation with state vector $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$ and output $y(t) \in \mathbb{R}$.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 2u(t).$$

Q12 **(1 Points)** Calculate the initial condition response $y_{\mathrm{IC}}(t)$ with the initial condition $x(0) = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$.

Q13 **(1 Points)** Calculate the forced response $y_{\mathrm{F}}(t)$, $t \geq 0$, to the (non-unit) step function $u(t) = 2$, $t \geq 0$.

2 Stability

We have seen that, if the A matrix is diagonalizable, the output will be given by some linear combination of **exponential terms** of the form:

$$y(t) = e^{\lambda_i t}$$

$$y(t) = e^{\sigma_i t} \sin(\omega_i t + \phi_i)$$

The growth of these terms is dictated by the **real part** of the eigenvalues of A. We can see that if the eigenvalues λ_i (or σ_i) have a positive real part, the output will grow exponentially over time, i.e become unstable ($y \to \infty$).
But how do we classify stability?

- **Lyapunov stability**: a system is called Lyapunov stable if, for any bounded initial condition, and zero input, the state remains bounded, i.e.:

$$\|x_0\| < \epsilon \text{ and } u = 0 \qquad \Rightarrow \qquad \|x(t)\| < \delta \quad \forall\, t \geq 0$$

- **Asymptotic stability**: a system is called asymptotically stable if, for any bounded initial condition, and zero input, the state converges to zero, i.e.:

$$\|x_0\| < \epsilon \text{ and } u = 0 \qquad \Rightarrow \qquad \lim_{t \to +\infty} \|x(t)\| = 0$$

- **Bounded-Input, Bounded-Output stability (BIBO)**: A system is called BIBO-stable if, for any bounded input, the output remains bounded, i.e.:

$$\|u(t)\| < \epsilon \quad \forall\, t \geq 0 \text{ and } x_0 = 0 \qquad \Rightarrow \qquad \|y(t)\| < \delta \quad \forall\, t \geq 0$$

A system is called **unstable** if not stable!

We can also check the stability by looking at the eigenvalues of an LTI system with matrix A:

- **Lyapunov stable**: if $\mathrm{Re}(\lambda_i) \leq 0 \quad \forall\, i$

  This is valid only for **diagonalizable** A **matrices**. Unfortunately, not all matrices are diagonalizable. For such systems the output is additionally composed of time scaled exponential functions (e.g $t^{m_i} e^{\lambda_i t}$). This means that we have to be a little bit more careful when looking at the Lyapunov stability.
  For non-diagonalizable A matrices, we can say that a system is Lyapunov stable iff $\mathrm{Re}(\lambda_i) \leq 0 \quad \forall i$ **AND** there are no repeated eigenvalues with 0 real part.

- **Asymptotically stable**: if $\mathrm{Re}(\lambda_i) < 0 \quad \forall\, i$

Important note: For minimal LTI systems **asymptotic stability** = **BIBO stability**.

Example: Swing

Last week we got that, after a linearization around the equilibrium point $x_e = (\pi, 0)$ and $u_e = 0$ the state space representation was:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{c}{mL^2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{mL} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Let's try to solve this together for the initial condition response, that is $x(0) = x_0$ and the input $u(t) = 0$.
Assume that the following values for the internal parameters are: $g = 10$, $L = 5$, $c = 25$, $m = 1$. The initial condition is now given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

We now have to solve:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x(t)$$

And we can easily find the eigenvalues and the eigenvectors:

$$\lambda_1 = 1, \quad \lambda_2 = -2, \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

This system is unstable since $\text{Re}(\lambda_1) = 1 > 0$.

Quickly looking at the other equilibrium point: $x_e = (0, 0)$ and $u_e = 0$.
Here we have to look at:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x(t)$$

The eigenvalues are given by:

$$\lambda_1 = \frac{1}{2}(-1 + j\sqrt{7}), \quad \lambda_2 = \frac{1}{2}(-1 - j\sqrt{7})$$

And we can state that system is asymptotically stable since $\text{Re}(\lambda_i) < 0$!

Stability of Non-Linear Systems

When studying real-world systems, we often use linearized LTI models obtained around an equilibrium point. However, the stability properties of such a linearized model are only valid in a small neighborhood of the equilibrium point where the linearization was performed.
The **Hartman–Grobman Theorem** provides a rigorous link between the stability of the non-linear system and that of its linearization:

> If the linearization of a non-linear system around an equilibrium point is (asymptotically) stable, then the non-linear system is also locally (asymptotically) stable at that equilibrium point.

Note: The Hartman–Grobman theorem applies only to points where all eigenvalues of the linearized system have non-zero real parts (hyperbolic: $\text{Re}(\lambda_i) \neq 0, \forall i$).
Therefore, no conclusions about Lyapunov stability can be drawn.

Exam Problem: HS23

Problem: Consider the linear time-invariant system G in state-space form given by,

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} -2 & -1 \end{bmatrix} x(t) + 5\,u(t),\end{aligned}$$

where $x(t)$ represents the state of the system, $u(t)$ the input and $y(t)$ the output. Note that the given state-space representation of G is a minimal realization.

Q13 **(0.5 Points)** Mark the correct answer.
The system G is,

A lyapunov stable, but not asymptotically stable.
B asymptotically stable.
C unstable.

Q14 **(0.25 Points)** Mark the correct answer.
The system G is BIBO stable.

A True B False

Exam Problem: HS22

Problem: You are given the following linear time-invariant system with state vector $x(t) \in \mathbb{R}^3$ and input $u(t) \in \mathbb{R}$:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot x(t)\,.$$

Q10 **(0.5 Points)** Mark the correct answer for each statement.

Statement	True	False
The system is asymptotically stable.		
The system is Lyapunov stable.		
The system is BIBO-stable.		