Exercise 5 - Transfer Functions

1 Time Response - Continued

Recall from last week that the solution of the state-space LTI system is given by:

$$\boxed{\begin{aligned} x(t) &= e^{\mathrm{A}t}x_0 + \textstyle\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau \\ y(t) &= \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}\textstyle\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau + \mathrm{D}u(t) \end{aligned}}$$

As we saw, this formula gives a complete characterization of the system's time response.

1.1 Forced Response

We have already seen how the system reacts to initial conditions and no input, i.e how the homogeneous solution $Ce^{\mathrm{A}t}x_0$ behaves. We know that this part of the solution describes the natural response of the system, more specifically its behavior in the absence of external inputs.
Now, we will focus on the forced response, i.e. the part of the solution that depends on the input $u(t)$, that is:

$$\mathrm{C}\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau + \mathrm{D}u(t)$$

As you can already imagine, this part is harder to compute, since it involves a convolution integral. To gain a better understanding of this term, we will consider the response to some "elementary inputs". Then, since we are dealing with LTI systems, we can later use the principle of superposition to construct the response to general inputs, expressed as a combination of these elementary inputs.

1.2 General Response

Since we are looking at linear systems, we can decompose any input $u(t)$ as a linear combination of elementary inputs. So we can have any input $u(t)$ expressed as:

$$u = \sum_i \alpha_i u_i$$

where u_i are elementary inputs and α_i are constants.
Then, by the principle of superposition, we can write the response as a sum of all outputs:

$$y = \sum_i \alpha_i y_i$$

It would be convenient if we could find some input u such that when linearly combined with itself, it can generate almost any other input. Luckily, we can use a tool from Analysis III to help us!

1.3 Laplace Transform

The Laplace Transform is a powerful integral transform that converts a time-domain function into a complex frequency-domain representation. It is defined as follows:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt$$

And it's inverse is given by:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds$$

where s $= \sigma + j\omega$ is a complex variable.

In Analysis III, you have seen that the Laplace Transform can be used to solve ODE's. In Control Systems I, we will use it to transform differential equations into algebraic equations, which are easier to manipulate and solve.
If we look closer at the inverse Laplace Transform, we can see that it is a linear combination of complex exponentials e^{st}, each weighted by $F(s)$, the Laplace Transform of $f(t)$. This means that if we can find the response of the system to these complex exponentials, we can use the principle of superposition to construct the response to almost any input, since it will be a linear combination of e^{st} terms.

Let's consider the input $u(t) = e^{st}$. The response of the system to this input is given by:

$$y(t) = \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau + \mathrm{D}u(t)$$

$$\Downarrow$$

$$y(t) = \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}e^{s\tau}d\tau + \mathrm{D}e^{st}$$

If we rearrange the integral, we get:

$$y(t) = \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}e^{\mathrm{A}t}\int_0^t e^{(s\mathbb{I}-\mathrm{A})\tau}\mathrm{B}\,d\tau + \mathrm{D}e^{st}$$

And if $(s\mathbb{I} - \mathrm{A})$ is invertible, we can compute the integral:

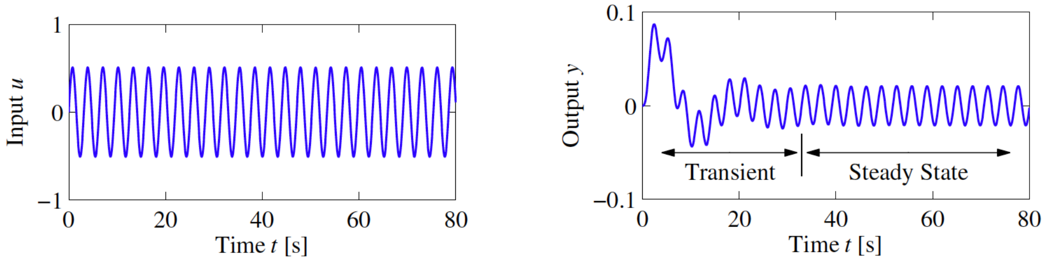
$$y(t) = \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}e^{\mathrm{A}t}[(s\mathbb{I}-\mathrm{A})^{-1}e^{(s\mathbb{I}-\mathrm{A})\tau}\mathrm{B}]_0^t + \mathrm{D}e^{st}$$

$$\Downarrow$$

$$y(t) = \mathrm{C}e^{\mathrm{A}t}x_0 + \mathrm{C}e^{\mathrm{A}t}(s\mathbb{I}-\mathrm{A})^{-1}(e^{(s\mathbb{I}-\mathrm{A})t} - \mathbb{I})\mathrm{B} + \mathrm{D}e^{st}$$

And if we rearrange the terms, we finally get:

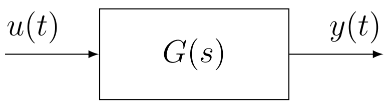
$$\boxed{y(t) = \underbrace{\mathrm{C}e^{\mathrm{A}t}\left(x_0 - (s\mathbb{I}-\mathrm{A})^{-1}\mathrm{B}\right)}_{\text{Transient response } (\to 0 \text{ if as. stable})} + \underbrace{(\mathrm{C}(s\mathbb{I}-\mathrm{A})^{-1}\mathrm{B} + D)e^{st}}_{\text{Steady-state response } y_{ss}}}$$



We can now generally say that the steady state response to the input $u(t) = e^{st}$ is given by:

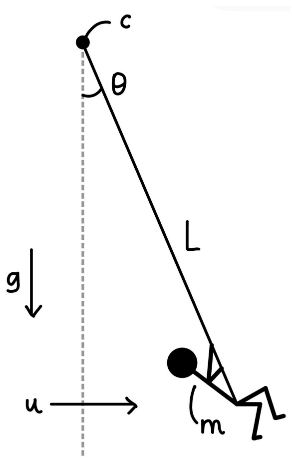
$$\boxed{y_{ss}(t) = G(s)e^{st} \quad \text{where} \quad G(s) = \mathrm{C}(s\mathbb{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D} \quad \in \mathbb{C}}$$

The complex function $G(s)$ is called the **transfer function** of the system and describes how a stable system G transforms an input $u(t) = e^{st}$ into an output $y(t) = G(s)e^{st}$ at steady state.
You can think of it as the Σ in the block diagrams.



Remember: the transient response goes to zero as t $\to \infty$ if the system is asymptotically stable, i.e. if all eigenvalues of A have strictly negative real parts.

Example - Swing



Recall the pendulum linearized around the equilibrium point $x_e = (0, 0)$ and $u_e = 0$.

The LTI state-space matrices were given by:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0]$$

If we want to compute the transfer function of this system, we can use the formula that we just derived:

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} + [0] =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + s + 2} \begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2+s+2} \begin{bmatrix} 0.2 \\ 0.2s \end{bmatrix}$$

Finally, we get:

$$G(s) = \frac{0.2}{s^2+s+2}$$

Exam Problem: HS23

Problem: Consider the linear time-invariant system in state-space representation given by,

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where,

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = -\frac{1}{2}.$$

Q17 **(1 Points)** Mark the correct answer. Which of the following transfer functions $G(s)$ corresponds to the given state-space representation above?

- A $G(s) = -\frac{s^2-5s+1}{(s^2-3s+1)}$
- B $G(s) = \frac{s^2-s+1}{2(s^2-3s+1)}$
- C $G(s) = \frac{s^2-s+1}{(s^2-3s+1)}$
- D $G(s) = \frac{s^2-5s+1}{2(s^2-3s+1)}$
- E $G(s) = \frac{-s^2+5s-1}{2(s^2-3s+1)}$
- F $G(s) = s - \frac{5}{2}$
- G $G(s) = s + \frac{5}{2}$

$$G(s) = C(sI - A)^{-1}B + D$$

$$\Rightarrow (sI - A)^{-1} = \begin{pmatrix} s-2 & 1 \\ 1 & s-1 \end{pmatrix}^{-1} = \frac{1}{\det(sI-A)} \begin{pmatrix} s-1 & -1 \\ -1 & s-2 \end{pmatrix} =$$

$$= \frac{1}{(s-2)(s-1) - 1^2} \begin{pmatrix} s-1 & -1 \\ -1 & s-2 \end{pmatrix}$$

$$\Rightarrow G(s) = \frac{1}{s^2 - 3s + 1} (1 \ \ -1) \begin{pmatrix} s-1 & -1 \\ -1 & s-2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2} =$$

$$= \frac{1}{s^2 - 3s + 1} (1 \ \ -1) \begin{pmatrix} s-1 \\ -1 \end{pmatrix} - \frac{1}{2} = \frac{s-1+1}{s^2-3s+1} - \frac{1}{2} =$$

$$\Rightarrow G(s) = \frac{2s - s^2 + 3s - 1}{2(s^2 - 3s + 1)} = \frac{-s^2 + 5s - 1}{2(s^2 - 3s + 1)}$$

Appendix: Matrix Inverses

Inverse of a 2×2 matrix.
For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

if $\det(A) = ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Inverse of a 3×3 matrix.
For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

if $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ is the adjugate matrix:

$$\operatorname{adj}(A) = \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}^T.$$

Alternative methods:

- **Gauss–Jordan:** Reduce $[A \,|\, \mathbb{I}]$ to $[\mathbb{I} \,|\, A^{-1}]$ by row operations.

1.3.1 Decompose inputs

At the beginning however, we said that we wanted to express a general input $u(t)$ as a linear combination of elementary inputs. But how do we decompose a general input into complex exponentials e^{st}?
Let's look at an example where our input is a sinusoid:

$$u(t) = \sin(\omega t)$$

We can use Euler's formula to express the sine function as a combination of complex exponentials:

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

So we can express the input as:

$$u(t) = \sum_i U_i e^{s_i t} \quad \text{where} \quad U_1 = \frac{1}{2j},\, s_1 = j\omega \quad \text{and} \quad U_2 = -\frac{1}{2j},\, s_2 = -j\omega$$

Then, by the principle of superposition, we can express the output as:

$$y(t) = \sum_i G(s_i)e^{s_i t}U_i = \frac{1}{2j}G(j\omega)e^{j\omega t} - \frac{1}{2j}G(-j\omega)e^{-j\omega t}$$

Furthermore, we can rewrite $G(j\omega)$ in polar form:

$$G(j\omega) = Me^{j\phi}$$

where $M = |G(j\omega)|$ is the magnitude and $\phi = \angle G(j\omega)$ is the phase.

Finally, we can express the output as:

$$y(t) = \frac{1}{2j}Me^{j(\omega t+\phi)} - \frac{1}{2j}Me^{-j(\omega t+\phi)}$$

$$\Downarrow$$

$$y(t) = M\sin(\omega t + \phi)$$

This means that:

A sinusoidal input produces a sinusoidal output at the same frequency, but with a different amplitude and phase.
The amplitude is scaled by the magnitude of the transfer function at that frequency, and the phase is shifted by the phase of the transfer function at that frequency.

In general:

$$u(t) = \sum_i U_i e^{s_i t} \quad \Rightarrow \quad y(t) = \sum_i G(s_i)e^{s_i t}U_i$$

Now, we are able to use the inverse Laplace Transform that generalizes this sum, such that we can express any input as a continuous sum of complex exponentials:

$$u(t) = \mathcal{L}^{-1}\left\{U(s)\right\} = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} U(s)e^{st}ds$$

$$y(t) = \mathcal{L}^{-1}\left\{G(s)U(s)\right\} = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} G(s)U(s)e^{st}ds$$

In other words, the Laplace Transform of the output is given by:

$$\boxed{Y(s) = G(s)U(s)}$$

By using the Laplace Transform, we have transformed the convolution integral in the time domain into a simple multiplication in the frequency domain. This is one of the main advantages of using the Laplace Transform in Control Systems.

Appendix: Magnitude and Phase calculations

Magnitude:
The magnitude of the transfer function at a given frequency is given by:

$$|G(j\omega)| = \sqrt{\mathrm{Re}(G(j\omega))^2 + \mathrm{Im}(G(j\omega))^2}$$

Some operations on the magnitude:

- $|G_1(j\omega) \cdot G_2(j\omega)| = |G_1(j\omega)| \cdot |G_2(j\omega)|$
- $\left|\frac{G_1(j\omega)}{G_2(j\omega)}\right| = \frac{|G_1(j\omega)|}{|G_2(j\omega)|}$

Phase:
The phase of the transfer function at a given frequency is given by:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right)$$

Some operations on the phase:

- $\angle(G_1(j\omega) \cdot G_2(j\omega)) = \angle G_1(j\omega) + \angle G_2(j\omega)$
- $\angle\left(\frac{G_1(j\omega)}{G_2(j\omega)}\right) = \angle G_1(j\omega) - \angle G_2(j\omega)$

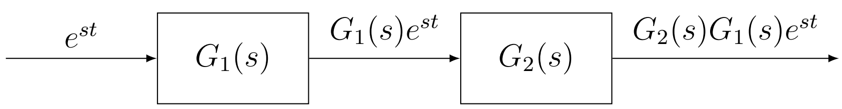
2 Transfer Functions

Recall that the transfer function a LTI system is given by:

$$G(s) = \mathrm{C}(s\mathbb{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D}$$

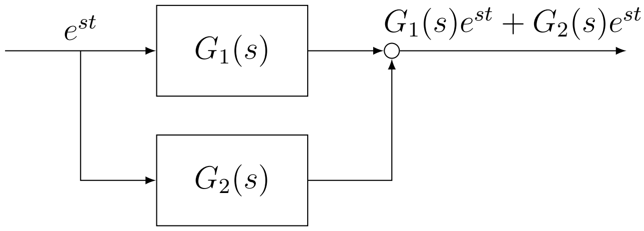
A very important property of transfer functions is that they allow to connect systems very easily.

- **Serial interconnection:** If we have two systems $G_1(s)$ and $G_2(s)$ connected in series, the overall transfer function is given by:



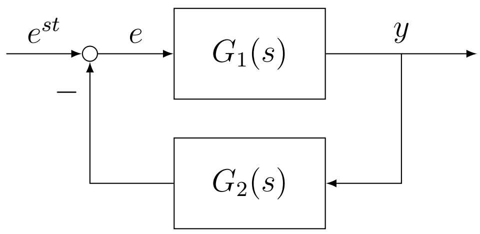
$$G(s) = G_2(s)G_1(s)$$

- **Parallel interconnection:** If we have two systems $G_1(s)$ and $G_2(s)$ connected in parallel, the overall transfer function is given by:



$$G(s) = G_1(s) + G_2(s)$$

- **(Negative) Feedback connection:** If we have a system $G_1(s)$ with a feedback loop containing $G_2(s)$, the overall transfer function is given by:



$$G(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

2.1 TF and State-space representation relation

Another important property of transfer functions is that they do not depend on the state-space representation. In other words, different state-space representations can lead to the same transfer function.
For this reason, we are interested in minimal realizations, i.e. matrices A, B, C, D of the smallest possible dimension that can represent the transfer function.
There are a few forms that are commonly used to catch the relationship between state-space and transfer functions and to find minimal realizations:

- **Partial Fraction Expansion:** If A is diagonal, the transfer function can be expressed as a sum of simpler fractions, where each fraction corresponds to the eigenvalues of A.

$$G(s) = \frac{p_1}{s - \lambda_1} + \frac{p_2}{s - \lambda_2} + \cdots + \frac{p_n}{s - \lambda_n} + d$$

$$\Updownarrow$$

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_n} \end{bmatrix}$$

$$C = \begin{bmatrix} \sqrt{p_1} & \dots & \sqrt{p_n} \end{bmatrix}, \quad D = d$$

- **Controllable Canonical Form:** If A is in controllable canonical form, the transfer function can be expressed as a ratio of two polynomials, where the denominator corresponds to the characteristic polynomial of A and the numerator corresponds to the coefficients of B and C.

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0} + d$$

$$\Updownarrow$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} \end{bmatrix}, \quad D = d$$

An important note is that the roots of the denominator of $G(s)$ are the eigenvalues of A, and they are called the **poles** of the transfer function. The roots of the numerator are called the **zeros** of the transfer function.
So if we want to analyze the stability of a system using its transfer function, we only need to look at the poles of the transfer function (next week we will look at poles and zeros in more detail).

We now have two equivalent representations of a system: the **state-space model** (time domain) and the **transfer function** (frequency domain). Each offers different insights, and converting between them allows a unified understanding of system behavior.

Exam Problem: FS24

Problem: Consider the transfer function $G(s)$ given by,

$$G(s) = \frac{1}{s-2} + \frac{2}{s+1} + \frac{\pi^2}{s+9} + 7.$$

Q15 **(0.5 Points)** Mark the correct answer. Which of the following state-space representations (A, B, C, D) is a realization of $G(s)$?

A $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \sqrt{2} \\ \pi \end{bmatrix}$, $C = \begin{bmatrix} 1 & \sqrt{2} & \pi \end{bmatrix}$, $D = 7$

B $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -9 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \sqrt{2} \\ \pi \end{bmatrix}$, $C = \begin{bmatrix} 1 & \sqrt{2} & \pi \end{bmatrix}$, $D = 7$

C $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\pi^2 \end{bmatrix}$, $B = \begin{bmatrix} \sqrt{2} \\ 1 \\ 3 \end{bmatrix}$, $C = \begin{bmatrix} \sqrt{2} & 1 & 3 \end{bmatrix}$, $D = 7$

D $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -9 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \sqrt{2} \\ \pi \end{bmatrix}$, $C = \begin{bmatrix} 1 & \sqrt{2} & \pi \end{bmatrix}$, $D = 0$

Solution:
$\lambda_1 = 2 \quad \lambda_2 = -1 \quad \lambda_3 = -9$
$d = 7$
$\Rightarrow$ only by looking at these variables we know that it must be option B

$\Rightarrow$ then we can also confirm by looking at matrices B and C

Exam Problem: HS22

Q17 **(1 Points)**

Given:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -4 & -1 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{2} & -2 & 0 & -\frac{1}{3} & 0 \end{bmatrix}, D = 0.$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

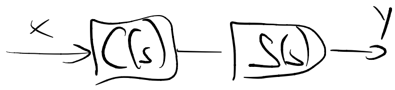
- [ ] $G(s) = \frac{\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3}}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$
- [x] $G(s) = \frac{-\frac{1}{3}\cdot s^3 - 2\cdot s + \frac{1}{2}}{s^5 + 6\cdot s^4 + s^3 + 4\cdot s^2 + 2}$
- [ ] $G(s) = \frac{-\frac{1}{3}\cdot s^3 - 2\cdot s + \frac{1}{2}}{s\cdot(s^4 + 6\cdot s^3 + s^2 + 4\cdot s + 2)}$
- [ ] $G(s) = \frac{s\cdot(\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3})}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$

- only by looking at $a_0 = 2$ (and $d = 0$) we can exclude option (1) and option (4)
- then looking at $a_1 = 0$ we see that we should not have a constant multiplied by s in the denominator

$\Rightarrow$ it must be option (2)

$\Rightarrow$ CHECK FOR "TRICKS", DON'T LOSE TIME COMPUTING THE WHOLE TF
E.G. LOOKING AT THE NUMERATOR IN THIS CASE IS USELESS

2.2 Dynamic Compensator



We can think of a dynamic compensator as another transfer function that is defined in such a way that it helps achieve the control objective.
It can be described by a transfer function of the form:

$$C(s) = k + \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}$$

Some common types of compensators are:

- **Proportional control:** $C(s) = K$
- **PI control:** $C(s) = K_P + \frac{K_I}{s}$
- **Lead-lag control:** $C(s) = k\frac{s+z}{s+p}$
- **PD control:** $C(s) = K_P + K_D s$

We will look at these compensators in more detail in the next exercises.