O MAIOR CONTROLADOR DE SISTEMAS DE SEMPRE

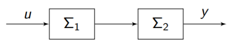
1 Definitions

1.1 System Classification

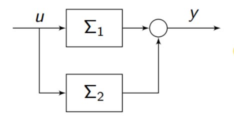
SISO Single Input/Output (scalar)	**MIMO** Multiple Input/Output (vector)
Linear $U = \alpha U_1 + \beta U_2$, $\alpha y_1 + \beta y_2$ $y(t) \propto u(t)$ Superposition is allowed: $\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) = \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$ $y(t) = u(t^2)$ $y(t) = u(t-a) + u(t-b)$ $y(t) = \frac{d}{dt}u(t)$ $y(t) = \int_0^t u(\tau)d\tau$ $y(t) = a \cdot x(t) + u(t)$	**Non-linear** Superposition it **not** allowed: $\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) \neq \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$ $y(t) = a \cdot u(t) + b$ $y(t) = sin(t)$ $y(t) = x(t) \cdot u(t)$ $y(t) = u^2(t)$ $y(t) = u^2(t)$ $y(t) = u(t) + a$ integration and differentiation are linear operations
causal a system is causal if and only if the future input does not affect the present output. (physical realizable systems, degree of numerator $n <$ degree of denominator m of TF)	**non-causal/acausal** Depend on future inputs, are not realizable and not physically possible $y(t) = u(t+5)$ $\int_{-\infty}^{t+1} u(t)dt$
static or memoryless The output at the present time depends only on the value at the present time. $y(t) = 3 \cdot u(t)$ $y(t) = \sqrt{u(t)}$ $y(t) = 2^{-(t-1)}u(t)$	**dynamic** this system (an integrator) remembers everything that happened in the past or depends on inputs from times different from the present. Diff Eqs. $\rightarrow$ dynamic $y(t) = \int_0^t u(\tau)d\tau$ $y(t) = u(t-\tau)$ $\forall\tau \neq 0$ $y(t) = u(t^2)$ $y(t) = \dot{u}(t)$
time invariant A time-invariant system is a time-independent map between in and output which doesn't depend on time used. The system responds to the input x in a way that does not depend on when the system is used. $y(t) = \frac{d}{dt}u(t)$ $y(t) = 3 \cdot u(t)$ $y(t) = \int_0^t u(\tau)d\tau$ $(G_T u)(t) = u(t-\tau)$ $\Sigma G_T u = G_T \Sigma u = G_T y$	**Time-varying** If $f(t)$ is summed or multiplied with $u(t)$ $y(t) = f(t) \cdot u(t)$ $u(3t+a)$ $y(t) = u(t) + f(t)$ $y(t) = sin(t) \cdot u(t)$ $\lambda) \; t \to t+\tau$ $y(t) = u(t) + t$ $y(t^2) = u(t)$ $2) \; u(t-\tau)$ $y(t) = 3 \cdot u(sin(t))$

Dimension of a System: The dimension of a causal (and therefore realizable) system is the minimal number of variables sufficient to describe the system's state

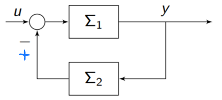
1.2 Block Diagrams



serial interconnection: $\Sigma = \Sigma_2\Sigma_1$



parallel interconnection: $\Sigma = \Sigma_1 + \Sigma_2$



(negative) feedback interconnection: $\Sigma = \Sigma_1(I + \Sigma_1\Sigma_2)^{-1}$

2 Linearisation

2.1 LTI Matrices

Linear Time-Invariant Matrices

Non-linear systems are often linearized around equilibrium point (x_e, u_e) and are only a good approximation near that point. The **deviation**, denoted by the deviation variable δ tells us how far away the linear system is from the actual system, slightly deviated from the equilibrium points (First term of Taylor Expansion). Note that usually the δ is omitted.

$f = \delta\dot{x}(t) = A\delta x(t) + B\delta u(t)$ $\quad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}$

$g = \delta y(t) = C\delta x(t) + D\delta u(t)$ $\quad C \in \mathbb{R}^{1\times n}, D \in \mathbb{R}$

$$A \overset{def}{=} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x_e, u_e}$$

dimension n of the state vector is equal to dimension of the system

$$B \overset{def}{=} \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}_{x_e, u_e}$$

$\dot{x}(t) = f(x(t), u(t))$

equilibrium point (x_e, u_e)

$f(x_e, u_e) = 0$

$$C \overset{def}{=} \begin{bmatrix} \frac{\partial g_0}{\partial x_1} & \cdots & \frac{\partial g_0}{\partial x_n} \end{bmatrix}_{x_e, u_e} \qquad D \overset{def}{=} \begin{bmatrix} \frac{\partial g_0}{\partial u} \end{bmatrix}_{x_e, u_e}$$

2.2 Similarity Transformation (Linear Algebra)

2.2.1 Matrix Exponential

For quadratic matrices the matrix exponential is defined as follows:

$e^{At} = T \cdot e^{\Lambda t} T^{-1}$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \ldots$$

$A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ $\quad A^2 = diag(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2)$

$A^n = diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_n^n)$

$e^{At} = diag(1 + \lambda_1 t + \frac{1}{2}\lambda_1^2 t^2 + \ldots, 1 + \lambda_2 t + \frac{1}{2}\lambda_2^2 t^2 + \ldots, \ldots)$

2.2.2 Repeated Eigenvalues

If repeated eigenvalues appear, the matrix cannot be diagonalized, but it can be brough into Jordan form. In general the homogeneous response of a state-space model is a linear combination of $exp(\lambda t)$ and $t^m exp(\lambda t)$, where m is the mulipicity of the eigenvalue.

For a 2^{nd} order system:

$$y(t) = Ce^{At}x_0 = Cexp(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix})x_0$$

$= c_1 exp(\lambda t)x_{0,1} + c_1 t exp(\lambda t)x_{0,1} + c_2 exp(\lambda t)x_{0,2}$

Often, repeated eigenvalues occur at $\lambda = 0$:

- $y(t) = c_1 x_{0,1} + c_1 t x_{1,0} + c_2 x_{0,2}$

System modeling: $\frac{d}{dt}storage = \Sigma inflow - \Sigma outflow$
- endogenous inputs: can be manipulated by the designer
- exogenous: generated by the environment e.g. disturbances
- outputs: measured or performance (not directly measurable, but controlled e.g. fuel consump.)
- standard form: $\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t), u(t))$

Jordan Form J	Matrix e^{tJ}
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix} = e^{\lambda_1 t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix} = e^{\lambda_1 t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

2.2.3 Diagonalisation

The choice of a state-space model for a given system is not unique and in most cases A is neither diagonal nor in Jordan form. Often a coordinate transformation is performed via diagonalisation.

Given $\begin{cases} \dot{x}(t) = Ax + Bu \\ y(t) = Cx + Du \end{cases}$

only 2x2: $= \lambda^2 - spur(A)\cdot\lambda + det(A)$

1. Calculate eigenvalues λ_i of A with $det(A - \lambda\mathbb{I})$
2. Calculate the eigenvectors v_i of A with $(A - \lambda_i\mathbb{I})v_i = 0$
3. The transformation matrix is: $T = (v_1, v_2, \ldots, v_n)$
4. Calculate T^{-1} (see section 2.3.3)
5. $\tilde{A} = T^{-1}AT = diag(\lambda_1, \ldots, \lambda_n)$
6. Compute $\tilde{B} = T^{-1}B$ and $\tilde{C} = CT$, which completes the coordinate transformation $\tilde{X} = T^{-1}x$ and therefore $x = T\tilde{x}$

2.2.4 Computing Inverse of a Matrix

If a matrix M has full rank, then it is invertible. For Control System we usually use Cramer's Rule.

- $M^{-1} = \frac{adj(M)}{det(M)}$
- Adjoint 2x2: $adj\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Adjoint 3x3: $adj\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$

Alternatively, if Cramer's Rule fails use the **Gauss-Jordan Algorithmus** from Linear Algebra:

To use this method, write the matrix and the identity matrix side by side and perform the Gauss algorithm on both sides simultaneously so that the identity matrix appears on the left side at the end.

Here's a tip: First, create a right triangular matrix on the left side by "Gaussing down," then create a diagonal matrix on the left side by "Gaussing up," and finally create the identity matrix on the left side by multiplying rows.

E.g.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

3 General Solution LTI Systems

3.1 Time Response of Linear time-invariant systems

Given $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1} \\ y(t) = Cx(t) + Du(t) & C \in \mathbb{R}^{1\times n}, D \in \mathbb{R} \\ \begin{cases} x(0) = x_{IC}(0) = x_0 \\ u_{IC}(t) = 0, \; t \geq t_0 \end{cases} & \begin{cases} x_F(0) = 0 \\ u_F(t) = u(t), \; t \geq t_0 \end{cases} \end{cases}$

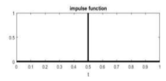
Solution: ↳ initial condition ↳ forced

- Initial condition response: $x_{IC} = e^{At}x_0$
- Forced Response: $\int_0^t e^{A(t-\tau)}Bu(\tau)$
- $x(t) = x_{IC} + x_F$
  $= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)$
- Initial condition output: $y_{IC} = Ce^{At}x_0$ ← initial response
- Forced Output: $y_F = C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$ ↳ forced response
- $y(t) = y_{IC} + y_F$
  $= Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du$ ← feedthrough

Also remember (state-transition function) : $\varphi(t-\tau) := e^{A(t-\tau)}$

3.2 Impulse Response

For an input of an Impulse Function of the form:



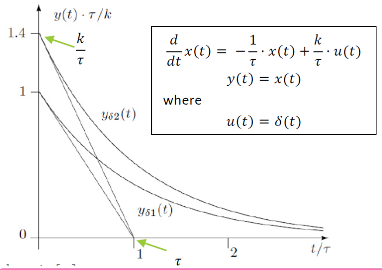
with the input

$$u(t) = \delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

We receive the general solution:

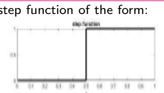
$$\boxed{y_\delta(t) = e^{\frac{-t}{\tau}} \cdot \left(x_0 + \frac{k}{\tau}\right)}$$

with time constant τ and coefficient k. Graphically it is displayed as:



3.3 Step response

For an input of a step function of the form:



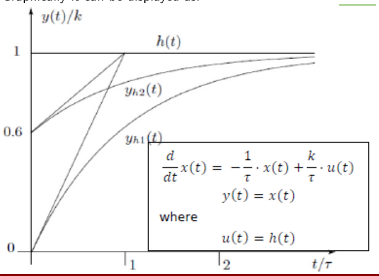
with the input:

$$u(t) = h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

We receive the general solution:

$$\boxed{y_h(t) = e^{\frac{-t}{\tau}} \cdot x_0 + k \cdot \left(1 - e^{\frac{-t}{\tau}}\right)}$$

Graphically it can be displayed as:



4 Stability

4.1 Asymptotic Stability

A **system is called asymptotically stable** if, for any bounded initial condition, and zero input, the state remains bounded and converges to zero:

- $||x_0|| < \epsilon$, and $u(t) = 0 \longrightarrow \lim_{t\to\infty} ||x(t)|| = 0$

For a linearised system with A,B,C,D matrices:

- A **system is asymptotically stable** if $Re(\lambda_i)$ of $A < 0$ $\forall i$

output linked to eigen value of A

For a scalar system:

- $a < 0 \longrightarrow$ **system is asymptotically stable** ↳ $f\,\lambda > 0$ output will grow e^{λ}

4.2 Lyapunov Stability

A **system is called Lyapunov stable** if, for any bounded initial condition, and zero input, the state remains bounded:

- $||x_0|| < \epsilon$, and $u(t) = 0 \longrightarrow ||x(t)|| < \delta$ for all $t \geq 0$

For a linearised system with A,B,C,D matrices:

- A **system is Lyapunov stable** if $Re(\lambda_i)$ of $A \leq 0$ $\forall i$

For a scalar system:

- $a \leq 0 \longrightarrow$ **system is Lyapunov stable**

Note: If a system is asymptotically stable it is also Lyapunov stable.

4.3 BIBO-Stability

BIBO: Bounded Input, Bounded OUtput:

A system is called **BIBO-stable** if, for any bounded input, the output remains bounded

- $||u(t)|| < \epsilon$ $\forall t \geq 0$, and $x_0 = 0 \longrightarrow ||y(t)|| < \delta$ for all $t \geq 0$

Note: For **minimal LTI system asymptotically stability = BIBO stability.** If there is a bounded continuous input, the output also needs to be continous.

If the input is an **impulse (dirac delta δ-function)** than the **impulse response (output)** $\sigma(t)$ needs to have a finite area:

- $\int_{-\infty}^{\infty} |\sigma(t)|dt < \infty$
- $\sigma(t) = \mathcal{L}^{-1}(G(s) \cdot \mathcal{L}(\delta(t))$
- Quick way: BIBO-stable if all poles of TF $G(s)$ have negative real part.

4.4 Unstable

A **system is unstable if its state moves away from the equilibrium point**

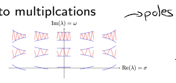
- A **system is unstable** if $\exists Re(\lambda_i)$ of $A > 0$

5 Frequency Domain

5.1 Laplace Transform

In general it is much simpler to calculate responses and design a control system in the frequency domain. We transform the response from the time domain into the frequency domain with the variable $s = j\omega$ and are interest in the frequency response.

- differential equations turn into algebraic equations
- convolutions in signals turn into multiplcations
- system can be analysed easily
- there is no information loss
- the laplace transform is linear



5.2 Input Signals

reference signal	$r(t)$	$R(s)$
impulse	$\delta(t)$	1
step	$h(t)$	$\frac{1}{s}$
ramp	$t \cdot h(t)$	$\frac{1}{s^2}$
harmonic	$c(t) = cos(\omega t) \cdot h(t)$ with $sin(\omega t)$	$\frac{s}{s^2+\omega^2}$ substitute s with ω

Some useful Laplace expressions

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st}f(t)\,dt$$

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \cdot \mathcal{L}(f(t)) + \beta \cdot \mathcal{L}(g(t))$$

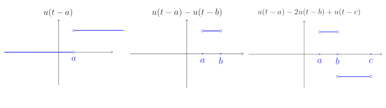
$$\mathcal{L}\int_0^t f(x)\,dx = \frac{1}{s}F(s)$$

$$\begin{aligned} \mathcal{L}(f')(s) &= s\mathcal{L}(f) - f(0) \\ \mathcal{L}(f'')(s) &= s^2\mathcal{L}(f) - sf(0) - f'(0) \\ \mathcal{L}(f''')(s) &= s^3\mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0) \end{aligned}$$

5.2.1 t-shifting, Heaviside Funktion

If $a \geq 0$, $\quad u(t-a) := \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$ $\quad L(u(t-a)) = \frac{e^{-as}}{s}$

$h(t) = u(t-0)$



$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$$L(f(t)u(t-a)) = e^{-as}L(f(t+a))$$

5.3 Transfer Functions

5.3.1 Output Response to $u(t) = e^{st}$

Exponential signals play an important role in linear systems. Many signals can be represented as exponential or as sum of exponentials. A constant signal is simply $e^{\alpha t}$ with $\alpha = 0$. Damped sine and cosine signals can be represented by: $e^{\sigma + j\omega t} = e^{\sigma t}[cos(\omega t) + jsin(\omega t)]$

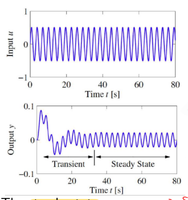
Given $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$

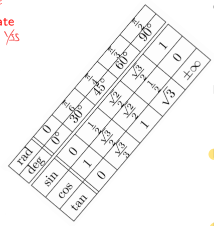
Set the input to be $u(t) = e^{st}$

The output response is comprised of two terms:

- $y(t) = Ce^{At}[x_0 - (s\mathbb{I} - A)^{-1}B] + [C(s\mathbb{I} - A)^{-1}B + D]e^{st}$

transient response → 0 if asymptotically stable; steady state yss





The **steady state**: so $G(s) = C(s\mathbb{I} - A)^{-1}B + D$

- $y_{ss} = [C(s\mathbb{I} - A)^{-1}B + D]e^{st} = G(s)e^{st}$ where $G(s)$ is a transfer function.

Note: If we set $s = j\omega$ the system response force by $u(t) = sin(\omega t)$ or $u(t) = cos(\omega t)$ can be determined.

5.3.2 State Space (time) ⟶ Transfer Function (frequency)

The transfer function $G(s)$ of $\Sigma(s)$ describes a system in the frequency domain.

$$\begin{aligned} G(s) = \frac{Y(s)}{U(s)} &= C \cdot (s \cdot I - A)^{-1} \cdot B + D \\ &= \frac{C \cdot Adj(s \cdot I - A) \cdot B}{det(s \cdot I - A)} + D \\ &= \frac{b_m \cdot s^m + \ldots + b_1 \cdot s + b_0}{s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0} + D \end{aligned}$$

$$adj\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $Y(s)$ and $U(s)$ are the Laplace transform of the output $y(t)$ and input $u(t)$.

- $Y(s) = G(s) \cdot U(s)$ $\rightarrow$ $U(t) = \sum U_i e^{st}$
- $\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}(G(s) \cdot U(s)) = y(t)$

Categories of Transfer Functions

$deg(N(s)) := m \quad deg(D(s)) := n$ where N is the numerator and D is the denominator of the transfer function

- Proper Transfer Function: $n \geq m$ (causal)
- Strictly Proper Transfer Function: $n > m$
- Improper Transfer Function: $n < m$ (acausal)

5.3.3 Transfer Function (frequency) ⟶ Controllable Canonical Form (time)

Given any transfer function $G(s)$ with no common poles/zeros (no pole-zero cancellation), it is straightforward for a system in a state-space form that realizes that transfer function. This is known as the controller canonical form and is the minimal realization of the system. Given the transfer function:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m \cdot s^m + \ldots + b_1 \cdot s + b_0}{s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0} + d$$

the **state-space representation of the system can be calculated as:**

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ -a_0 & -a_1 & -a_2 & \ldots & \ldots & -a_{n-1} & 1 \\ \hline b_0 & b_1 & b_2 & \ldots & \ldots & \ldots & d \end{array}\right]$$

if A like this we can calculate directly the TF

5.3.4 Ways to write TF

Given general transfer function:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m \cdot s^m + \ldots + b_1 \cdot s + b_0}{s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0}$$

Root-Locus Form:

- $G(s) = k_{rl}\frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$ zeros; $p_i < 0$ unstable poles; ↳ poles (eigenvalues of A)

Bode Form: Bode gain

- $G(s) = \frac{k_{Bode}}{s^q}\frac{(\frac{s}{-z_1} + 1)(\frac{s}{-z_2} + 1)\ldots(\frac{s}{-z_m} + 1)}{(\frac{s}{-p_1} + 1)(\frac{s}{-p_2} + 1)\ldots(\frac{s}{-p_{n-q}} + 1)}$

Partial Fraction Expansion

- $G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \ldots + \frac{r_n}{s - p_n} + r_0$ $= D$

r_i: residues # 6.2

Use **Cover-up method** to determine residues:

For **non-repeated poles** p_i:

- $r_i = \lim_{s\to p_i}(s - p_i)G(s)$

if a zero concide with the pole they cancel out → problem if the pole is unstable

For **repeated poles** of order m:

- $r_i = \frac{1}{(m-1)!}\lim_{s\to p_i}\frac{d^{m-1}}{ds^{m-1}}((s - p_i)^m)G(s)$

5.4 Sinusoidal or Frequency Response

The **steady-state frequency response of a linear system can be computed from its transfer function by setting** $s = j \cdot \omega$,

$$u(t) = e^{j\cdot\omega\cdot t} = \cos(\omega \cdot t) + j \cdot \sin(\omega \cdot t)$$

The resulting steady-state output ($t \to \infty$) has the form:

$$\begin{aligned} y(t) &= G(j \cdot \omega)e^{j\cdot\omega\cdot t} \\ &= M \cdot e^{j\cdot\omega\cdot t + \phi} \\ &= M \cdot \cos(\omega \cdot t + \phi) + j \cdot M \cdot \sin(\omega \cdot t + \phi) \end{aligned}$$

Where M and ϕ are the magnitude and phase of $G(j \cdot \omega)$

$$M = |G(j \cdot \omega)|$$

$$\phi = \angle(G(j \cdot \omega)) = \arctan\left(\frac{Im(G(j \cdot \omega))}{Re(G(j \cdot \omega))}\right)$$

The frequency response $G(j \cdot \omega)$ can thus be represented by two curves: The magnitude and the phase curve which can be represented on a log/log(dB) scale. These types of plots are called the Bode Plot (see chapter 8).

5.5 Initial & Final Value Theorem

Initial Value Theorem

- $$y(t \to 0) = \lim_{s\to\infty} sG(s)U(s)$$

Final Value Theorem → determine steady state value

- $$y(t \to \infty) = \lim_{s\to 0} sG(s)U(s)$$

To apply the final value theorem to G(s) **these conditions must be met:**

1. **all poles of the transfer function must have negative real parts** → *must be stable*
2. **at most one pole at origin.**

→ **can't be applied to unstable systems**

6 ROOT LOCUS → Using open loop TF

6.1 Definition

6.1.1 Angle & Magnitude Rules

Basic Operation for complex numbers

- $|a \cdot b| = |a| \cdot |b|$
- $|\frac{a}{b}| = \frac{|a|}{|b|}$
- $\angle(a \cdot b) = \angle a + \angle b$
- $\angle(\frac{a}{b}) = \angle a - \angle b$

$\angle(a+jb) = \begin{cases} \arctan(\frac{b}{a}) & a>0 \\ \arctan(\frac{b}{a})+\pi, & a<0, b\geq 0 \\ \arctan(\frac{b}{a})-\pi, & a<0, b<0 \end{cases}$

$\arctan(\pm 1) = \pm\frac{\pi}{4}$
$\arctan(0) = 0$
$\arctan(\pm\infty) = \pm\frac{\pi}{2}$

Angle Rule: Calculation of angles of all components of a transfer function.

- $\angle(s-z_1)+\angle(s-z_2)+...-\angle(s-p_1)-\angle(s-p_2)-... =$
  $\angle(-\frac{1}{k}) = \begin{cases} 180^\circ(\pm q \cdot 360^\circ) \text{if} & k>0 \\ 0^\circ(\pm q \cdot 360^\circ) \text{if} & k<0 \end{cases}$

Magnitude Rule: Calculation of magnitudes of all components of a transfer function

- $\frac{|s-z_1| \cdot |s-z_2| \cdot ... \cdot |s-z_m|}{|s-p_1| \cdot |s-p_2| \cdot ... \cdot |s-p_n|} = \frac{1}{|k|}$

TF of closed loop $T(s) = \frac{kL(s)}{1+kL(s)}$ complementary sensitivity

Characteristic equation of closed loop: $\frac{N(s)}{D(s)} = -\frac{1}{k}$

6.1.2 Real Axis

Since the characteristic equation has real coefficients, any zeros must occur in complex conjugate pairs (which are symmetric about the real axis).

- The root locus is symmetric about the real axis (complex poles/zeroes come in conjugate pairs).
- all points on the real axis are on the root locus
- all points on the real axis to the left of an **odd** number of poles/zeroes are on the **positive** k root locus
  ↳ we start with - - - and then line after we approach × ○
- All points on the real axis to the left of an **even** number of poles/zeroes (or none) are on the **negative** k root locus
- When two branches come together on the real axis, there will be "breakaway" or "break-in" points

6.1.3 Asymptotes

If the system has more open-loop poles than zeroes and $k \to \infty$, the excess pole will "explode" and go "to infinity". To determine in which direction these excess poles go to inifity consider: **Asymptote angles** δ_i:

- For $k_{rl} > 0$ $\delta_i = \frac{\pi \cdot (2i+1)}{n-m}, i = \{0, ..., n-m-1\}$ (poles n, zeros m)
- For $k_{rl} < 0$ $\delta_i = \frac{\pi \cdot 2i}{n-m}, i = \{0, ..., n-m-1\}$

6.2 Breakaway Points

For simplicity only consider the breakaway points that are on the root locus. Given a TF $G(s) = \frac{N(s)}{D(s)}$ Breakaway points can be computed with:

internal instability: pole-zero cancellation
well posed system when $T(s) = \frac{4}{3}$
interconnection → no sense

- $N(s)\frac{dD(s)}{ds} - D(s)\frac{dN(s)}{ds} = 0$ ↳ stability conserved

integrator: $G(s) = \frac{4}{s}$ add a pole
differentiator: $G(s) = s$ add a zero
pole at the origin: "negative derivative" ↳ steeper start
⇒ minimum phase zero → zero in the LHP; zero in the RHP
↳ add an "anticipatory effect" → so done dove andare
⇒ non-minimum phase zero → output moves initially in the wrong direction

6.3 Recipe

1. Sketch the root locus on the real axis according to 6.1.2
2. Closed loop poles approach the open loop poles (×) as $k \to 0$. Start drawing from open loop poles (×)
3. Closed loop poles approach the open loop zeroes (○) as $k \to \infty$. Certain branches of the root locus end at the open-loop zeroes (one-to-one rule)
4. The $(n-m)$ "excess poles will diverge on asymptotes to infinity. Two poles on the real axis meet in breakaway points. Such poles rotate about 90° when they meet (angle rule). Compute the asymptotes and the breakaway point.
5. Compute the "centre of mass", which is the origin point of the asymptotes σ_a
   $\sigma_a = \frac{1}{n-m}(\sum_{i=1}^{n} Re(p_i) - \sum_{j=1}^{m} Re(z_J))$
6. Calculate asymptote angles δ_i according to 6.1.3
7. Calculate breakaway points according to 6.2

7 BODE

7.1 Definition

The Bode plot is a tool to visualize the frequency response of a transfer function $G(j \cdot \omega)$, which includes a magnitude and phase curve, plotted in log/log scale, called the decibel scale. If two transfer functions $G_1(s)$ and $G_2(s)$ have the same bode plots, they are the same transfer function ($G_1(s) = G_2(s)$)

7.2 Decibel Scale

- $\mathbb{X}_{dB} = 20 \cdot log_{10}(\mathbb{X})$
- $\mathbb{X} = 10^{\frac{\mathbb{X}_{dB}}{20}}$

$\mathbb{X}$	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	10	100
$\mathbb{X}_{dB}$	−60	−40	−20	≈ −6	≈ −3	0	≈ 3	≈ 6	20	40

- $|G_1(s)| \cdot |G_2(s)| = |G_1(s)|_{dB} + |G_2(s)|_{dB}$
- $|G(s)^{-1}| = -|G(s)|_{dB}$ frequency in rad

7.3 Rules for Drawing Bode Plots

- Bode plots are additive (use superposition principle for complicated TF)
- If an element is added to a transfer function, it will affect a certain frequency. Past this frequency the magnitude plot will have permanently shifted. In the case of the phase plot, the change in phase is felt over two decades, one before aforementioned frequency and one after.
- Note to differentiator and integrator: Magnitude line passes through 0 at $\omega = \frac{1}{\tau}$
- Starting point of bode plot is affected by differnetiators and integrators.

Consider a complex conjugate element:

complex conjugate pole $p_c = \frac{1}{(\frac{2}{\omega_0})^2 + 2\xi(\frac{s}{\omega_0}) + 1}$ with
$p_{1,2} = -\xi \cdot \omega_0 \pm j \cdot \omega_0\sqrt{1-\xi^2}$
complex conjugate zero $z_c = (\frac{2}{\omega_0})^2 + 2\xi(\frac{s}{\omega_0}) + 1$

Element type	Formally	Magnitude	Phase
single stable pole	$Re(p_i) \leq 0$	$-20\frac{dB}{dec}$	-90°
single unstable pole	$Re(p_i) > 0$	$-20\frac{dB}{dec}$	$+90^\circ$
cmplx stable pole	$Re(p_i) \leq 0$	$-40\frac{dB}{dec}$	-180°
cmplx unstable pole	$Re(p_i) > 0$	$-40\frac{dB}{dec}$	$+180^\circ$
single mnm phs zero	$Re(z_i) \leq 0$	$+20\frac{dB}{dec}$	$+90^\circ$
single non-mnm phs zero	$Re(z_i) > 0$	$+20\frac{dB}{dec}$	-90°
cmplx non-mnm phs zero	$Re(z_i) > 0$	$+40\frac{dB}{dec}$	-180°
cmplx mnm phs pole	$Re(z_i) \leq 0$	$+40\frac{dB}{dec}$	$+180^\circ$
positive constant	$k \geq 0$	$0\frac{dB}{dec}$	0°
negative constant	$k < 0$	$0\frac{dB}{dec}$	$\pm 180^\circ$
differentiator	$\tau \cdot s$	$+20\frac{dB}{dec}$	$+90^\circ$
integrator	$\frac{1}{\tau \cdot s}$	$-20\frac{dB}{dec}$	-90°
time delay	$e^{s \cdot \tau}$	$0\frac{dB}{dec}$	$-\omega \cdot \tau$

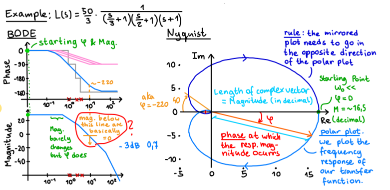
7.4 Recipe

1. Bring $G(s)$ into BODE form, calculate poles and zeroes and draw them in the bode diagrams (magnitude and phase).
   $G(s) = \frac{k_{Bode}}{s^q}\frac{(\frac{1}{-z_1}+1)(\frac{1}{-z_2}+1)...(\frac{1}{-z_m}+1)}{(\frac{1}{-p_1}+1)(\frac{1}{-p_2}+1)...(\frac{1}{-p_{n-q}}+1)}$
2. Find starting point at $\omega = 0$. If there are no integrators $\frac{1}{s}$ or differentiators s compute the dB value of k_{bode} and draw a horizontal line. Phase graph in this case starts at 0° if $k_{bode} > 0$ or at -180° if $k_{bode} < 0$.
   Integrator $\frac{1}{s}$: Slope is instantly $-20\frac{dB}{dec}$. Phase starts at -90°
   Differentiator s: Slope is instantly $+20\frac{dB}{dec}$. Phase starts at $+90^\circ$
   ↳ dB of k int through 0dB at $\omega = 1$
   We take the absolute position of poles and zeros
3. Draw the rough graphs
4. Smooth out the graphs (see examples)

I can verify stability only if it is open-loop stable

- Use magnitude and phase to sketch the polar plot, then mirror it around the Re-axis

Example: $L(s) = \frac{50}{3} \cdot \frac{(\frac{s}{4}+1)}{(\frac{s}{3}+1)(s+1)}$

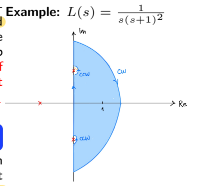


8.4 Open-Loop Poles on Imaginary Axis /D-Contour

If there are open-loop poles on the imaginary axis, make small "indentations" into the Nyquist contour to exclude these imaginary poles. This means you avoid the imaginary poles with a half-circle with radius ε.

Note: When moving counterclockwise around the poles, close the Nyquist plot clockwise "towards infinity".

Example: $L(s) = \frac{1}{s(s+1)^2}$



Avoid this complex conjugate pole with a circle with radius ε: $\varepsilon \cdot e^{j\theta}, -\pi/2 \leq \theta \leq \pi/2$ Computing $\lim_{\varepsilon\to 0} \angle(L(\varepsilon \cdot e^{j\theta}))$ gives the phase of the circle, which depending on $L(s)$ can be a multiple of θ, which lies between $-\pi/2$ and $\pi/2$ or might be shifted by an angle.

$\lim_{\varepsilon\to 0} \angle(L(\varepsilon \cdot e^{j\theta})) =$
$\lim_{\varepsilon\to 0} \angle(\frac{1}{\varepsilon \cdot e^{j\theta}(\varepsilon^2 \cdot e^{2j\theta} + 2\varepsilon \cdot e^{j\theta} + 1)}) =$
$\lim_{\varepsilon\to 0} \angle(\frac{1}{\varepsilon e^{j\theta}}) =$
$\lim_{\varepsilon\to 0} \angle(\frac{1}{\varepsilon}e^{-j\theta}) = -\theta$

· Calcolo limite
· inserisco $\theta_1 = -\frac{\pi}{2}$ $\theta_2 = \frac{\pi}{2}$
· $\theta_2 - \theta_1$ = rotazione CW

8 NYQUIST

8.1 Definition

The Nyquist plot is a parametric polar plot of a frequency response of a transfer function $G(j\omega)$, where $r = |G(j\omega)|$ and $\varphi = \angle G(j\omega)$. It is the most reliable method to construct the behaviour of a closed loop system from its corresponding open loop system. With the Nyquist plot stability can be checked. If two transfer functions $G_1(s)$ and $G_2(s)$ have the same Nyquist plot, then they are not necessarily the same transfer function ($G_1 \neq G_2(s)$ in general)

8.2 Nyquist Condition

Consider a closed-loop system with an open loop transfer function $kL(s)$, which has P poles in the region enclosed by the Nyquist contour (which are the unstable open-loop poles). Let N be the net number of **clockwise** (counterclockwise counts negative) encirclements of $-\frac{1}{k}$ by $L(s)$ when s moves along the Nyquist contour in the clockwise direction. The system then has $Z = N + P$ unstable closed-loop poles. For a stable closed-loop system $Z = 0$

8.3 Drawing the Nyquist Plot

Quick Tools

For drawing a nyquist plot there are 4 useful limits we have to use:

- $\lim_{\omega\to 0} G(j\omega)$

Defines where on the positive real axis the starting point is positioned (where transfer function converges to for $\omega \to 0$). The result we get from this has to be converged back into the linear scale. (Gain Margin)

- $\lim_{\omega\to 0} \angle G(j\omega)$

This limit defines which angle the starting position has.

- $\lim_{\omega\to\infty} |G(j\omega)|$

Defines the convergence of ω towards infinity. (Frequency phase when intercepting the unit circle)

- $\lim_{\omega\to\infty} \angle G(j\omega)$

Defines the angle at which the plot reaches the center of the graph. (frequency magnitude when crossing -180°)

Finally, check if there are further intersections with the real axis:

- $Im(L(j \cdot \omega_a)) \stackrel{!}{=} 0$
  Plug this frequency ω_a into magnitude of transfer function: $|L(j \cdot \omega_a)|$ = real-axis intercept

If a more accurate plot is necessary

- First draw BODE plots and use the information there to plot frequency response in the complex plane

9 Time Domain Specifications

9.1 Steady-State Error

The static steady-state error is defined as :

$e_{ss} = \lim_{t\to\infty} r(t) - y(t) = \lim_{s\to 0} s \cdot S(s) \cdot R(s) = \lim_{s\to 0}\frac{1}{1+L(s)}$

Unit ramp of order q:

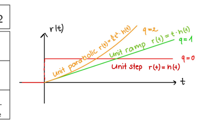
$r(t) = \frac{1}{q!}t^q$ sensitivity $S(s) \cdot \frac{1}{TF}$

We distinguish between types of system and to do that we bring any standard feedback loop $L(s)$ in the BODE-Form and let the frequency tend to zero:

$\lim_{s\to 0}\frac{k_{Bode}}{s^m}$ → type of the system → numbers of poles at the origin

,where m gives us the "number of integrators" of the OPEN-loop system (type m)

e_{ss}	$q=0$	$q=1$	$q=2$
Type 0	$\frac{1}{1+k_{\text{Bode}}}$	∞	∞
Type 1	0	$\frac{1}{k_{\text{Bode}}}$	∞
Type 2	0	0	$\frac{1}{k_{\text{Bode}}}$



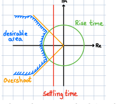
9.2 Step Response of a 1st Order System

Note: Very similar information in section 3.3.
Suppose we have a stable first-order system:

$G(s) = \frac{1}{\tau \cdot s + 1} \longleftrightarrow \begin{cases} \dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}u \\ y = x \end{cases}$

Then if we set the initial condition $x_0 = x(t=0) = 0$ then our step response is

$y(t) = 1 - e^{-\frac{t}{\tau}}$



Settling Time

- $T_d = \tau log(\frac{100}{d})$

where $\tau = -\frac{1}{p}$ is the time constant of the real pole p and d is the percentage of tolerance towards the steady state.

9.3 Step Response of a 2nd Order System

Suppose we have the following 2^{nd} order system:

$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Leftrightarrow \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u \\ y = x; \end{cases}$

For an underdamped system ($\zeta < 1$) with a zero initial condition ($x_0 = 0$) the step response is

If the system has more than two poles (higher order) → we can approximate by the slower pole → closer to imaginary axis (more dominant effect factor)

$y(t) = A - \frac{1}{cos(\varphi)}e^{\sigma t}cos(\omega t + \varphi), t \geq 0$

for example $G(s) = \frac{1}{(s+1)(s+10)}$ ↳ factor ensures steady state stays the same
↳ approximate the most dominant ones (dominant pole approximation)

Settling time (d %)

- $T_d = \frac{-1}{\sigma}ln\frac{d}{100}$

Time to peak

- $T_p = \frac{\pi}{\omega}$

Rise Time $T_{100\%}$

- $T_{100\%} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$

Peak overshoot (ratio)

- $M_p = e^{\frac{\sigma\pi}{\omega}} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$
- $\zeta^2 = \frac{(ln(M_p))^2}{\pi^2 + (ln(M_p))^2}$ per max peak overshoot $\zeta \geq \zeta^*$
- $M_p \approx \frac{71^\circ - \varphi}{117^\circ}$

Rise Time $T_{90\%}$

- $T_{90\%} = (0.14 + 0.4\zeta)\frac{2\pi}{\omega_n}$

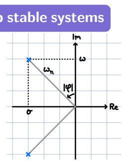
10 Frequency Domain Specifications

10.1 Robustness Margins for open-loop stable systems

Gain Margin:
↳ defined positive downwards

- $g_m = \frac{1}{|L(j\omega_{pc})|}$

$\omega_n = \sqrt{\sigma^2 + \omega^2}$
$|\sigma| = \zeta\omega_n$
$\varphi = \arctan(\frac{\sigma}{\omega})$
$\zeta = sin(|\varphi|)$

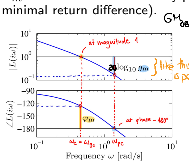


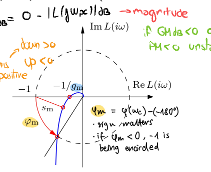
Phase Margin:

- $\varphi_m = 180^\circ + \angle(j\omega_{gc})$

ω_{pc}: phase crossover frequency at which the phase $\angle L(j\omega)$ crosses $\pm 180^\circ$.
ω_{gc}: gain crossover frequency at which the magnitude $|L(j\omega)| = 1$
S_m: minimum distance of Nyquist plot to −1 (also known as the minimal return difference).

$GM_{dB} = 0 - |L(j\omega_{pc})|$ → magnitude
if $GM_{dB} < 0$ or $PM < 0$ unstable





10.2 Bode Plot Obstacle Course

Closed-loop bandwidth: $|S(j\omega)| << 1$ at low frequencies for disturbance rejection and good command tracking

- maximum ω s.t. $|T(j\omega)| > \frac{1}{\sqrt{2}} = -3.0103dB$
- output can track the commands(inputs) to within a factor of ≈ 0.7

$|T(j\omega)| << 1$ at high frequencies for noise rejection

Beware that values in rad/s need to be divded by 2π to get Hz
If the phase margin is about 90° then $L(j\omega_c) = -j$ where ω_c is the crossover frequency. It then holds that

remember $S + T = 1$

- $|T(s)| = |\frac{L(j\omega_c}{1+L(j\omega)}| = |\frac{-j}{1-j}| = \frac{1}{\sqrt{2}}$
- The open-loop crossover frequency is therefore approximately equal to the bandwidth of the closed-loop system.

The bandwidth represents the frequencies that pass through the closed-loop system, but don't get filtered away or amplified.

10.3 Loop Shaping

Integrators

- Add as many integrators as needed to be able to track an m-th order ramp input with zero steady state error (see 10.1)
- This increases the magnitude of at low frequencies and decreases it at high frequencies
- This also decreases the by # integrators ·90'o everywhere, which is problematic for the phase margin

Proportional gain k

- Choose the gain k s.t. the low freuquency asymptote clears the disturbance rejection specification ($W_1(s)$)
- k shifts the magnitude of the plot up or down
- The phase plot is unaffected
- An increased gain improves command tracking (increase magnitude at low frequencies) and also increased closed-loop bandwidth (moves the crossover frequency to the right)

for stable OL small k yield stable CLs

Lead Compensator → pole-zero pair
↳ stability can be compromised

- $C_{lead}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a} \cdot \frac{s+a}{s+b}$ with $0 < a < b$

Effects → main use: increase phase margin

- Increases the phase around $\sqrt{ab}$ by
  $\varphi_{max} = 2 \cdot arctan(\sqrt{\frac{b}{a}}) - 90^\circ$
- Magnitude and phase at lower frequencies are unaffected
- Increases the slope of the magnitude at frequencies between a and b by $+20\frac{dB}{decade}$
  Increases magnitude at high frequencies by factor $\frac{b}{a}$ (which increased noise sensitivity, which is bad)

Lag Compensator → main use: improve command tracking and disturbance rejection

- $C_{lag}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a} \cdot \frac{s+a}{s+b}$ with $0 < b < a$

$\frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{\frac{s+a}{a}}{\frac{s+b}{b}} = \frac{b}{a}\frac{s+a}{s+b}$

Effects

decrease magnitude at high frequency by $\frac{b}{a}$

- Decreases the phase around $\sqrt{ab}$ by
  $-\varphi_{min} = 2 \cdot arctan(\sqrt{\frac{b}{a}}) - 90^\circ$
- Magnitude at lower frequencies are unaffected
- Decreases the slope of the magnitude at frequencies between a and b by $-20\frac{dB}{decade}$

side effect: reduction of phase margin

11 PID Control

Criterio	Poli Instabili ($P > 0$)	Poli Stabili ($P = 0$)
Segmento	Il diagramma di Nyquist deve **circondare il segmento** P volte	Il diagramma di Nyquist **non deve intersecare** il segmento
Cerchio	Il diagramma di Nyquist deve **circondare il cerchio** P volte	Il diagramma di Nyquist **non deve intersecare** il cerchio

Proportional Gain k_P

Pros

- faster response
- decrease of steady-state error
- increase of closed-loop bandwidth
- decrease in disturbance sensitivity

Cons

- increase in noise sensitivity
- decrease in phase margin in 2nd or higher order systems
- more oscillations in 2nd or higher order systems

Integral Gain $\frac{k_I}{s}$

Pros

- Eliminates the steady-state error to a unit step input (as long as the closed-loop is stable)

Cons

- The response becomes more oscillatory
- Reduces stability margins
- Response is slower
- Phase margin decreases

* if feedback loop is self-sustaining oscillate indefinitely and causes a limit cycle ↳ draw $-\frac{1}{N(A)}$ check intersections

Derivative Gain

Pros

- acts like a damper against increases in error - reduces overshoot

Cons

- increase in noise sensitivity
- potentially slower response
- response becomes less oscillatory (improves phase margin oscillations of 2nd or higher order systems)
- Phase margin increase
- increases stability margins

12 Time Delays

12.1 Definition

- $y(t) = u(t-T)$, where $T \geq 0$ is the amount of delay

Suppose the input to a system is $u(t) = e^{st}$. Then it holds that

- $y(t) = e^{s(t-T)} = e^{-sT}u(t)$
- Time delay TF: e^{-sT} *

$PM = PM_{\text{without delay}} - \omega_{gc}T$ ↳ crossover frequency
$\angle G(j\omega) - \omega T$

This TF is linear but not rational. It has no poles or zeroes, so the root-locus method cannot be applied. Furthermore it holds:

- $|e^{-sT}| = 1$
- $\angle(e^{j\omega T}) = -\omega T$

effect of a time delay is a phase shift of $-\omega T$

which implies that the gain crossover frequency ω_{gc} is unaffected, when including a delay term into a TF. → phase margin <<

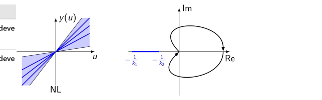
13 Nonlinearities

13.1 Absolute Stability

13.1.1 Necessary Condition (segment)

A necessary condition for absolute stability of the feedback is that the Nyquist plot encircles the segment $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ counterclockwise a number of times equal to the number of poles of $L(s)$ with positive real part (modified Nyquist condition for segment instead of point $-\frac{1}{k}$)

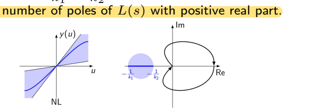
* $e^{-sT} \approx \frac{\frac{2}{T}-s}{\frac{2}{T}+s}$ Padé approximation → non-minimum phase zero



Necessary: There are some choices of NL within the sector, which make the closed-loop system stable. If not fulfilled no stability possible.

13.1.2 Sufficient Condition (circle)

A sufficient condition for absolute stability of the feedback system is that the number of times the Nyquist plot of $L(s)$ encircles the circle with diameter $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ counterclockwise a number of times equal to the number of poles of $L(s)$ with positive real part.



Sufficient: Ensures that NL is stable no matter the choice. If not fulfilled, no conclusion can be drawn on absolute stability of the closed-loop system.

13.2 Describing Functions

Frequency Response of a static nonlinearity (example)
Suppose you apply sinusoidal input to a static nonlinearity:

$u(t) = Asin(\omega t)$

examples of static NL: saturation, switch/relay, deadzone, quantizer

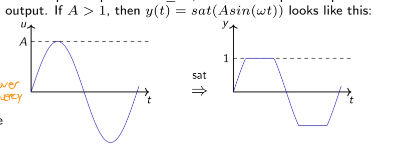
The output will be of the form:

$y(t) = f(Asin(\omega t))$

All we can say is that the output y will be a periodic signal with the same frequency as the input. Take for example the **saturation nonlinearity**:

$sat(u) = \begin{cases} 1 & if \ u \geq 1 \\ u & if \ -1 < u < 1 \\ -1 & if \ u \leq 1 \end{cases}$

If the input amplitude $A \leq 1$, then the output is equal to the output. If $A > 1$, then $y(t) = sat(Asin(\omega t))$ looks like this:



The output of the non-linearity can be approximated by its first harmonic:
Odd non-linearity: $y(t) \approx b_1 sin(\omega t)$
Even non-linearity: $y(t) \approx a_1 cos(\omega t)$
The ratio of a_1 or b_1 to A is called the **describing function**:

$N_{odd}(A) = \frac{b_1}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)sin(n\omega t)d(\omega t)$

$N_{even}(A) = \frac{a_1}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)cos(n\omega t)d(\omega t)$

If the input has both odd and even components a more general form of the describing function is given by.

$N(A, \omega) = \frac{c_1(A, \omega)}{A}e^{j\phi_1(A,\omega)}$

The new loop transfer function utilizing the describing function is approximated as:

$L'(A, s) \approx N(A)L(s)$