Exercise 6 - Poles and Zeros

1 Transfer Functions (continued)

1.1 Different ways to write transfer functions

Last week, we saw that we can write transfer functions as **strictly proper rational functions**, plus possibly a "**direct feedthrough**" term (controllable canonical form):

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d$$

We can also write them in other forms, as the **partial fraction expansion** (useful to understand how different modes contribute to the response):

$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} + r_0$$

Remember that the poles p_i are the roots of the characteristic polynomial $\det(sI - A)$, which are the eigenvalues of the system matrix A. Additionally, the variables r_i are called the *residues*.
Furthermore, we can factorize the numerator and the denominator in slightly different ways to get two important forms of transfer functions:

- **Root-locus form**:

$$G(s) = \frac{k_{rl}}{s^q} \frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_{n-q})}$$

- **Bode form**:

$$G(s) = \frac{k_{bode}}{s^q} \frac{\left(\frac{s}{-z_1} + 1\right)\left(\frac{s}{-z_2} + 1\right)\dots\left(\frac{s}{-z_m} + 1\right)}{\left(\frac{s}{-p_1} + 1\right)\left(\frac{s}{-p_2} + 1\right)\dots\left(\frac{s}{-p_{n-q}+1}\right)}$$

These two forms are particularly useful for control design techniques such as root-locus and frequency response (Bode plot) methods, which we will cover later in the course.

Note that in all forms above, p_i are the poles of the system, while z_i are the zeros. But what insights do the TF, poles, and zeros provide about the system's behavior? Let's explore this further.

1.2 Steady-state response to a unit step

Let's consider a stable LTI system with transfer function

$$G(s) = 2\frac{s+5}{s^2+8s+12},$$

and analyze its steady-state response to a unit step input $u(t) = h(t) = e^{0t} = 1,\ t \geq 0$. We saw last week that

$$y_{ss} = G(s)e^{st},$$

plugging in $s = 0$ for the unit step, we get

$$y_{ss} = G(0)e^{0t} = G(0) = 2\frac{0+5}{0^2+8\cdot 0+12} = \underset{\substack{\downarrow \\ k_{Bode}}}{\frac{5}{6}}$$

You can also obtain the same result using the **Final Value Theorem**:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,G(s)U(s) = \lim_{s\to 0} s\,G(s)\frac{1}{s} = G(0) = \frac{5}{6}$$

where we used the fact that $\mathcal{L}(u(t)) = \frac{1}{s}$.

This means that the system will eventually settle at a value of $\frac{5}{6}$ in response to a unit step input.

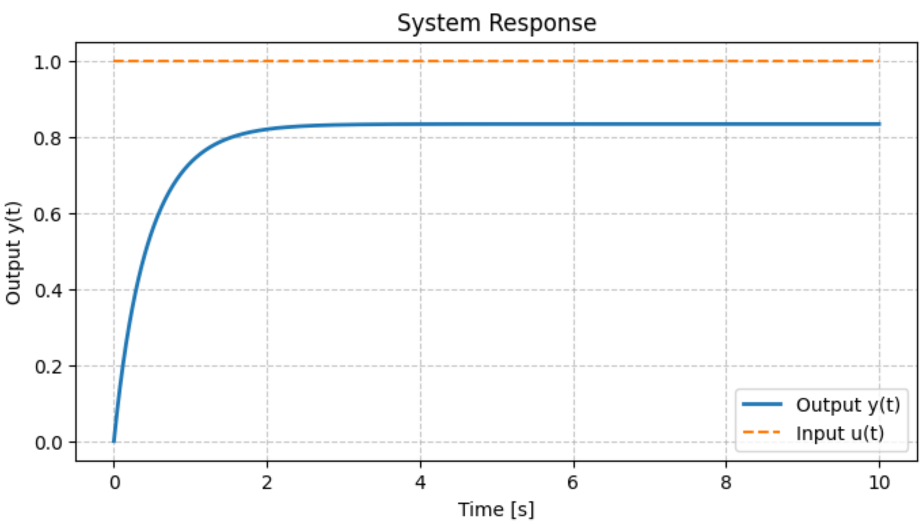
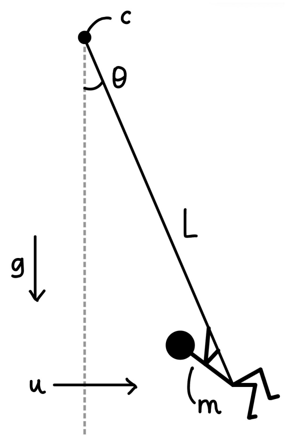


Figure 1: Steady-state response to a unit step input.

1.3 Steady-state response to a sinusoidal input

Consider now again the swing/pendulum system that we already saw many times.



$$\Longrightarrow \qquad G(s) = \frac{0.2}{s^2 + s + 2}$$

We want to analyze the steady-state response of this system to a sinusoidal input $u(t) = \sin(t)$. As we saw last week, we can express the sinusoidal input in terms of complex exponentials:

$$u(t) = \sin(t) = \frac{e^{jt} - e^{-jt}}{2j}$$

We also saw that we can get the steady-state response with $y_{ss} = |G(j)| \sin(t + \angle G(j\omega))$. Therefore we need to compute $|G(j)|$ and $\angle G(j)$:

$$G(j) = \frac{0.2}{(j)^2 + j + 2} = \frac{0.2}{2 + j - 1} = \frac{0.2}{1 + j}$$

Calculating the magnitude and phase:

$$|G(j)| = \left| \frac{0.2}{1 + j} \right| = \frac{|0.2|}{|1 + j|} = \frac{0.2}{\sqrt{1^2 + 1^2}} = \frac{0.2}{\sqrt{2}} \approx 0.14$$

$$\angle G(j) = \angle \left[\frac{0.2}{1 + j} \right] = \angle 0.2 - \angle(1 + j) = 0 - \frac{\pi}{4} = -\frac{\pi}{4} = -45^\circ$$

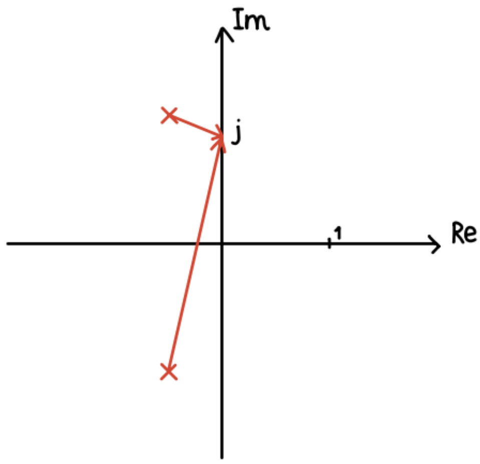
Thus, the steady-state response is:

$$y_{ss}(t) = |G(j)| \sin(t + \angle G(j)) = 0.14 \sin\left(t - \frac{\pi}{4}\right)$$

We could get the same result by doing it graphically with a pole-zero map, in this case we could write the TF as

$$G(s) = \frac{0.2}{(s^2 + s + 2)} = \frac{0.2}{\left(s + \frac{1+\sqrt{-7}}{2}\right)\left(s + \frac{1-\sqrt{-7}}{2}\right)}$$

Plotting the poles in the complex plane, we can evaluate the distance from the poles to the point j (since the input frequency is $\omega = 1$ rad/s).



The magnitude of the transfer function at this frequency is given by the ratio of the distances from the zeros to $j\omega$ over the distances from the poles to $j\omega$. Since there are no zeros in this case, we only consider the poles:

$$|G(j)| = \frac{0.2}{|j - p_1||j - p_2|} = \frac{0.2}{\sqrt{(1-1.32)^2 + (0+0.5)^2} \cdot \sqrt{(1+1.32)^2 + (0+0.5)^2}} \approx 0.14$$

The phase can also be determined graphically by measuring the angles from the poles to the point $j\omega$:

$$\angle G(j) = -(\angle(j - p_1) + \angle(j - p_2)) = -\frac{\pi}{4}$$

Thus, we arrive at the same steady-state response:

$$y_{ss}(t) = 0.14 \sin\left(t - \frac{\pi}{4}\right)$$

Note that as we saw last week, the steady-state response to a sinusoidal input is a sinusoid at the same frequency as the input, but scaled in amplitude and shifted in phase.

Important: Remember that these two responses are valid only for stable systems, otherwise the output will diverge.

Exam Problem: HS24

Problem: Consider a linear time-invariant system with transfer function $G(s)$ that is given by a root-locus gain $k_{\text{rl}} = 2$ and by three poles and two zeros as shown in Figure 4.

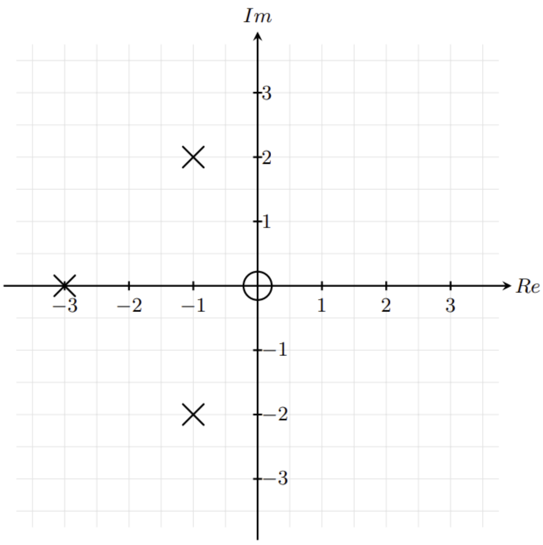


Figure 4: Poles ($\times$) and zeros ($\circ$) of the LTI system $G(s)$.

Q15 (1 Points)
What is the magnitude of $G(s)$ when $s = j$?

Q16 (1 Points)
What is the phase of $G(s)$ when $s = j$?

2 Poles and Zeros

We know that we can write transfer function as ratios of polynomials in s:

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Additionally, we now know that the roots of the numerator are called **zeros** of the system, while the roots of the denominator are called **poles** of the system.
These two concepts are fundamental in Control Systems theory that provide insights into the behavior of systems.

2.1 Poles

In order to understand the effect of poles on system behavior, let's consider the response of a system to an impulse input $u(t) = \delta(t)$, where

$$\int_0^{\epsilon} \delta(t)dt = 1,\ \epsilon > 0 \quad \text{and} \quad \int_0^t f(\tau)\delta(\tau)d\tau = f(0),\ \forall t > 0$$

We can solve the general output equation for $y(t)$ by assuming that $D = 0$, $x(0) = 0$ and $u(t) = \delta(t)$:

$$y(t) = \cancel{\mathrm{C}e^{\mathrm{A}t}x(0)} + \int_0^t \mathrm{C}e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau + \cancel{\mathrm{D}u(t)} =$$

$$= \int_0^t \mathrm{C}e^{\mathrm{A}(t-\tau)}\mathrm{B}\,\delta(\tau)d\tau = \mathrm{C}e^{\mathrm{A}t}\mathrm{B}$$

Now, thanks to our knowledge of Laplace transforms, we can apply the Laplace transform to $y(t)$ and $u(t)$, and since $Y(s) = G(s)U(s)$ we can get the TF $G(s)$.
Let's do it step by step considering a first order system:

$$y(t) = ce^{at}b \quad \Longrightarrow \quad Y(s) = \frac{cb}{s-a}$$

$$u(t) = \delta(t) \quad \Longrightarrow \quad U(s) = 1$$

Now we have that $G(s) = Y(s) = \frac{r}{s-a}$

And that $y(t) = re^{at}$

$x(t)$	$X(s)$
impulse: $\delta(t)$	1
step: $h(t)$	$\frac{1}{s}$
$h(t)t^n$	$\frac{n!}{s^{n+1}}$
$h(t)e^{at}$	$\frac{1}{s-a}$

We can see that the pole at $s = a$ directly influences the time-domain response $y(t) = re^{at}$.
Extending this to higher-order systems, we find that the transfer function can be expressed as:

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}$$

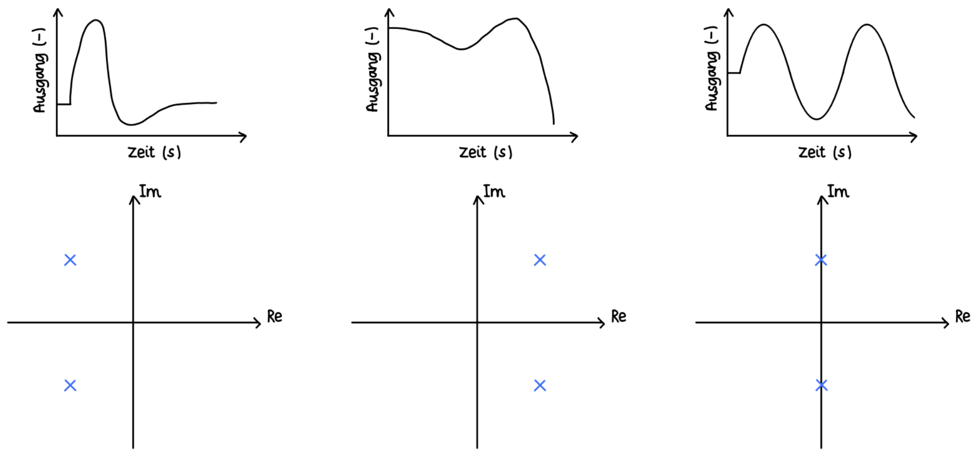
which leads to the time-domain response:

$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}$$

This shows that each pole p_i contributes an exponential term $e^{p_i t}$ to the system's response, scaled by its corresponding residue r_i. For real poles, this results in exponential growth or decay, while complex poles lead to oscillatory behavior.

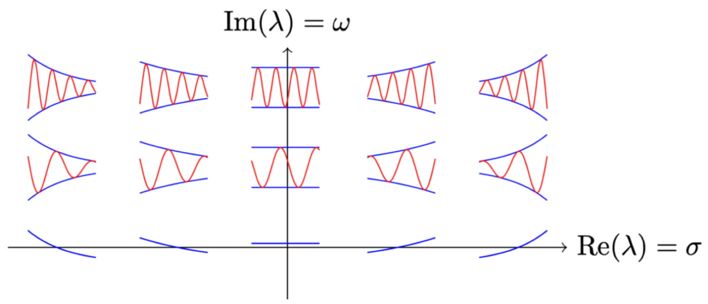
We are essentially "decomposing" the system's response into a sum of first order systems, each associated with a pole of the transfer function.

Example



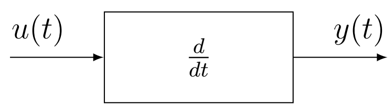
From the plots above, we can see how the location of poles in the complex plane affects the system's time-domain response.

Cool picture about effect of poles



2.2 Zeros

To understand the effect of zeros we will look at the differentiator system:



If we apply a general input $u(t)$ to this system, the output will be $y(t) = \frac{du(t)}{dt}$.
Now, let's analyze the system's response to a general input $u(t) = e^{st}$:

$$y(t) = \frac{du(t)}{dt} = \frac{d}{dt}e^{st} = se^{st}$$

We can then see that the transfer function of the differentiator is:

$$G(s) = \frac{Y(s)}{U(s)} = s$$

Note: for an integrator system, we would have $G(s) = \frac{1}{s}$.

We can say that by multiplying the input by s in the Laplace domain, i.e. adding a zero to the original TF, we are introducing a derivative action in the time domain.
This usually can be seen has an *anticipatory effect*.

To understand better this concept, let's analyze the step response of a system with and without a zero. For this case, in contrast to the previous section, we will look more at the plots than the equations.
Consider the plots below, where we have the step response for two systems:

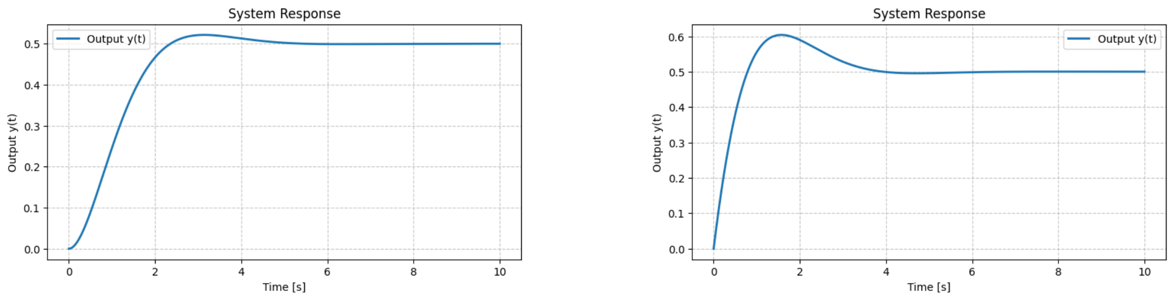
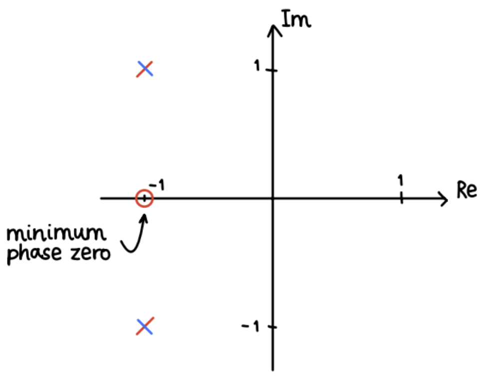


Figure 2: Step response of a system without zeros (left) and with a zero (right).

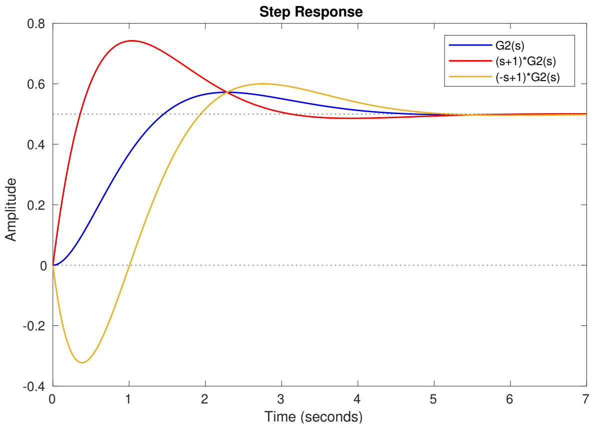
Both systems have the same poles, but the system on the right has a zero at $s = -1$ as we can see from the two transfer functions:

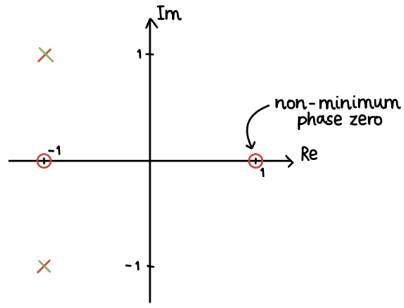
$$G_1(s) = \frac{1}{s^2 + 2s + 2} \qquad \text{and} \qquad G_2(s) = \frac{s+1}{s^2 + 2s + 2}$$

Zeros in the left half plane (LHP) are called **minimum phase zeros** and as we said before, they introduce an anticipatory effect, making the system respond faster to changes in the input.



However, we can also have zeros in the right half plane (RHP), i.e. with positive real parts. For the poles we said that RHP poles make the system unstable, but what about RHP zeros?
Let's see the step response of a system with a RHP zero:





For this case, we observe that the stability of the system is not affected by the RHP zero, but the response exhibits an inital movement in the opposite direction of the final value, a phenomenon known as non-minimum phase behavior.
The RHP zeros are called **non-minimum phase zeros**.

2.3 Pole-zero cancellation

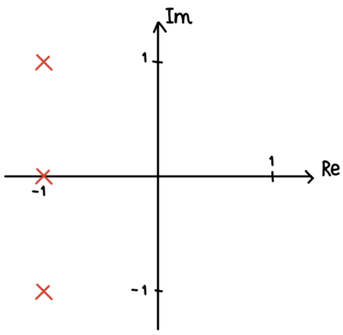
In some cases, a system may have poles and zeros that are very close to each other in the complex plane. When a pole and a zero are exactly at the same location, they can cancel each other out in the transfer function. This is known as **pole-zero cancellation**.
But what does this mean for the system behavior? In other words, if we have add a zero near a pole, what happens? Let's analyze this situation.
Consider the TF:

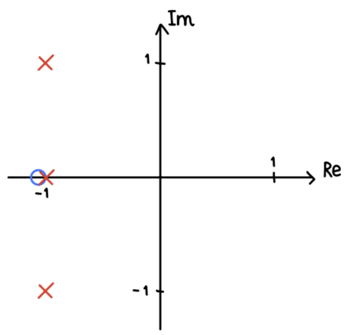
$$G_1(s) = \frac{1}{(s+1)(s+1+j)(s+1-j)}$$

Now, let's add a zero very close to one of the poles, for example near the pole $s+1$:

$$G_2(s) = \frac{s+1+\epsilon}{(s+1)(s+1+j)(s+1-j)}$$

The two complex representations of the systems are:





Remember from earlier that in the partial fraction expansion form, we can understand the influence of each pole on the system response:

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} \quad \Rightarrow \quad y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}$$

So let's compute the partial fraction expansion of both transfer functions to see how the residues change. We can use the **cover-up method**:

$$r_i = \lim_{s \to p_i} (s - p_i) G(s)$$

For $G_1(s)$:

$$r_1 = \lim_{s \to -1} (s+1) G_1(-1) = \lim_{s \to -1} \frac{\cancel{(s+1)}}{\cancel{(s+1)}(-1+1+j)(-1+1-j)} = \frac{1}{(j)(-j)} = 1$$

$$r_2 = \lim_{s \to -1+j} (s+1-j) G_1(1+j) = \frac{1}{(-1+j+1)(-1+j+1+j)} = \frac{1}{(j)(2j)} = -\frac{1}{2}$$

$$r_3 = \lim_{s \to -1-j} (s+1+j) G_1(-1-j) = \frac{1}{(-1-j+1)(-1-j+1-j)} = \frac{1}{(-j)(-2j)} = -\frac{1}{2}$$

Thus we have:

$$G_1(s) = \frac{1}{s+1} + \frac{-1/2}{s+1-j} + \frac{-1/2}{s+1+j}$$

Similarly, for $G_2(s)$ we get in the end:

$$G_2(s) \approx \frac{\epsilon}{s+1} + \frac{-1/2j}{s+1-j} + \frac{1/2j}{s+1+j}$$

This means that the smaller the value of ϵ, the smaller the contribution of the pole at $s = -1$ to the overall system response. In other words, as the zero approaches the pole, it effectively cancels out its influence on the system dynamics.
If $\epsilon = 0$, we have a perfect pole-zero cancellation, and the system behaves as if that pole does not exist.
This phenomenon can be useful in control system design, where we may want to eliminate the effect of certain poles to achieve desired performance characteristics.

It's important to note that there is no concern if the pole that is being cancelled is stable (i.e., in the LHP). However, if the pole to be cancelled lies in the RHP, the situation becomes critical. In that case, even though the pole and the zero algebraically cancel in the transfer function (input $\rightarrow$ output), the underlying unstable mode (state) is still present in the system's realization. As a result, the system may still behave as unstable internally — any small modelling error, noise or non-zero initial condition can excite that hidden unstable mode, causing the system to diverge.

Exam Problem: FS24

Problem: Consider the step responses of four linear time-invariant systems G_i, $i = 1, 2, 3, 4$ shown in Figure 3. In Figure 4 the pole-zero maps corresponding to G_i, $i = 1, 2, 3, 4$, are shown in randomized order.

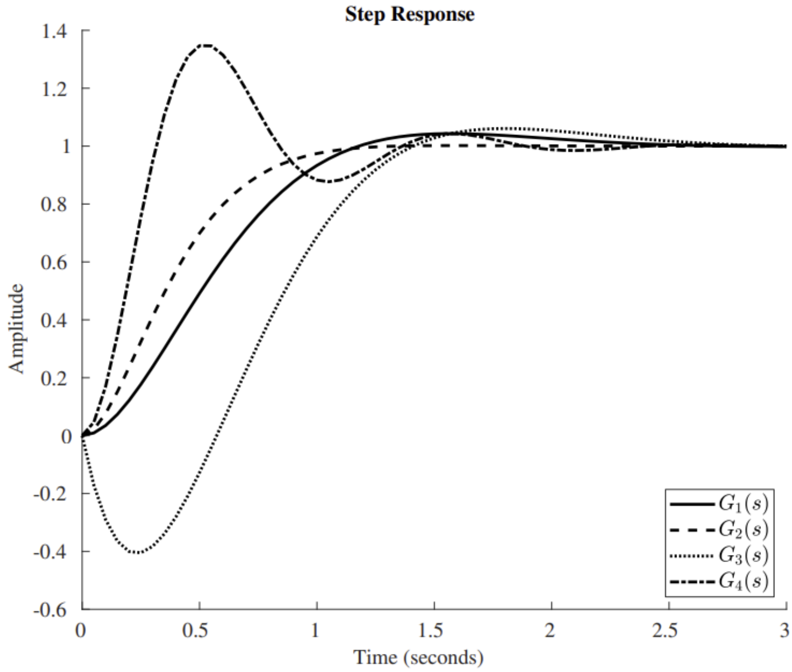
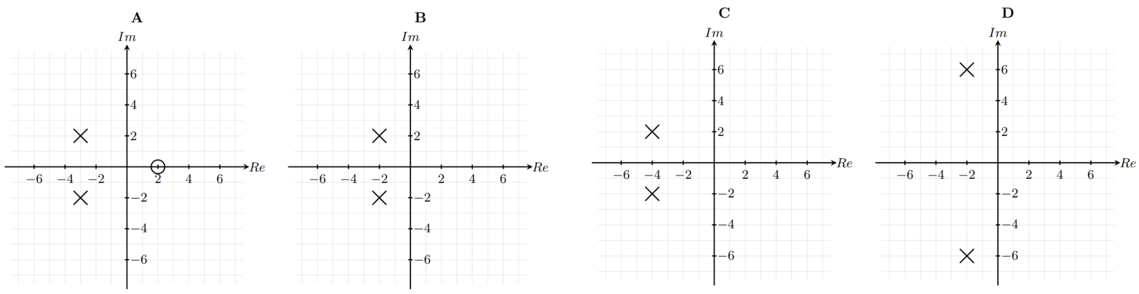


Figure 3: Step responses of G_i, $i = 1, 2, 3, 4$.



Q19 (1 Points) Mark the correct answer.
Mark the correct pole-zero map and transfer function pairings.

- A (A, G_2), (B, G_4), (C, G_1), (D, G_3)
- B (A, G_3), (B, G_1), (C, G_2), (D, G_4)
- C (A, G_2), (B, G_3), (C, G_1), (D, G_4)
- D (A, G_3), (B, G_2), (C, G_4), (D, G_1)

3 Initial and Final Value Theorems

The Initial Value Theorem (IVT) and Final Value Theorem (FVT) are useful tools in control systems for determining the initial and final values of a time-domain signal based on its Laplace transform.

3.1 Initial Value Theorem

The Initial Value Theorem states that if $y(t)$ is a time-domain function and $Y(s)$ is its Laplace transform, then:

$$\boxed{\lim_{t\to 0^+} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} sG(s)U(s)}$$

This theorem is only valid for asymptotic stable systems, i.e., systems whose poles are all in the left half of the complex plane (LHP).

3.2 Final Value Theorem

The Final Value Theorem states that if $y(t)$ is a time-domain function and $Y(s)$ is its Laplace transform, then:

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)U(s)}$$

This theorem is only valid for asymptotic stable systems, i.e., systems whose poles are all in the left half of the complex plane (LHP).

Exam Problem (didn't find the year)

What are the initial y_0 and final values y_∞ of an *impulse response* for the following input to output transfer function?

$$G(s) = \frac{s+1}{2s^2 + 0.5s + 1}$$

A $y_0 = 0.5, \quad y_\infty = 0$

C $y_0 = 0, \quad y_\infty = 1$

B $y_0 = 1, \quad y_\infty = 0$

D $y_0 = 0, \quad y_\infty = 0.5$