Exercise 7 - Root Locus

1 Open Loop vs Closed Loop

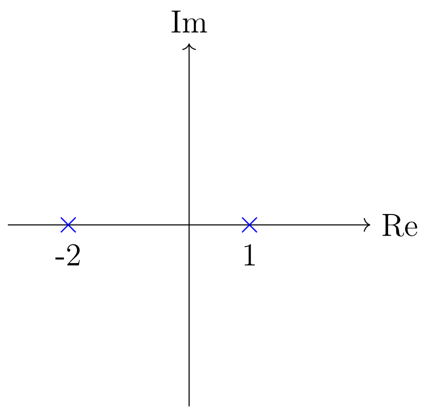
Before designing a controller, it is important to understand *what a controller or compensator actually is* and what role it plays in a feedback system. We can start by building some intuition — when we think of a controller, we might imagine a *computer or software* that adjusts the input to a physical system based on its output.
In this section, we will begin by addressing these fundamental questions by looking at the simplest and most intuitive type of controller: the **proportional (P) controller**, which multiplies the error signal by a gain k.

Example - Swing

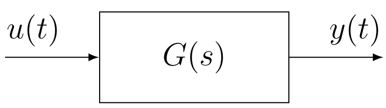
To understand better this concept, let's consider our swing/pendulum example from the previous exercises. This time, we will look in more detail at the inverted pendulum, which is unstable without control. Therefore, we want to design a controller that stabilizes it.
Our original system has the following transfer function: open-loop TF

$$G(s) = \frac{0.2}{s^2 + s - 2} = \frac{0.2}{(s-1)(s+2)} = L(s)$$

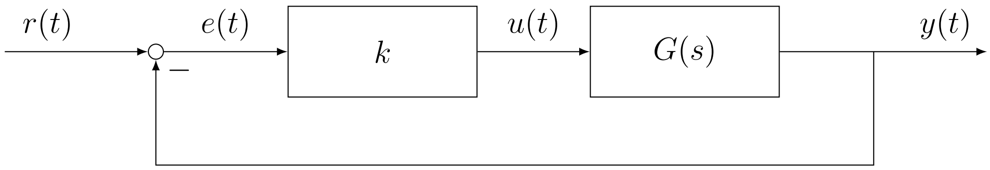
We identify that the system has a pole in $s = 1$ (unstable) and a pole in $s = -2$ (stable), that can be represented in a pole-zero map as:



Furthermore the system can be represented in a block diagram as:



In order to control the system (i.e. to stabilize it), we will add a proportional controller $C(s)$ with gain k. The new block diagram now looks like:



For now, the controller $C(s)$ will simply be a constant k that multiplies the error $e(t)$. This, as we said before, is also called gain and it is denoted by k.

The closed-loop transfer function of the system is given by:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{k\frac{0.2}{s^2+s-2}}{1 + k\frac{0.2}{s^2+s-2}} = \frac{0.2k}{s^2 + s - 2 + 0.2k}$$

Note: $T(s)$ is also called the complementary sensitivity.

As we can see from the TF, we can change the denominator and therefore the position of the poles of the closed-loop system by changing the value of k.
This means that we can change the stability of the system by changing k, thus, in this case, stabilizing the system by bringing the poles to the left half of the complex plane.

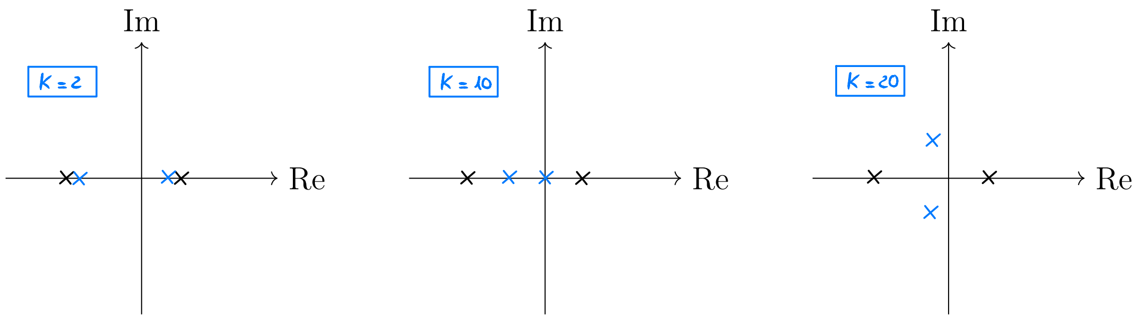
Let's see how the poles of the closed-loop system change as we vary k. The characteristic polynomial of the closed-loop system is given by:

$$s^2 + s - 2 + 0.2k = 0$$

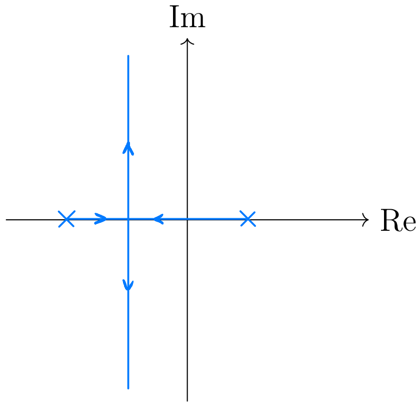
Solving for s, we get:

$$s = \frac{-1 \pm \sqrt{1 - 4(0.2k - 2)}}{2} = \frac{-1 \pm \sqrt{9 - 0.8k}}{2}$$

If we vary k from 0 to 20, we can see how the poles move in the complex plane:



We could analyze further the behavior of the closed-loop poles by doing algebra, but this would be tedius and not very insightful. Instead, we can use a graphical method called **Root Locus** to visualize how the poles of the closed-loop system move as we vary k.
If we plot the path of the closed-loop poles for our example we get the following root locus plot:



We can now see that as we increase k, the unstable pole moves from its original positions (for $k = 0$) towards the left half of the complex plane, stabilizing the system for sufficiently high values of k.

Note that thanks to the root locus plot we can easily visualize the position of the closed-loop poles based on the position of the open loop poles and zeros and the value of k. As a result we can quickly see if the controller we designed is feasible or not.

2 Root Locus

As we said before, ideally we want to be able to assess the properties of the closed-loop system based on the open-loop system, and avoiding complex calculations. Luckily, there are some rules that we can follow that allow us to draw the root locus plot more easily.

Root Locus Rules:

1. The root locus is symmetric with respect to the real axis.
2. The number of closed-loop poles is equal to the number of open-loop poles (by increasing the gain k we do not "create" or "destroy" poles).
3. The closed-loop poles approach the open-loop poles as $k \to 0$ (i.e. the root locus starts at the open-loop poles).
4. All points on the real axis are part of the positive root locus if the number of poles and zeros to its right is odd (for this reason we always sketch the root locus from right to left).

5. If there are as many open-loop zeros as open-loop poles, then the closed-loop poles approach the open-loop zeros as $k \to \infty$ (i.e. the root locus ends at the open-loop zeros). If there are more open-loop poles than open-loop zeros, then the remaining closed-loop poles go to "infinity" along asymptotes.

6. The asymptotes of the root locus branches that go to infinity have angles given by:
$$\theta_m = \frac{(2m+1)\cdot 180°}{\#\text{poles} - \#\text{zeros}}, \quad m = 0, 1, 2, \ldots, (\#\text{poles} - \#\text{zeros} - 1)$$
The asymptotes intersect the real axis at:
$$s_{com} = \frac{\sum x_{\text{poles}} - \sum x_{\text{zeros}}}{\#\text{poles} - \#\text{zeros}}$$
Note: x_{poles} and x_{zeros} are the real parts of the poles and zeros, respectively.

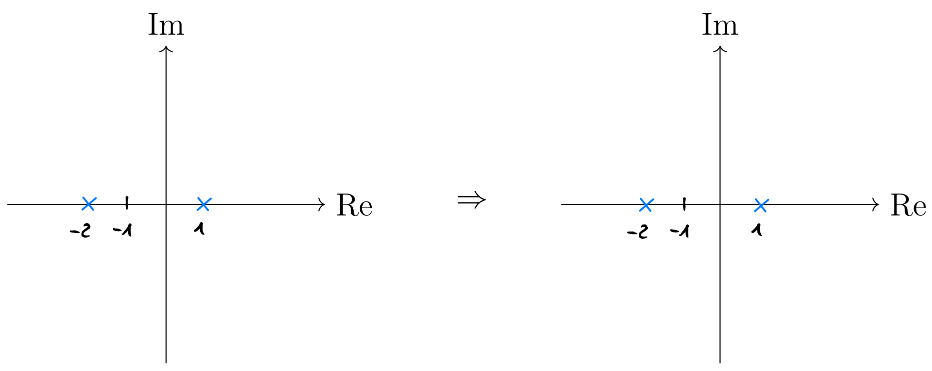
7. When two root locus branches meet on the real axis, they will break away from the real axis at a breakaway point.

8. You can find the breakaway points by solving:
$$N'(s)D(s) - N(s)D'(s) = 0$$
where $N(s)$ is the numerator and $D(s)$ is the denominator of the open-loop transfer function.

To better understand these rules, let's apply them to some examples.

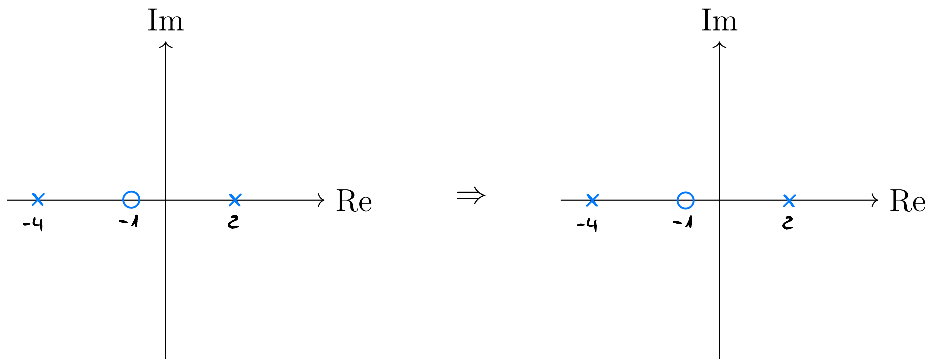
Examples

Let's consider the example from before:

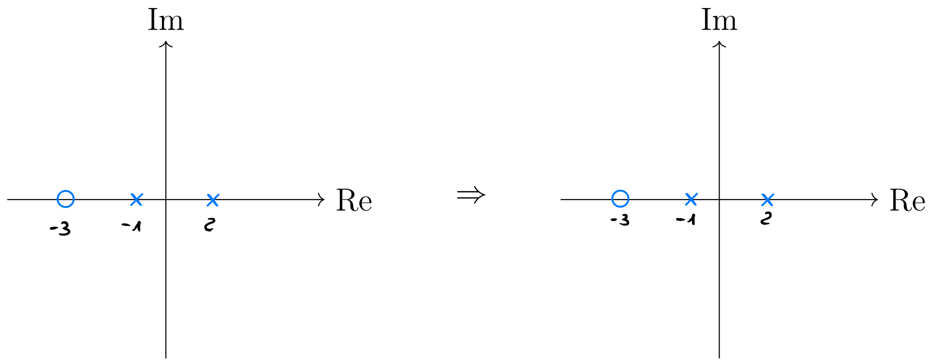


Contact point of asymptotes: $s_{com} = \frac{1+(-2)}{2-0} = -0.5$
Asymptote angles: $\theta_0 = \frac{1}{2} \cdot 180° = 90°$, $\theta_1 = \frac{3}{2} \cdot 180° = 270°$

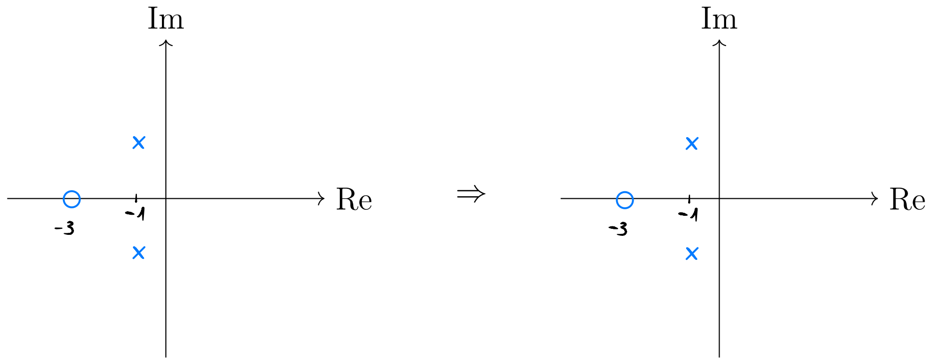
Now let's add a zero in different locations and see how the root locus changes with some more examples:



$$\#\text{Poles} = 2,\ \#\text{Zeros} = 1$$
$$\text{Contact point of asymptotes: } s_{com} = \frac{-4+2+1}{2-1} = -1$$
$$\text{Asymptote angle: } \theta = \frac{2 \cdot 0+1}{2-1} \cdot 180^\circ = \pm 180^\circ$$

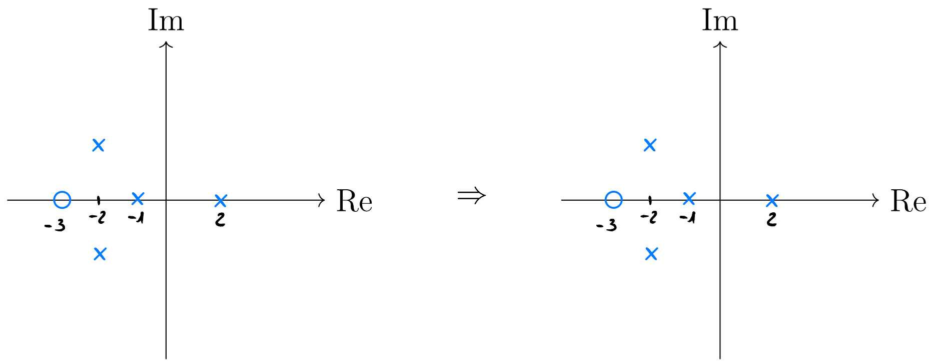


$$\#\text{Poles} = 2,\ \#\text{Zeros} = 1$$
$$\text{Contact point of asymptotes: } s_{com} = \frac{-1+2+3}{2-1} = 4$$
$$\text{Asymptote angle: } \theta = \frac{2 \cdot 0+1}{2-1} \cdot 180^\circ = \pm 180^\circ$$



$$\#\text{Poles} = 2,\ \#\text{Zeros} = 1$$
$$\text{Contact point of asymptotes: } s_{com} = \frac{-1-1+3}{2-1} = 1$$
$$\text{Asymptote angle: } \theta = \frac{2 \cdot 0+1}{2-1} \cdot 180^\circ = \pm 180^\circ$$

Finally let's consider a case with 4 poles and 1 zero:



$$\#\text{Poles} = 4,\ \#\text{Zeros} = 1$$
$$\text{Contact point of asymptotes: } s_{com} = \frac{-1-2-2+2+3}{4-1} = 0$$
$$\text{Asymptote angles: } \theta_0 = \frac{0+1}{4-1} \cdot 180° = 60°,\ \theta_1 = \frac{2+1}{4-1} \cdot 180° = 180°,\ \theta_2 = \frac{4+1}{4-1} \cdot 180° = 300°$$

You can find a useful online tool for plotting the root locus of a system at: `https://lpsa.swarthmore.edu/Root_Locus/RLDraw.html`.

Exam Problem: FS24

Problem: Consider the closed-loop system T shown in Figure 6, where L represents a linear time-invariant system and $k \in \mathbb{R}$. The root locus of L for $k > 0$ is shown in Figure 7.

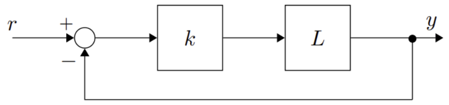


Figure 6: Closed-loop system T, open-loop system L and proportional gain k.

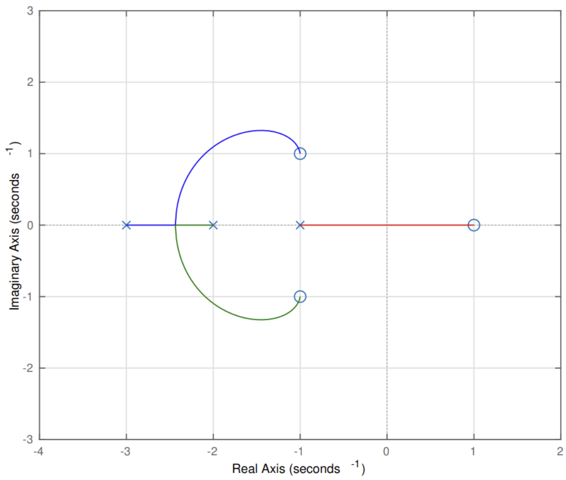


Figure 7: Root Locus of L. Poles are represented by $\times$ symbols, zeros by o symbols.

Q21 **(0.5 Points)** Mark the correct answer.
The open-loop system L is a minimum-phase system.

A True B False

Q22 **(0.5 Points)** Mark the correct answer.
The open-loop system L is asymptotically stable.

A True B False

Q23 **(0.75 Points)** Mark the correct answer.
There exists a gain $k^\star$ such that for all k such that $0 < k \le k^\star$ the closed-loop system T is asymptotically stable and all poles of the closed-loop system have zero imaginary part, i.e. are real numbers.

A True B False

Exam Problem: FS24

Problem: Consider the transfer functions,

$$G_1(s) = \frac{s^2 - s}{s^3 + s - 1},\; G_2(s) = \frac{s+2}{s^2 - s + 1},\; G_3(s) = -\frac{s^2 - s}{s^3 + s - 1},\; G_4(s) = \frac{s+2}{s(s-5)}.$$

In Figure 5 the root locus plots for $k > 0$ of the aforementioned transfer functions are shown in a randomized order.

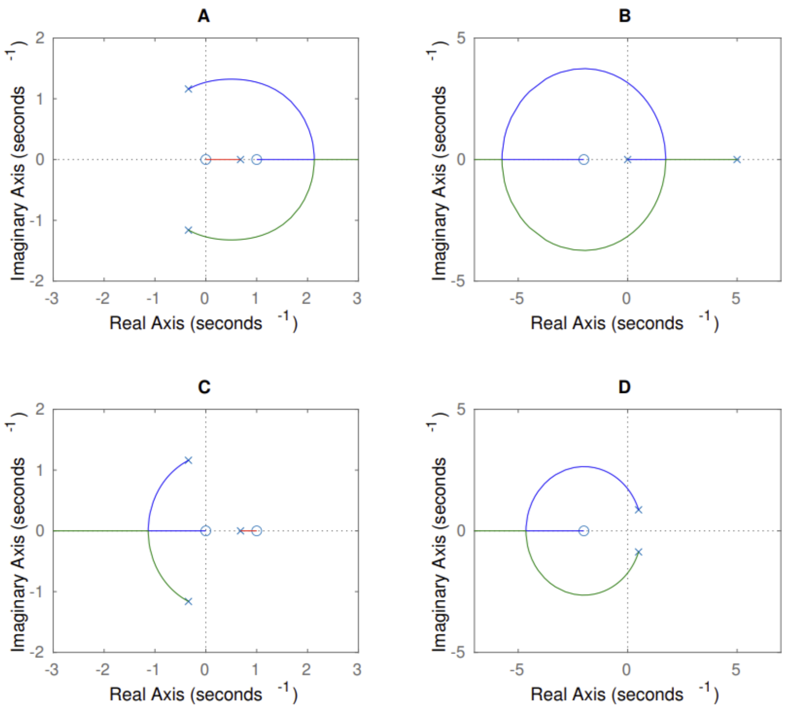


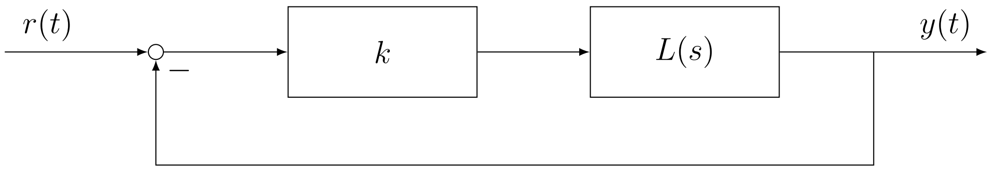
Figure 5: Root locus plots of G_1, G_2, G_3 and G_4 in random order. Poles and zeros are represented by $\times$ and o symbols respectively.

Q20 **(1 Points)** Mark the correct answer.
Mark the correct root-locus plot and transfer function pairings?

A (A, G_3), (B, G_4), (C, G_2), (D, G_1)
B (A, G_3), (B, G_2), (C, G_1), (D, G_4)
C (A, G_1), (B, G_2), (C, G_3), (D, G_4)
D (A, G_3), (B, G_4), (C, G_1), (D, G_2)

2.1 Dynamic Compensators

Now, let's go back a little bit and look at the block diagram again:



$L(s)$ will be some rational transfer function of the form $L(s) = \frac{N(s)}{D(s)}$ that represents our open-loop system.
The closed-loop transfer function of the system is given by:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{k\frac{N(s)}{D(s)}}{1 + k\frac{N(s)}{D(s)}} = \frac{kN(s)}{D(s) + kN(s)}$$

Therefore, as we already saw, the poles of the closed-loop system are given by the roots of the characteristic polynomial:

$$D(s) + kN(s) = 0$$

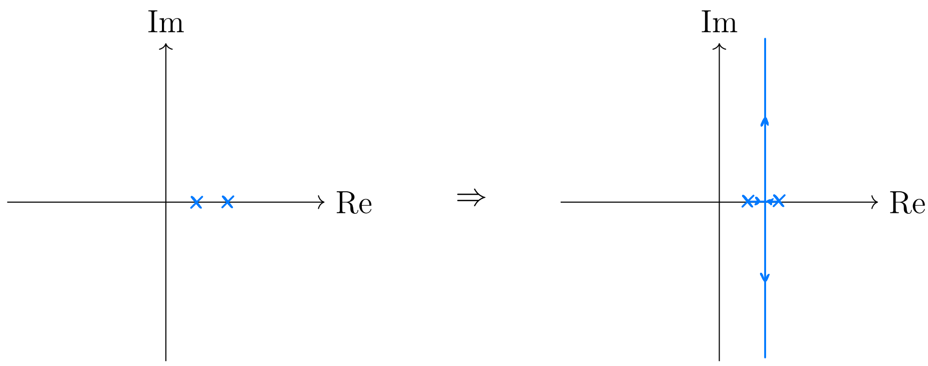
However, often a proportional controller is not sufficient to achieve the desired performance. In these cases, we can use a more complex controller, called a **dynamic compensator**, that has its own dynamics and can be represented by a transfer function $C(s)$.
The open loop transfer function of the system is now given by:

$$L(s) = C(s)P(s)$$

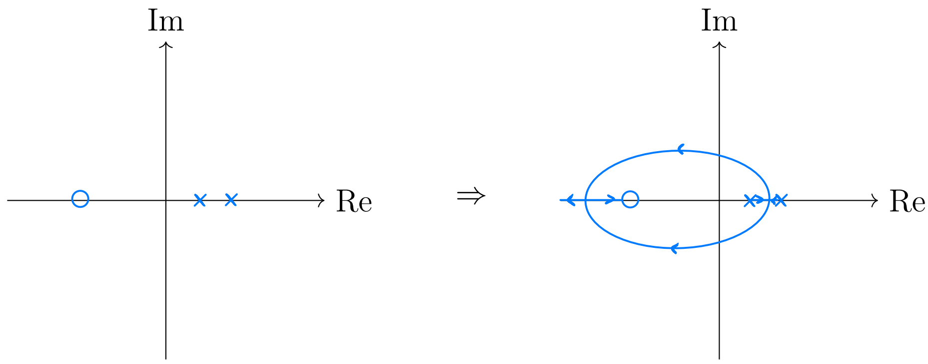
Examples of dynamic compensators include:

- Proportional-Integral (PI) Controller: $C(s) = k_P + \frac{k_I}{s}$
- Proportional-Derivative (PD) Controller: $C(s) = k_D s + k_P$
- Proportional-Integral-Derivative (PID) Controller: $C(s) = k_D s + k_P + \frac{k_I}{s}$
- Lead Compensator: $C(s) = \frac{s+z}{s+p}$ with $z < p$
- Lag Compensator: $C(s) = \frac{s+z}{s+p}$ with $z > p$

Looking at an example with two unstable poles:



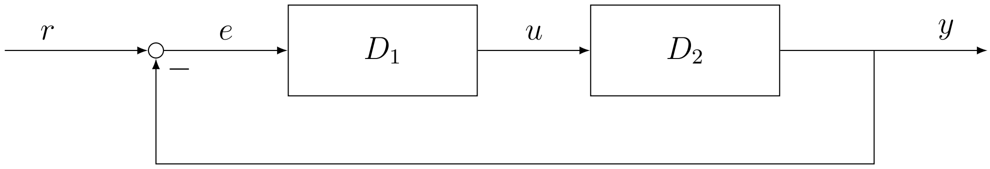
We can see that with a proportional controller alone, it is not possible to stabilize the system. However, by adding a zero to the controller (PD controller), we can change the root locus and stabilize the system:



2.2 Techical Concerns

Well-posedness

If we consider the block diagram below



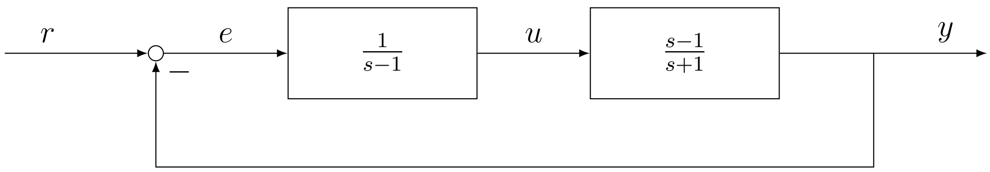
and assume D_1 and D_2 to be scalars, the denominator of the closed-loop transfer function is given by:

$$1 + D_1 D_2$$

For the system to be well-posed, we need to ensure that the denominator is not zero, i.e. $1 + D_1 D_2 \neq 0$. If this condition is not met, the system is said to be not well-posed (or ill-posed).

Stability

If we consider again another block diagram



we can see that the closed-loop transfer function is given by:

$$T(s) = \frac{\frac{1}{s-1} \cdot \frac{s-1}{s+1}}{1 + \frac{1}{s-1} \cdot \frac{s-1}{s+1}} = \frac{1}{s+2}$$

this seems to be stable, however, in such interconnections, we need to be careful about hidden unstable poles. In this case, the controller has a pole in $s = 1$ (unstable) that

is cancelled by a zero of the system. This means that even though the closed-loop transfer function appears stable, the internal dynamics of the controller are unstable, leading to an overall unstable system.
Conclusion: Internal stability requires that all closed-loop transfer functions between any two signals must be stable!