Katholieke
Universiteit
Leuven

Institute of
Astronomy

INTRODUCTION TO GENERAL RELATIVITY - EXERCISE SETS - SOLUTIONS

Author:
David Petit (r0818065)

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Contents

1 Exercise Set 1 - Review of Special Relativity

1.1 Summation conventions

Prompt:

Problem 1. Determine if the summation convention is applied correctly and that the indices make sense.

a) $F_\mu = m a_\nu$.

b) $\mathrm{d}s^2 = \hat{g}_{mn} \mathrm{d}y^m \mathrm{d}y^n + g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$.

c) $\partial_\mu g^{\alpha\beta} = 0$.

d) $g^{\alpha\mu} \partial_\alpha F_{\mu\nu} = \partial^\mu F_{\alpha\nu}$

e) $\partial^\mu F_{\mu\nu} - A^\mu F_{\mu\nu} = j_\nu$.

See also Example 7.1 in Hartle for more examples.

Solutions: a) Nonsense, there are two unbalanced free indices. μ and ν can be different and then you have 1-sided equations
$F_\mu = m a_\mu$ would work well

b) Yes, good

c) Yes, good

d) No, the α and μ are summed over (and drop out) on the left hand side, leaving only a lower ν index; but the right hand side has the ν as well as a μ and α which don't have existence on the left hand side.
$g^{d\mu} \partial_d F_{\mu\nu} = \partial^\mu F_{\mu\nu}$ would work well

e) Yes, good

1.2 (anti)symmetry of tensors

Prompt:

Problem 2. A tensor is said to be totally symmetric if any two indices can be interchanged. For example if $S_{\mu\nu}$ is symmetric, then

$$S_{\mu\nu} = S_{\nu\mu} .$$

Similarly, a tensor is said to be totally antisymmetric if exchange of any two indices gives *minus* it self, i.e.

$$H_{\mu\nu\rho} = -H_{\nu\mu\rho} .$$

a) Show that if $A_{\mu\nu}$ is symmetric and $B^{\rho\sigma}$ is antisymmetric, then $A_{\mu\nu}B^{\mu\nu} = 0$.

b) Let ϵ_{ijk} be a totally antisymmetric tensor in three dimensions. Say $\epsilon_{123} = 1$, determine ϵ_{111}, ϵ_{112}, ϵ_{321} and ϵ_{231}.

c) Let $\mathbf{A}$ and $\mathbf{B}$ be two vectors in three dimensions with components A^i and B^j, show that the components of the cross product $\mathbf{A} \times \mathbf{B}$ can be written using the ϵ_{ijk} as follows

$$(\mathbf{A} \times \mathbf{B})^i = \delta^{ij}(\epsilon_{jkl}A^k B^l) ,$$

where δ^{ij} is the Kronecker-delta.

d) Show that if $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $\partial_\mu A_\nu$ is a shorthand notation for $\partial A_\nu / \partial x^\mu$, then

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 .$$

Solution: a) start with symmetry

$$A_{\mu\nu}B^{\mu\nu} = A_{\mu\nu}B^{\mu\nu}$$

A is symmetric, swap one pair of indices

$$A_{\nu\mu}B^{\mu\nu} = A_{\mu\nu}B^{\mu\nu}$$

B is anti-symmetric, swap one pair and add a negative

$$-A_{\nu\mu}B^{\nu\mu} = A_{\mu\nu}B^{\mu\nu}$$

Switch indices (this is allowed because they are arbitrarily ordered/named)

$$-A_{\mu\nu}B^{\mu\nu} = A_{\mu\nu}B^{\mu\nu}$$

This is the negative of what we started with. So, like X = -X, this is only true when X = 0, therefore

$$A_{\mu\nu}B^{\mu\nu} = 0$$

b) We are given,

$$\epsilon_{123} = 1$$

ϵ_{111} is like ϵ_{ijk}, but if we shuffle ijk around then ϵ_{111} doesn't change. So $\boldsymbol{\epsilon_{111}}$ = - $\boldsymbol{\epsilon_{111}}$, which means it equals 0 (has to be 0 or cannot be anything non-0). From this

$$\epsilon_{111} = 0$$

We change one index by one to get ϵ_{112}, even though there is a two and it invites more consideration, focus on any point of (anti-)symmetry: The first two indices (1 and 1) can be swapped and the ϵ will become $-\epsilon$, but before and after the swap, they are both of the form ϵ_{112}. So,

$$\epsilon_{112} = -\epsilon_{112} \ (\textit{where the } 1 \textit{ indices have switched})$$

This implies

$$\epsilon_{112} = 0$$

Now for the last two, take a different approach. Start again with,

$$\epsilon_{123} = 1$$

Swapping an index position changes the sign,

$$\epsilon_{321} = -1$$

Swapping an index position changes the sign again,

$$\epsilon_{231} = 1$$

c)
d)

1.3 four-vectors (of massive particles)

Prompt: Show that the four-vector of a massive particle is a time-like unit-vector,

$$u \cdot u = -1$$

Then show that this implies that $a \cdot u = 0$ where a is the acceleration four-vector.

Solution:
Start with definition of 4-velocity,

$$\partial_\mu = \frac{dx_\mu}{d\tau}$$

where

$$d\tau = -\eta_{\alpha\beta} dx_\mu dx_\nu d\tau^2$$

rewrite these *'d and/or write the length element is defined as 4-velocity times small displacement in time

$$dx_\mu = u_\mu \cdot d\tau$$

plug this into **

$$d\tau^2 = -\eta_{\alpha\beta}$$

$$dx_\mu = uu_\alpha d\tau u_\beta d\tau$$

and...???

$$-\eta_{\alpha\beta}u^\alpha u^\beta d\tau^2$$

where

$$u^\alpha u^\beta d\tau = 1$$

and...?

$$-\eta_{\alpha\beta}u^\alpha u^\beta = 1$$

and this is

$$u \cdot u = 1$$

---

start the next part with a definition of acceleration ("a" is derivative of 4-velocity)... ... ... WHAT IS THIS SUPPOSED TO BE???

$$a_\mu = \frac{d_{\mu\nu}}{d\tau}$$

the force is Newtonian-like

$$F\nu = (m \cdot a)\nu$$

claim that $a \cdot u = 0$; show with, normalized 4-velocity??????? and derivative of a constant is 0

$$\frac{d}{d\tau}\eta_{\alpha\beta}u^\alpha u^\beta = \frac{d}{d\tau}(-1) = 0$$

act with derivative operator (product rule)

$$\frac{d}{d\tau}(\eta_{\alpha\beta})u^\alpha u^\beta + \eta\frac{d}{d\tau}(u^\alpha)u^\beta + \eta u^\alpha\frac{d}{d\tau}(u^\beta) = 0$$

add... ... ... WHY ARE DERIVATIVES THE SAME

$$\frac{d}{d\tau}(\eta_{\alpha\beta})u^\alpha u^\beta + 2\eta u^\alpha u^\beta = 0$$

Can expand from η to g, as long as you only consider locally flat environments.

1.4 Force on mass in GR

Prompt: A particle moves on the x-axis along a world line described parametrically by

$$t(\sigma) = a^{-1}sinh(\sigma), \qquad x(\sigma) = a^{-1}cosh(\sigma)$$

Where a is a constant.
Re-express the world line in terms of proper time.
Computer the force experienced by the particle according to Newton's second law.

Soluiton:
Before, we used τ to parameterize a world line, now we use σ (or $\sigma(\tau)$)
Start with,

$$\tau = \int\sqrt{-\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}d\sigma$$

infer τ as a differential equation,

$$\frac{d\tau}{d\sigma} = -\sqrt{\eta_{\alpha\beta} \cdot \frac{dx^\alpha}{d\sigma} \cdot \frac{dx^\beta}{d\sigma}}$$

In flat space-time,

$$\eta_{\alpha\beta} = -1111$$

know all (above), combine

$$-\sqrt{derivative of the sinh and cosh f}$$

which is

$$-a^-2cosh^2(\sigma) + a^-2sinh^2(\sigma)$$

Express equations in terms of proper time.

1.5 Lorentz Transformation Matrix

Prompt: Show that the matrix

$$\Lambda = \begin{pmatrix} cosh\theta & -sinh\theta & 0 & 0 \\ -sinh\theta & cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a valid Lorentz transformation matrix. Show that this Lorentz transformation can be obtained as an example of a specific coordinate transformation $x'^{0\alpha}(x)$.
Solution:

1.6 Change coordinates, Penrose Diagram in flat space

Prompt: The way to construct Penrose diagrams, which are a way to write down an infinite space-time on a finite space (a sheet of paper) Go from t and r to t' and r' such that it captures the entire behavior of light in a space-time (a null trajectory, which is ds = 0). **Solution**: Consider a radial geodesic, so ϕ and θ are 0.

$$0 = -dt^2 + dr^2$$

so,

$$dt = + - dr$$

Draw a picture: Always 45 degree lines on the r-t coordinates–straight lines Now move all of this information into a finite box.
stuff...
substitute for r $1/2r^2 - \omega^2$

$$d\omega_2^2 = -d\theta^2 + sin^2\theta d\phi^2 \quad (the\ metric\ on\ a\ 2 - sphere)$$

Looking at minkowski space Each point is a 2-sphere of some radius Forget all by acting on ... We rotated the space-time description of t, and r to u, and v. ... Aligned Light rays are easy to understand in the new coordinates Now, check the domain of the coordinates t is from -infinity to infinity r is from 0 to infinity Condition V ¿= U Now the clever trick: compactify the coordinates (with new bounded coordinates) For example (many can work), use the arctan: map U and V to arctan Let,

$$= tan(u\prime)$$

and ... Now compute differentials

$$dU\prime =$$

Now compute the V - U

$$Vtan - Utan$$

Expand everything
Plug into mathematica

$$= V\prime - ...$$

...
Conformally related; by a scale factor
$V - U >= 0$ implies ... $v\prime >= u\prime$ $u\prime = arctanu$ and $v\prime = arctanv$. and ... Mapped an infinite region to a finite region in R2. Last thing, go back to a familiar t and r picture. Take $t\prime$ and $r\prime$ to go back $u\prime = 1/2t\prime - r\prime$ $v\prime = 1/2t\prime + r\prime$??? Have metric and conformal factor... Computer du/dv go to $dt/dr\prime$

$$becomes \omega_2\prime$$

insert new coordinates

$$dt\prime^2 ... + sin^2(r)???$$

compare to original coordinates/set-up
remember ($u\prime$ and $v\prime$???) are bound...

$$0 < t\prime < \pi - r$$

$$0 < r\prime < \pi???$$

... time from 0 to pi radius from ??? Makes a triangle (from 0,1 to 1,0 to 0,-1)
Follow-up: Make archs from bottom to top, moving away from the 0,0 origin–"scries."

1.7 Coordinate transformation for two-sphere

Prompt: Find a coordinate transformation that transforms the line element for a two-sphere

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

to the form

$$d\Omega^2 = f(x, y)(dx^2 + dy^2)$$

Solution:

1.8 Penrose Diagram - Constant curves

Prompt: Problem 7.4 in Hartle. In the Penrose diagram for flat space spanned by the coordinates (t0 , r 0) (also (u0 , v0)), make a rough sketch of the following a) curves of constant r and b) curves of constant t.
Solution:

1.9 Four-sphere, change coordinates, metric

Prompt: The surface of a sphere of radius R in four flat Euclidean dimensions is given by X 2 + Y 2 + Z 2 + W 2 = R2 . a) Show that points on the sphere may be located by the coordinates (χ, θ, ϕ), where X = R sin χ sin θ cos ϕ , Z = R sin χ cos θ , Y = R sin χ sin θ sin ϕ , W = R cos χ . b) Find the metric describing the geometry on the surface of the sphere in these coordinates. c) Calculate the volume element of the metric found in (b).

Solution: a) b) c)

2 Exercise Set 2 - Geodesics

2.1 1.) Christoffel symbols

2.1.1 1.) Calculation

Prompt: Problem 1. (Problem 8.2 in Hartle) In usual spherical coordinates, the metric on a two-dimensional sphere is

$$ds^2 = a^2 d\theta^2 + sin^2\theta d\phi^2$$

where a is a constant. a) Calculate the Christoffel symbols "by hand". b) Show that a great circle is a solution of the geodesic equation. (Hint: Make use of the freedom to orient the coordinates so the equation of a great circle is simple.)
Solution: A Christoffel symbol is denoted as $\Gamma^{\alpha}_{\mu\nu}$. To calculate one, use the formula,

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}$$

Note: we usually see the g with the subscript terms, the g with a superscript is the inverse. Now, from the provided metric,

$$g_{\alpha\beta} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & sin^2\theta \end{pmatrix}$$

Matrix inversion has many steps, refresh one's linear algebra; however a (diagonal) matrix inversion is easy, just take the inverse of each of the terms. Do this to the above, and get

$$g^{\alpha\beta} = a^{-2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2\theta} \end{pmatrix}$$

With two dimensions ($\theta \ and \phi$), there are eight Christoffel symbols:

$$\Gamma^{\theta}_{\theta\theta}, \Gamma^{\theta}_{\theta\phi}, \Gamma^{\theta}_{\phi\theta}, \Gamma^{\theta}_{\phi\phi}, \Gamma^{\phi}_{\theta\theta}, \Gamma^{\phi}_{\theta\phi}, \Gamma^{\phi}_{\phi\theta}, \Gamma^{\phi}_{\phi\phi}$$

Now calculate them one by one,

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\theta}(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}) + \frac{1}{2} g^{\theta\phi}(\partial_\theta g_{\phi\theta} + \partial_\theta g_{\phi\theta} - \partial_\phi g_{\theta\theta})$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\theta}(\partial_\theta(1) + \partial_\theta(1) - \partial_\theta(1)) + \frac{1}{2}(0)(\partial_\theta g_{\phi\theta} + \partial_\theta g_{\phi\theta} - \partial_\phi g_{\theta\theta})$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\theta}(0 + 0 - 0) + \frac{1}{2}(0)$$

$$\Gamma^{\theta}_{\theta\theta} = 0$$

$$\Gamma^{\theta}_{\theta\phi} =$$

$$\Gamma^{\theta}_{\phi\phi} =$$

$$\Gamma^{\phi}_{\phi\theta} =$$

$$\Gamma^{\phi}_{\theta\theta} =$$

$$\Gamma^{\phi}_{\phi\phi} =$$

2.1.2 2.) Great circles as solutions of geodesics

Have memorized and start with,

$$\frac{du^\mu}{d\sigma} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$

In this case of two dimensions, the general equation takes 6 particular forms:

4 of them are not insightful ($0 = 0$) The 2 remaining are,

$$\frac{du^\theta}{d\sigma} + \Gamma^\theta_{\phi\phi} u^\phi u^\phi + \Gamma^\theta_{\theta\theta} u^\theta u^\theta = 0$$

Why no mixing of the theta and phi terms?

$$\frac{du^\theta}{d\sigma} + (-sin\theta)(cos\theta)(u^\phi)(u^\phi) + 0 = 0$$

and

$$\frac{du^\phi}{d\sigma} + \Gamma^\phi_{\phi\phi} u^\phi u^\phi + \Gamma^\theta_{\theta\theta} u^\theta u^\theta = 0$$

More terms above?

$$\frac{du^\phi}{d\sigma} + 2cot\theta(u^\theta)(u^\phi) = 0$$

For optimal simplicity, orient the coordinates such that $\theta = \pi/2$. Then,

$$\frac{d}{d\theta}(\theta) = \frac{d}{d\theta}(\pi/2) = 0$$

But note that the definition of $\frac{d}{d\theta} = u^\theta$, so

$$u^\theta = 0$$

Substitute

$$\frac{du^\phi}{d\sigma} + 2cot\theta \cdot (0) \cdot u^\phi = 0$$

$$\frac{du^\phi}{d\sigma} + (0) = 0$$

$$\frac{du^\phi}{d\sigma} = 0$$

Integrate both sides

$$u^\phi = constant = P$$

Also,

$$u^\phi = \frac{d\phi}{d\tau}$$

So,

$$\frac{d\phi}{d\tau} = P$$

$$d\phi = Pd\tau$$

$$\phi = P\tau + C$$

2.2 2.) Lagrangian

Prompt: Problem 2. (Problem 8.3 in Hartle) A three-dimensional spacetime has the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

a) Find the explicit Lagrangian for the variational principle for geodesics in this spacetime in these coordinates. b) Using the result of (a) write out the components of the geodesic equation by computing them from the Lagrangian. c) Read of the nonzero Christoffel symbols for this metric from your results in (b).

Solution: Let,

$$f = (1 + \frac{2M}{r})$$

The Lagrangian will be obtained from the proper time, which is

$$\int d\tau = S = \int d\sigma \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}$$

So,

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = -g_{\mu\nu}\dot{x^\mu}\dot{x^\nu}$$

And the g term comes from the metric

$$g = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & -f^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin\theta \end{pmatrix}$$

$$L = \sqrt{f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2}$$

Memorized and use the Euler–Lagrange equation:

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \left(\frac{\partial L}{\partial x^\mu}\right) = 0$$

Note that, in this class, $\dot{x}^\mu$ and x^μ are independent, so

$$L = -g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = f(\dot{x^\mu})$$

And

$$L \neq f(x^\mu)$$

So,

$$\left(\frac{\partial L}{\partial x^\mu}\right) = 0$$

The remaining portion of the E-L equation here is then

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0$$

Solve this for 3 variables: t, r, and phi

For t,

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{d}{d\sigma}\left(\frac{\partial}{\partial \dot{t}}(L)\right) = 0$$

Substitute in for L

$$\frac{d}{d\sigma}\left(\frac{\partial}{\partial \dot{t}}\sqrt{f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2}\right) = 0$$

Chain rule,

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\left(f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)^{-1/2}\frac{\partial}{\partial \dot{t}}\left(f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)\right) = 0$$

Notice, the middle-left part is the inverse of the original L ($L^{-1} = (f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2)^{-1/2}$), so

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\frac{1}{L}\left(\frac{\partial}{\partial \dot{t}}f\dot{t}^2 - \frac{\partial}{\partial \dot{t}}f^{-1}\dot{r}^2 - \frac{\partial}{\partial \dot{t}}r^2\dot{\phi}^2\right)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2L}\left(\frac{\partial}{\partial \dot{t}}f\dot{t}^2 - 0 - 0\right)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2L}\left(f\cdot 2\dot{t}\right)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{f\dot{t}}{L}\right) = 0$$

$$\frac{d}{d\sigma}\left(\frac{f(r)\dot{t}}{L}\right) = 0$$

For r,

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{d\sigma}\left(\frac{\partial}{\partial \dot{r}}(L)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\left(f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)^{-1/2}\frac{\partial}{\partial \dot{r}}\left(f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\frac{1}{L}\left(\frac{\partial}{\partial \dot{r}}f\dot{t}^2 - \frac{\partial}{\partial \dot{r}}f^{-1}\dot{r}^2 - \frac{\partial}{\partial \dot{r}}r^2\dot{\phi}^2\right)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\frac{1}{L}\left((0) - \frac{\partial}{\partial \dot{r}}f^{-1}\dot{r}^2 - \frac{\partial}{\partial \dot{r}}r^2\dot{\phi}^2\right)\right) = 0$$

For phi,

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{d\sigma}\left(\frac{\partial}{\partial \dot{\phi}}(L)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\left(f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)^{-1/2}\frac{\partial}{\partial \dot{\phi}}\left(f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)\right) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{1}{2}\frac{1}{L}\left(\frac{\partial}{\partial \dot{\phi}}f\dot{t}^2 - \frac{\partial}{\partial \dot{\phi}}f^{-1}\dot{r}^2 - \frac{\partial}{\partial \dot{\phi}}r^2\dot{\phi}^2\right)\right) = 0$$

2.3 3.) Constant norm of four-velocity

Prompt: Problem 3. (Problem 8.6 in Hartle) Show by direct calculation from the geodesic equation

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$$

that the norm of the four-velocity u · u is a constant along a geodesic.
Solution:

2.4 4.) Hyperbolic geodesics

Prompt: Problem 4. (Problem 8.12 in Hartle) The hyperbolic plane defined by the metric

$$ds^2 = \frac{1}{y}\left(dx^2 + dy^2\right),\ y \geq 0$$

is a classic example of a two-dimensional surface. a) Show that points on the x-axis are an infinite distance from any point (x, y) in the upper half-plane. b) Write out the geodesic equations. c) show that the geodesics are semicircles centered on the x-axis or vertical lines. d) Solve the geodesic equations to find x and y as functions of the length S along these curves.
Solution:

2.5 5.) Geodesics and Killing vectors

Prompt: Problem 5. We denote the inner product of two 4-vectors with the dot:

$$u \cdot v = g_{\mu\nu} u_\mu u \nu$$

a) Using the geodesic equation, show that

$$\frac{d}{d\tau}(K \cdot u) = u^\mu u^\nu (\partial_\mu K_\nu - \Gamma^\sigma_{\mu\nu} K_\sigma)$$

where uµ is the 4-velocity. A vector K^μ that satisfies $d\tau(K \cdot u) = 0$ for all u is called a Killing vector. b) Using the above show that for a metric $g_{\mu\nu}$ for which the metric components do not depend on one of the coordinates, say x (i) , then the vector

$$K^\mu = \eta^{\mu(i)}$$

is Killing.
Solution:

2.6 6.) Killing vectors in the SZ metric

Prompt: Problem 6. The Schwarzschild metric is given by

$$ds^2 = -F(r)c^2 dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega_2^2$$

where

$$F(r) = 1 - \frac{2M}{r}$$

a) Identify two Killing vectors for the Schwarzschild metric. b) Construct three conserved quantities along any geodesic, what is the interpretation of these conserved quantities? c) Use the conserved quantities to obtain an equation for $u^r = dr/d\tau$ which reads

$$\epsilon = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

where ϵ is a constant and Veff is the effective potential. Determine ϵ and Veff .
Solution:

2.7 7.) Particle orbits in SZ geometry

Prompt: Problem 7. (Problem 9.5 in Hartle) Sketch the qualitative behaviour of a particle orbit that comes in from infinity with a value of E exactly equal to the maximum of the effective potential, Veff . How does the picture change of the value of E is a little bit larger than the maximum or a little bit smaller? Problem 8. (Problem 9.6 in Hartle) An observer falls radially inward toward a black hole of mass M whose exterior geometry is the Schwarzschild geometry, starting with a zero kinetic energy at infinity. How much time does it take, as measured on the observer's clock, to pass between 6M and 2M.
Solution:

3 Exercise Set 3 - Schwarschild, geodesics, Eddington-Finkelstein

3.1 1.) Light in SZ metric

Prompt: Problem 1. (Problem 9.6 in Hartle) An observer falls radially inward toward a black hole of mass M whose exterior geometry is the Schwarzschild geometry, starting with a zero kinetic energy at infinity. How much time does it take, as measured on the observer's clock, to pass between 6M and 2M.
Solution: In Schwarzschild geometry/metric/line element, the geodesic of a null (particle) is (a photon/light):

$$\frac{1}{b^2} = \frac{1}{e^2} \cdot \frac{dr}{d\lambda} + W_{eff}$$

Note that we have to use λ and not τ, because the propertime felt by light is 0, or timelike geodesics are $d\tau = ds = 0$ for light Prove the above equation!
We know in Schwarzschild geometry

$$f(r) = 1 - \frac{2M}{r}$$

Let,

$$\theta = \frac{\pi}{2}$$

Use Killing vectors in Schwarzschild geometry

$$\xi_t^\mu = (-1, 0, 0, 0)$$

$$\xi_\phi^\mu = (0, 0, 0, 1)$$

... ... From the solutions of the Schwarzschild geometry

$$E = g_{\mu\nu}\xi_t^\mu u^\nu(\lambda)$$

$$E = f(r)\frac{dt}{d\lambda} = f(r)\dot{t}$$

and

$$l = r^2\dot{\phi}$$

Now, use the knowledge of null geodesics

$$ds^2 = 0$$

Substitute

$$g_{\mu\nu}\xi_t^\mu u^\mu u^\nu$$

... ...

$$-f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\phi^2$$

Substitute $\dot{t}$ and $\dot{\phi}$ with expressions of E and l
... ...
Solve for $\dot{r}$
... ...

Let $b = l/E$

...

So,

$$\frac{1}{e^2}(\frac{dr}{d\lambda})^2 + \frac{1}{r^2}(1 - \frac{2M}{r}) = \frac{1}{b^2}$$
$$\frac{1}{b^2} = \frac{1}{e^2}(\frac{dr}{d\lambda})^2 + W_{eff}$$

3.2 2.) Geodesic of light in SZ geometry

Prompt: Problem 2. Following the steps outlined above find the a geodesic for a massless particle in the Schwarzschild geometry. Express your answer in terms of an effective potential as follows 1 1 = 2 2 b ‘
dr dλ 2 + Weff , where b = ‘/e is the impact parameter and Weff is the effective potential experienced by massless particles.

Solution: A timelike case

Start with,

$$\frac{1}{2}\dot{r}^2 + V_{eff} = \epsilon = \frac{E^2 - 1}{2}$$
$$\frac{1}{2}\dot{r}^2 + \left(-\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\right) = \epsilon = \frac{E^2 - 1}{2}$$

Consider circular stable orbits

$$\dot{r} = \frac{dr}{dt} = 0$$
$$\ddot{r} = \frac{d^2r}{dt^2} = 0$$

... some how... get

$$V_{eff} = 0$$
$$\left(-\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\right) = 0$$

So,

$$-r^3\left(-\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\right) = -r^3 \cdot 0$$
$$Mr^2 - l^2r + 3Ml^2 = 0$$

This is a quadratic equation, use the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and solve to get

$$r_{+-} = \frac{l^2}{2M}(1 + -\sqrt{1 - \frac{12M^2}{l^2}})$$

R_- is the unstable inner circular orbit, so use the R_+ solution

$$r_+ = \frac{l^2}{2M}(1 + \sqrt{1 - \frac{12M^2}{l^2}})$$

Solve for the minimum r; this occurs when the square root term is 0.

$$\sqrt{1 - \frac{12M^2}{l^2}}) = 0$$

$$1 - \frac{12M^2}{l^2}) = 0$$

$$\frac{12M^2}{l^2}) = 1$$

$$l = \sqrt{12}M$$

Plug this back into the r_+ equation

$$r_+ = \frac{12M^2}{2M}(1 + \sqrt{1 - \frac{12M^2}{(12M^2)}})$$

$$r_+ = 6M(1 + \sqrt{1-1})$$

$$r_+ = 6M$$

For *null* geodesics (light rays), $W'_{eff} = 0$, so

$$-2r + 6M = 0$$

$$r_{min} = 3M$$

3.3 3.) Innermost orbits

Prompt: Problem 3. Find the innermost stable circular orbit for massive particles and the innermost unstable orbit of massless ones.
Solution: Start with the two:

$$b = \frac{l}{E}$$

$$b = r \cdot sin\phi$$

Before the comet starts to interact with the mass, it is said that $r \to \infty$

$$b = r \cdot \phi$$

$$\phi = \frac{b}{r}$$

Take the derivative

$$\frac{d\phi}{dr} = -\frac{b}{r^2}$$

$$\frac{d\phi}{dr} = \frac{d\phi}{d\lambda}\frac{d\lambda}{dr} = -\frac{b}{r^2}$$

Memorize that in SC geometry the two:

$$\dot{\phi} = \frac{l}{r^2}$$

$$\frac{1}{2}\dot{r}^2 + V_{eff} = \frac{1}{2}\dot{r}^2 + \left(-\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\right) = \epsilon = \frac{E^2-1}{2}$$

??? ??? ???

$$\dot{r}^2 = 2\epsilon$$

$$\dot{r} = \sqrt{2\epsilon}$$

??? ??? ???

$$b = \frac{l}{\sqrt{2\epsilon}}$$

so,

$$l = b\sqrt{2\epsilon}$$

Now consider the point of closest approach, call this point at a distance R from the mass. At this point the movement is tangential to the direct of the mass, so

$$\dot{r}|_R = 0$$

???

$$V_{eff}|_R = \epsilon$$

or

$$V_{eff}(r = R) = \epsilon$$

Now consider the 4-velocity at the point R,

$$U^{\mu}_{constant}|_R = (t, 0, 0, \phi)$$

??? ??? ???

$$U^{\mu}_{constant}|_R = (\frac{E}{1 - \frac{2M}{r}}), 0, 0, \frac{l}{R^2})$$

We want the 4-velocity from the perspective of a stationary observer

$$U^{\mu}_{observer}|_R = (U^t, 0, 0, 0)$$

??? ??? ??? "by manipulation"

$$U_{obs} \cdot U_{obs} = -1$$

$$-(U^t_0)^2 \cdot (1 - \frac{2M}{R}) = -1$$

$$U_{observer}|_R = (\frac{1}{\sqrt{1 - \frac{2M}{R}}}, 0, 0, 0)$$

The observer is at rest; the comet isn't. Use a Lorentz transformation!

$$\Lambda^{\mu}_{\nu} \cdot^{\mu}_{constant} = U^{\mu}_{observer}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

??? ??? ???

$$-\gamma = g_{\mu\nu} U^{\mu}_{observer} \cdot U^{\nu}_{constant}$$

$$-\gamma = -\left(1 - \frac{2M}{R}\right)\left(\frac{1}{\sqrt{1 - \frac{2M}{R}}}\right)\left(\frac{E}{1 - \frac{2M}{R}}\right)$$

$$-\gamma = -(1)\left(\frac{1}{\sqrt{1 - \frac{2M}{R}}}\right)(\frac{E}{(1)})$$

$$\gamma = \left(\frac{E}{\sqrt{1 - \frac{2M}{R}}}\right)$$

3.4 4.) Velocity of closest approaches in SZ geometry

Prompt: Problem 4. (Problem 9.12 in Hartle) A comet starts at infinity, goes around a relativistic star of mass M and goes out to infinity. The impact parameter at infinity is b. The Schwarzschild radius of closest approach is R. What is the speed of the comet at closest approach as measured by a stationary observer at that point?
Compute the angle variation ϕ along an orbit (calculate the perihelion).

Solution: Start with,

$$\frac{1}{2}\dot{r}^2 + V_{eff} = \frac{1}{2}\dot{r}^2 + \left(-\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\right) = \epsilon = \frac{E^2 - 1}{2}$$

$$l = \dot{\phi} r^2$$

Substitute the $\dot{r}$ term,

$$\frac{d\phi}{dr}^2 = \frac{\dot{\phi}^2}{\dot{r}^2} = \frac{-2(V_{eff} - \epsilon)}{l^2/r^2}$$

Take square root and inverse

$$\frac{d\phi}{dr}^{-1} = +- \frac{r^2}{l}\sqrt{2(\epsilon - V_{eff})}$$

Solve for $d\phi$

$$d\phi = +- \frac{l \cdot dr}{\sqrt{2(\epsilon - V_{eff})}}$$

One period is from the closest point to the farthest point, and back to the closest point. This means the integral of ϕ and r is bound between,

$$\Delta\phi = 2l \int_{r_1}^{r_2} \frac{dr}{r^2}(r^2(1 - \frac{2GM}{c^2 r})^{-1/2}[\frac{c^2 E^2}{1 - \frac{2GM}{c^2}} - (c^2 + \frac{l^2}{r^2})]^{-1/2}$$

Now expand all square root terms; $\frac{GM}{c^2 r} << 1$

$$\Delta\phi \sim 2l \int_{r_1}^{r_2} \frac{dr}{r^2} \frac{1 + \frac{GM}{c^2 r}}{\sqrt{c^2(E^2 - 1) + \frac{2E^2 GM}{r} - \frac{1}{r^2}(l^2 - \frac{4E^2 G^2 M^2}{c^2})}}$$

Now, U substitution. Use, $u = \frac{1}{r}$,

$$\Delta\phi \sim 2l \int_{u_1}^{u_2} \frac{du(1 + \frac{GM}{c^2}u)}{\sqrt{c^2(E^2 - 1) + u^2 E^2 GM - u^2(l^2 - (\frac{2EGM}{c})^2)}}$$

Simplify with

$$A = l^2 - (\frac{2EGM}{c})^2)$$

$$B = 2E^2 GM$$

$$D = c^2(E^2 - 1)$$

Substitute them in

$$\Delta\phi \sim 2l \int_{u_1}^{u_2} \frac{du(1 + \frac{GM}{c^2}u)}{\sqrt{D + Bu - Au^2)}}$$

Memorize and use,

$$u = \alpha x + \beta$$

$$\alpha = \sqrt{\frac{D}{A} + \frac{\beta^2}{4A^2}}$$

$$\beta = \frac{\beta}{2A}$$

This equation right above must be a typo... it would be A = 1/2... Substitute these terms in

$$\Delta\phi \sim 2l \int_{x_1}^{x_2} \frac{dx(1 + \frac{GM}{c^2}\beta + \frac{\alpha GM}{c^2}x)}{\sqrt{A(1 - x^2)}}$$

which is from $D + Bu - Au^2 = A\alpha^2(1 - x^2)$??? ??? ???

$$\Delta\phi = \frac{2l}{\sqrt{A}}[(1 + \frac{GM\beta}{c^2}) \int_{-1}^{1} \frac{dx}{\sqrt{(1 - x^2)}}]$$

Maybe another U-substitution???

$$\Delta\phi = \frac{2l}{\sqrt{A}}[(1 + \frac{GM\beta}{c^2})\pi]$$

Maybe another taylor expansion

$$\Delta\phi = 2\pi + 6\pi(\frac{GM}{c \cdot l})^2 + O(\frac{1}{c^2})$$

$$\Delta\phi \sim 2\pi + 6\pi(\frac{GM}{c \cdot l})^2$$

This is the movement of the perihelion

3.5 5.) Precession of the Perihelion

Skipped

Prompt: Problem 5. (Problem 9.15 in Hartle) Precession of the Perihelion of a Planet To find the first order in $1/c^2$ relativistic correction to the angle $\Delta\phi$ swept out in one bound orbit, one might be tempted to expand the integrand in

$$\Delta\phi = 2l \int_{r_1}^{r_2} \frac{dr}{r^2} \left[c^2(e^2 - 1) + \frac{2GM}{r} - \frac{l^2}{r^2} + \frac{2GMl^2}{c^2r^3} \right]^{-1/2}$$

in the small quantity $2GN`2/c2r3$ and keep only the first two terms. R r This would be a mistake because the resulting integral would diverge near a turning point such as $2dr/(r2-r)^3/2$, whereas the original 2integral is finite. There are several ways of rewriting the integrand so it can be expanded. One trick is to factor $(1 - 2GM/c2r)$ out of the denominator so that it can be written

$$\Delta\phi = 2l \int_{r_1}^{r_2} \frac{dr}{r^2} \left(1 - \frac{2GM}{c^2r}\right)^{-1/2} \left[c^2e^2 \left(1 - \frac{2GM}{c^2r}\right)^{-1} - \left(c^2 + \frac{l^2}{r^2}\right) \right]^{-1/2}$$

. The factor in the brackets is then still the square root of a quantity quadratic in 1/r to order 1/c2 . To derive the expression
GM 2

$$\delta\phi_{prec} = 6\pi \left(\frac{GM}{cl}\right)^2$$

evaluate this expression as follows. (a) Expand the factors of $(1 - 2GM/c2r)$ in the preceding equation in powers of $1/c^2$, keeping only the $1/c^2$ correction to Newtonian quantities and using $e^2 = 1 + 2E_{Newt}/mc^2 + \cdot\ \cdot\ \cdot\ \cdot$ (b) Introduce the integration variable $u = 1/r$, and show that the integral can be put in the form

$$\Delta\phi = \left[1 + 2(\frac{GM}{cl})^2\right] 2 \int_{u_2}^{u_1} \frac{du}{[(u_1 - u)(u - u_2)]^{1/2}} + \frac{2GM}{c^2} \int_{u_2}^{u_1} \frac{u \cdot du}{[(u_1 - u)(u - u_2)]^{1/2}} + \frac{1}{c^{3+}}$$

(c) The first integral (including the 2) equals 2π. Show that the second integral gives $(\pi/2)(u_1 + u_2)$ and that this equals $\pi GM/l^2$ to lowest order in $1/c^2$. (d) Combine these results to derive (3).
Solution: Skipped

3.6 6.) Light moving near SZ BH

Prompt: Problem 6. (Example 9.2 in Hartle) A stationary observer is stationed at a radius $R < 3M$ outside a black hole of mass M. He sends out light rays in various directions in the equatorial plane $\theta = \pi/2$ making angles Φ with the radial direction. Radial light with $\Phi = 0$ have impact parameter $b = 0$ and escape. (a) Find an orthonormal basis $\mathbf{e}_{\hat{\alpha}}$ associated with the laboratory of the observer. Let the three space-like vectors point along the coordinate axes r, θ, ϕ. (b) Write the initial four velocity u_0 of a light ray sent out by the observer in terms of the constants $\mathbf{e}$ and $\mathbf{l}$. (c) Show that the initial angle $tan\Phi = u_0^{\hat{\phi}}/u^{\hat{r}}$ can be related directly to the impact parameter $b = l^2/e^2$ by the equation

$$\tan\Phi = \frac{1}{R}\left(1 - \frac{2M}{R}\right)^{1/2} \left[\frac{1}{b^2} - \frac{1}{R^2}\left(1 - \frac{2M}{R}\right)\right]^{-1/2}$$

(d) Determine the critical angle Φ_{crit} below which the light rays can escape to infinity. Furthermore determine the radius R so that the critical angle is $\pi/2$. What does the sky look like for the observer stationed at that radius?
Find orthonormal basis $(e^{\mu}_{\hat{a}})$ associated with the observer!
Solution:
Start with the $e^{\mu}_{\hat{a}}$ term:

$$\mu \in \{t, r, \theta, \phi\}$$

$$\hat{\alpha} \in \{0, 1, 2, 3\}$$

Definition of orthonormal (basis):

$$g_{\mu\nu} e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$$

Now, find a frame associated with a stationary observer at R_μ

$$u^{\mu}_{obs} = (u_t, 0, 0, 0)$$

WHY??????? The first vector in the basis is:

$$e^{\mu}_{\hat{t}} = u^{\mu}_{obs}$$

WHY??????? $\Downarrow$

$$u^{\mu}u^{\nu}g_{\mu\nu} = -1$$

... ...

$$-(u^t)^2\ (1 - \frac{2M}{R}) = -1$$

WHY???????

$$e^{\mu}_{\hat{t}} = (\frac{1}{\sqrt{1-\frac{2M}{R}}}, 0, 0, 0)$$

Repeat this process for r, θ, ϕ! We can also reuse the procedure and more simply get:

$$e^{\mu}_{\hat{r}} = (0, \alpha, 0, 0)$$

$$e^{\mu}_{\hat{\theta}} = (0, 0, \alpha, 0)$$

$$e^{\mu}_{\hat{\phi}} = (0, 0, 0, \alpha_2)$$

WHAT IS THE PATTERN??? The above terms are then calculated as: For r,

$$e^{\mu}_{\hat{r}}e^{\nu}_{\hat{r}}g_{\mu\nu} = 1$$

$$(\alpha^2)(\frac{1}{1-\frac{2M}{R}}) = 1$$

$$\alpha = \sqrt{1 - \frac{2M}{R}}$$

$$e^{\mu}_{\hat{r}} = (0, \sqrt{1-\frac{2M}{R}}, 0, 0)$$

For θ,

$$e^{\mu}_{\hat{\theta}} = (0, 0, \frac{1}{R}, 0)$$

For ϕ,

$$e^{\mu}_{\hat{\phi}} = (0, 0, 0, \frac{1}{R})$$

Now conisder an initial 4-velocity

$$u^{\mu}_{initial} = u^{\mu}_0 = (t, r, \theta, \phi)$$

??? ??? ???

$$\dot{t} = \frac{E}{1-\frac{2M}{R}}$$

$$\dot{\phi} = \frac{l}{R^2}$$

??? ??? ???

$$u^{\mu}_0 u^{\nu}_0 g_{\mu\nu} = 0$$

... ...

$$-(1-\frac{2M}{R})\dot{t}^2+\frac{\dot{r}^2}{1-\frac{2M}{R}}+R^2\dot{\phi}^2=0$$

solve for r dot,

$$\dot{r}=\sqrt{E^2-(1-\frac{2M}{R})\frac{l^2}{R^2}}$$

??? ??? ???

$$u_0^\mu=(\frac{E}{1-\frac{2M}{R}},\frac{1}{\sqrt{1-\frac{2M}{R}}},0,\dot{\phi})$$

$$u_0^\mu=\sqrt{1-\frac{2M}{R}}\dot{t}e_{\hat{t}}^\mu+\frac{1}{\sqrt{1-\frac{2M}{R}}}\dot{r}e_{\hat{r}}^\mu+0+R\dot{\phi}e_{\hat{\phi}}^\mu$$

Plug into t dot, r dot, and phi dot,

$$u_0^\mu=\frac{E}{\sqrt{1-\frac{2M}{R}}}e_{\hat{t}}^\mu+\frac{\sqrt{E^2-(1-\frac{2M}{R})\frac{l^2}{R^2}}}{\sqrt{1-\frac{2M}{R}}}e_{\hat{r}}^\mu+\frac{l}{R}e_{\hat{\phi}}^\mu$$

Recap: Begin...

$$u_0^\mu=(\dot{t},\dot{r},\dot{\theta}=0,\dot{\phi})\rightarrow\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\rightarrow\{e_{\hat{\phi}}^\mu\}$$

Now calculate tangents,

$$tan\phi=\frac{u_0^{\hat{\phi}}}{u_0^{\hat{r}}}$$

$$tan\phi=\frac{\frac{l}{R}\sqrt{1-\frac{2M}{R}}}{\sqrt{E^2-(1-\frac{2M}{R})(\frac{l^2}{R^2})}}$$

Substitute in for E,

$$tan\phi=\frac{\sqrt{1-\frac{2M}{R}}}{R\sqrt{\frac{1}{b^2}-(1-\frac{2M}{R})(\frac{1}{R^2})}}$$

The rest is in the book...

3.7 7: Novel metrics around stars

Prompt: Problem 7. (Problem 9.18 in Hartle) Suppose in another theory of gravity (not Einstein's general relativity) the metric outside a spherical star is given by

$$ds^2=\left(1-\frac{2M}{r}\right)\left[-dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right]$$

Calculate the deflection of light by a spherical star in this theory assuming that photons move along null geodesics in this geometry and following the steps that led to (9.78) in Hartle. When you get the answer see if you can find a simpler way to do the problem.
Solution:

3.8 8: Particle geodesics programming plots

Prompt: Problem 8. (Problem 9.19 in Hartle) Write a program for the null geodesics in the Schwarzschild geometry analogous to the one on Hartle's website

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http://web.physics.ucsb.edu/~gravitybook/mathematica.html
```

for particle geodesics. Use this program to illustrate the orbits with impact parameter a little above and a little below the critical impact parameter for circular orbit.
Solution:

4 Exercise Set 4 - Lens, ...

4.1 Gravitational Lensing path length difference

Prompt: (Problem 11.4 in Hartle) Derive the path length difference in (11.12) in Hartle:

$$\Delta D \approx 2\beta\theta EDS$$

$$\beta << \theta_E << 1$$

Solution: 1.) Thin lens approximation

$$L(\theta) = L_A + L_B\theta_{+-} = \frac{1}{2}(\beta + \sqrt{2^2 + 4\theta_E^2})$$

Assume

$$\theta_E >> \beta$$

Define the path length difference,

$$\Delta L = L(\theta_+) + L(\theta_-) = \beta\theta_E D_s$$

Now expand the square root WHERE IS THE SQRT????????????????

4.2 Lensing by mass distributions (extended lens)

Prompt: (Problem 11.2 in Hartle) An odd number of gravitational lens images Realistic gravitational lenses are not point sources, as assumed in the discussion in Section 11.1 in Hartle, but rather a mass distribution. A lens that is a distribution of mass produces an odd number of images. For a simple model, assume that the gravitational lens is a transparent disk of radius r* and constant surface mass density σ oriented perpendicularly to the line of sight. Using the thin lens approximation show that, in addition to the two images given by (11.6 in Hartle), there is a third image inside the angle subtended by the disk and find its angular position θ. Assume only the mass inside the deflection radius affects the bending of light.
Solution: 2.) Extended lens Memorize or know how to derive,

$$\theta = \beta + \frac{\theta_E^2}{\theta}$$

In this exercise,

$$M(r) = \{\pi\sigma r^2, r <\}$$

... ...

4.3 3.) Limits of lensing

Prompt: (Problem 11.3 in Hartle) When the line of sight to a star is far from the line of sight to a gravitational lens, the effects of lensing should become negligible. Show that when $\beta\theta_E, \theta_+ \approx \beta, \theta_- \approx 0, I_+/I_* \approx 1, and I_- \approx 0$. Explain why these results mean that gravitational lensing is negligible.
Solution: 3.) Do Tayler expansions

4.4 4.) Luminosity, calculate L

Prompt: An X-ray source with luminosity $L = 3 \times 1036 erg/s$ is powered by accretion onto black hole with mass $6M_{sun}$. Assuming all the radiation is released at the innermost stable circular orbit, estimate the rate $\dot{M}$ at which mass is being accreted by the black hole in M_{sun}/yr.
Solution: 4.) Luminosity, calculate L

$$L = \frac{\Delta E}{\Delta t} = \dot{E}\ Use\ Newtonian\ potential\ energy$$

$$\Delta E \sim \frac{GM\Delta m}{R}$$

Divide by Δt

$$\frac{\Delta E}{\Delta t} = \frac{GM}{R}\frac{\Delta m}{\Delta t}$$

$$\dot{E} = \frac{GM}{R}\dot{m}$$

The left hand side is the power, and here the Luminosity!!!

$$L = \frac{GM\dot{m}}{R}$$

Don't know m dot, but Memorize and use:

$$R = 6M$$

$$L = \frac{GM\dot{m}}{(6M)}$$

$$L = \frac{G\dot{m}}{6}$$

Use natural units, where $G = 1$,

$$L = \frac{\dot{m}}{6}$$

4.5 5.) Analysis of a novel line element

Prompt: (Problem 12.4 in Hartle) Consider the spacetime specified with the line element

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2$$

Except for $r = M$, the coordinate t is always timelike and the coordinate r is spacelike. (a) Find a transformation to new coordinates (v, r, θ, ϕ) analogous to the Eddington-Finkelstein coordinates that sets $g_{rr} = 0$ and show that the geometry is not singular at r = M. 2(b) Sketch a $(\hat{t}, r)$ diagram analogous to the Eddington-Finkelstein diagram showing the world lines of ingoing and outgoing light rays and the light cones. (c) Is this the geometry of a black hole?
Solution: 5.) Eddington-Finkelstein and black holes Given the following metric, (a.) Find a new set of coordinates to show r=M is not a physical singularity,

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + (1 - \frac{M}{r})^- 2dr^2 + r^2 d\Omega^2$$

Note, thus is similar but not the same as the Schwarschild metricLet (a new radial coordinate be defined as),

$$\frac{dr_*}{dr} = \frac{1}{(1 - \frac{M}{r})^2}$$

(In SC it was...) Now solve for r_*,

$$dr_* = \frac{dr}{(1 - \frac{M}{r})(1 - \frac{M}{r})}$$

$$dr_* = \frac{dr}{(1 - \frac{2M}{r} + \frac{M^2}{r^2})}$$

$$dr_* = \frac{dr}{(r^2 - 2Mr + M^2)}$$

$$dr_* = \frac{r^2 dr}{(r - M)^2}$$

Rewrite the numerator is a very particular form,

$$dr_* = \frac{r^2 - 2Mr + 2Mr - 2M^2 + 2M^2 + M^2 - M^2 dr}{(r - M)^2}$$

Rearrange,

$$dr_* = \frac{r^2 - 2Mr + M^2 + 2Mr - 2M^2 + 2M^2 - M^2 dr}{(r - M)^2}$$

Factor,

$$dr_* = \frac{(r - M)^2 + 2M(r - M) + 2M^2 - M^2 dr}{(r - M)^2}$$

Divide terms,

$$dr_* = \frac{(r - M)^2}{(r - M)^2} dr + \frac{2M(r - M)}{(r - M)^2} dr + \frac{2M^2 - M^2}{(r - M)^2} dr$$

Simplify,

$$dr_* = (1)dr + \frac{2M}{(r - M)} dr + \frac{M^2}{(r - M)^2} dr$$

Integrate,

$$\int dr_* = \int (1)dr + \int \frac{2M}{(r - M)} dr + \int \frac{M^2}{(r - M)^2} dr$$

$$r_* = r + 2M \int \frac{1}{(r - M)} dr + M^2 \int \frac{1}{(r - M)^2} dr$$

$$r_* = r + 2Mln(r - M) + M^2((-1)\frac{1}{(r - M)}$$

Now memorize and define,

$$t = v - r_*$$

Take (simple) derivative,

$$dt = dv - dr_*$$

Plug into metric,

$$ds^2 = (1 - \frac{M}{r})^2(-dv^2 + 2dvdr_*) + r^2 d\Omega^2$$

HOW DOES THIS BECOME: ???

$$ds^2 = -(1 - \frac{M}{r})^2 dv^2 + 2dvdr_* + r^2 d\Omega^2$$

So coordinates v, r_*, θ, ϕ ???MAKE???,

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{M}{r})^2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin\theta \end{pmatrix}$$

Now use this metric, and calculate the behavior of light (null rays)

$$ds^2 = 0 = -(1 - \frac{M}{r})^2 dv^2 + 2dvdr_* + r^2 d\Omega^2$$

Consider radial light rays,

$$ds^2 = 0 = -(1 - \frac{M}{r})^2 dv^2 + 2dvdr_* + (0)$$

$$0 = -(1 - \frac{M}{r})^2 dv^2 + 2dvdr_*$$

$$0 = dv(2dr_* - (1 - \frac{M}{r})^2 dv)$$

There are two solutions

$$dv = 0$$

$$2dr_* - (1 - \frac{M}{r})^2 dv = 0$$

Rearrange the second solution,

$$dv = \frac{2dr_*}{(1-\frac{M}{r})^2}$$

Solve for v in each solution

$$dv_1 = 0 \rightarrow\rightarrow\rightarrow v_1 = constant$$

$$\int dv_2 = \int \frac{2dr}{(1-\frac{M}{r})^2}$$

Let $u = 1 - \frac{M}{r}$ and $du = -(-1)r^{-2}dr = \frac{1}{r^2}dr$

$$\int dv_2 = \int \frac{2r^2 du}{(1-\frac{M}{r})^2}$$

... HOW TO SOLVE THIS INTEGRAL???

$$v = constnat = v - 2r_*$$

Now consider time (and radius) Memorize (WHY????????)

$$\tilde{t} = u - v$$

For ingoing light rays (WHY?????)

$$\tilde{t} = constant - r$$

For outgoing light rays (WHY??????)

$$\tilde{t} = constant - r + 2r_*$$

substitute into r_*,

$$\tilde{t} = constant - r + 4Mln(\frac{r}{m} - 1) - \frac{2M^2}{r - m}$$

Plot this to see light cones that are tilted towards the black hole when outside (but open up away from it), and light cones that are wide open within the region when inside the black hole (there is freedom of movement within, but is asymptotic to non-movement at the horizon radius).

4.6 6.) Time in SZ BH

Prompt: Problem 6. (Problem 12.5 in Hartle) An observer falls radially into a spherical black hole of mass M. The observer starts from rest relative to a stationary observer at a Schwarzschild coordinate radius of 10M. How much time elapses on the observer's own clock before hitting the singularity?

Solution:

Radial infall. Memorize and start with schwarschild geometry's standard conclusions:

$$\frac{1}{2}\dot{r}^2 = V_{eff} = \epsilon$$

Consider radial infall,

$$l = f(\dot{\phi}) = 0$$

So, the general solution is (What's the definition of V eff???):

$$(\frac{dr}{d\tau})^2 = 2(\epsilon + \frac{M}{r})$$

The initial conditions in the problem are distance away, and initial velocity:

$$r(\tau = 0) = 10M$$

$$\frac{dr}{d\tau} = \dot{r}|_{R=10M} = 0$$

Substitute,

$$0 = 2(\epsilon + \frac{M}{10M})$$

$$0 = (\epsilon + \frac{1}{10})$$

$$\epsilon = -\frac{1}{10}$$

Now plug this into the general solution and solve for τ,

$$(\frac{dr}{d\tau})^2 = 2((-\frac{1}{10}) + \frac{M}{r})$$
$$(\frac{dr}{d\tau})^2 = (-\frac{2}{10}) + \frac{2M}{r}$$
$$(\frac{dr}{d\tau})^2 = -\frac{2}{10} + \frac{2M}{r}$$
$$(\frac{dr}{d\tau}) = \sqrt{-\frac{1}{5} + \frac{2M}{r}}$$
$$dr = d\tau \cdot \sqrt{-\frac{1}{5} + \frac{2M}{r}}$$
$$d\tau = \frac{dr}{\sqrt{-\frac{1}{5} + \frac{2M}{r}}}$$
$$\int d\tau = \tau = \int \frac{dr}{\sqrt{-\frac{1}{5} + \frac{2M}{r}}}$$

HOW TO SOLVE THIS INTEGRAL???

$$\tau = 5\pi M\sqrt{5}$$

Next part: consider being within a black hole, maximize your time away from a singularity. Start at the edge or on the horizon, $r = 2M$, rearrange the metric (ds) to be:

$$ds^2 = -\frac{dr^2}{2\frac{M}{r} - 1} + (\frac{2M}{r} - 1)dt^2 + r^2 d\Omega^2$$

Use

$$d\tau = -ds$$

So,

$$d\tau^2 = \frac{dr^2}{2\frac{M}{r} - 1} - (\frac{2M}{r} - 1)dt^2 - r^2 d\Omega^2$$

This numbers are always positive, so the maximum τ occurs when:

$$d\tau_{max}^2 = \frac{dr^2}{2\frac{M}{r} - 1} - (0) - (0)$$

Take square root,

$$d\tau_{max} = \sqrt{\frac{dr^2}{2\frac{M}{r} - 1}}$$
$$d\tau_{max} = \frac{dr}{\sqrt{2\frac{M}{r} - 1}}$$
$$\int d\tau_{max} = \tau = \int_{r=2M}^{r=0} \frac{dr}{\sqrt{2\frac{M}{r} - 1}}$$

Solve the integral (use Taylor expansion?) ... ...

$$d\tau_{max} \sim 2\pi M$$

4.7 7: Information while in a BH

Prompt: Problem 7. (Problem 12.8 in Hartle) Can an observer who falls into a spherical black hole receive information about events that take place outside? Is there any region of spacetime outside the black hole that an interior observer cannot eventually see? Analyse these questions using the Eddington-Finkelstein diagram.
Solution:

4.8 8: If there was negative mass...

Prompt: Problem 8. (Problem 12.11 in Hartle) Negative mass does not occur in nature. But just as an exercise, analyse the behaviour of radial light rays in a Schwarzschild geometry with a negative value of M. Sketch the Eddington-Finkelstein diagram showing these light rays. Is the negative mass Schwarzschild geometry a black hole?
Solution:

4.9 9: Proper time maximization in a BH

Prompt: Problem 9. (Problem 12.14 in Hartle) Once across the event horizon of a black hole, what is the longest proper time the observer can spend before being destroyed in the singularity?
Solution:

4.10 10: Penrose diagram limits from Kurskal's SZ geometry

Prompt: Problem 10. (Problem 12.26 in Hartle) Show that the boundaries of the Penrose diagram for the Kurskal extension of the Schwarzschild geometry are as given in Box 12.5 in Hartle.
Solution:

5 Exercise Set 5 - Black Holes

5.1 Transformation: Schwarszchild to Kruskal

Prompt: Problem 1. (Problem 20 in Hartle) Explicitly carry out the transformation from Schwarzschild to Kruskal coordinates defined in (12.13). Find the metric in Kruskal coordinates for both $r > 2M$ and $r < 2M$.
Solution: Memorize the SZ metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

Consider light rays, ignore massive particles ...

$$t(r) = -r - 2M \ln |\frac{r}{2M} - 1|, \quad for\ ingoing\ rays$$

$$t(r) = r + 2M \ln |\frac{r}{2M} - 1|, \quad for\ outgoing\ rays$$

The plot of these shows light cones of a modified SZ metric. The map does not cover all of the territory, so change coordinates! Have the idea that light rays of lines of slope (-)1, and the space takes whatever geometry to accommodate that. Use Eddington-Finklestein coordinates of the form:

$$\bar{t}(r) = t + 2M \ln \frac{r}{2M} - 1$$

for ingoing light:

$$\bar{t}(r) = -r + \delta t$$

This ingoing light now has the slope = -1 characteristic desired. Kruskel coordinates imples does the same procedure for both ingoing and outgoing light and setting the infinite limits to values of 1. Here define the tortoise coordinate in two equations, dr_*/dr and r_*:

$$\frac{dr_*}{dr} = \frac{1}{\left(1 - \frac{2M}{r}\right)}, \qquad r > 2M$$

$$r_* = r + 2M \ln |\frac{2}{2M} - 1|$$

where $r_{(r_* \to \infty)} \to \infty$ and $r_{(r_* \to -\infty)} \to 2M$.
This approach is wrong... avoid it

No. 1

$$dr_* = \frac{dr}{\left(1 - \frac{2M}{r}\right)}$$

Let $u = 1 - \frac{2M}{r}$, $du = 0 + 2M(r^{-2})dr = \frac{2M}{r^2} dr$, and $dr = \frac{r^2}{2M} du$.

$$dr_* = \frac{()}{(u)}$$

No. 2

$$dr_* = \frac{dr}{\left(1 - \frac{2M}{r}\right)}$$

$$dr_* = \frac{dr}{\left(\frac{r}{r} - \frac{2M}{r}\right)}$$

$$dr_* = \frac{dr}{\left(\frac{r-2M}{r}\right)}$$

$$dr_* = \frac{r\ dr}{r - 2M}$$

$$dr_* = \frac{r}{r - 2M} dr$$

Let $u = r - 2M$, $du = dr$.

$$dr_* = \frac{u + 2M}{u} du$$

$$\int dr_* = \int \frac{u + 2M}{u} du$$

$$r_* = \int 1\ du + \int \frac{2M}{u} du$$

$$r_* = \int du + 2M \int \frac{1}{u} du$$

$$r_* = (u) + 2M(\ln u)$$

$$r_* = (r - 2M) + 2M(\ln r - 2M)$$

$$r_* = r - 2M + 2M(\ln r - 2M)$$

Here define coordinates u and v:

$$u = t + r_*$$

$$v = t - r_*$$

$$du = dt + dr_*$$
$$dv = dt - dr_*$$

WHAT'S NEXT? Substitute these into the (light) SZ metric:

$$0 = -\left(1 - \frac{2M}{r}\right)(dv + dr_*)^2 + \left(1 - \frac{2M}{r}\right)(dr)^2 + r^2 d\Omega^2$$

$$0 = -\left(1 - \frac{2M}{r}\right)(dv + dr_*)^2 + \left(1 - \frac{2M}{r}\right)(dr\frac{dr_*}{dr_*})^2 + r^2 d\Omega^2$$
$$0 = -\left(1 - \frac{2M}{r}\right)(dv + dr_*)^2 + \left(1 - \frac{2M}{r}\right)(\frac{dr}{dr_*}dr_*)^2 + r^2 d\Omega^2$$
$$0 = -\left(1 - \frac{2M}{r}\right)(dv + dr_*)^2 + \left(1 - \frac{2M}{r}\right)((1 - \frac{2M}{r})dr_*)^2 + r^2 d\Omega^2$$

... ... This is supposed to yield

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) du\ dv + r^2 d\Omega^2, \quad where\ r = f(u, v)\ only$$

Note: The metric is still degenerate at $r = 2M$, because: $\frac{v-u}{2} = r_2 = r + 2M \ln|\frac{r}{2M} - 1|$ At this point, define a new pair of coordinates, let:

$$X = e^{-u/4M}$$

$$Y = e^{v/4M}$$

so,

$$dX = -\frac{1}{4M}e^{-u/4M}du = -\frac{du}{4M}X$$
$$dY = \frac{1}{4M}e^{v/4M}dv = \frac{dv}{4M}Y$$

WHAT'S NEXT??? Do something with the two

$$X \cdot Y = e^{\frac{v-u}{4M}} = \left(\frac{r}{2M} - 1\right) e^{r/2M}$$

To turn the previous metric

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) du\ dv + r^2 d\Omega^2$$

Into

$$ds^2 = 0 = \left(1 - \frac{2M}{r}\right)(4M)^2 \frac{dX\ dY}{XY} + r^2 d\Omega^2$$
$$ds^2 = 0 = (4M)^2 \frac{\left(1 - \frac{2M}{r}\right)}{\left(\frac{r}{2M} - 1\right)} e^{-r/2M} dX\ dY + r^2 d\Omega^2$$
$$ds^2 = 0 = \frac{32M^3}{r} e^{-r/2M} dX\ dY + r^2 d\Omega^2$$

- "r" is defined implicitly as $f(x, y)$

- so far, this is for $r > 2M$
- now, extend the coordinates
  - this is known as the "kruskel extension"
  - (x, y) ϵ (-inf, inf)
    - done!
- compare to typical situations:

$$X = U + V$$
$$Y = U - V$$

so,

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} dX\ dY + r^2 d\Omega^2$$

and

$$U^2 - V^2 = (\frac{r}{2M} - 1)e^{r/2M}$$

This is graphically the Kruskel diagram with most of the action happening in the first quadrant and light moving on slope = +/- 1 lines.
The horizon is located at $r = 2M$, which is $U^2 - V^2 = 0$, or $U = \pm V$
The singularity is located at $r = 0$, which is $U^2 - V^2 = 1$, or $U = \pm\sqrt{1 + V^2}$

5.2 Make Penrose diagram

Prompt: Problem 2. Explicitly carry out the coordinate transformation in Box 12.5 in Hartle and construct the Penrose diagram. What are the coordinate ranges for the Penrose coordinates u0 , v0 ? In Penrose coordinates, where is the event horizon and singularity located?
Solution: Turn part 1 into a penrose diagram... Start with the old (X, Y) where,

$$XY = 0, \quad \textit{is the horizon}$$

$$X \cdot Y = -1, \quad \textit{is the singularity}$$

Compact(ify) them,

$$u' = -\arctan Y$$
$$v' = \arctan X$$

Within the domain

$$u', v' \quad \epsilon \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(WHAT IS HAPPENING???) At the horizon,

$$X = Y = 0$$

so,

$$u' = v' = 0$$
$$U' = \frac{v' - u'}{2}$$

$$V' = \frac{v' + u'}{2}$$

(WHAT IS HAPPENING???) At the singularity,

$$\tan v' \tan u' = 1$$

(HOW TO KNOW TO???) Use boundary conditions

$$u' = \mp\frac{\pi}{2} + v'$$

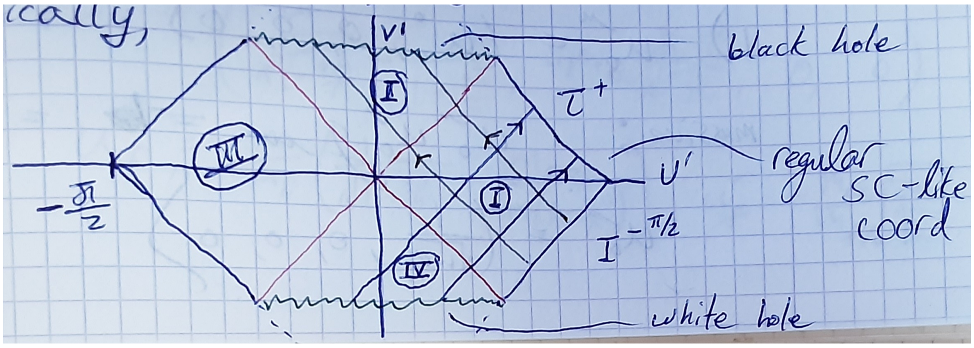
$$U' = \pm\frac{\pi}{2} - V'$$

$$\tan(v' + u')\tan(v' - u') = 1$$

Solve this to get

$$\boxed{cos(2v') = 0}$$

Graph it as



5.3 Asymptote travel

Prompt: Problem 3. Is it possible for an observer to travel between the two asymptotic regions on the Kruskal diagram?
Solution: No, it is not. Requires exceeding the speed of light

5.4 Normal vector, null vector on SZ BH three-surface

Prompt: Problem 4. (Problem 12.12 in Hartle) Check that the normal vector to the horizon three-surface of a Schwarzschild black hole is a null vector.
Solution: Look at $r = 2M$; go to a coordinate system which is regular at $r = 2M$: (u, r, θ, ϕ). It's metric:

$$ds^2 = -(1 - \frac{2M}{r})du^2 + 2dudr + r^2 d\Omega^2$$

Comes from the first exercise in this set. Now,

- Find 3 tangent vectors to the horizon
- $m^u = (m^0, m^1, 0, 0)$
- Check allowed values for m^0, m^1

... ...

5.5 Energy to escape gravity well of SZ BH

Prompt: Problem 5. (Problem 12.15 in Hartle) A spaceship whose mission is to study the enviroment around a black hole is hovering at a Schwarzschild coordinate radius R outside a spherical black hole of mass M. To escape back to infinity, crew must eject a part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction f of the rest mass that can escape to infinity? What happens to this fraction as R approaches 2M?

Solution: Let $m = originalmass$, $m_{esc} = massatR = \infty, t = \infty$, and $m_f = massoffuelejected$ There is movement away from the BH, use the 4-vector,

$$u_0^\mu = (u_0^0, 0, 0, 0)$$

It's a massive ship,

$$u_0^\mu u_0^\nu g_{\mu\nu} = -1$$

$$u_0 u_0 = -1$$

$$g =$$

$$u_0^\mu = \left(\frac{1}{\sqrt{1 - \frac{2M}{R}}}, 0, 0, 0 \right)$$

assume m goes to $r = \infty$ on a <u>radial geodesic</u>:

$$l = d\Omega = 0$$

Memorize from earlier problem set (2?) the two equations:

$$\dot{t}^2 = \frac{E}{1 - \frac{2M}{r}}$$

$$\dot{r}^2 = \frac{2M}{r} + E^2 - 1$$

so,

$$u_{esc}^\mu = \left(\frac{E}{1 - \frac{2M}{r}}, \dot{r}, 0, 0 \right)$$

(IS NEXT TO???) Plug in $\dot{r}$,

$$u_{esc}^\mu = \left(\frac{E}{1 - \frac{2M}{r}}, \dot{\sqrt{\frac{2M}{r} + E^2 - 1}}, 0, 0 \right)$$

At $r = \infty$, $E = 1$ (from SZ derivation where $\varepsilon = 0 = (E^2 - 1)/2$), and $r = R$. Keep the positive square root, it's physical.

$$u_{esc}^\mu = \left(\frac{E}{1 - \frac{2M}{r}}, \dot{\sqrt{\frac{2M}{R} + 1 - 1}}, 0, 0 \right)$$

$$u_{esc}^\mu = \left(\frac{E}{1 - \frac{2M}{r}}, \dot{\sqrt{\frac{2M}{R}}}, 0, 0 \right)$$

We have: (m, u_0^μ) and (m_{esc}, u_{esc}^μ).
We want: (m_f, u_f^μ)
Use conservation of 4-momentum ($\Delta p = 0 = \Sigma p_f - p_i$)

$$mu_0^\mu = m_{esc}u_{esc} + m_f u_f^\mu$$

$$m_f u_f^\mu = mu_0^\mu - m_{esc}u_{esc}$$

$$m_f u_f^\mu = \left(\frac{M}{\sqrt{1-\frac{2M}{r}}} - \frac{M}{1-\frac{2M}{r}}, -m_{esc}\sqrt{\frac{2M}{R}}, 0, 0 \right)$$

$$\frac{1}{m_f}\left[m_f u_f^\mu = \left(\frac{M}{\sqrt{1-\frac{2M}{r}}} - \frac{M}{1-\frac{2M}{r}}, -m_{esc}\sqrt{\frac{2M}{R}}, 0, 0 \right) \right]$$

$$u_f^\mu = \left(\frac{M}{m_f\sqrt{1-\frac{2M}{r}}} - \frac{M/m_f}{1-\frac{2M}{r}}, -\frac{m_{esc}}{m_f}\sqrt{\frac{2M}{R}}, 0, 0 \right)$$

$$u_f^\mu = \left(\frac{M}{m_f}\left(\frac{1}{\sqrt{1-\frac{2M}{r}}} - \frac{1}{1-\frac{2M}{r}} \right), -\frac{m_{esc}}{m_f}\sqrt{\frac{2M}{R}}, 0, 0 \right)$$

$$\boxed{u_f^\mu = \frac{M}{m_f}\left(\frac{1}{\sqrt{1-\frac{2M}{r}}} - \frac{1}{1-\frac{2M}{r}} \right), -\frac{m_{esc}}{m_f}\sqrt{\frac{2M}{R}}, \quad 0, \quad 0}$$

$$u_f^t = \frac{M}{m_f}\left(\frac{1}{\sqrt{1-\frac{2M}{r}}} - \frac{1}{1-\frac{2M}{r}} \right)$$

$$u_f^r = -\frac{m_{esc}}{m_f}\sqrt{\frac{2M}{R}}$$

Use

$$u_0^\mu u_0^\nu g_{\mu\nu} = \boxed{-1 = -\left(\frac{M}{m_f}\right)^2 \left(1 - \frac{m_{exc}}{M}\frac{1}{\sqrt{1-2M/R}}\right)^2 + \left(\frac{m_{esc}}{m_f}\right)\left(\frac{2M}{R-2M}\right)}$$

This is the end of the theory. We can compute with the following:

$$X = \frac{m_{esc}}{M}$$

$$Y = \frac{m_f}{M}$$

$$1 = \frac{1}{Y^2}\left(1 - \frac{X}{\sqrt{1-2M/R}}\right)^2 + \frac{X^2}{Y^2}\frac{2M}{R-2M}$$

Shape it into a quadratic,

$$X^2 - (\frac{2}{\sqrt{1-2M/R}})X + (1-Y^2) = 0$$

Solve quadratic,

$$X = \pm\frac{1}{\sqrt{1-2M/R}} \pm \sqrt{\frac{1}{1-2M/R} - 1 + Y^2}$$

Because it's a rocket with finite fuel, use $X/Y\epsilon[0,1]$; the "+" solution violates this

$$X = \frac{1}{\sqrt{1-2M/R}} - \sqrt{\frac{1}{1-2M/R} - 1 + Y^2}$$

Consider the perfect propulsion: $Y \to 0$. All fuel is made to E and ejected away perfectly as light.

$$X_{max} = \frac{m_{esc}}{M}|_{max,y=0} = \frac{1-\sqrt{\frac{2M}{R}}}{\sqrt{1-\frac{2M}{R}}}$$

Here, as $R \to 2M$, then $M \to 0$; this mean the closer to the horizon you are the closer to 100% of your mass you must eject.

6 Exercise Set 6 - Gravitational Waves

6.1 GW space-time and (3) Killing vectors

Prompt: Show that the gravitational wave space-time (16.2 in Hartle) has three Killing vectors: (0, 1, 0, 0), (0, 0, 1, 0), and (1, 0, 0, 1).
Solution: Memorize and write down the GW metric:

$$ds^2 = -dt^2 + [1 + f(t-z)]dx^2 + [1 - f(t-z)]dy^2 + dz^2$$

ds is not a function of x or of y (the dx^2, and dy^2 terms do not count). Therefore, the x and y vectors

$$\boxed{\xi_x^\mu = (0,1,0,0)}$$

and

$$\boxed{\xi_y^\mu = (0,0,1,0)}$$

are Killing vectors.

And for the third (1, 0, 0, 1) vector: Make the change of coordinate

$$t \to t + \delta$$

$$z \to z + \delta$$

The deltas will appear in the $f_*(t-z)$ terms, but they will cancel out. Thus the 3rd killing vector is

$$\boxed{\xi_3^\mu = (1,0,0,1)}$$

... ...
the metric does not depend on z. So, in the coordinate system (u, x, y, z), we have the Killing vector ξ'^μ = (0, 0, 0, 1).

(HOW??? ??? ???) We transform this vector to the original coordinate system, and we get ξ^μ = (1, 0, 0, 1).

Alternate solution: (1, 0, 0, 1) is a Killing vector because the metric is invariant under the transformation (t, x, y, x) $\to$ (t + δ, x, y, z + δ).

This is. $x^\mu \to x^\mu + \delta\xi^\mu$, where ξ^μ is the Killing vector.

6.2 GW on one test mass

Prompt: Consider a test mass at rest at position x = (X, Y, Z) in flat spacetime. A gravitational wave (16.2 in Hartle) passes over the test mass. Show that to leading order in the amplitude of the wave the coordinate position of the test mass remains (X, Y, Z). Compare the solution below with the section 16.2 of Hartle's Gravity.
Solution:
Consider the test mass at (X, Y, Z). If it moves, let's call its new coordinates $(x + \delta x, y + \delta y, z + \delta z) = (x', y', z')$. "the amplitude of the wave the coordinate position" means movement, which means geodesic equation:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

We're considering small changes in coordinates, thus:

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}\frac{d\delta x^\alpha}{d\tau}\frac{d\delta x^\beta}{d\tau}$$

The δs in the right-hand terms are approximately negligible,

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}\frac{d\cancel{\delta}x^\alpha}{d\tau}\frac{d\cancel{\delta}x^\beta}{d\tau}$$

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}u^\alpha u^\beta$$

At rest, we have the general vector $u^\alpha_{rest} = (1, 0, 0, 0)$

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{tt}(1)$$

Also, given a gravitational wave, we have

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{tt} = d\Gamma^i_{\alpha\beta} = \frac{1}{2}\eta^{i\alpha}(\partial_\alpha h_{\mu\alpha} + \partial_\beta h_{\mu\beta} - \partial_i h_{\alpha\beta})$$

I think there are many 0 terms as we go from the above general equation to the below particular equation,

$$\delta\Gamma^i_{tt} = \frac{1}{2}\eta^{i\mu}(2\ \partial_t h_{t\mu} - \partial_\mu h_{tt} + 0 + 0...)$$

(WHY DOES???) This equals zero,

$$\delta\Gamma^i_{tt} = \frac{1}{2}\eta^{i\mu}(2\ \partial_t h_{t\mu} - \partial_\mu h_{tt}) = 0$$

So,

$$\frac{d^2\delta x^i}{d\tau^2} = 0$$

Take an integral,

$$\frac{d\delta x^i}{d\tau} = C$$

(WHY IS???) This being zero,

$$\frac{d\delta x^i}{d\tau} = 0$$

Also implies constant, unchanged coordinates, so

$$\boxed{X', Y', Z' = X, Y, Z}$$

6.3 GW on two test masses

Prompt: Consider the gravitational wave in (16.2 in Hartle) and two test masses, one at the origin and the other at a location (X, Y, Z) in the Cartesian coordinates used in (16.2). Show that the 1change in distance between the masses produced by the wave is given by

$$\delta L(t) = L - L_* = \frac{1}{2}\int_0^{L_*} h_{ij}(t - \lambda n^z)n^i n^j d\lambda$$

Where

$$n^i = \left(\frac{X}{L_*}, \frac{Y}{L_*}, \frac{X}{L_*} \right)$$

Here n^i is the unit tangent vector to the straight-line path between the test masses and L_* is the unperturbed distance between them

Solution: Memorize and begin with parametrizing the straight line joining the origin with the point (X, Y, Z) as

$$x^\mu(\lambda) = (0, \lambda n^x, \lambda n^y, \lambda n^z) \rightarrow \dot{x}^\mu = (0, n^x, n^y, n^z)$$

from

$$\vec{x} = (t, x, y, z)$$

The unperturbed distance L_* is

$$L_* = \sqrt{X^2 + Y^2 + Z^2}$$

The parameter λ is the space inside of the $0 \rightarrow$ L (we'll integrate it calculate it) and it takes the values $0 \leq \lambda \leq L_*$.

Memorize and use the normal/generic GW Minkowski perturbation metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(t - z)$$

Note: the $h_{\mu\nu}$ is a function of $(t - z)$, it is **NOT** multiplied by the factor $(t - z)$ Memorize and use the definition of L (geodesic equation):

$$L = \int_0^{L_*} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$L = \int_0^{L_*} \sqrt{(\eta_{\mu\nu} + h_{\mu\nu}(t - z)) \dot{x}^\mu \dot{x}^\nu} d\lambda$$

Convert index notation, so before $\mu\nu \rightarrow ij$ now. Use

$$\dot{x}^i = n^i$$

$$\eta_{ij} \dot{x}^i \dot{x}^j = 1$$

$$||n^i|| = 1$$

The movement of the wave is **entirely** in the z direction, so the general z and $\eta_{\mu\nu}$ term take the form

$$z = \lambda n^z$$

$$\eta_{\mu\nu} = 1$$

$$L = \int_0^{L_*} \sqrt{1 + h_{ij}(t - \lambda n^z) n^i n^j} d\lambda$$

$$L = \int_0^{L_*} \sqrt{1 + h_{ij} n^i n^j} d\lambda$$

Within the square root, $|h_{ij}(t - \lambda n^z) n^i n^j| << 1$. Use a Taylor expansion for the square root; which is in general

$$\sqrt{1 + x} \sim 1 + \frac{x}{2}$$

and in this particular instance

$$\sqrt{1+h_{ij}(t-\lambda n^z)n^in^j} \sim 1+\frac{1}{2}(h_{ij}(t-\lambda n^z)n^in^j)$$

and

$$L=\int_0^{L_*}\sqrt{1+h_{ij}(t-\lambda n^z)n^in^j}d\lambda \sim \int_0^{L_*}[1+\frac{1}{2}(h_{ij}(t-\lambda n^z)n^in^j)]d\lambda$$

$$L=\int_0^{L_*}[1+\frac{1}{2}(h_{ij}(t-\lambda n^z)n^in^j)]d\lambda$$

$$L=\int_0^{L_*}(1)d\lambda+\int_0^{L_*}\frac{1}{2}(h_{ij}(t-\lambda n^z)n^in^j)d\lambda$$

$$L=(L_*-0)+\int_0^{L_*}\frac{1}{2}(h_{ij}(t-\lambda n^z)n^in^j)d\lambda$$

$$L-L_*=\frac{1}{2}\int_0^{L_*}(h_{ij}(t-\lambda n^z)n^in^j)d\lambda$$

$$L-L_*=\frac{1}{2}\int_0^{L_*}h_{ij}(t-\lambda n^z)n^in^jd\lambda$$

$$L-L_*=\frac{1}{2}\int_0^{L_*}h_{ij}n^in^jd\lambda$$

Shown. Now continue this by solving for a reasonable h_{ij} (first half of next problem) and substituting it into this equation (second half of next problem).

6.4 GW Polarization

Prompt: Express $\frac{\delta L(t)}{L_*}$ for a "+" polarized wave of definite frequency ω and amplitude a.

Solution: Memorize beginning with h_{ij}

$$h_{ij}=\begin{pmatrix} h_{tt} & h_{tx} & h_{ty} & h_{tz} \\ h_{xt} & h_{xx} & h_{xy} & h_{xz} \\ h_{yt} & h_{yx} & h_{yy} & h_{yz} \\ h_{zt} & h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

Let the z-axis be linearized and aligned with the direction of the wave's propagation. This means

$$h_{ij}=\begin{pmatrix} h_{tt} & h_{tx} & h_{ty} & h_{tz} \\ h_{xt} & h_{xx} & h_{xy} & h_{xz} \\ h_{yt} & h_{yx} & h_{yy} & h_{yz} \\ h_{zt} & h_{zx} & h_{zy} & h_{zz} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{yx} & h_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_{ij}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{yx} & h_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_x(t-z) & 0 \\ 0 & f_x(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Use the "+" wave polarization

$$h_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_x(t-z) & 0 \\ 0 & f_x(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & 0 & 0 \\ 0 & 0 & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Use the ω and a terms

$$h_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & 0 & 0 \\ 0 & 0 & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a\ cos(\omega(t-z)) & 0 & 0 \\ 0 & 0 & -a\ cos(\omega(t-z)) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or more compactly,

$$h_{xx} = -h_{yy} = a\ cos(\omega(t-z))$$

Turn $(t-z)$ into $(t-\lambda n^z)$ by aligning z-axis to wave direction. Combine this and the above with this solution from the last exercise

$$L-L_* = \frac{1}{2}\int_0^{L_*} h_{ij}(t-\lambda n^z)n^i n^j d\lambda \quad = \frac{1}{2}\int_0^{L_*}[h_{xx}n^x n^x + h_{xy}n^x n^y + h_{yy}n^y n^y + h_{xz}n^x n^z + ...]d\lambda$$

to get:

$$L - L_* = \frac{1}{2}\int_0^{L_*}(a\ cos(\omega(t-z)))(t-\lambda n^z)n^i n^j\ d\lambda$$

$$\delta L = \frac{1}{2}\int_0^{L_*}(a\ cos(\omega(t-z)))(t-\lambda n^z)n^i n^j\ d\lambda$$

simplify; only two of the n's are non-zero (n^{xx} and n^{yy})

$$\delta L = \frac{1}{2}\int_0^{L_*}(a\ cos(\omega(t-\lambda n^z)))((n^x)^2 - (n^y)^2)\ d\lambda$$

$$\delta L = \frac{1}{2}((n^x)^2 - (n^y)^2)\int_0^{L_*}(a\ cos(\omega(t-\lambda n^z)))\ d\lambda$$

$$\delta L = \frac{a}{2}((n^x)^2 - (n^y)^2)\int_0^{L_*}(cos(\omega(t-\lambda n^z)))\ d\lambda$$

let,

$$u = \omega(t-\lambda n^z),\ so\ \ du = -\omega n^z d\lambda\ \ and\ \ d\lambda = -\frac{du}{\omega n^z}$$

substitute,

$$\delta L = \frac{a}{2}((n^x)^2 - (n^y)^2)\int_0^{L_*}(cos(u))\ \frac{du}{-\omega n^z}$$

$$\delta L = -\frac{a}{2\omega n^z}((n^x)^2 - (n^y)^2)\int_0^{L_*}(cos(u))\ du$$

$$\delta L = -\frac{a}{2\omega n^z}((n^x)^2 - (n^y)^2)(sin(u))\Big|_0^{L_*}$$

$$\delta L = -\frac{a}{2\omega n^z}((n^x)^2 - (n^y)^2)(sin(\omega(t - \lambda n^z)))\Big|_0^{L_*}$$

$$\delta L = -\frac{a}{2\omega n^z}((n^x)^2 - (n^y)^2)(sin(\omega(t - (L_*)n^z)) - sin(\omega(t - (0)n^z)))$$

$$\delta L = -\frac{a}{2\omega n^z}((n^x)^2 - (n^y)^2)(sin(\omega(t - (L_*)n^z)) - sin(\omega t))$$

$$\delta L = \frac{a}{2\omega n^z}((n^x)^2 - (n^y)^2)(sin(\omega t) - sin(\omega(t - (L_*)n^z)))$$

$$\delta L = \frac{a}{2\omega n^z}[(n^x)^2 - (n^y)^2][sin(\omega t) - sin(\omega(t - L_*n^z))]$$

Let $Z = L_*n^z$

$$\delta L = \frac{a}{2\omega n^z}[(n^x)^2 - (n^y)^2][sin(\omega t) - sin(\omega(t - Z))]$$

Use $x^i = (X, Y, Z)$ and $n^i = \frac{n^i}{L_*}$, which is: $n^x = \frac{X}{L_*}$, $n^y = \frac{Y}{L_*}$, and $n^z = \frac{Z}{L_*}$

$$\delta L = \frac{a}{2\omega \frac{Z}{L_*}}[(\frac{X}{L_*})^2 - (\frac{Y}{L_*})^2][sin(\omega t) - sin(\omega(t - Z))]$$

Now make the ratio,

$$\frac{\delta L_{(t)}}{L_*} = \frac{\frac{a}{2\omega \frac{Z}{L_*}}[(\frac{X}{L_*})^2 - (\frac{Y}{L_*})^2][sin(\omega t) - sin(\omega(t - Z))]}{L_*}$$

$$\frac{\delta L_{(t)}}{L_*} = (\frac{1}{L_*})[\frac{a}{2\omega \frac{Z}{L_*}}[(\frac{X}{L_*})^2 - (\frac{Y}{L_*})^2][sin(\omega t) - sin(\omega(t - Z))]]$$

$$\frac{\delta L_{(t)}}{L_*} = (\frac{1}{L_*})[\frac{aL_*}{2\omega Z}[(\frac{X}{L_*})^2 - (\frac{Y}{L_*})^2][sin(\omega t) - sin(\omega(t - Z))]]$$

$$\frac{\delta L_{(t)}}{L_*} = \frac{a}{2\omega Z}[(\frac{X}{L_*})^2 - (\frac{Y}{L_*})^2][sin(\omega t) - sin(\omega(t - Z))]$$

$$\frac{\delta L_{(t)}}{L_*} = \frac{a}{2\omega Z}[\frac{1}{L_*^2}(X^2 - Y^2)][sin(\omega t) - sin(\omega(t - Z))]$$

$$\frac{\delta L_{(t)}}{L_*} = \frac{a(X^2 - Y^2)}{2\omega Z L_*^2}[\frac{1}{1}][sin(\omega t) - sin(\omega(t - Z))]$$

$$\boxed{\frac{\delta L_{(t)}}{L_*} = \frac{a(X^2 - Y^2)}{2\omega Z L_*^2}[sin(\omega t) - sin(\omega(t - Z))]}$$

Use this equation with a specified initial condition (ring of resting particles) to solve exercise 5

6.5 GW deformation (circle to ellipse)

Prompt: The equation for an ellipse is $x^2/\hat{a}^2 + y^2/\hat{b}^2 = 1$ where $\hat{a}$ is the semi-major axis and $\hat{b}$ is the semi-minor axis if $\hat{a} > \hat{b}$. Show that an initial circle of test particles distorts into an ellipse according to (16.13 in Hartle) to lowest order in a and compute the semi-major and semi-minor axes as a function of time

Solution Consider particles initially in the circle at

$$x^2 + y^2 = R^2 \;, \qquad z = 0$$

and a gravitational wave with (for instance) "+" polarization in the z direction. According to the previous exercise, the distance between those particles and the origin (0, 0, 0) when the gravitational wave passes is written from the above exercise

$$\frac{\delta L_{(t)}}{L_*} = \frac{\delta L(t)}{R} = \frac{a(X^2 - Y^2)}{2\omega Z L_*^2}[sin(\omega t) - sin(\omega(t - Z))]$$

memorize the simplification $Z \to 0$ ... ...

$$\frac{\delta L_{(t)}}{L_*} = \frac{\delta L(t)}{R} = \frac{a(x^2 - y^2)}{2R^2} cos(\omega t)$$

Where does this (below) equation belong???

$$\frac{\delta R}{R} = \frac{a(X^2 - X^2)}{2R^2} cos(\omega t)$$

Now begin the derivation knowing that we're going from a circle $x^2 + y^2 = R^2$ to an ellipse of the shape

$$x^2 + y^2 = (R + \delta R)^2$$

This implies a change in the length of size δL, or

$$L_* \to L_* + \delta L$$

Next, use the equation of the above equation to find

$$\left.\frac{\delta R}{R}\right|_{fixed\ z>0} = \frac{a(X^2 - Y^2)}{2\omega Z L_*^2}[sin(\omega t) - sin(\omega(t - Z))]$$

$$\left.\frac{\delta R}{R}\right|_{fixed\ z>0} = \frac{a(X^2 - Y^2)}{2\omega Z L_*^2}[sin(\omega t) - sin(\omega(t - Z))]$$

Now let everything go to infinitesimally small quantities

$$\lim_{Z\to 0}\left(\frac{\delta R}{R}\right) = \lim_{Z\to 0}\left(\frac{a(X^2 - Y^2)}{2\omega Z L_*^2}[sin(\omega t) - sin(\omega(t - Z))]\right)$$

$$\lim_{Z\to 0}\left(\frac{\delta R}{R}\right) = \frac{a(X^2 - Y^2)}{2\omega L_*^2}\lim_{Z\to 0}\left(\frac{sin(\omega t) - sin(\omega(t - Z))}{Z}\right)$$

The far right term is the definition of a derivative

$$\left.\left(\frac{\delta R}{R}\right)\right|_{Z=0} = \frac{a(X^2 - Y^2)}{2\omega L_*^2}\frac{d}{dZ}(sin(\omega t))$$

$$\left(\frac{\delta R}{R}\right)\bigg|_{Z=0} = \frac{a(X^2 - Y^2)}{2\omega L_*^2} cos(\omega t)$$

Now continue with... ... ...

$$x^2 + y^2 = (R + \delta L(t))^2 = R^2\ [1 + (\frac{a(x^2 - y^2)}{2R^2} cos(\omega t))\ ]^2$$

$$\frac{1}{R^2}(x^2 + y^2) = [\ 1 + (\frac{a(x^2 - y^2)}{2R^2} cos(\omega t))\ ]^2$$

Maybe at this point, expand the square, and drop the 2nd order (3rd) term, then take the square root **ASK Ludovico Machet!**

$$\sqrt{\frac{1}{R^2}(x^2 + y^2) = [\ 1 + (\frac{a(x^2 - y^2)}{2R^2} cos(\omega t))\ ]^2}$$

Do Taylor series expansion

$$\sqrt{\frac{1}{R^2}(x^2 + y^2)} \approx\ 1 + 2\left(\frac{a(x^2 - y^2)}{2R^2} cos(\omega t)\right)^2 + (...^2)^2 + ...$$

Assume higher order terms are negligible

$$\sqrt{\frac{1}{R^2}(x^2 + y^2)} \approx\ 1 + 2\left(\frac{a(x^2 - y^2)}{2R^2} cos(\omega t)\right)^2$$

$$\sqrt{\frac{1}{R^2}(x^2 + y^2)} \approx\ 1 + 2\left(\frac{1}{2}\right)\left(\frac{a(x^2 - y^2)}{2R^2} cos(\omega t)\right)$$

$$\frac{1}{R}\sqrt{(x^2 + y^2)} =\ 1 + \left(\frac{a(x^2 - y^2)}{2R^2} cos(\omega t)\right)$$

$$\frac{1}{R}\sqrt{(x^2 + y^2)} =\ 1 + \left(\frac{a(x^2 - y^2)}{2R^2} cos(\omega t)\right)$$

$$R^2\left[\frac{1}{R}\sqrt{(x^2 + y^2)} =\ 1 + \left(\frac{a(x^2 - y^2)}{2R^2} cos(\omega t)\right)\right]$$

$$R\sqrt{(x^2 + y^2)} =\ R^2 + \left(\frac{a(x^2 - y^2)}{2} cos(\omega t)\right)$$

This still doesn't go towards the answer...
working backwards from the answer:

$$x^2 + y^2 = R^2 + a\ cos(\omega t)(x^2 - y^2)$$

$$x^2 + y^2 = R^2 + cos(\omega t)(x^2 a - y^2 a)$$

$$x^2 + y^2 = R^2 + x^2 a\ cos(\omega t) - y^2 a\ cos(\omega t)$$

$$x^2 - x^2 a\ cos(\omega t) + y^2 + y^2 a\ cos(\omega t) = R^2$$

$$(x^2 - x^2 a\ cos(\omega t)) + (y^2 + y^2 a\ cos(\omega t)) = R^2$$

$$x^2(1 - a\ cos(\omega t)) + y^2(1 + a\ cos(\omega t)) = R^2$$

Do we assume that x and y are small and that the x^4 and y^4 terms will be negligible??? No... This would lead to a perfect circle all the time.
Somehow the above becomes

$$\frac{1}{R^2}[\ x^2(1 - a\ cos(\omega t)) + y^2(1 + a\ cos(\omega t))\ ] = 1$$

This is of the ellipse form, $x^2/a^2 + y^2/b^2 = 1$ which means that

$$a^2 = R^2/(1 - a\ cos(\omega t))$$

$$b^2 = R^2/(1 + a\ cos(\omega t))$$

and

$$a = R/\sqrt{(1 - a\ cos(\omega t))}$$

$$b = R/\sqrt{(1 + a\ cos(\omega t))}$$

For is the 1 term dominates, use Taylor expansion and get

$$a \sim R/(1 - \frac{1}{2}a\ cos(\omega t))$$

$$b \sim R/(1 + \frac{1}{2}a\ cos(\omega t))$$

A similar logic applies to the other "x" polarization.

6.6 Rotation as superposition for GWs

Prompt: In Section 16.3 in Hartle we produced a gravitational wave with $\times$ polarization by rotating the + polarization (16.2) by 45o . Show that a rotation by an arbitrary angle theta doesn't give another independent solution but rather one that could be written as a superposition of + and $\times$. This is one way of seeing that there are only two linearly independent polarization of a gravitational wave.
Solution. Start with the memorized ***rotation*** matrix in the z direction (in the end, nothing (important) changes
... ...

6.7 7.) Radiation, quadrupole, and more

Prompt:

Problem 7. In the discussion above we considered gravitational waves fluctuating in empty space. In this case the perturbed metric satisfies

$$\Box(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h^{\mu\nu}\eta_{\mu\nu}) \equiv (-\partial_t^2 + \nabla^2)(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h^{\mu\nu}\eta_{\mu\nu}) = 0\,,$$

from now on we will denote the object in the bracket by $\bar{h}_{\alpha\beta}$. Verify that (1) is a solution of this equation. Einstein's field equations relate the deformation of geometry to the energy and matter present in the universe in such a way that the above equation is modified to

$$\Box\bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta}\,,$$

where $T_{\alpha\beta}$ is the energy momentum tensor which is symmetric in $\alpha\beta$ and satisfies the continuity equation

$$\partial_t T^{tt} + \partial_k T^{kt} = 0\,.$$

Using the standard retarded solution to the wave equation in the long distance limit ($r \to \infty$)

$$\bar{h}^{\alpha\beta} = 4\int d^3\mathbf{x}' \frac{T^{\alpha\beta}(t - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \to \frac{4}{r}\int d^3x' T^{\alpha\beta}(t-r,\mathbf{x}')\,,$$

and using the continuity equation above, show that spacial part of the fluctuation $\bar{h}^{ij}$ can be expressed solely in terms of the *second mass moment*

$$I^{ij}(t-r) \equiv \int d^3\mathbf{x}'\ T^{tt}(t-r,\mathbf{x}')x'^i x'^j\,.$$

Show that the *spacial* part of the fluctuation $\bar{h}^{ij}$ is $l^{ij}(t-r) \equiv \int d^3x'T^{tt}(t-r,\mathbf{x}')x'^i x'^j$
Solution.
Start with the first thing given,

$$\Box\left(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h^{\mu\nu}\eta_{\mu\nu}\right) = 0$$

where,

$$\Box = -\partial_t^2 + \nabla^2$$

check this with $h_{\mu\nu}(t-z)$, which is only in the z-direction:

$$\Box = -\partial_t^2 + \partial_z^2$$

so,

$$\Box h_{\mu\nu(t-z)} = -\partial_t^2 h_{ij} + \partial_z^2 h_{ij}$$

(WHY IS THIS???) Set to zero,

$$-\partial_t^2 h_{ij} + \partial_z^2 h_{ij} = 0$$

$$\partial_t^2 h_{ij} = \partial_z^2 h_{ij}$$

(WHY ALWAYS DOES) This implies,

$$\boxed{\Box h_{\mu\nu}(t-z) = 0}$$

The "0" implies pure vacuum. Use the other equation given:

$$\Box\bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta}$$

If $T^{\alpha\beta} \neq 0$, use an Einstein equation soluiton:

$$\bar{h}^{\alpha\beta} = 4\int d^3x \frac{T^{\alpha\beta}(t-|x-x'|, x')}{|x-x'|}$$

Consider the long distance,

$$r \to |x - x'|$$

$$\bar{h}^{\alpha\beta} \sim \frac{h}{r}\int d^2x' T^{\alpha\beta}(t-r, x')$$

(WHY DOES ONE???) Use conservation of $T^{\alpha\beta}$

$$\partial_\alpha T^{\alpha\beta} = 0$$

This provides two (final) soltuions:

$$i.)\quad d_t T^{tt} + d_t T^{ti} = 0$$

$$ii.)\quad d_i T^{it} + d_i T^{ij} = 0$$

Consider the components,

$$h^{t\alpha} = 0$$

$$z^\mu \to x^\mu + \xi^\mu$$

The next step is because of guage freedomes (WHAT ARE THOSE???)

$$h^{ij} = \frac{h}{r}\int d^3x^i T^{ij}$$

For T^{ij}, use:

$$T^{ij} = \partial_k(x^{|i|}T^{k|j|}) - x^{|i|}\partial_n T^{k|j}$$

(An exercise is to check this). Apparently next is,

$$h^{ij} = \frac{k}{r}\int d^3x' \left(\partial'_x(x^{|i|}T^{k|j|}) - x^{|i|l}\partial_z T^{i|j}\right)$$

$$h^{ij} = \frac{k}{r}\int d^3x' \left(\partial'_x\cancel{(x^{|i|}T^{k|j|})} - x^{|i|l}\partial_x T^{i|j}\right)$$

$$h^{ij} = \frac{k}{r}\int d^3x' \left(-x^{|i|l}\partial_x T^{i|j}\right)$$

$$h^{ij} = -\frac{k}{r}\int d^3x' x^{|i|l}\partial_k T^{i|j}$$

$$h^{ij} = -\frac{k}{r}\int d^3x' x^{l(i)}\partial_k T^{k|j}$$

then

$$h^{ij} = -\frac{k}{r}\int d^2x' x^{l(i)}\partial_k T^{(lj)}$$

Now compute many parts of this general equation...

$$x^{l(i)}\partial_k T^{(lj)} = \frac{1}{2}\partial_x(x^i x^j \partial_t T^{tk}) - \frac{1}{2}x^i x^j \partial_t \partial_k T^{tk}$$

Maybe 8 more computations to go???
In the end we have:

$$\bar{h}^{ij} = \frac{2}{r}\int d^3x' x'^i x^{ij}\partial_t^2 T^{tt}$$

$$\bar{h}^{ij} = \frac{2}{r}\frac{d^2}{dt^2}\left[\int d^3x x^i x^j \partial_t^2 T^{tt}\right]$$

$$\bar{h}^{ij} = \frac{2}{r}\frac{d^2}{dt^2}I^{ij}$$

... and thus is shown.

6.8 Energy in GWs

Prompt: The first detection of gravitational waves by the LIGO gravitational wave detector was at frequencies of $\sim$ 200 Hz that caused a strain of $\delta L/L \sim 10^{-21}$. What was the flux of energy of the wave incident to Earth? If it came from 410 Mpc away, how fast was the source losing energy to gravitational waves when it was emitted? Express the answer in terms of the mass of the sun.

Solution: Memorize the definition of power and flux as,

$$Power = P = Area \cdot flux = A \cdot f_{GW}$$

$$f_{GW} = \frac{\omega^2 a^2 c^3}{32\pi G}$$

$$a \sim \frac{\delta L}{L}$$

You can estimate f_{GW} from

$$f \sim \frac{E}{A \cdot t} = \frac{mc^2}{A \cdot t}$$

And getting units to work out with the exponents of constants G and c.

$$P = (4\pi R^2)\frac{\omega^2 a^2 c^3}{32\pi G}$$

$$P = \frac{\pi R^2 \omega^2 (\delta L/L)^2 c^3}{8\pi G}$$

$$P = \frac{(410\ Mpc)^2 (200\ Hz)^2 (10^{-21})^2 c^3}{8G}$$

$$P = \frac{(410\ Mpc)^2 (200\ Hz)^2 (10^{-21})^2 (3 \cdot 10^8\ m/s)^3}{8(6.7 \times 10^{-11} m^3 kg^{-1} s^{-2})}$$

$$P = 3 \cdot 10^{47}\ W$$

$$P = 2\ c^2\ M_{\odot}/s$$

7 Exercise Set 7 - Cosmology!

7.1 1.) (Problem 18.4 in Hartle)

Prompt: Suppose the present value of the Hubble constant is 72(km/s)/M pc and that the universe is at critical density. A photon is emitted from our galaxy now. What is the redshift of this photon when it is received in another galaxy 10 billion years in the future, assuming it continues to be matter dominated.
Solution: Memorize and begin with the Friedmann equation as:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

From the problem statement (matter-dominated and critical denisty), use

$$\rho = \rho_m = \rho_{crit}$$

$$k = 0$$

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{0}{a^2} = \frac{8\pi G}{3}\rho_m$$

Solve for ρ

$$\frac{3H^2}{8\pi G} = \rho_m$$

$$\rho_m = \rho_{crit} = \frac{3H^2}{8\pi G}$$

This relationship will be true for other times and places, so also:

$$\rho_{m,0} = \frac{3H_0^2}{8\pi G}$$

Goal compute redshift, z, from $z = a(t_o)/a(t_e) - 1$. Let's analyze ρ, in general, at constant mass

$$\rho = \frac{m}{l^3}$$

$$\rho l^3 = m$$

$$\rho_0 l_0^3 = m = \rho l^3$$

$$\rho = m = \rho_0 \left(\frac{l_0^3}{l^3}\right)$$

Traditionally, l is a fixed length (or rigid body), but here it's expanding over time, so that is to be rewritten such that

$$l = a(t)$$

and

$$\rho = \rho_0 (\frac{a_0}{a})^3$$

Substitute this ρ into the $\rho_{m,0}$ from before

$$\rho_{m,0} = \rho_0 (\frac{a_0}{a})^{-3} = \frac{3H_0^2}{8\pi G}$$

$$H_0^2 = \frac{8\pi G}{3}\rho_0(\frac{a_0}{a})^3$$

Memorize and use the definition of the Hubble constant, H

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2$$

So,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0(\frac{a_0}{a})^3$$

Get the a to be a function of the given H_0

$$H^2(a^2) = \left(\frac{\dot{a}}{1}\right)^2 = \frac{8\pi G}{3}\rho_0(\frac{a_0}{a})^3(a^2)$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_0(\frac{a_0}{a})^3(a^2)$$

$$\dot{a}^2 = \left(\frac{8\pi G}{3}\rho_0\right)(\frac{a_0}{a})^3(a^2)$$

$$\dot{a}^2 = H_0^2(\frac{a_0}{a})^3(a^2)$$

$$\dot{a}^2 = H_0^2(\frac{a_0^3}{a})$$

HOW DOES THIS BECOME??? ??? ???

$$\dot{a}^2 = H_0^2(\frac{a_0^2}{a})$$

From that unknown change, we continue with

$$\sqrt{\dot{a}^2 = H_0^2(\frac{a_0^2}{a})}$$

$$\dot{a} = H_0\frac{a_0}{\sqrt{a}}$$

$$\frac{da}{dt} = H_0\frac{a_0}{\sqrt{a}}$$

$$\frac{\sqrt{a}}{a_0}da = H_0 \cdot dt$$

$$\int \frac{\sqrt{a}}{a_0}da = \int H_0 \cdot dt$$

$$\int \frac{a^{1/2}}{a_0}da = \int H_0 \cdot dt$$

HOW DOES THIS INTEGRATE TO BECOME THIS??? ??? ???

$$\frac{3}{2}(\frac{a}{a_0})^{3/2} = \int H_0 \cdot dt$$

$$\frac{3}{2}(\frac{a}{a_0})^{3/2} = H_0 t + C$$

Solve for the constant C with, $a(t=0)=0$

$$\frac{3}{2}(\frac{(0)}{a_0})^{3/2} = H_0(0) + C$$
$$\frac{3}{2}(0)^{3/2} = 0 + C$$
$$0 = C$$

WHAT DO WE DO WITH THE C ??? ??? ???

Somehow (maybe with the C), the fraction gets flipped... WHY???

$$\frac{2}{3}(\frac{a}{a_0})^{3/2} = H_0 t + 0$$

$$(\frac{a}{a_0})^{3/2} = \frac{3}{2} H_0 t$$
$$(\frac{a}{a_0}) = (\frac{3}{2} H_0 t)^{2/3}$$
$$(\frac{a}{a_0}) = (\frac{3}{2} H_0 t)^{2/3}$$
$$a = a_0 \left(\frac{3}{2} H_0 t\right)^{2/3}$$

Finally, consider times for the scale factors (a) to get z...

$$z = \frac{a(t_e)}{a(t_0)} - 1$$

$$z = \frac{a(t_0 + \Delta t)}{a(t_0)} - 1$$

Skipping steps...
from the expression

$$a(t) = a_0(t) \left(\frac{3}{2} H_0 t\right)^{2/3}$$

$$a(t=t_0) = a_0(t_0) \left(\frac{3}{2} H_0 t_0\right)^{2/3}$$

$$t_0 = \frac{2}{3H_0} \left(\frac{a(t_0)}{a_0(t_0)}\right)^{3/2}$$

Memorize that (WHY??)

$$t_0 = \frac{2}{3H_0} (\Delta t)^{3/2}$$

Skipping steps (plug this t_0 into the z equation)... WHAT ARE THESE STEPS??? ??? ???

$$z = \left(1 + \frac{3}{2} H_0 \Delta t\right)^{2/3} - 1$$

$$z = (1 + \frac{(3(72km/s/MPc))}{(2)}(10GYr))^{2/3} - 1$$

$$z = (1 + 2.04)^{2/3} - 1$$

$$z = (3.04)^{2/3} - 1$$

$$z = 2.1 - 1$$

$$\boxed{z = 1.1}$$

All done! Next exercise :-)

This is an alternative (partial) solution from an earlier year's handout/file

$$H_0 t = \frac{1}{a_0}(\frac{3}{2}a^{3/2}) - C$$

$$H_0 t = \frac{1}{a_0^{3/2}}\frac{3}{2}a^{3/2} + 0$$

$$H_0 t = \frac{1}{1}\frac{3a^{3/2}}{2a_0^{3/2}} + 0$$

$$H_0 t = \frac{3a^{3/2}}{2a_0^{3/2}} + 0$$

$$\frac{2H_0 t}{3} = \frac{a^{3/2}}{a_0^{3/2}}$$

$$(\frac{2H_0 t}{3})^{2/3} = \frac{a}{a_0}$$

Have memorized the behavior of redshift z as:

$$z + 1 = \frac{a(t_{obs})}{a(t_{emit})}$$

Combine these two equations (for $0 \rightarrow$ emit, and $\_$ for observe)

$$z + 1 = \frac{a(t_{obs})}{a(t_{emit})} = \frac{a}{a_0} = \frac{(\frac{2H_0 t}{3})^{2/3}}{(\frac{2H_0 t_0}{3})^{2/3}}$$

$$z + 1 = \frac{(\frac{2H_0 t}{3})^{2/3}}{(\frac{2H_0 t_0}{3})^{2/3}}$$

$$z + 1 = \frac{(t)^{2/3}}{(t_0)^{2/3}}$$

$$z + 1 = (\frac{(t)}{(t_0)})^{2/3}$$

$$z = (\frac{(t)}{(t_0)})^{2/3} - 1$$

$$z = (\frac{(t_0 + \Delta t)}{(t_0)})^{2/3} - 1$$

$$z = (1 + \frac{(\Delta t)}{(t_0)})^{2/3} - 1$$

WHY DOES $1/t_0 = 3H_0/2$??? ??? ???

$$z = (1 + \frac{(3H_0)}{(2)}\Delta t)^{2/3} - 1$$

Plug in values

$$z = (1 + \frac{(3(72km/s/MPc))}{(2)}(10GYr))^{2/3} - 1$$

$$z = 1.1$$

7.2 2.) (Problem 18.5 in Hartle)

Prompt: The cosmic microwave background radiation has been propagating to us since the universe became transparent at a temperature of approximately 3000 K. Its temperature today is 2.73 K. What is the red-shift, z of the radiation?
Solution:
Memorize and begin with the definition of redshift,

$$z = \frac{a(t_{obs})}{a(t_{emit})} - 1$$

Have memorized the relationship

$$T \propto \frac{1}{a}$$

As an aside, this comes (unintuitively) from:

$$\rho_{rad} \propto a^{-4}$$

$$\rho_{rad} \propto T^{-4}$$

This means between obs and emit a and T are inverse substitutions:

$$z = \frac{\frac{1}{T_{obs}}}{\frac{1}{T_{emit}}} - 1$$

$$z = \frac{T_{emit}}{T_{obs}} - 1$$

Plug in values

$$z = \frac{3000K}{2.73K} - 1$$

$$z = 1099 - 1$$

$$\boxed{z = 1098}$$

7.3 3.) (Problem 18.6 in Hartle)

Prompt: A type Ia supernova has a red-shift, z = 1.1. The observed brightness rises and falls on a timescale of two months. (More precisely let's say the difference in times between when the supernova is at half peak brightness is two months). What is the timescale for the rise and fall in the supernova's rest frame as would be seen by a hypothetical observer close to the supernova and at rest with respect to it?
Solution: Start with the cosmology metric and derive the red-shift

$$ds^2 = -dt^2 + a(t)^2 dx^2$$

The phenomena in light (and null geodesics), so,

$$0 = -dt^2 + a(t)^2 dx^2$$

Isolate distance traveled,

$$dt^2 = a(t)^2 dx^2$$

$$dx^2 = \frac{dt^2}{a(t)^2}$$

$$dx = \frac{dt}{a(t)}$$

$$D = R = \int dx = \int \frac{dt}{a(t)}$$

This is true for any times/distances in an inertial frame, which makes for different integration limits:

$$D = R = \int dx = \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)}$$

(This came from calculating a new integral that is slightly (right) shifted from an original integral: new = old - left + right)

$$\int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)} = \int_{t_e}^{t_o+\Delta t_o} \frac{dt}{a(t)}$$

If Δt is small, than the ratio (and rectangular integral area) between dt/a_1 and dt/a_2 will be constant, so:

How do "small Δt_e and Δt_o" lead to integral removal?:

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_o}{a(t_o)}$$

Isolate Δt_e

$$\Delta t_e = a(t_e)\frac{\Delta t_o}{a(t_o)}$$

$$\Delta t_e = \Delta t_o \frac{a(t_e)}{a(t_o)}$$

$$\Delta t_e = \Delta t_o(\frac{1}{z+1})$$

Plug in values

$$\Delta t_e = (2\ months)(\frac{1}{(1.1)+1})$$

$$\Delta t_e = (2\ months)(\frac{1}{2.1})$$

$$\boxed{\Delta t_e = 0.95\ months}$$

7.4 4.) (Problem 18.8 in Hartle)

Prompt: In Section 9.2 the red-shift of a photon in the Schwarzschild geometry was derived using the conservation law arising from time-translation symmetry. Show that the cosmological red-shift (18.10) can be derived from the space translation symmetry of the metric (18.1) in a similar way.

Solution: Memorize and begin with the Friedmann–Lemaître–Robertson–Walker (FLRW, FRW, RW, or FL) Metric:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

space translation symmetry implies Killing vectors, which are of the form:

$$\xi_x^\mu = (0, 1, 0, 0), \quad \xi_y^\mu = (0, 0, 1, 0), \quad \xi_z^\mu = (0, 0, 0, 1)$$

This implies that

$$a(t)^2 p^x = const.$$

because

$$\xi_x^\mu p^\nu g_{\mu\nu} = const.$$

Therefore,

$$p^x = \frac{const.}{a^2(t)}$$

$$p^x \propto \frac{1}{a^2(t)}$$

We will use this in a minute.
Now consider the momentum as a vector quantity that, for light rays here, is null:

$$0 = p^\mu p^\nu g_{\mu\nu}$$

$$0 = -(p^t)^2 + a(t)^2(p^x)^2$$

$$(p^t)^2 = a(t)^2(p^x)^2$$

$$\sqrt{(p^t)^2 = a(t)^2(p^x)^2}$$

$$p^t = a(t)|p^x|$$

We will use this in a minute.

$$\frac{p^t}{a(t)} = |p^x|$$

$$|p^x| \propto \frac{1}{a(t)}$$

Have memorized and use the energy of a 4-velocity photon (wrt a static observe) as,

$$u^\mu = (1, 0, 0, 0)$$

Memorize the definition/idea that, for a p^μ observed w.r.t. a u^μ, E is defined as:

$$E \equiv -g_{\mu\nu} u^\mu p^\nu = p^t$$

(similarly, get the $p^t \propto a^{-1}(t)$ from the metric with a null (light) vector as:)

$$p^\mu p^\nu g_{\mu\nu} = 0$$

$$-(p^t)^2 + a^2(t)(p^x)^2 = 0$$

$$(p^t)^2 = a^2(t)(p^x)^2$$

$$p^t = a(t)|p^x|$$

$$p^t = a(t)(1/a^2(t))$$

The E is known from Q.M. as well as,

$$E = \hbar\omega$$

Substitute,

$$\omega = \frac{E}{\hbar} = \frac{1}{\hbar}p^t = \frac{1}{\hbar}\left(a(t)|p^x|\right) = \frac{1}{\hbar}\left(a(t)\frac{const.}{a^2(t)}\right) \propto \left(a(t)\frac{1}{a^2(t)}\right)$$

Simplify $a(t)$ terms

$$\omega \propto \frac{1}{a(t)}$$

so,

$$\frac{\omega_1}{\omega_2} = \frac{a(t_1)}{a(t_2)}$$

which is the cosmological redshift derived from space symmetry and Killing vectors.

7.5 5.) (Problem 18.19 in Hartle) De Sitter space

Prompt: Solve the Friedman equation for the scale factor as a function of time for a closed FRW models that have only vacuum energy ρv. Do these models have an initial big bang singularity?

Solution: Begin with constant isolated vacuum energy:

$$\rho_v = \rho = const.$$

Memorize the conservation of energy as:

$$\Delta E = -w$$

$$\frac{d}{dt}(\rho a^3) = -P\frac{d}{dt}(a^3)$$

Use the simplist case:

$$\rho\frac{d}{dt}(a^3) = -P\frac{d}{dt}(a^3)$$

$$\rho = -P$$

Memorize the ______ and vacuum energy density equation as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_v - \frac{1}{a^2}$$

$$\rho_v = \frac{\Lambda}{8\pi}$$

Substitute, solve for a wrt t:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}(\frac{\Lambda}{8\pi}) - \frac{1}{a^2}$$

$$\frac{1}{a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}(\frac{\Lambda}{1}) - \frac{1}{a^2}\right]$$

$$(\dot{a})^2 = \frac{\Lambda}{3}a^2 - 1$$

$$\dot{a} = \sqrt{\frac{\Lambda}{3}a^2 - 1}$$

$$\frac{da}{dt} = \sqrt{\frac{\Lambda}{3}a^2 - 1}$$

$$da = \sqrt{\frac{\Lambda}{3}a^2 - 1dt}$$

$$dt = \frac{1}{\sqrt{\frac{\Lambda}{3}a^2 - 1}}da$$

$$\int dt = \int \frac{1}{\sqrt{\frac{\Lambda}{3}a^2 - 1}}da$$

Solve with a substitution: let,

$$a = \sqrt{\frac{3}{\Lambda}}cosh(x)$$

$$x = arccosh(\sqrt{\frac{\Lambda a^2}{3}})$$

$$da = \sqrt{\frac{3}{\Lambda}}sinh(x)dx$$

$$\int dt = \int \frac{1}{\sqrt{\frac{\Lambda}{3}(\sqrt{\frac{3}{\Lambda}}cosh(x))^2 - 1}}(\sqrt{\frac{3}{\Lambda}}sinh(x)dx)$$

$$\int dt = \int \frac{\sqrt{\frac{3}{\Lambda}}sinh(x)}{\sqrt{\frac{\Lambda}{3}(\sqrt{\frac{3}{\Lambda}}cosh(x))^2 - 1}}dx$$

$$\int dt = \int \frac{\sqrt{\frac{3}{\Lambda}}sinh(x)}{\sqrt{\frac{\Lambda}{3}(\frac{3}{\Lambda}cosh^2(x)) - 1}}dx$$

$$\int dt = \int \frac{\sqrt{\frac{3}{\Lambda}}sinh(x)}{\sqrt{cosh^2(x) - 1}}dx$$

Memorize and use the identity:
$\cosh^2 x - \sinh^2 x = 1$

$$\int dt = \int \frac{\sqrt{\frac{3}{\Lambda}}sinh(x)}{\sqrt{sinh^2(x)}}dx$$

$$\int dt = \int \frac{\sqrt{\frac{3}{\Lambda}}sinh(x)}{sinh(x)}dx$$

$$\int dt = \int \frac{\sqrt{\frac{3}{\Lambda}}}{1}dx$$

$$\int dt = \int \sqrt{\frac{3}{\Lambda}}dx$$

$$\int dt = \sqrt{\frac{3}{\Lambda}} \int dx$$

$$t = \sqrt{\frac{3}{\Lambda}}x$$

$$t = \sqrt{\frac{3}{\Lambda}} \left(arccosh(\sqrt{\frac{\Lambda a^2}{3}}) \right)$$

$$\sqrt{\frac{\Lambda}{3}}t = arccosh \left(\sqrt{\frac{\Lambda a^2}{3}} \right)$$

$$cosh \left(\sqrt{\frac{\Lambda}{3}}t \right) = \sqrt{\frac{\Lambda a^2}{3}}$$

$$cosh^2 \left(\sqrt{\frac{\Lambda}{3}}t \right) = \frac{\Lambda a^2}{3}$$

$$\Lambda a^2 = 3 \cdot cosh^2 \left(\sqrt{\frac{\Lambda}{3}}t \right)$$

$$a^2 = \frac{3}{\Lambda}cosh^2 \left(\sqrt{\frac{\Lambda}{3}}t \right)$$

$$\boxed{a = \sqrt{\frac{3}{\Lambda}}cosh \left(\sqrt{\frac{\Lambda}{3}}t \right)}$$

Plot this a as a function of t. It's a $\cup$ with minimum at $t = 0$; the simplest "bouncing cosmology" (no B.B., no singularity)

A similar alternative derivation: Memorize the equations for the cosmological constant and Friedmann equations:

$$\rho_v = -p_v = 8\pi G\Lambda = const.$$

$$\dot{a}^2 = a^2\frac{\Lambda}{3} - f(k)$$

In a closed FRW model, $f(k) = 1$ and

$$\dot{a}^2 = a^2\frac{\Lambda}{3} - 1$$

$$\dot{a} = \sqrt{a^2\frac{\Lambda}{3} - 1}$$

$$\frac{da}{dt} = \sqrt{a^2\frac{\Lambda}{3} - 1}$$

$$\frac{da}{\sqrt{a^2\frac{\Lambda}{3} - 1}} = dt$$

$$\int \frac{da}{\sqrt{a^2\frac{\Lambda}{3} - 1}} = \int dt = t$$

This integral might be solved with a u-substitution...investigate this... Or memorize the solution as:

$$t = \frac{1}{\sqrt{\Lambda/3}}cosh(t\sqrt{\Lambda/3})$$

The "cosh" function is always positive (with minimum value of 1 and no singularities/asymptotes), no matter the t. The solution implies a collapsing and then expanding cosmology.

7.6 6.) Spatially curved FLRW models (geometry)

Prompt: The spatially curved FLRW models are described by the same line element (1) but now

$$dL^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

where k = -1, 0, 1. Change coordinates for k = -1 such that the metric takes the form

$$d\chi^2 + sinh^2\chi d\Omega^2$$

Show that this metric is a metric of a hyper-surface described by the equation

$$-T^2 + X^2 + Y^2 + Z^2 = -1$$

Do the same for k = +1, i.e. change coordinates to obtain

$$d\chi^2 + sin^2\chi d\Omega^2$$

which is the metric of the three-sphere

$$X^2 + Y^2 + Z^2 + W^2 = 1$$

Solution: Note: Before using the FLWR metric, consider it's origins–The Einstein Equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$
$$geometry = matter$$

The $T_{\mu\nu}$ is complex. Simplify it: Assume homogeneous & isotropic universe. Now the $T_{\mu\nu}$ makes spacial slices with six Killing vectors (J_1, J_2, J_3, P_1, P_2, P_3); three rotations, three translations. This gives rise to a spacial (!= f(t)) metric, here shown with r, θ, & ϕ (x,y,z is messy):

$$\gamma_{ab} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix}$$

This can go into the spacial corner of a 4x4 $g_{\mu\nu}$ metric with $g_{tt} = -1$; but let's just consider the line element dL from the spacial γ space alone for now:

$$dL^2 = \frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta \; d\phi^2$$

Now consider the three possible cases of k: k=-1 $\rightarrow$ hyperboloid space-time; k=0 $\rightarrow$ flat space-time (ignore: k=1 $\rightarrow$ 3-sphere). Use the 4D metric to set up the change of coordinates:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-kr^2}a^2_{(t)} & 0 & 0 \\ 0 & 0 & r^2 a^2_{(t)} & 0 \\ 0 & 0 & 0 & r^2 a^2_{(t)} sin^2\theta \end{pmatrix}$$

First case: k=0

$$g_{\mu\nu,1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & r^2a^2(t) & 0 \\ 0 & 0 & 0 & r^2a^2(t)sin^2 \end{pmatrix}$$

$$ds_1^2 = -dt^2 + a^2(t)dr^2 + a^2(t)r^2d\Omega^2$$

First case: k=-1

$$g_{\mu\nu,2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1+r^2} & 0 & 0 \\ 0 & 0 & r^2a^2(t) & 0 \\ 0 & 0 & 0 & r^2a^2(t)sin^2 \end{pmatrix}$$

$$ds_2^2 = -dt^2 + \frac{a^2(t)}{1+r^2}dr^2 + a^2(t)r^2d\Omega^2$$

Consider the space (without time and (increasing) scale factor)

$$dL_2^2 = \frac{1}{1+r^2}dr^2 + r^2d\Omega^2$$

Let,

$$r = \sinh\chi$$

$$dr = \cosh\chi d\chi$$

So,

$$dL_2^2 = \frac{1}{1+(\sinh\chi)^2}(\cosh\chi d\chi)^2 + (\sinh\chi)^2d\Omega^2$$

$$dL_2^2 = \frac{\cosh^2\chi}{1+\sinh^2\chi}(d\chi)^2 + \sinh^2\chi\ d\Omega^2$$

Memorize and use the identity: $\cosh^2 x - \sinh^2 x = 1$

$$dL_2^2 = \frac{\cosh^2\chi}{\cosh^2\chi}(d\chi)^2 + \sinh^2\chi\ d\Omega^2$$

$$dL_2^2 = d\chi^2 + \sinh^2\chi\ d\Omega^2$$

Now find a good embedding into (T, X, Y, Z), which is an auxiliary space-time. Claim: This is a hyperboloid (and is satisfied if T,X,Y,Z is a 4D Minkowski space),

$$-T^2 + X^2 + Y^2 + Z^2 = -1$$

Use the following definitions (of a hyperboliod):

$$T = \cosh\chi, \quad X = \sin\chi\sin\theta\cos\phi, \quad Y = \sinh\chi\sin\theta\sin\phi, \quad Z = \sinh\chi\cos\theta$$

And the metric:

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$$

First compute the derivatives of (T, X, Y, Z):

$$dT = \sinh\chi d\chi$$

$$dX = \cos\chi\cos\theta(-1)\sin\phi\ d\chi d\theta d\phi$$
$$dY = \cosh\chi\cos\theta\cos\phi\ d\chi d\theta d\phi$$
$$dZ = \cosh\chi(-1)\sin\theta\ d\chi d\theta$$

Substitute,

$$ds^2 = -(\sinh\chi d\chi)^2 + (-\cos\chi\cos\theta\sin\phi\ d\chi d\theta d\phi)^2 + (\cosh\chi\cos\theta\cos\phi\ d\chi d\theta d\phi)^2 + (-\cosh\chi\sin\theta\ d\chi d\theta)^2$$

$$ds^2 = -\sinh^2\chi d\chi^2 + \cos^2\chi\cos^2\theta\sin^2\phi\ d\chi^2 d\theta^2 d\phi^2 + \cosh^2\chi\cos^2\theta\cos^2\phi\ d\chi^2 d\theta^2 d\phi^2 + \cosh^2\chi\sin^2\theta\ d\chi^2 d\theta^2$$

Pull out the χ, θ to isolate the ϕ,

$$ds^2 = -\sinh^2\chi d\chi^2 + \cos^2\chi\cos^2\theta d\chi^2 d\theta^2(\sin^2\phi + \cos^2\phi)d\phi^2 + \cosh^2\chi\sin^2\theta\ d\chi^2 d\theta^2$$

DOES THE $d\phi$ TERM ALSO GET ABSORBED BY THE 1??? ??? ???

$$ds^2 = -\sinh^2\chi d\chi^2 + \cos^2\chi\cos^2\theta d\chi^2 d\theta^2(1) + \cosh^2\chi\sin^2\theta\ d\chi^2 d\theta^2$$

HOW TO PROCEED FROM HERE... NON-LINEAR DEPENDENT TERMS... ??? ??? ??? ...
...
...
This should end with recovery of $dL_2^2 = d\chi^2 + \sinh^2\chi\ d\Omega^2$
Thus ends the class-given solution.

An old solution (from another year's notes)
Let

$$r = sinh\chi$$

Do a change of variables within the give

$$dL^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$
$$dr = cosh\chi d\chi$$
$$dL^2 = \frac{(cosh\chi d\chi)^2}{1-k(sinh\chi)^2} + (sinh\chi)^2 d\Omega^2$$

+

7.7 7.) (Problem 18.24 in Hartle) The Einstein static universe

Prompt: Consider a closed ($k = +1$) FRW model containing a matter density ρm, a vacuum energy density corresponding to a positive cosmological constant Λ, and no radiation.
(a) Show that for a given value of Λ, there is a critical value of ρm for which the scale factor does not change with time. Find this value.
(b) What is the spatial volume of this universe in terms of Λ?
(c) If ρm differs slightly from this value, the scale factor will vary in time. Does the evolution remain close to the static universe or diverge from it?

Solution:
(a)
A closed universe implies

$$k = 1$$

A matter and vacuum energy density imply:

$$\rho_m = \rho_{m,0}\frac{a_0^3}{a^3}$$

$$\rho_v = 8\pi G\Lambda$$

$$\rho_{total} = \rho = \rho_m + \rho_v$$

Memorize and use the Friedmann and ___ equations of the form ($G = 1$):

$$H^2 = \frac{\Lambda}{3} + \frac{8\pi}{3}\rho_m - \frac{1}{a^2}$$
$$\dot{a}^2 + V_{eff}(a) = -1$$

Rewrite,

$$V_{eff}(a) = -1 - \dot{a}^2$$

HOW DOES THIS BECOME??? ??? ???

$$V_{eff}(a) = -a^2\frac{\Lambda}{3} - a^2\frac{8\pi}{3}$$

FOR WHAT REASON??? ??? (STATIC CONSTANT UNIVERSE?)

$$0 = -a^2\frac{\Lambda}{3} - a^2\frac{8\pi}{3}$$

$$0 = \frac{\Lambda}{3} + \frac{8\pi}{3}$$

SHOULD THERE BE A ρ_v TERM IN THERE??? ??? ???

$$0 = \frac{\Lambda}{3} + \frac{8\pi}{3}\rho_v$$

$$\rho_v = -\frac{\Lambda}{8\pi}$$

Recall $\rho = \rho_m + \rho_v$, and get ρ_m now

$$\rho_m \propto \frac{1}{a^3}$$

Therefore

$$\rho_m = \rho_{m,0}(\frac{a_0}{a})^3$$

Plug this into the Veff equation above

$$V_{eff}(a) = -a^2\frac{\Lambda}{3} - \frac{8\pi\rho_m}{3}a^2 = -a^2\frac{\Lambda}{3} - \frac{8\pi\rho_{m,0}}{3a^3}a^2a_0^3$$

$$V_{eff}(a) = -a^2\frac{\Lambda}{3} - \frac{8\pi\rho_{m,0}}{3a}a_0^3$$

Find a static stable point

$$V'_{eff}(a) = 0 = \frac{d}{da}\left[-a^2\frac{\Lambda}{3} - \frac{8\pi\rho_{m,0}}{3a}a_0^3\right]$$

$$0 = -2a\frac{\Lambda}{3} - \frac{d}{da}\left[\frac{8\pi\rho_{m,0}}{3a}a_0^3\right]$$

$$0 = -2a\frac{\Lambda}{3} - \frac{8\pi\rho_{m,0}}{3}a_0^3\frac{d}{da}\frac{1}{a}$$

HOW DOES a BECOME a_0 HERE??? ??? ???

$$0 = -2a\frac{\Lambda}{3} - \frac{8\pi\rho_{m,0}}{3}a_0^3\left[(-1)a_0^{-2}\right]$$

$$0 = -2a_0\frac{\Lambda}{3} + \frac{8\pi\rho_{m,0}}{3}a_0$$

$$\frac{-3}{a_0}\left[0 = -2a_0\frac{\Lambda}{3} + \frac{8\pi\rho_{m,0}}{3}a_0\right]$$

$$0 = 2\Lambda - 8\pi\rho_{m,0}$$

$$8\pi\rho_{m,0} = 2\Lambda$$

$$4\pi\rho_{m,0} = \Lambda$$

$$\boxed{\rho_{m,0} = \frac{\Lambda}{4\pi}}$$

(b)
Use the dL and effective potential from problem 6:

$$dL^2 = a_0^2\left(\frac{1}{1-r}dr^2 + r^2 d\Omega_2^2\right)$$

$$\dot{a}|_{a=a_0}^{r} + V_{eff}(a = a_0) = -1$$

$$\dot{a}|_{a_0}^{r} + V_{eff}(a_0) = -1$$

$$0 + V_{eff}(a_0) = -1$$

$$V_{eff}(a_0) = -1$$

HOW DOES THIS IMPLY??? ??? ???

$$a_0 = \frac{1}{\sqrt{\Lambda}}$$

Maybe use:

$$a_0^2(d\chi^2 + \sin^{\chi} d\Omega_2^2)$$

$$g_{\mu\nu} = a_0^2\begin{pmatrix}1 & 0 & 0\\ 0 & \sin^2\chi & 0\\ 0 & 0 & \sin^2\chi\sin^2\theta\end{pmatrix}$$

Memorize or see homework (sheet) 1, exercise 9 for:

$$Volume = V = \int d^3 \times \sqrt{-g}$$

$$Volume = V = 2\pi^2 R^3$$

$$Volume = V = 2\pi^2\Lambda^{-3/2}$$

$$V_{eff} = -a^2\frac{\Lambda}{3} - \frac{8\pi}{3}\rho m\ a^2$$

$$Volume = V = \int \sin^2\chi \sin\theta d\chi d\theta d\phi$$

$$Volume = V = (1)(\pi)(2\pi)$$

$$Volume = V = 2\pi^2$$

(C)
Unlike before, now:

$$\rho_m < \Lambda/4\pi G$$

$$\rho_m = \Lambda/4\pi G$$

$$\rho_m > \Lambda/4\pi G$$

Plot Veff(a) wrt a. This yields 3 $\cap$ functions (from the ρ's above) with 3 maxima–one above -1, one on -1, and 1 below -1.

The large ρ_m leads to the smallest Veff. The maximum is always below -1. Expansion is always happening (and started from an $a \to 0$, B.B. beginning).

The middle (critical) ρ_m leads to the Critical static solution. The maximum reaches -1 at one point. DOES EXPANSION ASYMTOPTE TO STATIC AT THE POINT???

The small ρ_m solution leads to the largest Veff, which crosses the -1 value at two points. The function above -1 is nonphysical (imaginary $i = \sqrt{-1}$ answers), so the universe approaches Veff=-1 along the left less than -1 line, and then recedes in the opposite direction towards a=0 (big-bang, a maximum size, then big crunch); or along the right less than -1 line (shrinks for some time, reaches a minimum size, then expands forever).

The large ρ and the two small ρ solutions are all unstable, so leaving from the middle ρ value is an unstable change

An old solution (from another year's notes)
Substitute and simplify so that we get a particle-geodesic-like equation of motion.

$$H^2 = \frac{\Lambda}{3} + \frac{8\pi G}{3}(\rho_{m,0}\frac{a_0^3}{a^3}) - \frac{1}{a^2}$$

$$H^2 = \frac{\Lambda}{3} + \frac{8\pi G\rho_{m,0}}{3}\frac{a_0^3}{a^3} - \frac{1}{a^2}$$

... ... HOW DOES THIS GO TO THE EQUATION OF:

$$\dot{a}^2 - a^2\frac{\Lambda}{3} - \frac{8\pi G a_0^3\rho_{m,0}}{3a} = -1$$

Make the equation look simple: let, $V_{eff}(a) = -a^2\frac{\Lambda}{3} - \frac{8\pi G a_0^3\rho_{m,0}}{3a}$, and

$$\dot{a}^2 + V_{eff}(a) = -1$$

This equation is like a particle-geodesic equation of motion. Memorize that there a static solution at the point where the potential is at a maximum:

$$V'_{eff,\ sol\underline{n}}(a = a_{maximum} = a_0) = 0$$

There is already an a_0 in the previous equation, DOES THIS POSE AN ISSUE??? ??? ??? RENAME THE SOLUTION TO SOMETHING OTHER THAN a_0???

$$V'_{eff,\ sol^{\underline{n}}}(a_0) = 0$$

$$(-a^2\frac{\Lambda}{3} - \frac{8\pi G a_0^3 \rho_{m,0}}{3a})'_{(a_0)} = 0$$

$$(-a^2\frac{\Lambda}{3} - \frac{8\pi G a_0^3 \rho_{m,0}}{3a})'_{(a_0)} = 0$$

Solve for the energy density of mass, initial $\rho_{m,0}$:

(b)

Memorize and use the volume of a 3-sphere (first homework sheet, exercise 9):

$$V = 2\pi R^3$$

Get R from the particle-geodesic-like equation of motion and the metric of the universe:

$$\dot{a}^2 + V_{eff}(a) = -1$$

$$ds^2 = -dt^2 + \frac{1}{\Lambda}d\Omega^2_{3\text{-}sphere} = -dt^2 + R^2 d\Omega^2_{3\text{-}sphere}$$

Using the previous *static* solution, $\dot{a} = 0$, the top equation becomes:

$$0 + V_{eff}(a_0) = -1$$

$$V_{eff}(a_0) = -1$$

$$-a_0^2\frac{\Lambda}{3} - \frac{8\pi G a_0^3 \rho_{m,0}}{3a_0} = -1$$

HOW DOES THIS LEAD TO THE EXTREMELY SIMPLE: ??? ??? ???

$$a_0 = \frac{1}{\sqrt{\Lambda}}$$

(AND IS SOLVING FOR a_0 NECESSARY???) Get R as $f(\Lambda)$:

$$ds^2 = -dt^2 + \frac{1}{\Lambda}d\Omega^2_{3\text{-}sphere} = -dt^2 + R^2 d\Omega^2_{3\text{-}sphere}$$

$$\frac{1}{\Lambda}d\Omega^2_{3\text{-}sphere} = R^2 d\Omega^2_{3\text{-}sphere}$$

$$\frac{1}{\Lambda} = R^2$$

$$R = \sqrt{\frac{1}{\Lambda}}$$

$$R = \frac{1}{\sqrt{\Lambda}}$$

Calculate the volume,

$$V = 2\pi R^3 = 2\pi(\frac{1}{\sqrt{\Lambda}})^3$$

$$V = 2\pi(\frac{1}{\Lambda^{3/2}})$$

$$\boxed{V = 2\pi\Lambda^{-3/2}}$$

7.8 Unnumbered: Derive Hubble's Law from the RW metric

Prompt:

Solution: Memorize and start with the length, l

$$l_{(r,t)} = \int_0^r \frac{a \cdot dr}{\sqrt{1 - kr^2}}$$

Take the derivative wrt time, t

$$\frac{d}{dt} l_{(r,t)} = \frac{d}{dt} \int_0^r \frac{adr}{\sqrt{1 - kr^2}}$$

a is a function of t, but neither depends on r, so it comes outside of the integral

$$\frac{dl}{dt} = \frac{da}{dt} \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

$$\dot{l} = \dot{a} \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

$$\dot{l} = \frac{a}{a} \dot{a} \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

$$\dot{l} = \frac{1}{a} \dot{a} \int_0^r \frac{a \cdot dr}{\sqrt{1 - kr^2}}$$

The integral term is what we started with, so

$$\dot{l} = \frac{1}{a} \dot{a}(l)$$

$$\dot{l} = \frac{\dot{a}}{a} \cdot l$$

Substitute appropriate terms/variables as

$$\dot{l} = v_H, \ and \ \frac{\dot{a}}{a} = H$$

$$\boxed{v_H = H \cdot l}$$

This is the Hubble law for proper distance

7.9 Unnumbered: Light rays in cosmology (redshift?)

Prompt:

Solution: Memorize and begin with RW Universe Metric

$$ds^2 = -dt^2 + a_{(t)}^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right)$$

The open, flat, and closed universes

$$k = 1, 0, -1$$

And the Friedmann equation.

$$\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - kr$$

Consider light emitted and observed at different times, and wavelengths (frequencies), $t_{obs}, \lambda_{obs}, \omega_{obs}, t_e, \lambda_e, \omega_e$ Let the light be emitted (and observed) in regular pulses, the intervals are $\delta t_e, \delta t_{obs}$ Now solve the set of equations for these δt's. It's light, so, $ds = 0$

$$ds^2 = -dt^2 + a_{(t)}^2(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2)$$

$$0 = -dt^2 + a_{(t)}^2(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2)$$

$$\sqrt{dt^2 = a_{(t)}^2(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2)}$$

Consider simple, radially traveling light, so $d\Omega = 0$

$$dt = \sqrt{a_{(t)}^2(\frac{dr^2}{1-kr^2} + (0))}$$

$$dt = a_{(t)}\sqrt{\frac{dr^2}{1-kr^2}}$$

$$\frac{dt}{a_{(t)}} = \frac{dr}{\sqrt{1-kr^2}}$$

Ok, set the limits on the integral at an initial time and a pulse later time:

$$\int_{t_e}^{t_{obs}} \frac{dt}{a_{(t)}} = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

$$\int_{t_e+\delta t_e}^{t_{obs}+\delta t_{obs}} \frac{dt}{a_{(t)}} = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

The right hand sides of the two equations above are equal, so

$$\int_{t_e}^{t_{obs}} \frac{dt}{a_{(t)}} = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e+\delta t_e}^{t_{obs}+\delta t_{obs}} \frac{dt}{a_{(t)}}$$

$$\int_{t_e}^{t_{obs}} \frac{dt}{a_{(t)}} = \int_{t_e+\delta t_e}^{t_{obs}+\delta t_{obs}} \frac{dt}{a_{(t)}}$$

$$\int_{t_e}^{t_{obs}} \frac{dt}{a_{(t)}} - \int_{t_e+\delta t_e}^{t_{obs}+\delta t_{obs}} \frac{dt}{a_{(t)}} = 0$$

multiply by negative one

$$-\int_{t_e}^{t_{obs}} \frac{dt}{a_{(t)}} + \int_{t_e+\delta t_e}^{t_{obs}+\delta t_{obs}} \frac{dt}{a_{(t)}} = 0$$

maybe memorize this (boundary) transformation

$$\int_{t_e}^{t_e+\delta t_e} \frac{dt}{a_{(t)}} - \int_{t_{obs}}^{t_{obs}+\delta t_{obs}} \frac{dt}{a_{(t)}} = 0$$

certainly memorize this simplification

$$\frac{1}{a_{(t_e)}}\delta t_e - \frac{1}{a_{(t_{obs})}}\delta t_{obs} = 0$$

$$\frac{\delta t_e}{a_{(t_e)}} = \frac{\delta t_{obs}}{a_{(t_{obs})}}$$

$$\delta t_e = \frac{a_{(t_e)}}{a_{(t_{obs})}}\delta t_{obs}$$

$$\frac{\delta t_e}{\delta t_{obs}} = \frac{a_{(t_e)}}{a_{(t_{obs})}}$$

Memorize and recognize that the redshift Z is defined inversely in

$$1 + Z = \frac{\delta t_{obs}}{\delta t_e}$$

so

$$\boxed{Z = \frac{a_{(t_{obs})}}{a_{(t_e)}} - 1 = \frac{\delta t_{obs}}{\delta t_e} - 1}$$

7.10 Unnumbered: Express distance to horizon as a function of time. Evaluate this in FLRW matter-dominated universe

Prompt:

Solution:
Memorize starting with coordinates in FLRW model where light rays move at 45 degree angles.

$$dt = a_{(t)} \cdot d\eta$$

This is also known as "conformal time." Originally, the line element was

$$ds^2 = a^2_{(t)}[-dt^2 + dr^2 + r^2 d\Omega^2]$$

now in terms of η it is

$$ds^2 = a^2_{(t)}[-(a_{(t)} \cdot d\eta)^2 + dr^2 + r^2 d\Omega^2]$$

HOW DOES THIS BECOME??? ??? ???

$$ds^2 = a^2_{(\eta)}[-d\eta^2 + dr^2 + r^2 d\Omega^2]$$

Now simplify: light rays, $ds = 0$, moving radially $d\Omega = 0$

$$0 = a^2_{(\eta)}[-d\eta^2 + dr^2 + 0]$$

$$\frac{1}{a^2}[0 = a^2(-d\eta^2 + dr^2)]$$

$$0 = -d\eta^2 + dr^2$$

$$d\eta^2 = dr^2$$

$$dr = d\eta$$

Now plug back in the original value of η as $f(t)$

$$\int dr = \int \frac{dt'}{a_{(t')}}$$

Use the integral limits

$$\int_0^{r_{max}=r_{horizon}} dr = \int_0^t \frac{dt'}{a_{(t')}}$$

$$r_{horizon} = \int_0^t \frac{dt'}{a_{(t')}}$$

The horizon is proportional to time (more time for B.B. to reach observer). Memorize that the physical distance, d to the horizon at an observed time t is:

$$d = a \cdot r$$

$$d_{horizon\ (t)} = a_{(t)} \cdot r_{horizon}$$

$$d_{horizon\ (t)} = a_{(t)} \cdot \int_0^t \frac{dt'}{a_{(t')}}$$

Evaluate this in FLRW matter-dominated universe This implies

$$\Omega_m = 1,\ and\ \Omega_r = \Omega_v = 0$$

COPY AND PASTE FROM EARLIER EXERCISE

$$a(t) = (\frac{t}{t_0})^{2/3}$$

WHY CAN WE CHANGE COORDINATES LIKE THIS??? use, change t' to t, and substitute into both a's of

$$d_{horizon\ (t)} = a_{(t)} \cdot \int_0^t \frac{dt}{a_{(t')}}$$

to get

$$d_{horizon\ mat\ (t)} = ((\frac{t}{t_0})^{2/3}) \cdot \int_0^t \frac{dt}{(\frac{t}{t_0})^{2/3}}$$

Do calculus. The t_0s are constant and cancel out

$$d_{horizon\ (t)} = (t^{2/3}) \cdot \int_0^t \frac{dt}{(t)^{2/3}}$$

$$d_{horizon\ (t)} = t^{2/3} \int_0^t t^{-2/3} dt$$

$$d_{horizon\ (t)} = t^{2/3} (\frac{t^{1/3}}{1/3}) \Big|_0^t$$

$$d_{horizon\ (t)} = t^{2/3} \cdot \frac{3}{1} (t^{1/3}) \Big|_0^t$$

$$d_{horizon\ (t)} = t^{2/3}(3)(t^{1/3} - 0)$$

$$d_{horizon\ (t)} = 3t^{2/3}(t^{1/3})$$

$$\boxed{d_{horizon\ mat\ (t)} = 3t}$$

Evaluate this in FLRW radiation-dominated universe, This implies

$$\Omega_r = 1,\ and\ \Omega_m = \Omega_v = 0$$

COPY AND PASTE FROM EARLIER EXERCISE

$$a(t) = (t/t_0)^{1/2}$$

use the distance of horizon equation

$$d_{horizon\ rad\ (t)} = a_{(t)} \cdot \int_0^t \frac{dt}{a_{(t')}}$$

turn t' into t and substitute. WHY??? ??? ???

$$d_{horizon\ (t)} = (\frac{t}{t_0})^{1/2} \cdot \int_0^t \frac{dt}{(\frac{t}{t_0})^{1/2}}$$

Do calculus. The t_0s are constant and cancel out

$$d_{horizon\ (t)} = (\frac{t}{1})^{1/2} \cdot \int_0^t \frac{dt}{(\frac{t}{1})^{1/2}}$$

$$d_{horizon\ (t)} = (t)^{1/2} \cdot \int_0^t t^{-1/2} dt$$

$$d_{horizon\ (t)} = t^{1/2} \cdot \frac{t^{1/2}}{\frac{1}{2}}\Big|_0^t$$

$$d_{horizon\ (t)} = 2t^{1/2} \cdot (t^{1/2} - 0)$$

$$\boxed{d_{horizon\ rad\ (t)} = 2t}$$

7.11 Unnumbered: Current universe age and size as function of (now) Hubble constant in matter-dominated universe

Prompt:

Solution: Memorize matter-dominated universe's scale factor

$$a_{(t)} = (t/t_0)^{2/3}$$

$$\frac{d}{dt}[a_{(t)} = (t/t_0)^{2/3}]$$

$$\frac{da}{dt} = \frac{2}{3t_0^{2/3}}(t)^{\frac{2}{3}-1}]$$

$$\frac{da}{dt} = \frac{2}{3t_0^{2/3}} t^{-1/3}$$

$$\dot{a} = \frac{2t^{-1/3}}{3t_0^{2/3}}$$

$$\frac{1}{a}[\dot{a} = \frac{2t^{-1/3}}{3t_0^{2/3}}]$$

$$\frac{\dot{a}}{a} = \frac{2t^{-1/3}}{3t_0^{2/3}} \cdot \frac{1}{a}$$

$$\frac{\dot{a}}{a} = \frac{2t^{-1/3}}{3t_0^{2/3}} \cdot \frac{1}{(t/t_0)^{2/3}}$$

$$\frac{\dot{a}}{a} = \frac{2t^{-1/3}}{3t_0^{2/3}} \cdot (\frac{t_0}{t})^{2/3}$$

$$\frac{\dot{a}}{a} = \frac{2t^{-1/3}}{3} \cdot (\frac{1}{t})^{2/3}$$

$$\frac{\dot{a}}{a} = \frac{2t^{-1/3}}{3t^{2/3}}$$

$$\frac{\dot{a}}{a} = \frac{2}{3t}$$

Memorize $\dot{a}/a = H$, the Hubble constant

$$H = \frac{2}{3t}$$

$$H_0 = \frac{2}{3t_0}$$

$$\frac{3H_0}{2} = \frac{1}{t_0}$$

$$\frac{2}{3H_0} = t_0$$

How to end this exercise??? ??? ??? Why is $H_0 = 1/t_H$???

$$\boxed{t_0 = \tfrac{3}{2} t_H}$$

Combine this with the boxed answer before of $d_{horizon\ mat\ (t)} = 3t$ for

$$d_{horizon\ mat\ (t_0)} = 3t_0$$

$$d_{horizon\ mat\ (t_0)} = 3(\frac{2}{3H_0})$$

$$d_{horizon\ mat\ (t_0)} = \frac{2}{H_0}$$

7.12 Unnumbered: Show energy density +3p is positive in FLRW models

Prompt:

Solution: Show that in FLRW models if "$\rho + 3p$" is always positive, there will always be a singularity at some time in the past. Is this the case for our universe.
Have memorized and start with the **fluid equation** in density form,

$$\dot{\rho} + \frac{3\rho\dot{a}}{a} + \frac{3P\dot{a}}{c^2 a} = 0$$

and the **Friedmann** equation,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

rewrite the Friedmann equation,

$$(\dot{a})^2 = \frac{8\pi G\rho}{3}a^2 - kc^2$$

Derivative

$$\frac{d}{dt}[\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - kc^2]$$

$$2(\dot{a})\frac{d}{dt}(\dot{a}) = \frac{d}{dt}(\frac{8\pi G\rho}{3}a^2 - kc^2)$$

$$2(\dot{a})\frac{d}{dt}(\dot{a}) = \frac{d}{dt}(\frac{8\pi G\rho}{3}a^2) - \frac{d}{dt}(kc^2)$$

$$2(\dot{a})\frac{d\dot{a}}{dt} = \frac{8\pi G}{3}\frac{d}{dt}(\rho a^2) - (0)$$

$$2(\dot{a})\ddot{a} = \frac{8\pi G}{3}(\frac{d\rho}{dt}(a^2) + \rho(2a(\frac{da}{dt}))$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}(a^2) + \rho(2a(\dot{a}))$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\dot{a})$$

Rewrite the fluid equation

$$\dot{\rho} = -\frac{3\rho\dot{a}}{a} - \frac{3P\dot{a}}{c^2 a}$$

Substitute this into the derivative of the Friedmann equation,

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}((-\frac{3\rho\dot{a}}{a} - \frac{3P\dot{a}}{c^2 a})a^2 + 2\rho a\dot{a})$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}((-3\rho a\dot{a} - \frac{3Pa\dot{a}}{c^2}) + 2\rho a\dot{a})$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(-3\rho a\dot{a} - \frac{3Pa\dot{a}}{c^2} + 2\rho a\dot{a})$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(-\rho a\dot{a} - \frac{3Pa\dot{a}}{c^2})$$

$$\frac{1}{\dot{a}}[2\dot{a}\ddot{a} = -\frac{8\pi G}{3}(\rho a\dot{a} + \frac{3Pa\dot{a}}{c^2})]$$

$$2\ddot{a} = -\frac{8\pi G}{3}(\rho a + \frac{3Pa}{c^2})$$

$$\frac{1}{2a}[2\ddot{a} = -\frac{8\pi G}{3}(\rho a + \frac{3Pa}{c^2})]$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3 \cdot 2a}(\rho a + \frac{3Pa}{c^2})$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\frac{\rho a}{a} + \frac{3Pa}{c^2 \cdot a})$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + \frac{3P}{c^2})$$

Let units be natural, $G = c = 1$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P)$$

This is the acceleration equation. A positive $(\rho + 3P)$ means $\ddot{a}$ will be negative; this implies there was a singularity in the past.

7.13 Unnumbered: Show what happens when the universe is flat, and matter-dominated

Prompt:

Solution: Memorize and start with this (the approximate solution for the Friedmann-Lemaitre equation for a flat universe with Radiation, Matter and Vacuum density):

$$\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - \frac{k}{a^2}$$

it's flat, so $k = 0$

$$\dot{a}^2 = \frac{8\pi G\rho}{3}a^2$$

$$\dot{a}^2 - \frac{8\pi G\rho}{3}a^2 = 0$$

$$\frac{1}{2H_0^2}[\dot{a}^2 - \frac{8\pi G\rho}{3}a^2 = 0]$$

$$\frac{1}{2H_0^2}\dot{a}^2 - \frac{1}{2H_0^2}\frac{8\pi G\rho}{3}a^2 = 0$$

$$\frac{\dot{a}^2}{2H_0^2} - \frac{8\pi G\rho a^2}{2H_0^2(3)} = 0$$

rearrange

$$\frac{\dot{a}^2}{2H_0^2} - (\frac{8\pi G}{3H_0^2})\frac{\rho a^2}{2} = 0$$

Now define things: let,

$$\frac{1}{\rho_{crit}} = \frac{8\pi G}{3H_0^2}$$

$$U_{eff} = -\frac{\rho a^2}{2\rho_{crit}} = -\frac{8\pi G}{3H_0^2}\frac{\rho a^2}{2} = -\frac{4\pi G \rho a^2}{3H_0^2}$$

substitute them in to get

$$\frac{\dot{a}^2}{2H_0^2} + U_{eff(a)} = \frac{\dot{a}^2}{2H_0^2} - \frac{\rho a^2}{2\rho_{crit}} = 0$$

memorize the ratio of $\frac{\rho}{\rho_{crit}}$ is

$$\frac{\rho}{\rho_{crit}} = \Omega_\nu + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^2}$$

plug this into the "Ueff" above and get the similar

$$U_{eff} = -\frac{\rho a^2}{2\rho_{crit}} = -\frac{a^2}{2}(\Omega_\nu + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^2}) = -\frac{8\pi G}{3H_0^2}\frac{\rho a^2}{2} = -\frac{4\pi G \rho a^2}{3H_0^2}$$

keep the parts we'll use after the simplifications

$$-\frac{a^2}{2}(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^2}) = -\frac{4\pi G \rho a^2}{3H_0^2}$$

make the first simplification: matter dominated universe. This means $\Omega_r = \Omega_v = 0$ and $\Omega_m = 1$, so,

$$-\frac{a^2}{2}((0) + \frac{(1)}{a^3} + \frac{(0)}{a^2}) = -\frac{4\pi G \rho a^2}{3H_0^2}$$

$$-\frac{a^2}{2}(\frac{(1)}{a^3}) = -\frac{4\pi G \rho a^2}{3H_0^2}$$

$$-\frac{1}{2a} = -\frac{4\pi G \rho a^2}{3H_0^2}$$

$$U_{eff} = -\frac{1}{2a} = -\frac{4\pi G \rho a^2}{3H_0^2}$$

plug this $U_{eff,matter}$ into the flat Friedmann equation

$$\frac{\dot{a}^2}{2H_0^2} + U_{eff(a)} = 0 = \frac{\dot{a}^2}{2H_0^2} + (-\frac{1}{2a})$$

now do calculus, for a

$$\frac{\dot{a}^2}{2H_0^2} = \frac{1}{2a}$$

$$(\frac{da}{dt})^2 = \frac{2H_0^2}{2a}$$

$$\sqrt{(\frac{da}{dt})^2 = \frac{H_0^2}{a}}$$

$$(\frac{da}{dt}) = \frac{H_0}{\sqrt{a}}$$

$$\sqrt{a}da = H_0 dt$$

$$\int a^{1/2} \cdot da = \int H_0 \cdot dt$$
$$\frac{3}{2}a^{3/2} = H_0(t + constant)$$
$$a^{3/2} = \frac{2}{3}H_0(t + C)$$
$$a = (\frac{2}{3}H_0(t + C))^{2/3}$$
$$a = (\frac{2H_0t}{3} + \frac{2H_0C}{3})^{2/3}$$
$$a = (Dt + E)^{2/3}$$
$$\boxed{a_{(t)} \sim t^{2/3}}$$

Now consider a radiation dominated universe Mathematically, this means that

$$\Omega_r = 1$$

Have memorized and use:

$$U_{eff,rad} = -\frac{1}{2a^2}$$

Have memorized the general definition of $U_{eff} = -\dot{a}^2/2H_0^2$ and substitute it in,

$$-\frac{\dot{a}^2}{2H_0^2} = -\frac{1}{2a^2}$$

$$2H_0^2a^2[-\frac{\dot{a}^2}{2H_0^2} = -\frac{1}{2a^2}]$$
$$-\frac{a^2\dot{a}^2}{1} = -\frac{H_0^2}{1}$$
$$\sqrt{a^2\dot{a}^2 = H_0^2}$$
$$a\dot{a} = H_0$$
$$a\frac{da}{dt} = H_0$$
$$a \cdot da = H_0 \cdot dt$$
$$\int a \cdot da = \int H_0 \cdot dt$$
$$\int a \cdot da = H_0 \int dt$$
$$\frac{1}{2}a^2 = H_0(t + constant)$$

Consider time, assume it goes from t_0 to t

$$\frac{1}{2}a^2 = H_0t|_{t_0}^{t}$$

$$a^2 = 2H_0(t - t_0)$$
$$a = (2H_0(t - t_0))^{1/2}$$
$$a = (2H_0 t - 2H_0 t_0)^{1/2}$$
$$a = (Bt - D)^{1/2}$$
$$a_{(t)} \sim t^{1/2}$$

Now consider a vacuum dominated universe Mathematically, this means that

$$\Omega_v = 1$$

have memorized and use:

$$U_{eff,vac} = -\frac{a^2}{2}$$

Have memorized the general definition of $U_{eff} = -\dot{a}^2/2H_0^2$ and substitute it in,

$$-\frac{\dot{a}^2}{2H_0^2} = -\frac{a^2}{2}$$

$$2\dot{a}^2 = 2H_0^2 a^2$$

$$\frac{\dot{a}^2}{a^2} = H_0^2$$

$$\frac{\dot{a}}{a} = H_0$$

$$\frac{da}{adt} = H_0$$

$$\frac{da}{a} = \frac{H_0}{dt}$$

$$\int \frac{da}{a} = \int H_0 \cdot dt$$

$$ln(a) = H_0 \int dt$$

$$ln(a) = H_0(t + constant)$$

consider time, assume it goes from t_0 to t

$$ln(a) = H_0(t|_{t_0}^{t})$$

$$ln(a) = H_0(t - t_0)$$
$$a = e^{ln(a)} = e^{H_0(t-t_0)}$$

This can be further manipulated using the definition of $H_0^2 = H^2 = 8\pi G\rho_\nu/3 = \Lambda/3$ to get

$$a_{(t)} = e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}$$

done... ... ... BUT WHAT IS THE U_{eff} OF THE MATTER-DOMINATED UNIVERSE???

Exercise Set	Status
1	maybe done, double check
2	maybe done, double check
3	maybe done, double check
4	maybe done, double check
5	nothing
6	maybe done, double check
7	maybe done, double check

Table 1: Solutions typed, and to be typed