

Buckle your seatbelt Dorothy, 'cause Kansas... is going bye-bye! — Cypher

1 Introduction

“In mathematics, a *matrix* is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns,” as discussed on Wikipedia¹. Matrices are widely used in linear algebra, the branch of mathematics concerning linear relationships.

This assignment is to write public class `FractionMatrix` implements `FractionMatrixI` to represent an immutable rectangular matrix of `Fractions`. The class includes methods for the basic operations of addition, scalar multiplication, transposition, and matrix multiplication, as well as mutually recursive methods for calculating the *determinant* and *cofactors* of a square matrix.

The following links provide additional explication.

- ▶ [http://en.wikipedia.org/wiki/Matrix_\(mathematics\)#Basic_operations](http://en.wikipedia.org/wiki/Matrix_(mathematics)#Basic_operations) — Basic matrix operations.
- ▶ http://en.wikipedia.org/wiki/Invertible_matrix#Analytic_solution — Matrix inversion through calculation of the adjugate matrix.
- ▶ http://en.wikipedia.org/wiki/Adjugate_matrix — Calculation of the adjugate matrix through the transpose of the cofactor matrix.
- ▶ https://en.wikipedia.org/wiki/Determinant#Laplace_expansion — Laplace expansion and the adjugate matrix.
- ▶ http://en.wikipedia.org/wiki/Laplace_expansion — Calculation of the determinant by Laplace's expansion by cofactors.
- ▶ <https://www.khanacademy.org/math/linear-algebra/matrix-transformations/inverse-of-matrices/v/linear-algebra-3x3-determinant> — Kahn Academy 3x3 example of Laplace's expansion.

2 Laplace expansion

An $n \times n$ (square) matrix \mathbf{A} is *invertible* if there exists \mathbf{A}^{-1} such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad (2.1)$$

where \mathbf{I}_n is the *identity matrix* (an $n \times n$ matrix with ones on the main diagonal and zeros everywhere else).

If it exists, the inverse matrix \mathbf{A}^{-1} can be calculated by:

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{\text{comatrix}(\mathbf{A})^T}{\det(\mathbf{A})} \quad (2.2)$$

The cofactor matrix (or *comatrix*) of \mathbf{A} is the matrix of cofactors of \mathbf{A} , where the element in the i th row and j th column is given by:

$$C_{ij}(\mathbf{A}) = \det(\text{minor}_{ij}(\mathbf{A}))(-1)^{i+j} \quad (2.3)$$

The minor matrix of \mathbf{A} for an $n \times n$ matrix is the $(n-1) \times (n-1)$ matrix created by removing row i and column j from \mathbf{A} . (Note: $n > 1$ or the minor matrix is undefined.)

In general, for an $n \times n$ matrix:

$$\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} C_{ij} \quad \text{expansion on row } i \quad (2.4)$$

$$= \sum_{i=1}^n a_{ij} C_{ij} \quad \text{expansion on column } j \quad (2.5)$$

Because the calculation of the determinant by Laplace's expansion by cofactors is a mutually recursive algorithm (the determinant is defined in terms of the cofactor and the cofactor is defined in terms of the determinant of a smaller matrix)

¹As is usual with Wikipedia, the pages on mathematics and computer science are good sources of reliable information

it requires base cases.

$$C_{ij}([a]) = 1 \quad \text{for a } 1 \times 1 \text{ matrix} \quad (2.6)$$

$$\det([a]) = a \quad \text{for a } 1 \times 1 \text{ matrix} \quad (2.7)$$

Base case (2.7) is a consequence of base case (2.6) and definitions (2.4) or (2.5) when there is only one row and one column. For a 1×1 matrix $\mathbf{A} = [a]$:

$$\det(\mathbf{A}) = a_{11}C_{11} = (a)(1) = a \quad (2.8)$$

For a 2×2 matrix, the formula for the determinant is usually given as:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc \quad \text{for a } 2 \times 2 \text{ matrix} \quad (2.9)$$

The general formulas (2.4) and (2.5) for the determinant yield the same result as (2.9) for expansion on any row or column.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = +a \det([d]) - b \det([c]) \quad \text{row 1} \quad (2.10)$$

$$= -c \det([b]) + d \det([a]) \quad \text{row 2} \quad (2.11)$$

$$= +a \det([d]) - c \det([b]) \quad \text{column 1} \quad (2.12)$$

$$= -b \det([c]) + d \det([a]) \quad \text{column 2} \quad (2.13)$$

$$= ad - bc \quad (2.14)$$

3 Example

It is useful to explore Laplace's expansion through an example. Given the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \quad (3.1)$$

²(3.6) is an example of where it is advantageous to calculate along a row or column with zeros, because there is no need to calculate the cofactor for the zero elements. That is a potential code optimization.

$$\text{adj}(\mathbf{A}) = \begin{bmatrix} + \begin{vmatrix} 5 & 6 \\ 8 & 0 \end{vmatrix} & - \begin{vmatrix} 4 & 6 \\ 7 & 0 \end{vmatrix} & + \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ - \begin{vmatrix} 2 & 3 \\ 8 & 0 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 7 & 0 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \\ + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix}^T \quad (3.2)$$

$$= \begin{bmatrix} -48 & +42 & -3 \\ +24 & -21 & +6 \\ -3 & +6 & -3 \end{bmatrix}^T \quad (3.3)$$

$$= \begin{bmatrix} -48 & +24 & -3 \\ +42 & -21 & +6 \\ -3 & +6 & -3 \end{bmatrix} \quad (3.4)$$

Then calculate $\det(\mathbf{A})$ along the first row of matrix \mathbf{A} and comatrix $\text{adj}(\mathbf{A})$ using (3.1) and (3.3):

$$\det(\mathbf{A}) = (1 \times -48) + (2 \times +42) + (3 \times -3) = 27 \quad (3.5)$$

Or, calculate $\det(\mathbf{A})$ along the third column²:

$$\det(\mathbf{A}) = (3 \times -3) + (6 \times +6) + 0 = 27 \quad (3.6)$$

Using (2.2) yields:

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{48}{27} & +\frac{24}{27} & -\frac{3}{27} \\ +\frac{42}{27} & -\frac{21}{27} & +\frac{6}{27} \\ -\frac{3}{27} & +\frac{6}{27} & -\frac{3}{27} \end{bmatrix} \quad (3.7)$$

4 Invertible matrix

An invertible $n \times n$ matrix \mathbf{A} has interesting properties.

► $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$, e.g.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \begin{bmatrix} -\frac{48}{27} & +\frac{24}{27} & -\frac{3}{27} \\ +\frac{42}{27} & -\frac{21}{27} & +\frac{6}{27} \\ -\frac{3}{27} & +\frac{6}{27} & -\frac{3}{27} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.1)$$

► The columns of \mathbf{A} (and \therefore the rows of \mathbf{A}^T) are linearly in-

dependent³.

- ▶ The rows of \mathbf{A} (and \therefore the columns of \mathbf{A}^T) are linearly independent⁴.
- ▶ $\mathbf{A}\vec{x} = \vec{b}$ has exactly one solution for each \vec{b} in \mathbb{R}^n ⁵.

5 Determinant

The determinant of $n \times n$ matrix \mathbf{A} has interesting properties.

- ▶ $\det(\mathbf{A}) \neq 0 \iff \mathbf{A}$ is invertible
- ▶ $\det(\mathbf{A}^T) = \det(\mathbf{A})$
- ▶ $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = \det(\mathbf{A})^{-1}$ ($\det(\mathbf{A}) \neq 0$)
- ▶ $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ (\mathbf{A}, \mathbf{B} are $n \times n$)
- ▶ $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$

Appendix

This is the FractionMatrixI.java interface file, including all the methods to be implemented in FractionMatrix.java.

```
1  /*
2  * FractionMatrixI.java
3  *
4  * Interface for a matrix of fractions. http://www.thematrix101.com/
5  *
6  * @author David C. Petty // http://j.mp/psb_david_petty
7  */
8
9  public interface FractionMatrixI
10 {
11     int numRows(); // number of rows
```

```
12     int numberColumns(); // number of columns
13     boolean isSquare(); // true if matrix is square
14     int getDimension(); // dimension of square matrix
15     Fraction get(int row, int col); // get element at [row][col]
16     FractionMatrix scalarMultiply(Fraction scalar); // matrix multiplied by scalar
17     FractionMatrix add(FractionMatrix that); // matrix sum of this with that
18     FractionMatrix multiply(FractionMatrix that); // matrix product of this with that
19     FractionMatrix transpose(); // matrix transpose
20     FractionMatrix getMinor(int row, int col); // minor matrix removing row and col
21     Fraction cofactor(int row, int col); // cofactor for row and col
22     Fraction determinant(); // matrix determinant
23     FractionMatrix adjugate(); // matrix adjugate
24     FractionMatrix inverse(); // matrix inverse
25 }
```

Listing 1: FractionMatrixI.java

This is the FractionI.java interface file, including all the methods to be implemented in Fraction.java.

```
1  /*
2  * FractionI.java
3  *
4  * Interface for Fraction.
5  *
6  * @author David C. Petty // http://j.mp/psb_david_petty
7  */
8
9  public interface FractionI
10 {
11     // instance methods
12     int getNumerator();
13     int getDenominator();
14     Fraction add(Fraction f);
15     Fraction add(int n);
16     Fraction multiply(Fraction f);
17     Fraction multiply(int n);
18     double doubleValue();
19
20     // instance methods to add
21     Fraction negate();
22     Fraction subtract(Fraction f);
23     Fraction subtract(int n);
24     Fraction reciprocal();
25     Fraction divide(Fraction f);
26     Fraction divide(int n);
27 }
```

Listing 2: FractionI.java

³ \therefore if $T(\vec{x}) = \mathbf{A}\vec{x}$:

$$\mathbf{A} = \begin{bmatrix} \uparrow & \dots & \uparrow \\ \vec{x}_1 & \dots & \vec{x}_n \\ \downarrow & \dots & \downarrow \end{bmatrix} \implies \text{im}(\mathbf{A}) = \text{span}\{\vec{x}_1, \dots, \vec{x}_n\} = \mathbb{R}^n \quad (4.2)$$

⁴ \therefore if $T(\vec{x}) = \mathbf{A}\vec{x}$:

$$\mathbf{A} = \begin{bmatrix} \leftarrow & \vec{x}_1 & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \vec{x}_n & \rightarrow \end{bmatrix} \implies \text{im}(\mathbf{A}) = \text{span}\{\vec{x}_1, \dots, \vec{x}_n\} = \mathbb{R}^n \quad (4.3)$$

⁵ \therefore to solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 18 \\ 4x_1 + 5x_2 + 6x_3 = 36 \\ 7x_1 + 8x_2 = -9 \end{cases}$ for \vec{x} where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 18 \\ 36 \\ -9 \end{bmatrix}$:

$$\mathbf{A}\vec{x} = \vec{b} \quad (4.4)$$

$$\mathbf{A}^{-1}\mathbf{A}\vec{x} = \mathbf{A}^{-1}\vec{b} \quad (4.5)$$

$$\mathbf{I}_n\vec{x} = \mathbf{A}^{-1}\vec{b} \quad (4.6)$$

$$\vec{x} = \mathbf{A}^{-1}\vec{b} = \begin{bmatrix} -\frac{48}{27} & +\frac{24}{27} & -\frac{3}{27} \\ +\frac{42}{27} & -\frac{21}{27} & +\frac{6}{27} \\ -\frac{3}{27} & +\frac{6}{27} & -\frac{3}{27} \end{bmatrix} \begin{bmatrix} 18 \\ 36 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \text{ is the solution, because } \begin{cases} (1) + 2(-2) + 3(7) = 18 \\ 4(1) + 5(-2) + 6(7) = 36 \\ 7(1) + 8(-2) = -9 \end{cases} \quad (4.7)$$