`Buckle your seatbelt Dorothy, 'cause Kansas... is going bye-bye!` — Cypher

1 Introduction

"In mathematics, a *matrix* is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns," as discussed on Wikipedia[1]. Matrices are widely used in linear algebra, the branch of mathematics concerning linear relationships.

This assignment is to write `public class FractionMatrix implements FractionMatrixI` to represent an immutable rectangular matrix of `Fraction`s. The class includes methods for the basic operations of addition, scalar multiplication, transposition, and matrix multiplication, as well as mutually recursive methods for calculating the *determinant* and *cofactors* of a square matrix.

The following links provide additional explication.

- http://en.wikipedia.org/wiki/Matrix_(mathematics)#Basic_operations — Basic matrix operations.
- http://en.wikipedia.org/wiki/Invertible_matrix#Analytic_solution — Matrix inversion through calculation of the adjugate matrix.
- http://en.wikipedia.org/wiki/Adjugate_matrix — Calculation of the adjugate matrix through the transpose of the cofactor matrix.
- https://en.wikipedia.org/wiki/Determinant#Laplace_expansion — Laplace expansion and the adjugate matrix.
- http://en.wikipedia.org/wiki/Laplace_expansion — Calculation of the determinant by Laplace's expansion by cofactors.
- https://www.khanacademy.org/math/linear-algebra/matrix-transformations/inverse-of-matrices/v/linear-algebra-3x3-determinant — Kahn Academy 3x3 example of Laplace's expansion.

2 Laplace expansion

An $n \times n$ (square) matrix $\mathbf{A}$ is *invertible* if there exists $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I_n} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{2.1}$$

where $\mathbf{I_n}$ is the *identity matrix* (an $n \times n$ matrix with ones on the main diagonal and zeros everywhere else).

If it exists, the inverse matrix $\mathbf{A}^{-1}$ can be calculated by:

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{\text{comatrix}(\mathbf{A})^T}{\det(\mathbf{A})} \tag{2.2}$$

The cofactor matrix (or *comatrix*) of $\mathbf{A}$ is the matrix of cofactors of $\mathbf{A}$, where the element in the ith row and jth column is given by:

$$C_{ij}(\mathbf{A}) = \det(\text{minor}_{ij}(\mathbf{A}))(-1)^{i+j} \tag{2.3}$$

The minor matrix of $\mathbf{A}$ for an $n \times n$ matrix is the $n—1 \times n—1$ matrix created by removing row i and colums j from $\mathbf{A}$. (Note: $n > 1$ or the minor matrix is undefined.)

In general, for an an $n \times n$ matrix:

$$det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij}C_{ij} \qquad \text{expansion on row } i \tag{2.4}$$

$$= \sum_{i=1}^{n} a_{ij}C_{ij} \qquad \text{expansion on column } j \tag{2.5}$$

Because the calculation of the determinant by Laplace's expansion by cofactors is a mutually recursive algorithm (the determinant is defined in terms of the cofactor and the cofactor is defined in terms of the determinant of a smaller matrix)

[1] As is usual with Wikipedia, the pages on mathematics and computer science are good sources of reliable information

it requires base cases.

$$C_{ij}\left(\begin{bmatrix}a\end{bmatrix}\right) = 1 \qquad \text{for a } 1\times 1 \text{ matrix} \tag{2.6}$$
$$\det\left(\begin{bmatrix}a\end{bmatrix}\right) = a \qquad \text{for a } 1\times 1 \text{ matrix} \tag{2.7}$$

Base case (2.7) is a consequence of base case (2.6) and definitions (2.4) or (2.5) when there is only one row and one column. For a 1×1 matrix $\mathbf{A} = \begin{bmatrix}a\end{bmatrix}$:

$$\det(\mathbf{A}) = a_{11}C_{11} = (a)(1) = a \tag{2.8}$$

For a 2×2 matrix, the formula for the determinant is usually given as:

$$\det\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = ad - bc \qquad \text{for a } 2\times 2 \text{ matrix} \tag{2.9}$$

The general formulas (2.4) and (2.5) for the determinant yield the same result as (2.9) for expansion on any row or column.

$$\det\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = +a\det\left(\begin{bmatrix}d\end{bmatrix}\right) - b\det\left(\begin{bmatrix}c\end{bmatrix}\right) \qquad \text{row 1} \tag{2.10}$$
$$= -c\det\left(\begin{bmatrix}b\end{bmatrix}\right) + d\det\left(\begin{bmatrix}a\end{bmatrix}\right) \qquad \text{row 2} \tag{2.11}$$
$$= +a\det\left(\begin{bmatrix}d\end{bmatrix}\right) - c\det\left(\begin{bmatrix}b\end{bmatrix}\right) \qquad \text{column 1} \tag{2.12}$$
$$= -b\det\left(\begin{bmatrix}c\end{bmatrix}\right) + d\det\left(\begin{bmatrix}a\end{bmatrix}\right) \qquad \text{column 2} \tag{2.13}$$
$$= ad - bc \tag{2.14}$$

3 Example

It is useful to explore Laplace's expansion through an example. Given the matrix:

$$\mathbf{A} = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 0\end{bmatrix} \tag{3.1}$$

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} +\begin{vmatrix}5 & 6\\ 8 & 0\end{vmatrix} & -\begin{vmatrix}4 & 6\\ 7 & 0\end{vmatrix} & +\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix} \\ -\begin{vmatrix}2 & 3\\ 8 & 0\end{vmatrix} & +\begin{vmatrix}1 & 3\\ 7 & 0\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 7 & 8\end{vmatrix} \\ +\begin{vmatrix}2 & 3\\ 5 & 6\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 4 & 6\end{vmatrix} & +\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}\end{bmatrix}^T \tag{3.2}$$
$$= \begin{bmatrix}-48 & +42 & -3\\ +24 & -21 & +6\\ -3 & +6 & -3\end{bmatrix}^T \tag{3.3}$$
$$= \begin{bmatrix}-48 & +24 & -3\\ +42 & -21 & +6\\ -3 & +6 & -3\end{bmatrix} \tag{3.4}$$

Then calculate $\det(\mathbf{A})$ along the first row of matrix $\mathbf{A}$ and $\operatorname{comatrix}(\mathbf{A})$ using (3.1) and (3.3):

$$\det(\mathbf{A}) = (1\times -48) + (2\times +42) + (3\times -3) = 27 \tag{3.5}$$

Or, calculate $\det(\mathbf{A})$ along the third column[2]:

$$\det(\mathbf{A}) = (3\times -3) + (6\times +6) + 0 = 27 \tag{3.6}$$

Using (2.2) yields:

$$\boxed{\mathbf{A}^{-1} = \begin{bmatrix}-\frac{48}{27} & +\frac{24}{27} & -\frac{3}{27}\\ +\frac{42}{27} & -\frac{21}{27} & +\frac{6}{27}\\ -\frac{3}{27} & +\frac{6}{27} & -\frac{3}{27}\end{bmatrix}} \tag{3.7}$$

4 Invertible matrix

An *invertible* $n\times n$ matrix $\mathbf{A}$ has interesting properties.

- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I_n}$, *e.g.*

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 0\end{bmatrix}\begin{bmatrix}-\frac{48}{27} & +\frac{24}{27} & -\frac{3}{27}\\ +\frac{42}{27} & -\frac{21}{27} & +\frac{6}{27}\\ -\frac{3}{27} & +\frac{6}{27} & -\frac{3}{27}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{4.1}$$

- The columns of $\mathbf{A}$ (and $\therefore$ the rows of $\mathbf{A}^T$) are linearly in-

[2] (3.6) is an example of where it is advantageous to calculate along a row or column with zeros, because there is no need to calculate the cofactor for the zero elements. That is a potential code optimization.

dependent[3].

- The rows of $\mathbf{A}$ (and $\therefore$ the columns of $\mathbf{A}^T$) are linearly independent[4].
- $\mathbf{A}\vec{x} = \vec{b}$ has exactly one solution for each $\vec{b}$ in $\mathbb{R}^n$ [5].

5 Determinant

The determinant of $n \times n$ matrix $\mathbf{A}$ has interesting properties.

- $\det(\mathbf{A}) \neq 0 \iff \mathbf{A}$ is invertible
- $\det(\mathbf{A}^T) = \det(\mathbf{A})$
- $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = \det(\mathbf{A})^{-1}\ (\det(\mathbf{A}) \neq 0)$
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})\ (\mathbf{A}, \mathbf{B}$ are $n \times n)$
- $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$

Appendix

This is the `FractionMatrixI.java` interface file, including all the methods to be implemented in `FractionMatrix.java`.

```java
/*
 * FractionMatrixI.java
 *
 * Interface for a matrix of fractions. http://www.thematrix101.com/
 *
 * @author David C. Petty // http://j.mp/psb_david_petty
 */

public interface FractionMatrixI
{
    int numberRows();                               // number of rows
    int numberColumns();                            // number of columns
    boolean isSquare();                             // true if matrix is square
    int getDimension();                             // dimension of square matrix
    Fraction get(int row, int col);                 // get element at [row][col]
    FractionMatrix scalarMultiply(Fraction scalar); // matrix multiplied by scalar
    FractionMatrix add(FractionMatrix that);        // matrix sum of this with that
    FractionMatrix multiply(FractionMatrix that);   // matrix product of this with that
    FractionMatrix transpose();                     // matrix transpose
    FractionMatrix getMinor(int row, int col);      // minor matrix removing row and col
    Fraction cofactor(int row, int col);            // cofactor for row and col
    Fraction determinant();                         // matrix determinant
    FractionMatrix adjugate();                      // matrix adjugate
    FractionMatrix inverse();                       // matrix inverse
}
```

Listing 1: `FractionMatrixI.java`

This is the `FractionI.java` interface file, including all the methods to be implemented in `Fraction.java`.

```java
/*
 * FractionI.java
 *
 * Interface for Fraction.
 *
 * @author David C. Petty // http://j.mp/psb_david_petty
 */

public interface FractionI
{
    // instance methods
    int getNumerator();
    int getDenominator();
    Fraction add(Fraction f);
    Fraction add(int n);
    Fraction multiply(Fraction f);
    Fraction multiply(int n);
    double doubleValue();

    // instance methods to add
    Fraction negate();
    Fraction subtract(Fraction f);
    Fraction subtract(int n);
    Fraction reciprocal();
    Fraction divide(Fraction f);
    Fraction divide(int n);
}
```

Listing 2: `FractionI.java`

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[3] $\therefore$ if $T(\vec{x}) = \mathbf{A}\vec{x}$:

$$\mathbf{A} = \begin{bmatrix} \uparrow & \dots & \uparrow \\ \vec{x}_1 & \dots & \vec{x}_n \\ \downarrow & \dots & \downarrow \end{bmatrix} \implies \mathrm{im}(A) = \mathrm{span}\left\{\vec{x}_1, \dots, \vec{x}_n\right\} = \mathbb{R}^n \tag{4.2}$$

[4] $\therefore$ if $T(\vec{x}) = \mathbf{A}\vec{x}$:

$$\mathbf{A} = \begin{bmatrix} \leftarrow & \vec{x}_1 & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \vec{x}_n & \rightarrow \end{bmatrix} \implies \mathrm{im}(A) = \mathrm{span}\left\{\vec{x}_1, \dots, \vec{x}_n\right\} = \mathbb{R}^n \tag{4.3}$$

[5] $\therefore$ to solve $\left\{\begin{array}{rrrcr} x_1 & +2x_2 & +3x_3 & = & 18 \\ 4x_1 & +5x_2 & +6x_3 & = & 36 \\ 7x_1 & +8x_2 & & = & -9 \end{array}\right\}$ for $\vec{x}$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 18 \\ 36 \\ -9 \end{bmatrix}$:

$$\mathbf{A}\vec{x} = \vec{b} \tag{4.4}$$

$$\mathbf{A}^{-1}\mathbf{A}\vec{x} = \mathbf{A}^{-1}\vec{b} \tag{4.5}$$

$$\mathbf{I}_n\vec{x} = \mathbf{A}^{-1}\vec{b} \tag{4.6}$$

$$\vec{x} = \mathbf{A}^{-1}\vec{b} = \begin{bmatrix} -\frac{48}{27} & +\frac{24}{27} & -\frac{3}{27} \\ +\frac{42}{27} & -\frac{21}{27} & +\frac{6}{27} \\ -\frac{3}{27} & +\frac{6}{27} & -\frac{3}{27} \end{bmatrix} \begin{bmatrix} 18 \\ 36 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \text{ is the solution, because } \left\{\begin{array}{rrrcr} (1) & +2(-2) & +3(7) & = & 18 \\ 4(1) & +5(-2) & +6(7) & = & 36 \\ 7(1) & +8(-2) & & = & -9 \end{array}\right\}. \tag{4.7}$$

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