An optical ray-tracer

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Abstract

A computer model coded in Python was set up to investigate ray-tracing. Lenses with both spherical and planar surfaces were added to the model. The phenomenon of spherical aberration was investigated for ray bundles of varying radius by looking at the geometric focus (the root mean square) of the ray positions at the paraxial focus, and comparing this to the theoretically predicted diffraction limit. The performance of a biconvex lens was also investigated with respect to minimising the geometric focus. Finally, reflection and dispersion were also added to the model.

Introduction

Ray tracing is an important technique in the field of computer graphics, able to generate an image with a high degree of realism by recursively tracing the path individual rays take through a system. Hence an understanding of the techniques involved is well worth having.

Setup / Method

In creating the model, several mathematical methods had to be looked at.

Firstly, interception with a sphere. A sphere with radius R centred at $\boldsymbol{O}$ has vector equation $|\boldsymbol{x}-\boldsymbol{O}|^2=R^2$, and a line starting at point $\boldsymbol{P}$ with direction vector $\boldsymbol{k}$ has vector equation $\boldsymbol{x}=\boldsymbol{P}+l\,\boldsymbol{k}$, where l is the distance from $\boldsymbol{P}$ along the line.

Combining these leads us to solve the equation $|\boldsymbol{P}+l\,\boldsymbol{k}-\boldsymbol{O}|^2=R^2$:

$$(\boldsymbol{P}+l\,\boldsymbol{k}-\boldsymbol{O}).(\boldsymbol{P}+l\,\boldsymbol{k}-\boldsymbol{O})=R^2$$
$$l^2(\boldsymbol{k}.\boldsymbol{k})+2l(\boldsymbol{k}.(\boldsymbol{P}-\boldsymbol{O}))+(\boldsymbol{P}-\boldsymbol{O}).(\boldsymbol{P}-\boldsymbol{O})-R^2=0$$

A quadratic may be observed: $a\,l^2+b\,l+c=0$ where $a=\boldsymbol{k}.\boldsymbol{k}$, $b=2(\boldsymbol{k}.(\boldsymbol{P}-\boldsymbol{O}))$ and $c=(\boldsymbol{P}-\boldsymbol{O}).(\boldsymbol{P}-\boldsymbol{O})-R^2$.

Solving this gives:

$$l=\frac{-(\boldsymbol{k}.(\boldsymbol{P}-\boldsymbol{O}))\pm\sqrt{(\boldsymbol{k}.(\boldsymbol{P}-\boldsymbol{O}))^2-\boldsymbol{k}^2((\boldsymbol{P}-\boldsymbol{O})^2-R^2)}}{\boldsymbol{k}^2}.$$

Because $\boldsymbol{k}$ is a unit vector, the denominator may be omitted.

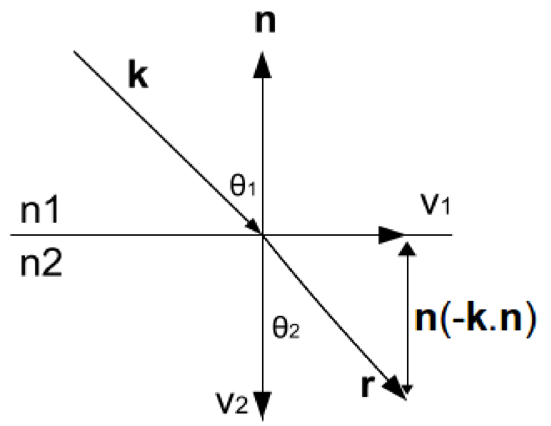

Fig. 1: Refraction using Snell's Law.

Secondly, application of Snell's Law in 3D. From Fig. 1, a 2D view of the plane containing the incident and normal unit vectors, it can be seen that $\boldsymbol{r}=v_2\cos\theta_2+v_1\sin\theta_2$, where v_1 and v_2 are unit vectors perpendicular and antiparallel to the normal respectively. It follows that $v_2=-\boldsymbol{n}$ since $\boldsymbol{n}$ is already normalised, and $v_1=\frac{\boldsymbol{k}+\boldsymbol{n}(-\boldsymbol{k}.\boldsymbol{n})}{|\boldsymbol{k}+\boldsymbol{n}(-\boldsymbol{k}.\boldsymbol{n})|}$.

Knowing that $\boldsymbol{n}(-\boldsymbol{k}.\boldsymbol{n})=\cos\theta_1\boldsymbol{n}$, the equation then becomes $\boldsymbol{r}=-\boldsymbol{n}\cos\theta_2+\frac{\boldsymbol{k}+\cos\theta_1\boldsymbol{n}}{|\boldsymbol{k}+\cos\theta_1\boldsymbol{n}|}\sin\theta_2$. We can simplify this greatly using Snell's Law, $\sin\theta_2=\eta\sin\theta_1$ where $\eta=\frac{n_1}{n_2}$, and other steps.

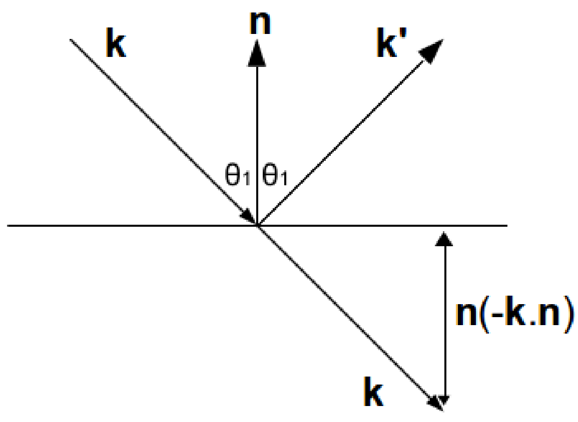

Fig. 2: Reflection

It can be seen that $\boldsymbol{k}+\cos\theta_1\boldsymbol{n}=\boldsymbol{k}\sin\theta_1$, hence $|\boldsymbol{k}+\cos\theta_1\boldsymbol{n}|=\sin\theta_1$. Using trig identities and Snell's Law, $\cos\theta_2=\sqrt{1-\sin^2\theta_2}=\sqrt{1-\eta^2\sin^2\theta_1}=\sqrt{1-\eta^2(1-\cos^2\theta_1)}$.

The final equation to solve is thus $\boldsymbol{r}=\eta\boldsymbol{k}+(\eta\cos\theta_1-\sqrt{1-\eta^2(1-(\cos\theta_1)^2)})\boldsymbol{n}$. This is numerically easy to calculate as $\cos\theta_1=-\boldsymbol{n}.\boldsymbol{k}$, which is a simple dot product.[1]

Finally, reflection in 3D. This follows the same principles as refraction but is significantly easier to solve, as it is obvious from a cursory examination of Fig. 2 that $k'=k+2\boldsymbol{n}(-\boldsymbol{k}.\boldsymbol{n})$, or $k'=k-2\boldsymbol{n}(\boldsymbol{k}.\boldsymbol{n})$.

Results / Discussion:

Lenses were modeled as spherical surfaces centred on the z axis with a refractive index of 1.5168 (crown glass). The intercept between the surface and the z axis (in mm), and the curvature of the surface (in mm^{-1})were given as parameters, with positive curvature representing a convex surface and negative curvature representing a concave one. Zero curvature was treated as a special case representing a plane. A circular bundle of rays was passed through the lens and their paths were traced. Fig.s 3 and 4 above show the convex and concave lens respectively behaving as expected.

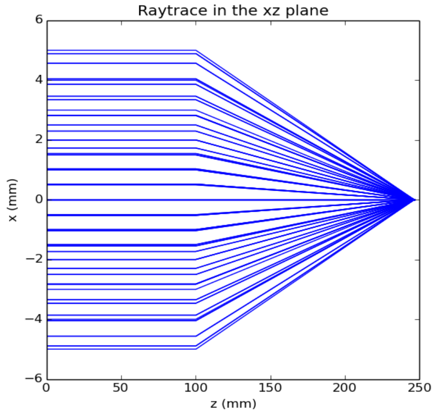

Fig. 3: Lens with z_0 = 100, curvature = 0.02 and n_2 = 1.5168

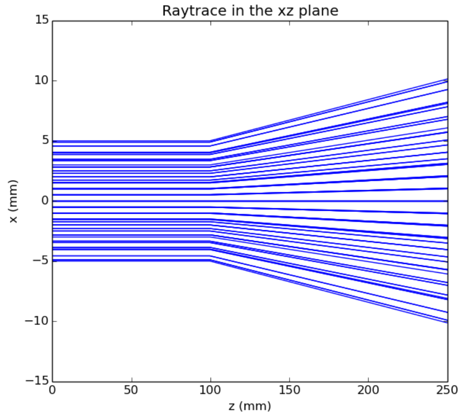

Fig. 4: Lens with z_0 = 100, curvature = -0.02 and n_2 = 1.5168

All spherical lenses exhibit a property called spherical aberration. This is where incoming rays are not focused at a single point, but spread out along the optical axis. This is because a ray will be refracted more when it intercepts a lens nearer its edge. This was investigated by finding the paraxial focus of the system under investigation (seeing where a ray a short distance of 0.1mm from the optical axis was focused) and plotting the intercepts of all rays with this plane, called a spot diagram. A measure of quality of the lens is the size of the spot diagram (or geometrical focus), as a larger size means a greater degree of spherical aberration. The metric chosen to quantify the geometrical focus was the root mean square (RMS) of the distances between the intercepts and the optical axis.

A planoconvex lens of width 5mm, refractive index 1.5168 and having curvature 0.02 on one side was investigated with both the plane side facing the incoming rays and then the convex side facing the rays. The paraxial focus was found, the geometrical focus was plotted and the RMS was calculated for each of these configurations.

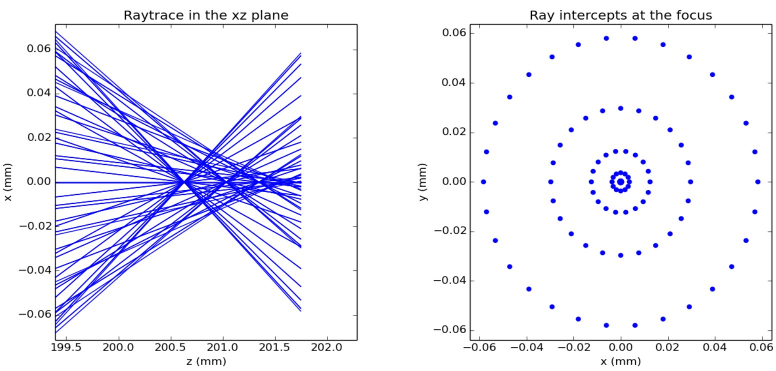

Fig. 5: Planoconvex lens, plane side first

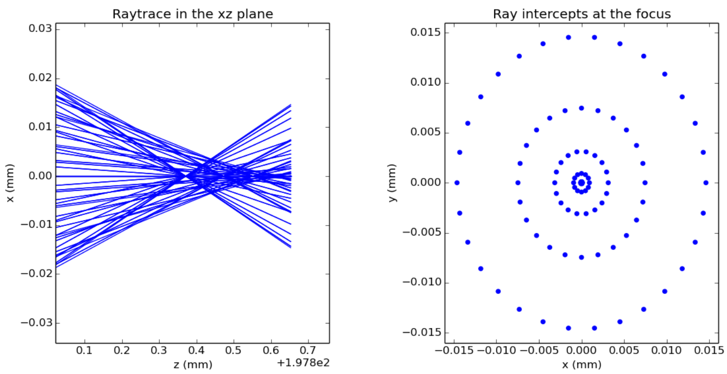

Fig. 6: Planoconvex lens, convex side first

Fig.s 5 and 6 show this lens plane side and convex side first respectively. The raytrace is zoomed in close to the paraxial focus so the spherical aberration phenomenon may be more easily observed. The RMS was calculated, and found to be 0.0372mm for the plane side first and 0.0093mm for the convex side first. Hence the planoconvex lens performs better when it is placed convex side first.

The performance of these lenses in the model was compared to the theoretical predictions of the geometrical focus size for light waves (not rays!). This is given by the Airy Pattern, calculated by $r \sim \frac{\lambda f}{D}$ [2], where λ is the wavelength of the light (588nm in this case), *f* is the focal length (the paraxial focus) and *D* is the aperture diameter (the width of the input beam).

The RMS of the geometrical focus and the Airy Pattern size were plotted for ray bundles of varying radii for both orientations of the planoconvex lens.

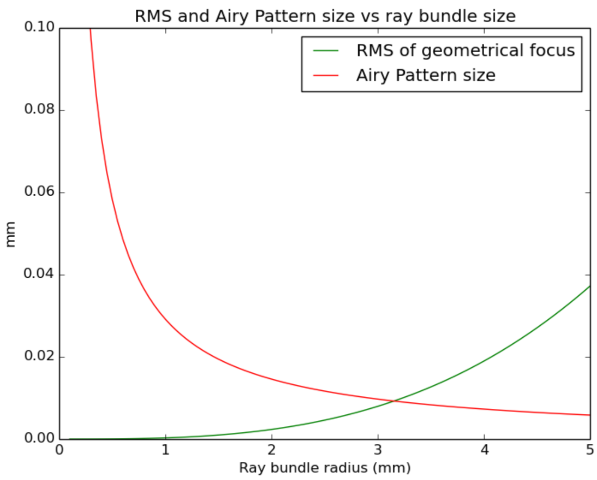

Fig. 7: RMS and Airy Pattern size for plane side first

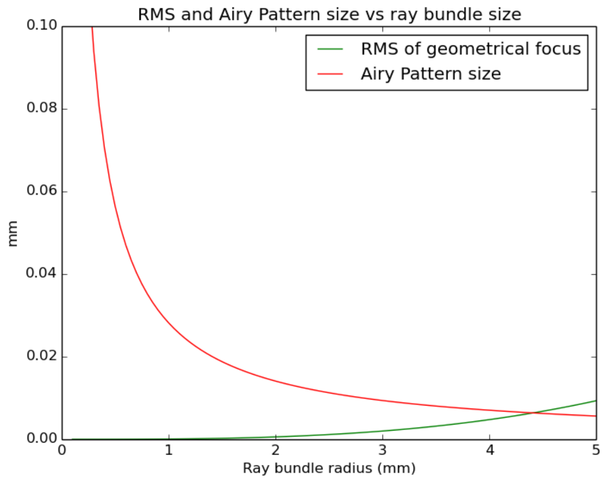

Fig. 8: RMS and Airy Pattern side for convex side first

Fig. 7 shows the RMS increases at a faster rate for the planoconvex lens with its plane side first than it does for the convex side first, as we expect. However, these figures illustrate another phenomenon of interest: the diffraction limit. This is the smallest distance over which features can be distinguished before they blur together, and for a circular aperture is given by the size of the Airy Pattern.[3]

This shows that for radii below the intercepts on Fig.s 7 and 8, the model is not ideal for predicting the behaviour of light waves (as it predicts a focus smaller than the theoretical limit). At the intercepts, the systems are 'diffraction limited' – that is, they predict a focus as good as the theoretical diffraction limit. Above the intercepts, the model can be used to predict the behaviour of light waves, so for general investigation a ray bundle radius above these intercepts must be used to produce valid observations, and indeed all ray bundles used except for in this section had a radius of 5mm.

In optics, a lens is often designed so that its curvatures produce the smallest possible spherical aberration for given object and image distances. This was modeled for a biconvex lens by optimising a function that accepted the two curvatures as parameters and returned the RMS at a chosen distance (150mm along the optical axis). The function scipy.optimize.fmin_tnc was used. Below are the results returned and the RMS obtained for various initial estimates of the best curvatures.

Lens 1 Initial Curv.	Lens 2 Initial Curv.	Lens 1 Returned Curv.	Lens 2 Returned Curv.	Optimised RMS (mm)
0.03	-0.03	0.03500101	-0.00539009	0.03492858
0.025	-0.025	0.03335925	-0.0072387	0.03523673
0.02	-0.02	0.03508153	-0.0052989	0.03492812
0.015	-0.015	0.03500202	-0.00538853	0.03492791
0.01	-0.01	0.03163932	-0.00915682	0.03617317
0.005	-0.005	0.03502108	-0.00536686	0.03492755

The optimisation function performs well, with all the optimised RMS values being just over 0.001 mm from each other. It appears that for object image at -∞ and image distance at 47.5mm, the best form lens is biconvex with curvatures of 0.03502108mm^{-1} and $-0.00536686\text{mm}^{-1}$.

Reflecting surfaces were added to the model, and found to work. Fig.s 9 and 10 show a convex and concave mirror respectively.

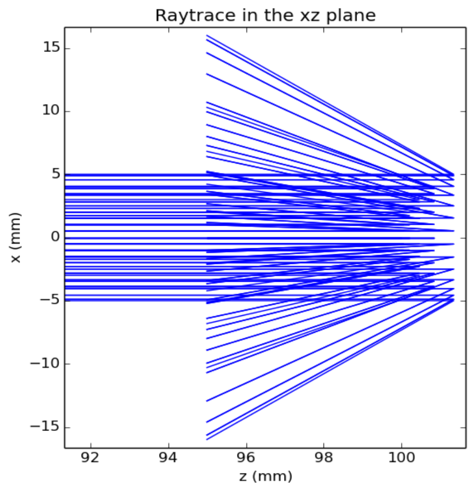

Fig. 9: Mirror with z_0 = 100 and curvature = 0.1

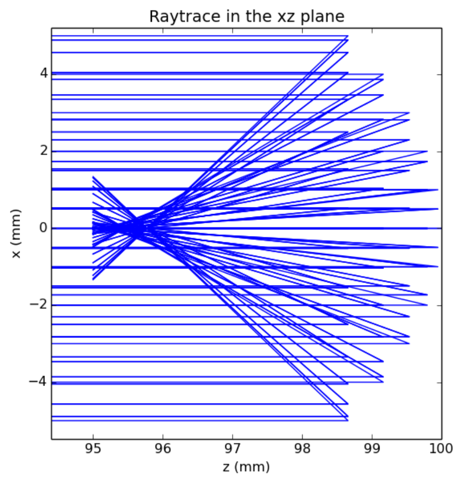

Fig. 10: Mirror with z_0 = 100 and curvature = -0.1

Finally, dispersion was also added to the model, and found to work to a degree. The Sellmeier equation was used to calculate the refractive index of the glass based upon the wavelength, with constants chosen for BK7 glass:

$$n=\sqrt{1+\frac{1.03961212\times\lambda^2}{\lambda^2-0.00600069867}+\frac{0.231792344\times\lambda^2}{\lambda^2-0.0200179144}+\frac{1.01046945\times\lambda^2}{\lambda^2-103.560653}}$$ [4]

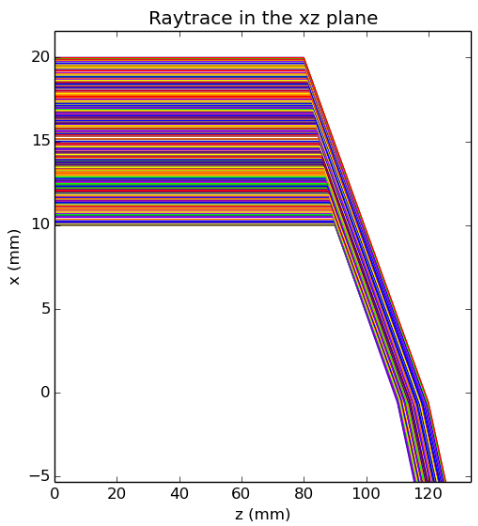

Fig. 11: Rainbow bundle, prism view

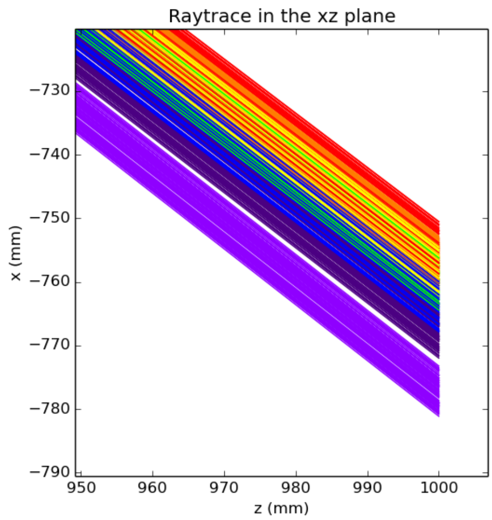

Fig. 12: Rainbow bundle, output view

A special bundle of rays, each one being given a random wavelength corresponding to the colours of the rainbow[5], was passed through a prism, to show the effects of dispersion. A rainbow-like pattern can clearly be seen at the output, though it is not as clear as was hoped.

Conclusion

In this lab, the performance of planoconvex and biconvex lenses was investigated. It was found that a planoconvex lens performs best at minimising spherical aberration when the convex side is first. In addition, a best form biconvex lens was designed using an optimisation function.

Future work could include the complete simulation of a rainbow, intensity of reflected/refracted light, the modeling of different objects etc.

Bibliography

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4: 'http://www.refractiveindex.info/?group=GLASSES&material=BK7', accessed 16/2/14
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